Geometrie

I Spații vectoriale

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Spații vectoriale

1. Def. Exemple

Def: $V \neq \emptyset$, K – corp comutativ

$+ : V \times V \to V$ adunarea vectorilor
(op. internă)

$\cdot : K \times V \to V$ înmulțirea vect. cu scalari
(op. externă)

I $(V, +)$ grup comutativ

II 1. $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$

2. $(\alpha_1 + \alpha_2) v = \alpha_1 v + \alpha_2 v$, $(\forall)\ v, v_1, v_2 \in V$

3. $\alpha_1(\alpha_2 v) = (\alpha_1 \alpha_2) v$ $\quad \alpha, \alpha_1, \alpha_2 \in K$

4. $1 v = v$

$(V/_K, +, \cdot)$ sp. vect. peste corpul comutativ K

$v \in V$, $\alpha \in K$
vectori, scalari

$K = \mathbb{R} \to$ sp. vect. real

$K = \mathbb{C} \to$ sp. vect. complex

Exemple. 1) $(K/_K, +, \cdot)$ – sp. vect. $\quad \mathbb{C}/\mathbb{C}$; $\mathbb{R}/\mathbb{R}$; $\mathbb{Q}/\mathbb{Q}$; $\mathbb{Z}_p/\mathbb{Z}_p$ (p prim)

• $H \subseteq K \to (K/_H, +, \cdot)$ sp. vect. $\quad \mathbb{C}/\mathbb{R}$, $\mathbb{C}/\mathbb{Q}$, $\mathbb{R}/\mathbb{Q}$
subcorp

• $K^n = K \times K \times \ldots \times K = \{(x_1, \ldots, x_n) \,/\, x_i \in K, (\forall) i = \overline{1,n}\}$

$+ : K^n \times K^n \to K^n$

$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) \overset{def}{=} (x_1 + y_1, \ldots, x_n + y_n)$, $(\forall)\ (x_1, \ldots, x_n), (y_1, \ldots, y_n) \in K^n$

$\cdot : K \times K^n \to K^n$

$k(x_1, \ldots, x_n) \overset{def}{=} (k x_1, \ldots, k x_n)$, $(\forall)\ (x_1, \ldots, x_n) \in K^n$, $k \in K$

$(K^n/_K, +, \cdot)$ sp. vect. $\quad \mathbb{C}^n/\mathbb{C}$, $\mathbb{R}^n/\mathbb{R}$, $\mathbb{Q}^n/\mathbb{Q}$, $\mathbb{Z}_p^n/\mathbb{Z}_p$

2) $(\mathcal{M}_{(m,n)}(K)/_K, +, \cdot)$ sp. vect.

3) • $(K[X]/_K, +, \cdot)$ sp. vect.

• $(K_n[X]/_K, +, \cdot)$ sp. vect.

$K_n[X] = \{P \in K[X] \mid \text{grad } P \leq n\}$

4) Fie $A \in \mathcal{M}_{(m,n)}(K)$

$S(A) = \{X \in K^n \mid AX = 0\}$ (mulțimea sol. unui sistem omogen)

→ sp. vect. peste K ($+, \cdot$ din K^n)

<u>Obs</u>. Este primul exemplu de sp. vectorial care apare ca o parte stabilă a unui alt sp. vectorial. → $\boxed{\text{subsp. vectorial}}$

Liniar independență. Liniar dependență

<u>Def</u>. 1) Fie $S = \{v_1, v_2, \dots, v_n\} \subset V$

S s.n. sistem de vectori <u>liniar indep.</u> dacă:

$(\forall)\ \alpha_1, \dots, \alpha_n \in K$, $\underbrace{\alpha_1 v_1 + \dots + \alpha_n v_n}_{\substack{\text{comb. lin. a vect. } \{v_1, \dots, v_n\} \\ \text{cu scalari } \{\alpha_1, \dots, \alpha_n\}}} = 0_V \Rightarrow \alpha_1 = \dots = \alpha_n = 0$

i.e [$(\forall)$ comb. lin. nulă se realizează <u>numai</u> cu scalari nuli]

• Dacă S – s.v.l.i. atunci $(\forall)$ subsistem al său este, de asemenea, lin. ind.

• Dacă $v \neq 0_V \to \{v\}$ lin. ind.

2) S s.n. sistem de vectori <u>liniar dep.</u> dacă: <u>nu</u> este lin. indep.

i.e. [$(\exists)$ comb. lin. nulă care se realizează <u>și</u> cu scalari <u>nu toți</u> nuli]

<u>Exemplu</u>: $(\mathbb{R}^n/_{\mathbb{R}}, +, \cdot)$

$S = \{e_1 = (1, 0, \dots, 0), \dots e_i = (0, \dots, 0, \underset{(i)}{1}, 0 \dots 0), \dots, e_n = (0, \dots, 1)\}$

→ s.v. lin. ind.

Sisteme de generatori

Def: Fie $S \subset V$

Mulțimea tuturor comb. lin. finite de vectori din S s.n. acoperirea (închiderea) liniară a lui S (not $\overline{S}$)

$$\overline{S} = \{ v \in V \,/\, v = \sum_{i=1}^{k} \alpha_i v_i,\ v_i \in S,\ \alpha_i \in K,\ k \in \mathbb{N} \}$$

Obs: $S \subseteq \overline{S}$

Exemplu: 1) $S' = \{ v_1 = (1,0,0),\ v_2 = (1,1,1) \} \subset \mathbb{R}^3$

$\overline{S'} = \{ (\alpha_1 + \alpha_2, \alpha_2, \alpha_2) / \alpha_1, \alpha_2 \in \mathbb{R} \}$

$\overline{S} \subsetneq \mathbb{R}^3$

2) $S = \{ 1, X, X^2, \dots, X^n, \dots \} \subset K[X]$

$\overline{S} = K[X]$

3) $S' = \{ 1, X, X^2, \dots, X^n \}$

$\overline{S} = K_n[X] = \{ P \in K[X] / \text{grad } P \leq n \}$

Def: Fie $S \subseteq V$ a.î. $\overline{S} = V$.

