Public

Introduction to Monte Carlo Simulation in Finance

Simulating Interest Rates Derivatives Profiles for Counterparty Credit Risk Management

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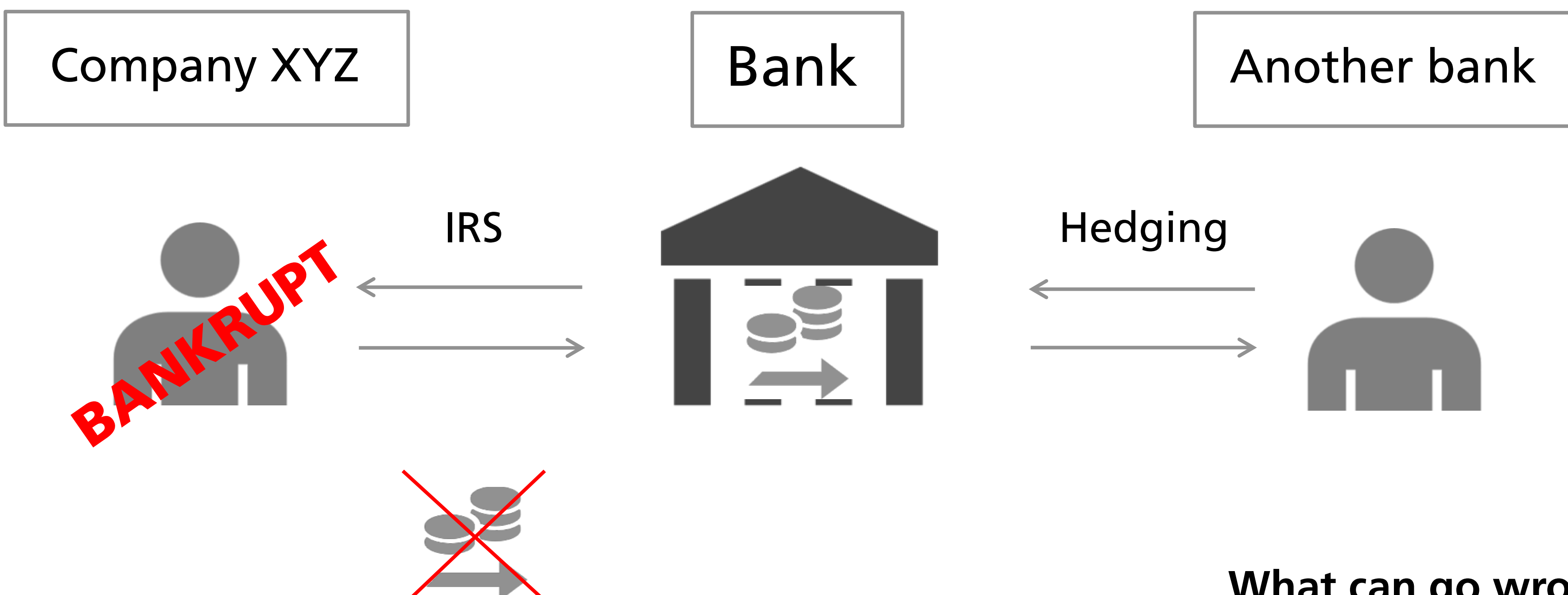
Section 1

Counterparty Credit Risk - Introduction

Counterparty Credit Risk - Introduction

- UBS, and other IBs, trade derivatives with various counterparties worldwide;
- Typical scenario:
  - Company XYZ issues $1bi of 15y bonds at fixed rate;
  - Company XYZ enters into a swap with UBS to transform its fixed rate liability into a floating one, ie, UBS pays fixed rate to XYZ Co. vs receiving floating for 15y in $1bi;
  - To hedge market risk, UBS enters into offsetting transactions with another Bank.



What can go wrong?

Counterparty Credit Risk - Introduction

Expected loss – simple example

Example:

$100mio 5y loan to a company

Company has no tangible assets

Probability of default over next 5y is 5%

Expected Loss = EAD x PD x LGD

- EAD: Exposure at Default
- PD: Probability of Default
- LGD: Loss Given Default

1/ What is the EAD?

2/ What is the LGD roughly?

For a corporate loan, it is easy to determine the EAD (Exposure at Default). But how can we do it for a portfolio of derivatives?

Counterparty Credit Risk - Introduction

Counterparty credit risk

Counterparty credit risk is a risk that a counterparty will not be able to meet its obligations (payments). It is different from the credit risk of bonds and loans.

- The loss amount is not known until the time of bunkruptcy
- The risk is typically bilateral, both parties take the risk.

Instruments generating counterparty risk:

OTC Derivatives	Financing transactions
forwards	repo
swaps	reverse repo
CDS	security lending
European/American/Asian Options	

In financial institutions this risk impacts in particular two areas:

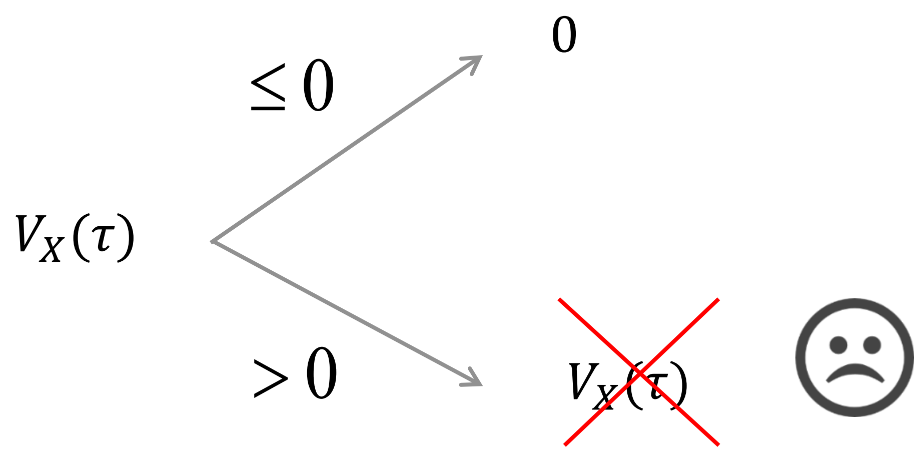
pricing and capital requirements.

