Variance Reduction Techniques

Techniques to improve convergence in Monte Carlo Simulations

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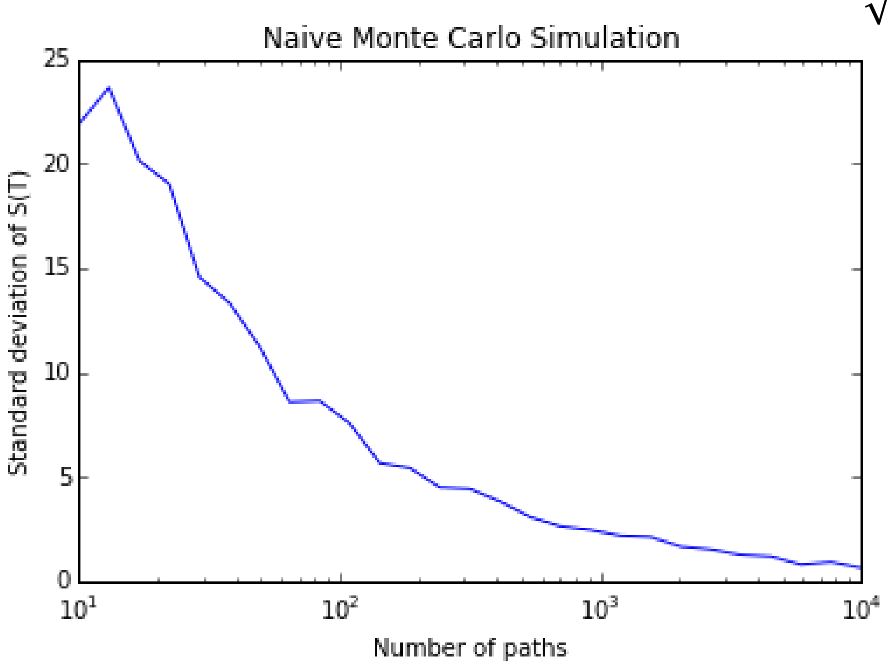
Section 1

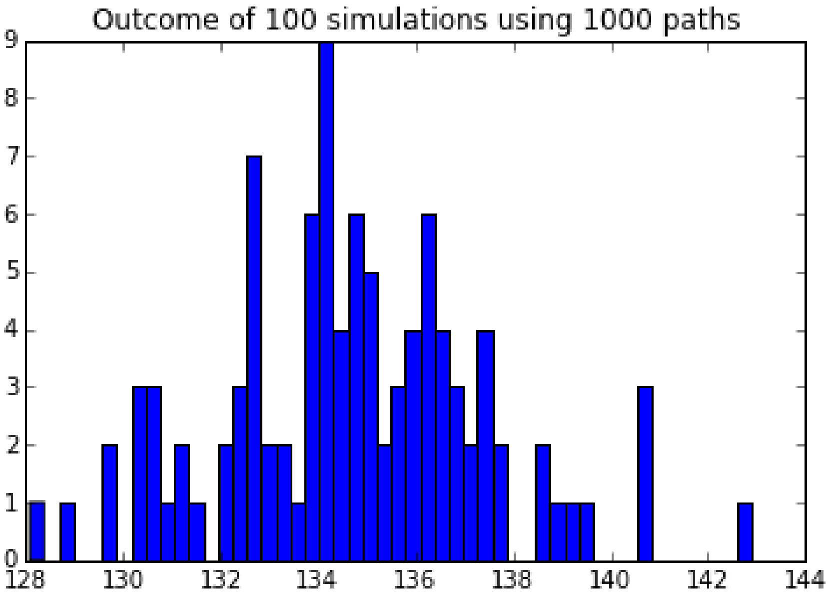
The Problem at Hand

Monte Carlo Convergence Speed

- We are going to simulate a stock price S, with given drift mu, volatility sigma for a horizon T;
- In order to perform this simulation, we are going to use m time steps and n paths which we can choose – in practice we'll use the maximum number of m and n our computer allows;
- The standard deviation of S at time T will be of the order of:

