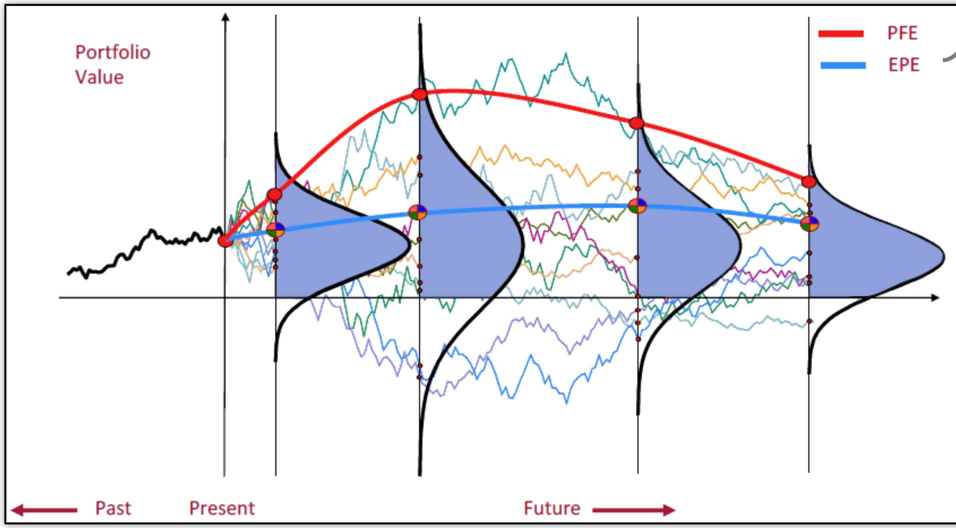
Pricing path dependent products

Why we use American Monte Carlo?

April 11, 2022

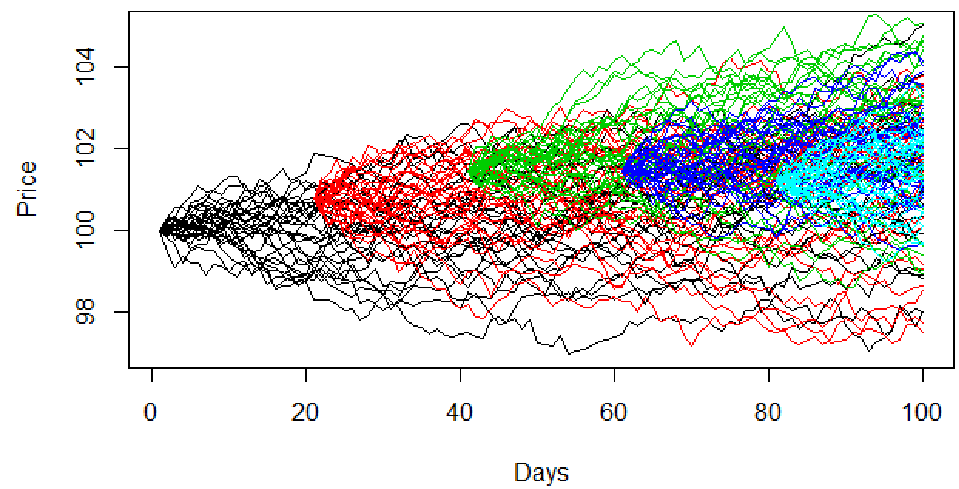
Motivation



Exposure At Default

$$CVA = \underbrace{LGD}_{\text{Loss Given Default}} \times \int_0^T \overbrace{EAD}^{\text{Exposure At Default}} \times \underbrace{PD}_{\text{Probability of Default}} du$$

- Calculating CVA requires knowledge of TV distribution at each point in time
- MC simulation can provide very general solution to problem of pricing a derivatives regardless of it's complexity
- For consistency we need to treat the portfolio s a whole , which force us to price trades on scenario by scenario basis



Issue: scenario consistency, path driven products

- That would require **nesting MC** for each time and scenario we want to price
- Nesting the MC gives rise to high usage of computational usage.

$$10{,}000 \rightarrow 10 \times 10{,}000 \times 10{,}000 = 10^9$$

- The solution to this is a **American Monte Carlo** algorithm

Origins

- AMC originated as a solution to pricing callable derivatives, e.g. American Options.
- Problem of finding the optimal exercise point
- The Idea of Tsitsiklis and Van Roy is translate:

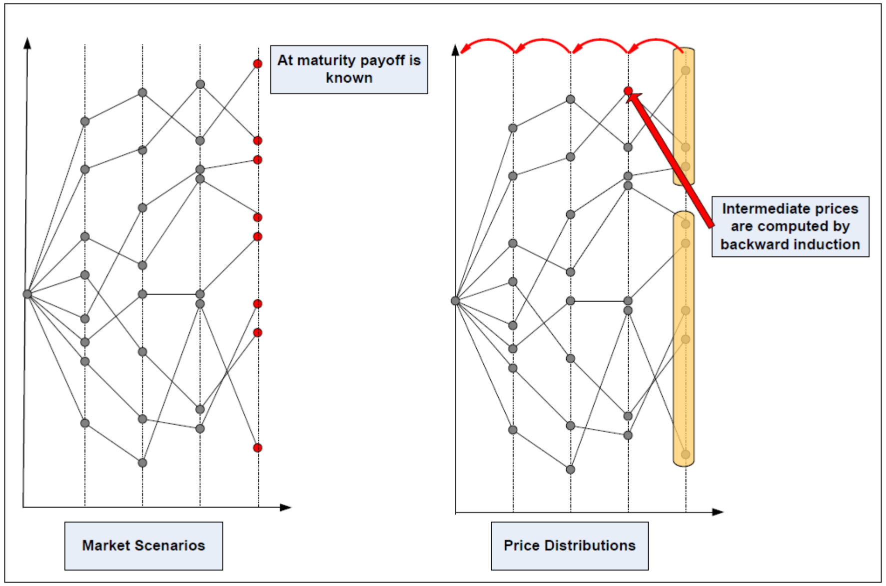
$$V = \sup_{\tau \in T} E[(K - S_\tau)^+ \exp(-\int_0^\tau r_t dt)]$$