Counterparty Credit Risk - Introduction

Credit exposure

We have a derivative X having value $V_X(t)$ at time t.

If the counterparty defaults at time τ. Our loss is:



Credit exposure at time t is the replacement value of the contract if it is positive and zero otherwise.

$$E_X(t) = max(V_X(t), 0)$$

We don't know the future value of $V_X(t)$, the exposure is random!

Counterparty Credit Risk - Introduction

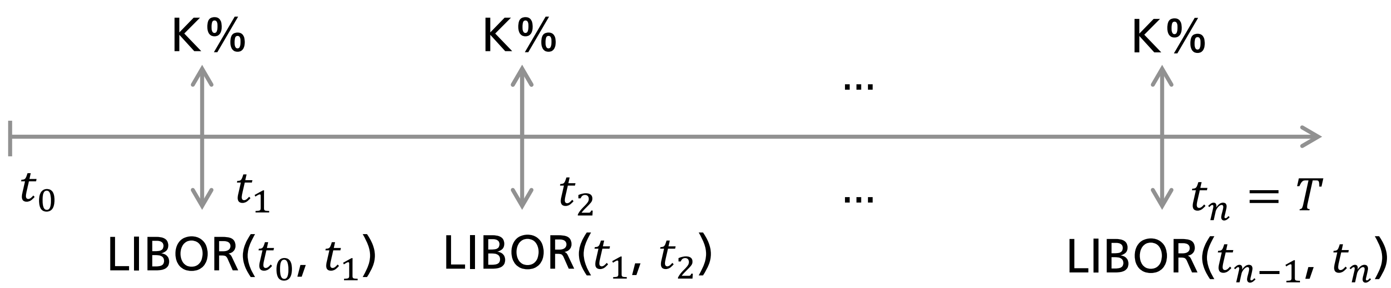
Profiles' definitions

Profile	Description	How to compute
$\mathbf{EE}_t$	Expected Exposure: Expectation of the portfolio value floored at zero.	$EE_t = \mathbb{E}(\max(\mathrm{V_t}, 0))$
$\mathbf{PFE}_t$	Potential Future Exposure: 97.5% quantile of the portfolio value at time t	$PFE_t = \inf\{x: P(\mathrm{V_t} \leq x) \geq \alpha)\}$, α=97.5%
$\mathbf{Rev\_EE}_t$	Reverse Expected Positive Exposure: Equivalent to $\mathrm{EPE_t}$ but from the counterparty's point of view	$Rev\_EE_t = \mathbb{E}(\max(-\mathrm{V_t}, 0))$
$\mathbf{TCM}_{t,\delta}$	Total Claim Mean: For a given close-out period of length δ, also called margin period of risk (MPR), the TCM is given by the formula on the RHS.	$TCM_{t,\delta} = \mathbb{E}(\max(\mathrm{V}_{t,\delta} - \mathrm{V_t}, 0)^+$

Counterparty Credit Risk - Introduction

Example (interest-rate swap)

Interest Rate Swap (IRS) is a contract, in which two counterparties to agree to exchange future interest payments.



K – swap rate which is set at the beginning of the contract, usually so that the initial value of the contract is zero: $V_{swap}(0) = 0$.

$V_{swap}(t) = \sum_{i=t+1}^{T} DF(i)(K - F(t))$, where $DF(i)$ is a discount factor,

$F(t)$ - forward swap rate (starts at t and expires at T).

Swaption is an option that gives its holder right to enter into a swap.

Counterparty Credit Risk - Introduction

Example (interest-rate swap) – cnt'd

For an interest rate swap the expected exposure is:

$$\mathbb{E}(max(V_X(t),0)) = \underbrace{\mathbb{E}(max\,(\textstyle\sum_{i=t+1}^{T} DF(i)(K-F(t)),0))}_{\text{price of the swaption on the remaining part of the swap}}$$

Calculation of the expected exposure profile for an interest rate swap can be brought down to calculation of the prices of relevant swaptions. To calculate them we would need to choose a model and know volatility of the forward rates.

