$\vec{n_1} = (2,-1,3)$
$\vec{n_2} = (3,1,1)$ > vectori normali ai celor 2 plane ce definesc

Directia dr. d este data de:

$$v = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 3 & 1 & 1 \end{vmatrix} = (-4,+7,5)$$

$$\vec{v} = \vec{n_\pi} \Rightarrow \pi : -4x+7y+5z+d=0$$

$$M(1,2,-1) \in \pi \Rightarrow -4+14-5+d=0 \Rightarrow d=-5$$

$$\pi: -4x+7y+5z-5=0$$

* $\boxed{4x-7y-5z+5=0}$

Ap1 Să se scrie ec planului determinat de dreptele paralele

$$(d_1): \frac{x+1}{2} = \frac{y-2}{3} = \frac{z+3}{2}$$

$$(d_2): \frac{x-3}{2} = \frac{y+1}{3} = \frac{z-1}{2}$$

Rez: Avem: $\vec{v_1} = (2,3,2)$

$$\left.\begin{array}{l} P_1(-1,2,-3) \in d_1 \\ P_2(3,-1,1) \in d_2 \end{array}\right| \Rightarrow \vec{v_2} = \vec{P_1P_2} = (4,-3,4)$$

$$\pi: \begin{vmatrix} x+1 & y-2 & z+3 \\ 2 & 3 & 2 \\ 4 & -3 & 4 \end{vmatrix} = 0$$

$$\Leftrightarrow (x+1)\cdot 18 - (y-2)\cdot 0 + (z+3)(-18) = 0$$

$$x+1-z-3=0$$

$$\boxed{x-z-2=0}$$

$\boxed{Apl.}$ Să se determine vectorul director al dreptei

(d) $\begin{cases} 2x-y+z+4=0 \\ x+4y+3z-1=0 \end{cases}$

şi un punct de pe dreapta (d).

Rez: (V1) $\vec{v} = \vec{n_1} \times \vec{n_2}$, unde $\vec{n_1} = (2,-1,1)$

$\vec{n_2} = (1,4,3)$

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 4 & 3 \end{vmatrix} = (-7,-5,9)$$

(V2) Rezolvăm sistemul: $\begin{cases} 2x-y+z+4=0 \\ x+4y+3z-1=0 \end{cases}$

$$\begin{cases} 2x-y+z=-4 \\ x+4y+3z=1 \end{cases}$$

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 4 & 3 \end{pmatrix}$$

$$\Delta_2 = \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 9 \neq 0 \Rightarrow \operatorname{rg} A = 2$$

x, y nec. principale

$z = t$ nec. secundară

$$\begin{cases} 2x-y=-4-t \mid \cdot 4 \\ x+4y = 1-3t \mid \cdot(-2) \\ z=t, t\in\mathbb{R} \end{cases} \Rightarrow 9x = -15-7t$$

$x = -\frac{5}{3} - \frac{7}{9}t$

$-9y = -6+5t$

$y = \frac{2}{3} - \frac{5}{9}t$

$$\Rightarrow \begin{cases} x=-\frac{5}{3}-\frac{7}{9}t \\ y=\frac{2}{3}-\frac{5}{9}t, \quad t\in\mathbb{R}. \\ z=t \end{cases}$$

Obs: $\langle(-7,-5,9)\rangle = \langle(-\frac{7}{9},-\frac{5}{9},1)\rangle$

$\vec{v} = (-\frac{7}{9}, -\frac{5}{9}, 1)$ $\quad t=0 \Rightarrow P_0(-\frac{5}{3}, \frac{2}{3}, 0) \in d.$

$\boxed{Apl.}$ Să se scrie ec. implicite (sub formă de rapoarte) ale dreptei:

$$(d) \begin{cases} x-2y+3z-1=0 \\ 2x+y-2z-3=0 \end{cases}$$

Rez: Vectorul director al dr. (d) este: $\vec{v}=\vec{n_1}\times\vec{n_2}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 2 & 1 & -2 \end{vmatrix}$

$=(1,+8,5)$

Luăm: $z=0 \Rightarrow \begin{cases} x-2y=1 \\ 2x+y=3 \;/\cdot 2 \end{cases} \xrightarrow{/\cdot(-2)} 5x=7 \Rightarrow x=\frac{7}{5}$

