The (Casual) Causality Course
Introduction, some notations and fundamentals

Gherardo Varando

IPL

02 May 2023

The course

- **When:** Tuesday and Thursday from 10:30 to 13:00 for the next 4 weeks
- **Where:** In person IPL hall and (limited) virtual at zoom link `https://uv-es.zoom.us/j/94222318081?pwd=NkNGOForRDYrNEtuUHR2cGFmQWN6Zz09`
- **Who:** Gherardo Varando (gherardo.varando@uv.es) and Emiliano Díaz (emiliano.diaz@uv.es)
- **Material** available on github `https://github.com/IPL-UV/casual_causality_course` if you are in the IPL you should have access to the private repo, otherwise write me an email with your github email and username and I will grant you access.

Content

Week 1 **Introduction and notations**

- Tue 02 Course introduction and first concepts
- Thur 04 Causal frameworks and framing causal problems

Week 2 **Causal Discovery**

- Tue 09 Classical approaches
- Thur 11 Continuous optimization methods and NN parametrizations

Week 3 **Causal Inference**

- Tue 16 Causal effect estimation and possible biases
- Thur 18 Machine learning methods for causal effect estimation

Week 4 **Applications**

- Tue 23
- Thur 25

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- Tue 02 Course introduction and first concepts
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Week 2 **Causal Discovery**

- Tue 09 Classical approaches
- Wed 10? Practical session on causal discovery?
- Thur 11 Continuous optimization methods and NN parametrizations

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Statistical Models: a very short story

- A mathematical model of the data generating process
- A description of the probability of data
- A collection of statistical assumptions
- Allows us to compute probabilities of events

Definition

Formally we can define a statistical model as a pair $(\mathcal{S}, \mathcal{P})$ where

- $\mathcal{S}$ is a sample space, formally $\mathcal{S} = (\Omega, \mathcal{F})$ with $\mathcal{F}$ a σ-algebra on Ω a set
- $\mathcal{P}$ is a collection of probability distributions on $\mathcal{S}$

Usually $\mathcal{P}$ is indexed by some finite-dimensional parameter θ, in that case the model is said to be **parametric**, if instead the parameter is infinite dimensional (or there is no parameter) the model is said to be **non-parametric**.

[Wasserman, 2004]

Inference and statistical problems

- modelling the data

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- prediction or forecasting

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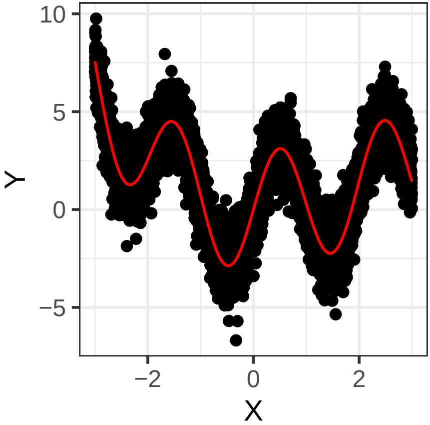
Example

image classification, weather forecasting, stock price prediction, crop yield prediction, crop detection, cloud detection, gap-filling

Regression models

Given data (X_i, Y_i)

- find best function that approximate $Y = f(X)$



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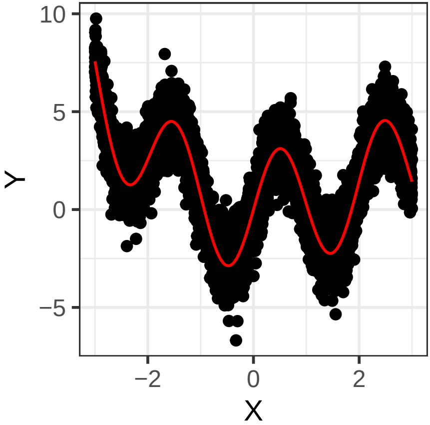