⇩

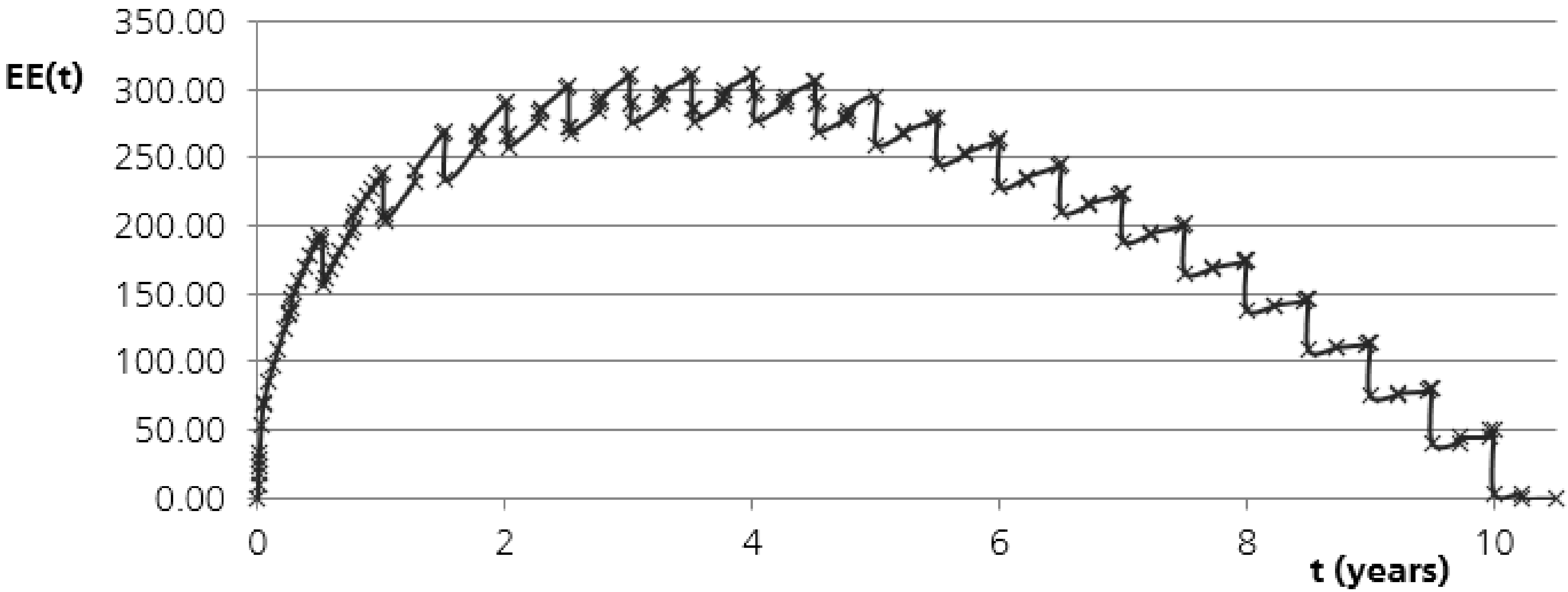
Even for the simplest products calculation of the expected exposure

gives rise to model risk!

Counterparty Credit Risk - Introduction

Example exposure profile of an at-the-money interest-rate swap

For an at-the-money 10Y swap, where we receive fixed payments the EE profile looks as follows:



The shape of the profile is the results of combination of two effects :

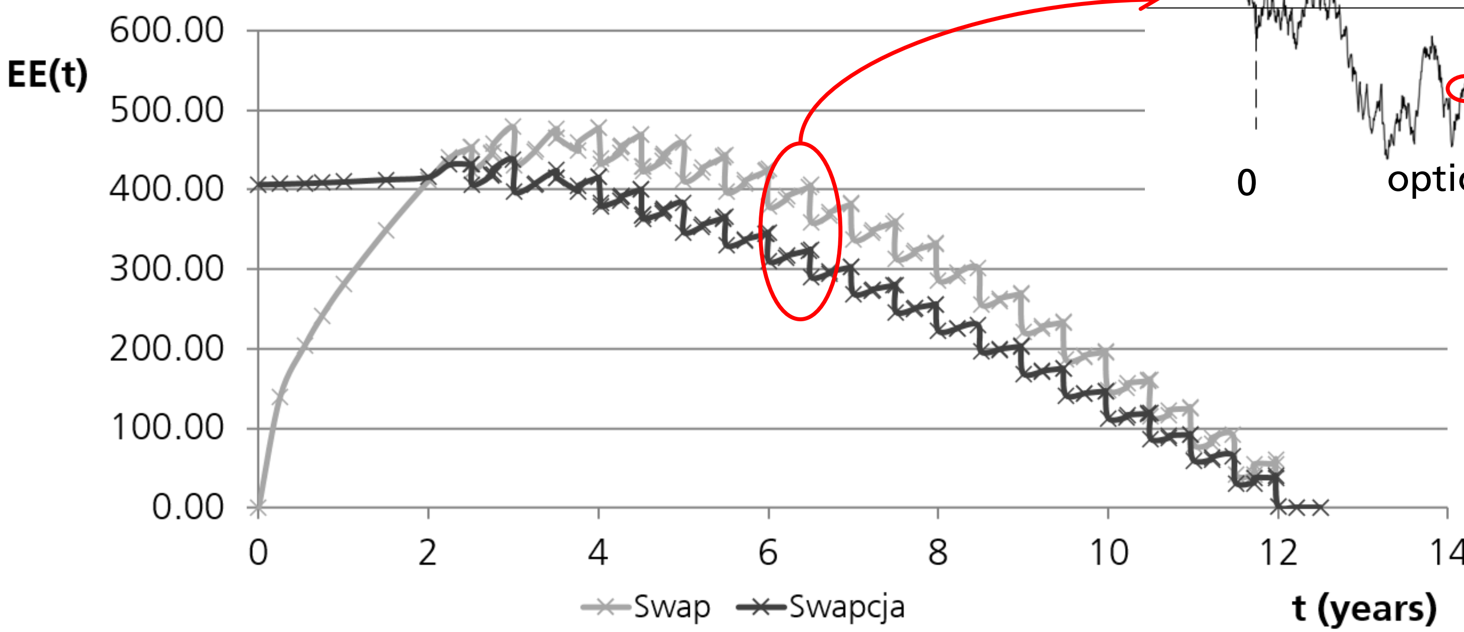
- Increasing uncertainty regarding the future cashflows,
- Number of future cashflows, decreasing in time.

