AR1_Random_Walk

What is a Random Walk?

A *random walk* is a time series process where each value is the sum of the previous value and a random noise term. It can be described mathematically as:

$X_t = X_{t-1} + \epsilon_t$

Here: - (X_t): The value at time (t), - (X_{t-1}): The value at time (t-1), - (ϵ_t): A random noise term, typically assumed to follow a normal distribution with mean 0 and variance (σ^2).

Key characteristics of a random walk:

1. *Non-Stationarity*: A random walk is not stationary because its mean and variance change over time. The variance grows with time, making it unpredictable in the long run.
2. *Memory*: Each step depends directly on the previous value, so the process exhibits strong persistence over time.

AR(1) Model

The *AR(1)* model is defined as:

$X_t = \phi * 1 * X_{t-1} + \epsilon_t$

Here: - ($\phi * 1$): Autoregressive coefficient, which determines the influence of ($X * t - 1$) on (X_t). - (ϵ_t): White noise.

The AR(1) process is stationary if ($|\phi_1| < 1$). For ($|\phi_1| \geq 1$), the process becomes non-stationary.

How ($\phi_1 = 1$) Makes AR(1) a Random Walk

When ($\phi_1 = 1$), the AR(1) model becomes:

$X_t = X_{t-1} + \epsilon_t$

This is exactly the definition of a random walk. In this case: - Each value (X_t) is the previous value (X_{t-1}) plus a random noise term. - The time series lacks a tendency to revert to a mean, as there is no dampening factor to bring values back to a central level.

Key Implications of ($\phi_1 = 1$):

1. *Non-Stationarity*:
   - For ($|\phi_1| < 1$), the AR(1) process is stationary and has constant variance.
   - For ($\phi_1 = 1$), the variance of (X_t) grows linearly with time: $\text{Var}(X_t) = t \cdot \sigma^2$ This makes the process non-stationary.
2. *Unbounded Growth*:
   - As time progresses, the values of (X_t) can drift arbitrarily far from their starting point due to the cumulative effect of (ϵ_t).
3. *Lack of Mean Reversion*:
   - In a stationary AR(1) process (($|\phi_1| < 1$)), the series tends to revert to its mean. For ($\phi_1 = 1$), there is no mean reversion, as the process has no equilibrium level.

Visualization of Random Walk vs Stationary AR(1):

1. *Random Walk* ($\phi_1 = 1$):
   - The series drifts up or down without a predictable pattern.
   - Variance increases over time, and the path depends heavily on the noise term.
2. *Stationary AR(1)* ($|\phi_1| < 1$):
   - The series oscillates around a fixed mean with a constant variance.
   - Noise contributes to deviations, but the process is "pulled back" to the mean by the coefficient (ϕ_1).

Understanding this relationship is crucial for time series modeling, as it helps determine whether a process needs differencing or detrending to achieve stationarity before analysis or forecasting.