Pricing path dependent products

Why we use American Monte Carlo?

April 11, 2023

Path dependent products

A **path-dependent product** is a contract with payoff that depends not just on the value of the underlying asset at maturity but on the entire history of the asset over the duration of the contract.

- **Barrier option** – it's payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price.

Option type	Initial Condition	Payoff
Up-And-In	$S < B$	$1_{\left\{\max_{t\in[0,T]} S_t \geq B\right\}} H$
Up-And-Out	$S < B$	$1_{\left\{\max_{t\in[0,T]} S_t < B\right\}} H$
Down-And-In	$S > B$	$1_{\left\{\min_{t\in[0,T]} S_t \leq B\right\}} H$
Down-And-Out	$S > B$	$1_{\left\{\min_{t\in[0,T]} S_t > B\right\}} H$

where

$$H = \begin{cases} (S_T - K)^+, & \text{for call option,} \\ (K - S_T)^+, & \text{for put option.} \end{cases}$$

Path dependent products

- **Asian option** - payoff is determined by the average underlying price over some period of time
  - *Arithmetic average*

$$H_T^{\text{call}} = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)^+$$

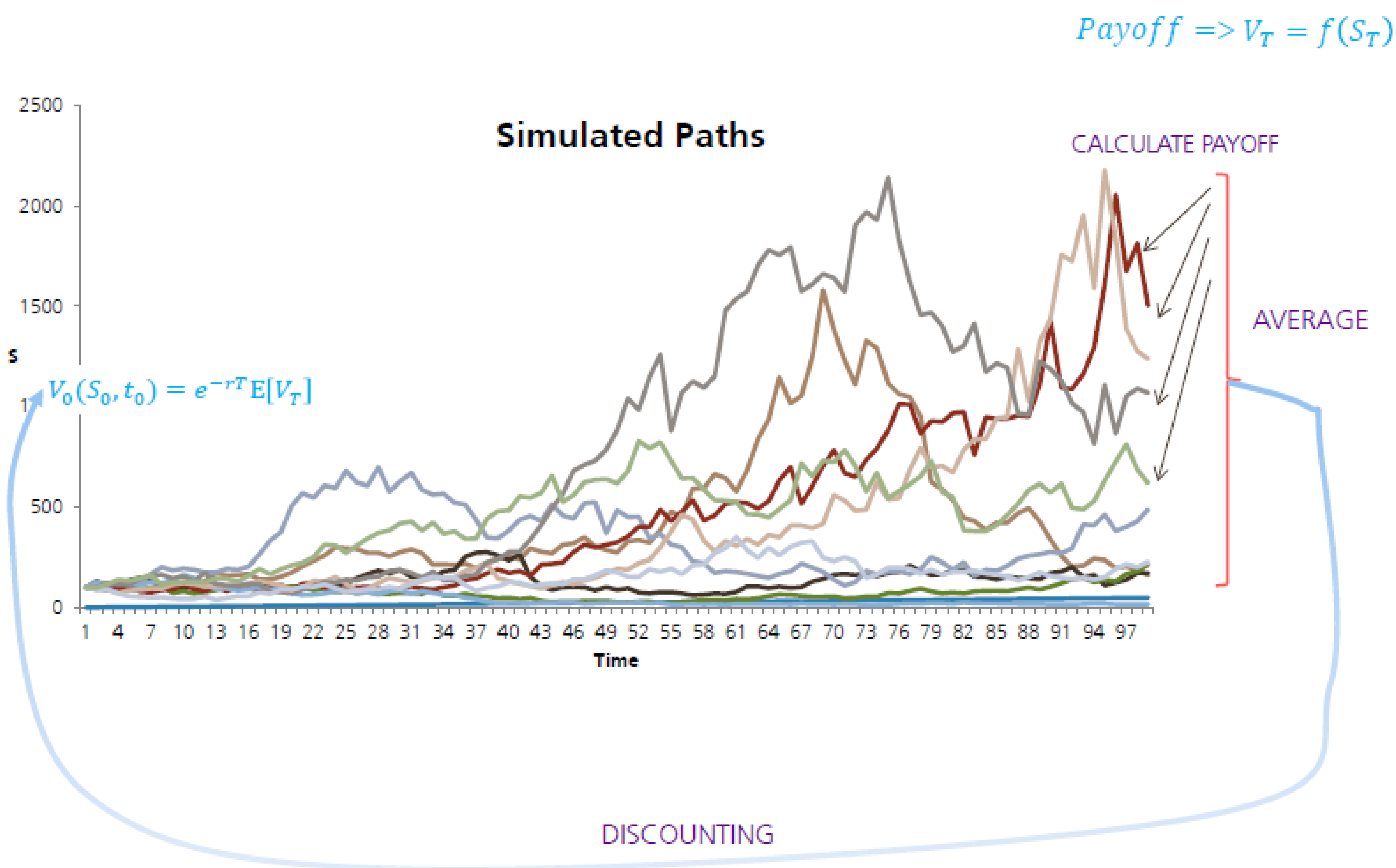
$$\text{(Discrete case)} \qquad \left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K\right)^+$$

