Ho-Lee model calibration

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Agenda

- Short rate models
- Fitting the curve
- Pricing swaptions
- Calibration
- Pricing Bermudans
- Implementation
- Exercises

Interest rate curve

Discounting with annual rate r: $$B(0,T) = \left(1 + \frac{1}{r}\right)^{-T}$$

With continuous compounding: $$B(0,T) = e^{-rT}$$

In general r is different for different T:

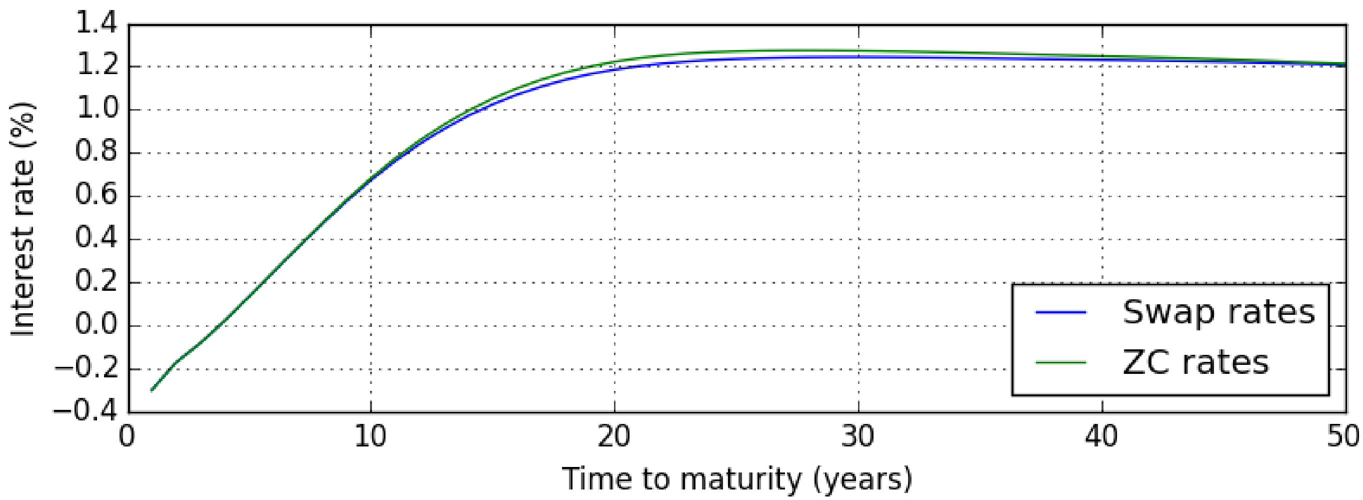

Figure 1: EURIBOR yield curve. Swap rates are the numbers quoted on the market and zero-coupon rates are obtained by bootstrapping and used for discounting.

Which rate should we model?

Instantaneous rate

Instantaneous rate is the annualized rate used for discounting for an infinitesimally short maturity:

$$r(t) = \lim_{T \to t^+} -\frac{\ln B(t,T)}{T-t}$$

Hence:

$$B(0,T) = \mathbb{E}\left(e^{-\int_0^T r(t)dt}\right)$$

- Not used directly for discounting any product
- Not quoted on the market
- Not corresponding to any ,,real" object
- Convenient theoretical concept
- Single parameter for entire curve
- Closed-form solutions in many models

