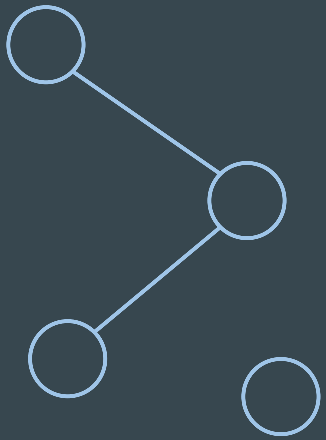
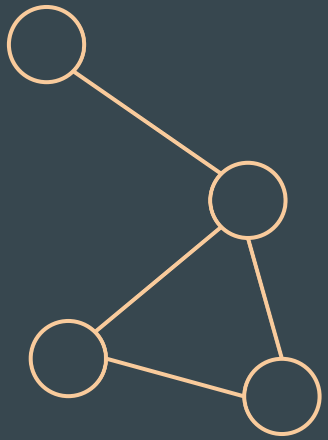
Graph Theory 4

Minimum Spanning Tree

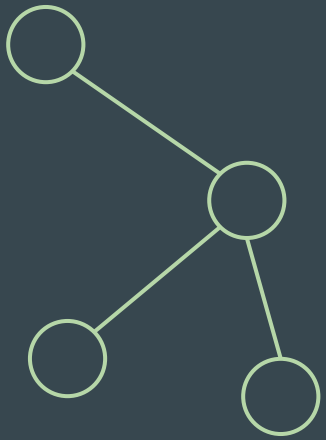
Reminder : what is a tree ?



acyclic graph



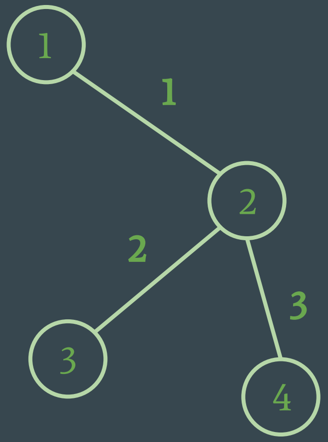
connected graph



tree = connected + acyclic

Some properties of trees...

- A tree has N-1 edges for N vertices
- A tree is a connected & acyclic graph
- Remove any edge from the tree and it disconnects the graph
- In the general case vertice can have any degree
    - Some trees are more specific : binary trees, k-trees



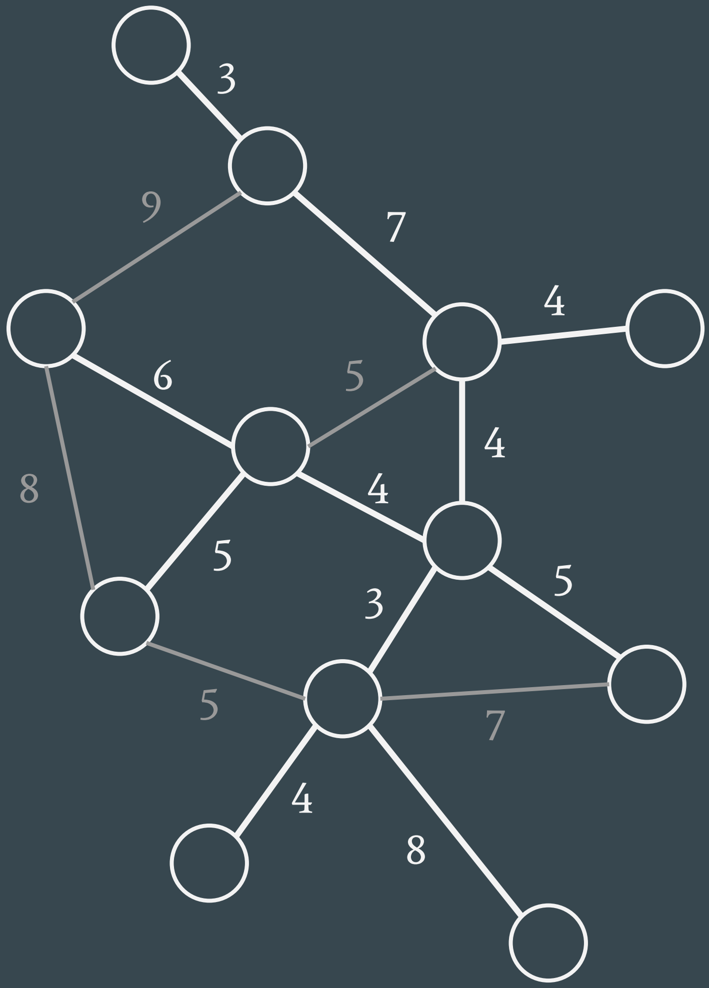
Minimum spanning tree

We are given a connected graph with weighted edges

→ what is the minimum cost to connect all the nodes?