  - *Geometric average*

$$H_T^{\text{call}} = \left(\exp\left\{\frac{1}{T}\int_0^T \ln S(t)\,dt\right\} - K\right)^+$$

$$\text{(Discrete case)} \qquad \left(\sqrt[n]{\prod_{i=1}^{n} S_{t_i}} - K\right)^+$$

Motivation



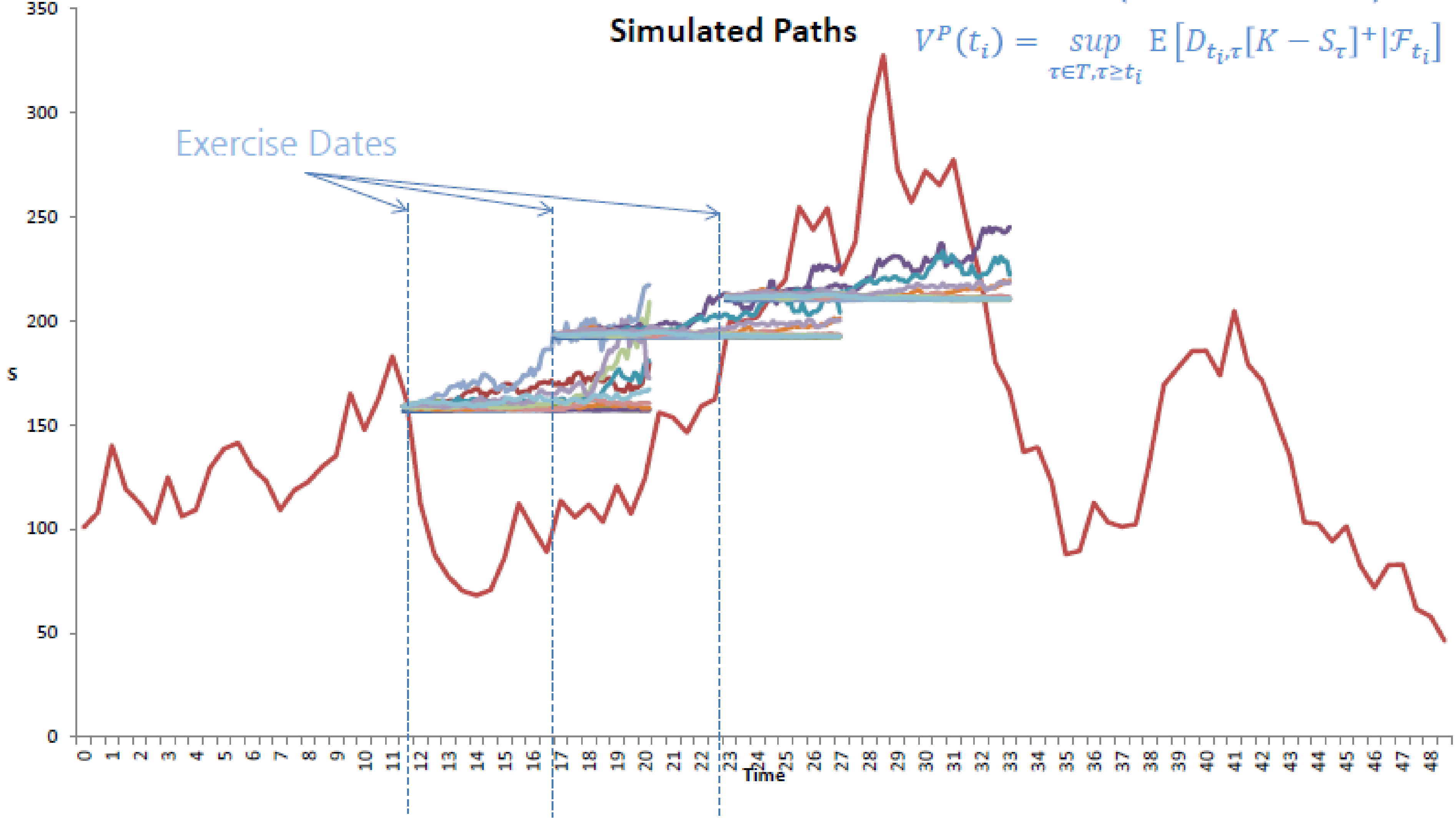
$Payoff \Rightarrow V_T = f(S_T)$
Simulated Paths
CALCULATE PAYOFF
AVERAGE
$V_0(S_0, t_0) = e^{-rT}\mathrm{E}[V_T]$
2500
2000
1500
500
0
S
1 4 7 10 13 16 19 22 25 28 31 34 37 40 43 46 49 52 55 58 61 64 67 70 73 76 79 82 85 88 91 94 97
Time
DISCOUNTING