Short-rate models

one-factor models without jumps: $dr_t = a(t, r_t)\,dt + b(t, r_t)\,dW_t$

Model	Dynamics (SDE)	$r > 0$	$r \sim$	AB	AO
Vašiček	$dr_t = (\theta - \alpha r_t)dt + \sigma dW_t$	N	$\mathcal{N}$	Y	Y
CIR	$dr_t = (\theta - \alpha r_t)dt + \sigma\sqrt{r_t}dW_t$	Y*	$NC\chi^2$	Y	Y
RB	$dr_t = \theta r_t dt + \sigma r_t dW_t$	Y	$L\mathcal{N}$	Y	N
Ho-Lee	$dr_t = \theta_t dt + \sigma_t dW_t$	N	$\mathcal{N}$	Y	Y
Hull-White	$dr_t = (\theta_t - \alpha r_t)dt + \sigma_t dW_t$	N	$\mathcal{N}$	Y	Y
BDT	$d\ln r_t = (\theta_t + \frac{\sigma_t'}{\sigma_t}\ln r_t)dt + \sigma_t W_t$	Y	$L\mathcal{N}$	N	N
BK	$d\ln r_t = (\theta_t + \phi_t \ln r_t)dt + \sigma_t W_t$	Y	$L\mathcal{N}$	N	N

AB $=$ analytical formula for zero-coupon bond pricing

AO $=$ analytical formula for bond option pricing

For more details & examples see e.g.: D. Brigo, F. Mercurio, Interest Rate Models - Theory and Practice [1].

* Assuming the Feller condition is satisfied, i.e. $\theta \geq \frac{1}{2}\sigma^2$

Vašiček model:

$$dr(t) = (\theta - \alpha r(t))dt + \sigma dW(t)$$

θ - drift parameter. Calculated from swap rates. A term structure $\theta(t)$ needed to fit the full interest rate curve.

σ - volatility. Calibrated to swaptions. A term structure $\sigma(t)$ needed to fit all ATM swaptions or all co-terminal swaptions for a Bermudan.

α - mean reversion rate. Determines the short rate's autocorrelation:

$$\mathrm{Corr}(r(s), r(t)) = \sqrt{\frac{e^{2\alpha s} - 1}{e^{2\alpha t} - 1}}, \qquad t > s > 0$$

higher α - lower autocorrelation.

Distribution of $r(t)$ in Ho-Lee [3]

Dynamics:

$$dr(t) = \theta(t)dt + \sigma(t)dW(t)$$

By integrating from s to t, we get:

$$r(t) = r(s) + \int_s^t \theta(u)du + \int_s^t \sigma(u)dW(u)$$

Hence, using Itô's isometry and the martingality of Itô's integrals (see [10]):

$$\mathbb{E}(r(t)) = r(0) + \int_0^t \theta(u)du$$

$$\mathrm{Var}(r(t)) = \int_0^t \sigma(u)^2 du$$

Finally, $r(t) \sim \mathcal{N}\left(r(0) + \int_0^t \theta(u)du, \int_0^t \sigma(u)^2 du\right)$

Distribution of $r(t)$ in Hull-White [5] and Vašiček [11]

Dynamics:

$$dr(t) = (\theta(t) - \alpha r(t))dt + \sigma(t)dW(t) \tag{1}$$

This cannot be integrated directly because of the $\alpha r(t)dt$ term on the RHS. Note that by Itô's lemma

$$d\left(e^{\alpha t} r(t)\right) = e^{\alpha t}\left(dr(t) + \alpha r(t)dt\right)$$

Hence, using (1), we get

$$d\left(e^{\alpha t} r(t)\right) = e^{\alpha t}\theta(t)dt + e^{\alpha t}\sigma(t)dW(t)$$

Integrating from s to t and multiplying by $e^{-\alpha t}$, we obtain

$$r(t) = r(s)e^{-\alpha(t-s)} + \int_s^t \theta(u)e^{-\alpha(t-u)}du + \int_s^t \sigma(u)e^{-\alpha(t-u)}dW(u)$$

Distribution of $r(t)$ in Hull-White and Vašiček - continued

Hull-White:

$$\begin{aligned} \mathbb{E}(r(t)) &= r(0)e^{-\alpha t} + \int_0^t \theta(u) e^{-\alpha(t-u)} du \\ \mathrm{Var}(r(t)) &= \int_0^t \sigma(u)^2 e^{-2\alpha(t-u)} du \end{aligned} \tag{2}$$

In Vašiček this becomes (assuming $\alpha \neq 0$):

$$\begin{aligned} \mathbb{E}(r(t)) &= r(0)e^{-\alpha t} + \theta \cdot \frac{1 - e^{-\alpha t}}{\alpha} \\ \mathrm{Var}(r(t)) &= \sigma^2 \cdot \frac{1 - e^{-2\alpha t}}{2\alpha} \end{aligned}$$

As before, $r(t) \sim \mathcal{N}\left(\mathbb{E}(r(t)), \mathrm{Var}(r(t))\right)$.

