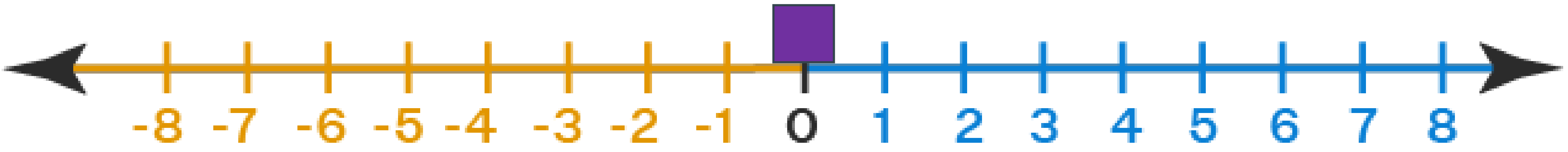
Applications of Differentiation

-- Contextual Applications

Straight-Line Motion

Considering the straight-line motion where an object moves along a straight line: When is the particle moving to the right? Or moving to the left? Or when it's at rest? When it's speeding up, slowing down?...

Position, velocity, acceleration…

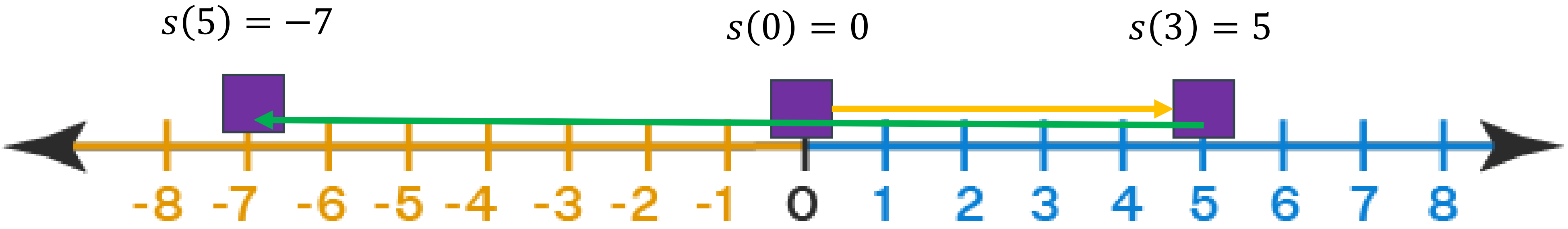


Position Function…

Position function $\boldsymbol{s(t)}$: tells you the location of the particle

(Total) Distance:

Displacement:



Velocity Function…

$$\text{Average velocity} = \frac{\textit{Final position} - \textit{initial position}}{\textit{time inerval}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

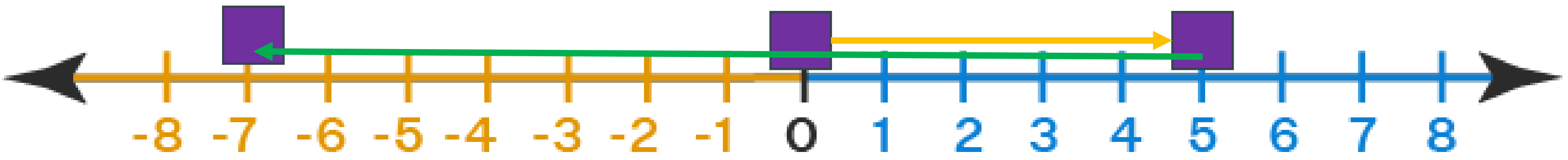
Velocity function v(t) =

- Moving forward (to the right) when $v(t)$ ____ 0

  Moving backward (to the left) when $v(t)$ ____ 0

  object **stopped** when $v(t)$ ____ 0

$s(5) = -7$ $s(0) = 0$ $s(3) = 5$



Example 1:

For $s(t) = t^2 - 2t - 3$, show its position on the number line for $t = 0,1,2,3,4$.

Find the displacement, distance and average velocity of the particle on the interval $[0,4]$.

Speed and velocity

Speed =

The speed of an object must either be positive or zero (meaning the object has stopped).

Speed up when ________________

Slow down when ______________

Acceleration

Acceleration $a(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h} = v'(t) = s''(t)$

- a(t) > 0: object accelerating to the right, v(t) ______
- a(t) < 0: object accelerating to the left, v(t) ______
- a(t) =0: v(t) ____________________

<u>Example 2</u>: A particle moves along the *x*-axis with position function $s(t) = t^2 - 4t + 2$.

