Module 1: The Basics of Input-Output Analysis

2.3 Input-Output Models: Linkages and Multipliers

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Input-Output Models: Linkages and Multipliers

Technical Coefficients Matrix (A Matrix)

- **The standard Input-Output Modeling technique is the <u>demand-pull model</u>**
  - Also named the "Leontief Model"

- **The model establishes linear relations between the required inputs and total output for each industry**
  - For each industry j displayed in the columns of the Z Matrix, the value of each input employed in production is divided by the total output of industry j
    - **The result of this operation is an "$a_{i,j}$ coefficient" that indicates the ratio of industry i's input to industry j's output**

$$a_{i,j} = \frac{z_{i,j}}{x_j} \qquad (1)$$

- **The complete matrix of $a_{i,j}$ coefficients is named the <u>Technical Coefficients Matrix (A Matrix)</u>**

$$A = Z\hat{x}^{-1} \qquad (2)$$

Input-Output Models: Linkages and Multipliers

Technical Coefficients Matrix (A Matrix)

- **Let's go back to our example and calculate its Technical Coefficients Matrix:**

	Wheat	Iron	Final Demand	Total Output
Wheat	280	120	175	575
Iron	240	160	0	400
Imports	10	20	0	0
Value Added (Imports + Factors' of Production Remuneration)	55	120	=	
Total Output	575	400		975

- **The A Matrix will be calculated as follows:**

	Wheat	Iron
Wheat	280/575 = 0.49	120/400 = 0.3
Iron	240/575 = 0.42	160/400 = 0.4

- **Note: The $a_{i,j}$ coefficients are interpretable only when the A Matrix is read following its columns (and not rows)**
  - e.g. the iron industry needs 0.3 units of wheat and 0.4 units of its own product (iron) to produce 1 unit of output

Input-Output Models: Linkages and Multipliers

Total Requirements Matrix (L Matrix)

- **However, in order to produce its output, a given industry k not only requires direct inputs from other industries, but also indirect inputs from the productive network**
  - Indirect inputs comprise all the upstream network of production that is not directly supplying industry k
    - **e.g. the inputs provided to the industries that supply industry k are indirect inputs**

- **Starting from the Technical Coefficients Matrix (A Matrix), we can calculate the Total Requirements Matrix (L Matrix) that displays all the direct and indirect inputs needed by each sector to produce its output:**

$$\boldsymbol{L} = (\boldsymbol{I} - \boldsymbol{A})^{-1} \qquad \textbf{(3)}$$

  - **In which I consists of an identity matrix**
  - **The L Matrix is also usually named as the Leontief Inverse**

- **In the Total Requirements Matrix each coefficient $l_{i,j}$ represents the direct and indirect inputs that an industry requires to produce 1 unit of its output**

Input-Output Models: Linkages and Multipliers

The Demand-pull/Leontief Model

- **Back to our example, the L Matrix would look like the following:**

	Wheat	Iron
Wheat	3.3	1.6
Iron	2.3	2.8

- **Note: Like the A Matrix, the $l_{i,j}$ coefficients are interpretable only when the L Matrix is read following its columns**
  - e.g. the iron industry requires 1.6 direct and indirect units of wheat and 2.8 direct and indirect units of its own product (iron) to produce 1 unit of output

- **The Total Requirements Matrix (L) is a component of the Demand-pull/Leontief Model presented in equation 6:**

$$Ax + y = x \qquad (4)$$

$$(I - A)^{-1} y = x \qquad (5)$$

Substituting equation 3 into 5 and reorganizing:

$$x = Ly \qquad (6)$$

Input-Output Models: Linkages and Multipliers

The Demand-pull/Leontief Model

- **The Demand-pull/Leontief Model (equation 6) establishes the relationship between industrial output and final demand in an economy**
  - It establishes the amount of industrial output production required to satisfy a certain level and structure of final demand
    - **It is a demand-led model by definition**

- **The L Matrix is a representation of the interindustry relations of the economy**
  - In the literature, the A Matrix of technical coefficients is understood as a representation of the technological structure of an economy
  - Studies on economic structural change often assess the changes of the A Matrix of technical coefficients

- **Key assumptions of the Demand-pull/Leontief Model:**
  - Demand determines total output of the economy
  - Constant returns to scale
  - Factors of production are complementary and not substitutable
  - Prices are held constant

Input-Output Models: Linkages and Multipliers

Allocation Coefficients Matrix (B Matrix)

- **In the same way that the Leontief Model studies the backward linkages of an economy (input structure), the Ghosh Model allows us to study the forward linkages**
  - The focus is on how industrial output is allocated throughout the productive structure
  - Conversely to the Leontief approach, the Ghosh approach interprets the IO Table horizontally!