$5y=1 \Rightarrow y=\frac{1}{5}$

$P_0\left(\frac{7}{5},\frac{1}{5},0\right)\in d$

$$\boxed{d: \frac{x-\frac{7}{5}}{1}=\frac{y-\frac{1}{5}}{8}=\frac{z-0}{5}}$$

$\boxed{Apl.}$ Să se scrie ec. dr. care trece prin pct. $M(2,-1,1)$ și este paralelă cu dr: $(d) \begin{cases} x+y-z=0 \\ x+2y+z-1=0 \end{cases}$

Rez: $\vec{v_d}=\vec{n_1}\times\vec{n_2}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}=(3,-2,1)$

$d\parallel d' \Rightarrow \vec{v_{d'}}=\vec{v_d}$

$$d': \frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-1}{1}$$

Ap1 Să se scrie ec. perpendicularei dusă din pct. $M(-1,2,3)$ pe dreapta $(d): \frac{x+2}{3}=\frac{y-4}{1}=\frac{z}{1}$

Rez: Scriem ec. planului (P) ce trece prin pct. M și este perpendiculară pe dr (d).

$$\vec{n}_P=\vec{v}_d=(3,1,1)$$

$$P: 3x+y+z+d=0$$

$$M\in P \Rightarrow -\cancel{3}+2+\cancel{3}+d=0 \Rightarrow d=-2$$

$$P: 3x+y+z-2=0$$

Proiecția pct. M pe dr. (d) este intersecția dr. (d) cu planul (P).

$$d\cap P \begin{cases} 3x+y+z-2=0 \\ \frac{x+2}{3}=\frac{y-4}{1}=\frac{z}{1}(=t)\end{cases} \quad \begin{cases} 3x+y+z-2=0 \\ x=3t-2 \\ y=t+4 \\ z=t\end{cases}$$

$$3(3t-2)+(t+4)+t-2=0 \Rightarrow 11t=4 \Rightarrow t=\frac{4}{11}$$

$$\Rightarrow M_0\left(-\frac{10}{11}, \frac{48}{11}, \frac{4}{11}\right)$$

$$MM_0: \frac{x+1}{-\frac{10}{11}+1}=\frac{y-2}{\frac{48}{11}-2}=\frac{z-3}{\frac{4}{11}-3}$$

$$\boxed{\frac{x+1}{1}=\frac{y-2}{26}=\frac{z-3}{-29}}$$

Ap1. Dați o reprezentare parametrică dreptei (d)

$$(d): \begin{cases} x+y-z+1=0 \\ 2x-y+3z-4=0\end{cases}$$

Rez: (V_1) Rezolvăm sistemul $\begin{cases} x+y-z=-1 \\ 2x-y+3z=4\end{cases}$

$$A=\begin{pmatrix}1 & 1 & -1\\ 2 & -1 & 3\end{pmatrix}$$

$$\Delta_p=\Delta_2=\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}=-3\neq 0 \Rightarrow \operatorname{rg} A=2 \rightarrow x,y \text{ nec. principale}, \; z=t, t\in\mathbb{R} \text{ nec. secundară}$$

$$\begin{cases}x+y=1+t\\ 2x-y=4-3t\\ z=t, t\in\mathbb{R}\end{cases} \rightarrow 3x=5-2t \Rightarrow x=\frac{5}{3}-\frac{2}{3}t$$

$$y=1+t-\frac{5}{3}+\frac{2}{3}t=-\frac{2}{3}+\frac{5}{3}t$$

ec. parametrice $\begin{cases}x=\frac{5}{3}-\frac{2}{3}t\\ y=-\frac{2}{3}+\frac{5}{3}t, \; t\in\mathbb{R}\\ z=t\end{cases}$ $\quad P_0\left(\frac{5}{3},-\frac{2}{3},0\right)\in d$, $\quad \vec{v}_d=\left(-\frac{2}{3},\frac{5}{3},1\right)$

$\boxed{\text{Apl.}}$ Să se arate că dreapta

$$(d)\quad \frac{x-2}{3}=\frac{y+2}{-1}=\frac{z-3}{4}$$

este situată în planul $P: 2x+2y-z+3=0$

Rez: $\dfrac{x-2}{3}=\dfrac{y+2}{-1}=\dfrac{z-3}{4}\;(=t)$

$$\Rightarrow \begin{cases}x=2+3t\\ y=-2-t, \; t\in\mathbb{R}\\ z=3+4t\end{cases}$$

$$2(2+3t)+2(-2-t)-(3+4t)+3=0$$

$$\Leftrightarrow \cancel{6t}+\cancel{4}-\cancel{4}-\cancel{2t}-\cancel{3}-\cancel{4t}+\cancel{3}=0$$

$0=0$ (A)

Deci $d\subset P$.