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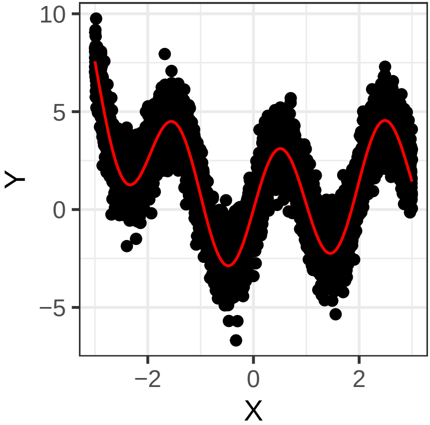
- linear case $Y = \sum \beta_i X_i + \beta_0 + \varepsilon$ we can do inference on the parameters β_i: confidence intervals, hypothesis testing



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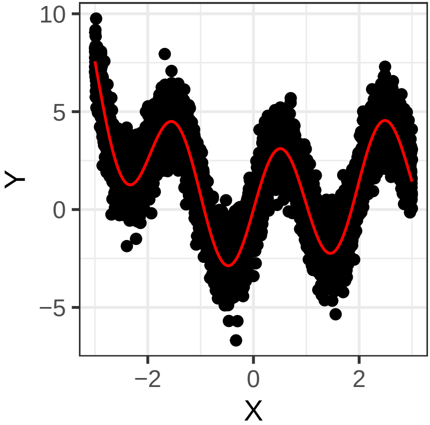
- non-additive noise models, $Y = f(X, \varepsilon)$



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How we can expand a regression model to obtain a full statistical model for the joint probability $P(X, Y)$?

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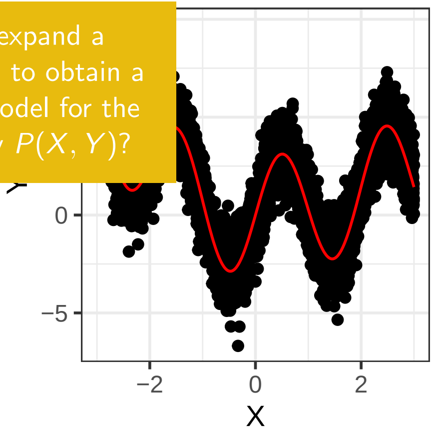

Bayesian Networks

Definition

A Bayesian Network (BN) over random variables $X_1, X_2, \ldots, X_p$ is a pair (G, P) where

- G is a DAG over p nodes (indexed as the r.vs)
- P is a joint probability over $X_1, \ldots, X_p$ such that $P = \prod_{i=1}^{p} P(X_i | X_{pa(i)})$

- BNs are an example of probabilistic graphical models
- define statistical models

[Lauritzen, 1996, Koller and Friedman, 2009, Maathuis et al., 2018]

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[Wasserman, 2004] and others think that the term Bayesian Network is misleading and poor terminology since BN do not have anything to do with Bayesian methods

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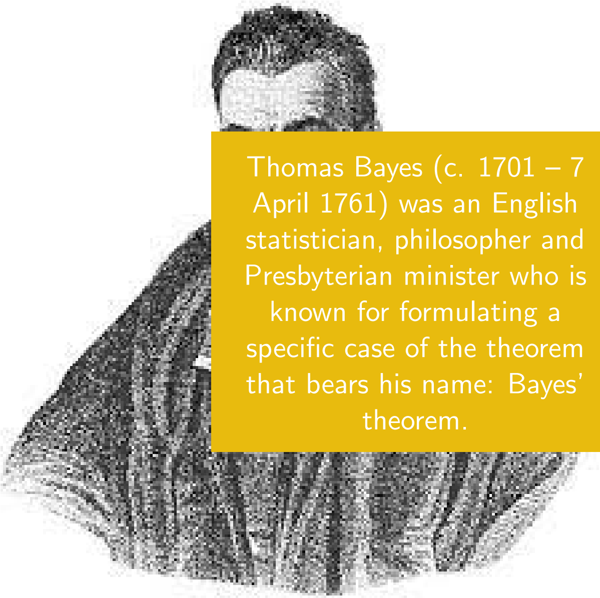
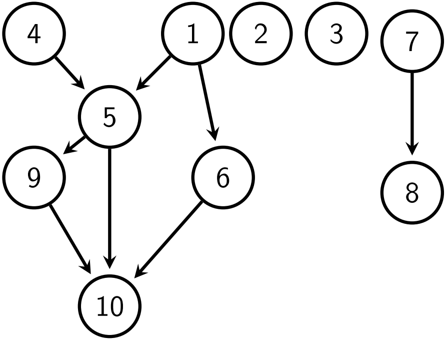


