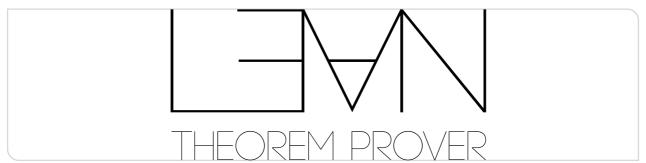




Theorembeweiserpraktikum

Quotients, Axioms, Booleans

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- maybe a bit long, sorry about that!
- make sure that your proofs are properly structured:
 - simp / simp_all (without only) only at the end of a tactic block (e.g. using have)
 - don't close multiple goals at different points of a single tactic block (e.g. using focus / case)
- prefer have over theorem for use-once subproofs
 - see also Zulip for some improvements around it, among others
- doing case analysis on something like h: a ∈ b:: bs is surprisingly hard for Lean!
 - match can only do it if the inductive indices are variables (e.g. on h: a = b)
 - uses cases instead
- make sure your proofs and terms are properly indented
- as always, see the sample solution for concise solutions and other tricks

Reminder: Projects Are upon Us



- 1.6. 5.7.2021
- in groups of two or three people
- verification of a simple compiler optimization
- alternative for mathematicians: Eulerian graphs

Quotients: Motivation



- Sometimes, not enough elements of a type are equal.
- Want to give an arbitrary binary predicate and *make* them equal.
- In mathematics: The real numbers as a quotient on Cauchy sequences.





Quotients in Lean work like in set theory: Given

- A type α : Sort u and
- a binary relation $r : \alpha \rightarrow \alpha \rightarrow Prop$ on α ,

we get

- the quotient type Quot r,
- with a constructor Quot.mk $r : \alpha \rightarrow Quot r$,
- an axiom stating

```
\textbf{axiom sound :} \ \forall \ \{\alpha : \textbf{Sort u}\} \ \{\textbf{r} : \alpha \rightarrow \alpha \rightarrow \textbf{Prop}\} \ \{\textbf{a} \ \textbf{b} : \alpha\}, \ \textbf{r} \ \textbf{a} \ \textbf{b} \rightarrow \textbf{Quot.mk r} \ \textbf{a} = \textbf{Quot.mk r} \ \textbf{b}
```

and an induction principle which lifts functions on the base type to functions on the quotient:

```
Quot.lift: \{\alpha: Sort u_1\} \rightarrow \{r: \alpha \rightarrow \alpha \rightarrow Prop\} \rightarrow \{\beta: Sort u_2\} \rightarrow \{f: \alpha \rightarrow \beta\} \rightarrow \{\emptyset: \alpha
```

Quotients in Lean



- Note that we didn't require r to be an equivalence relation.
- Objects of Quot r are still the equivalence classes of the reflexive-transitive-symmetric closure of r since equality itself is reflexive, transitive, and symmetric.
- Main advantage of using Quot r and =: We cannot rewrite along the relation r, but along =.

Propositional Extensionality



- sound for quotients was an axiom postulating an equality which otherwise could not been proved.
- The concept of extensionality: Two objects should be equal if they have the same observable properties.
- The only observable property of a proposition is its truth value, so propositional extensionality takes the form

```
axiom propext {a b : Prop} : (a \leftrightarrow b) \rightarrow a = b
```

Functional Extensionality



- The observable property of a function is that we can evaluate it at any point.
- So we have functional extensionality equating functions which are pointwise equal:

```
theorem funext \{f_1, f_2 : \forall (x : \alpha), \beta x\} (h : \forall x, f_1 x = f_2 x) : f_1 = f_2 := --...
```

- Proof idea:
 - **1** Quotient $\gamma := \forall (x : \alpha), \beta x$ by pointwise equality \sim .
 - ② Construct a function $e: \gamma/\sim \to \gamma$ such that $e([f]) \equiv f$.
 - Have

$$f_1 \equiv e([f_1]) = e([f_2]) \equiv f_2.$$

The Axiom of Choice



- Set-theoretic version: For every set X of non-empty sets there exists a function f such that $f(A) \in A$ for every $A \in X$.
- If we want to formalize this in type theory, we fail since non-emptiness is a proposition which we cannot match on if our goal is not a proposition!
- So in type theory, the axiom states exactly that we can extract the element of a non-empty type:

```
namespace Classical
axiom choice \{\alpha : Sort u\} : Nonemptv \alpha \rightarrow \alpha
```

The law of excluded middle em is then derived via a construction called Diaconescu's theorem.

A Remark on Bool and Prop



- In programming, the type Bool is used for propositions, why can't we do that?
- Constructive logic with Bool is always decidable!
- Lean still has Bool valued logical connectives or, and, ...
- There is a Bool valued equality consisting of a type class

```
class BEq (\alpha : Type u) where beq : \alpha \to \alpha \to Bool
```

- It is used by some of Lean's own implementations like filter.
- Can be overwritten with a new instance, to modify it.
- If equality on a type is decidable, we have the instance

```
instance [DecidableEq α] : BEq α where
beq a b := decide (Eq a b)
```



Other Helpful Stuff: Options

```
set_option pp.notation false
#check [] ++ [1, 2] -- HAppend.hAppend List.nil (List.cons 1 (List.cons 2 List.nil)) : List Nat
```

pp.explicit	show implicit parameters using @
pp.all	show everything
trace.Meta.Tactic.simp	show applied simp theorems
	can be used to move from simp to simp only

Can also be done inside a term or tactic block using set_option opt val in