Curs 2

Geometrie și Algebră liniară

Teorema Laplace
(regula lui Laplace)

Fie $A \in \mathcal{M}_n(\mathbb{C})$ și $I = \{i_1, \dots, i_k\} \subset \{1, 2, \dots, n\}$

Atunci: $\det A = \sum_{\substack{J \subset \{1,2,\dots,n\} \\ \text{card } J = k}} M_{IJ} \cdot M'_{IJ}$,

unde $M'_{I,J} = (-1)^{i_1 + \dots + i_k + j_1 + \dots + j_k} M_{\overline{I},\overline{J}}$

$M_{I,J} \overset{\text{not}}{=} \det A_{I,J}$

C.P.: $I = \{i\}$, card $I = 1$

$\to$ obținem dezv. după linia "i" a $\det A$

Analog, se obține dezvoltarea după coloana "j" a $\det A$.

Exemplu:

Calculați $\det A$, unde $A = \begin{pmatrix} a & b & c & d & e & f \\ 0 & g & h & i & j & 0 \\ 0 & 0 & k & l & 0 & 0 \\ 0 & 0 & m & n & 0 & 0 \\ 0 & p & q & r & s & 0 \\ t & u & x & y & z & w \end{pmatrix}$

a) dezv. după prima coloană

b) folosind regula lui Laplace pt. $I = \{3, 4\}$

Rez: a) $\det A = a \cdot (-1)^2 \begin{vmatrix} g & h & i & j & 0 \\ 0 & k & l & 0 & 0 \\ 0 & m & n & 0 & 0 \\ p & q & r & s & 0 \\ u & x & y & z & \boxed{w} \end{vmatrix} + t \cdot (-1)^7 \begin{vmatrix} b & c & d & e & \boxed{f} \\ g & h & i & j & 0 \\ 0 & k & l & 0 & 0 \\ 0 & m & n & 0 & 0 \\ p & q & r & s & 0 \end{vmatrix}$

$$= aw \underbrace{\begin{vmatrix} g & h & i & j \\ 0 & k & l & 0 \\ 0 & m & n & 0 \\ p & q & r & s \end{vmatrix}}_{\overset{\text{Not}}{=}\Delta} - tf \underbrace{\begin{vmatrix} g & h & i & j \\ 0 & k & l & 0 \\ 0 & m & n & 0 \\ p & q & r & s \end{vmatrix}}_{=\Delta} = \Delta(aw - tf)$$

$$\Delta = \begin{vmatrix} g & h & i & j \\ 0 & k & l & 0 \\ 0 & m & n & 0 \\ p & q & r & s \end{vmatrix} \overset{\text{Dezv.}}{\underset{C_1}{=}} g(-1)^2 \begin{vmatrix} k & l & 0 \\ m & n & 0 \\ q & r & \boxed{s} \end{vmatrix} + p \cdot (-1)^5 \begin{vmatrix} h & i & \boxed{j} \\ k & l & 0 \\ m & n & 0 \end{vmatrix}$$

$$= g \cdot s \cdot (kn - ml) - pj(kn - ml) = (gs - jp)(kn - lm)$$

Deci: $\boxed{\det A = (gs - jp)(kn - lm)(aw - tf)}$

b) Folosind regula lui Laplace pt. $I = \{3, 4\}$

$I = \{3, 4\}$

$\text{card } J = 2 = \text{card } I$, $J \subset \{1, 2, ..., 6\}$

Deci, dezv. are $C_6^2 = 15$ termeni.

Așadar: $J \in \{ \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,4\}, \{2,5\},$
$\{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}, \{4,5\}, \{4,6\}, \{5,6\} \}$

$$\det A = \sum_{\substack{J \subset \{1,2,\dots,6\} \\ \text{card } J = 2}} \det A_{I,J} \cdot (-1)^{i_1+i_2+j_1+j_2} \det A_{\overline{I},\overline{J}}$$

$$\det A = \underset{\boxed{J=\{3,4\}}}{} \underbrace{\begin{vmatrix} k & l \\ m & n \end{vmatrix}}_{kn-ml} \cdot (-1)^{3+4+3+4} \begin{vmatrix} a & b & e & f \\ 0 & g & j & 0 \\ 0 & p & s & 0 \\ t & u & z & w \end{vmatrix}$$

$$\underset{I=\{2,3\}}{=} (kn - ml) \begin{vmatrix} g & j \\ p & s \end{vmatrix} \cdot (-1)^{2+3+2+3} \begin{vmatrix} a & f \\ t & w \end{vmatrix} = (gs - jp)(kn - ml) \cdot (aw - tf)$$ $\boxed{\text{q.e.d}}$