$$\frac{\sigma\sqrt{T}}{\sqrt{n}}$$





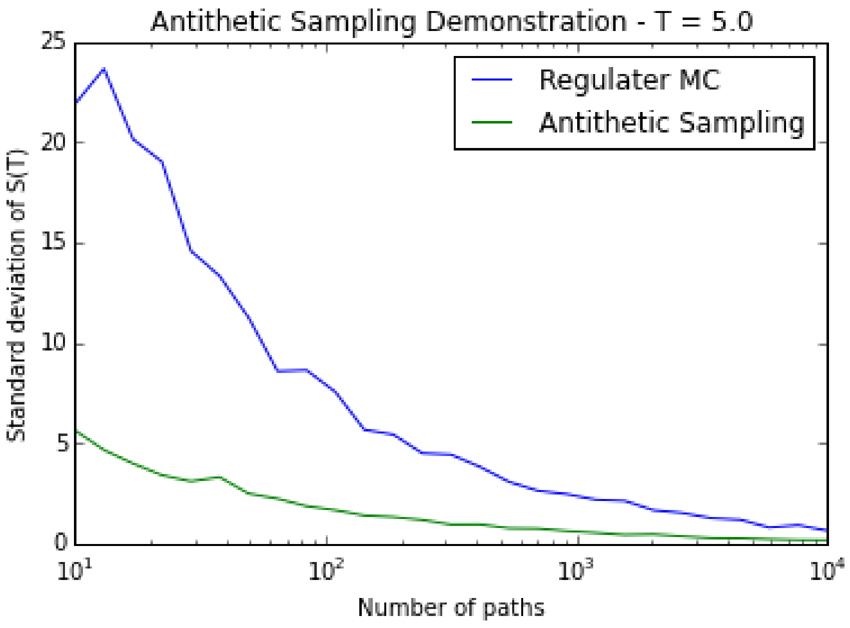
   – This precision is not enough to price derivatives, therefore, we need to find ways to improve convergence speed

