Modeling Equity Prices

INTQuant 2022, Katowice

Financial Markets and Derivatives

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The simplest example of a derivative is a **forward contract** which is an agreement between two parties to but or sell an asset at a specified price on predefined expiry date. The party agreeing to buy the underlying asset in the future assumes a *long position*, and the party agreeing to sell the asset in the future assumes a *short position*. The price agreed upon is called the *delivery price* and is given by the following formula:

$$F_0 = S_0 e^{rT},$$

where S_0 is the current stock price, r is the risk-free rate at which we can borrow and deposit money in the bank and T is the contract maturity expressed in years.

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As a result, at time T we have:

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Remark. In the above example and in the rest of the presentation, we assume that the shares do not pay dividends (part of the profit of company that is paid to shareholders).

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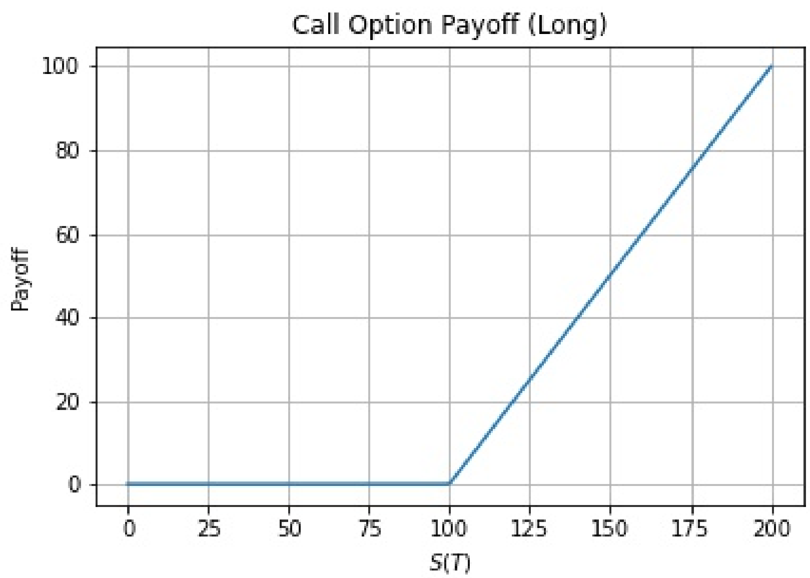
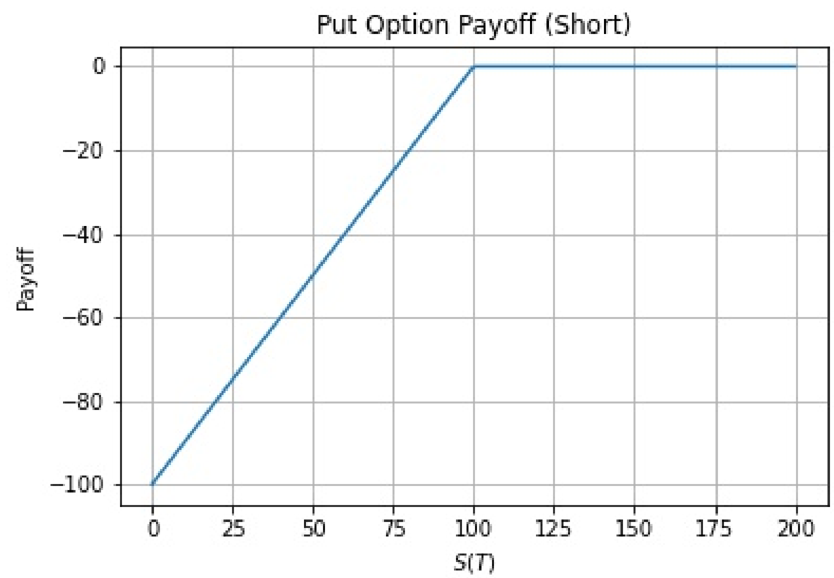


(a) Payoff from long position in European Call Option.



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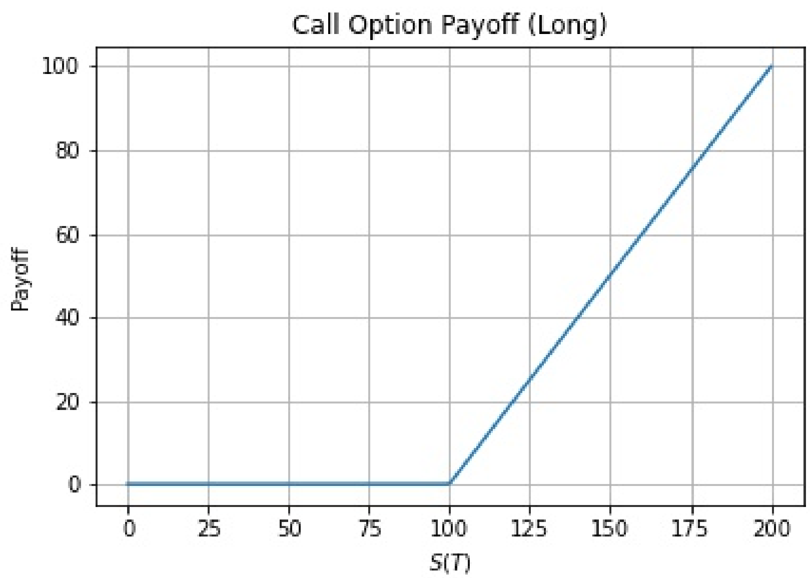

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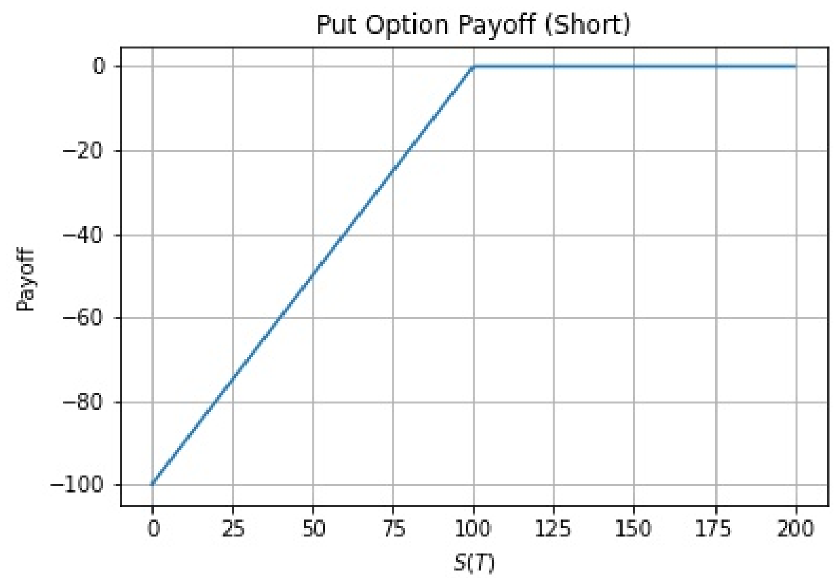

(b) Payoff from long position in European Put Option.

Question: What is the price of such an option?

One-Step Binomial Tree

Example. Assume that stock price is currently $\$100$ and that at the end of 3 months it will be either $\$120$ or $\$90$. What is the price of European Call Option with maturity 3 months and strike price $\$100$?

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1. If $S(T) = 120$ (price goes up), then the value of shares is 120Δ and the value of option is 20, hence the value of portfolio is $120\Delta - 20$.

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We have

$$120\Delta - 20 = 90\Delta \Rightarrow 30\Delta = 20 \Rightarrow \boxed{\Delta = \frac{2}{3}},$$

so the riskless portfolio is: long position in $\frac{2}{3}$ shares of stock and short position in 1 option.

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It's easy to verify that:

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In the absence of arbitrage opportunities, the riskless portfolio must earn the risk-free rate of interest (assume that $r = 5\%$ per annum, continuously compounded). The value of the portfolio today must be the present value of 60, i.e. $60e^{-0.05\times\frac{1}{4}} = 59.25$.

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If we now denote the value of option by f, then the value of the portfolio today is

$$100 \times \frac{2}{3} - f = 59.25 \Rightarrow \boxed{f = 7.41}$$

Risk-Neutral Valuation

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A risk-neutral world has two features that simplify the pricing of derivatives:

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The real-world measure is sometimes referred to as the $\mathbb{P}-$measure, while the risk-neutral world measure is referred to as the $\mathbb{Q}-$measure.

Let's back to our example and denote by p the probability of an upward movement in the stock price in a risk neutral world. We can argue that the expected return on the stock in a risk-neutral world must be the risk-free rate of 5%. This means that p must satisfy

$$120p + 90\,(1-p) = 100e^{0.05\times\frac{3}{12}} \Rightarrow \boxed{p = 0.375}$$

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Discounting at the risk-free rate we get

$$f = 7.5e^{-0.05\times\frac{3}{12}} = 7.41.$$

General Binomial Tree

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Remark! As $\Delta t \to 0$, the Binomial Model option price converges to Black-Scholes price!

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It should be noted that, in practice, we do not observe stock prices following continuous-variable, continuous-time processes. Stock prices are restricted to discrete values (e.g., multiples of a cent) and changes can be observed only when the exchange is open for trading. Nevertheless, the continuous-variable, continuous-time process proves to be a useful model for many purposes.