Zero-coupon bond pricing

We want to calculate

$$B(t,T) = \mathbb{E}\left(e^{-\int_t^T r(s)ds} | \mathcal{F}_t\right) \tag{3}$$

For $s > t$:

$$r(s) = r(t)e^{-\alpha(s-t)} + \int_t^s \theta(u)e^{-\alpha(s-u)}du + \int_t^s \sigma(u)e^{-\alpha(s-u)}dW(u)$$

We integrate from t to T. Note that $t \leq u \leq s \leq T$.

$$\int_t^T \int_t^s \theta(u)e^{-\alpha(s-u)}duds = \int_t^T \theta(u)\left(\int_u^T e^{-\alpha(s-u)}ds\right)du$$

$$\int_t^T \int_t^s \sigma(u)e^{-\alpha(s-u)}dW(u)ds = \int_t^T \sigma(u)\left(\int_u^T e^{-\alpha(s-u)}ds\right)dW(u)$$

Zero-coupon bond pricing - continued

Let's denote

$$D(u,T) = \int_u^T e^{-\alpha(s-u)} ds \tag{4}$$

After integration:

Hull-White: $D(u,T) = \dfrac{1 - e^{-\alpha(T-u)}}{\alpha}$

Ho-Lee: $D(u,T) = T - u$

$$\int_t^T r(s)ds = r(t)D(t,T) + \int_t^T \theta(u)D(u,T)du + \int_t^T \sigma(u)D(u,T)dW(u)$$

$$\int_t^T r(s)ds - r(t)D(t,T) \sim \mathcal{N}\left(\int_t^T \theta(u)D(u,T)du, \int_t^T \sigma(u)^2 D(u,T)^2 du\right)$$

Zero-coupon bond pricing - continued

$$X \sim \mathcal{N}(m, \sigma^2) \Rightarrow \mathbb{E}(e^{-X}) = e^{-m+\frac{1}{2}\sigma^2}$$

Hence, using (3):

$$B(t,T) = \exp\left(-r(t)D(t,T) - \int_t^T \theta(u)D(u,T)du + \frac{1}{2}\int_t^T \sigma(u)^2 D(u,T)^2 du\right) \tag{5}$$

Ho-Lee:

$$B(t,T) = \exp\left(-r(t)(T-t) - \int_t^T \theta(u)(T-u)du + \frac{1}{2}\int_t^T \sigma(u)^2 (T-u)^2 du\right)$$

Vašiček:

$$B(t,T) = \exp\left(-r(t)D(t,T) - \theta\int_t^T D(u,T)du + \frac{1}{2}\sigma^2\int_t^T D(u,T)^2 du\right)$$

Fitting the curve

$B(0,T)$ - implied from the yield curve

$\int_0^T \theta(u) D(u,T) du$ - chosen to fit $B(0,T)$

In Vašiček we only can get approximate fit by manipulating θ and α.

