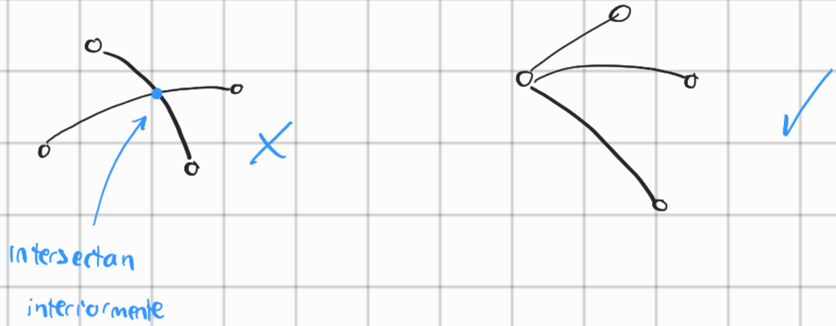
Definición. Un grafo (simple) G es llamado plano si tiene por lo menos una representación en un plano de tal manera que cualesquiera dos aristas no se intersecten interiormente

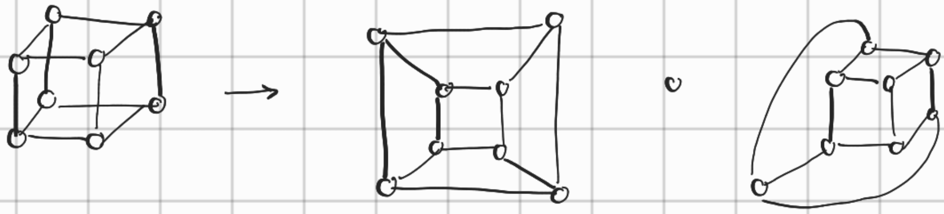


1) K_4 (completo de 4 vértices)

representación plana

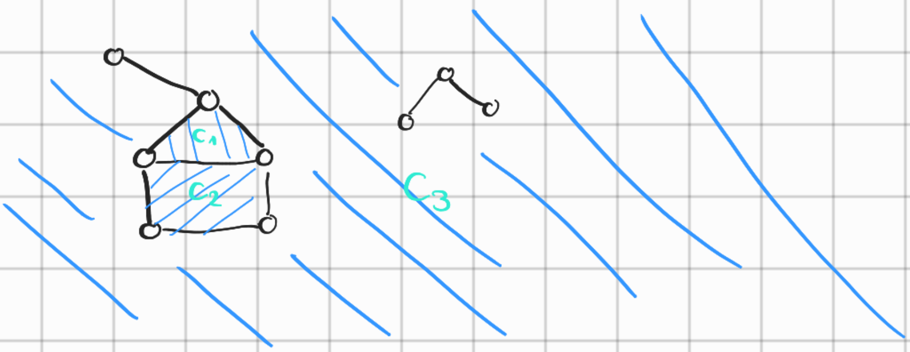
$\therefore$ K_4 es plano.

2) Q_3



$\therefore$ Q_3 es plano

Definición. Dada una representación plana, definimos una cara como la región del plano que está limitada por las aristas. La región exterior también se considera como una cara.



Teorema (Fórmula de Euler) Sea G un grafo conexo que tiene una representación plana con C caras, V vértices y A aristas. Se cumple que $C + V = A + 2$.

En consecuencia, todas las representaciones del mismo grafo conexo tienen el mismo número de caras.

Prueba. Por inducción en el número de aristas, para el caso base $A = 0$ tenemos

$V = 1$ y $C = 1$ y $C + V = 1 + 1 = 0 + 2 = A + 2$.

Supongamos que ya está demostrado para representaciones planas de grafos conexos que tienen k aristas.

Consideramos la representación plana de un grafo G conexo que tiene $k+1$ aristas.

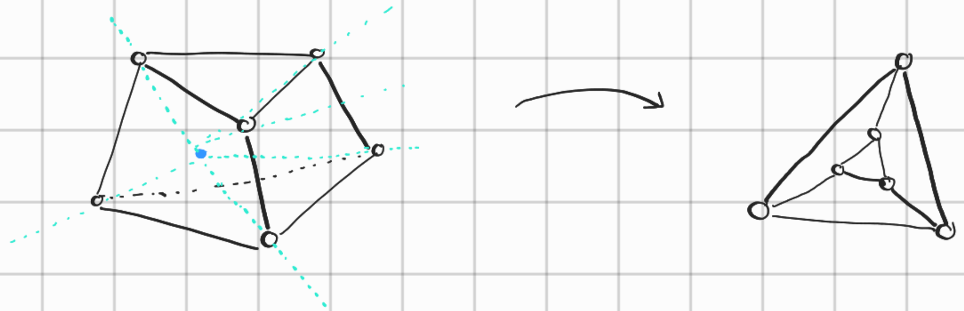
Si G es un árbol, entonces $C = 1$, $V = k+2$ y cumple $1 + (k+2) = (k+1) + 2$.