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Suppose that the price of a stock is $100 now. If the stock price follows a Markov process, our predictions for the future should be unaffected by the price one week ago, one month ago, or one year ago. The only relevant piece of information is that the price is now $100. **Predictions for the future are uncertain and must be expressed in terms of probability distributions**. The Markov property implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past.

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The Markov property of stock prices is consistent with the weak form of market efficiency. This states that the present price of a stock impounds all the information contained in a record of past prices. If the weak form of market efficiency were not true, technical analysts could make above-average returns by interpreting charts of the past history of stock prices. There is very little evidence that they are in fact able to do this.

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Consider the change in the value of W during a relatively long period of time, T. This can be denoted by $W\left(T\right) - W\left(0\right)$. It can be regarded as the sum of the changes in W in N small time intervals of length Δt, where $N = T\Delta t$ Thus,

$$W\left(T\right) - W\left(0\right) = \sum_{i=1}^{N} \epsilon_i\sqrt{\Delta t}$$

where the ϵ_i $\{i = 1, 2, ..., N\}$ are distributed $N\left(0, 1\right)$. We know from the second property of Wiener processes that the ϵ_i are independent of each other. It follows from equation that $W(T) - W(0)$ is normally distributed, with mean of 0 variance of $N\Delta t = T$, standard deviation of $\sqrt{T}$.

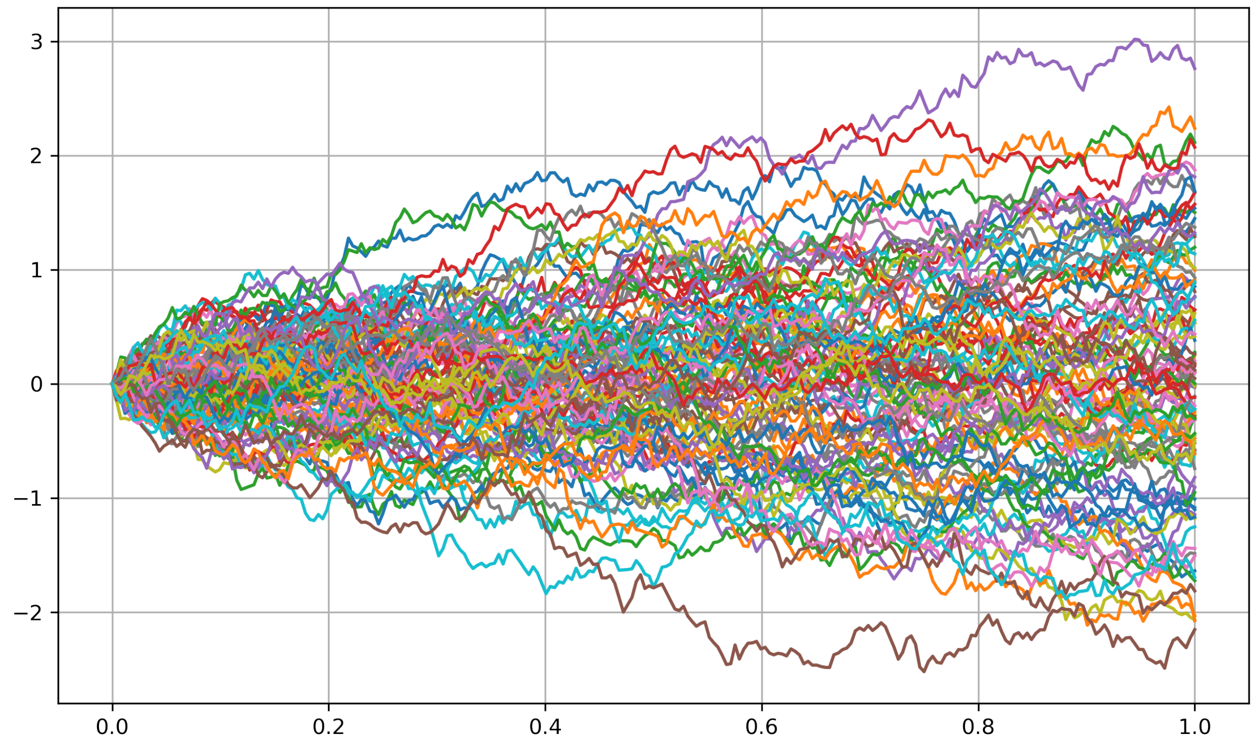

Figure 1: Paths of Wiener Process.

Differential notation

In ordinary calculus, it is usual to proceed from small changes to the limit as the small changes become closer to zero. Thus, $dx = adt$ is the notation used to indicate that $\Delta x = a\Delta t$ in the limit as $\Delta t \to 0$. We use similar notational conventions in stochastic calculus. So, when we refer to dW as a Wiener process, we mean that it has the properties for ΔW given above in the limit as $\Delta t \to 0$

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The mean change per unit time for a stochastic process is known as the **drift rate** and the variance per unit time is known as the **variance rate**. The basic Wiener process, dW, that has been developed so far has a drift rate of zero and a variance rate of 1.0. The drift rate of zero means that the expected value of W at any future time is equal to its current value. The variance rate of 1.0 means that the variance of the change in W in a time interval of length T equals T. A **generalized Wiener process** for a variable X can be defined in terms of dW as

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To understand above equation, it is useful to consider the two components on the right-hand side separately. The adt term implies that x has an expected drift rate of a per unit of time. Without the bdW term, the equation is $dX = adt$, which implies that $dX/dt = a$. Integrating with respect to time, we get $X = X_0 + at$ where X_0 is the value of X at time 0. In a period of time of length T, the variable X increases by an amount aT. The bdW term on the right-hand side can be regarded as adding noise or variability to the path followed by X. The amount of this noise or variability is b times a Wiener process. A Wiener process has a variance rate per unit time of 1.0. It follows that b times a Wiener process has a variance rate per unit time of b^2 . In a small time interval Δt, the change ΔX in the value of X is

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Thus ΔX has a normal distribution with mean of $\Delta X = a\Delta t$ variance of $\Delta X = b^2\Delta t$ and standard deviation of $\Delta X = b\sqrt{\Delta t}$.

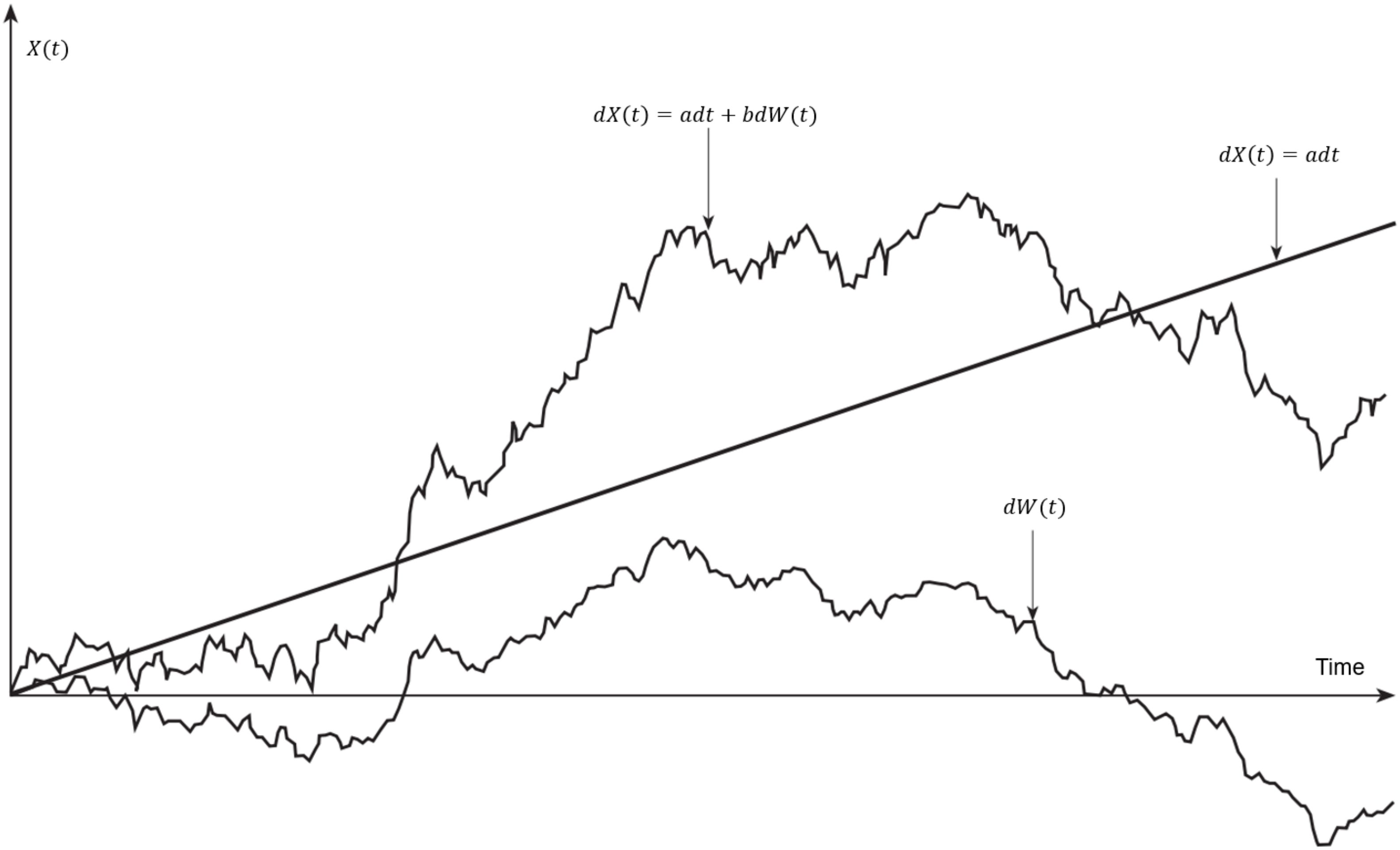

Figure 2: Path of generalized Wiener process ([1, page 321]).