$v(t) =$ ______________________ $a(t) =$ ______________________

t	s(t)	v(t)	$\|v(t)\|$	a(t)	what direction the particle is moving	speeding up or slowing down
0						
2						
4						
6						

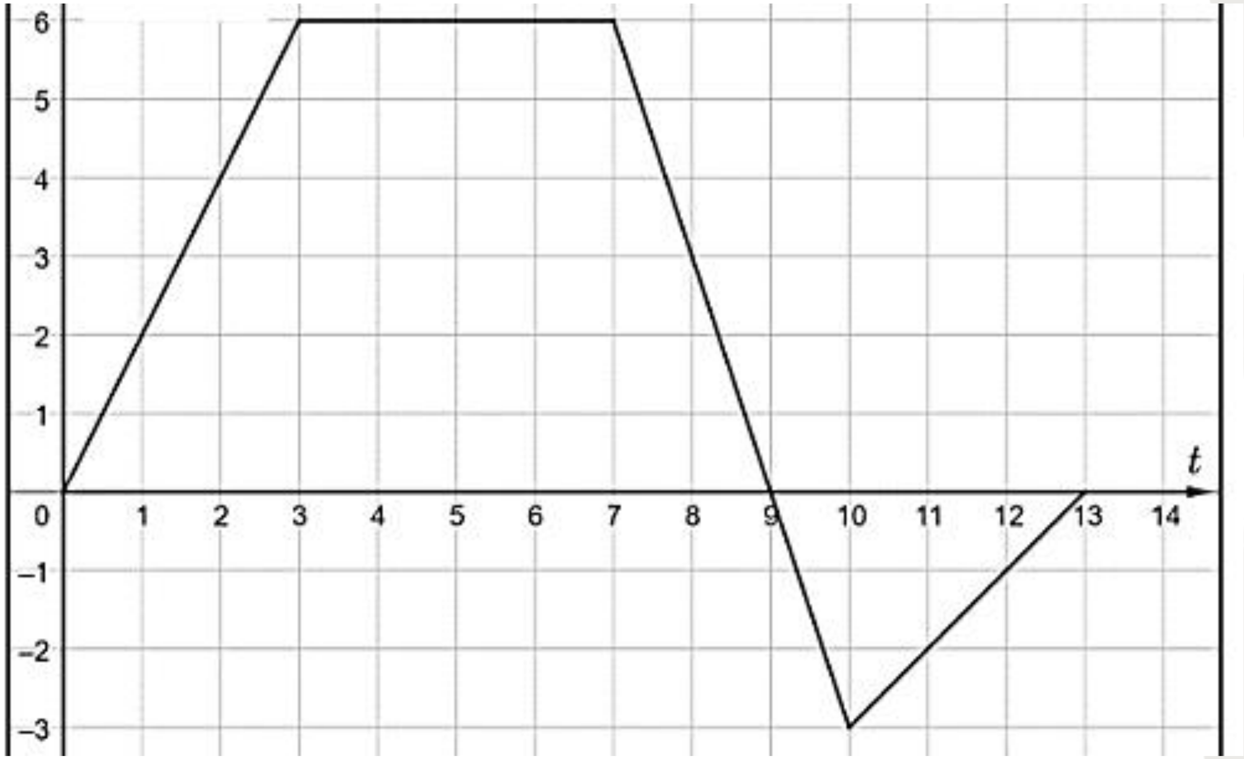
- ✧ Speed up when a(t) and v(t) have the ______________ sign(s)
- ✧ Slow down when a(t) and v(t) have the ______________ sign(s)

Relationship Between Velocity and Acceleration

How are we moving?	$a(t) > 0$	$a(t) < 0$	$a(t) = 0$
$v(t) > 0$			
$v(t) < 0$			

<u>Example 3</u>: The graph below models the velocity of a bug on the interval $0 \leq t \leq 13$.

a. Find $v(3)$ and $v(11)$.
b. Find $a(1)$, $a(5)$, and $a(9)$.
c. At what time does the bug turn around?
d. On what interval does the bug have a negative acceleration?



A particle starts moving at time $t = 0$ and moves along the x-axis so that its position at time $t \geq 0$ is given by $x(t) = t^3 - \frac{9}{2}t^2 + 7$.

(a) Find the velocity of the particle at any time $t \geq 0$.

(b) For what values of t is the particle moving to the left.

(c) Find the values of t for which the particle is moving but its acceleration is zero.

(d) For what values of t is the speed of the particle decreasing?

Example 5: A particle is moving along a horizontal line with position function $s(t) = t^3 - 9t^2 + 24t + 4$ for $t > 0$. Do an analysis of the particle's direction, acceleration, motion (speeding up or slowing down), and position.

[Hint]

- Direction: $v(t)>0$ right $v(t)<0$ left → Find the sign of $v(t)$
- Speeding up/slowing down:
  - Method 1: graph of $|v(t)|$
  - Method 2: observe the sign of $a(t)v(t)$

Summary

Statement	Translation
The bug is stopped…	$v(t) = 0$
The bug is moving to the right	$v(t) > 0$
The bug is moving to the left	$v(t) < 0$
The bug turns around	$v(t)$ changes signs
The bug is speeding up	$\lvert v(t)\rvert$ increasing or $v(t)a(t) > 0$
The bug is slowing down	$\lvert v(t)\rvert$ decreasing or $v(t)a(t) < 0$