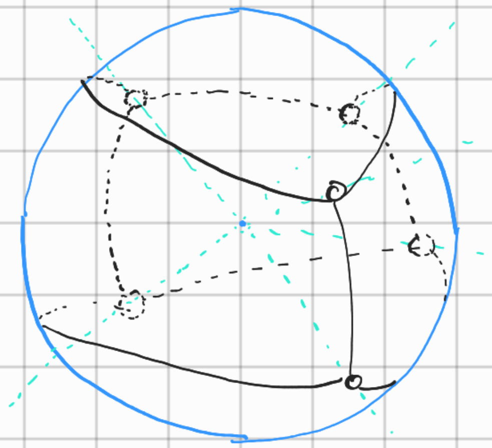
Si G no es un árbol, entonces contiene por lo menos un ciclo.

Consideremos $G - \{a\}$ que es conexo y tiene k aristas. Además, tiene exactamente una cara menos. La fórmula también se cumple. ■

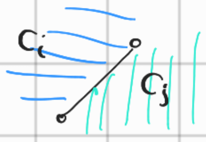
Observaciones:

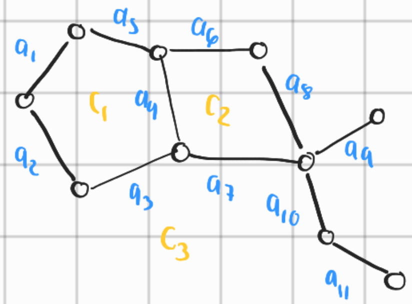
- Si el grafo tiene n componentes conexas, se cumple que $C + V = A + n + 1$.
- La fórmula de Euler también se puede aplicar a poliedros convexos.





Teorema. Si una representación plana de un grafo conexo con al menos tres vértices tiene A aristas y C caras, se cumple que $2A \geq 3C$.

Ejemplo.  i puede ser igual a j



C_1 : $a_1 a_2 a_3 a_4 a_5$

C_2 : $a_4 a_7 a_8 a_6$

C_3 : $a_3 a_7 a_{10} a_{11} a_{11} a_{10} a_9 a_9 a_8 a_6 a_5 a_1 a_2$

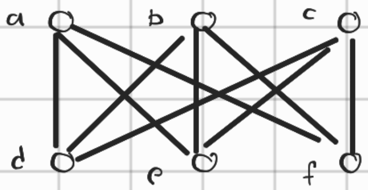
Cada arista aparece exactamente dos veces, y cada cara está rodeada por al menos 3 aristas, luego $2A \geq 3C$.

Corolario. Si además de las condiciones anteriores, se cumple que el grafo no contiene triángulos, tenemos la desigualdad $A \geq 2C$.

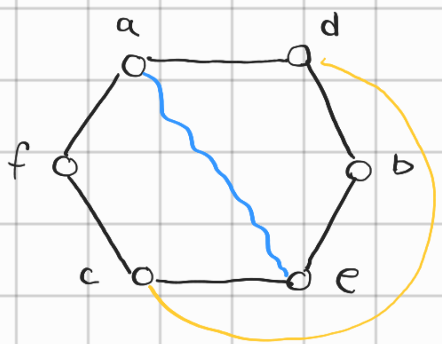
Corolario. El grafo $K_{3,3}$ no es plano.

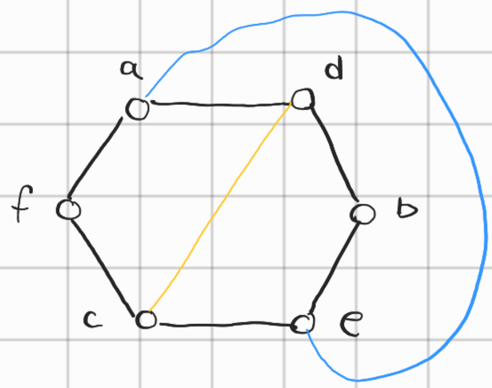
Prueba 1. Sabemos que $A = 9$ y $V = 6$. Supongamos que $K_{3,3}$ es plano. Por el teorema de Euler, $C = 5$. Por otro lado, como $K_{3,3}$ es bipartito, $A = 9 < 10 = 2C$ lo cual es imposible. Así, $K_{3,3}$ no es plano. ■

Prueba 2.



Primero, el ciclo adbecf siempre tiene que estar. Coloquemos las aristas ae y cd.

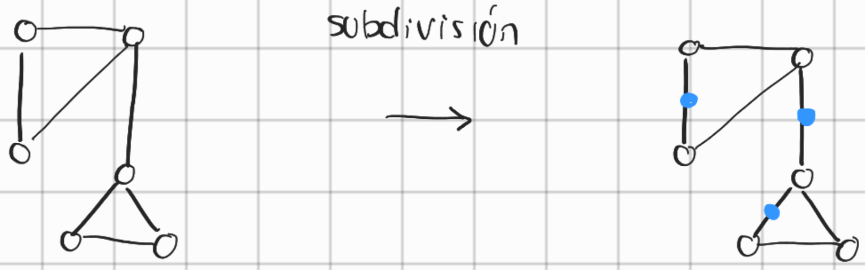


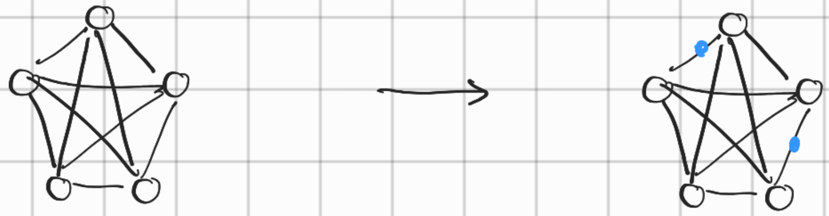


Luego no podemos colocar fb sin intersectar alguna de la aristas previamente colocadas. ■