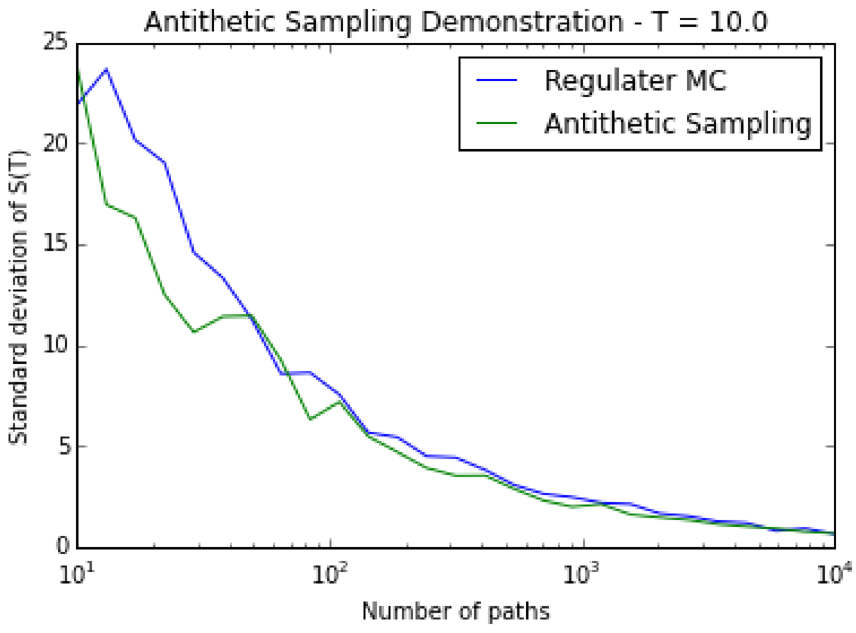
Section 2

Antithetic Sampling and Moment Matching Techniques

Antithetic Sampling

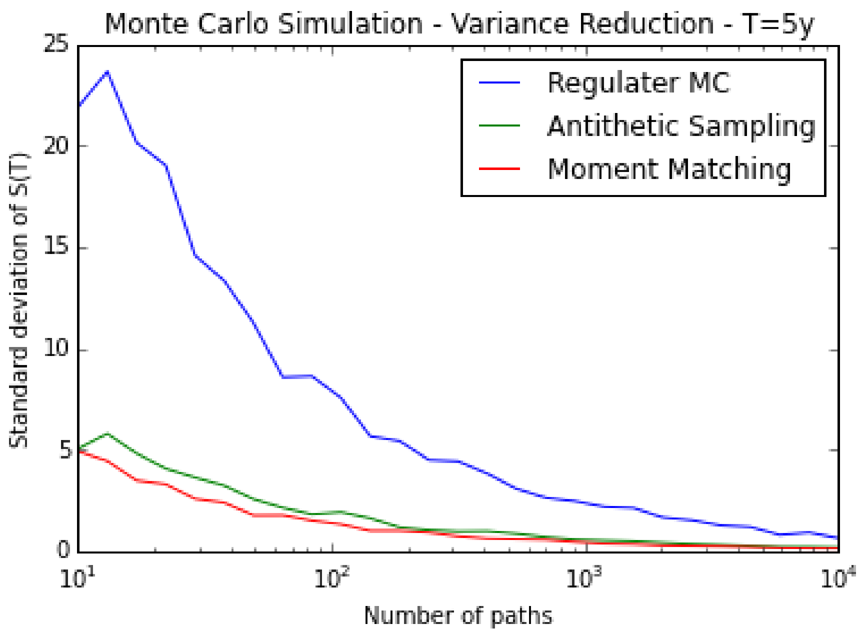
- The idea is to use mirrored increments, ie, instead of generating n paths Brownian increments for each time step, we generate only n/2 and for the n/2 remaining paths, we use increments of same magnitude but with different sign
- It works best for smaller values of T. Why is this the case?

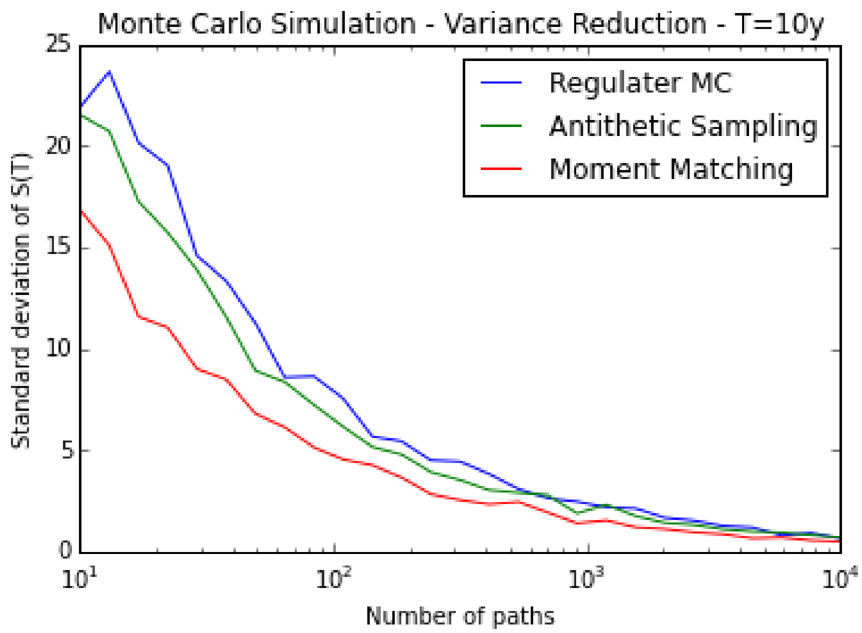




Moment Matching – part 1

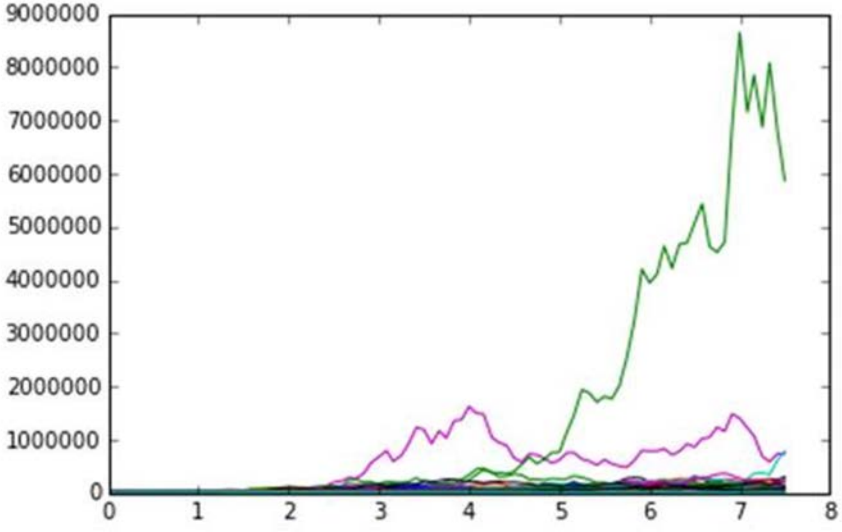
- The Brownian increments generated are not necessarily N(0, 1)
- Moment matching involves re-normalizing the Brownian increments at each time step to force them to conform to N(0,1)
  - $dW'(t) = [dW(t) - \text{mean}(dW(t))] / \text{stdev}(dW(t))$
- This renormalization introduces some dependency between paths, but nothing to be concerned about