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It's important to highlight that the equation above involves a small approximation. It assumes that the drift and variance rate of X remain constant, equal to their values at time t, in the time interval between t and $t + \Delta t$.

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- Variability of the return in a short period of time, Δt, is the same regardless of the stock price. In other words, an investor is just as uncertain about the return when the stock price is \$50 as when it is \$10. This suggests that the standard deviation of the change in a short period of time Δt should be proportional to the stock price

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The variable ΔS is the change in the stock price S in a small time interval Δt, and as before ϵ has a standard normal distribution (i.e., a normal distribution with a mean of zero and standard deviation of 1.0). The parameter μ is the expected rate of return per unit of time from the stock. The parameter σ is the stock price volatility.

Remark! The model in equation (2) represents the stock price process in the real world. In a risk-neutral world, μ equals the risk-free rate r.

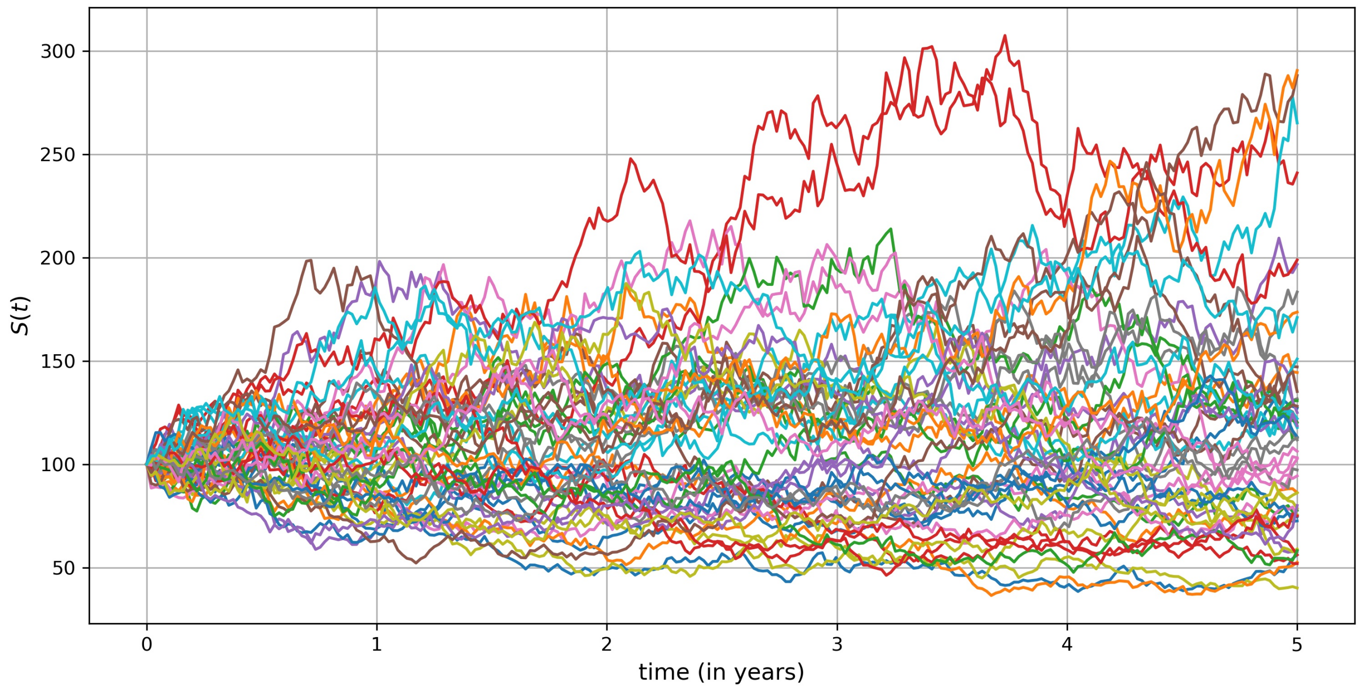

Figure 3: Paths of Geometric Brownian Motion ($\mathbb{E}[S_t] = S_0 e^{\mu t}$)

Impact of σ

$$\boxed{dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad \mu, \sigma \in \mathbb{R}, S_0 > 0.}$$

The parameter σ, the stock price volatility, is critically important to the determination of the value of many derivatives. Typical values of σ for a stock are in the range 0.15 to 0.60 (i.e., 15% to 60%).

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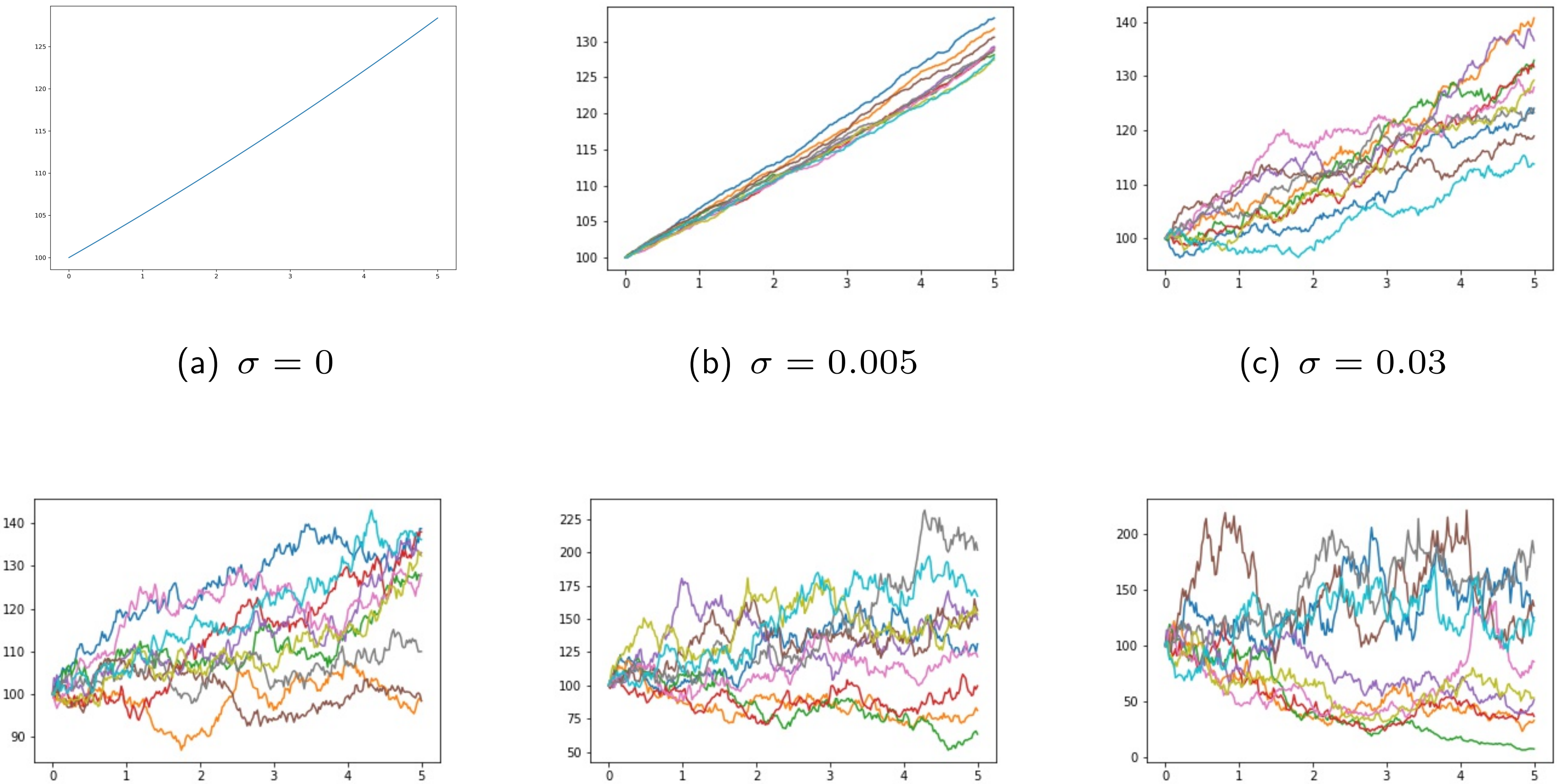
Note that if $\sigma = 0$, then the stock price process is:

$$\begin{aligned} \sigma &= 0 \Rightarrow dS_t = \mu S_t dt \Rightarrow \boxed{S_t = S_0 e^{\mu t}} \\ \text{since} &\Rightarrow \left(dS_t = \frac{\partial}{\partial t}\left(S_0 e^{\mu t}\right) dt = \mu S_t dt \right) \end{aligned}$$



(a) $\sigma = 0$ (b) $\sigma = 0.005$ (c) $\sigma = 0.03$

(d) $\sigma = 0.08$ (e) $\sigma = 0.2$ (f) $\sigma = 0.4$

Figure 4: Impact of σ.