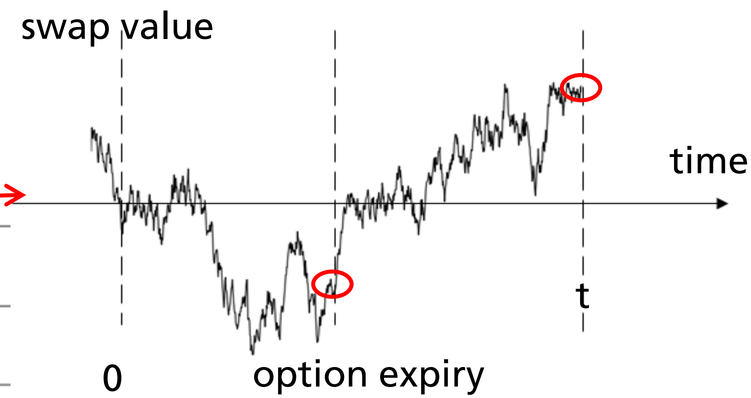
Counterparty Credit Risk - Introduction

Example expected exposure profile of an at-the-money interest-rate swaption

Exposure profile of a 2Yx10Y swaption

and the underlying forward swap:





- Until the expiry of the option its value is at least the intrinsic value.
- After the expiry in some of the scenarios the option will not be exercised.

Section 4

Monte Carlo Simulations

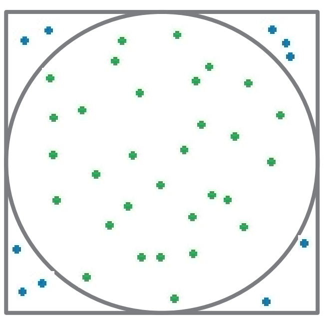
Monte Carlo Simulations

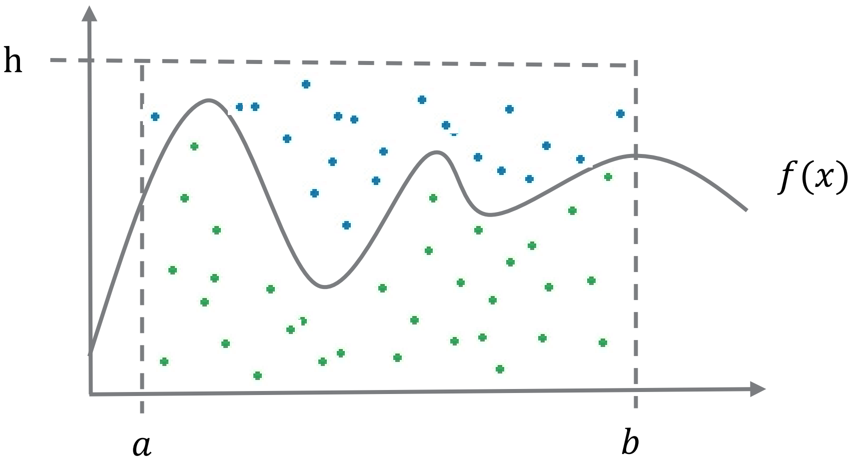
Monte Carlo methods - introduction

Monte Carlo methods – a class of algorithms that rely on repeated random sampling to obtain numerical results.

Some famous examples:

- approximation of π
- approximation of the value of an integral





$$\pi \approx 4 * \frac{number\ of\ dots\ falling\ in\ the\ circle}{total\ number\ of\ dots}$$

$$\int_a^b f(x)dx \approx (b-a) * h * \frac{num.\ of\ dots\ below\ the\ graph}{total\ number\ of\ dots}$$

Monte Carlo Simulations

Monte Carlo methods – introduction cnt'd

Monte Carlo is commonly used for estimation of statistics (like expected value and high quantiles) of a random variable, when we know that it is a function of another random variable with a known distibution.

If we have X and want to calculate expected value of f(X) we can estimate it with

$\hat{\alpha}_N = \frac{1}{N}\sum_{i=0}^{N} f(X_i)$, where X_i are independently sampled from the distribution of X.

From The Law of Large Numbers we get that: $\mathbb{E}(\hat{\alpha}_N) \to \mathbb{E}\big(f(X)\big)$.

A concrete example:

S – stock price process following Black-Scholes model

We can price a European call option: $C(0,t) \approx e^{-rT}\frac{1}{N}\sum_{i=0}^{N} \max\big(S_i^T - K, 0\big)$.

What is the precision of the estimate?

From Central Limit Theorem: $\hat{\alpha}_N \xrightarrow{d} N\left(\mathbb{E}\big(f(X)\big), \frac{s_N}{\sqrt{N}}\right)$, so we have confidence interval for $\mathbb{E}\big(f(X)\big)$:

$\left(\hat{\alpha}_N - z_{\frac{\delta}{2}}\frac{s_N}{\sqrt{N}}, \hat{\alpha}_N + z_{\frac{\delta}{2}}\frac{s_N}{\sqrt{N}}\right)$, where $z_\delta : N(z_\delta) = 1-\delta$, s_N - std. dev. of $f(x_i)$

Monte Carlo Simulations

Stochastic process path generation example

We need to generate the whole paths for:

- Pricing of path-dependent derivatives (Asian options, American options, barriers, …)
- Pricing in short rate models
- Calculation of the exposure profiles

Let $t_0 = 0 < t_1 < \ldots < t_n = T$,

The stock price process in Black-Scholes model:

$$dS(t) = rS(t)dt + \sigma\, S(t)dW(t), \qquad S(0) = S_0$$

The solution is:

$$S(t) = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\}, \qquad S(0) = S_0$$

Monte Carlo Simulations

Stochastic process path generation example cnt'd

After simple transformations we get:

$$S(t_{i+1}) = \mathrm{S}(t_i)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_{i+1}\right\},$$

A recipe to generate Black-Scholes stock price path:

1) Take $S(t_0) = S_0$
2) Generate a random numer from the standard normal distribution – Z_1
3) $S(t_1) = \mathrm{S}(t_0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(t_1 - t_0) + \sigma\sqrt{t_1 - t_0}Z_1\right\}$
4) Generate another random numer from the standard normal distribution – Z_2
5) $S(t_2) = \mathrm{S}(t_1)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(t_2 - t_1) + \sigma\sqrt{t_2 - t_1}Z_2\right\}$

…

[Proceed recursevely like that until you reach $t_n = T$.]

