

# Uncertainty\_calculation

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## 1 Error bars in $\Delta T$ calculations with RCMIP data:

Following is a brief description of the method used to produce the uncertainty bars in the figures.

### 1.0.1 General about computing $\Delta T$ :

We compute the change in GSAT temperature ( $\Delta T$ ) from the effective radiative forcing (ERF) estimated from the RCMIP models (Nicholls et al 2020), by integrating with the impulse response function (IRF(t-t')) (Geoffroy et al 2013). See Nicholls et al (2020) for description of the RCMIP models and output.

For any forcing agent  $x$ , with estimated  $ERF_x$ , the change in temperature  $\Delta T$  is calculated as:

$$\Delta T_x(t) = \int_0^t ERF_x(t') IRF(t - t') dt'$$

**The Impulse response function (IRF):** In these calculations we use the impulse response function (Geoffroy et al 2013):

$$IRF(t) = 0.885 \cdot \left( \frac{0.587}{4.1} \cdot \exp\left(\frac{-t}{4.1}\right) + \frac{0.413}{249} \cdot \exp\left(\frac{-t}{249}\right) \right)$$
$$IRF(t) = \frac{1}{\lambda} \sum_{i=1}^2 \frac{a_i}{\tau_i} \cdot \exp\left(\frac{-t}{\tau_i}\right)$$

with  $\frac{1}{\lambda} = 0.885$  (K/Wm<sup>-2</sup>),  $a_1 = 0.587$ ,  $\tau_1 = 4.1$ (yr),  $a_2 = 0.413$  and  $\tau_2 = 249$ (yr) (note that  $i = 1$  is the fast response and  $i = 2$  is the slow response and that  $a_1 + a_2 = 1$ )

### 1.1 Sources of uncertainty:

In these calculations we are considering the following variabilities/uncertainties 1. ERF estimate uncertainty: represented as the variability in the RCMIP models 2. Equilibrium climate sensitivity(ECS) uncertainty/IRF uncertainty

ERF uncertainty is represented as the spread in the ERF produced by the considered RCMIP models. ECS uncertainty is the uncertainty by which the ERF is translated into changes in temperature.

Starting with:

$$\Delta T_x(t) = \int_0^t ERF_x(t') IRF(t - t') dt'$$

Now, define  $\Delta_x$  as follows:

$$\Delta_x = \lambda \int_0^t ERF_x(t') IRF(t - t') dt' \quad (1)$$

$$= \lambda \int_0^t ERF_x(t') \sum_{i=1}^2 \frac{a_i}{\tau_i \cdot \lambda} \cdot \exp\left(\frac{-(t - t')}{\tau_i}\right) dt' \quad (2)$$

$$= \int_0^t ERF_x(t') \sum_{i=1}^2 \frac{a_i}{\tau_i} \cdot \exp\left(\frac{-(t - t')}{\tau_i}\right) dt' \quad (3)$$

$$(4)$$

So, then:

$$\Delta T_x(t) = \frac{1}{\lambda} \cdot \Delta_x(t) \quad (5)$$

This means that the uncertainty in  $\Delta T$  can be calculated according to the propagated uncertainty/variability in the product of parameter  $\frac{1}{\lambda} = \alpha$  and uncertainty in  $ERF_x$ .

**NB:** From here we make the assumption that it is good enough to represent the uncertainty of the IRF through the value of  $\frac{1}{\lambda} = \alpha$ , and not to consider the values of  $\tau_i$  and  $a_i$ .