Ito's Lemma

The price of a stock option is a function of the underlying stock's price and time. More generally, we can say that the price of any derivative is a function of the stochastic variables underlying the derivative and time. A serious student of derivatives must, therefore, acquire some understanding of the behavior of functions of stochastic variables. An important result in this area was discovered by the mathematician K. Ito in 1951, and is known as Ito's lemma.

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Suppose that the value of a variable X follows the Ito process

$$dX = a\left(X, t\right) dt + b\left(X, t\right) dW \tag{3}$$

where dW is a Wiener process and a and b are functions of X and t. The variable X has a drift rate of a and a variance rate of b^2 .

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Ito's lemma shows that a function G of X and t follows the process

$$\boxed{dG = \left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)dt + \frac{\partial G}{\partial X}bdW}$$

where the dW is the same Wiener process as in equation (3).

General assumptions

The general assumptions of the Black–Scholes model are:

1. The stock price follows the process

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad \mu, \sigma \in \mathbb{R}, S_0 > 0.$$

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6. Security trading is continuous.
7. The risk-free rate of interest, r, is constant and the same for all maturities.

Black-Scholes Model

In the Black-Scholes model, the dynamics of the Stock Price process S and Bank Account B are given by the following SDE's

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \qquad && \mu, \sigma \in \mathbb{R}, S_0 > 0 \\ dB_t &= r B_t dt, \quad B_0 = 1 && \Rightarrow B_t = e^{rt} \end{aligned}$$

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Theorem. The solution of (4) is (see Appendix)

$$\boxed{S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{\mathbb{Q}}}}.$$

Risk-Neutral Valuation

Consider a derivative that provides a payoff at one particular time. It can be valued using **risk-neutral valuation** by using the following procedure:

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In more mathematical terms, the price of financial derivative with payoff H_T is given by the following formula

$$\boxed{V_0 = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT} H_T\right]}$$

Call and Put option prices in Black-Scholes model

Using Risk-Neutral Valuation, it can be shown (see Appendix) that the prices of call and put options in the Black-Scholes model are given by the following formulas:

$$\begin{aligned} C_0 &= S_0\Phi\left(d_+\right) - Ke^{-rT}\Phi\left(d_-\right) \\ P_0 &= Ke^{-rT}\Phi\left(-d_-\right) - S_0\Phi\left(-d_+\right) \end{aligned}$$

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\begin{aligned}
C_0 &= S_0\Phi\left(d_+\right) - Ke^{-rT}\Phi\left(d_-\right) \\
P_0 &= Ke^{-rT}\Phi\left(-d_-\right) - S_0\Phi\left(-d_+\right)
\end{aligned}
$$

For another way of looking at the Black–Scholes equation for the value of a European call option, note that it can be written as

$$
C_0 = e^{-rT}\Phi\left(d_-\right)\left[S_0 e^{rT}\frac{\Phi\left(d_+\right)}{\Phi\left(d_-\right)} - K\right]
$$

The terms here have the following interpretation:

- e^{-rT}: Present value factor,
- $\Phi\left(d_-\right)$: Probability of exercise (in a risk-neutral world),
- $S_0 e^{rT}\frac{\Phi(d_+)}{\Phi(d_-)}$: Expected stock price in a risk-neutral world if option is exercised,
- K: Strike price paid if option is exercised.

Call-Put Parity

Theorem. In absence of arbitrage, the prices of the call and put options with the same strike K and maturity T satisfies the following equation:

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$$C_T - P_T = (S_T - K)^+ - (K - S_T)^+ = \begin{cases} S_T - K, & S_T \geq K \\ S_T - K, & S_T < K \end{cases} = S_T - K.$$

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So to avoid arbitrage, we need to have

$$C_0 - P_0 = S_0 - K^{-rT} \qquad \blacksquare$$

Other derivatives

The other popular financial derivatives are:

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- Barrier option: type of derivative where the payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price.

Option type	Initial Condition	Payoff
Up-And-In	$S < B$	$1_{\left\{\max_{t\in[0,T]} S_t \geq B\right\}} H$
Up-And-Out	$S < B$	$1_{\left\{\max_{t\in[0,T]} S_t < B\right\}} H$
Down-And-In	$S > B$	$1_{\left\{\min_{t\in[0,T]} S_t \leq B\right\}} H$
Down-And-Out	$S > B$	$1_{\left\{\min_{t\in[0,T]} S_t > B\right\}} H$

Table 1: Payoff from Barrier options.

where

$$H = \begin{cases} (S_T - K)^+, & \text{for call option,} \\ (K - S_T)^+, & \text{for put option.} \end{cases}$$

- Asian Option: payoff is determined by the average underlying price over some period of time:

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    - Arithmetic Average:

$$
\begin{aligned}
H_T^{\text{call}} &= \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)^+ \\
\text{(Discrete case)} \quad & \left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K\right)^+
\end{aligned}
$$

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    - Geometric Average

$$H_T^{\text{call}} = \left(\exp\left\{\frac{1}{T}\int_0^T \ln S(t)\,dt\right\} - K\right)^+$$

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\text{(Discrete case)} &\quad \left(\sqrt[n]{\prod_{i=1}^{n} S_{t_i}} - K\right)^+
\end{aligned}
$$

- Equity Swap: a set of future cash flows are agreed to be exchanged between two counterparties at set dates in the future. The two cash flows are usually referred to as "legs" of the swap; one of these "legs" is usually pegged to a floating rate such as LIBOR. This leg is also commonly referred to as the "floating leg". The other leg of the swap is based on the performance of either a share of stock or a stock market index. This leg is commonly referred to as the "equity leg".

Volatility

The one parameter in the Black–Scholes–Merton pricing formulas that cannot be directly observed is the volatility of the stock price. In practice, traders usually work with what are known as **implied volatilities**. These are the volatilities implied by option prices observed in the market, i.e. if the market price of European Call Option is $C^{*}(K,T)$, then the Black-Scholes Implied Volatility for a given K and T is such value of $\sigma(K,T)$ for which the Black-Scholes model returns the market price, i.e.

$$C^{\mathrm{BS}}\left(S_0, K, T, r, \sigma(K,T)\right) = C^{*}(K,T)$$

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Implied volatilities are used to monitor the market's opinion about the volatility of a particular stock. **Whereas historical volatilities (determined from historical data) are backward looking, implied volatilities are forward looking**. Traders often quote the implied volatility of an option rather than its price. This is convenient because the implied volatility tends to be less variable than the option price. The implied volatilities of actively traded options on an asset are often used by traders to estimate appropriate implied volatilities for other options on the asset.

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The CBOE publishes indices of implied volatility. The most popular index, the SPX VIX, is an index of the implied volatility of 30-day options on the S&P 500 calculated from a wide range of calls and puts. It is sometimes referred to as the “fear factor.” An index value of 15 indicates that the implied volatility of 30-day options on the S&P 500 is estimated as 15%

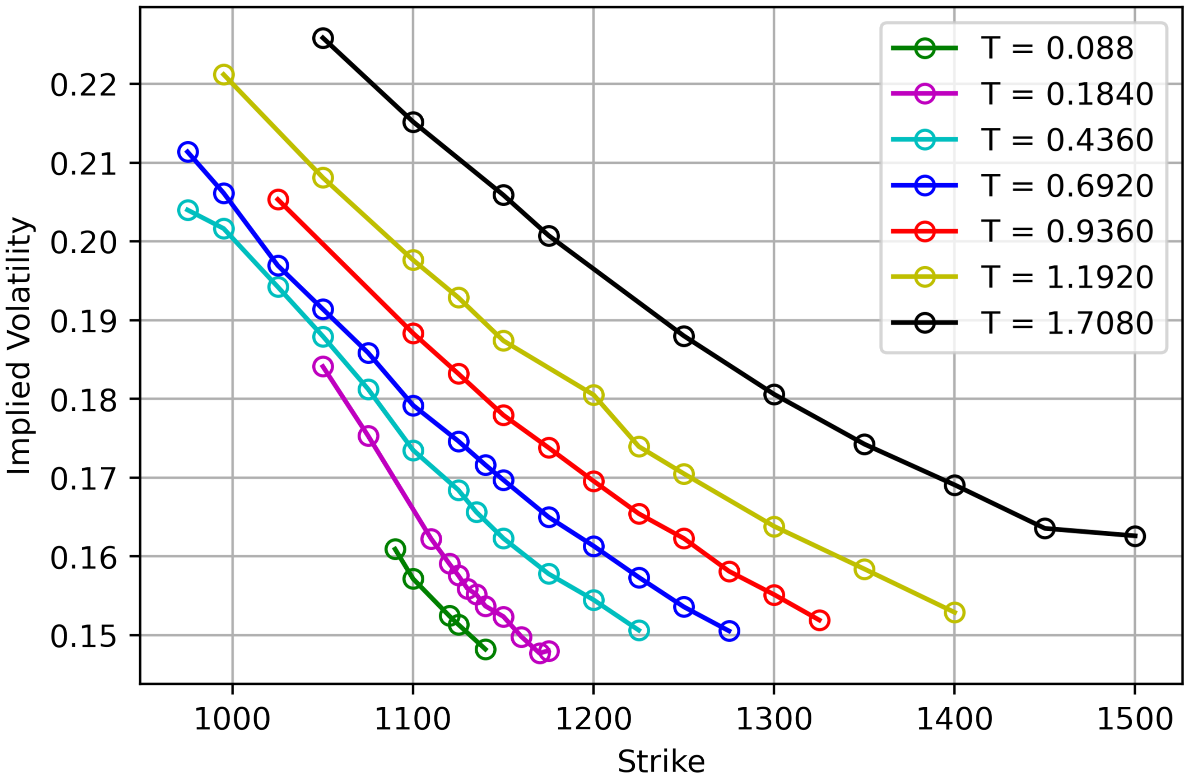

Figure 5: Implied Volatility Surface.