Motivation

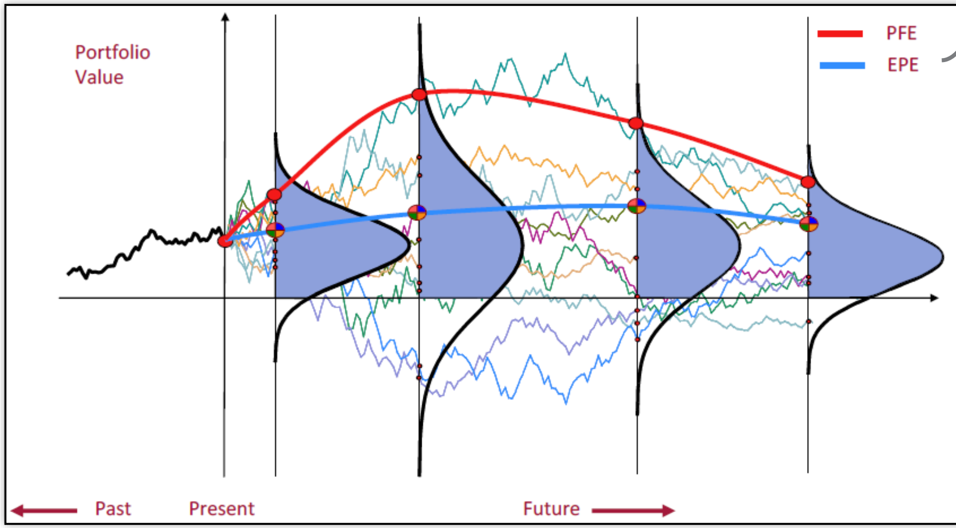
American Payoff

$$V(t_i) = \mathrm{Max}\left(\left[K - S_{t_i}\right]^+, V^P(t_i)\right)$$

$$V^P(t_i) = \sup_{\tau \in T, \tau \geq t_i} \mathrm{E}\left[D_{t_i,\tau}[K - S_\tau]^+ | \mathcal{F}_{t_i}\right]$$