(on this graph: 53)

/!\ Multiple MST are possible



Prim's algorithm

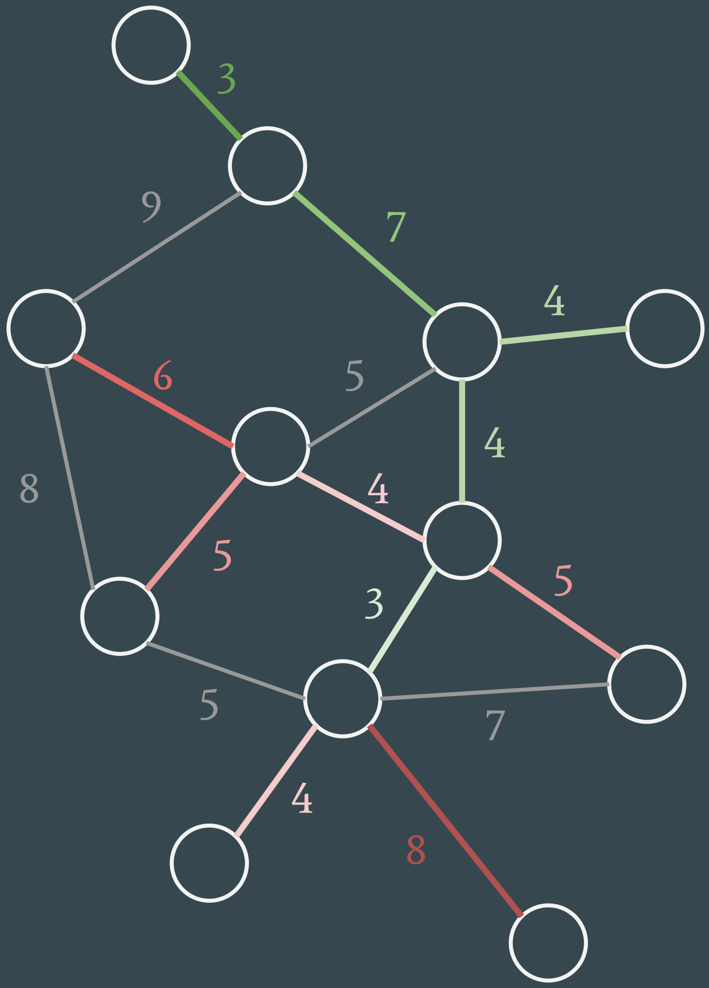
Greedy algorithm:

- put a random starting node in the tree
- while the tree doesn't have all the nodes:
  - select the node closest to the tree
  - add the corresponding edge in the tree

Implemented with a heap (remember Dijkstra last week ?)

→ Complexity : O((|V| + |E|) log |E|)
with a binary heap

(on the right, from green to red, order of addition in the tree)