Det

Determinanți triagonali:

Calculați determinantul $\Delta_n = \begin{vmatrix} \alpha & \beta & 0 & & \\ \gamma & \alpha & \beta & & 0 \\ 0 & \gamma & \alpha & & \\ & & & \alpha & \\ & 0 & & & \alpha \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} \alpha & \beta \\ \gamma & \alpha \end{vmatrix} = \alpha^2 - \beta\gamma$

$\vdots$

$$\Delta_n = \alpha \cdot \Delta_{n-1} - \beta \cdot \underbrace{\begin{vmatrix} \boxed{\gamma} & \beta & 0 & \dots & 0 \\ 0 & \alpha & \beta & \dots & 0 \\ 0 & \gamma & \alpha & \beta \dots & 0 \\ \vdots & & & \alpha & \beta \\ 0 & \dots & \dots & \gamma & \alpha \end{vmatrix}}_{n-1 \text{ col.}} = \alpha \Delta_{n-1} - \beta \cdot \gamma \cdot \Delta_{n-2}$$

$\Delta_n - \alpha \Delta_{n-1} + \beta\gamma \Delta_{n-2} = 0$

Ec. caracteristică asociată: $\lambda^2 - \alpha\lambda + \beta\gamma = 0$

P.p. că are răd. $\lambda_1 \neq \lambda_2$

$\begin{cases} \alpha = \lambda_1 + \lambda_2 \\ \beta\gamma = \lambda_1\lambda_2 \end{cases}$ $\qquad \Delta_1 = \alpha = \lambda_1 + \lambda_2 = \dfrac{\lambda_1^2 - \lambda_2^2}{\lambda_1 - \lambda_2}$

În acest caz $(\lambda_1 \neq \lambda_2)$ $\qquad \Delta_n = \dfrac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}$

$\Delta_2 = \alpha^2 - \beta\gamma = (\lambda_1 + \lambda_2)^2 - \lambda_1\lambda_2 = \lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 = \dfrac{\lambda_1^3 - \lambda_2^3}{\lambda_1 - \lambda_2}$

Inducție matematică:

$P(n)$: $\Delta_n = \dfrac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}$, $(\forall)\ n \geq 2$

I $P(2)$ $\Delta_2 = \dfrac{\lambda_1^3 - \lambda_2^3}{\lambda_1 - \lambda_2}$ (A)

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II Pp. $P(k)$ adev. pt. $2 \le k < n$ $\Rightarrow P(n)$ adev.

$$\Delta_n = \alpha \Delta_{n-1} - \beta\gamma \Delta_{n-2} = (\lambda_1+\lambda_2)\cdot\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2} - \lambda_1\lambda_2 \cdot \frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}$$

$$= \frac{\lambda_1^{n+1} + \cancel{\lambda_2\lambda_1^n} - \cancel{\lambda_1\lambda_2^n} - \lambda_2^{n+1} - \cancel{\lambda_1^n\lambda_2} + \cancel{\lambda_1\lambda_2^n}}{\lambda_1-\lambda_2} = \frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}$$

$\boxed{T^*}$ Fie $A, B \in \mathcal{M}_n(K)$. {K − corp comutativ}

Atunci: $\boxed{\det(AB) = \det(A)\det(B)}$

$\boxed{C}$ $\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA) \Rightarrow \boxed{\det(AB) = \det(BA)}$

Preliminarii

{Consecințe ale th. LAPLACE} 2'

I Fie $A = \begin{pmatrix} M & N_{m,p} \\ O_{m,p} & P \end{pmatrix}$, $M \in \mathcal{M}_m(\mathbb{C})$, $P \in \mathcal{M}_p(\mathbb{C})$, $A \in \mathcal{M}_{m+p}(\mathbb{C})$

$\Rightarrow \boxed{\det A = \det(M)\cdot\det(P)}$

II Fie $A = \begin{pmatrix} M & N \\ P & O \end{pmatrix}$, $N \in \mathcal{M}_n(\mathbb{C})$, $P \in \mathcal{M}_p(\mathbb{C})$, $A \in \mathcal{M}_{n+p}(\mathbb{C})$

$\Rightarrow \boxed{\det A = (-1)^{p\cdot n} \det(N)\cdot\det(P)}$

T Binet–Cauchy:

Fie $A \in \mathcal{M}_{n,p}(\mathbb{C})$, $B \in \mathcal{M}_{p,r}(\mathbb{C})$, Fie $k \le \min\{n,p,r\}$

Fie $I = \{i_1, \dots, i_k\} \subset \{1,2,\dots,n\}$, $L = \{l_1,\dots,l_k\} \subset \{1,\dots,r\}$.

