Subspații vectoriale

Def: Fie $V/_K$ sp. vect. și $V' \subseteq V$, $V' \neq \emptyset$

V' s.n. subsp. vect. al lui V dacă: e închisă (stabilă) la adunarea vect. și la înmulțirea cu scalari

i.e. $\begin{bmatrix} (\forall)\ v_1, v_2 \in V' \Rightarrow v_1 + v_2 \in V' \\ (\forall)\ \alpha \in K, v \in V' \Rightarrow \alpha v \in V' \end{bmatrix}$

$\boxed{P}$ $V' \subseteq V$

ssp. vect. $\Leftrightarrow \begin{bmatrix} (\forall)\ v_1, v_2 \in V' \Rightarrow \alpha_1 v_1 + \alpha_2 v_2 \in V' \\ \alpha_1, \alpha_2 \in K \end{bmatrix}$

Exemple: ① $\{0_V\}$, $V \subseteq V$

ssp. vect. improprii

② $\mathbb{R}_n[x] \subset \mathbb{R}[x]$

ssp. vect.

③ Fie $V/_K$ sp. vect.

$W \subseteq V$

submulț. $\not\ni 0_V \Rightarrow W$ nu este subsp. vect.

$\boxed{\text{Apl}}$ Fie $\mathbb{R}^n/_{\mathbb{R}}$ sp. vect. real

$$S(A) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,/\, \sum_{j=1}^{n} a_{ij} x_j = 0, \ (\forall)\, i=\overline{1,m}\}$$

$$A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}, \quad m \leq n, \quad \operatorname{rg} A = m$$

Dem. că: $S(A) \subset \mathbb{R}^n$ ssp. vect. $\left(\dim_{\mathbb{R}} S(A) = n-m\right)$

Sol: Fie $x, y \in S(A)$, $\alpha, \beta \in \mathbb{R}$ $\Rightarrow \sum_{j=1}^{n} a_{ij} x_j = 0$, $\sum_{j=1}^{n} a_{ij} y_j = 0$, $(\forall)\, i = \overline{1,m}$ $\Rightarrow$

$$\Rightarrow \sum_{j=1}^{n} a_{ij}(\alpha x_j + \beta y_j) = \alpha \sum_{j=1}^{n} a_{ij} x_j + \beta \sum_{j=1}^{n} a_{ij} y_j = 0$$

$\Rightarrow \alpha x + \beta y \in S(A) \Rightarrow S(A) \subset \mathbb{R}^n$ ssp. vect.

$\dim_{\mathbb{R}} S(A) = n - m = n - \operatorname{rg} A$ (cogt)

Consecințe:

1) $\mathbb{R}^2/_{\mathbb{R}}$

- $\{0_{\mathbb{R}^2}\}$, $\mathbb{R}^2$ ssp. vect. triviale (improprii) $\rightarrow$ 0, resp. 2-dim.
- $d = \{(x_1, x_2) \in \mathbb{R}^2 \,/\, a_1 x_1 + a_2 x_2 = 0\} \subset \mathbb{R}^2$ ssp. vect. 1-dim (dr. vect $\ni 0_{\mathbb{R}^2}$)

$\operatorname{rg}(a_1\ a_2) = 1$ (i.e. $a_1^2 + a_2^2 > 0$)

2) $\mathbb{R}^3/\mathbb{R}$

· $\{0_{\mathbb{R}^3}\}$, $\mathbb{R}^3$ ssp. vect. improprii (de dim. 0, resp. 3)

· $d = \{(x_1,x_2,x_3) \in \mathbb{R}^3 / \begin{cases} a_1x_1 + a_2x_2 + a_3x_3 = 0, \\ b_1x_1 + b_2x_2 + b_3x_3 = 0 \end{cases}$ $\operatorname{rg}\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = 2$ $\} \subset \mathbb{R}^3$ ssp. vect. 1-dim

(dr. vect. $\ni 0_{\mathbb{R}^3}$)

