To Teacher

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I kindly ask if you could revisit Problem 2 in class.
My classmates and I found it quite challenging to grasp, even with the solution provided.
Your further explanation would be greatly appreciated.
Thank you!
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NOTE

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To prove that a language is NP-complete, we usually
take 2 steps of provement:
proving that it is NP firstly, and proving that
it can be reduced from the NP-complete language
(SAT,3SAT,clique and hampath)
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Problem 1

Let *DOUBLE-SAT* $= \{\langle\phi\rangle \mid \phi$ has at least two satisfying assignments$\}$. Show that *DOUBLE-SAT* is NP-complete.
[Hint: reduce SAT to DOUBLE-SAT by a map that uses an extra variable x and the construct $\wedge (x \vee \overline{x})$]

1. DOUBLE-SAT is in NP

Given a formula ϕ, guess two distinct assignments and verify in polynomial time that both satisfy ϕ. Thus, DOUBLE-SAT $\in$ NP.

2. DOUBLE-SAT is NP-hard (Reduction from SAT)

Let ϕ be a SAT instance. Define:

$$\psi = \phi \wedge (x \vee \neg x)$$

where x is a fresh variable not in ϕ.

Equivalence:

- If $\phi \in SAT$, then any satisfying assignment of ϕ can be extended to two distinct assignments of ψ by setting $x = \text{True}$ or False. Hence $\psi \in DOUBLE - SAT$.
- If $\psi \in DOUBLE - SAT$, then ϕ must be satisfiable (since $x \vee \neg x$ is always true).

The reduction is polynomial-time.

3. Conclusion

Since DOUBLE-SAT $\in$ NP and is NP-hard, it is **NP-complete**.

Problem 2

A ***coloring*** of a graph is an assignment of colors to its nodes so that no two adjacent nodes are assigned the same color. Let

$$\mathit{3COLOR} = \{\langle G\rangle |\ G \text{ is colorable with 3 colors}\}.$$

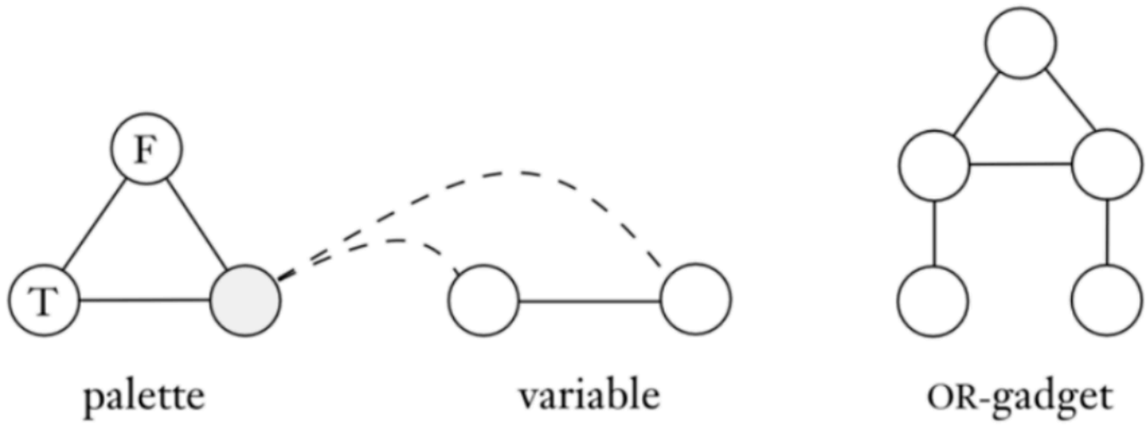
3COLOR is in NP because a coloring can be verified in polynomial time. We want to show that 3COLOR is in NP-complete by showing that 3SAT is polynomially reducible from 3COLOR.

Provement

Let $\phi = c_1 \wedge c_2 \wedge \cdots \wedge c_l$ be a 3CNF formula over variables $x_1, x_2, \cdots, x_m$, where c_i represents the clauses. We construct a graph G_ϕ comprising $2m + 6l + 3$ nodes: 2 nodes for each variable, 6 nodes for each clause, and 3 additional nodes. We will describe this graph using the following "gadgets":

1. Gadget Types and Graph Construction

- The graph G_ϕ includes:
  - A variable gadget for each variable x_i.
  - Two OR - gadgets for each clause.
  - One palette gadget.



- The four bottom nodes of the OR - gadgets will be merged with other nodes in the graph as follows:
  - Label the nodes of the palette gadget as T, F, and R.
  - Label the nodes in each variable gadget as $+$ and $-$, and connect each to the R node in the palette gadget.
  - For each clause, connect the top of one of the OR - gadgets to the F node in the palette.
  - Merge the bottom node of that OR - gadget with the top node of the other OR - gadget.
  - Merge the three remaining bottom nodes of the two OR - gadgets with corresponding nodes in the variable gadgets. Specifically, if a clause contains the literal x_i, merge one of its bottom nodes with the $+$ node of x_i. If the clause contains the literal $\overline{x_i}$, merge one of its bottom nodes with the $-$ node of x_i.

With this kind of Graph G_ϕ, we have:

1. When 3SAT holds, 3Color also holds

Assume there is a 3SAT formula ϕ with a satisfiable assignment. For the graph G_ϕ to be 3 - colored:

- **Coloring variable nodes**: Color the nodes in the variable gadgets based on the variable assignments. If variable x_i is assigned true, color the $+$ node in the variable gadget with the color corresponding to "true" in the palette (e.g., T color). If x_i is assigned false, color the $+$ node with the color representing "false" (e.g., F color), and color the $-$ node with the opposite color. Since the $+$ and $-$ nodes in the variable gadget are connected to the R node in the palette gadget, this coloring method ensures that the colors of the nodes adjacent to the R node comply with the 3 - coloring rules.
- **Coloring OR - gadgets**: Since the 3SAT formula is satisfiable, each clause is true. For the two OR - gadgets corresponding to each clause, based on their construction and connection methods (e.g., the top of an OR - gadget is connected to the F node in the palette), their nodes can be reasonably assigned colors. As the clause is true, the output of the OR - gadget meets the

requirements, ensuring that adjacent nodes have different colors. Thus, the entire graph G_ϕ can be colored with three colors, meaning 3Color holds.

2. When 3Color holds, 3SAT also holds

Assume the graph G_ϕ can be colored with three colors:

- **Determining variable assignments**: Based on the semantic definition of colors in the palette gadget (T represents true and F represents false), observe the color of the $+$ node in the variable gadget. If the $+$ node is colored with T color, assign the corresponding variable x_i as true. If the $+$ node is colored with F color, assign x_i as false.
- **Verifying clauses are true**: Due to the 3 - colorability of the graph, the coloring of nodes in each OR - gadget satisfies the rule that adjacent nodes have different colors. According to the connection method between OR - gadgets and variable gadgets (the merging rules of nodes corresponding to literals in clauses), it can be ensured that the output of each OR - gadget corresponding to a clause meets the requirements, that is, each clause is true. So the entire 3SAT formula ϕ is satisfiable, meaning 3SAT holds.

In conclusion, 3SAT holds if and only if 3Color holds. Through this reduction method, it is proved that 3Color is an NP - complete problem.

2. Conclusion

From the above construction, it can be concluded that 3COLOR is NP - complete.