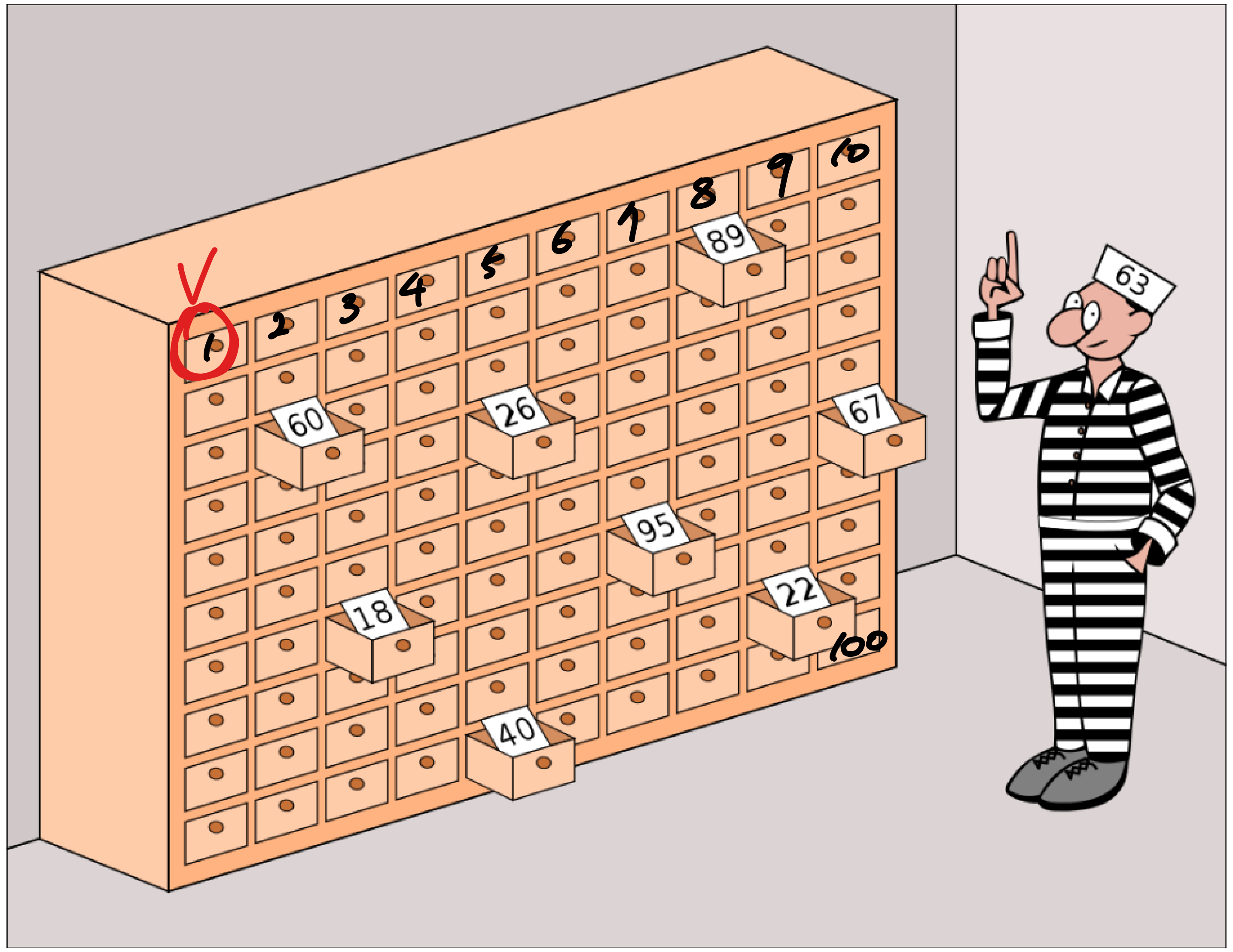
Combinatoric Problems

The 100 prisoners



박스 100개, 안에 100개의 숫자

죄수 100명 존재

박스 50개 오픈 ⇒ 자기 번호 존재해야함.

