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Simulations in Python, Part I

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Monte Carlo Method: Motivation

The price of a European call option at $t = 0$ is

$$\alpha = e^{-rT} E[max(S_T - K, 0)]$$

where r is the risk free interest rate, S_T is the stock price on the maturity date T, and, K is the strike price. The Black-Scholes-Merton formula is

$$\alpha = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

where $d_1 = \frac{ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = \frac{ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}$ provided that S_t is follows a geometric Brownian motion (among other conditions) In general, S_t follows a complex random process that has no explicit solution.

Monte Carlo Method: Principle

As a remedy, we generate a sample $(S_{Ti})_{i=1}^{n}$ so that we can estimate α with

$$\widehat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} e^{-rT} max(S_{Ti} - K, 0)$$

Let $h(S_{Ti}) := e^{-rT} max(S_{Ti} - K, 0)$
From the low of large number $\widehat{\alpha} \xrightarrow{p} \alpha$ and the central limit theorem

$$\sqrt{n}(\widehat{\alpha} - \alpha) \xrightarrow{d} N(0, \sigma_h^2)$$

where $\sigma_h^2 = Var[h(S_{Ti})]$.
Then we can build a confidence interval as follows,

$$Pr(\widehat{\alpha} - z_{\delta/2}\frac{s}{\sqrt{n}} \leq \alpha \leq \widehat{\alpha} + z_{\delta/2}\frac{s}{\sqrt{n}}) \xrightarrow{p} 1 - \delta$$

where s is a consistent estimator of σ_h

Simulation of Paths: continuous case

The stock price process in the Black-Scholes follows the geometric Brownian motion,

$$dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

with solution,

$$S(t) = S(0)exp\left\{(r - \frac{1}{2}\sigma^2)t + \sigma W(t)\right\}$$

where $W(t)$ follows a Wiener process (e.i. $W(t+u) - W(t) \sim N(0, u)$, among other properties)

Discretization of stochastic process: Euler Scheme

The integral form of

$$dS(t) = a(S(t))dt + b(S(t))dW(t)$$

is

$$\begin{aligned} S(t+h) - S(t) &= \int_t^{t+h} a(S(u))du + \int_t^{t+h} b(S(u))dW(u) \\ &\approx a(S(t))h + b(S(t))(W(t+h) - W(t)) \\ &= a(S(t))h + b(S(t))\sqrt{h}N(0,1) \end{aligned}$$

Based on this, the Euler scheme is

$$\widehat{S}(t_{i+1}) = \widehat{S}(t_i) + a(\widehat{S}(t_i))(t_{i+1} - t_i) + b(\widehat{S}(t_i))\sqrt{t_{i+1} - t_i}Z_{i+1}$$

Because $W(t_{i+1}) - W(t_i) = \sqrt{t_{i+1} - t_i}Z_{i+1}$ where $Z_{i+1} \sim N(0,1)$

Discretization of stochastic process: Milstein Scheme

We aim to improve the approximation for the second term of

$$S(t+h) - S(t) \approx a(S(t))h + \int_t^{t+h} b(S(u))dW(u)$$

From

$$\begin{aligned}\int_t^{t+h} b(S(u))dW(u) &\approx b(S(t))(W(t+h) - W(t)) \\ &+ \frac{1}{2}b'(S(t))b(S(t))h(N(0,1)^2 - 1)\end{aligned}$$

the Milstein Scheme is

$$\begin{aligned}\widehat{S}(t_{i+1}) &= \widehat{S}(t_i) + a(\widehat{S}(t_i))(t_{i+1} - t_i) + b(\widehat{S}(t_i))\sqrt{t_{i+1} - t_i}Z_{i+1} \\ &+ \frac{1}{2}b'(\widehat{S}(t_i))b(\widehat{S}(t_i))(t_{i+1} - t_i)(Z_{i+1}^2 - 1)\end{aligned}$$

where $Z_{i+1} \sim N(0,1)$

Discretization of the geometric Brownian motion

Exact discretization:
From $S(t_i) = S(0)exp\left\{(r - \frac{1}{2}\sigma^2)t_i + \sigma W(t_i)\right\}$

$$S(t_{i+1}) = S(t_i)exp\left\{(r - \frac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_{i+1}\right\}$$

From $dS(t) = rS(t)dt + \sigma S(t)dW(t)$ (i.e. $a(S(t)) = rS(t)$ and $b(S(t)) = \sigma S(t)$) Euler Scheme:

$$\widehat{S}(t_{i+1}) = \widehat{S}(t_i) + r\widehat{S}(t_i)(t_{i+1} - t_i) + \sigma\widehat{S}(t_i)\sqrt{t_{i+1} - t_i}Z_{i+1}$$

Milstein Scheme:

$$\begin{aligned}\widehat{S}(t_{i+1}) \quad = \quad & \widehat{S}(t_i) + r\widehat{S}(t_i)(t_{i+1} - t_i) + \sigma\widehat{S}(t_i)\sqrt{t_{i+1} - t_i}Z_{i+1} \\ & + \frac{1}{2}\sigma^2(t_{i+1} - t_i)(Z_{i+1}^2 - 1)\end{aligned}$$

The Cox–Ingersoll–Ross (CIR) model

The CIR process is

$$d(X(t)) + \kappa(\theta - X(t))dt + \sigma\sqrt{X(t)}dW(t)$$

Euler Scheme:

$$\widehat{X}(t_{i=1}) = \widehat{X}(t_i) + \kappa(\theta - \widehat{X}(t_i))(t_{i+1} - t_i) + \sigma\sqrt{\widehat{X}(t_i)}\sqrt{t_{i+1} - t_i}Z_{i+1}$$

Milstein Scheme:

$$\begin{aligned}\widehat{X}(t_{i+1}) &= \widehat{X}(t_i) + \kappa(\theta - \widehat{X}(t_i))(t_{i+1} - t_i) + \sigma\sqrt{\widehat{X}(t_i)}\sqrt{t_{i+1} - t_i}Z_{i+1} \\ &\quad + \frac{1}{4}\sigma^2(t_{i+1} - t_i)(Z_{i+1}^2 - 1)\end{aligned}$$

Correlated Wiener Processes

Let $W_1(t)$ and $W_2(t)$ be two independent Wiener processes and $\rho \in (-1, 1)$, then

$$\begin{aligned} \tilde{W}_1(t) &= W_1(t) \\ \tilde{W}_2(t) &= \rho W_1(t) + \sqrt{1-\rho^2} W_2(t) \end{aligned}$$

are correlated Wiener processes with correlation ρ. In general, $\tilde{W}(t) = [\tilde{W}_1(t) \ldots \tilde{W}_k(t)]^T$, $W(t) = [W_1(t) \ldots W_k(t)]^T$ where J is a $k \times k$ matrix. Then, we can simulate k correlated Wiener processes

$$\begin{aligned} \tilde{W}(t_{i+1}) &= JW(t_{i+1}) \\ &= JW(t_i) + \sqrt{t_{i+1} - t_i} JZ_{i+1} \end{aligned}$$

where $JZ \sim N(0, JJ^T)$