In Hull-White and Ho-Lee we calculate from (5)

$$\int_0^T \theta(u) D(u,T) du = -\ln B(0,T) - r(0) D(0,T) + \frac{1}{2} \int_0^T \sigma(u)^2 D(u,T)^2 du$$

Inputs in the formula:

$B(0,T)$, $r(0)$	- implied from yield curve
$\sigma(u)$	- model parameter, calibrated to swaptions
$D(u,T)$	- depends on choice of α

θ not directly used for pricing - only $\int_t^T \theta(u) D(u,T) du$ is needed

Fitting the curve - continued

From (4), we get:

$$D(u,T) = D(u,t) + e^{-\alpha(u-t)}D(t,T)$$

Applying this to (5) we get after some calculations:

$$\int_t^T \theta(u)D(u,T)du = \ln\frac{B(0,t)}{B(0,T)} - f(0,t)D(t,T) + \frac{1}{2}\int_t^T \sigma(u)^2 D(u,T)^2 du + \frac{1}{2}D(t,T)^2 \int_0^T \sigma(u)^2 e^{-2\alpha(u,t)}du \tag{6}$$

where $f(t,T)$ is the instantaneous forward rate for T at time t:

$$f(t,T) = -\frac{\partial \ln B(t,T)}{\partial T}$$

$f(0,T)$ - obtained directly from the yield curve

$f(t,t) = r(t)$

Bond option pricing

Differentiating (5) we get:

$$dB(t,T) = r(t)B(t,T)dt - \sigma(t)D(t,T)B(t,T)dW(t)$$

Solving analogically to Black-Scholes model, we get option prices:

$$\begin{aligned}\mathbf{BC}(0;S,T,K) &= B(0,T)N(d_+) - KB(0,S)N(d_-)\\ \mathbf{BP}(0;S,T,K) &= KB(0,S)N(-d_-) - B(0,T)N(-d_+)\end{aligned} \tag{7}$$

where

$$d_\pm = \frac{\ln \frac{B(0,T)}{B(0,S)K} \pm \frac{1}{2}\nu(0,S)}{\sqrt{\nu(0,S)}},$$

$$\begin{aligned}\nu(0,S) &= \mathrm{Var}(\ln B(S,T)) \stackrel{(5)}{=} D(S,T)^2\,\mathrm{Var}(r(S))\\ &\stackrel{(2)}{=} D(S,T)^2 \int_0^S \sigma(u)^2 e^{-2\alpha(S-u)}du\end{aligned} \tag{8}$$

Pricing swaptions

Fixed leg:

$$N \cdot \sum_{i=1}^{n} K \cdot \tau_i \cdot B(t, T_i)$$

Floating leg:

$$N \cdot (1 - B(t, T_n))$$

- N - notional
- K - strike (fixed coupon)
- T_0 - exercise of the option
- $T_1, \ldots, T_n$ - fixed coupon payments
- $\tau_i = T_{i-1} - T_i$

Value of Payer Swaption at time T_0:

$$\mathbf{PS}(T_0; N, K, \mathbf{T}) = N \cdot \left(1 - K \cdot \sum_{i=1}^{n} \tau_i \cdot B(T_0, T_i) - B(T_0, T_n)\right)^{+} \tag{9}$$

Pricing swaptions - Jamshidian's trick [6]

Express (9) as sum of options. Note that all $B(T_0, T_i)$ are decreasing functions of a single factor: $r(T_0)$.

Denote:

$$B(t, T) = F(t, r(t); T)$$

Find the break-even rate $\tilde{r}$, for which:

$$1 - K \cdot \sum_{i=1}^{n} \tau_i \cdot F(T_0, \tilde{r}; T_i) - F(T_0, \tilde{r}; T_n) = 0 \tag{10}$$

Option is exercised if and only if $r(T_0) \geq \tilde{r}$. In such case

$$B(T_0, T_i) \leq K_i := F(T_0, \tilde{r}; T_i), \qquad i = 1, 2, \ldots, n$$

Finally,

$$\mathbf{PS}(T_0; N, K, \mathbf{T}) = K \cdot \sum_{i=1}^{n} \tau_i \cdot (K_i - B(T_0, T_i))^+ + (K_n - B(T_0, T_n))^+$$

Pricing swaptions - continued

Payer swaption:

$$\mathbf{PS}(0; N, K, \mathbf{T}) = K \cdot \sum_{i=1}^{n} \tau_i \cdot \mathbf{BP}(0; T_0, T_i, K_i) + \mathbf{BP}(0; T_0, T_n, K_n) \tag{11}$$

Receiver swaption:

$$\mathbf{RS}(0; N, K, \mathbf{T}) = K \cdot \sum_{i=1}^{n} \tau_i \cdot \mathbf{BC}(0; T_0, T_i, K_i) + \mathbf{BC}(0; T_0, T_n, K_n) \tag{12}$$

where bond option values are given by (7)

For more details on all the calculations, see:
D. McInerney, T. Zastawniak, Stochastic Interest Rates [9].