1번 : $\frac{50}{100} = \frac{1}{2}$

⋮

100번 : $\frac{1}{2}$

생존 확률 : $\left(\frac{1}{2}\right)^{100} = ?$

1번죄수

1번 박스 열기

(가정)

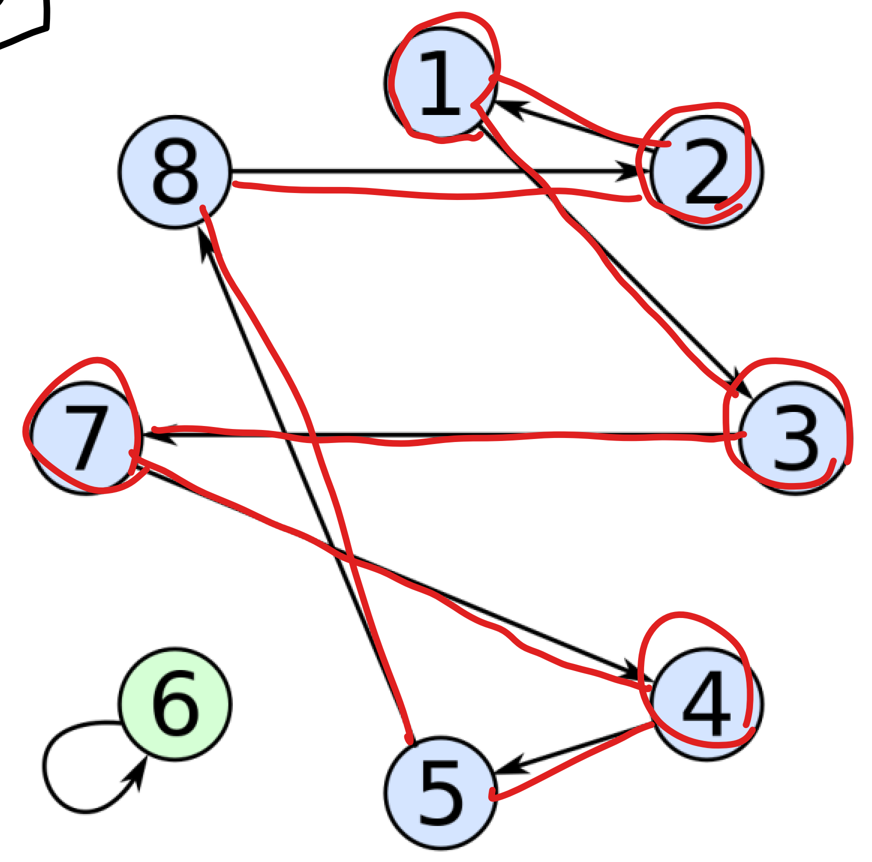
1번 박스에 36 이 존재

⇒ 36번 박스 열기

Max 50 번

The 100 prisoners

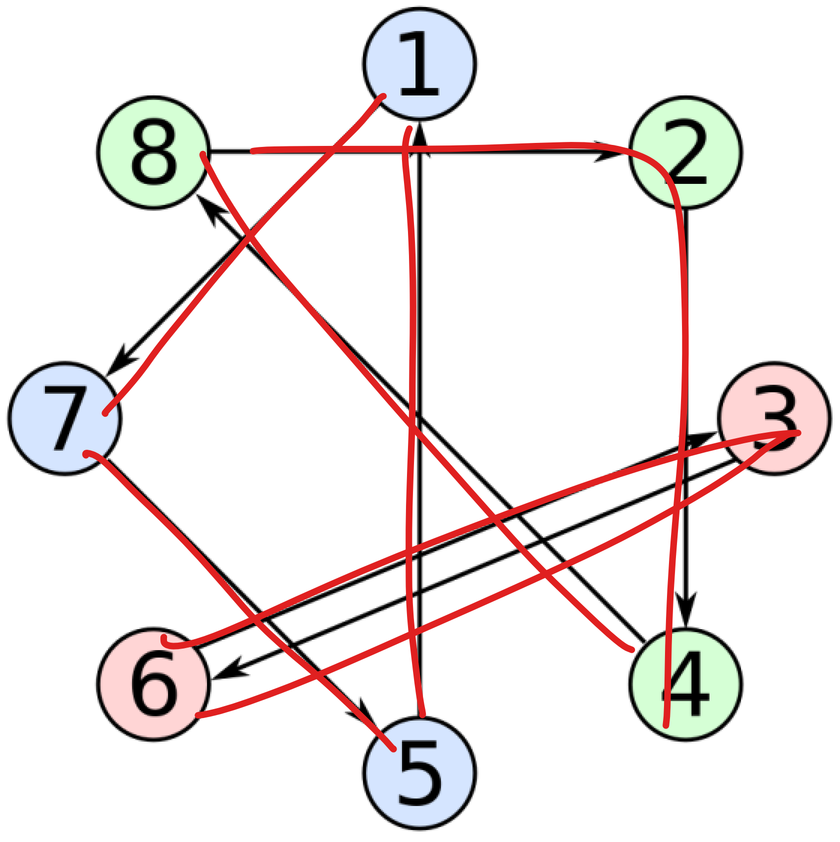
8명 → 4번 open가능



전체 사망

4 이하

3, 3, 2



The 100 prisoners

Cycle length 최대 50 $\Longleftrightarrow$ 모든 죄수가 자신의 번호를 본다.

