Monte Carlo Simulation

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April 2020

1 Simulation from distribution

For most of known distributions there are built in functions for random number generations (of course, pseudo random). For example in R it is: rnorm(), runif(), rexp().
Most of them are based on the inverse of the distribution function.

Example 1. Simulate N numbers from exponetial distribution with $\lambda = 1.5$:

1. Simulate N numbers from uniformal distribution $U[0, 1]$
2. Calculate inverse of distribution function:

$$F(x) = 1 - e^{-\lambda x}$$

$$F^{-1}(p) = \frac{\log(1-p)}{-\lambda}$$

3. For each element from (1) calculate $F^{-1}(p)$

Problem 1. *Write R function rsin(N,λ) which generates N numbers defined by probability density function $f(x) = \frac{sin(x)}{2}$, for $x \in [0, \pi]$*

1.1 Wiener process generation

Wiener process $W(t, w)$ (or usually W_t) is stochastic process with the following properties:

1. $W(0, w) == 0$
2. Almost all trajectories are continuous
3. $(W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, ..., W_{t_k} - W_{t_{k-1}})$ are independent random variables for any k and set $\{t_1, ..., t_k\}$
4. For any $t > s > 0$: $W_t - W_s \sim N(0, t - s)$

In order to satisfy all of the assumptions the easiest way to generate one path is based on increments generation. Algorithm will be the following:

1. Pick time horizon T and large enough N, set $\delta = T/N$
2. Define $t_0 = 0, \quad t_1 = \delta, \quad ..., \quad t_k = k\delta, \quad ,..., \quad t_N = N\delta = T$
3. Set $W_0 = 0$
4. $W_{i+1} = W_i + \sqrt{t_{k+1} - t_k} Z_i$, where $Z_i \sim N(0, 1)$
5. Repeat (4) N times

Problem 2. *Write R function wiener(K, T, N), which generates K Wiener trajectories on the horizon* $[0, T]$ *with N steps. (Suggest to return result as matrix, where rows are trajectories)*

2 Integrals approximation

Suppose we want to approximate value of the integral:

$$\int_a^b f(x)dx$$

2.1 Numerical approach (most basic)

Based on the Riemann integral approach:

1. Pick large enough N
2. Define $\delta = \frac{b-a}{N}$
3. Set $x_0 = a, \quad x_1 = a + \delta, \quad ..., \quad x_N = a + N\delta = b$
4. Calculate:
$$\sum_{i=1}^{N} f(x_i)\delta$$

2.2 Probabilistic approach

Let's rewrite our integral first:

$$\int_a^b f(x)dx = \int_a^b \frac{f(x)}{h(x)} h(x)dx$$

Then if $h(x)$ is probability density function, such that $\frac{f(x)}{h(x)} < \infty$. Hence:

$$\int_a^b \frac{f(x)}{h(x)} h(x)dx = E\left(\frac{f(x)}{h(x)}\right)$$

Implies simple algorithm for estimation:

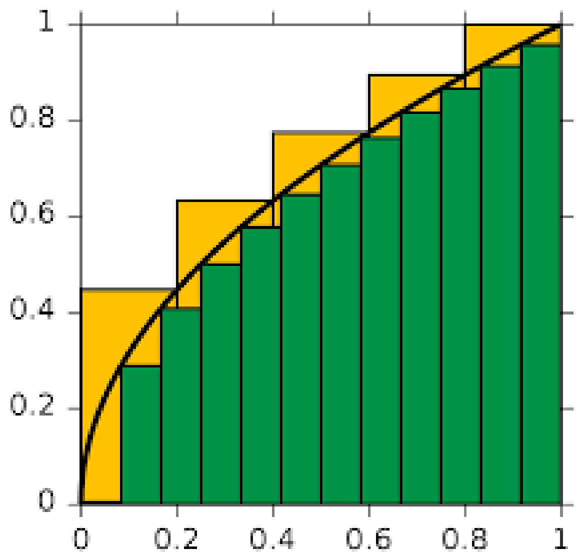

Figure 1: Numerical integral calculation

1. Pick large enough N
2. Generate N random numbers $(x_1, ..., x_N)$ with distribution $h(x)$
3. Calculate value

$$\frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{h(x_i)}$$

Problem 3. *Write R function int(f,a,b,N), which calculates integral* $\int_a^b f(x)dx$ *using* $U[a,b]$ *as distribution for* $h(x)$