· $\mathcal{P} = \{(x_1,x_2,x_3) \in \mathbb{R}^3 / a_1x_1 + a_2x_2 + a_3x_3 = 0,$ $\operatorname{rg}(a_1\ a_2\ a_3) = 1\} \subset \mathbb{R}^3$ ssp. vect. 2-dim

(plan vect. $\ni 0_{\mathbb{R}^3}$)

Ap 1 Fie sp. vect. $\mathbb{R}^3/\mathbb{R}$ și $U = \{(x,y,z) \in \mathbb{R}^3 / x+2y+3z=0\} \subset \mathbb{R}^3$

a) Stabiliți dacă $\underset{\text{ssp. vect.}}{U \subset \mathbb{R}^3}$

b) Determinați $\dim_{\mathbb{R}} U = ?$

Rez a) Fie $v_1, v_2 \in U$ $\Rightarrow$ $v_1 = (x_1, y_1, z_1)$, $x_1 + 2y_1 + 3z_1 = 0$

$\alpha_1, \alpha_2 \in \mathbb{R}$ $\qquad v_2 = (x_2, y_2, z_2)$, $x_2 + 2y_2 + 3z_2 = 0$

Ar. că: $\alpha_1 v_1 + \alpha_2 v_2 \in U$

$\alpha_1 v_1 + \alpha_2 v_2 = (\underbrace{\alpha_1 x_1 + \alpha_2 x_2}_{x}, \underbrace{\alpha_1 y_1 + \alpha_2 y_2}_{y}, \underbrace{\alpha_1 z_1 + \alpha_2 z_2}_{z})$

$x + 2y + 3z = \alpha_1 x_1 + \alpha_2 x_2 + 2(\alpha_1 y_1 + \alpha_2 y_2) + 3(\alpha_1 z_1 + \alpha_2 z_2) =$

$= \alpha_1 \underbrace{(x_1 + 2y_1 + 3z_1)}_{=0} + \alpha_2 \underbrace{(x_2 + 2y_2 + 3z_2)}_{=0} = 0 \Rightarrow$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 \in U$

Deci $\underset{\text{ssp. vect.}}{U \subset \mathbb{R}^3}$

b) (V1) Folosim aplicația (teoretica) anterioară:

În cazul nostru $A = (1\ 2\ 3) \in \mathcal{M}_{(1,3)}(\mathbb{R})$

$\underset{\text{ssp. vect}}{U = S(A) \subset \mathbb{R}^3}$ și $\dim_{\mathbb{R}} U = 3 - \operatorname{rg} A = 3 - 1 = 2$

$\Rightarrow U \subset \mathbb{R}^3$
subsp. vect. ($\ni 0_{\mathbb{R}^3}$)

(V_2) Determinăm, în mod explicit, o bază pt. U

Fie $v \in U$
arbitrar

$v = (x, y, z)$, $x + 2y + 3z = 0$
$\Rightarrow x = -2y - 3z$

$U \ni v = (-2y-3z, y, z) = (-2y, y, 0) + (-3z, 0, z)$
$= y\underbrace{(-2, 1, 0)}_{u_1} + z\underbrace{(-3, 0, 1)}_{u_2} = y u_1 + z u_2 \Rightarrow S' = \{u_1, u_2\} \subset U$
sist. de generatori

În plus, se poate arăta că $S' \subset U$
s.v. lin indep. + sist. de gen.

$\Rightarrow \underset{\text{bază}}{S' \subset U} \Rightarrow \dim_{\mathbb{R}} = 2$

$\boxed{T}$ Fie $U = \{(x, y, z) \in \mathbb{R}^3 / -x + 3y + z = 0\} \subset \mathbb{R}^3/_{\mathbb{R}}$

a) Stabiliți dacă $U \subset \mathbb{R}^3$
subsp. vect.

b) Determinați $\dim_{\mathbb{R}} U$.

T. dimensiunii (Grassmann)

Fie $V/_K$ sp. vect. (finit dimensional) și $V_1, V_2 \subseteq V$
subsp. vect.

Atunci $\boxed{\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)}$