50 초과 r개 선택 100 − r

$1 \sim 100$: r개 선택 , $r > 50$

$$\underbrace{\underbrace{\binom{100}{r}}_{\text{''}} \times \underbrace{(r-1)!}_{\text{''}} \times (100-r)! = \frac{100!}{r}}_{\text{실패 경우의 수.}}$$

$$\underbrace{1 - \frac{1}{100!}\sum_{r=51}^{100}\frac{100!}{r} = 1 - \sum_{r=51}^{100}\frac{1}{r}}$$

The 100 prisoners

$$1 - \frac{1}{100!}\sum_{r=51}^{100}\frac{100!}{r} = 1 - \sum_{r=51}^{100}\frac{1}{r}$$

$$H_n = \sum_{k=1}^{n}\frac{1}{k}$$

$$= 1 - (H_{100} - H_{50})$$

$$= 0.3118218207 \ldots > 30\%$$

31.18%

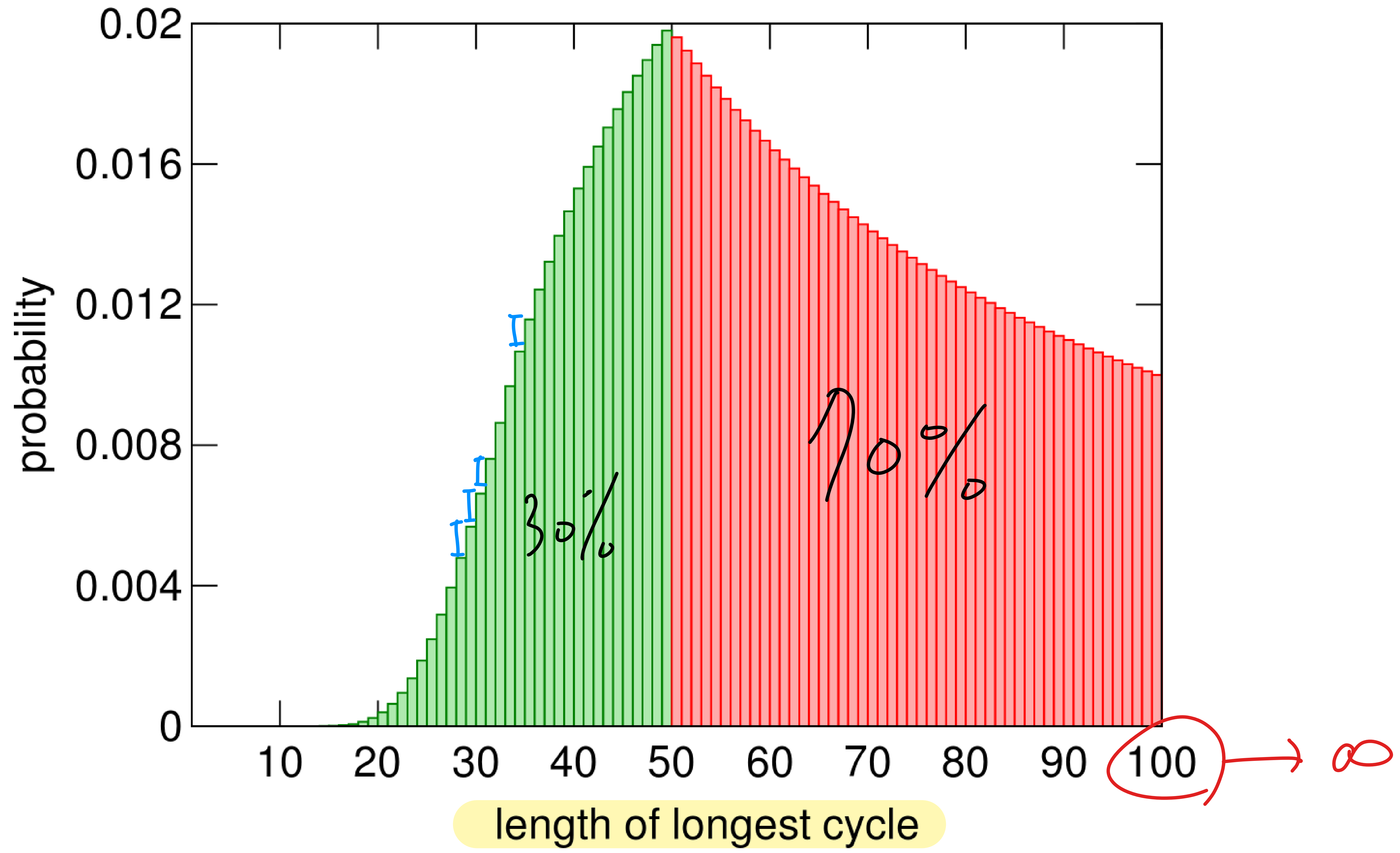
$n = 4$

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$

2번 뽑기. $1 - \sum_{r=3}^{4}\frac{1}{r} = 1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12} = 0.41\ldots$