Moment Matching - part 2

- We know that S(T), also called forward price of maturity T is given by:
  - S(T) = S(0) exp(mu T)
- Therefore, we can re-normalize our simulated stock price directly at each time step, instead of re-normalize the Brownian increments, to force the simulated mean to match the forward price;
- This "brutte force technique" is very powerful as the simulated price is forced to converge to its theoretical value;
- In most financial applications, this is a requirement, therefore this technique must be used;
- Now, think what happens in case we have one explosive path. What will happen to the other paths when we perform this normalization step? How to remediate it?



Section 3

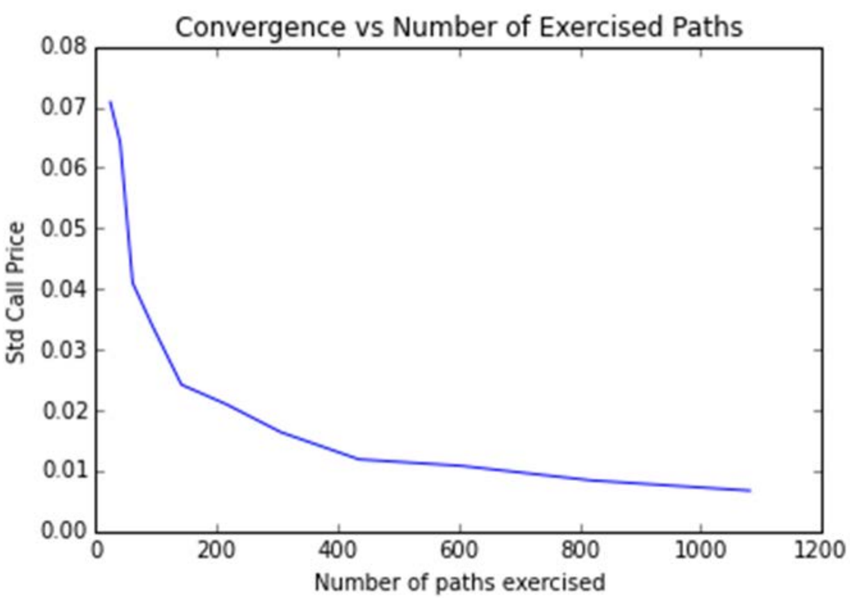
Importance Sampling

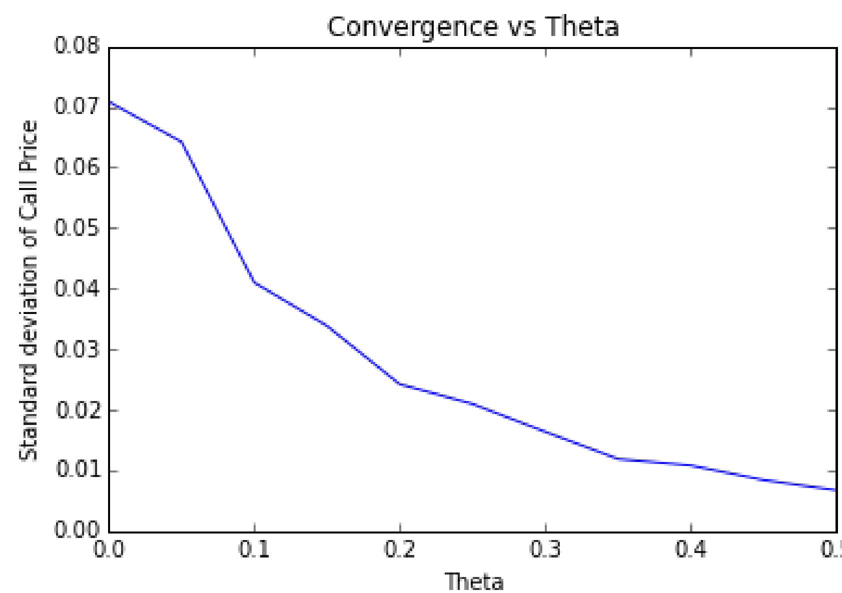
Options

- Options give you the right but not the obligation to buy the stock at a given price – strike, denoted K, at the moment of the option expiry T;
- To price an option using Monte-Carlo, we simulate the stock up to the expiry date;
- Then we compute:
  - Call price = mean(max(S(T) – K, 0) x exp(-rT) => where r is the risk free interest rate
  - Put price = mean(max(K – S(T), 0) x exp(-rT)
- To price options precisely, first we need the forwards to be priced very precisely;
- The option price

Option price convergence

- The convergence of option prices will depend on the number of paths whereby the stock is above (below) the strike K at expiry for calls (puts);