### 1.1.1 Distribution of a product of two independent variables:

For any two random variables X and Y:

$$Var(X \cdot Y) = Cov(X^2, Y^2) + [Var(X) + E(X)^2] \cdot [Var(Y) + E(Y)^2] - [Cov(X, Y) + E(X) \cdot E(Y)]^2 \quad (6)$$

$$= \sigma_{X^2, Y^2} + (\sigma_X^2 + \mu_X^2) \cdot (\sigma_Y^2 + \mu_Y^2) - (\sigma_{X, Y} + \mu_X \cdot \mu_Y)^2 \quad (7)$$

If X and Y are independent:

$$Var(X \cdot Y) = (\sigma_X^2 + \mu_X^2) \cdot (\sigma_Y^2 + \mu_Y^2) - \mu_X^2 \cdot \mu_Y^2 \quad (8)$$

Assuming  $\Delta_x$  and  $\alpha = \frac{1}{\lambda}$  are independent we get:

$$Var(\Delta T_x) = Var(\alpha \cdot \Delta_x) \quad (9)$$

$$= (Var(\alpha) + E(\alpha)^2)(Var(\Delta_x) + E(\Delta_x)^2) - E(\alpha)^2 E(\Delta_x)^2 \quad (10)$$

Let  $\sigma_x = \sqrt{Var(\Delta_x)}$ ,  $\mu_x = E(\Delta_x)$ ,  $\sigma_\alpha = \sqrt{Var(\alpha)}$  and  $\mu_\alpha = E(\alpha)$

$$Var(\Delta T_x) = (\sigma_x^2 + \mu_x^2)(\sigma_\alpha^2 + \mu_\alpha^2) - \mu_x^2 \mu_\alpha^2 \quad (11)$$

Since  $\sigma_x$  and  $\mu_x$  can be estimated from the RCMIP models output for each agent  $x$  and  $\sigma_\alpha$  and  $\mu_\alpha$  is estimated in chapter 7 of the WG1-AR6 report, we can calculate the uncertainty/variability in prediction with this formula above.

**NB:** The formula above does not assume anything about the distribution of  $\alpha$  and  $\Delta_x$ . However, by calculating the standard deviation and the mean, and not e.g. quantiles, we are assuming that these measures are good enough and hence that the distributions are not too skewed.

## 1.2 Sums and differences:

For the model outputted ERF, the ERF is decomposed into various component contributions (ozone, CH<sub>4</sub> etc). In the calculations of the temperature change (GSAT) for each component, we use the ERF for each component. If we want to look at differences or sums of temperature changes, e.g. the difference between methane contribution  $X_i$  and the total anthropogenic contribution  $Y$ , we would have some covariance between  $X$  and  $Y$ , because if one model has large  $X_i$  it would normally have large  $Y$  as well.

So either we can take this into account explicitly:

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$$

Here we use an alternative approach where we rather treat the sum or difference of the ERF as one stochastic variable and alpha as another and assume they are independent. The independence of the error on ECS and ERF is a good assumption here. Secondly, we do then not need to consider the covariance of ERF between different components because it is implicitly covered.

## 1.3 Method:

The following method is used: 1. Intra model variability from *ERF* from different models 2. Assume this is independent of the *IRF* 3. Combine these two uncertainties with  $Var(\Delta T_x) = (\sigma_x^2 + \mu_x^2)(\sigma_\alpha^2 + \mu_\alpha^2) - \mu_x^2 \mu_\alpha^2$

In other words:

Let  $\sigma_\alpha$  and  $\mu_\alpha$  be the standard deviation and mean for a normal distribution of the  $\alpha$  parameter in ECS. Secondly, let  $Z_{i=1,...,N}$  be defined as

$$Z_i = \lambda \int_0^t ERF_i(t') IRF(t - t') dt' \quad (12)$$

$$= \int_0^t ERF_i(t') \sum_{i=1}^2 \frac{a_i}{\tau_i} \cdot \exp\left(\frac{-(t - t')}{\tau_1}\right) dt' \quad (13)$$

$$(14)$$

where  $ERF_i$  is some difference or sum of different ERF components from model nr  $i$ .

We then calculate the population standard deviation as

$$\sigma_{Z_i} = \frac{\sqrt{\sum (Z_i - \mu_Z)^2}}{N} \quad (15)$$

and we can get

$$\sigma_T = (\sigma_{X_i} + \mu_{Z_i})(\sigma_\alpha + \mu_\alpha) - \mu_{Z_i} \mu_\alpha \quad (16)$$

Finally, we then use  $\sigma_T$  to represent an estimate of the uncertainty in the computed change in GSAT.