Thomas Bayes (c. 1701 – 7 April 1761) was an English statistician, philosopher and Presbyterian minister who is known for formulating a specific case of the theorem that bears his name: Bayes' theorem.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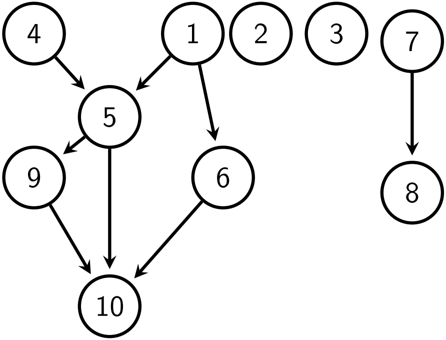
- 1. parents $pa(i)$
  2. children $ch(i)$
  3. descendants $de(i)$
  4. non-descendants $nde(i)$
  5. ancestrors $an(i)$



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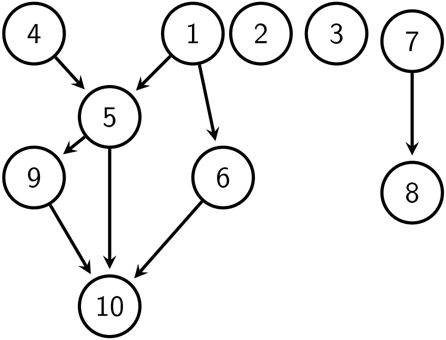

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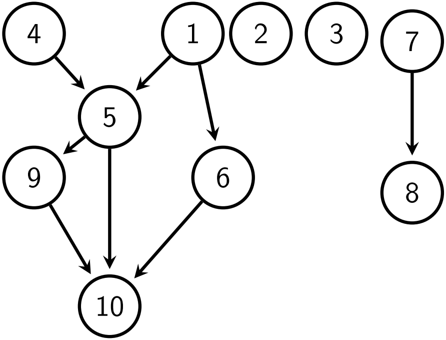

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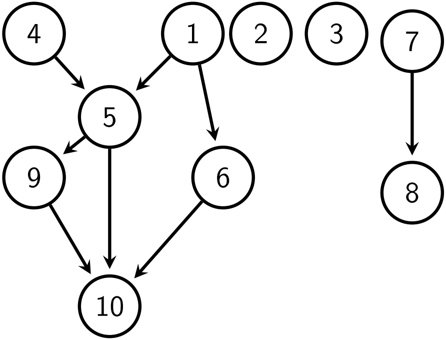

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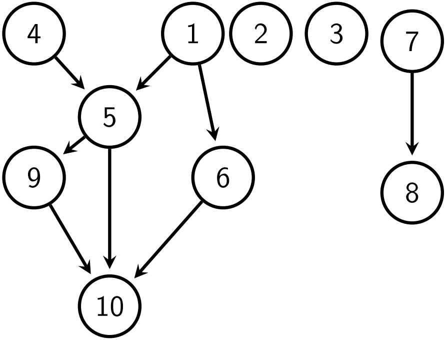

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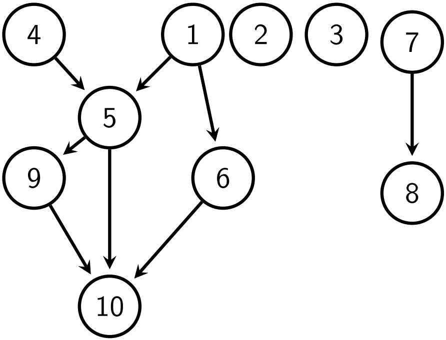
can you write the factorization of the joint probability associated with this graph?



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- which are the assumptions associated with a given BN (G, P)?

