# Examples of PIFs

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## 1 Degenerated

Let $P(X = k) = 1$ for some $k > 0$. Let $R(x;\theta) \geq 1$ y $f(x) > 0$ arbitrary . The Population Impact Fraction is:

$$\text{PIF} = 1 - \frac{E\left[R\left(f(x);\theta\right)\right]}{E\left[R(x;\theta)\right]} = 1 - \frac{R\left(f(k);\theta\right)}{R\left(k;\theta\right)} \tag{1}$$

## 2 Multivariate normal

Let $X \sim N_k(\mu, \Sigma)$ normal multivariate of dimension $k$. Let $R(X;\theta) = e^{\theta^T X}$ and $f(X) = AX + b$. The Population Impact Fraction is:

$$\text{PIF} = 1 - \frac{E\left[R\left(f(x);\theta\right)\right]}{E\left[x;\theta\right]} = 1 - \frac{\overbrace{\frac{1}{(2\pi)^k\sqrt{|\Sigma|}}\int\limits_{\mathcal{X}} R\left(f(x);\theta\right) e^{-\frac{(x-\mu)^T\Sigma^{-1}(x-\mu)}{2}}dx}^{\text{N}}}{\underbrace{\frac{1}{(2\pi)^k\sqrt{|\Sigma|}}\int\limits_{\mathcal{X}} R\left(x;\theta\right) e^{-\frac{(x-\mu)^T\Sigma^{-1}(x-\mu)}{2}}dx}_{\text{D}}} \tag{2}$$

where

$$D = \frac{1}{(2\pi)^k\sqrt{|\Sigma|}}\int\limits_{\mathcal{X}} R\left(x;\theta\right) e^{-\frac{(x-\mu)^T\Sigma^{-1}(x-\mu)}{2}}dx = e^{\mu^T\theta + \frac{1}{2}\theta^T\Sigma\theta} \tag{3}$$

on the other hand:

$$
\begin{aligned}
N &= \frac{1}{(2\pi)^k \sqrt{|\Sigma|}} \int_{\mathcal{X}} e^{\theta^T f(x)} e^{-\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}} dx \\
&= \frac{e^{\theta^T b}}{(2\pi)^k \sqrt{|\Sigma|}} \int_{\mathcal{X}} e^{\theta^T A X + b} e^{-\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}} dx \\
&= e^{(A\mu)^T \theta + \frac{1}{2}\theta^T A \Sigma A^T \theta + \theta^T b}
\end{aligned} \tag{4}
$$

Substituting in (2) we obtain:

$$
PIF = 1 - \frac{e^{\mu^T \theta + \frac{1}{2}\theta^T \Sigma \theta}}{e^{(A\mu)^T \theta + \frac{1}{2}\theta^T A \Sigma A^T \theta + \theta^T b}} \tag{5}
$$

# 3 Exponencial

Let $X \sim \mathrm{Exp}(\lambda)$ y $R(X;\theta) = 1 + \sum\limits_{i=1}^{n} \theta_1 x^i$. Let $f(x) = k_1 x + b$. The PIF is:

$$
\begin{aligned}
\mathrm{PIF} &= 1 - \frac{\int\limits_0^\infty \frac{1}{\lambda} e^{-x/\lambda} \left[1 + \sum\limits_{i=1}^{n} \theta_i f(x)^i\right] dx}{\int\limits_0^\infty \frac{1}{\lambda} e^{-x/\lambda} \left[1 + \sum\limits_{i=1}^{n} \theta_i x^i\right] dx} \\
&= 1 - \frac{1 + \int\limits_0^\infty e^{-x/\lambda} \left[\sum\limits_{i=1}^{n} \theta_1 f(x)^i\right] dx}{1 + \int\limits_0^\infty e^{-x/\lambda} \left[\sum\limits_{i=1}^{n} \theta_i x^i\right] dx} \\
&= 1 - \frac{1 + \sum\limits_{i=1}^{n} \theta_i \int\limits_0^\infty \left[e^{-x/\lambda} f(x)^i\right] dx}{1 + \sum\limits_{i=1}^{n} \theta_i \int\limits_0^\infty \left[e^{-x/\lambda} x^i\right] dx}
\end{aligned} \tag{6}
$$

where:

$$
\int\limits_0^\infty \left[e^{-x/\lambda} x^i\right] dx = (-\lambda)^{i+1} \Gamma(i+1, \frac{1}{\lambda} x)\Bigg|_{x=0}^{x\to\infty} = (-\lambda)^{i+1} \Gamma(i+1) \tag{7}
$$

on the other hand:

$$
\begin{aligned}
\int_0^\infty \left[e^{-x/\lambda} f(x)^i\right] dx &= \int_0^\infty \underbrace{\left[e^{-x/\lambda}(k_1x+b)^i\right]}_{(k_1x+b)^i=\sum\limits_{j=0}^{i}\binom{i}{j}(k_1x)^{i-j}b^i} dx \\
&= \sum_{j=0}^{i}\left[\binom{i}{j} b^i k_1^{i-j}\int_0^\infty \left(x^{i-j}e^{-x/\lambda}\right)dx\right] \\
&= \sum_{j=0}^{i}\left[\binom{i}{j} b^i k_1^{i-j}(-\lambda)^{i-j+1}\Gamma(i-j+1,\frac{1}{\lambda}x)\right]\Bigg|_{x=0}^{x\to\infty} \\
&= \sum_{j=0}^{i}\left[\binom{i}{j} b^i k_1^{i-j}(-\lambda)^{i-j+1}\Gamma(i-j+1)\right]
\end{aligned}
\tag{8}
$$