$\boxed{Apl}$

Fie dreapta $d: \begin{cases} x+y+z=0 \\ x-y+z=0 \end{cases}$ $M(1,-1,2)$

a) Scrieți ec. planului π a.î. $M \in \pi$ și $d \perp \pi$.

b) $d \cap \pi = \{P\}$

Rez: a) $\vec{v_d} = \vec{n_1} \times \vec{n_2}$

V1 $\vec{n_1} = (1,1,1)$

$\vec{n_2} = (1,-1,1)$

$$\vec{v_d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2\vec{i} - 0\vec{j} - 2\vec{k} = (2,0,-2)$$
$$= 2(1,0,-1)$$

$\vec{n_\pi} = (1,0,-1)$, $M(1,-1,2) \in \pi$

$\pi: 1(x-1) + 0(y+1) - 1(z-2) = 0$

$\pi: x - z + 1 = 0$

V2 $d \begin{cases} x+y+z=0 \\ x-y+z=0 \end{cases}$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

$\Delta_p = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0 \Rightarrow rg\, A = 2$

x, y nec. principale

$z = t$, nec. secundară

$t \in \mathbb{R}$

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$$\begin{cases} x+y=-t \\ x-y=-t \end{cases} \Rightarrow 2x=-2t \Rightarrow x=-t$$

$z=t,\ t\in\mathbb{R}$

$$\begin{cases} y=0,\ t\in\mathbb{R} \\ z=t \end{cases}$$

$$d: \frac{x}{-1}=\frac{y}{0}=\frac{z}{1}(=t)$$

b) $d \begin{cases} x=-t \\ y=0 \\ z=t \end{cases}$ $\pi: x-z+1=0$

$$d\cap\pi \begin{cases} x=-t \\ y=0 \\ z=t \\ x-z+1=0 \Rightarrow -t-t+1=0 \Rightarrow t=\frac{1}{2} \end{cases}$$

$\{P\}=d\cap\pi$

$P\left(-\frac{1}{2},0,\frac{1}{2}\right)$

$\boxed{\text{Apl.}}$ Fie $\pi: x-2y+z=1$

$P_0(1,1,3)$

a) Scrieți ec. normalei prin P_0 la planul π

b) Calculați $d(P_0,\pi)$

Rez: a) $\vec{n}=(1,-2,1)$

$$n_{P_0}: \frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-3}{1}$$

b) $d(P_0,\pi)=?$

$P_0(x_0,y_0,z_0)$

$\pi: ax+by+cz+d=0$

$$d(P_0,\pi)=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

$$d(P_0,\pi)=\frac{|1-2\cdot1+3-1|}{\sqrt{1^2+(-2)^2+1^2}}=\frac{1}{\sqrt{6}}$$

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$\boxed{Apl}$ Să se afle coordonatele simetricului pct. $M_0(1,1,1)$ față de dr. (d) $\frac{x-1}{2}=\frac{y+1}{2}=\frac{z+12}{-1}\;(=t)$

Rez. Scriem ec. planului P care trece prin pct. M_0 și este perpendicular pe dr. d

$P:\; 2(x-1)+2(y-1)-(z-1)=0$

$2x+2y-z-3=0$

Considerăm $\{A\}=P\cap d$ $\begin{cases}2x+2y-z-3=0\\ x=2t+1\\ y=2t-1\\ z=-t-12\end{cases}$ $\Leftrightarrow$

$\Leftrightarrow 2(2t+1)+2(2t-1)-(-t-12)-3=0$

$9t+9=0 \Rightarrow t=-1$

$A(-1,-3,-11)$

$M_0A=AM_0'$

$\begin{cases}\frac{x+x'}{2}=-1\\ \frac{y+y'}{2}=-3\\ \frac{z+z'}{2}=-11\end{cases} \Rightarrow \begin{cases}x'=-2-x\\ y'=-6-y\\ z'=-22-z\end{cases} \Rightarrow \begin{cases}x'=-2-1=-3\\ y'=-6-1=-7\\ z'=-22-1=-23\end{cases}$