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BN/DAG and conditional independences

The following statements are equivalent:

- (G, P) is a BN, that is P factorize recursively wrt the DAG G
- P satisfies the *local Markov property* wrt G, that is $X_i \perp\!\!\!\perp X_{nd(i)} | X_{pa(i)}$
- P satisfies the *global Markov property* wrt G, that is $X_A \perp\!\!\!\perp X_B | X_D$ whenever A and B are d-separated by D in DAG G (A and B are separated by D in $G^m_{an(A \cup B \cup D)}$)

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takeaway
BN/DAG are graphical ways to encode conditional independence statements

BN/DAG and co…

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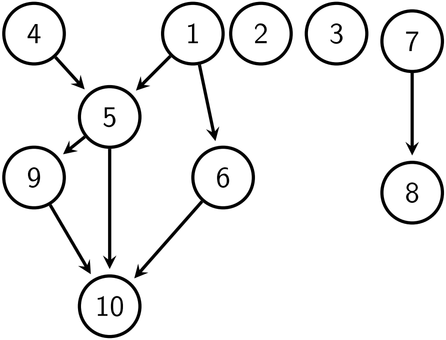
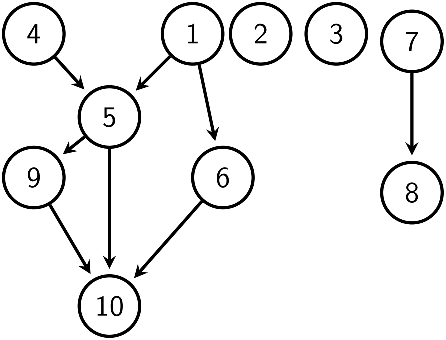
- d-separation, equivalent to separation in the moral graph of the ancestors of involved vertices



- d-separation, equivalent to separation in the moral graph of the ancestors of involved vertices
- equivalently i and j are d-separated by D if there exists no undirected path u between i and j such that
    1. every collider in u has a descendants in D
    2. no other vertex on u is in D



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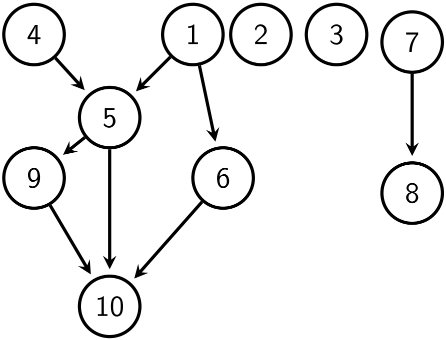
Can you list some d-separations in the graph? some are easy ...



Sampling from a BN

How can we obtain samples from a probability distribution associated with a BN?

Sampling from a BN

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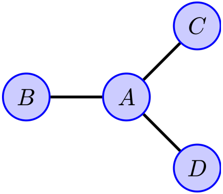
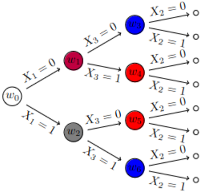
we can use the topological order to sample efficiently

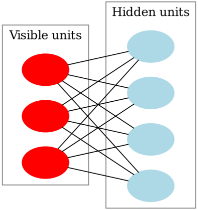
- pick a topological order of the nodes in G
- to generate each sample:
  1. start by sampling x_i from $P(X_i)$ for each nodes i without parents (there must be at least one)
  2. follow the topological order and sample from $P(X_i|X_{pa(i)} = x_{pa(i)})$ (since we follow the topological order $x_{pa(i)}$ is already sampled)

Other graphical models

- Markov networks, Markov random fields or undirected graphical models (e.g. Ising models in statistical physics) [Koller and Friedman, 2009, Lauritzen, 1996]
- Model based on event trees such as staged event trees or chain event graphs [Leonelli and Varando, 2023]
- Chain graphs
- restricted Boltzman machines