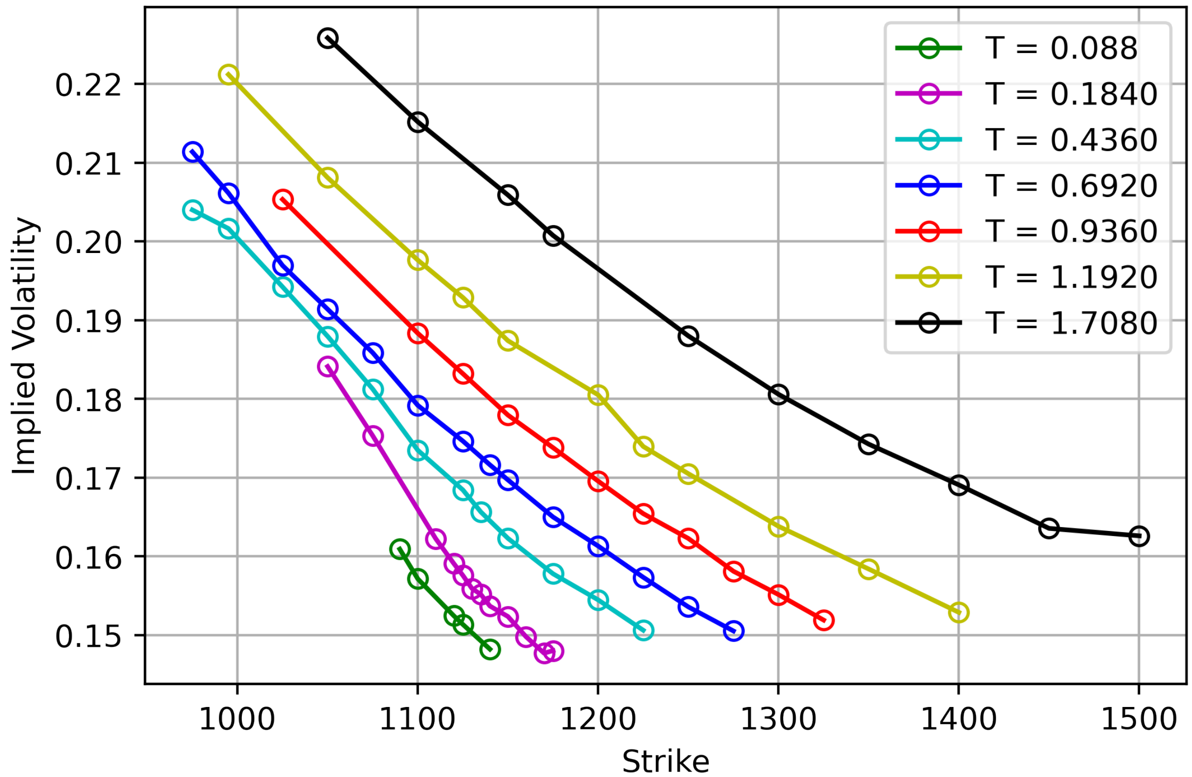

Figure 5: Implied Volatility Surface.

Remark! The main assumption of Black-Scholes model is that voltility σ is constant. However, as can be seen above, the implied volatilities depends on K and T. This is a major drawback of Black-Scholes model.

Limitations of the Black-Scholes model

We have already mentioned one limitation of the Black-Scholes model - constant volatility assumption. The other one is that log returns in BS model are normally distributed

$$L_t := \ln\left(\frac{S_t}{S_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right)$$

Empirical studies suggest that the distribution of log returns has usually heavier tails than the normal distribution (kurtosis is greater than 3) and is asymmetrical (negative skewness - price drops are greater than increases).

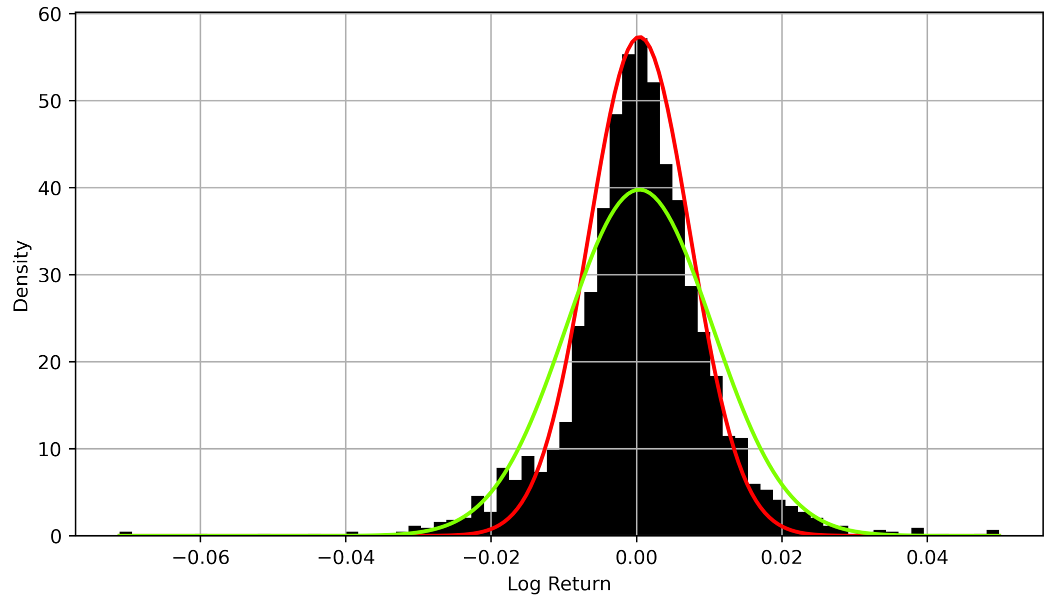

Figure 6: Distribution of log returns from S&P 500.

Model Calibration

The parameters of a given model are usually estimated using the current prices of options available in the market. This process is called **model calibration**. The purpose of calibration is to find the θ set of parameters ($\theta = \{\sigma\}$ in BS Model) of a given model, which minimizes the so-called the error function, i.e. the most often appropriately modified difference

$$|C^{(m)} - C^*(\theta)|$$

between the observed market price of option $C^{(m)}$ and the price of option $C^*(\theta)$ derived from the selected model. Our error function will be RMSE (root mean square error)

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} |C_i^{(m)} - C_i^*(\theta)|^2}$$

where N denotes the number of options used for calibration.

In Python, you can use `https://docs.scipy.org/doc/scipy/reference/optimize.html`.

