Geometrie și algebră liniară

Spații vectoriale

- Liniar indep. Liniar dep.
- Sist. de generatori
- Baze

Def: Fie $V/_K$ sp. vect.

$S = \{v_1, \dots, v_m\} \subset V$

a) S - s.v. lin. indep. dacă:

$(\forall)\ \alpha_1 v_1 + \dots + \alpha_m v_m = 0_V \Rightarrow \alpha_1 = \dots = \alpha_m = 0$

$\alpha_i \in K,\ i = \overline{1,m}$

b) S - s.v. lin. dep. dacă:

$(\exists)\ \alpha_i \in K,\ i = \overline{1,m}$ a.î. $\alpha_1 v_1 + \dots + \alpha_m v_m = 0_V$

nu toți nuli

$\boxed{Ap1.}$ Stabiliți dacă următoarele sist. vect. sunt liniar indep. sau liniar dependente

a) $S_1 = \{v_1 = (-1,1,1),\ v_2 = (1,-1,+1),\ v_3 = (+1,1,-1)\} \subset \mathbb{R}^3/_{\mathbb{R}}$

b) $S_2 = \{v_1 = (1,2,1),\ v_2 = (2,1,1),\ v_3 = (5,5,3)\} \subset \mathbb{R}^3/_{\mathbb{R}}$

c) $S_3 = \{v_1 = e^x,\ v_2 = e^{-x},\ v_3 = sh\, x\} \subset C^0(\mathbb{R})$

→ comb. lin. nulă, arbitrară

Rez: a) Fie $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{\mathbb{R}^3}$

$\alpha_i \in \mathbb{R},\ (\forall) i = \overline{1,3}$

$\alpha_1(-1,1,1) + \alpha_2(1,-1,1) + \alpha_3(1,1,-1) = (0,0,0)$

$$\Rightarrow S_1 \begin{cases} -\alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_1 - \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_2 - \alpha_3 = 0 \end{cases}$$

→ sistem liniar omogen

$\boxed{1}$

$$A = \left(\boxed{\begin{matrix}1\\1\\1\end{matrix}} \boxed{\begin{matrix}1\\1\\1\end{matrix}} \boxed{\begin{matrix}1\\1\\1\end{matrix}} \right)$$
$\quad v_1 \quad v_2 \quad v_3$

$\downarrow$ m. sist. $\mathcal{S}_1$

$\det A = 4 \neq 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$ sol. unică

$\Rightarrow S_1' = \{v_1, v_2, v_3\} \subset \mathbb{R}^3$
s.v. lin. indep.

b) Raționament analog

Fie $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{\mathbb{R}^3}$

$$\Rightarrow \mathcal{S}_2 \begin{cases} \alpha_1 + 2\alpha_2 + 5\alpha_3 = 0 \\ 2\alpha_1 + \alpha_2 + 4\alpha_3 = 0 \\ \alpha_1 + \alpha_2 + 3\alpha_3 = 0 \end{cases}$$ S. Liniar Omogen

$$A = \left(\boxed{\begin{matrix}1\\2\\1\end{matrix}} \boxed{\begin{matrix}2\\1\\1\end{matrix}} \boxed{\begin{matrix}5\\4\\3\end{matrix}} \right)$$
$\quad v_1 \quad v_2 \quad v_3$

$\downarrow$ m. sist. $\mathcal{S}_2$

$\det A = 0 \Rightarrow \mathcal{S}_2$ admite și sol. nenule

$\Rightarrow S_2' = \{v_1, v_2, v_3\} \subset \mathbb{R}^3$
s.v. lin. dep.

$\Delta_p = d_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 \neq 0 \Rightarrow \operatorname{rg} A = 2$

$\begin{cases} \alpha_1, \alpha_2 - \text{nec. princ.} \\ \alpha_3 = \lambda, \lambda \in \mathbb{R} \\ \text{nec. sec.} \end{cases}$

$\begin{cases} \alpha_1 + 2\alpha_2 = -5\lambda \\ 2\alpha_1 + \alpha_2 = -4\lambda \end{cases} \Bigg| \cdot(-2) \Rightarrow \begin{cases} \alpha_1 = -\lambda \\ \alpha_2 = -2\lambda \\ \alpha_3 = \lambda, \\ \lambda \in \mathbb{R} \end{cases}$