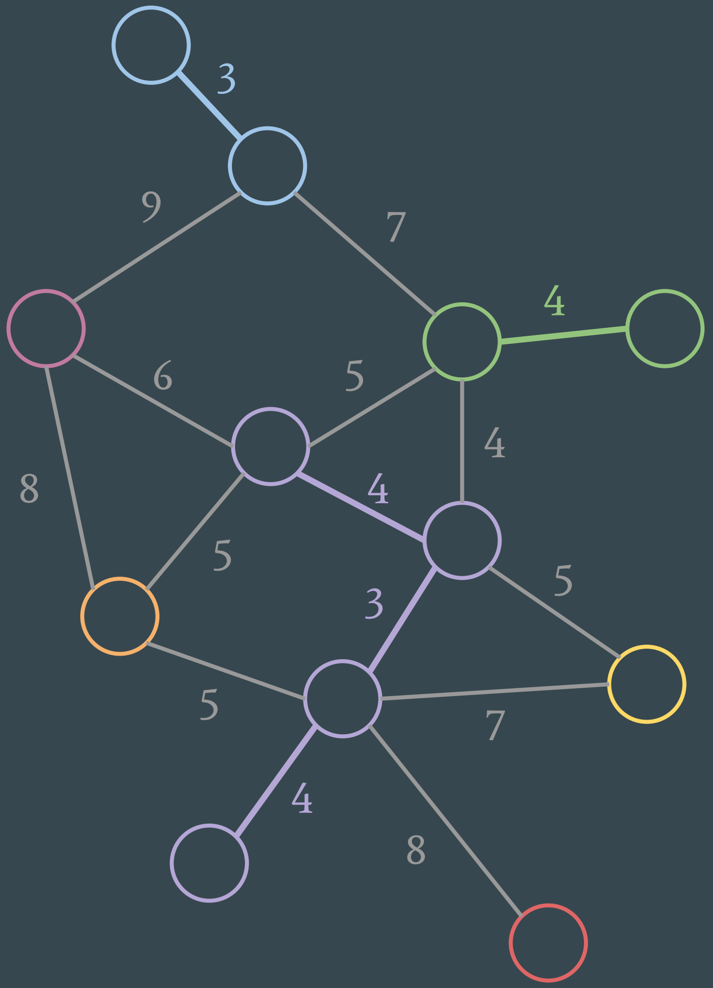
Kruskal's algorithm

Greedy algorithm:

- create a trivial forest (all nodes are alone)
- as long as we can, we use the smallest edge that creates a bridge between two trees

(on the right, example of a forest at some point during execution)

How to avoid making cycles ?

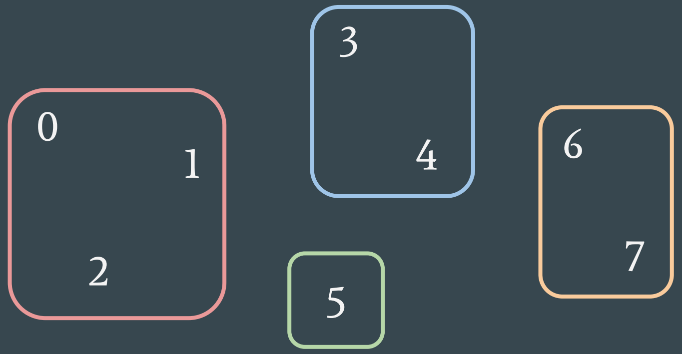


Disjoint sets (union-find d.s.)

We need a data structure to store disjoint sets with the following near-constant-time operations:

- add a new set
- merge two sets
- determine whether two elements are in the same set

Naive idea: let's use a list of sets



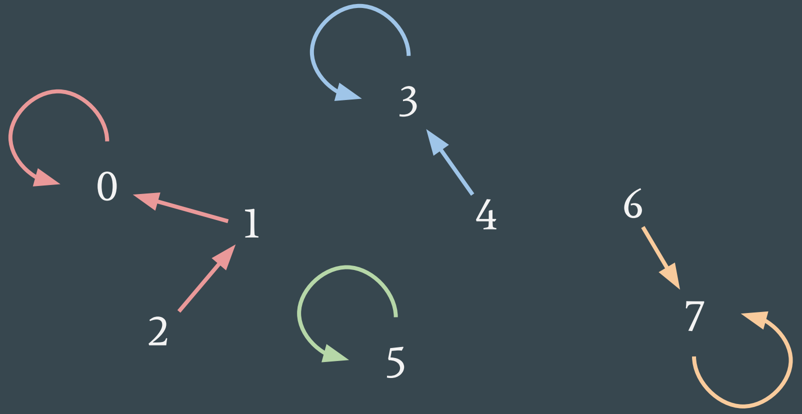
`[{0,1,2}, {3,4}, {5}, {6,7}]`

Add a new set	`O(1)`
Merge two sets	`O(n)`
Determine whether 2 elements are in the same set	`O(n)`

Disjoint sets (union-find d.s.)

Better idea: let's use a forest

- each node has a parent, the root of every tree is its own parent
- to merge two sets, point the root node of a set to any node of the other
  (you will usually use the *find* operation before, to check if the two sets are disjoint or the same)
- to determine whether 2 elements are in the same set, find the roots of their trees and compare them




```
{0:0, 1:0, 2:1, 3:3,
 4:3, 5:5, 6:7, 7:7}
```

Add a new set	$O(1)$
Merge two sets	$O(\|s_1\|+\|s_2\|)$
Determine whether 2 elements are in the same set	$O(\|s_1\|+\|s_2\|)$

Disjoint sets (union-find d.s.)

How to improve complexity?