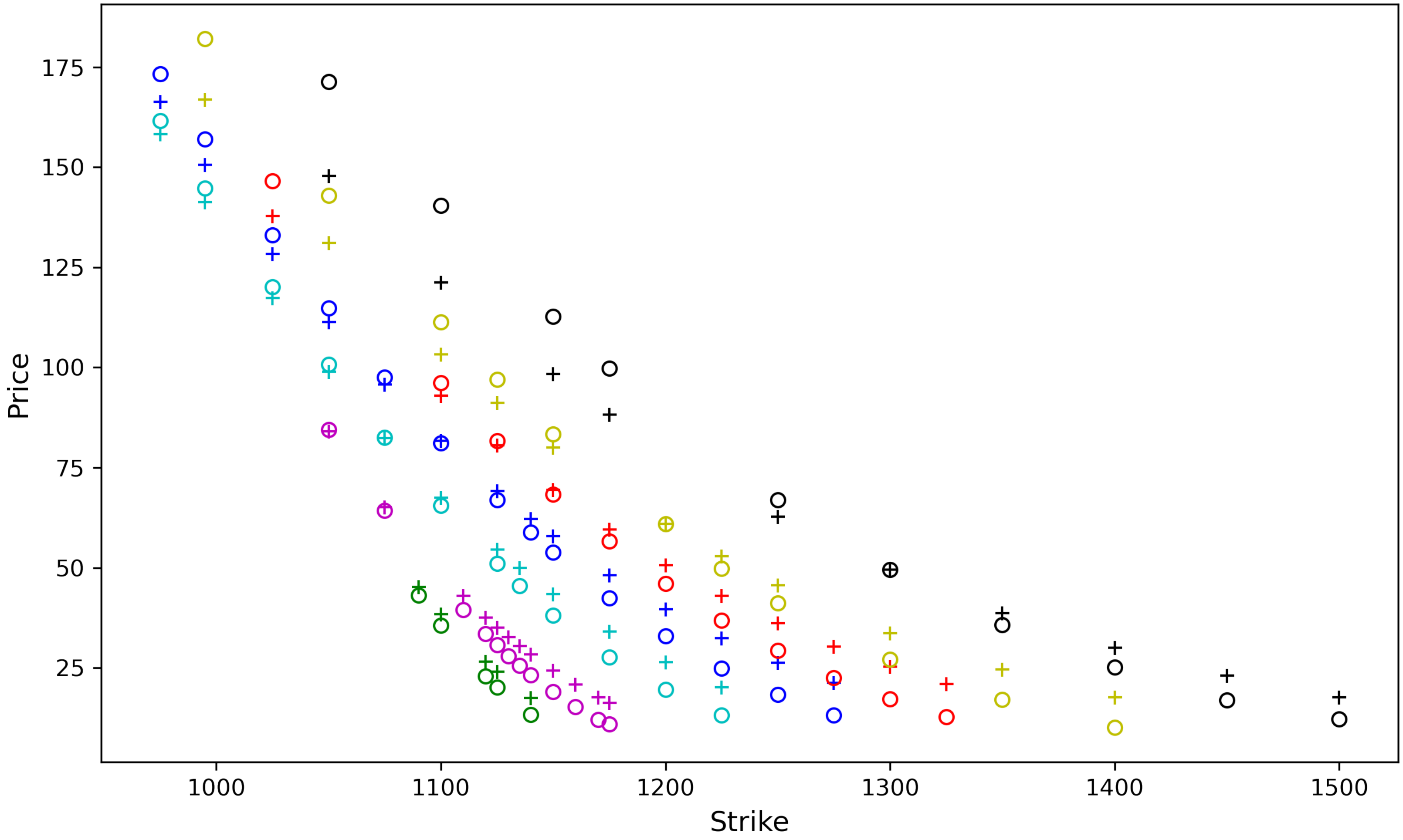

Figure 7: Example of Calibration results.

Greeks

The **Greeks** are the quantities representing the sensitivity of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. The aim of a trader is to manage the Greeks so that all risks are acceptable.

$$\epsilon = \begin{cases} 1, & \text{for call} \\ -1, & \text{for put} \end{cases}$$

Greek letter	**Derivative**	**Formula**
Δ (Delta)	$\frac{\partial}{\partial S}$	$\epsilon N\left(\epsilon d_{+}\right)$
Γ (Gamma)	$\frac{\partial^2}{\partial S^2}$	$\frac{n\left(d_{+}\right)}{S\sigma\sqrt{T-t}}$
$\mathcal{V}$ (Vega)	$\frac{\partial}{\partial \sigma}$	$Sn\left(d_{+}\right)\sqrt{T-t}$
ρ (Rho)	$\frac{\partial}{\partial r}$	$\epsilon K\left(T-t\right)e^{-r(T-t)}N\left(\epsilon d_{-}\right)$
Θ (Theta)	$\frac{\partial}{\partial t}$	$-\epsilon rKe^{-r(T-t)}N\left(\epsilon d_{-}\right)$ $-Sn\left(d_{+}\right)\frac{\sigma}{2\sqrt{T-t}}$

Table 2: Greeks.

Delta (Δ)

The delta Δ of an option is defined as the rate of change of the option price with respect to the price of the underlying asset. It is the slope of the curve that relates the option price to the underlying asset price:

$$\Delta = \frac{\partial V}{\partial S}$$

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Suppose that the delta of a call option on a stock is 0.5. This means that when the stock price changes by a small amount, the option price changes by about 50% of that amount.

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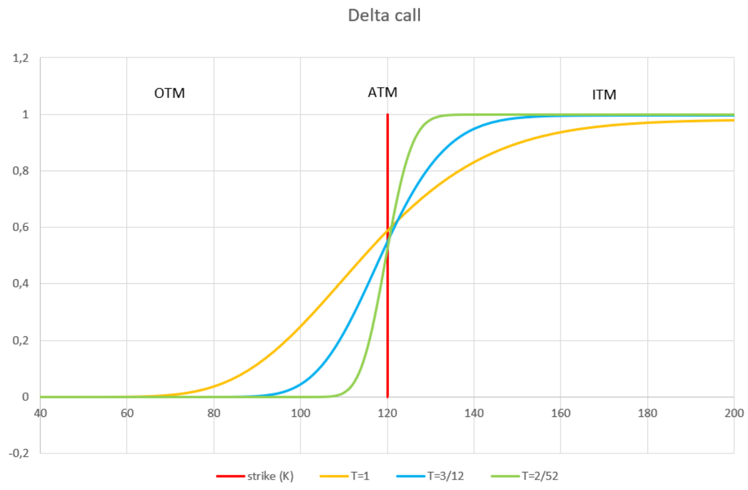
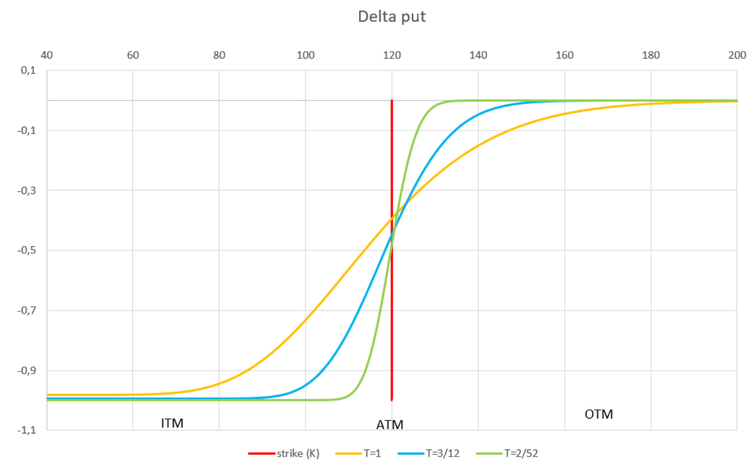
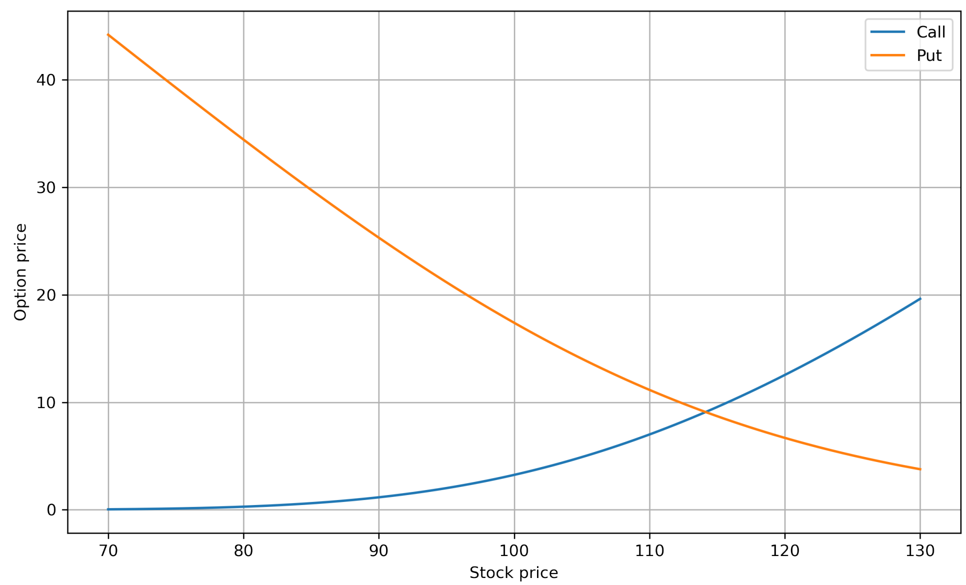
The delta of the stock position offsets the delta of the option position. A position with a delta of zero is referred to as *delta neutral*. It is important to realize that, since the delta of an option does not remain constant, the trader's position remains delta hedged (or delta neutral) for only a relatively short period of time. The hedge has to be adjusted periodically. This is known as rebalancing.



(a) Delta of a Call Option



(b) Delta of a Put Option



Gamma (Γ)

The gamma (Γ) of a portfolio of options on an underlying asset is the rate of change of the portfolio's delta with respect to the price of the underlying asset. It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}.$$

$$\Gamma_C = \Gamma_P = \frac{n\left(d_+\right)}{S\sigma\sqrt{T-t}}$$

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If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral need to be made only relatively infrequently. However, if gamma is highly negative or highly positive, delta is very sensitive to the price of the underlying asset. It is then quite risky to leave a delta-neutral portfolio unchanged for any length of time.