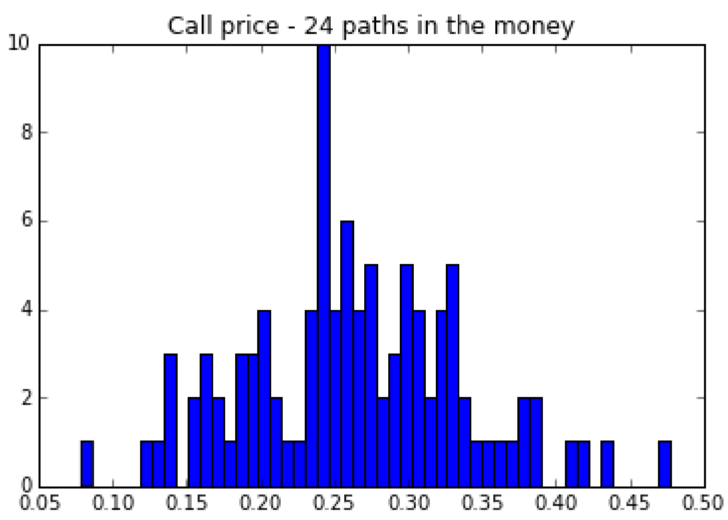
- As shown in the plot, the convergence of the call price depends on the number of paths in which the option is exercised;
- But how to vary the number of exercise paths?

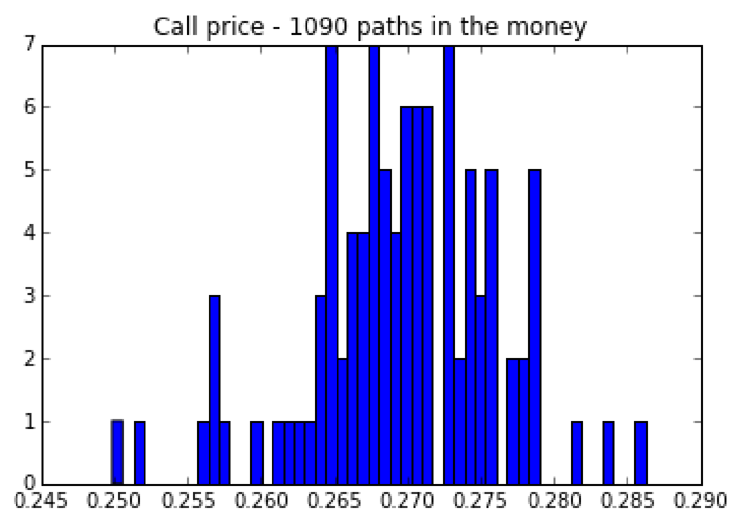
Change of Measure

- Use the change of measure technique
- Let $Z(t) = \exp(\theta \int_0^t dW_t + \frac{\theta^2}{2} t)$, where θ is an arbitrary positive number
- Where dW_t are Brownian increments with no drift
- Now, we perform a change of measure – see Cameron-Martin-Girsanov's Theorem (there are good lecture notes available online, some are easier to follow than others) ;
- In this new measure, a Brownian motion in risk neutral measure will have a drift different than zero;
  - It is this drift that will cause more paths to be exercised;
- To simulate the stock price in this new measure, we need to do the following:
  - Generate new Brownian increments:
    - $dW_t' = dW_t + \theta\ dt$
  - Simulate the stock price S'(t) using dW_t' instead of dW_t
  - The price of the call option will be given by:
    - $\text{Mean}(\max(S'(T) - K, 0)\ /\ Z(T)\ ) \times \exp(-rT)$
- Z(t) is an exponential martingale, it's **always** positive with expected value 1;
- Z(t) is the so called Radon-Nikodym derivative to change from the risk neutral measure (where numeraire is the bank account) to an alternative measure.