$\alpha_3 = \lambda, \lambda \in \mathbb{R}$

$-\lambda v_1 - 2\lambda v_2 + \lambda v_3 = 0_{\mathbb{R}^3}, \lambda \in \mathbb{R}$

e.p.
$\lambda = -1 \Rightarrow \underbrace{v_1 + 2v_2 - v_3 = 0_{\mathbb{R}^3}}$ → rel. de dep. liniară.

c). $\operatorname{sh} x = \dfrac{e^x - e^{-x}}{2}, (\forall) x \in \mathbb{R}$

$\downarrow$ sinus hiperbolic

$\Leftrightarrow e^x - e^{-x} = 2 \operatorname{sh} x, (\forall) x \in \mathbb{R}$

$\Leftrightarrow \boxed{v_1 - v_2 - 2v_3 = 0}$ → Rel. de dep. lin. $\Rightarrow S_3' = \{v_1, v_2, v_3\} \subset C(\mathbb{R})$
s.v. lin. dependent.

$\boxed{P_1}$ Fie $K^n/_K$ sp. aritmetic

$S' = \{v_1, \dots, v_m\} \subset K^n$, $A = \begin{pmatrix} \square & \square & & \square \\ v_1 & v_2 & \dots & v_m \end{pmatrix} \in \mathcal{M}_{(n,m)}(K)$

a) S' s.v. lin. indep. $\overset{dc}{\Longleftrightarrow}$ $\operatorname{rg} A = m \leq n$ (i.e. $\operatorname{rg} A$ max. posibil)

b) S' s.v. lin. dep. $\overset{dc}{\Longleftrightarrow}$ $\operatorname{rg} A \neq m$ $(m > n)$

$\boxed{T}$ Determinați valoarea parametrului real m a.î. s.v. următor să fie: a) liniar dependent

b) liniar independent

Sist. de generatori

Def: Fie $V/_K$ sp. vect. (finit generat)

$S' = \{v_1, \dots, v_m\} \subset V$

S' s.n. sistem de generatori pt. sp. vect. $V/_K$

dacă: $\langle S' \rangle = V$ i.e. $(\forall)\, v \in V$, $(\exists)\, \alpha_1, \dots, \alpha_m \in K$

a.î. $v = \alpha_1 v_1 + \dots + \alpha_m v_m$

$\boxed{A_p 1}$ Stabiliți dacă următoarele s.v. sunt sisteme de generatori pentru sp. vect. din care fac parte:

a) $S'_1 = \{v_1 = (1,1),\ v_2 = (0,1)\} \subset \mathbb{R}^2/_{\mathbb{R}}$

b) $S'_2 = \{v_1 = (1,2,1),\ v_2 = (3,1,2)\} \subset \mathbb{R}^3/_{\mathbb{R}}$

T c) $S'_3 = \{v_1 = (1,1,0),\ v_2 = (1,0,1),\ v_3 = (0,1,1)\} \subset \mathbb{R}^3/_{\mathbb{R}}$

T d) $S'_4 = \{v_1 = 1,\ v_2 = X-1,\ v_3 = (X-1)^2\} \subset \mathbb{R}_2[X]$

Rez: a) $S'_1 \subset \mathbb{R}^2/_{\mathbb{R}}$

sist. de gen $\Leftrightarrow$ $(\forall)\, v \in \mathbb{R}^2$, $(\exists)\, \alpha_1, \alpha_2 \in \mathbb{R}$

a.î. $v = \alpha_1 v_1 + \alpha_2 v_2$

Fie $v = (x, y) \in \mathbb{R}^2$
vector arbitrar

$v = \alpha_1 v_1 + \alpha_2 v_2 \iff (x, y) = \alpha_1 (1, 1) + \alpha_2 (0, 1)$

$\iff \begin{cases} \alpha_1 = x \\ \alpha_1 + \alpha_2 = y \end{cases} \Rightarrow \begin{cases} \alpha_1 = x \\ \alpha_2 = y - x \end{cases}$

Deci $(\exists) \begin{cases} \alpha_1 = x \\ \alpha_2 = y - x \end{cases}$ a.î. $(\forall)\ v = (x, y) \in \mathbb{R}$ $\Rightarrow$
$v = \alpha_1 v_1 + \alpha_2 v_2$