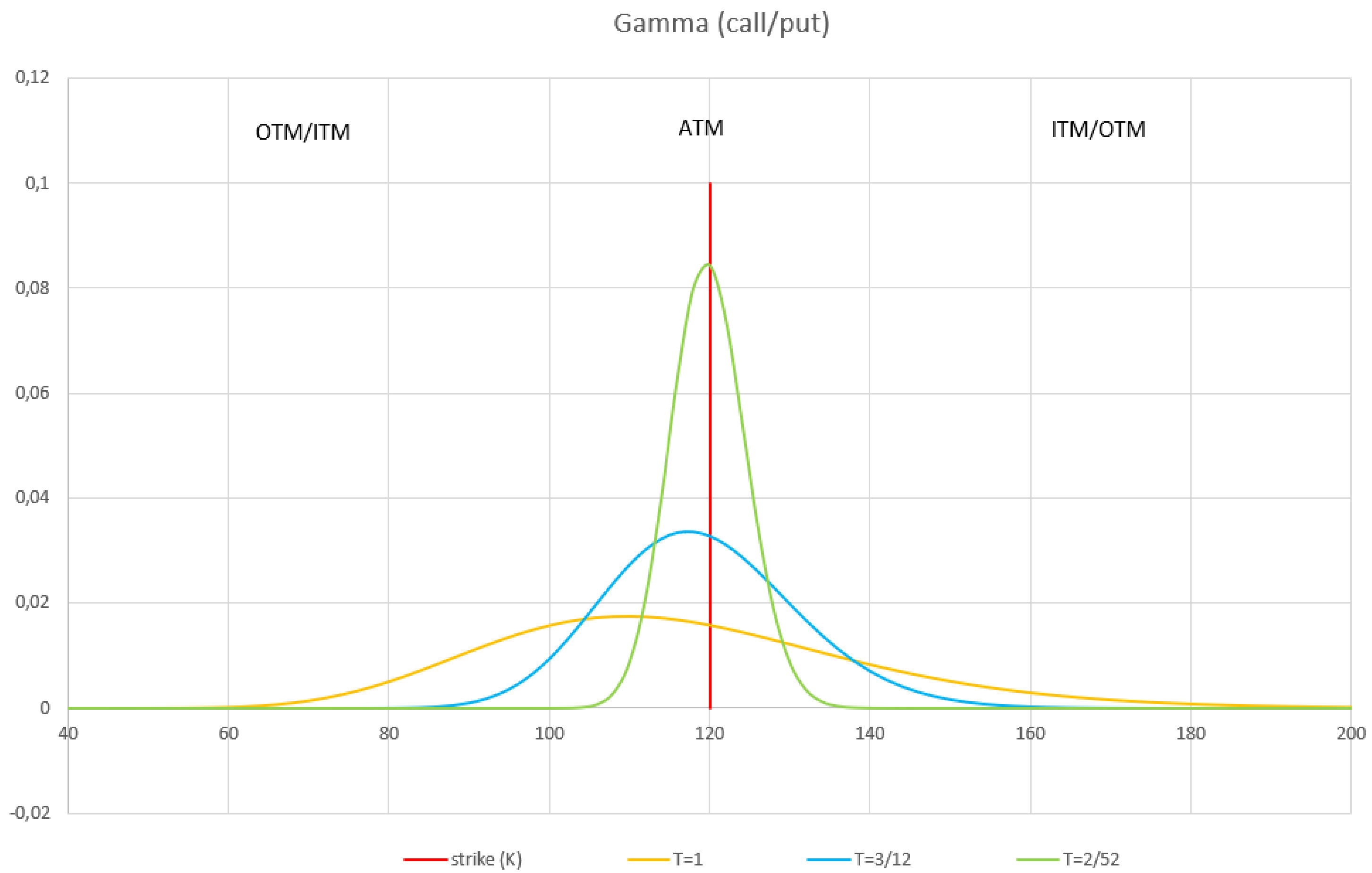

Figure 8: Gamma of the Call\Put option.

Vega ($\mathcal{V}$)

When Greek letters are calculated the volatility of the asset is in practice usually set equal to its implied volatility. The Black–Scholes–Merton model assumes that the volatility of the asset underlying an option is constant. This means that the implied volatilities of all options on the asset are constant and equal to this assumed volatility. But in practice the volatility of an asset changes over time. As a result, the value of an option is liable to change because of movements in volatility as well as because of changes in the asset price and the passage of time. The vega of an option, V, is the rate of change in its value with respect to the volatility of the underlying asset:

$$\mathcal{V}_C = \mathcal{V}_P = \frac{\partial}{\partial \sigma}(C) = Sn\left(d_+\right)\sqrt{T-t}$$

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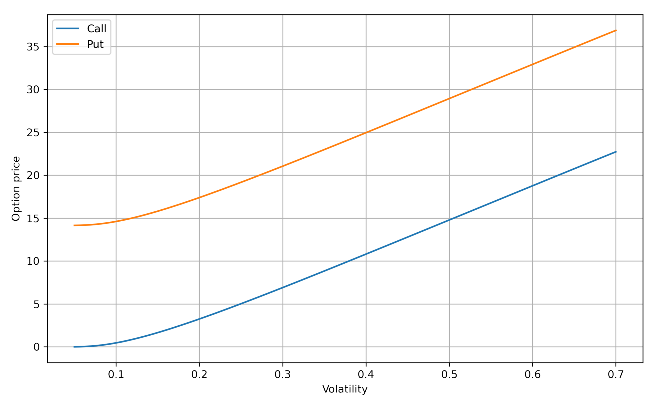
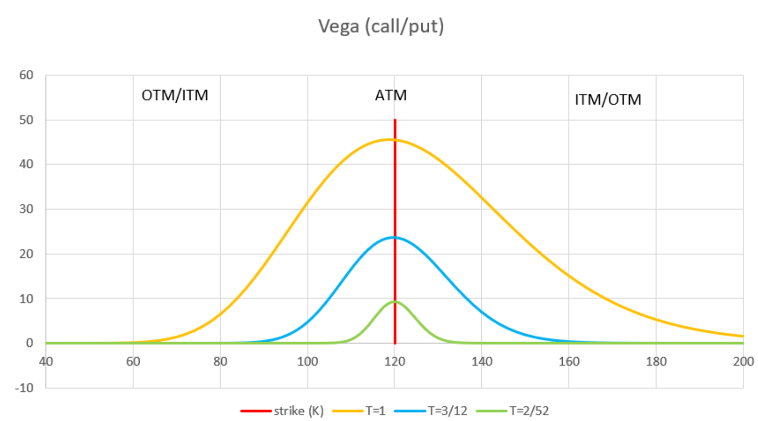
When vega is highly positive or highly negative, there is a high sensitivity to changes in volatility. If the vega of an option position is close to zero, volatility changes have very little effect on the value of the position. A position in the underlying asset has zero vega. Vega cannot therefore be changed by taking a position in the underlying asset. In this respect, vega is like gamma.



(a) Option price as a function of sigma.



(b) Vega of a Put Option

Local Volatility Model

In the **Local Volatility Model**, the dynamics of stock price is given by

$$dS_t = rS_t dt + \sigma_D(t, S_t) S_t dW_t$$

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The Dupire's Local Volatility $\sigma_D\left(t, S_t\right)$ is given by:

$$\sigma_D\left(t, S_t\right) = \sigma_D\left(T, K\right)\Big|_{T=t, K=S_t} = \frac{\frac{\partial C_{K,T}}{\partial T} + rK\frac{\partial C_{K,T}}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C_{K,T}}{\partial K^2}}$$

Heston model

Heston model belongs to the family of **stochastic volatility models.** The basic Heston model assumes that

$$dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S$$

where ν_t is the instantaneous variance and is given by CIR process

$$d\nu_t = \kappa\left(\theta - \nu_t\right)dt + \xi\sqrt{\nu_t}dW_t^{\nu}$$

and W_t^S, W_t^{ν} are Wiener processes with correlation ρ.

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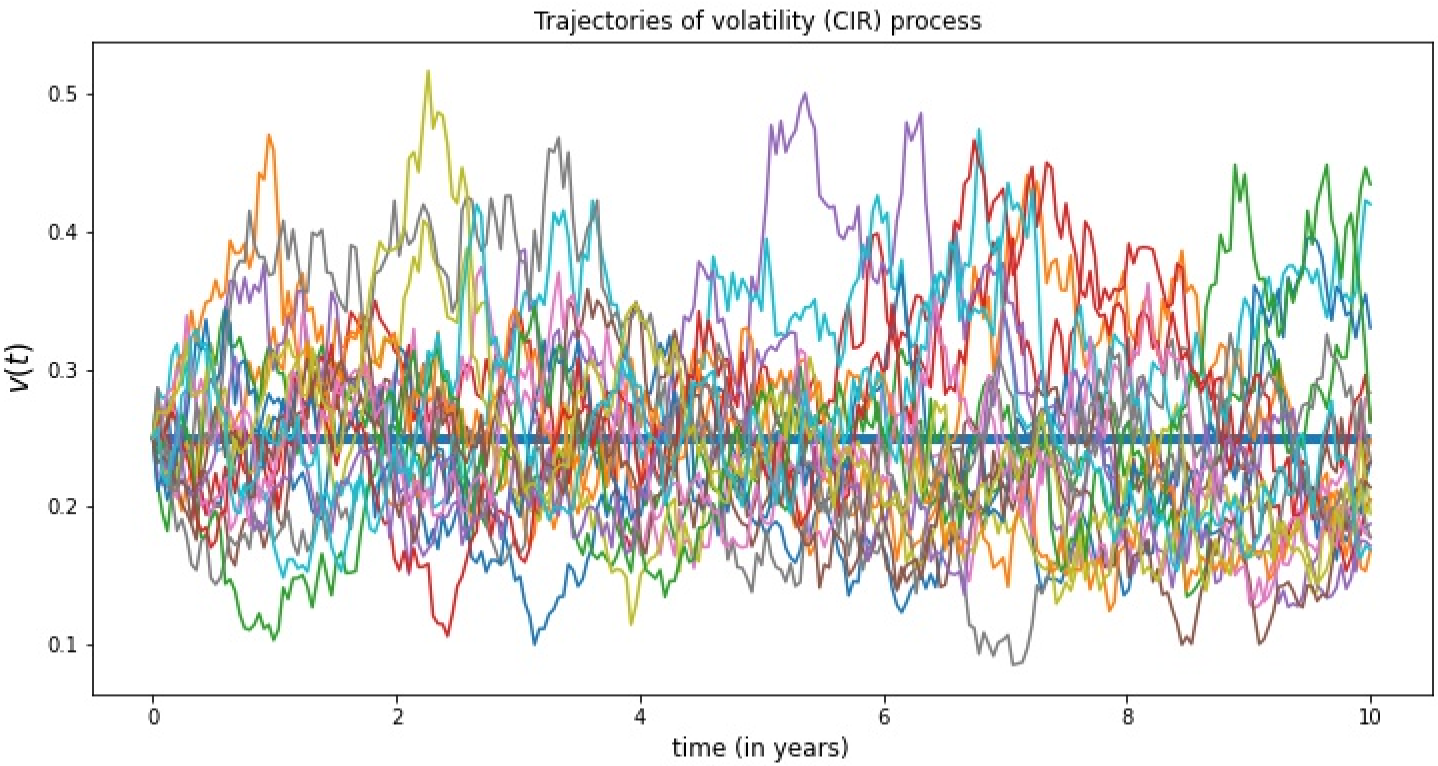

