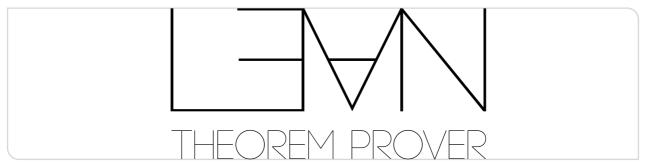




Theorembeweiserpraktikum

Inductive Types

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Outline



- Type families
- Dependent function types
- Recursive definitions on inductive types
- Structures

Type Families



- We have seen last time: *Predicates* take the form of functions $p: \alpha \to Prop$ for some type α .
- What about functions β : α → Type ? We call these *type families* on α .
- Mathematically, they can be used to model set-valued functions.
- Some common examples for type families:
 - The family Fin : Nat → Type of finite ordinal types.
 - The family Vec : $\alpha \rightarrow \text{Nat} \rightarrow \text{Type}$ of vectors (lists of a given length).
 - The family Bij : Type \rightarrow Type \rightarrow Type of bijections between types.

Dependent Functions



- What happens if we take the universal quantifier $\forall x, \beta x$ of such type families?
- lacktriangle We can see them as functions that assign to each x: α an element of the type β x.
- Example: The function which returns the vector repeating an element $x:\alpha$ a given number of times: repeat: $\alpha \to \forall$ n: Nat, Vec α n.
- Lean offers an alternative notation to the above: repeat : $\alpha \to (n : Nat) \to Vec \alpha n$.
- In definitions:

```
def repeat (\alpha : Type) (n : Nat) : Vec \alpha n := ...
```

Implicit Arguments



Suppose we have a function append which appends two vectors. Its type would be

```
def append (m n : Nat) (v : Vec \alpha n) (w : Vec \alpha m) : Vec \alpha (n + m) := ...
```

- The value of the first and second argument is uniquely determined by the third and fourth!
- We can use {...} to mark such arguments as *implicit*:

```
def append {m n : Nat} (v : Vec \alpha n) (w : Vec \alpha m) : Vec \alpha (n + m) := ...
```

to use the function as append v w instead of append m n v w .

- Single-letter undeclared variables are automatically turned into implicit arguments, so we could just leave out {m n : Nat} above.
- We can use @append to make all arguments explicit, and append (n := ...) to do so for n only.

Inductive Types: Natural Numbers



- So far, all inductive types we have defined were *non-recursive*.
- More interesting and especially infinite types can be constructed using recursive constructors.
- Example: The type of natural numbers Nat :

```
inductive Nat : Type where
| zero : Nat
| succ : Nat → Nat
```

Intuitively, the type is supposed to contain all terms which can be composed using the two constructors:

```
zero
succ zero
succ (succ zero)
succ (succ zero))
...
```



Selection of Technology

Inductive Types: Constructors and Recursors

- The presence of the constructors ensures that Nat contains at least those terms.
- How to state that Nat does not contain more? → Recursors!
- The definition of Nat postulates, besides Nat.zero and Nat.succ, the following function:

```
#check @Nat.rec
/- output:
Nat.rec : {motive : Nat \rightarrow Sort u_1} \rightarrow
motive Nat.zero \rightarrow ((n : Nat) \rightarrow motive n \rightarrow motive (Nat.succ n)) \rightarrow (t : Nat) \rightarrow motive t
-/
```

- Special cases:
 - motive := fun $_$ => α : Recursively defined functions on Nat .
 - motive := fun n => p n for a predicate p : Proof by induction.

Recursive match blocks



Recursors are best not used directly, Lean's match syntax is a convenience feature to do this for us:

```
def add (m n : Nat) : Nat :=
 match n with
   Nat.zero => m
   Nat.succ n => Nat.succ (add m n)
```

produces the same result as

```
def add' (m : Nat) : Nat → Nat := @Nat.rec (fun _ => Nat) m (fun _ r => Nat.succ r)
```

Note: This means that m + (n + 1) is definitionally equal to (m + n) + 1, but not (m + 1) + n to m + (1 + n)!

Indexed Inductive Types



- Like with predicates, we can define inductive type families with indices.
- Example: A definition of the type of vectors could look like this:

```
inductive Vec (\alpha: Type): Nat \rightarrow Type where
     nil : Vec α 0
    | cons : \{n : Nat\} \rightarrow \alpha \rightarrow Vec \alpha n \rightarrow Vec \alpha (n + 1)
```

• We now can define the append function as follows:

```
def append (v : Vec \alpha m) (w : Vec \alpha n) : Vec \alpha (m + n) :=
  match w with
    Vec.nil => v
  | Vec.cons a w => Vec.cons a (append v w)
```

Here the last line works because (m + n) + 1 is definitionally equal to the expected index m + (n + 1)

Structures



- You have already seen examples of inductive types with exactly one constructor (And , Iff).
- Those have some special properties and syntax in Lean. The code

```
structure Point (α : Type) where

x : α
y : α
```

means roughly the same as

```
inductive Point (\alpha : Type) : Type where | intro : (x : \alpha) \rightarrow (y : \alpha) \rightarrow Point \alpha
```

Structures



Structures offer notation access to the constructor arguments (fields), as you have seen in And.left:

```
def fivethree : Point Nat := { x := 5, y := 3 }
#eval fivethree.x -- prints 5
```

Structures can be extended by structures with more fields:

```
structure Triplet (\alpha : Type) extends Point \alpha where z : \alpha
```

contains the fields x, y, and z.

Namespaces



- Namespaces are a way to organize names of definitions hierarchically.
- Constructors and recursors of inductive types are automatically put into a namespace of the same name, see Vec.nil.
- We can open namespaces, to remove the prefix:

```
namespace Topic -- starts a namespace
def exampleDef := ...
end Topic -- ends a namespace
#check Topic.exampleDef

open Topic -- removes the necessity for the prefix "Topic."
#check exampleDef
```

Definitions marked with protected keep their prefix when the namespace is opened.