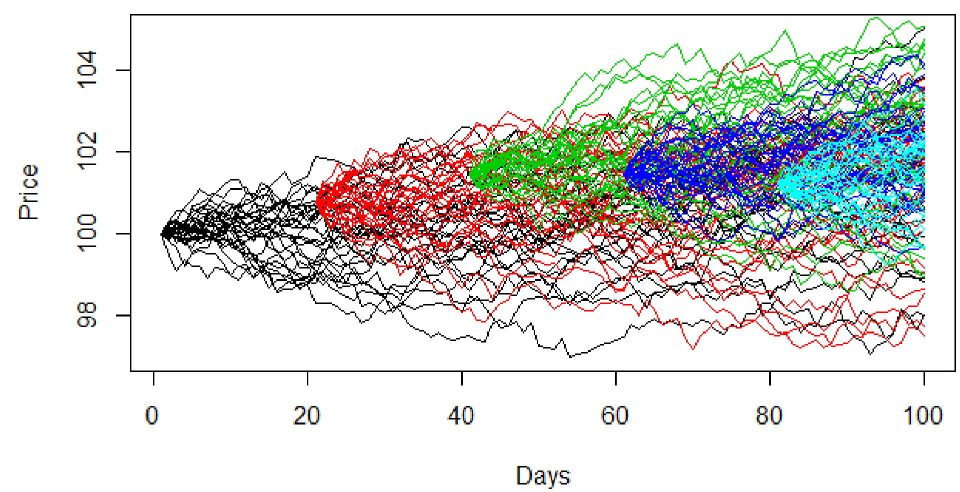


Motivation



$$\mathrm{CVA} = \underbrace{\mathrm{LGD}}_{\text{Loss Given Default}} \times \int_0^T \overbrace{EAD}^{\text{Exposure At Default}} \times \underbrace{PD}_{\text{Probability of Default}} \, du$$

- Calculating CVA requires knowledge of TV distribution at each point in time
- MC simulation can provide very general solution to problem of pricing a derivatives regardless of it's complexity
- For consistency we need to treat the portfolio s a whole , which force us to price trades on scenario by scenario basis



Issue: scenario consistency, path driven products

- That would require **nesting MC** for each time and scenario we want to price
- Nesting the MC gives rise to high usage of computational usage.

$$10{,}000 \rightarrow 10 \times 10{,}000 \times 10{,}000 = 10^9$$

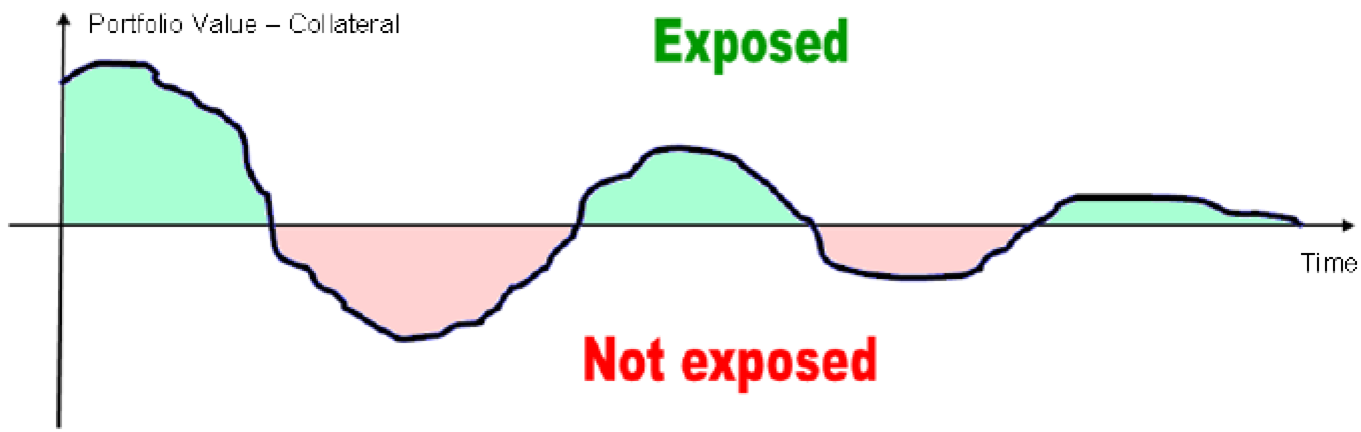
- The solution to this is an **American Monte Carlo** algorithm