Figure 9: Trajectories of the variance process, $\kappa = 1.0, \theta = 0.25, \xi = 0.2$

Jump-diffusion models

Jump-diffusion process is a stochastic process which involves both diffusion and jump component:

$$dX_t = \mu\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dW_t + \gamma\left(t, X_t\right) dJ_t, \quad J_t = \sum_{k=1}^{N_t} Z_t$$

Two popular examples are **Merton model** and **Kou model**.

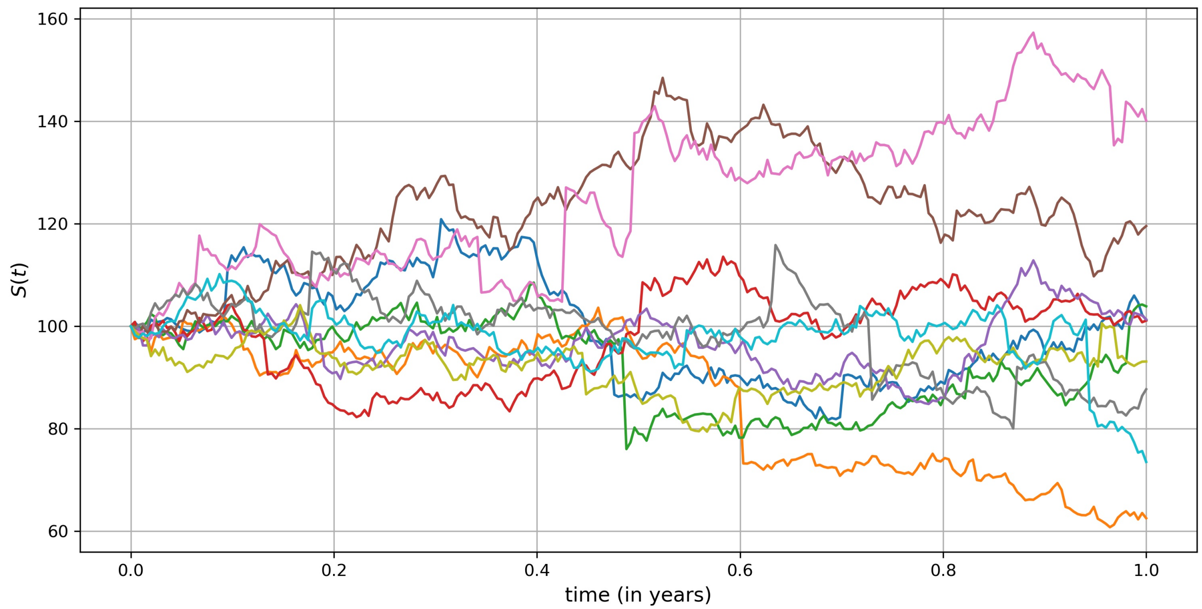

Figure 10: Paths of Jump-Diffusion Process.

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[3] Hilpisch, Y., *Python for Finance: Analyze Big Financial Data*, 2019.

[4] `https://www.youtube.com/c/quantpie`

Appendix

Black-Scholes SDE solution

We apply Ito's Lemma to a function

$$F(t, x) = \ln(x), x > 0.$$

The partial derivatives are

$$\frac{\partial F}{\partial t} = 0, \quad \frac{\partial F}{\partial x} = \frac{1}{x}, \quad \frac{\partial^2 F}{\partial x^2} = -\frac{1}{x^2}.$$

Next

$$\begin{aligned} d\ln(S_t) &= \left(rS_t \frac{1}{S_t} + \frac{\sigma^2 S_t^2}{2} \frac{-1}{S_t^2} \right) dt + \sigma S_t \frac{1}{S_t} dW_t \\ &= \left(r - \frac{\sigma^2}{2} \right) dt + \sigma dW_t \end{aligned}$$

Integrating both sides we get

$$
\begin{aligned}
\int_0^t d\ln\left(S_s\right) &= \int_0^t \left(r\text{-}\frac{\sigma^2}{2}\right) ds + \int_0^t \sigma dW_s \\
\ln S_t - \ln S_0 &= \left(r\text{-}\frac{\sigma^2}{2}\right) t + \sigma\left(W_s - W_0\right) \\
\ln S_t &= \ln S_0 + \left(r\text{-}\frac{\sigma^2}{2}\right) t + \sigma W_T \\
S_t &= S_0 e^{\left(r\text{-}\frac{\sigma^2}{2}\right)t+\sigma W_t} \blacksquare
\end{aligned}
$$

Exercise. Apply Ito's formula to function $F\left(t,x\right) = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma x}$ and process $dX_t = dW_t$ to get (4).

Derivation of Call\Put Option prices in BS Model

$$\boxed{S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x} - K \geq 0 \iff x \leq \frac{\ln\frac{S_0}{K} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} =: d}$$

$$\begin{aligned}
C_0 &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[(S_T-K)^+\right] = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\left(S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W_t} - K\right)^+\right] \\
&= e^{-rT}\int_{\mathbb{R}}\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x} - K\right)^+ \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx \\
\text{(from the frame above)} &= e^{-rT}\int_{-\infty}^{d} S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx - e^{-rT}\int_{-\infty}^{d} K\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx\} \\
&= S_0 e^{-\frac{\sigma^2}{2}T}\int_{-\infty}^{d}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2+\sigma\sqrt{T}}\,dx - e^{-rT}K\Phi(d) \\
&= S_0 e^{-\frac{\sigma^2}{2}T}\int_{-\infty}^{d}\frac{1}{\sqrt{2\pi}}e^{\frac{\sigma^2 T}{2}}e^{-\frac{1}{2}\left(x-\sigma\sqrt{T}\right)^2}\,dx - e^{-rT}K\Phi(d) \\
\left\{y = x-\sigma\sqrt{T}\right\} &= S_0\int_{-\infty}^{d-\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,dy - e^{-rT}K\Phi(d) \\
&= S_0\Phi\left(d-\sigma\sqrt{T}\right) - e^{-rT}K\Phi(d)
\end{aligned}$$

Note that

$$
\begin{aligned}
d_{-} \quad &:= \quad d = \frac{\ln \frac{S_0}{K} - \left(r - \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \\
d_{+} \quad &:= \quad d - \sigma\sqrt{T} = \frac{\ln \frac{S_0}{K} - \left(r - \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} - \sigma\sqrt{T} = \frac{\ln \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}
\end{aligned}
$$

Hence

$$
\boxed{C_0 = S_0 \Phi\left(d_{+}\right) - K e^{-rT} \Phi\left(d_{-}\right)}
$$

Put Option price

$$\boxed{C_0 = S_0\Phi\left(d_+\right) - Ke^{-rT}\Phi\left(d_-\right)}$$

$$\boxed{C_0 - P_0 = S_0 - Ke^{-rT}}$$

$$\boxed{\Phi\left(-x\right) = 1 - \Phi\left(x\right)}$$

From Call-Put Parity we have

$$\begin{aligned}
P_0 &= C_0 - S_0 + Ke^{-rT} \\
&= S_0\Phi\left(d_+\right) - Ke^{-rT}\Phi\left(d_-\right) - S_0 + Ke^{-rT} \\
&= S_0\left(\Phi\left(d_+\right) - 1\right) - Ke^{-rT}\left(\Phi\left(d_-\right) - 1\right) \\
&= S_0\left(1 - \Phi\left(-d_+\right) - 1\right) - Ke^{-rT}\left(1 - \Phi\left(-d_-\right) - 1\right) \\
&= \boxed{Ke^{-rT}\Phi\left(-d_-\right) - S_0\Phi\left(-d_+\right)}
\end{aligned}$$