Karlsruhe Institute of Technology

Theorembeweiserpraktikum

Introduction & Propositional Logic

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Outline

- What is interactive theorem proving?
- Terms and types
- The Curry-Howard correspondence
- The structure of a Lean file
- Some basic types in Lean
- Propositional logic in Lean

What is Interactive Theorem Proving (ITP)?

- Language and IDE supporting the user in writing *formal proofs* about certain structures which are then *verified* by the computer.
- ITP is *not* automated theorem proving (ATP)! User input is usually required to guide the system.
- Formal foundation of Lean: Type theory, inductive types (Calculus of Inductive Constructions, CIC).

Interactive Theorem Proving – How is it Useful?

- Verify proofs independently of an academic review system.
  - Puts the burden of proof verification on the author.
  - Less prone to human error.
- Write programs and verify them – in the same language!
- Alternative: Embed a different language and equip it with a formal semantics.

A Cherry-Picked List of Theorem Proving Projects

2005 *Gonthier & Werner:* four color theorem [Coq]

2014 *Hales et al:* Kepler conjecture [HOL Light/Isabelle]

2014+ *Leroy et al:* CompCert, a verified C compiler [Coq]

2020 *Han & van Doorn:* independence of the continuum hypothesis [Lean 3]

On Lean Versions

- This course is about *Lean 4*, a rewritten (in Lean!) and greatly extended implementation of Lean
- Lean 4 is **not** backwards-compatible with Lean 3, the latest stable release
- Most information you can find online will be about Lean 3
- Lean's sizable *mathlib* has mostly not been ported to Lean 4 yet
- If you find bugs or room for improvement, do tell us!

Terms and Types

- Martin-Löf type theory (MLTT) is a very strong, static type system.
- It is based on *type judgments*:

  $$t : \alpha$$

  means "t is an instance of α" for a term t and a type α.
- Everything has a fixed type, even types themselves: Universes are types that contain types:

$$\begin{array}{llll} 42 & : \mathbb{N} & & \\ & \mathbb{N} & : \mathsf{Sort}_1 & \\ & & \mathsf{Sort}_1 & : \mathsf{Sort}_2 \\ & & & \mathsf{Sort}_2 \quad : \ldots \end{array}$$

The Curry-Howard Correspondence

- Named after Haskell Curry and William Alvin Howard, 1969
- Paraphrasing:

  “**Proving** in intuitionistic (constructive) logic
  is the same as
  programming in typed lambda calculus.”
- We prove statements by constructing instances of certain types.

The Curry-Howard Correspondence

	Proving	Programming
$\alpha : \mathsf{Sort}_u$	α is a proposition	α is a data type
$t : \alpha$	t is a proof of α	t is an element of α
$\alpha \to \beta$	α implies β	type of functions from α to β
$\alpha \times \beta$	α and β	cartesian product of α and β
$\alpha + \beta$	α or β	disjoint sum of α and β

The Difference Between Proof and Data

- Some provers do not distinguish between proof and data
- The biggest difference in Lean is:

  We can treat instances $h : p$ and $h' : p$ for a proposition p as equal,

  but

  we need to distinguish instances $x : \alpha$ and $y : \alpha$ of a data type.
- Lean keeps propositions and data types in different universes, has aliases

$$
\begin{aligned}
\mathsf{Prop} &:\equiv \mathsf{Sort}_0 \\
\mathsf{Type} :\equiv \mathsf{Type}_0 &:\equiv \mathsf{Sort}_1 \\
\mathsf{Type}_1 &:\equiv \mathsf{Sort}_2 \\
\cdots \quad & \quad \cdots
\end{aligned}
$$

- Definitions of data are (conventionally) preceded by the keyword `def`, propositional definitions by `theorem`

The Structure of a Lean File

```
-- single-line comments start with "--"
import MyProject.Utils  -- import file `MyProject/Utils.lean` from current project/dependency

def add_three_mul (x y : Nat) : Nat :=  -- define a function with two arguments x and y
  x + 3 * y

theorem add_three_mul_gt : add_three_mul x y >= x :=  -- x and y are implicitly universally quantified
  ...

example : 5 + 3 * y >= 5 :=  -- an example is a theorem without a name
  add_three_mul_gt

#check add_three_mul_gt  -- check the type of our theorem
```

Propositional Logic in Lean: True and False

The canonical way to prove the proposition `True` is called `True.intro`:

```
example : True := True.intro
```

There is no way of proving false (hopefully!) but we can use the principle of “ex falso quodlibet” by using `False.elim`:

```
example (p : Prop) (fa : False) : p := False.elim fa
```

Propositional Logic in Lean: Implication

Implication is modelled by function types. So, applying an implication is function application:

```
section
variable (p q r : Prop)  -- automatically become arguments when referenced in the section

theorem modus_ponens (hpq : p → q) (hp : p) : q :=
  hpq hp  -- function application is written with a space instead of parentheses!

example (hpqr : p → q → r) (hp : p) (hq : q) : r :=  -- → has implicit parentheses on the right
  hpqr hp hq  -- function application has implicit parentheses on the left

example (htp : True → p) : p :=
  htp True.intro
end
```

We can input → writing \to.

Convention: We write *curried* functions: $\alpha \to \beta \to \gamma$ instead of $\alpha \times \beta \to \gamma$ or $\alpha \wedge \beta \to \gamma$!