- **The model establishes linear relations between the allocation of output and total output for each industry**
  - For each industry i displayed in the rows of the Z Matrix, the allocation of production to each other industry in the economy is divided by the total output of industry i
    - **The result of this operation is a "$b_{i,j}$ coefficient" that indicates the ratio of industry i's output that is allocated as an input to industry j**

$$b_{i,j} = \frac{z_{i,j}}{x_i} \qquad (7)$$

- **The complete matrix of $b_{i,j}$ coefficients is named the <u>Allocation Coefficients Matrix (B Matrix)</u>**

$$B = \hat{x}^{-1} Z \qquad (8)$$

Input-Output Models: Linkages and Multipliers

Allocation Coefficients Matrix (B Matrix)

- **Let's go back to our example and calculate its Allocation Coefficients Matrix:**

	Wheat	Iron	Final Demand	Total Output
Wheat	280	120	175	575
Iron	240	160	0	400
Imports	10	20	0	0
Value Added (Imports + Factors' of Production Remuneration)	55	120	=	
Total Output	575	400		975

- **The B Matrix will be calculated as follows:**

	Wheat	Iron
Wheat	280/575 = 0.49	120/575 = 0.2
Iron	240/400 = 0.6	160/400 = 0.4

- **Note: The $b_{i,j}$ coefficients are interpretable only when the B Matrix is read following its rows (and not columns)**
  - e.g. Given 1 unit of iron output, the iron industry allocates 0.6 units to the wheat industry and 0.4 units to its own iron industry
  - Note that the iron industry's b coefficients add to 1 as there is no allocation of iron to final demand in our simple example

Input-Output Models: Linkages and Multipliers

Output Inverse Matrix (G Matrix)

- **Due to the downstream network of production, like the in the Leontief approach, we can also separate direct and indirect output allocation**
  - e.g. the energy industry supplies to the mineral extraction industry that later supplies inputs to the manufacturing industry
    - **In this case, the energy industry directly allocates its output to the industry of mineral extraction and indirectly allocates it to the manufacturing industry**

- **Starting from the Allocation Coefficients Matrix (B Matrix), we can calculate the Output Inverse Matrix (G Matrix) that displays all the direct and indirect destination of 1 unit of production of an industry:**

$$\boldsymbol{G} = (\boldsymbol{I} - \boldsymbol{B})^{-1} \qquad (9)$$

- **In which I consists of an identity matrix**
- **The G Matrix is also usually named as the Ghosh Inverse**

- **In the Output Inverse Matrix each coefficient $g_{i,j}$ represents the direct and indirect allocation made by a given industry of 1 unit of its own output**

Input-Output Models: Linkages and Multipliers

The Ghosh Model

- **Back to our example, the G Matrix would look like the following:**

	Wheat	Iron
Wheat	3.3	1.1
Iron	3.3	2.8

- **Note: Like the B Matrix, the $g_{i,j}$ coefficients are interpretable only when the G Matrix is read following its rows**
  - e.g. Given 1 unit of output, the iron industry allocates 3.3 units of direct and indirect production to the wheat industry, and 2.8 units of direct and indirect production to itself

- **The Output Inverse Matrix (G) is a component of the Ghosh Model presented in equation 12:**

$$\boldsymbol{x}'\boldsymbol{B} + \boldsymbol{v} = \boldsymbol{x}' \qquad (10)$$

$$\boldsymbol{x}'(\boldsymbol{I} - \boldsymbol{B})^{-1} = \boldsymbol{v} \qquad (11)$$

Substituting equation 9 into 11 and reorganizing:

$$\boldsymbol{x}' = \boldsymbol{v}'\boldsymbol{G} \qquad (12)$$

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The Ghosh Model

- **The Ghosh Model (equation 12) establishes the relationship between value-added and industrial output in an economy**
  - It establishes the amount of industrial output production possible to be produced given the factors of production available (in the value-added vector)
    - **It is a supply-led model by definition**

- **Key assumptions of the Ghosh Model:**
  - Value-added determines total output of the economy
    - **Final demand adapts to output ensuring full employment**
  - Constant returns to scale
  - <u>Factors of production are substitutable among themselves</u>
  - The structure of output allocation is fixed

Input-Output Models: Linkages and Multipliers

Multipliers: Backward and Forward Linkages

- **The total backward and forward linkages of an industry are indicatives of the general importance of that industry to the whole productive network**
  - They are also called "upstream multiplier" and "downstream multiplier"

- **The upstream multiplier is calculated through the sum of the direct and indirect backward linkages of an industry using the L Matrix**
  - $\sum_{i=1}^{n} l_{i,j}$ **-> the upstream multiplier of an industry j is the sum of all its $l_{i,j}$ coefficients (in the column)**