EPE & PFE

Expected Exposure (EE)

Given a price distribution for the portfolio value (V_t) and the corresponding collateral (C_t), the standard definition of *Expected Exposure* is

$$EE_t = E[(V_t - C_t)^+ | \mathcal{F}_0]$$



Potential Future Exposure (PFE)

The *Potential Future Exposure* is defined as the maximum expected credit exposure calculated at some degree of statistical confidence. For example, the 97.5% PFE is the level of potential exposure that is exceeded with only 2.5% probability:

$$PFE_{\alpha,t} = \inf\{x \colon \mathbb{P}(V_t - C_t \leq x) \geq \alpha$$

Origins

- AMC originated as a solution to pricing callable derivatives, e.g. American Options.
- Problem of finding the optimal exercise point
- The Idea of Tsitsiklis and Van Roy is translate:

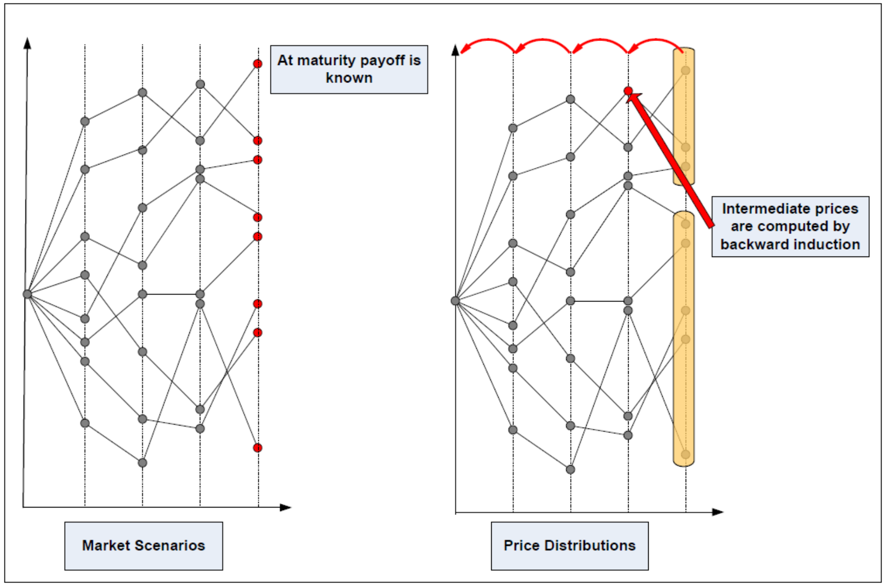
$$V = \sup_{\tau \in T} E[(K - S_\tau)^+ \exp(-\int_0^\tau r_t dt)]$$

- Exercise at maximal value

$$V_{i-1} = max\{\, h_{i-1}, E[D_{i-1,i} V_i \mid F_{i-1}\,]\}$$

- Make a decision to exercise or not at each step
- Require knowledge of the trade value at next step
- The price at maturity is known for each scenario