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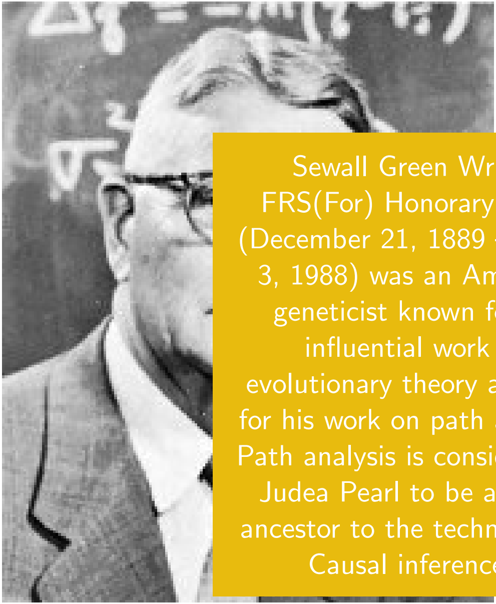
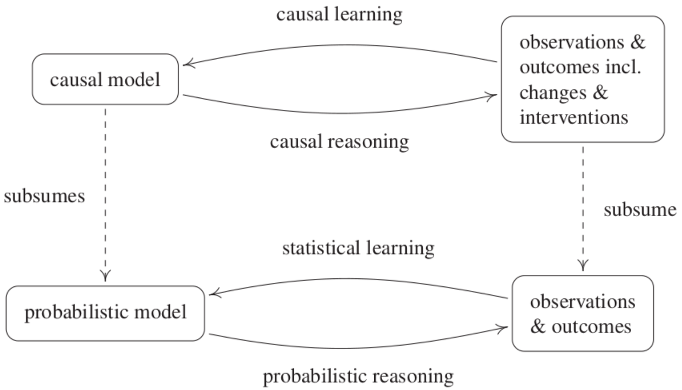
- statistical models (and ML models) represents association in the data
- stat/ML models are not mechanistic/physical/causal models of the data
- sometime in the sciences, but also in healthcare, econometrics, . . . we are actually interested in causal questions
- but what is *causality* ?



Sewall Green Wright FRS(For) Honorary FRSE (December 21, 1889 – March 3, 1988) was an American geneticist known for his influential work on evolutionary theory and also for his work on path analysis. Path analysis is considered by Judea Pearl to be a direct ancestor to the techniques of Causal inference.

(Probabilistic) Causal Models

- we consider a probabilistic definition for causality



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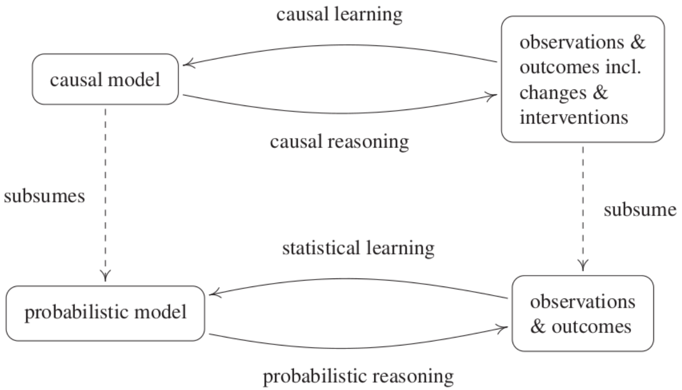
- *Roughly speaking, the statement "X causes Y" means that changing the value of X will change the distribution of Y* [Wasserman, 2004]



(Probabilistic) Causal Models

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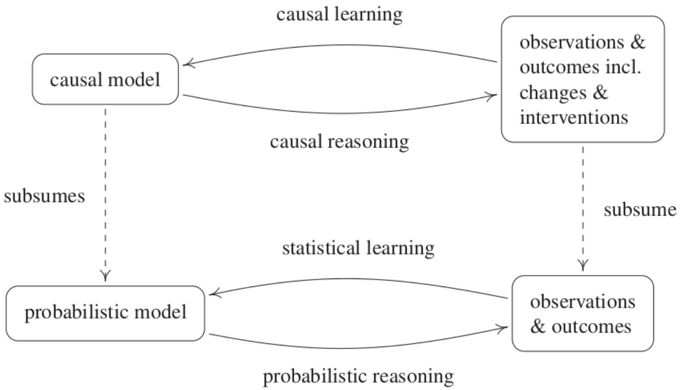
- *Roughly speaking, the statement "X causes Y" means that changing the value of X will change the distribution of Y* [Wasserman, 2004]
- Causal models contain more information than statistical models [Peters et al., 2017]



correlation does not imply causation

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Reichenbach's common cause principle[Peters et al., 2017]

If two random variables X and Y are statistically dependent ($X \not\perp\!\!\!\perp Y$), then there exists a third variable Z that causally influences both. (As a special case, Z may coincide with either X or Y.) Furthermore, this variable Z screens X and Y from each other in the sense that given Z, they become independent, $X \perp\!\!\!\perp Y | Z$.