Propositional Logic in Lean: Implication

Proving an implication is done by lambda abstraction:

```
section
variable (p q r : Prop)

example : p → p → p :=
  fun hp hp' => hp

example (hq : q) : p → q :=
  fun hp => hq

example (hpq : p → q) (hqr : q → r) : p → r :=
  fun hp => hqr (hpq hp)
end
```

The expression `fun a => b` is what mathematicians write as $a \mapsto b$!

Arguments vs. Implications

The following two Lean snippets are the same:

```
example (p q r : Prop) (hpq : p → q) (hqr : q → r) : p → r :=
  fun hp => hqr (hpq hp)
```

and

```
example (p q r : Prop) : (p → q) → (q → r) → p → r :=
  fun hpq hqr hp => hqr (hpq hp)
```

Parameters of a definition are just shorthand for iterated implications (functions)!

Propositional Logic in Lean: Disjunction

- The disjunction of two propositions is `p` and `q` is written `p ∨ q`.
- A case distinction on `p ∨ q` can be done by *matching* on the proof.

```
example (p q : Prop) (hp : p) : p ∨ q := Or.inl hp

example (p q r : Prop) (hpq : p ∨ q) (hpr : p → r) (hqr : q → r) : r :=
  match hpq with
  | Or.inl hp => hpr hp
  | Or.inr hq => hqr hq
```

Write `∨` as `\or`.

Propositional Logic in Lean: Conjunction

- The conjunction of two propositions `p` and `q` is written `p ∧ q`.
- A proof of `p ∧ q` is created using `And.intro`.
- A proof of `p` can be recovered using matching:

```
example (p q : Prop) (hpq : p ∧ q) : p :=
  match hpq with
  | And.intro hp hq => hp

example (p q : Prop) (hp : p) (hq : q) : p ∧ q := And.intro hp hq

example (p q r s : Prop) (hp : p) (hs : s) : (p ∨ q) ∧ (r ∨ s) :=
  And.intro (Or.inl hp) (Or.inr hs)
```

We can input `∧` using \and.

Algebraic Data Types and Propositions

Where does the strange `match` syntax come from? `Or` is defined as an algebraic data type:

```
inductive Or (p q : Prop) : Prop where
  | inl (hp : p) : Or p q
  | inr (hq : q) : Or p q
```

Each line starting with `|` is a *constructor* to `Or p q`. It states one way to construct an instance of the proposition.
`And` can be defined by the same mechanism, but only has *one* constructor with *multiple* parameters:

```
inductive And (p q : Prop) : Prop where
  | intro (hp : p) (hq : q) : And p q
```

Algebraic Data Types and Propositions

`True` and `False` are algebraic data types as well, with one and zero constructors, respectively!

```
inductive True : Prop where
  | intro : True

inductive False : Prop
```

The term **`nomatch`** lets us use the fact that `False` does not have any constructors, and `False.elim` is defined by using **`nomatch`**:

```
def False.elim (h : False) : α := nomatch h
```

Algebraic Data Types and Propositions

`And` and `Or` are given more convenient *infix notations* as we've seen.

```
infixr:35 " ∧ " => And
infixr:30 " ∨ " => Or
```

Both notations associate to the **r**ight, with `And` binding more tightly, i.e.

$$\texttt{a} \wedge \texttt{b} \wedge \texttt{c} \vee \texttt{d} \equiv (\texttt{a} \wedge (\texttt{b} \wedge \texttt{c})) \vee \texttt{d} \equiv \texttt{Or (And a (And b c)) d}$$

Algebraic Data Types and Propositions

Data types can be defined the same way, like this type of days of the week:

```
inductive Weekday : Type where
  | monday : Weekday
  | tuesday : Weekday
  | wednesday : Weekday
  | thursday : Weekday
  | friday : Weekday
  | saturday : Weekday
  | sunday : Weekday
```

This definition then enables us to *match* on variables of this data type:

```
def dayNumber (w : Weekday) : Nat :=
  match w with
  | Weekday.monday => 0
  | Weekday.tuesday => 1
  | Weekday.wednesday => 2
  | Weekday.thursday => 3
  | Weekday.friday => 4
  | Weekday.saturday => 5
  | Weekday.sunday => 6
```

Lean will throw an error whenever we forget one of the constructors!

Placeholders

Often, you will not write a proof in one go, but instead compose the proof term bit by bit. In this situation we can use the underscore `_` as a placeholder. Lean will try to infer its type and display it when hovering over the underscore as well as in the *Infoview*:

```
example (f : α → β) (a : α) : β :=
  f _  -- shows that we need to provide a term of type α
```

`have` Expressions

To create a local variable which abbreviates another expressions, we can use `have` :

```
example (f : β → β → γ) (g : α → β) (a : α) : γ :=
  have b : β := g a
  f b b
```

For longer proofs, you will find that using `have` is a way to make them more readable and concise!

Demo

Summary

type	*introduction*	*elimination*
`p`	`... ⊢ p`	`h : p, ... ⊢ ...`
	"how do I prove this"	"how do I use this"
implication/function type	*abstraction*	*application*
`q → r`	`fun hq => (_ : r)`	`h hq`
inductive type	*constructor app.*	*matching*

`q ∨ r`	`Or.inl hq` , `Or.inr hr`	`match h with`
		`| Or.inl hq => _`
		`| Or.inr hr => _`

`q ∧ r`	`And.intro hq hr`	`match h with` `| And.intro hq hr => _`
`True`	`True.intro`	–
`False`	–	`False.elim h` / `nomatch h`