Calibration of volatility

One needs to consider the following:

- Parametrization of $\sigma(t)$
- Calibration strategy
- Choice of instruments
- What about α?

A very good overview of these points is provided in:

- S. Gurrieri, M. Nakabayashi, T. Wong, Calibration Methods of Hull-White Model [2]

Parametrization of $\sigma(t)$

Recall ((5) and (2)) that $\sigma(t)$ is used in the below expressions which we would like to calculate analytically:

$$\int_t^T \sigma(u)^2 D(u,T)^2 du \quad \text{and} \quad \int_t^T \sigma(u)^2 e^{-2\alpha(T-u)} du$$

Idea 1: $\sigma(t) = \sigma$ (constant) - easy for calculation but cannot replicate the market swaption prices

Idea 2: Piecewise constant - if we're only interested in options expiring on dates $T_0, T_1, \ldots, T_n$, then shape of $\sigma(t)$ between these dates doesn't matter, as long as the total levels are calibrated

$$\sigma(t) = \begin{cases} \sigma_0, & t \leq T_0 \\ \sigma_i, & t \in (T_{i-1}, T_i], i = 1, \ldots, n \\ \sigma_n, & t > T_n \end{cases}$$

Idea 3: Piecewise linear or any other spline interpolation between points $T_0, T_1, \ldots, T_n$ - smoother shape of $\sigma(t)$ but no improvement on pricing with increased complexity of calibration

Calibration strategies - bootstrapping

- Choose a swaption (or set of swaptions) expiring at T_0. From market price obtain $\mathrm{Var}(r(T_0))$ and hence calculate σ_0
- From market price of a swaption expiring at T_1 obtain $\mathrm{Var}(r(T_1))$. It depends on σ_0 and σ_1. You already know σ_0, so you can imply σ_1
- Likewise, having calculated $\sigma_0, \ldots, \sigma_{i-1}$, you obtain σ_i from prices of swaptions expiring at T_i.

Pros:

- Computationally efficient - only 1-dim optimization needed
- Potential exact fit for chosen instruments

Cons:

- Results in spiky shape of $\sigma(t)$
- Tends to overfit short-term instruments at cost of long-term ones

Calibration strategies - least squares

- Choose a pool of swaptions $(\mathbf{S}_i)_{i=1}^m$
- Price each swaption using **(11)** and **(12)**
- Calculate the error as a function of $\overline{\sigma} = (\sigma_0, \ldots, \sigma_n)$:

$$\phi(\overline{\sigma}) = \sum_{i=1}^{m} \left| V_{\text{model}}(\mathbf{S}_i; \overline{\sigma}) - V_{\text{market}}(\mathbf{S}_i) \right|^2$$

- Use a numerical approach to find $\overline{\sigma}$ which minimizes $\phi(\overline{\sigma})$.
  A standard method for this is the Levenberg-Marquardt algorithm [7]. Open-source implementations exist in Python.

Computationally less efficient but generates more realistic results, reasonably fitting all instruments.

Choice of instruments

Volatilities can be implied from caps or swaptions.