	Wheat	**Iron**
Wheat	3.3	1.6
Iron	2.3	2.8
Backward linkages	5.6	4.4

  - In the example: 1 monetary unit of output produced by the wheat industry demands (pulls) the economy to generate 5.6 monetary units of value

Input-Output Models: Linkages and Multipliers

Multipliers: Backward and Forward Linkages

- **Conversely, the downstream multiplier is calculated through the sum of the direct and indirect forward linkages of an industry using the G Matrix**
  - **$\sum_{j=1}^{n} g_{i,j}$ -> the downstream multiplier of an industry j is the sum of all its $g_{i,j}$ coefficients (in the row)**

	Wheat	Iron	Forward linkages
Wheat	3.3	1.1	4.4
Iron	3.3	2.8	6.1

  - In the example: 1 monetary unit of output produced by the wheat industry supplies direct and indirect inputs to the production of a total of 4.4 monetary units in the whole economy

Input-Output Models: Linkages and Multipliers

Changes in Final Demand

- **The Demand-pull/Leontief Model provides a framework to assess how changes in final demand affect the level of output**
  - e.g. In the context of the ecological transition, it allows us to study and estimate the effects of the decarbonization
    - **Let us assume demand for high-emitting (sunset) industries fully declines. How does this drop in production affect the entire economy?**
  - Given the possibility to add more information to the satellite accounts, it is possible to study the effects of the ecological transition not only in output changes, but also in other social and ecological indicators (taxes, wages, employment, emissions, land use, etc.)

- **The basic equation employed for this analysis is the following one:**

$$\Delta \boldsymbol{x} = \boldsymbol{L}\Delta \boldsymbol{y} \qquad \textbf{(13)}$$

  - A major limitation of this approach is that all inputs employed in production are treated as equally critical and non-substitutable
  - Prices are held fixed

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On the limitations of the Ghosh Model

- **The Ghosh Model, on the other hand, is not a suitable tool to assess the effect of changes in value-added on the total output of the economy as it presents major limitations when it comes to understanding macroeconomic dynamics**
  - Final demand is completely elastic and adjustable to changes in value-added
  - The model allows industries to increase their output even in the case in which they do not receive more supplies from other industries
    - **e.g. in the Ghosh Model framework, an increase in the value-added of the industry of cars manufacturing will cause this industry to produce more cars even if it does not receive more supplies (such as manufactured metals) from other industries**
  - In the case of a reduction in output of a given industry, the standard Ghosh Model is also not able to address possible import substitution

- **Despite this criticism, the G Matrix provided by the ghoshian approach is still largely considered as the best tool to assess forward linkages**
  - Moreover, new and more sounding interpretations of the Ghosh model have been proposed in the literature, comparing it with Leontief Price Models (which we will address in the next module)

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An example: WIOD Brazilian IO Table for 2010

- **Once more we invite you now to watch the following videos where we present you a real IO Model using the World Input-Output Database (WIOD) 2013 Release (Timmer et al., 2015)**
  - In the videos we show how to calculate the A, L, B and G Matrices + Multipliers in Excel

- **Watch the videos here!**
  - The dataset can be downloaded here -> select the option of "National IO tables"
  - Open the file "BRA_NIOT_ROW_Sep12"

Suggested Readings:

Textbooks and further readings

- **Textbooks:**
  - Miller, R. E., & Blair, P. D. (2021). Input-Output Analysis: Foundations and Extensions (3rd ed.). Cambridge University Press. https://doi.org/10.1017/9781108676212
    - **Chapters 2 and 7**
  - Raa, T. ten (Ed.). (2017). Handbook of input-output analysis. Edward Elgar Publishing. https://doi.org/10.4337/9781783476329
    - **Chapter 4**

- **Recommended readings:**
  - Altimiras-Martin, A. (2024). A supply-driven model consuming simultaneously all primary inputs: Unfolding analytical potential beyond the Ghosh model. Economic Systems Research, 36(2), 249–264. https://doi.org/10.1080/09535314.2022.2137008
  - Dietzenbacher, E. (1997). In Vindication of the Ghosh Model: A Reinterpretation as a Price Model. Journal of Regional Science, 37(4), 629–651. https://doi.org/10.1111/0022-4146.00073
  - Ghosh, A. (1958). Input-Output Approach in an Allocation System. Economica, 25(97), 58. https://doi.org/10.2307/2550694
  - Oosterhaven, J. (1988). ON THE PLAUSIBILITY OF THE SUPPLY-DRIVEN INPUT-OUTPUT MODEL. Journal of Regional Science, 28(2), 203–217. https://doi.org/10.1111/j.1467-9787.1988.tb01208.x
  - Oosterhaven, J. (1996). Leontief versus Ghoshian Price and Quantity Models. Southern Economic Journal, 62(3), 750. https://doi.org/10.2307/1060892