<u>Corolario</u>. El grafo K_5 no es plano.

Prueba. Sabemos que $A = 10$ y $V = 5$. Supongamos que K_5 es plano. Por el teorema de Euler, $C = 7$. Por otro lado, $2A = 20 < 21 = 3C$ lo cual es imposible. Así, K_5 no es plano. ∎





<u>Teorema</u>. (Kuratowski) Un grafo no es plano si y solo si contiene como subgrafo a una subdivisión de $K_{3,3}$ o K_5.

MATH 228. Kuratowski's Theorem Carnegie Mellon University Mary Radcliffe

1 Introduction

<u>Proposition 1</u>. $K_{3,3}$ is not planar.

Proof. Done.

<u>Proposition 2</u>. K_5 is not planar.

Proof. Done.

2 Subdivisions and Subgraphs

Let us formally define this as a subdivision, as follows.

A graph H is said to be a subdivision of a graph G if H can be obtained from G by successively deleting an edge in G, and replacing that edge with a length 2 path (whose central vertex was not originally part of G). An edge that has been removed and replaced with a length 2 path is said to be subdivided in H.

Lemma 1. Let G be a graph. Then G is planar if and only if every subdivision of G is planar.

That is to say, the act of subdividing a graph does not change the planarity of the graph at all, since the fundamental shape has not change.
We note also here that quite trivially, if we have a planar graph, and we take a subgraph, it too must be planar. Indeed, we can simply take the original graph, embed it in the plane, and then remove any edges or vertices not present in the subgraph to produce a plane drawing of the desired subgraph. This is, certainly, a very trivial property.

Lemma 2. Let G be a planar graph. Then every subgraph of G is also planar.

3 Kuratowski's Theorem: Setup

Note that one direction here is made trivial by the lemmas presented in the previous section. Indeed, if G contains a nonplanar subgraph, then Lemma 2 immediately implies that G is nonplanar. But by the discussion in the introduction, we also know that $K_{3,3}$ and K_5 are nonplanar, so if G contains either of these, it should be nonplanar.
This is the crux of the theorem; the only way to be nonplanar is to have one of these two known bad shapes as a subgraph.

A cut-vertex in a graph G is a vertex v such that $G \setminus \{v\}$ has more components than G itself. That is, it is a vertex whose removal disconnects some part of the graph that used to be connected.

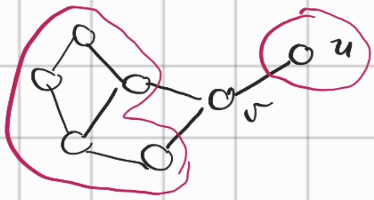
A block in a graph G is a subgraph B of G such that B has no cut vertices, but if we add any other vertex to B, it does have cut vertices (that is, it is a maximal subgraph in G having no cut vertices). In this way, we can view any graph G as being built of blocks, that are simply pasted together at cut-point.

A graph is called 2-connected if it is connected and has no cut-vertices. We can think of 2-connected as "if you want to disconnect it, you'll have to take away 2 things".

Lemma 3. Let G be a 2-connected graph and u, v vertices of G. Then there exists a cycle in G that includes both u and v.

Proof. We will prove this by induction on the distance between u and v.

First, note that the smallest distance is 1, which can be achieved only if u is adjacent to v. Suppose this is the case. Notice that u cannot have degree 1, since otherwise, it must be that v is a cut vertex.



Hence, u must have another neighbor in G, say w.

Let us consider the graph $G \setminus \{u\}$. Notice that this graph is still connected, by the definition of 2-connectedness, and hence, there exists a path in $G \setminus \{u\}$ between u and v. Moreover, this path cannot use the vertex u.

Adding u to this path on both ends in G creates a cycle in G, that contains both u and v.

Now, let us suppose the result is known for any u, v having distances at most $d-1$. Let $u, v \in V(G)$ have distance exactly d. Let Q be a path of length d between u and v. Take w to be the point in the path Q adjacent to v. Note that $d(u, w) = d-1$, so there exist a cycle C in G that contains both u and w.

If v is a member of this cycle, then we are done, as we have a cycle that contains both u and v.

If not, then v appears outside the cycle. Moreover, since $G \setminus \{w\}$ is connected, there exist a path P between u and v that does not include the vertex w. We note this path can contain vertices of the cycle C, just not the vertex w.