Deci: $M_0'(-3,-7,-23)$

$\boxed{Apl}$ Determinați proiecția pct. $M_0(-1,2,2)$ față de planul

$P: 2x-y+3z+23=0$

Rez. $\vec{n}_P=(2,-1,3)$

$n_{M_0}:\; \frac{x+1}{2}=\frac{y-2}{-1}=\frac{z-2}{3}\;(=t) \Rightarrow \begin{cases}x=2t-1\\ y=-t+2,\; t\in\mathbb{R}\\ z=3t+2\end{cases}$

$$n_{M_0} \cap P \quad \begin{cases} 2x - y + 3z + 23 = 0 \\ x = 2t - 1 \\ y = -t + 2 \\ z = 3t + 2 \end{cases} \Leftrightarrow 2(2t-1) - (-t+2) + 3(3t+2) + 23 = 0$$

$$\{M_0'\}$$

$$14t + 25 = 0 \Rightarrow t = -\frac{25}{14}$$

$$M_0'\left(-\frac{32}{7}, \frac{53}{14}, -\frac{47}{14}\right) \rightarrow \text{proiecția căutată}$$

$\boxed{Apl}$ Să se afle coordonatele simetricului pct. $M(1,3,-4)$ față de planul $P : 3x + y - 2z = 0$

Rez: $\vec{n_p} = (3, 1, -2)$

$$n_p : \frac{x-1}{3} = \frac{y-3}{1} = \frac{z+4}{-2} (= t)$$

$$n_p \cap P \quad \begin{cases} x = 3t + 1 \\ y = t + 3 \\ z = -2t - 4 \\ 3x + y - 2z = 0 \end{cases} \Leftrightarrow 3(3t+1) + (t+3) - 2(-2t-4) = 0$$

$$\{M'\}$$

$$14t + 14 = 0 \Rightarrow t = -1$$

$$M'(-2, 2, -2)$$

$$MM' = M'M''$$

$$\begin{cases} x' = \frac{x + x''}{2} \\ y' = \frac{y + y''}{2} \\ z' = \frac{z + z''}{2} \end{cases} \Rightarrow \begin{cases} x'' = 2x' - x \\ y'' = 2y' - y \\ z'' = 2z' - z \end{cases} \Rightarrow \begin{cases} x'' = -5 \\ y'' = 1 \\ z'' = 0 \end{cases}$$

Deci: $M''(-5, 1, 0)$.

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$\boxed{Ap\ 1.}$ Să se afle distanța de la pct. $M(-2,1,3)$ la planul P,

$(P)\ x+2y-3z-2=0$

Rez: $d(M,P)=\dfrac{|-2+2\cdot1-3\cdot3-2|}{\sqrt{1^2+2^2+(-3)^2}}=\dfrac{11}{\sqrt{14}}$

$\boxed{Ap\ 1.}$ Să se calculeze distanța dintre planele.

$P_1: 11x-2y-10z-15=0$

$P_2: 11x-2y-10z-45=0$

Rez: Obs: $P_1 \parallel P_2$

Considerăm $M\left(0,0,-\frac{3}{2}\right)\in P_1$

$$d(P_1,P_2)=\frac{\left|11\cdot0-2\cdot0-10\cdot\left(-\frac{3}{2}\right)-45\right|}{\sqrt{11^2+2^2+10^2}}=\frac{30}{15}=2$$

$\boxed{Ap\ 1.}$ Să se calculeze distanța de la pct. $M_0(3,-2,1)$ la dreapta $(d)\ \dfrac{x-1}{1}=\dfrac{y+2}{-2}=\dfrac{z-3}{3}$