Substituting in (6) we obtain:

$$
\text{PIF} = 1 - \frac{1+\sum\limits_{i=1}^{n}(-\lambda)^{i+1}\Gamma(i+1)}{1+\sum\limits_{i=1}^{n}\sum\limits_{j=0}^{i}\left[\binom{i}{j}b^i k_1^{i-j}(-\lambda)^{i+1-j}\Gamma(i+1-j)\right]}
\tag{9}
$$

# 4 Lognormal

Let $X \sim \text{Lognormal}(\mu,\sigma)$. Let $f(x)=k_1x+k_2$ with $k_1 \geq 1$ y sea $R(x;\theta)=e^{\theta x}$. Then:

$$\text{PIF} = 1 - \frac{\overbrace{E[e^{\theta k_1 x + k_2}]}^{=\infty}}{\underbrace{E[e^{\theta x}]}_{=\infty}} = 1 - e^{k_2} \frac{\int\limits_0^\infty e^{k_1 x} \frac{1}{x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} dx}{\int\limits_0^\infty e^{x} \frac{1}{x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} dx}$$

$$\begin{aligned}
&= 1 - e^{k_2} \lim_{t\to 0} \frac{\overbrace{\int\limits_0^{\frac{1}{t}} e^{k_1 x} \frac{1}{x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} dx}^{f(t)}}{\underbrace{\int\limits_0^{\frac{1}{t}} e^{x} \frac{1}{x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} dx}_{g(t)}} \overbrace{=}^{\text{l'Hôpital}} 1 - e^{k_2} \lim_{t\to 0} \frac{f'(t)}{g'(t)} \\
&= 1 - e^{k_2} \lim_{t\to 0} \frac{\left(\frac{1}{t^2}\right) e^{k_1 \frac{1}{t}} t e^{-\frac{(log(1/t) - \mu)^2}{2\sigma^2}}}{\left(\frac{1}{t^2}\right) e^{\frac{1}{t}} t e^{-\frac{(log(1/t) - \mu)^2}{2\sigma^2}}} \\
&= 1 - \lim_{t\to 0} e^{(k_1 - 1)/t} = \begin{cases} 1 - e^{k_2} & \text{if } k_1 = 1, \\ -\infty & \text{if } k_1 > 1, \\ 1 & \text{if } k_1 < 1. \end{cases}
\end{aligned} \tag{10}$$

# 5 Discrete finite

Let $X$ random variable that takes values in the set $\{x_1, x_2, \ldots, x_n\}$ with respective probabilities $p_1, p_2, \ldots, p_n$ such that $\sum_i p_i = 1$. Let $f(x)$ arbitrary and let $R(x) = \sum\limits_{j=0}^{l} \theta_j x^j$ with $\theta_0 = 1$. The PIF is:

$$\text{PIF} = 1 - \frac{E\left[R\left(f(x);\theta\right)\right]}{E\left[R(x;\theta)\right]} = 1 - \frac{\sum\limits_{k=1}^{n} p_k R\left(f(x_k);\theta\right)}{\sum\limits_{k=1}^{n} p_k R\left(x_k;\theta\right)} = 1 - \frac{\sum\limits_{k=1}^{n} \sum\limits_{j=0}^{l} p_k \theta_j f(x_k)^j}{\sum\limits_{k=1}^{n} \sum\limits_{j=0}^{l} p_k \theta_j x_k^j} \tag{11}$$

# 6 Poisson

Let $X \sim \text{Poisson}(\lambda)$ assume $R(X;\theta) = \sum\limits_{i=0}^{n} \theta_i x^i$ with $\theta_0 = 1$. Sea $f(x) = ax + b$ arbitrary. Then:

$$
\begin{aligned}
\mathrm{PIF} &= 1 - \frac{E\left[R\left(f(x);\theta\right)\right]}{E\left[R(x;\theta)\right]} = 1 - \frac{E\left[\sum\limits_{i=0}^{n} \theta_i f(x)^i\right]}{E\left[\sum\limits_{i=0}^{n} \theta_i x^i\right]} \\
&= 1 - \frac{\sum\limits_{i=0}^{n} \theta_i E\left[(ax+b)^i\right]}{\sum\limits_{i=0}^{n} \theta_i E\left[x^i\right]} \\
&= 1 - \frac{\sum\limits_{i=0}^{n} \theta_i E\left[\sum\limits_{k=0}^{i} \binom{i}{k} (ax)^{i-k} b^k\right]}{\sum\limits_{i=0}^{n} \theta_i E\left[x^i\right]} \\
&= 1 - \frac{\sum\limits_{i=0}^{n} \sum\limits_{k=0}^{i} \overbrace{\theta_i \binom{i}{k} a^{i-k} b^k}^{\pi_{i,k}} E\left[x^{i-k}\right]}{\sum\limits_{i=0}^{n} \theta_i E\left[x^i\right]} \\
&= 1 - \frac{\sum\limits_{i=0}^{n} \sum\limits_{k=0}^{i} \pi_{i,k} h^{(i-k)}(0)}{\sum\limits_{i=0}^{n} \theta_i h^{(i)}(0)}
\end{aligned}
\tag{12}
$$

where $h(t) = e^{\lambda(e^t - 1)}$ y $h^{(m)}$ denotes the $m$-th derivative of $h$.