ATM vol	1y	2y	5y	7y	10y	12y	15y	20y	30y
1y	19.9%	22.4%	23.1%	22.1%	21.2%	20.6%	19.9%	19.4%	19.2%
2y	25.4%	26.3%	24.9%	23.6%	22.4%	21.8%	21.1%	20.5%	20.2%
5y	27.7%	27.3%	25.0%	23.9%	22.8%	22.3%	21.5%	20.8%	20.6%
7y	26.0%	25.4%	23.6%	22.7%	21.9%	21.4%	20.8%	20.2%	20.1%
10y	22.7%	22.5%	21.8%	21.2%	20.6%	20.1%	19.5%	18.9%	19.1%
12y	21.6%	21.5%	20.8%	20.4%	19.8%	19.4%	18.8%	18.3%	18.6%
15y	20.4%	20.3%	19.7%	19.3%	18.8%	18.4%	18.0%	17.7%	18.0%
20y	18.7%	18.6%	18.3%	17.9%	17.5%	17.3%	17.1%	17.1%	17.3%
30y	17.9%	17.7%	17.5%	17.6%	17.7%	17.7%	17.6%	17.6%	17.5%

Figure 2: A typical swaption volatility matrix. The presented numbers are log-normal volatilities for ATM strikes

For each expiry we have multiple tenors quoted and only one parameter σ_i, so we cannot fit them all. Which of them to choose?

Idea 1: All of them. Put all available products to the least squares algorithm to get somewhere in the middle. You should get reasonable prices for all products but none of them is fit exactly. Good approach when the model needs to be used for many purposes without recalibration

Choice of instruments - hedging

Idea 2: Hedging instruments. For pricing an exotic one needs to correctly price the hedging instruments. E.g. to price a Bermudan swaption one should calibrate to its co-terminal European swaptions

Example: Bermudan swaption, first exercise: 10Y, then every 1Y, tenor 10Y.

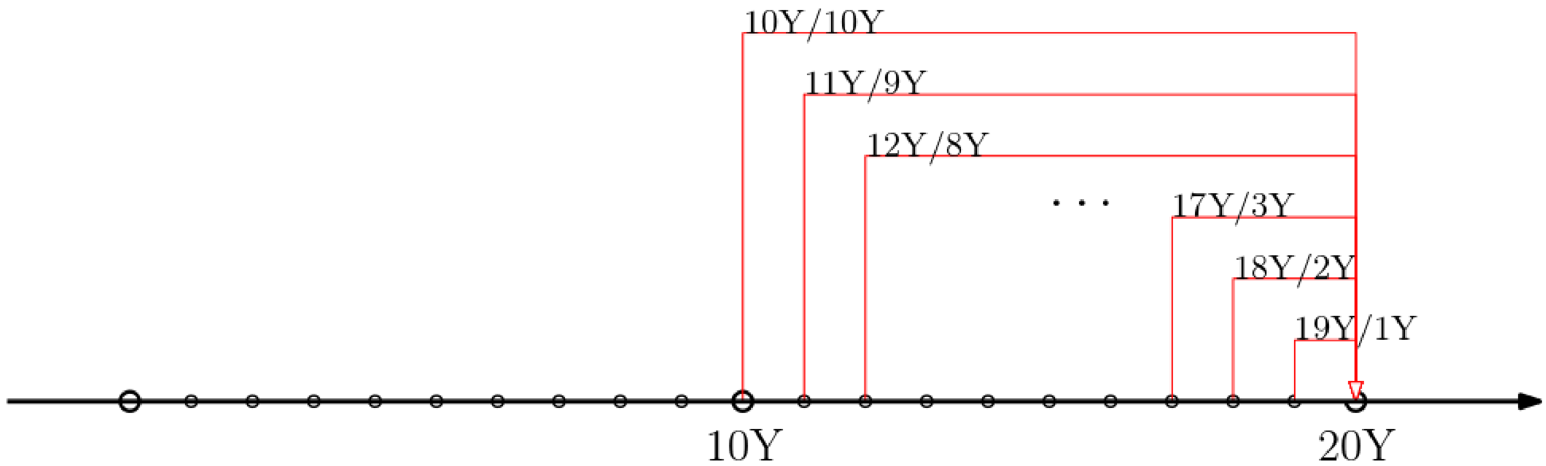

Figure 3: Co-terminal swaptions for a Bermudan

Hedging swaptions: 10Y/10Y, 11Y/9Y, 12Y/8Y, 13Y/7Y, . . ., 19Y/1Y
For e.g. 11Y/9Y data may not be available (Figure 2) - time interpolation needed
Alternatively, calibrate directly to the available instruments, e.g.:
10Y/10Y, 12Y/10Y, 12Y/7Y, 15Y/5Y, 20Y/1Y