- The distributions of future trade values provide us with the information on the current value.
- If we start from single starting point it is trivial (MC)
- At a given time ti we have multiple starting points: therefore the final distribution would be spread around the payoff function
- Multiple mean of estimating a true payoff
  - E.g. Backward induction step driven by regression



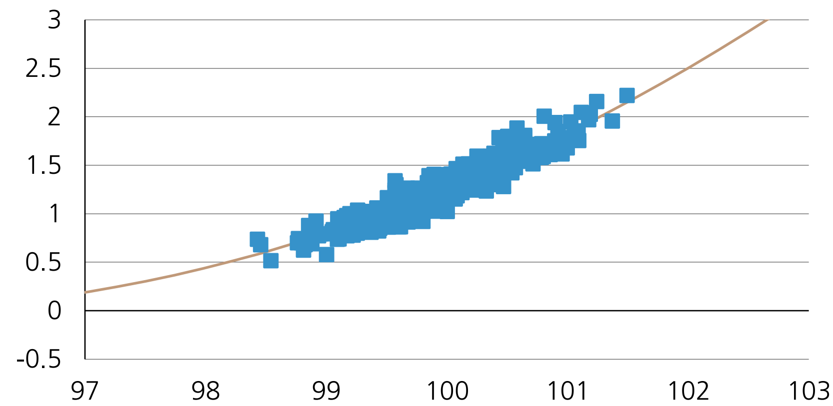
Regression and LS algorithm

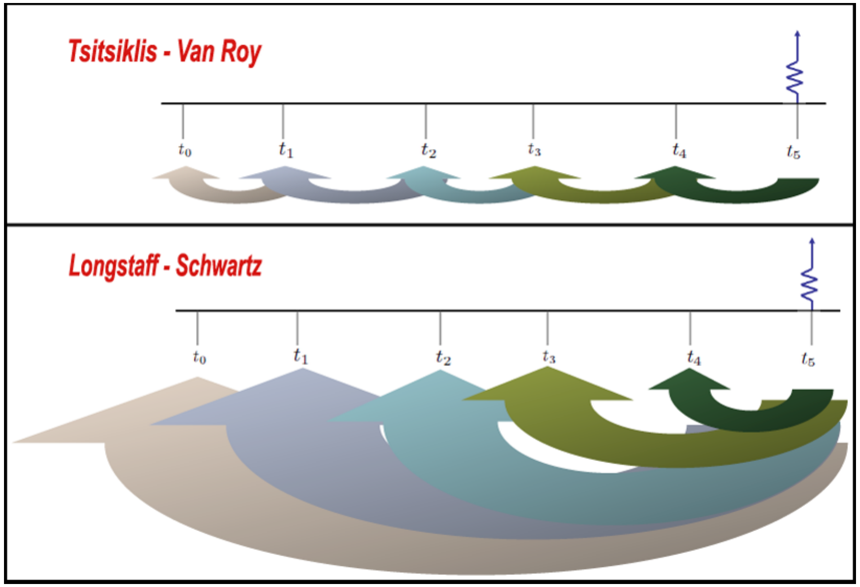
Driven by the chosen observabkes

- Polynomial regression is a natural choice

$$E[D_{i-1,i}V_i|F_{i-1}] = \alpha_0 + \sum_{i=1}^{N}\sum_{p=1}^{P_i} \alpha_{pi} x_i^p$$

  - For complex products need multidimensional fit
  - We need to carefully consider what are the observables
  - $\boldsymbol{\alpha_0}$ and $\boldsymbol{\alpha_{pi}}$ can be estimated with **Least Square** method
- Drawback: regression errors are accumulated with each regression step
- Improvement: **Longstaff-Schwartz** algorithm
  - Same principle except using realized cash-flows to regress against instead previous step's value,





AMC settings

1. **Observables** - for any trade, the observables (X’s uses in the regression) should be those risk drivers which possibly the best explains the value of the trade.