Modelling Counterparty Credit Risk

Stochastic process path generation – general recipe

How do we discretize and simulate a stochastic process in general?

$$dX(t) = a(X(t))dt + b(X(t))\, dW(t), \qquad X(0) = X_0$$

The Euler discretization scheme:

$$\hat{X}(t_{i+1}) = \hat{X}(t_i) + a\left(\hat{X}(t_i)\right)(t_{i+1} - t_i) + b\left(\hat{X}(t_i)\right)\sqrt{(t_{i+1} - t_i)}Z_{i+1},$$
$$i = 0, \dots, n-1, \qquad Z_i \ - i.i.d.\ \ N(0,1)$$

The estimation error is:

$$\mathbb{E}(|X(T) - \hat{X}(hn)|) \leq c_T\sqrt{h}$$

Increasing the precision by 10 mean increasing the numer 100.

Modelling Counterparty Credit Risk

Simulation of correlated processes

If $W_1(t)$ and $W_2(t)$ are two independent Wiener processes and $\rho \in [-1,1]$, then

$$\widetilde{W_1} = W_1$$

$$\widetilde{W_2} = \rho W_1 + \sqrt{1-\rho^2} W_2$$

Are correlated Wiener processes with correlation ρ.

In a generalized case, using the fact that if $Z \sim N_k(0, I)$ then $X = JZ \sim N_k(0, JJ^T)$

If we want to simulate a sample from multivariate normal distrubtion with covariance matrix Σ, we have to find J so that $JJ^T = \Sigma$.

We can transfer that to Wiener processes:

$[\widetilde{W_1}(t), \dots, \widetilde{W_k}(t)]^T = J[W_1(t), \dots, W_k(t)]^T$ are correlated and their covariance matrix is Σ, if J so that $JJ^T = \Sigma$.

Modelling Counterparty Credit Risk

Simulation of correlated processes

Methods of determining ``square-root’’ od a matrix

- Cholesky decomposition

  For a positively definite, symmetric matrix A there exists a mtrix L so that $A = L^T L$. Matrix L is a lower-triangle matrix, and it can be constructed with Cholesky-Banachiewicz algorithm.

- Singular-value decomposition (SVD)

  A symmetric, positevely semidefinite matrix A can be expressed as $A = VSV^T$,

  Where V is a ortonormal matrix, whose columns are eigenvectors of A, and matrix S=diag([$s_1, \ldots, s_k$]), where s_i are eigenvalues of A

  Then J $= VS^{1/2}V^T$, $S^{1/2}$= diag([$\sqrt{s_1}, \ldots, \sqrt{s_k}$]) is a symmetric, positevely semidefinite matrix and A= JJ^T holds.

Section 3

Modelling Counterparty Credit Risk

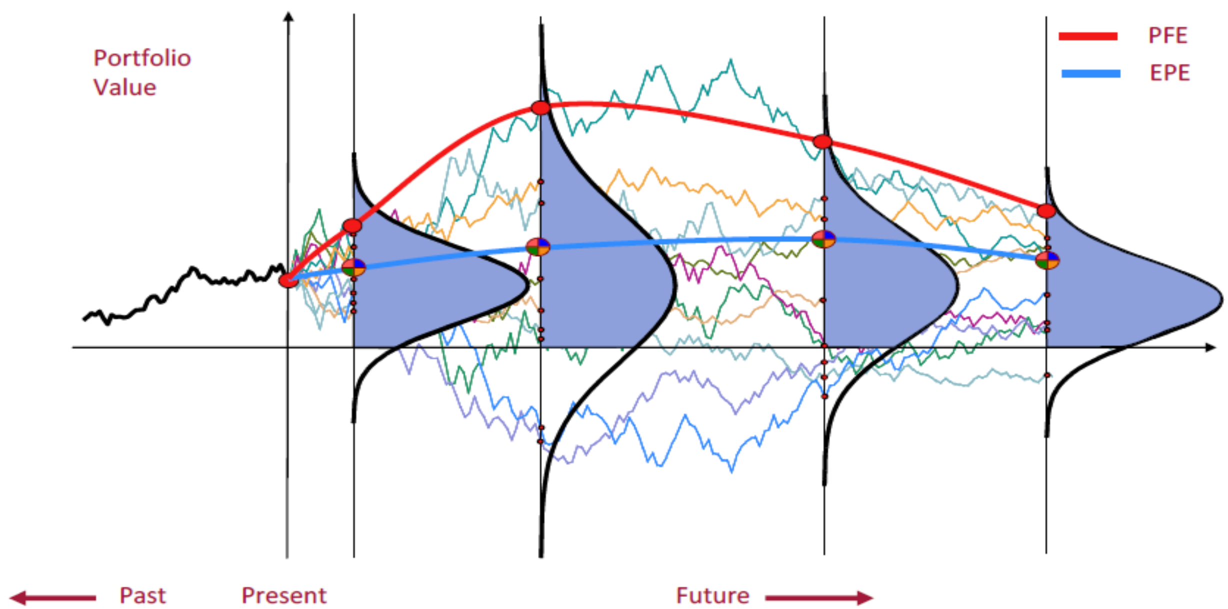
Modelling Counterparty Credit Risk

Expected Loss in the context of derivatives

We can simulate the evolution of the portfolio value over time

Potential Future Exposure (PFE): 97.5% quantile

Expected Positive Exposure (EPE): Average with values floored at 0



The equivalent of Expected Loss in derivatives world is called CVA (Credit Valuation Adjustments):

$$\boxed{\mathrm{CVA}_{0,T} = \overbrace{(1-R_V)}^{\text{LGD}} \int_0^T \mathbb{E}\left[\overbrace{N_u^{-1} \; V_u^+}^{\text{EAD part}} \; \overbrace{\mathbb{1}_{\tau=u}}^{\text{PD part}} \, du\right]}$$

approx.