### 1.3.1 Technical detail:

Since ERFs for different components are assumed to be linearly independent, so will the estimated changes in GSAT. Thus from calculating any set of  $\Delta T_c$ , where  $c$  signifies the components, we can calculate the change in GSAT from a sum or difference between components components,  $\Delta T_*$ , as follows:

$$\Delta T_* = \sum_{p \in P} \Delta T_p - \sum_{m \in M} \Delta T_m \quad (17)$$

Let  $\Delta T_{\alpha=\mu_\alpha}$  signify the same as above, but with each  $\Delta T_c$  calculated with the  $\alpha = \frac{1}{\lambda}$  factor in the IRF specified as equal to the expected value  $\mu_\alpha$ :

$$\Delta T_{\alpha=\mu_\alpha} = \sum_{p \in P} T_{p,\alpha=\mu_\alpha} - \sum_{m \in M} T_{m,\alpha=\mu_\alpha}. \quad (18)$$

If we then calculate  $\Delta T_{\alpha=\mu_\alpha,i}$  for each model  $i$ , we get :

$$Z_i = \frac{1}{\mu_\alpha} \Delta T_{\alpha=\mu_\alpha,i} \quad (19)$$

where the index  $i$  signifies the different models.

Now  $Z_i$  is as above and independent of the choice of  $\alpha = \frac{1}{\lambda}$ . Thus we can easily calculate

$$\sigma_Z = \frac{\sqrt{\sum (Z_i - \mu_Z)^2}}{N} \quad (20)$$

since

$$\mu_Z = \frac{1}{\mu_\alpha} \mu_{\Delta T_{\alpha=\mu_\alpha}} \quad (21)$$

we have

$$\sigma_Z = \frac{1}{\mu_\alpha} \sigma_{\Delta T_{\alpha=\mu_\alpha}}. \quad (22)$$

**Finally:** Let  $\Delta T = Z \cdot \alpha$  and assume  $Z$  and  $\alpha$  independent. Then

$$\sigma_{\Delta T}^2 = (\sigma_Z^2 + \mu_Z^2)(\sigma_\alpha^2 + \mu_\alpha^2) - \mu_Z^2 \mu_\alpha^2 \quad (23)$$

$$\sigma_{\Delta T}^2 = \frac{1}{\mu_\alpha^2} ((\sigma_{\Delta T_{\alpha=\mu_\alpha}}^2 + \mu_{\Delta T_{\alpha=\mu_\alpha}}^2)(\sigma_\alpha^2 + \mu_\alpha^2) - \mu_{\Delta T_{\alpha=\mu_\alpha}}^2 \mu_\alpha^2) \quad (24)$$

$$\sigma_{\Delta T} = \frac{1}{\mu_\alpha} ((\sigma_{\Delta T_{\alpha=\mu_\alpha}}^2 + \mu_{\Delta T_{\alpha=\mu_\alpha}}^2)(\sigma_\alpha^2 + \mu_\alpha^2) - \mu_{\Delta T_{\alpha=\mu_\alpha}}^2 \mu_\alpha^2)^{\frac{1}{2}} \quad (25)$$

```
[11]: def sigma_com(sig_DT, mu_DT, sig_alpha, mu_alpha):
      return (((sig_DT ** 2 + mu_DT ** 2) * (
          sig_alpha ** 2 + mu_alpha ** 2) - mu_DT ** 2 * mu_alpha ** 2) /
      ↪ mu_alpha ** 2) ** (.5)
```

In other words, it suffices to know

- a)  $\sigma_\alpha$  and  $\mu_\alpha$  and
- b)  $\Delta T_x$  calculated for a fixed  $\mu_\alpha$