2. **Basis functions** – typical choice is to use polynomial functions of a given order which itself may vary across the observables – for example we can use 3rd order polynomial for the first (the most important) observable and 1st order for the other observables.

3. **Bundling** – instead of doing regression on the entire sample, we can split it in into „bundles“ (each of which represents a similar market scenario) and perform regression within each bundle.

Walkthrough

Consider an American Put option, with strike of $100 and maturity of 1Y.

Walkthrough

Step 1 Simulate many realizations of the asset path from now to expiry

	Time					
Realizations	0	0.2	0.4	0.6	0.8	1
1	100.00	103.34	89.52	98.60	104.54	104.20
2	100.00	103.70	108.12	107.37	124.32	126.45
3	100.00	85.11	97.96	90.91	94.71	97.11
4	100.00	99.46	101.99	94.74	96.12	89.48
5	100.00	102.88	102.96	96.72	101.23	87.96
6	100.00	92.17	83.21	86.27	86.15	84.76
7	100.00	95.85	100.40	114.41	106.33	125.74
8	100.00	100.94	100.32	101.17	77.91	70.93
9	100.00	99.23	98.32	98.12	105.32	95.14
10	100.00	102.23	82.53	84.59	98.29	109.74

Walkthrough

Realizations	Time 0	0.2	0.4	0.6	0.8	1
1	100.00	103.34	89.52	98.60	104.54	104.20
2	100.00	103.70	108.12	107.37	124.32	126.45
3	100.00	85.11	97.96	90.91	94.71	97.11
4	100.00	99.46	101.99	94.74	96.12	89.48
5	100.00	102.88	102.96	96.72	101.23	87.96
6	100.00	92.17	83.21	86.27	86.15	84.76
7	100.00	95.85	100.40	114.41	106.33	125.74
8	100.00	100.94	100.32	101.17	77.91	70.93
9	100.00	99.23	98.32	98.12	105.32	95.14
10	100.00	102.23	82.53	84.59	98.29	109.74

Compute the Payoff At Expiration (assuming no exercise beforehand)

Cash Flow Matrix

0	0.2	0.4	0.6	0.8	1
	0	0	0	0	0
	0	0	0	0	0
	0	0	0	0	3
	0	0	0	0	11
	0	0	0	0	12
	0	0	0	0	15
	0	0	0	0	0
	0	0	0	0	29
	0	0	0	0	5
	0	0	0	0	0

Walkthrough

Step 2 Go to Next Time Step and Discount Cash Flows

$t = 0.8$

X	Y	Continuation Value
94.71	2.87	
96.12	10.42	
86.15	15.09	
77.91	28.78	
98.29	0.00	
0.00	0.00	
0.00	0.00	
0.00	0.00	
0.00	0.00	
0.00	0.00	

Only paths that are in the money at this time

Walkthrough

Step 3 Perform Regression of y against x

$t = 0.8$

X	Y	Continuation Value
94.71	2.87	5.78
96.12	10.42	4.48
86.15	15.09	15.69
77.91	28.78	28.54
98.29	0.00	2.66
0.00	0.00	0.00
0.00	0.00	0.00
0.00	0.00	0.00
0.00	0.00	0.00
0.00	0.00	0.00

Only paths that are in the money at this time

$$y = 0.24x^2 - 5.48x + 310.82$$

Walkthrough

Step 4 Evaluate the exercise Decision and Update the cash flow Matrix

Time

Realizations	0	0.2	0.4	0.6	0.8	1
1	100.00	103.34	89.52	98.60	104.54	104.20
2	100.00	103.70	108.12	107.37	124.32	126.45
3	100.00	85.11	97.96	90.91	94.71	97.11
4	100.00	99.46	101.99	94.74	96.12	89.48
5	100.00	102.88	102.96	96.72	101.23	87.96
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9	100.00	99.23	98.32	98.12	105.32	95.14
10	100.00	102.23	82.53	84.59	98.29	109.74