Present Value of EPE

$$\boxed{CVA = \sum_{n=1}^{N} LGD \; * \; PD(T_{n-1}, T_n) \; * \; EEPV \; (T_{n-1}, T_n)}$$

Modelling Counterparty Credit Risk

General CCR Modelling Recipe

Goal	**Obtain risk profiles (EPE, PFE) for a given portfolio in order to compute CVA, RWA and regulatory capital.**
Recipe	1. Choose a suitable risk factor model 2. Calibrate the model 3. Generate simulated paths, ie, simulate a variable of interest, eg.: short rate, stock price, spot FX; 4. Price the portfolio 5. Compute profiles 6. Compute measures like CVA, FVA, RWA, Regulatory Capital

Modelling Counterparty Credit Risk

Challenges

The task of the computation of the exposure profiles for a derivative portfolio seems analogous to Value-at-Risk calculation but can be far more complicated due to :

- far longer time horizon (often 30Y and beyond)
- the risk-mitigants involved (netting, collateral, break clauses)

- **250** counterparties
- **40** transactions on average
- **100** simulation steps
- **10,000** scenarios

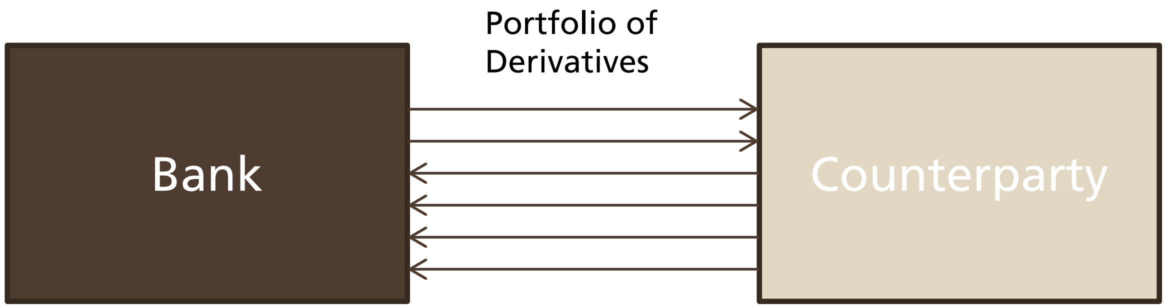
10 bln revaluations!

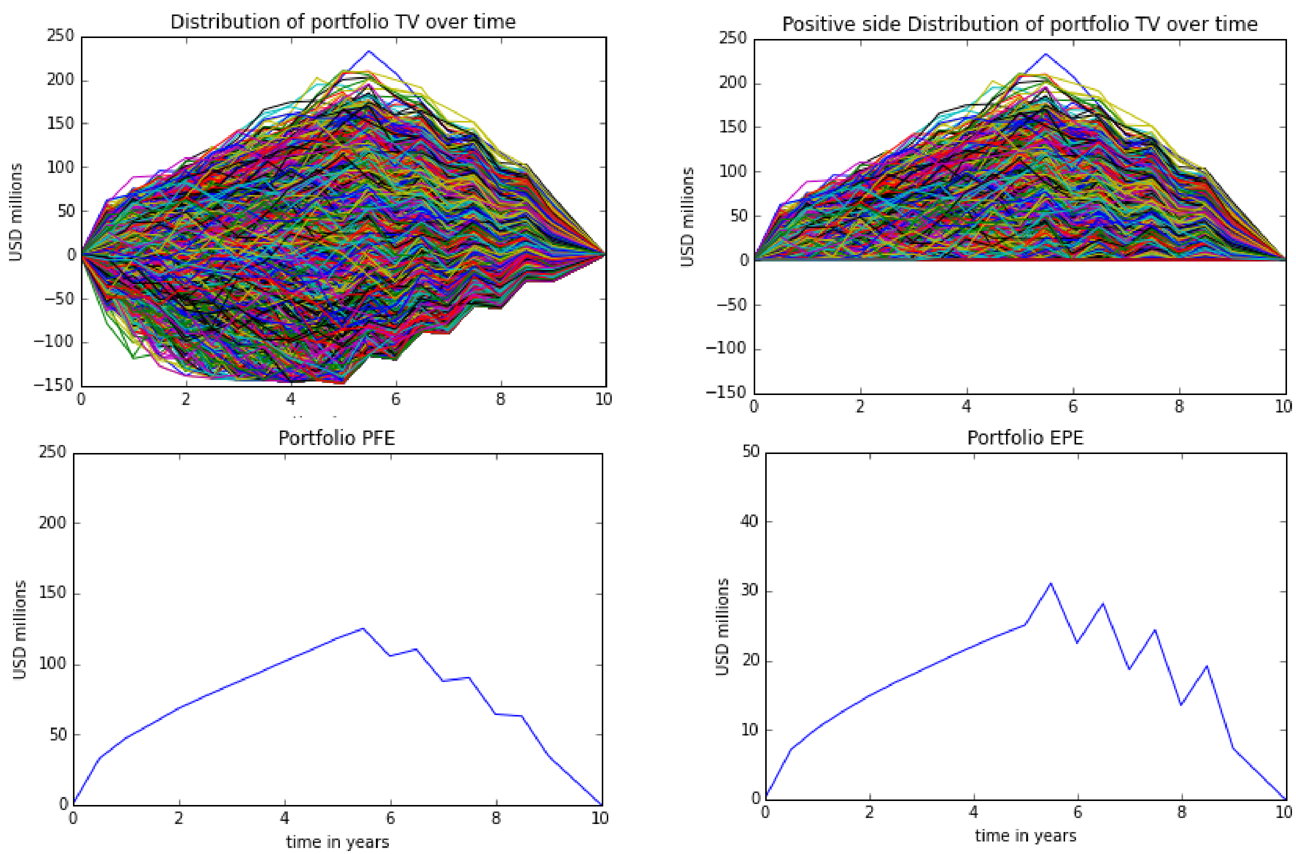
Efficiency of computation is very important

Algorithms and models involved	1. Correlated random number generation (SVD, Cholesky Decomposition) 2. Calibration – Bootstrapping, Minimum Squared Error, Closed Form Pricing 3. Pricing 4. Backward Induction / AMC

Modelling Counterparty Credit Risk

Portfolio Simulation





Section 4

Case Overview Revisited

Case Overview Revisited

What is our goal?