$\Rightarrow S_1 = \{v_1, v_2\} \subset \mathbb{R}^2/\mathbb{R}$
sist. de generatori (pt. $\mathbb{R}^2/\mathbb{R}$)

$\boxed{P_2}$ Fie $K^n/$ sp. aritmetic

$S' = \{v_1, \dots, v_m\} \subset K^n$, $A = \begin{array}{c} \left(\begin{array}{cccc} \square & \square & \dots & \square \end{array} \right) \\ \begin{array}{cccc} v_1 & v_2 & \dots & v_m \end{array} \end{array} \in \mathcal{M}_{(n,m)}(K)$

a) S' - sist. de generatori pt. K^n/K dc. $\operatorname{rg} A = n \leq m$ (i.e. $\operatorname{rg} A$ = max posibil)

b) S' - nu este sist. de gen. pt. K^n/K
dc. $\operatorname{rg} A \neq n$ $(n > m)$

b) Aplicăm P_2 :

$A = \begin{array}{c} \left(\begin{array}{cc} 1 & 3 \\ 2 & 1 \\ 1 & 2 \end{array} \right) \\ \begin{array}{cc} v_1 & v_2 \end{array} \end{array} \in \mathcal{M}_{(3,2)}(\mathbb{R}) \Rightarrow \operatorname{rg} A \leq 2 < 3 \Rightarrow$

$\xRightarrow[P_2]{\text{cf.}}$ S'_2 nu este sist. de gen. pt. sp. vect. $\mathbb{R}^3/\mathbb{R}$.

c), d) $\rightarrow$ temă pt. acasă.

Def: Fie $V/_K$ sp. vect. (finit generat)

$B = \{v_1, \dots, v_n\} \subset V$

B s.n. <u>bază</u> pt. sp. vect. $V/_K$ dacă $\begin{cases} 1) \ B \text{ s.v. lin. indep.} \\ 2) \ B \text{ s. de gen. pt. } V/_K \end{cases}$

$\boxed{P}$. 1) $(\forall)$ sp. vect. admite (mai multe) baze.

2) Fie $\underset{\text{baze}}{B_1, B_2} \subset V/_K \Rightarrow \operatorname{card} B_1 = \operatorname{card} B_2$

Def: $\dim_K V \overset{\text{def}}{=} \operatorname{card} B$, $\underset{\text{bază arb.}}{B \subset V}$

$\boxed{P_3}$ Fie $K^n/_K$ sp. aritmetic.

$B = \{v_1, \dots, v_n\} \subset K^n$

B bază pt. sp. vect. $K^n/_K \Leftrightarrow \begin{cases} 1) \ B \text{ s.v. lin. indep.} \\ 2) \ B \text{ s. de gen. pt. } K^n/_K \end{cases}$

$\Leftrightarrow \operatorname{rg} A = n \Leftrightarrow \det A \neq 0 \ (\in K^*)$

$A \in \mathcal{M}_n(K)$

$$A = \Big(\underset{v_1}{\square} \ \underset{v_2}{\square} \dots \underset{v_n}{\square} \Big)$$

$\boxed{Ap\,1}$ Fie vectorii $v_1 = (1, 2, 3) \in \mathbb{R}^3$

$v_2 = (2, -1, 1)$

Determinați $v_3 \in \mathbb{R}^3$ a.î. $B = \{v_1, v_2, v_3\} \subset \mathbb{R}^3$ <u>bază</u>

<u>Rez</u>: Aplicăm P_3:

$$A = \begin{pmatrix} \boxed{\begin{matrix}1\\2\\3\end{matrix}} & \boxed{\begin{matrix}2\\-1\\1\end{matrix}} & \boxed{\begin{matrix}x\\y\\z\end{matrix}} \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$$

$$\quad v_1 \quad v_2 \quad v_3$$

Considerăm: $v_3 = (x, y, z) \in \mathbb{R}^3$

$B = \{v_1, v_2, v_3\} \subset \mathbb{R}^3/_{\mathbb{R}}$

bază $\Longleftrightarrow$ $\det A \neq 0 \Longleftrightarrow 5x + 5y - 5z \neq 0 \Longleftrightarrow$

$\Longleftrightarrow x + y - z \neq 0.$

Deci: $(\forall)\ v_3 = (x, y, z)_{\in \mathbb{R}^3}$, unde $z \neq x + y$ este sol.

<u>Obs</u>: ∞ - infinitate de sol.