Mean-Field-Theory with Different Synaptic Time-Constants

Alexander van Meegen

We consider a LIF neuron in population α driven by input from multiple presynaptic populations β mediated by receptors with different time constants:

$$\tau_\alpha \dot{V}_\alpha = -(V_\alpha - V_L^\alpha) + \frac{\tau_\alpha}{C_\alpha} \sum_\beta I_{\alpha\beta}, \tag{1}$$

$$\tau_{\alpha\beta} \dot{I}_{\alpha\beta} = -I_{\alpha\beta} + \tau_{\alpha\beta} \sum_i J_i^{\alpha\beta} x_i^\beta(t). \tag{2}$$

Here, τ_α are the membrane time constants, C_α the membrane capacitances, and $\tau_{\alpha\beta}$ the synaptic time constants. Normally, $\tau_{\alpha\beta}$ only depends on β because the synaptic time constant is a property of the receptor, not the neuron; for generality, we keep $\tau_{\alpha\beta}$.

First, we shift $V_\alpha - V_L^\alpha \to V_\alpha$ and rescale the currents $\frac{\tau_\alpha}{C_\alpha} I_\alpha \to I_\alpha$ such that they are expressed in units of voltage,

$$\tau_\alpha \dot{V}_\alpha = -V_\alpha + \sum_\beta I_{\alpha\beta}, \tag{3}$$

$$\tau_{\alpha\beta} \dot{I}_{\alpha\beta} = -I_{\alpha\beta} + \tau_\alpha \frac{\tau_{\alpha\beta}}{C_\alpha} \sum_i J_i^{\alpha\beta} x_i^\beta(t). \tag{4}$$

Next, we define $\frac{\tau_{\alpha\beta}}{C_\alpha} J_{\alpha\beta} = \tilde{J}_{\alpha\beta}$ and approximate the input in mean-field approximation as a Gaussian white noise with mean and noise intensity

$$\mu_{\alpha\beta} = \tau_\alpha K_{\alpha\beta} \tilde{J}_{\alpha\beta} \nu_\beta, \tag{5}$$

$$\sigma_{\alpha\beta}^2 = \tau_\alpha K_{\alpha\beta} \tilde{J}_{\alpha\beta}^2 \nu_\beta. \tag{6}$$

This leads to the effective input currents

$$\tau_{\alpha\beta} \dot{I}_{\alpha\beta} = -I_{\alpha\beta} + \mu_{\alpha\beta} + \sigma_{\alpha\beta} \sqrt{\tau_\alpha} \xi_{\alpha\beta}(t), \tag{7}$$

where the $\xi_{\alpha\beta}$ are independent.

To incorporate the different input currents into one effective current, we follow [1, Sec. 5.5]. The currents have mean value $\langle I_{\alpha\beta} \rangle = \mu_{\alpha\beta}$, hence the total input current $I_\alpha = \sum_\beta I_{\alpha\beta}$ has mean $\langle I_\alpha \rangle = \sum_\beta \mu_{\alpha\beta}$. Their stationary variances are $\langle \Delta I_{\alpha\beta}^2 \rangle = \frac{\tau_\alpha}{2\tau_{\alpha\beta}} \sigma_{\alpha\beta}^2$, leading to the total variance $\langle \Delta I_\alpha^2 \rangle = \frac{\tau_\alpha}{2} \sum_\beta \frac{\sigma_{\alpha\beta}^2}{\tau_{\alpha\beta}}$. This can be rewritten as $\langle \Delta I_\alpha^2 \rangle = \frac{\tau_\alpha}{2\tilde{\tau}_\alpha} \sum_\beta \sigma_{\alpha\beta}^2$ with the effective synaptic time constant

$$\tilde{\tau}_\alpha = \frac{\sum_\beta \sigma_{\alpha\beta}^2}{\sum_\beta \frac{\sigma_{\alpha\beta}^2}{\tau_{\alpha\beta}}}, \tag{8}$$

compare [1, Eq. (5.49)]. Thus, the effective current $\tilde{I}$ obeying

$$\tilde{\tau}_\alpha \dot{\tilde{I}}_\alpha = -\tilde{I}_\alpha + \mu_\alpha + \sigma_\alpha \sqrt{\tau_\alpha} \xi_\alpha(t) \tag{9}$$

$$\mu_\alpha = \tau_\alpha \sum_\beta K_{\alpha\beta} \tilde{J}_{\alpha\beta} \nu_\beta \tag{10}$$

$$\sigma_\alpha^2 = \tau_\alpha \sum_\beta K_{\alpha\beta} \tilde{J}_{\alpha\beta}^2 \nu_\beta \tag{11}$$

has the same stationary, equal-time statistics as $I_\alpha = \sum_\beta I_{\alpha\beta}$ with the $I_{\alpha\beta}$ defined in Eq. (7).

For a neuron driven by the effective current Eq. (9),

$$\tau_\alpha \dot{V}_\alpha = -V_\alpha + \tilde{I}_\alpha, \tag{12}$$

we know the resulting rate from [1] if $\tilde{\tau}_\alpha < \tau_\alpha$. With the approximation $I_\alpha = \sum_\beta I_{\alpha\beta} \approx \tilde{I}_\alpha$, we kept the stationary equal-time statistics of the total current the same but slightly altered its temporal correlation structure.

Note that we can rewrite the effective synaptic time constant Eq. (8) as

$$\tilde{\tau}_\alpha = \frac{\tau_\alpha \sum_\beta K_{\alpha\beta} \tilde{J}^2_{\alpha\beta} \nu_\beta}{\tau_\alpha \sum_\beta \tilde{K}_{\alpha\beta} \tilde{J}^2_{\alpha\beta} \nu_\beta} \tag{13}$$

with $\tilde{K}_{\alpha\beta} = K_{\alpha\beta}/\tau_{\alpha\beta}$, i.e., we can reuse the algorithm for Eq. (11).

REFERENCES

[1] N. Fourcaud and N. Brunel, Neural Comput. **14**, 2057 (2002).