Calculate Risk Weighted Assets for the Bank using the Internal Model approach for three counterparties (Salzburg Bank, Bank of Cluj, Bank of Mazowsze) *as of 23rd February 2018. Assume the notional for all trades is 1.000.000.000 USD.*

Saltzburg Bank of Tyrol (ID)(netted) :

- 1y Call Option on SPDRM striked at 95 USD
- 2y Put Option on SPDRM striked at 115 USD

Bank of Cluj (ID)(non netted):

- 6m Call Asian Option on IRNMN striked at 570 USD
- 1.5y Put Asian Option on IRNMN striked at 450 USD

Bank of Mazowsze (ID)(netted):

- 1.5y Call American Option on AVNG striked at 4100 USD
- 9m Put Asian Option on AVNG striked at 4100 USD

* Any relations to the real tickers is purely coincidental

Section 6

Backward Induction and American Monte-Carlo Methods

Backward induction – 1st example

Let's consider a game where we roll a fair die three times. After the first and the second draw we can get a payoff equal to the number drawn or have a chance to roll the die one more time (forgeting about the previous result). If we choose to roll for the third time we get the result of the third rolling.

How do you assess what is the fair price for entering into the game?
At what level drawn the first or second time you should stop rolling and exercise the payoff and at what level you should continue rolling?

Using backward induction:

3rd rolling

we know that in the 3rd rolling if it happens the expected value is:

$$\frac{1}{6}\sum_{i=1}^{6} \mathrm{i} =$$

$$\frac{1}{6}(1+2+3+4+5+6) = 3.5$$

2nd rolling

it means that after the 2nd rolling the player should reject values less than 3.5 ie. 1, 2 and 3.

Then the expected value of the gain from the 2nd and 3rd rolling is:

$$\frac{1}{6}\sum_{i=1}^{6} \max(i, 3.5) =$$

$$\frac{1}{6}(3.5+3.5+3.5+4+5+6) = 4.25$$

1st rolling

hence at the first rolling we should reject values below 4.25 ie. $1, 2, 3, 4$.

Then the expected value of the gain from the whole game (price) is:

$$\frac{1}{6}\sum_{i=1}^{6} \max(i, 4.25) =$$

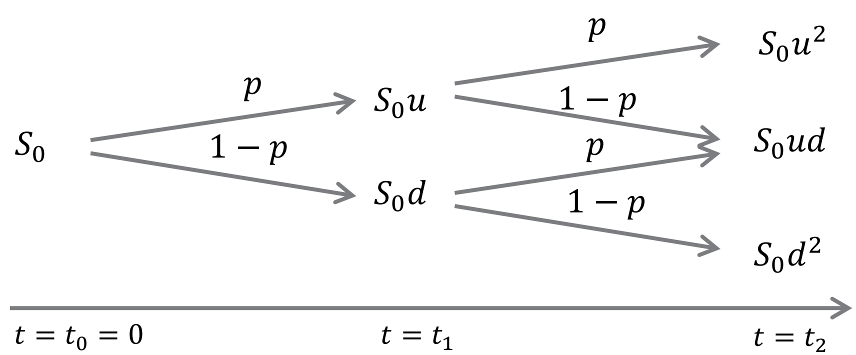
$$\frac{1}{6}(4.25+4.25+4.25+4.25+5+6) =$$

$$4.66$$

Backward Induction and American Monte-Carlo Methods

Backward induction – binomial tree

Example: stock price evolution – 2-step binomial tree



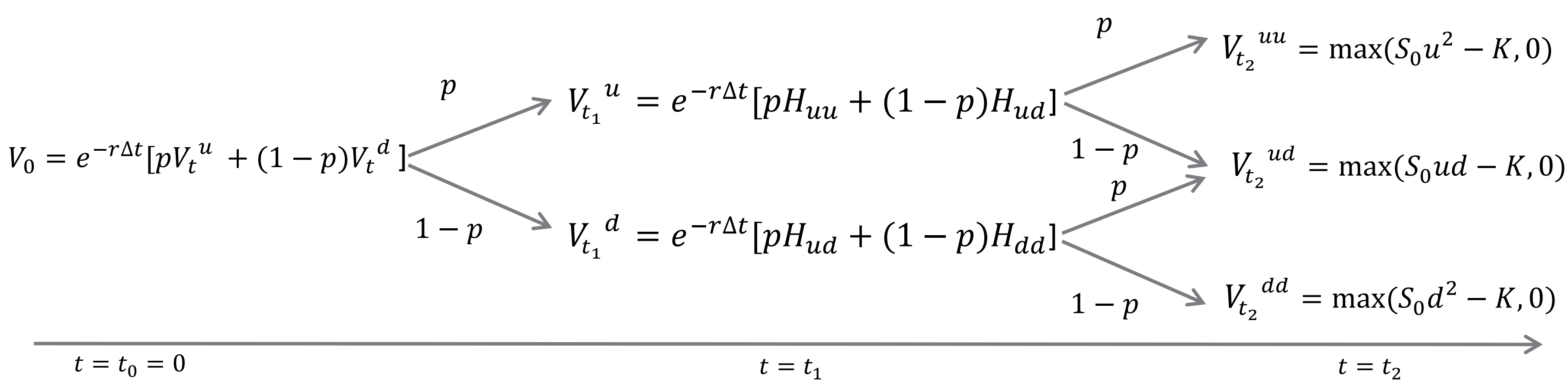
European option pricing (call with expiry t_2, strike K)

option price at $t = 0$

price of option at $t = t_1$:
$V_t = E\left(e^{-r\Delta t} H(t_2) \middle| F_{t_1}\right)$

price of option at $t = t_2$:

known – $\max(S_{t_2} - K, 0)$



Backward Induction and American Monte-Carlo Methods

Backward induction – binomial tree cnt'd

American option is an option that can be exercised at any time between the trade start and its contractual expiry.