- Exercise at maximal value

$$V_{i-1} = max\{\, h_{i-1}, E[D_{i-1,i} V_i \mid F_{i-1}\,]\}$$

- Make a decision to exercise or not at each step
- Require knowledge of the trade value at next step
- The price at maturity is known for each scenario

- The distributions of future trade values provide us with the information on the current value.
- If we start from single starting point it is trivial (MC)
- At a given time ti we have multiple starting points: therefore the final distribution would be spread around the payoff function
- Multiple mean of estimating a true payoff
  - E.g. Backward induction step driven by regression



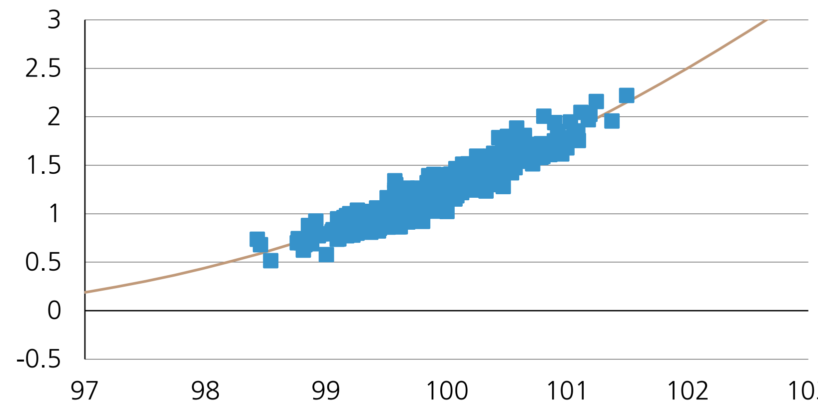
Regression and LS algorithm

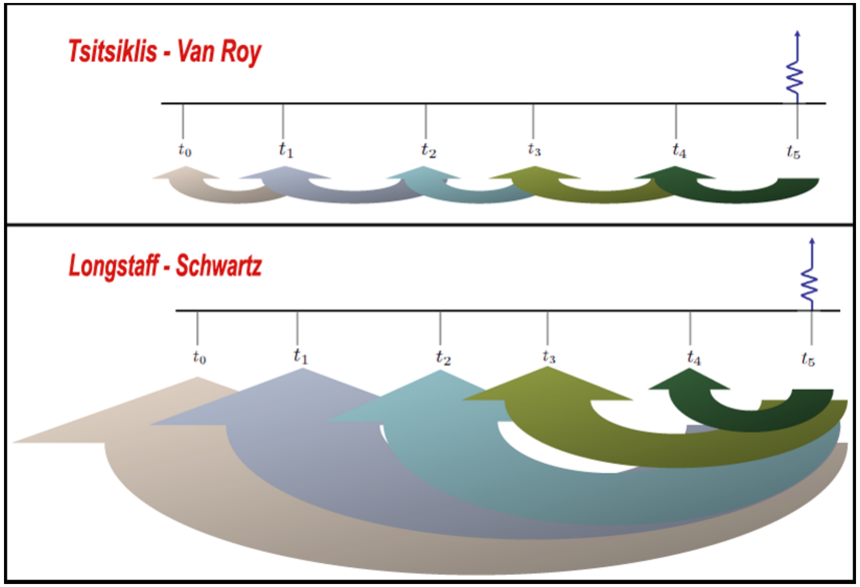
Driven by the chosen observabkes

- Polynomial regression is a natural choice

$$E[D_{i-1,i}V_i|F_{i-1}] = \alpha_0 + \sum_{i=1}^{N}\sum_{p=1}^{P_i} \alpha_{pi} x_i^p$$

  - For complex products need multidimensional fit
  - We need to carefully consider what are the observables
  - $\boldsymbol{\alpha_0}$ and $\boldsymbol{\alpha_{pi}}$ can be estimated with **Least Square** method
- Drawback: regression errors are accumulated with each regression step
- Improvement: **Longstaff-Schwartz** algorithm
  - Same principle except using realized cash-flows to regress against instead previous step's value,





Summary

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- Algorithm is widely used in the industry.
- With various modifications and improvements
- Algorithm is quite robust.
- Mind the usual limitations:
- Selected number of paths might not be enough in some cases
- The out of the money products might generate a lot of numerical noise

TODO:

- Implement the algorithm for simple IR swap.

To Read:

- Tsitsiklis, Van Roy; Optimal stopping of markov processes; IEEE Transactions on Automatic Control, Vol44, 1999.
- Longstaff, Schwartz; Valueing American Options in Simulations: A simple least-square aproach; The review of financial studies, Vol14, No1, 2001.
- Tilley; Valuing American Options in a Path Simulation Process; Transaction of Society of Actuaries, Vol 45,1993.