41% ↑

The 100 prisoners



The 100 prisoners

If n → ∞

$$
\begin{aligned}
\gamma &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)\\
&= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \int_1^n \frac{\mathrm{d}k}{k}\right)\\
&= \sum_{n=1}^{\infty}\left\{\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right\}\\
&= \int_1^{\infty}\left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)\mathrm{d}x\\
&\approx 0.577215664901532860600\cdots
\end{aligned}
$$

$$H_n = \sum_{k=1}^{n}\frac{1}{k}$$

$$\gamma = \lim_{n\to\infty}(H_n - \ln n)$$

100 명

$$1 - (H_{100} - H_{50})$$

2n 명 $(n \to \infty)$

$$
\begin{aligned}
1 - (H_{2n} - H_n) &= 1 - (H_{2n} - \ln 2n) + (H_n - \ln n) - \ln 2\\
&= 1 - \gamma + \gamma - \ln 2\\
&= 1 - \ln 2\\
&= 0.3068\cdots > 30\%
\end{aligned}
$$

$30.68\cdots\%$

Oddtown

Oddtown

n명 주민

n개 클럽!

club, 1) Every odd numbers of members.

2) every pair of clubs share an even number of members. (possibly none)

3명 주민 A B C

$C_1, \cdots, C_i \Rightarrow \max(i) = ?$

Oddtown

Let k be a club number.

$M_{ij} \in \mathbb{F}_2$ ex) $\overset{3\times 3}{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}}$

row : Clubs (C_i)

Column : inhabitants (x_j)

$\boxed{M_{ij}} = \begin{cases} 1, & x_j \in C_i \\ 0, & \text{otherwise.} \end{cases}$

(주민 3명)

	x_1	x_2	x_3
C_1	1	0	0
C_2	0	1	0
C_3	0	0	1

(주민 4명)

	x_1	x_2	x_3	x_4
C_1	1	1	0	1
C_2	1	0	1	1
C_3	1	1	1	0
C_4	0	1	1	1

Oddtown

Let S be a subset of n. $a_1 \cdots a_n$

$$\chi_S \in \mathbb{Z}_2^n \quad , \quad (1 \text{ or } 0)$$

Vector

$$\chi_S = (a_1, a_2, \cdots, a_n) \quad , \quad a_i = \begin{cases} 1 , & i \in S \\ 0 , & i \notin S \end{cases}$$

T. (another subset)

$$\chi_S \cdot \chi_T = \#(S \cap T)$$

$$\chi_S \cdot \chi_T = \begin{cases} 1 , & \text{if } \#(S \cap T) \text{ is odd.} \\ 0 , & \text{—— // —— is even.} \end{cases}$$

ex) $n = 3$

$\chi_S = (1, 1, 1)$, $\chi_T = (1, 0, 1)$

$\chi_S \cdot \chi_T = 0$

$S = (a_1, a_2, a_3)$ $\quad S \cap T = (a_1, a_3)$

$T = (a_1, \ , a_3)$ $\quad \#(S \cap T) = 2$: even

Oddtown

$$M_{ij} = \begin{cases} 1, & x_j \in C_{\underline{i}} \; {=k} \\ 0, & \text{otherwise.} \end{cases}$$

$M : (k \times \bigcirc)$ $M^T : (\bigcirc \times k)$

$MM^T = A$ $k \times k$ matrix.

$M \in \mathbb{F}_2$

$$M = \begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline C_1 & 1 & 1 & 0 & 1 \\ C_2 & 1 & 0 & 1 & 1 \\ C_3 & 1 & 1 & 1 & 0 \\ C_4 & 0 & 1 & 1 & 1 \end{array}$$

$$M^T = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$M \cdot M^T = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$\text{rank}(A) = k.$

Oddtown

Recall) B : $k \times m$. matrix.

C : $m \times n$

$\operatorname{rank}(BC) \leq \operatorname{rank}(B)$

$\quad " \quad \leq \operatorname{rank}(C)$

$\operatorname{tr}(MM^T) = \operatorname{tr}(I_k)$

$\operatorname{rank}(A) = k = \operatorname{rank}(MM^T) \leq \operatorname{rank}(M) \boxed{\leq n}$

$i \times j$

$i = k,\ j = n$

M : $k \times n$ matrix

THANK YOU!