Calibration of mean reversion (Hull-White)

Idea 1: Calibrate to correlation-dependent instruments, e.g. Bermudans - the higher auto-correlation, the lower switch value. Market data needed

Idea 2: Try to fit swaptions with different tenors and one expiry - term structure of α may be needed

Idea 3: Calibrate together with σ to fit the pool of swaptions

Idea 4: Choose a fixed a priori value based on historical observations

Whatever α you choose (including $\alpha = 0$), make sure to justify your choice.

Pricing Bermudan swaptions

At each exercise date T_i the option holder has a right to choose: continue or exercise. Choosing to exercise means getting an European swaption with exercise date T_i and maturity T_n:

$$\mathbf{Ex}(T_i) = \mathbf{PS}(T_i; N, K, (T_{i+1}, \dots, T_n))$$

Continuation means getting at time T_{i+1} another Bermudan swaption with one exercise date less:

$$\mathbf{Cont}(T_i) = B(T_i, T_{i+1})\mathbb{E}(\mathbf{Berm}(T_{i+1}; N, K, (T_{i+1}, \dots, T_n))|\mathcal{F}_{T_i}) \tag{13}$$

Therefore,

$$\mathbf{Berm}(T_i) = \max\{\mathbf{Ex}(T_i), \mathbf{Cont}(T_i)\}$$

We can calculate $\mathbf{Ex}(T_i)$ but $\mathbf{Cont}(T_i)$ looks like a vicious circle...
But for the last exercise $\mathbf{Cont}(T_{n-1}) = 0$. Apply the backward propagation.

Pricing Bermudan swaptions - continued

To calculate the expectation in (13) one needs to simulate $r(T_{i+1})$ knowing $r(T_i)$.

- Monte Carlo? - note that for each value of $r(T_i)$ we need a big number of paths for $r(T_{i+1})$ and the simulations get multiply nested. Even for 3-4 exercise dates the number of calculations becomes unmanageable.

- Longstaff-Schwartz method. In the simulations we already have realized payoffs of $\mathbf{Berm}(T_{i+1})$ for many different values of $r(T_i)$. By a clever use of least-squares method one can obtain the expectation of these payoffs as a function of $r(T_i)$. For more details see [8] or recall the lecture Introduction to MC Simulation in Finance.

- It is possible to explicitly calculate the conditional distribution of $r(T_{i+1})$ given $r(T_i)$. This however requires a numerical calculation of an improper integral. This is done using a discretization of possible values of r to a finite set $r_0, \ldots, r_N$ and calculating $\mathbf{Berm}(T_{n-1}; r_j)$ for $j = 1, \ldots, N$. Then, by backward propagation, one calculates $\mathbf{Berm}(T_{n-2}; r_j)$ for $j = 1, \ldots, N$, etc. Formulas on the next slide.

Pricing Bermudan swaptions - continued

$$\mathbb{E}(\mathbf{Berm}(T_{i+1}; r(T_{i+1}))|r(T_i)) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \int_{-\infty}^{\infty} \mathbf{Berm}(T_{i+1}; u) \exp\left(-\frac{(u-\tilde{\mu})}{2\tilde{\sigma}^2}\right) du$$

where, in Ho-Lee:

$$\begin{aligned} \tilde{\mu} &= (r(T_i) - f(0,T_i)) + f(0,T_{i+1}) + \tau_i \int_0^{T_i} \sigma^2(u) du \\ \tilde{\sigma}^2 &= \int_{T_i}^{T_{i+1}} \sigma(u)^2 du \end{aligned} \tag{14}$$