Atunci: $$\det(AB)_{I,L} = \sum_{\substack{J \subset \{1,\dots,p\} \\ \operatorname{card} J = k}} \det(A_{I,J})\cdot\det(B_{J,L})$$

c and $j = k$

<u>Dem:</u> $\boxed{T^*}$

$$C = \left(\begin{array}{c|c} A & O_n \\ \hline -I_n & B \end{array}\right) \rightsquigarrow C' = \left(\begin{array}{c|c} A & AB \\ \hline -I_n & O_n \end{array}\right)$$

$$\det C = \det C' = (-1)^{n^2} \det(AB) \cdot (-1)^n \cdot 1 = (-1)^{n^2+n} \cdot \det(AB)$$

n

$(\det A)(\det B)$

Deci: $\boxed{(\det A)(\det B) = \det(AB)}$

$\underline{K = \mathbb{C}}$

Def: Fie $A \in \mathcal{M}_n(\mathbb{C})$.

A s.n. matrice inversabilă dacă: $(\exists)\, B \in \mathcal{M}_n(\mathbb{C})$ a.î.

$$AB = BA = I_n$$

T A matrice inversabilă $\Longleftrightarrow \det A \neq 0$

Dacă $(\exists)$, $A^{-1} = \frac{1}{\det A} A^* \quad \{ \Longleftrightarrow A^* A = \det(A) \cdot I_n \}$

- Fie $A \in \mathcal{M}_2(\mathbb{C})$ inversabilă $\Longleftrightarrow \det(A) \neq 0$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det A = ad - bc \rightsquigarrow {}^tA = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \rightsquigarrow A^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Rightarrow \boxed{A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Exp:

Dacă: $A=\begin{pmatrix}\cos\theta & +\sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \underset{\det A = 1\neq 0}{\Longrightarrow} (\exists) A^{-1} = \begin{pmatrix}\cos\theta & -\sin\theta\\ +\sin\theta & \cos\theta\end{pmatrix} = {}^tA$

[P] Fie $A \in \mathcal{M}_n(\mathbb{C})$ m. inversabilă și $X \in \mathcal{M}_n(\mathbb{C})$

Atunci: $\boxed{\det(AXA^{-1}) = \det(X)}$

Dem: Avem: $\det(AXA^{-1}) = \det(A^{-1}AX) = \det(I_nX) = \det X$

- $\det: \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$
  $A \to \det A \in \mathbb{C}$

Obs: ! $\det(A+B) \neq \det A + \det B$

$A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $B = \begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$

$A+B = \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}$

$\left[\begin{array}{l}\det(A+B) = 0 \\ \det A + \det B = 1 + = 2\end{array}\right.$ ✗

Def: Fie $A \in \mathcal{M}_n(\mathbb{C})$ $\{A = (a_{ij})_{i,j=\overline{1,n}}\}$

$\operatorname{Tr} A \overset{def}{=} \sum_{i=1}^{n} a_{ii}$

urma matricei A

Exp: $\operatorname{Tr} I_n = n$

PROPRIETĂȚI: Fie $A, B \in \mathcal{M}_n(\mathbb{R})$.

Avem:

1) $Tr(A+B) = Tr(A) + Tr(B)$

2) $Tr(\alpha A) = \alpha Tr A$

3) $Tr(AB) = Tr(BA)$

Dem: 3) $Tr(AB) = (AB)_{11} + (AB)_{22} + \dots + (AB)_{nn} = \sum_{k=1}^{n} (AB)_{kk} =$

$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{kj} b_{jk} \right) = \sum_{j=1}^{k} \sum_{k=1}^{n} a_{kj} b_{jk} = \sum_{j=1}^{k} \left(\sum_{k=1}^{n} b_{jk} a_{kj} \right) =$

$= \sum_{j=1}^{n} (BA)_{jj} = Tr(BA)$

- Arătați că nu există $(\nexists)$ $A, B \in \mathcal{M}_n(\mathbb{R})$ a.î. $AB - BA = I_n$, $n > 1$

Dem: Pp. RA. că $(\exists)$ A, B. a.î. $AB - BA = I_n \overset{Tr}{\Longrightarrow} Tr(AB - BA) = Tr(I_n)$

$\overset{(1)}{\Longrightarrow} Tr(AB) + Tr(-BA) = n \overset{(2)}{\Longrightarrow} Tr(AB) - Tr(BA) = n \overset{3}{\Longrightarrow} 0 = n > 1$

$\Longrightarrow$ pp. făcută este falsă. $\downarrow$

Deci: $(\nexists)$ A, B a.î. $AB - BA = I_n$, $n > 1$ q.e.d.