option price at $t = 0$

price of option at $t = t_1$:

price of option at $t = t_2$:

known – $\max(S_{t_2} - K, 0)$

$$H_{uu} = \max(S_0 u^2 - K, 0)$$

$$V^u = \max(e^{-r\Delta t}[pH_{uu} + (1-p)H_{ud}], \max(Su - K, 0))$$

$$V_0 = \cdots$$

$$H_{ud} = \max(S_0 ud - K, 0)$$

$$V^d = \max(e^{-r\Delta t}[pH_{ud} + (1-p)H_{dd}], \max(Sd - K, 0))$$

$$H_{dd} = \max(S_0 d^2 - K, 0)$$

p, $1-p$

$t = t_0 = 0$ $\qquad$ $t = t_1$ $\qquad$ $t = t_2$

At $t = t_1$ we can either exercise the option or not.

Whether we do this depends on the information we have up until time $t = t_1$.

If the **continuation value** (price of the option at node $t = t_1$) is higher than what would be our payoff in case of immediate exercise, we hold the option, if not we exercise it.

Backward Induction and American Monte-Carlo Methods

American Monte Carlo

$V_n = h_n$ payoff at expiry

$V_i = \max\{h_i, V_i{}^P\}$ $i < n$

h_i − payoff if we exercise at t_i

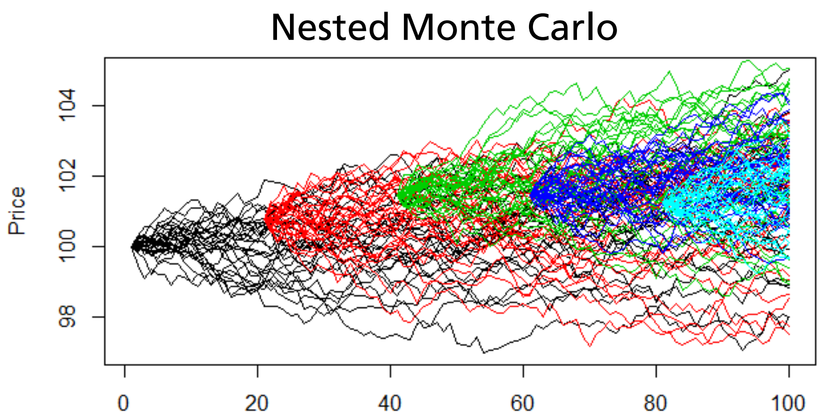
How to compute the continuation value $V_i{}^P$?

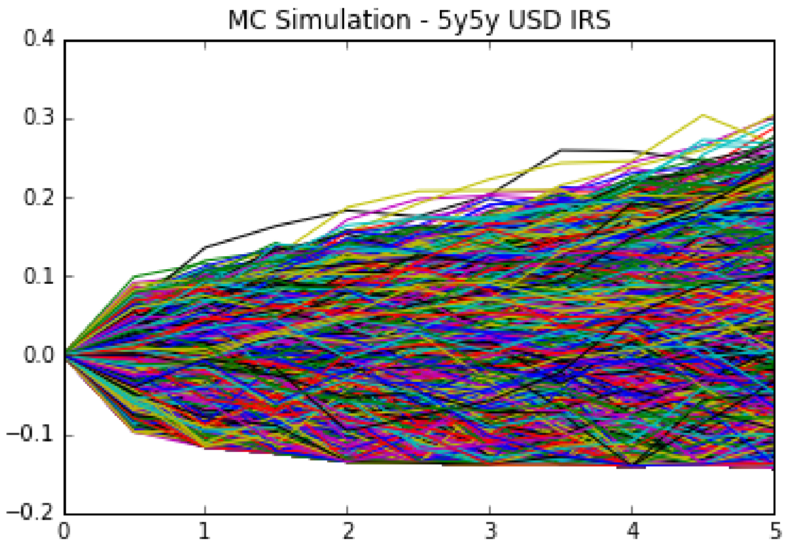
- Polynomial linear regression trick

$V_i \approx P(t_i, t_{i+1})(\psi(O_i))$, where :

ψ is a polynomial function

O_i are what we call the set of observables





Backward Induction and American Monte-Carlo Methods

American Monte Carlo – step-by-step

1) Simulate observables' paths (ex. zero coupon bonds / LIBOR rates, etc.)
2) Set the trade values at $t = t_n = T$ to the payoff at the end of the trade
3) Go one step back to t_{n-1} and calculate the continuation value by regressing the trade value at t_n against the observables.
4) If the product can be exercised at t take the maximum of the payoff you would get at t and the calculated continuation value, otherwise take the continuation value. Don't forget about the cashflows arising from the trade.

[Proceed recursevily backwards until the t=t0 is reached.]

*Easy to follow example given at:
Longstaff, Schwartz; Valueing American Options in Simulations: A simple least-square aproach; (The review of financial studies, Vol14, No1, 2001.)

Backward Induction and American Monte-Carlo Methods

AMC Regression Results