Rez: Fie $M_1(1,-2,3)\in d$, $\vec{v}=(+1,-2,3)$

$$d(M_0,d)=\frac{\|\overrightarrow{M_1M_0}\times\vec{v}\|}{\|\vec{v}\|}$$

$\overrightarrow{M_1M_0}=(-2,0,-2)$

$$\overrightarrow{M_1M_0}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & 0 & -2\\ 1 & -2 & 3\end{vmatrix}=(-4,+4,4)$$

$$d(M_0,d)=\frac{\|(4,4,4)\|}{\|(1,-2,3)\|}=\frac{4\sqrt{3}}{\sqrt{14}}$$

Perpendiculara comună a 2 dr. necoplanare

Fie dreptele $d_k : \frac{x-x_k}{\alpha_k} = \frac{y-y_k}{\beta_k} = \frac{z-z_k}{\gamma_k} (=t_k)$, $k=\overline{1,2}$

$M_1(x_1, y_1, z_1) \in d_1 \quad \vec{v_1}(\alpha_1, \beta_1, \gamma_1)$

$M_2(x_2, y_2, z_2) \in d_2 \quad \vec{v_2}(\alpha_2, \beta_2, \gamma_2)$

$\overrightarrow{M_1M_2} = (x_2-x_1, y_2-y_1, z_2-z_1)$

d_1, d_2 necoplanare $\Leftrightarrow \begin{vmatrix} \alpha_1 & \alpha_2 & x_2-x_1 \\ \beta_1 & \beta_2 & y_2-y_1 \\ \gamma_1 & \gamma_2 & z_2-z_1 \end{vmatrix} \neq 0$

$d = \perp$ comuna

$d \cap d_k = \{P_k\}$, $k=\overline{1,2}$

$P_1(x_1+\alpha_1 t_1, y_1+\beta_1 t_1, z_1+\gamma_1 t_1) \in d_1$

$P_2(x_2+\alpha_2 t_2, y_2+\beta_2 t_2, z_2+\gamma_2 t_2) \in d_2$

$\overrightarrow{P_1P_2} = (x_2-x_1+\alpha_2 t_2-\alpha_1 t_1, y_2-y_1+\beta_2 t_2-\beta_1 t_1, z_2-z_1+\gamma_2 t_2-\gamma_1 t_1)$

$\begin{cases} \langle \overrightarrow{P_1P_2}, \vec{v_1} \rangle = 0 \\ \langle \overrightarrow{P_1P_2}, \vec{v_2} \rangle = 0 \end{cases} \Rightarrow$ sist. cu nec. t_1, t_2

$\Rightarrow P_1, P_2$

$d = P_1P_2$

$dist(d_1, d_2) = \|\overrightarrow{P_1P_2}\|$

$\boxed{\text{Apl.}}$

Fie $d_1 : \frac{x-2}{1} = \frac{y}{2} = \frac{z-3}{1} \ (=t_1)$

$d_2 : \frac{x-1}{2} = \frac{y-3}{1} = \frac{z}{1} \ (=t_2)$

---

a) d_1, d_2 necoplanare

b) ec. $\perp$ comune (d)

c) dist. (d_1, d_2)

Rez: $v_1 = (1,2,1)$ $\qquad v_2 = (2,1,1)$

a) $M_1(2,0,3) \in d_1$ $\qquad M_2(1,3,0) \in d_2$

$\overrightarrow{M_1M_2} = (-1, 3, -3)$

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & -3 \end{vmatrix} = 11 \neq 0 \Rightarrow d_1, d_2 \text{ necoplanare}$$

b) $P_1(2+t_1, 2t_1, 3+t_1) \in d_1$

$P_2(1+2t_2, 3+t_2, t_2) \in d_2$

$\overrightarrow{P_1P_2} = (-1+2t_2-t_1, 3+t_2-2t_1, -3+t_2-t_1)$

$$\begin{cases} \langle \overrightarrow{P_1P_2}, v_1 \rangle = 0 \\ \langle \overrightarrow{P_1P_2}, v_2 \rangle = 0 \end{cases} \Leftrightarrow \begin{cases} -6t_1 + 5t_2 = -2 \\ -5t_1 + 6t_2 = 2 \end{cases} \Rightarrow t_1 = t_2 = 2$$

$P_1(4,4,5) \in d$ $\quad \overrightarrow{P_1P_2} = (1,1,-3)$

$P_2(5,5,2)$

$d : \frac{x-4}{1} = \frac{y-4}{1} = \frac{z-5}{-3}$ ec. perpendicularei comune

c) $\text{dist}(d_1, d_2) = \text{dist}(P_1, P_2) = \sqrt{11}$