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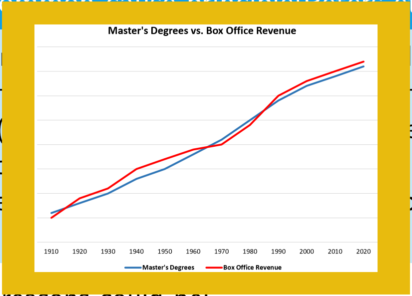

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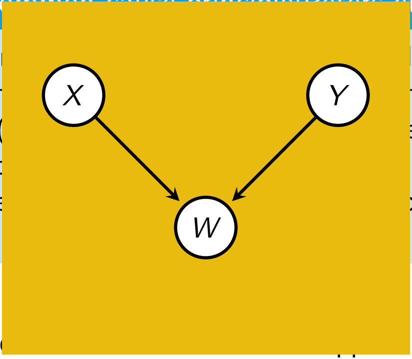

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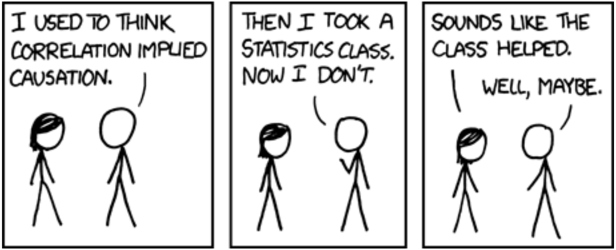
- **time dependence** and thus X and Y appear correlated
- conditioned on others (**selection bias**)
- statistical and finite sample problems



[1]

[1] `https://xkcd.com/552`

Causal models desiderata

- Represent data, similar to a statistical model
- Model what happen when changes/experiment/interventions
- Reason on and explore the causal relationships
- Represent causal realtionships

Causal regression models

- given a regression model $Y = f(X, \varepsilon)$ we could interpret this as a causal model
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- given a regression model $Y = f(X, \varepsilon)$ we could interpret this as a causal model
- in the sense that we immagine the physical/true generating process to be modeled by this equation
- If we make *experiments* by changing the value of X we know that the associated value for Y is generated accordingly to $f(X, \varepsilon)$

Structural Causal Models

Definition [Peters et al., 2017]

A SCM over variables $X_1, \ldots, X_p$ with noise variables $\varepsilon_1, \ldots, \varepsilon_p$ is a collection of **structural assignments**:

$$X_i = f_i(X_{pa(i)}, \varepsilon_i)$$

where ε_i are assumed jointly independent and f_i are fixed deterministic functions.

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- we assume the associated graph G to be a DAG
- A SCM defines a unique distribution P over the variables $X_1, \dots, X_p$

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A SCM over variables $X_1, \ldots, X_p$ with noise variables $\varepsilon_1, \ldots, \varepsilon_p$ is a collection of **structural assignments**:

$$X_i = f_i(X_{pa(i)}, \varepsilon_i)$$

where ε_i are assumed jointly independent and f_i are fixed deterministic functions.

- $X_{pa(i)}$ are called the parents or the **direct causes** of X_i
- we say X_i is a direct effect of its direct causes
- we assume the associated graph G to be a DAG
- A SCM defines a unique distribution P over the variables $X_1, \ldots, X_p$
- (G, P) is a Bayesian network

Structural Causal Models

Definition [Peters et al., 2017]

A SCM over varia… les $\varepsilon_1, \ldots, \varepsilon_p$ is a collection of **str**…

where ε_i are assu… e fixed deterministic func…

From a mathematical point of view there is no difference between a BN/DAG model and a SCM. It is the interpretation that is different, in a SCM we assume that the *regression models* defined for each variables are causal

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- the do-operator notation is due to Pearl

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