Atunci S s.n. sistem de generatori (ai lui V).

- Un sp. vectorial care admite un sistem finit de generatori s.n. **finit generat**.

Obs: 1) $\mathbb{R}^n$ s.p. vect. finit generat

2) $K[X]$ sp. vect. care **nu** este finit generat.

Def: 1) Fie V/K sp. vect.

$B \subset V$ s.n. bază pt. sp. vect. V dacă $\begin{cases} 1) & B \text{ s.v.l.i.} \\ 2) & B \text{ s.g. ai lui } V \ (\overline{B} = V) \end{cases}$

2) O bază ordonată s.n. **reper**.

C2

Exemple:

Geometrie

1) $\mathbb{R}^n/\mathbb{R}$

$B_0 = \{e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 1)\} \subset \mathbb{R}^n$

baza canonică

2) $\mathcal{M}_{(m,n)}(\mathbb{R})$

$B_0 = \{E_{ij}\}_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$, unde $E_{ij} = \begin{array}{cc} & j \\ i & \begin{pmatrix} 0 & & 0 \\ & \boxed{1} & \\ 0 & & 0 \end{pmatrix} \end{array}$

bază canonică

3) $K[X]$

$B_0 = \{1, X, X^2, \dots, X^n, \dots\} \subset K[X]$

bază canonică

4) $\mathbb{C}/\mathbb{R}$

$B_0 = \{1, i\}$, $\boxed{i^2 = -1}$

bază canonică

[P] Dacă $\underset{\text{bază}}{B} \overset{=\{v_1, \dots, v_n\}}{\subset} V$ atunci: $(\forall) v \in V$, $(\exists)!\ \alpha_1, \dots, \alpha_n \in K$ a.î.

$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

Dem: B - bază $\Rightarrow \overline{B} = V \Rightarrow (\forall) v \in V$, $(\exists) \alpha_1, \dots, \alpha_n \in K$

a.î. $v = \alpha_1 v_1 + \dots + \alpha_n v_n$

Pp. $(\exists) \beta_1, \dots, \beta_n \in K$ a.î. $v = \beta_1 v_1 + \dots + \beta_n v_n$

$\Rightarrow \sum_{i=1}^{n} (\alpha_i - \beta_i) v_i = 0 \underset{\text{l. ind.}}{\overset{B\ s.v.}{\Longrightarrow}} \alpha_i - \beta_i = 0$, $(\forall) i = \overline{1,n} \Rightarrow \alpha_i = \beta_i$, $(\forall) i = \overline{1,n}$ q.e.d.

Def: $(\alpha_1, \dots, \alpha_n) \overset{not}{=} [v]_B \to$ coord. vect. v în raport cu baza B

T. (schimbului) [Steinitz]: Fie V sp. vect. finit generat

$\underset{\text{s. de gen.}}{G} = \{g_1, \dots, g_s\} \subset V$

$\underset{\text{s.v. lin. ind.}}{A} = \{f_1, \dots, f_r\} \subset V$

Atunci: $\begin{cases} 1)\ r \le s \\ 2)\ (\exists)\ A' \subset G \text{ a.î. } A \cup A' = \underset{\text{bază}}{B} \subset V \end{cases}$

T: a) Orice sp. vect. are (mai multe) baze.

b) Oricare 2 baze ale unui sp. vect. f.g au acelaşi cardinal.

Dem: b) Fie $B, B' \subset V$ 2 baze

$$\left.\begin{array}{l} \left.\begin{array}{l} B - \text{s.v.l.i} \\ B' - \text{s.g.} \end{array}\right| \Rightarrow \operatorname{card} B \le \operatorname{card} B' \\ \left.\begin{array}{l} B' - \text{s.v.l.i} \\ B - \text{s.g.} \end{array}\right| \Rightarrow \operatorname{card} B' \le \operatorname{card} B \end{array}\right| \Rightarrow \operatorname{card} B = \operatorname{card} B'.$$

Def: Cardinalul comun al bazelor unui sp. vect. s.n. dimensiune $\dim_K V$

Exemple: $\dim_{\mathbb{R}} \mathbb{R}^n = n$; $\dim_{\mathbb{R}} \mathcal{M}_{(m,n)}(\mathbb{R}) = mn$

$\dim_K K[x] = +\infty$; $\dim_K K_n[x] = n+1$; $\dim_{\mathbb{R}} \mathbb{C} = 2$

Exerc: Ar. că: dacă $\{f_1, \dots, f_n\} \subset \mathbb{R}^n/_{\mathbb{R}}$ bază oarecare $\Rightarrow \{f_1, if_1, \dots, f_n, if_n\}$ bază $\subset \mathbb{C}^n/_{\mathbb{R}}$

i.e. $\dim_{\mathbb{R}} \mathbb{C}^n = 2n$

Corolar: Fie $\dim V = n$.

Atunci: 1) $(\forall)$ s.v. lin. indep. are cel mult n vect.

2) $(\forall)$ s. de gen. are cel puţin n vect.