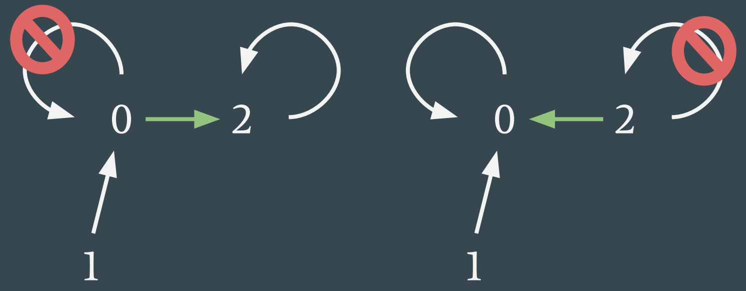
→ try to have shallow trees

→ this is achieved with ***union by rank*** (or *by size*) and ***path compression*** (or path *halving* or *splitting*)

Add a new set	`O(1)`
Merge two sets	`O(α(n))` *
Determine whether 2 elements are in the same set	`O(α(n))` *

* `α(n)` < 5 for any n that can be written in this physical universe, so that's basically O(1) (see inverse Ackermann function)

path compression on the path from a node to its root during a *find* operation



arbitrary union (left) VS *union by rank* (right)

Union-find in Python

- Bad news : this structure isn't implemented by default in Python
- Good news : Louis Sugy made a library for us 3 years ago ! (if lazy, just do pip install unionfind)
- Create a Union-find data structure with O(1) approximate complexities
- Use find(element) to get the root of an element
- Use union(e1, e2) to merge the sets containing e1 and e2
- Use is_same_set(e1, e2) to determine whether e1 and e2 are in the same set

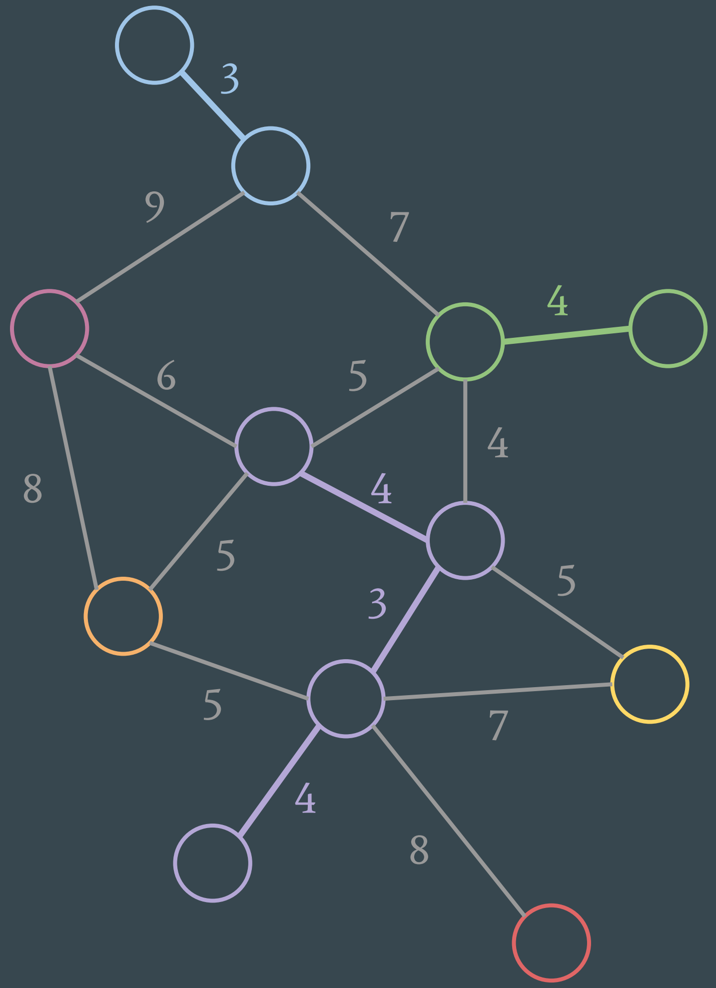
Back to Kruskal

Greedy algorithm:

- create a trivial forest (all nodes are alone)
- as long as we can, we use the smallest edge that creates a bridge between two trees

Complexity?

→ `O(|E| log |E|)` with an optimized Union-Find data structure and a heap to select the next edge



Note: Bernard Chazelle has found a `O(|E|α(m,n))` solution based on soft heaps, but that's a story for another time

Credits

Slides: Louis Sugy for INSAlgo

Edited in 2022 by Goll Sebastien and in 2023 by Lecorché Adriaan

Thanks to:

- Cormen, Thomas H.; Leiserson, Charles E.; Rivest, Ronald L.; Stein, Clifford (2009) [1990]. *Introduction to Algorithms* (3rd ed.), chapter 23
- Wikipedia