

#### BIOST 546: Machine Learning for Biomedical Big Data

Ali Shojaie

Lecture 12: Clustering - Part II Spring 2017

## Recap

- Clustering basics
- Hierarchical clustering (and bi-clustering)

## Today's Class

- K-means clustering
- Model based clustering
- Spectral clustering

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- In partition-based methods, data is partitioned into a number of a priori given clusters
- No need for intervention by analyst; however, number of clusters needs to be specified.

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So, we need to solve:

$$\min_{C_1, \dots, C_K} \sum_{k=1}^K \left\{ \frac{1}{|C_k|} \sum_{i, i' \in C_k} \sum_{j=1}^p (X_{ij} - X_{i'j})^2 \right\}$$

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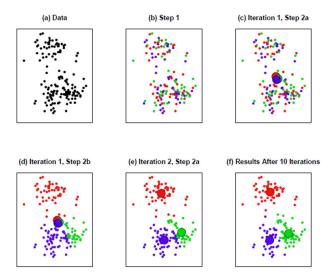
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    - (b) Assign each observation to the cluster with closest centroid (based on Euclidean distance).

## *K*-Means Clustering: An Example with Three Clusters



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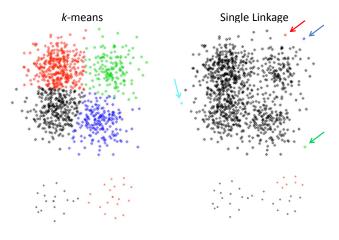
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- May result in artificially small clusters (a possible solution is to eliminate outliers)

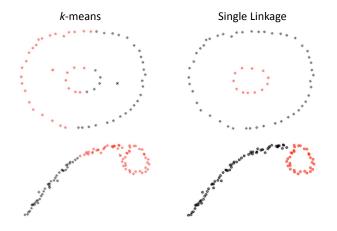
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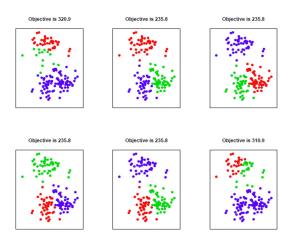


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#### Cluster separation

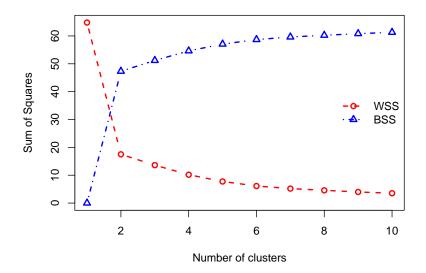
- Between sum of squares (BSS)
- ► For each cluster the "error" is the distance between the cluster centroid and the *grand mean*:

$$BSS = \sum_{k=1}^{K} d^2(m_k; m)$$



#### BSS vs WSS

For *K*-means,  $d^2(X_i; m_k) = \sum_{j=1}^p (X_{ij} - m_{kj})^2$  (Euclidean dist.)



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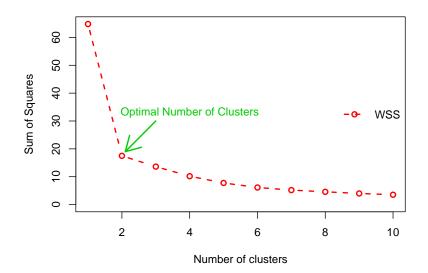
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- Therefore, need to look for an elbow in the plot of WSS over K

#### WSS for K-Means



## Other Methods for Choosing *K* (Number of Clusters)

 Gap Statistics (Tibshirani et al, 2001): Compares the curve of log WSS to the curve obtained from data uniformly distributed (i.e. no clusters), and estimates the optimal number of clusters to be the place where the gap between the two curves is largest.

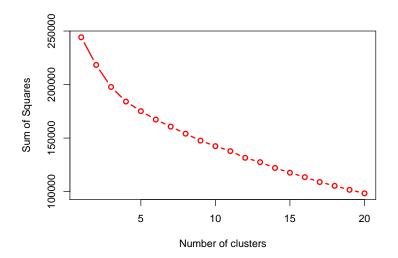
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- The Silhouette Coefficient (Rousseeuw, 1986): Combines homogeneity and separation, can find the best number of clusters by minimizing this coefficient over range of values of K.