Importance Sampling - Results

- In order to demonstrate the power of this technique, we priced a call option struck very very far out of the money;
- $K = S0 \exp(\mu T + 2.5\ \sigma \sqrt{T})$
- Using regular Monte Carlo simulation, on average, 24 paths expire in the money, ie, S(T) > K;
- Applying the change of measure technique, for some values of theta, we can have more than 1000 paths being exercised;
- Below, we can see the dispersion of prices obtained;





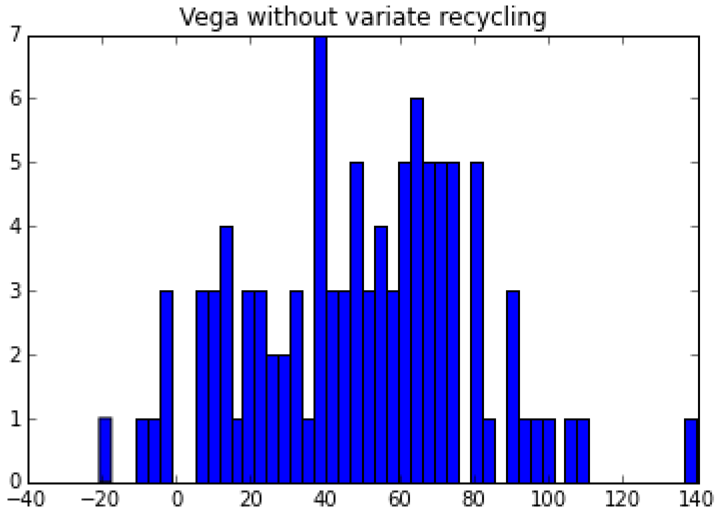
- Drawback: requires knowledge of the product to be priced. In this example, pricing a very OTM put, would require a very different choice of theta.

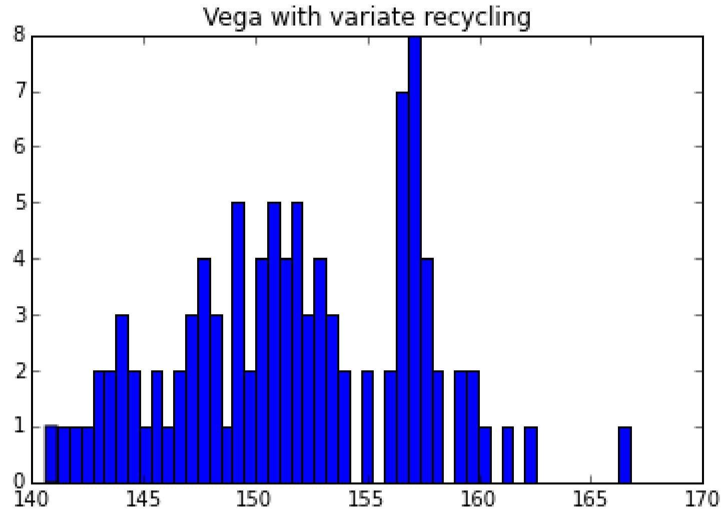
Section 4

Variate Recycling

Variate Recycling - Computing Sensitivities

- In finance, not only we are interested in the price, but also on the sensitivity of the price to certain parameters;
- In the context of options, two very important sensitivities are delta and vega;
  - $delta = \frac{\partial c}{\partial s}$ $\qquad vega = \frac{\partial c}{\partial \sigma}$
- In order to compute sensitivities, we normally bump the parameter up and down and reprice the derivative – bump and reprice;
- In order to speed the convergence of the sensitivity, it is key to use the same Brownian increments when bumping and repricing – this is called **Variate Recycling**;
- Below, we see the vegas of a 10y call struck at the money, computed using 10,000 paths for 100 different seeds with and without variate recycling





Bibliography

[1] $P. Jaeckel, Monte\ Carlo\ Methods\ in\ Finance, Wiley, 2002$