in Hull-White:

$$\begin{aligned} \tilde{\mu} &= (r(T_i) - f(0,T_i))e^{-\alpha\tau_i} + f(0,T_{i+1}) + \int_0^{T_i} \frac{\sigma^2(u)}{\alpha}(e^{-\alpha(\tau_i - 2u)} - e^{-2\alpha(T_{i+1}-u)}) du \\ \tilde{\sigma}^2 &= \int_{T_i}^{T_{i+1}} \sigma(u)^2 e^{-2\alpha(T_{i+1}-u)} du \end{aligned} \tag{15}$$

Implementation

- Curve bootstrapping
- Monte Carlo
- Updating θ
- Pricing swaptions

Implementation - curve bootstrapping

Input: swap rates
Output: zero rates, discount factors

- Make sure what is the convention of swap: payment frequencies, day count conventions, etc.
- For T_0 convert the annual (or other appropriate) compounding to the continuous one.
- For T_i use precomputed zero rates for $T_0, \dots, T_{i-1}$ and swap rate for T_i to compute the zero rate for T_i. E.g. if swap is annual and $T_i = i + 1$, then

$$r_{\text{swap}}(T_i) \sum_{k=0}^{i} B(0, T_k) = 1 - B(0, T_i)$$

- Use some interpolation method for zero rates between T_i and T_{i+1}

Implementation - Monte Carlo

Input: model, number of paths, time step
Output: matrix of simulated r's for each time and path

- For short time steps you can use directly (1):

```
dt = t[i+1] - t[i]
r[i + 1, path] = (theta[i] - alpha*r[i, path])*dt
                 + sigma[t]*randn(mean=0, stddev=sqrt(dt))
```

  More time steps = longer computations

- If you're only interested in nodes $T_0, \ldots, T_n$ you can skip the intermediate time steps by using the forward distribution of r given by (14) or (15). The integrals are easily calculated if θ and σ are piecewise constant. This approach is more efficient but cannot be used e.g. for American-style options.

To price any product calculate its value for each path using `r[n, path]` and then take average for all paths.

MC can be used for all calculations and is easy to implement. However, it requires huge amount of calculations to get a reasonable precision.

Implementation - updating θ

Assume $\theta(t)$ is piecewise constant between nodes $T_0, T_1, \ldots, T_n$. From (6) we get (for Ho-Lee model):

$$\theta_i \cdot \int_{T_{i-1}}^{T_i} (T_i - u)du = \ln \frac{B(0, T_{i-1})}{B(0, T_i)} - f(0, T_{i-1})\tau_i$$

$$+ \frac{1}{2}\sigma_i \int_{T_{i-1}}^{T_i} (T_i - u)^2 du + \frac{1}{2}\tau_i^2 \sum_{j=0}^{i} \sigma_j^2 \tau_j$$

with $\tau_i = T_i - T_{i-1}$ and $\tau_0 = T_0$. After integration:

$$\theta_i = 2\tau_i^{-2} \ln \frac{B(0, T_{i-1})}{B(0, T_i)} - 2\tau_i^{-1} f(0, T_{i-1}) + \frac{1}{3}\sigma_i \tau_i + \sum_{j=0}^{i} \sigma_j^2 \tau_j$$

Note that θ needs to be always updated when either

- curve moves - then $B(0, T)$ and $f(0, T)$ change
- σ is changed (also during the calibration)

Implementation - pricing swaptions

- For any times $0 < t < T$ and any hypothetical value of $r(t)$ calculate $F(t, r(t); T)$, using (5)
- Use any numerical root-finding algorithm (e.g. Newton or Brent) to find $\tilde{r}$ solving (10).
- Use formulas (11) and (12) along with (7) to get the price

You can use Monte Carlo instead which is easier to implement but less efficient.

In any case, make sure you always use the correct θ, marked to the yield curve and depending on σ.

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