Cash Flow Matrix

0	0.2	0.4	0.6	0.8	1
				0	0
				0	0
				0	3
				0	11
				0	12
				0	15
				0	0
				0	29
				0	5
				0	0

Walkthrough

REPEAT STEPS 2 - 4

Realizations \ Time	0	0.2	0.4	0.6	0.8	1
1	100.00	103.34	89.52	98.60	104.54	104.20
2	100.00	103.70	108.12	107.37	124.32	126.45
3	100.00	85.11	97.96	90.91	94.71	97.11
4	100.00	99.46	101.99	94.74	96.12	89.48
5	100.00	102.88	102.96	96.72	101.23	87.96
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7	100.00	95.85	100.40	114.41	106.33	125.74
8	100.00	100.94	100.32	101.17	77.91	70.93
9	100.00	99.23	98.32	98.12	105.32	95.14
10	100.00	102.23	82.53	84.59	98.29	109.74

Cash Flow Matrix

0	0.2	0.4	0.6	0.8	1
			0	0	0
			0	0	0
			0	0	2.89466
			0	0	10.52246
			0	0	12.04344
			13.73055	0	0
			0	0	0
			0	0	29.06963
			0	0	4.855155
			15.4135	0	0

Walkthrough

REPEAT STEPS 2 - 4

Realizations	Time 0	0.2	0.4	0.6	0.8	1
1	100.00	103.34	89.52	98.60	104.54	104.20
2	100.00	103.70	108.12	107.37	124.32	126.45
3	100.00	85.11	97.96	90.91	94.71	97.11
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9	100.00	99.23	98.32	98.12	105.32	95.14
10	100.00	102.23	82.53	84.59	98.29	109.74

Cash Flow Matrix

0.0	0.2	0.4	0.6	0.8	1.0
		10.48	0.00	0.00	0.00
		0.00	0.00	0.00	0.00
		0.00	0.00	0.00	2.89
		0.00	0.00	0.00	10.52
		0.00	0.00	0.00	12.04
		16.79	0.00	0.00	0.00
		0.00	0.00	0.00	0.00
		0.00	0.00	0.00	29.07
		0.00	0.00	0.00	4.86
		17.47	0.00	0.00	0.00

Walkthrough

REPEAT STEPS 2 - 4

Realizations	0	0.2	0.4	0.6	0.8	1
1	100.00	103.34	89.52	98.60	104.54	104.20
2	100.00	103.70	108.12	107.37	124.32	126.45
3	100.00	85.11	97.96	90.91	94.71	97.11
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9	100.00	99.23	98.32	98.12	105.32	95.14
10	100.00	102.23	82.53	84.59	98.29	109.74

Time

Cash Flow Matrix

0.0	0.2	0.4	0.6	0.8	1.0
	0.00	10.48	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	0.00
	14.89	0.00	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	10.52
	0.00	0.00	0.00	0.00	12.04
	0.00	16.79	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	29.07
	0.00	0.00	0.00	0.00	4.86
	0.00	17.47	0.00	0.00	0.00

Day Zero AMC TV

$11.23

Summary

Summary

- Algorithm is widely used in the industry.
- With various modifications and improvements
- Algorithm is quite robust.
- Mind the usual limitations:
- Selected number of paths might not be enough in some cases
- The out of the money products might generate a lot of numerical noise

TODO:

- Implement the algorithm for simple IR swap.

To Read:

- Tsitsiklis, Van Roy; Optimal stopping of markov processes; IEEE Transactions on Automatic Control, Vol44, 1999.
- Longstaff, Schwartz; Valueing American Options in Simulations: A simple least-square aproach; The review of financial studies, Vol14, No1, 2001.
- Tilley; Valuing American Options in a Path Simulation Process; Transaction of Society of Actuaries, Vol 45,1993.

Thank you!

Additional Q&A