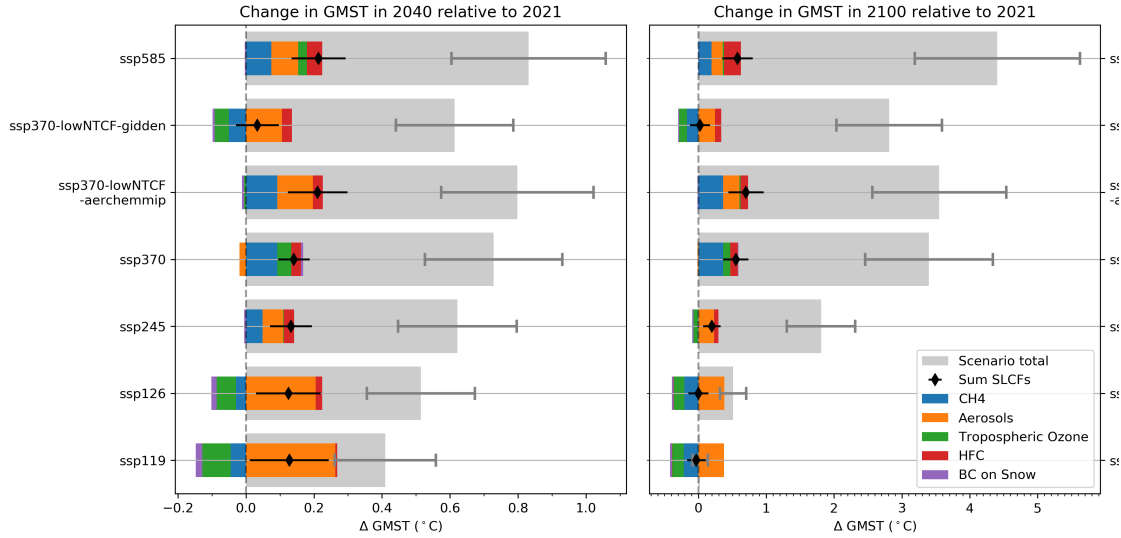
to compute the uncertainty bars.

#### 1.4 Example with $\sigma_\alpha = 0.24$ and $\mu_\alpha = 0.885$ :

```
[12]: from ar6_ch6_rcmipfigs.constants import RESULTS_DIR
      from IPython.display import Image

      fn = RESULTS_DIR+ '/figures/stack_bar_influence_years_horiz_errTot.png'
      Image(filename=fn)
```

[12]:



## 2 Using percentiles from RCMIP:

We have percentiles from at least two models. The distribution is unknown. To combine the percentiles from these these models, we can e.g.:

- Calculate the percentwise deviation from the median of all models. For a model  $i$ , calculate  $q_{1,i} = Q_{1,i}/Q_{2,i}$  and  $q_{3,i} = Q_{3,i}/Q_{2,i}$
- Take average over models:  $\overline{q_1}, \overline{q_3}$
- Visualize interval:  $(\overline{q_1}\overline{X}, \overline{q_3}\overline{X})$  where  $\overline{X}$  is the mean over ALL the models.

## 3 Referances:

- Geoffroy, O., Saint-Martin, D., Olivié, D. J. L., Voldoire, A., Bellon, G., and Tytéca, S. (2013). Transient Climate Response in a Two-Layer Energy-Balance Model. Part I: Analytical Solution and Parameter Calibration Using CMIP5 AOGCM Experiments. J. Clim. 26, 1841--1857. doi:10.1175/JCLI-D-12-00195.1.
- Nicholls, Z. R. J., Meinshausen, M., Lewis, J., Gieseke, R., Dommenges, D., Dorheim, K., Fan, C.-S., Fuglestad, J. S., Gasser, T., Golüke, U., Goodwin, P., Kriegler, E., Leach, N. J., Marchegiani, D., Quilcaille, Y., Samset, B. H., Sandstad, M., Shiklomanov, A. N., Skeie, R. B., Smith, C. J., Tanaka, K., Tsutsui, J., and Xie, Z.: Reduced complexity model intercomparison project phase 1: Protocol, results and initial observations, Geosci. Model Dev. Discuss., <https://doi.org/10.5194/gmd-2019-375>, in review, 2020. Nicholls Z. et al (2020), ``Reduced complexity

model intercomparison project phase 1: Protocol, results and initial  
observations'', <https://www.geosci-model-dev-discuss.net/gmd-2019-375/>