# Example: K-Means Clustering on NCI60 Data

```
## K-means clustering for the NCI60 data
mvdata <- read.table('nci.data')</pre>
mydata <- t(mydata)
mydata <- scale(mydata, center=F, scale=T)</pre>
Kmax < -20
                         #maximum number of clusters
Nstrt <- 25
                         #Number of starting partitions
wss <- numeric(K)
bss <- numeric(K)
tss <- numeric(K)
for(k in 1:Kmax){
  cl <- kmeans(dat, k, nstart=Nstrt)</pre>
  tss[k] <- cl$totss
  wss[k] <- cl$tot.withinss
  bss[k] <- cl$betweenss
## Plot the results
par(mar=c(4.5,4.5,1,1))
plot(1:Kmax, wss, type='b', lty=1, lwd=2, col=2,
    vlab='Sum of Squares', xlab='Number of clusters')
```

### How Many Clusters?



#### **Exercise**

- Repeat the analysis without normalizing the data. What do you see?
- Repeat the clustering with different values of  $nstart \in \{1,5,10,50\}$
- How do the results change?
- What is the optimal number of clusters?
- Compare the optimal number of clusters with what you observed using hierarchical clustering.

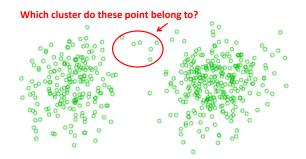
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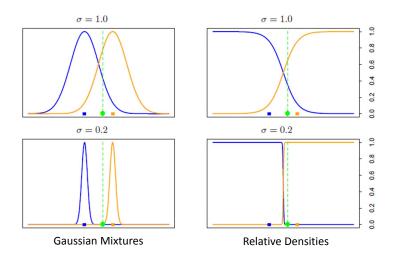
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- Function Mclust in the R-package mclust gives an implementation of this method



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- Similar ideas are used in the PageRank algorithm used by Google

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  - edges of G show positive similarities;
  - in other words, the similarity matrix is the adjacency matrix of the graph;
  - ideally the similarity graphs should represent the local neighborhood relationships.

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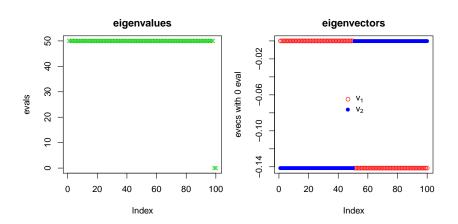
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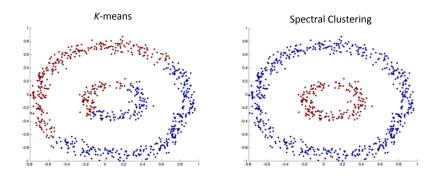
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- In practice, we find *approximate* partitioning of the graph by selecting *K* eigenvectors, and dropping the nodes corresponding to "small" entries

# Spectral Clustering



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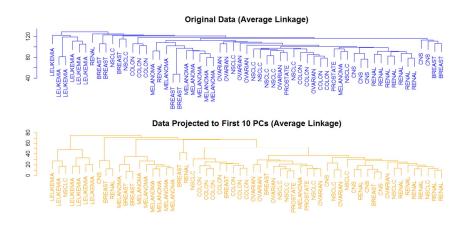
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- Basically, in this case, the affinity matrix is the sample covariance matrix, and eigenvectors are PCs
- The type of similarity graph (connected or nearest neighbors) as well as number of eigenvectors are important choices in spectral clustering.

#### NCI60 Data, again



The second approach is basically a version of spectral clustering, using the correlation matrix

Although many clustering algorithms are available, this is still a challenging problem:

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- Each clustering criterion imposes a certain structure on the data, and if the data happen to conform to the requirements of a particular criterion, the true clusters are recovered.
- It is very hard to evaluate how well a clustering algorithm performs on typical omics data sets.

### Caveats of Cluster Analysis

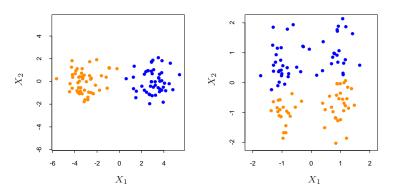
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- Clustering is a data exploration step, and its results should be interpreted that way; cannot make strong claims...