Optimization Problems

Guidelines for Solving Optimization Problems

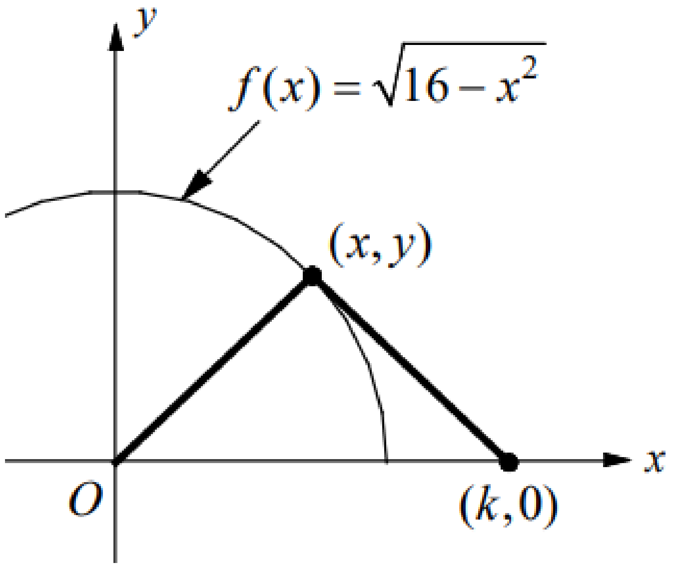
1. Read the problem carefully until you understand it.
2. In most problems it is useful to draw a picture. Label it with the quantities given in the problem.
3. Assign a variable to the unknown quantity and write an equation for the quantity that is to be maximized (or minimized), since this equation will usually involve two or more variables.
4. Use the given information to find relationships between these variables. Use these equations to eliminate all but one variable in the equation.
5. Use the first and second derivatives tests to find the critical points.

Optimization Problems

Let $f(x) = \sqrt{16 - x^2}$. An isosceles triangle, whose base is the line segment from (0,0) to (k,0), where k>0, has its vertex on the graph of f as shown in the figure.

(a) Find the area of the triangle in terms of k .

(b) For what values of k does the triangle have a maximum area?



Optimization Problems

Find the points on the curve $f(x) = \sqrt{x}$ that is nearest to the point

Optimization Problems

The point on the curve $y = 2 - \mathrm{x}^2$ nearest to $(3,2)$ is __________

Optimization Problems

What is the area of the largest rectangle that has its base on the x-axis and its other two vertices on the parabola $y = 6 - x^2$? Additional Condition: for positive y-values

Related Rate

<u>Example 1</u>: Water runs into a conical tank at a rate of $0.5\ \mathrm{m}^3/min$. The tank stands point down and has a height of $4m$ and a base radius of $2m$. How fast is the water level rising when the water is 2.5m deep?

✧ **Guideline for solving the related rate problem.**

Step 1: Read the problem and make a sketch if possible.

Step 2: Write down the rates that are given.

Write down the rate you are trying to find.

Step 3: Find an equation that ties your variables together.

Step 4: Differentiate your equation with respect to time t.

Remember, you are <u>implicitly</u> differentiating with respect to t.

Step 5: Substitute the given numerical information into the resulting equation and solve for the unknown rate.

Step 6: Label your answers in terms of the correct units (very important) and be sure you answered the question asked.

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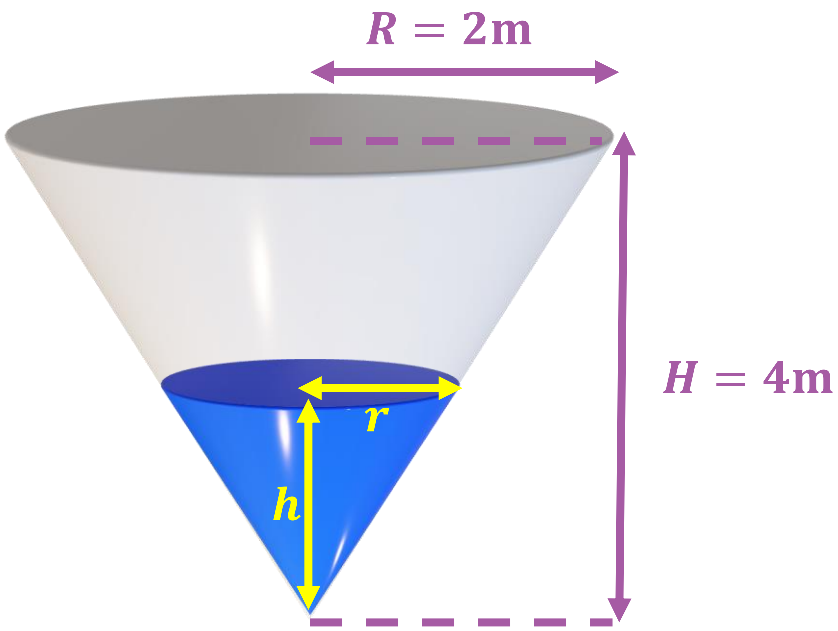

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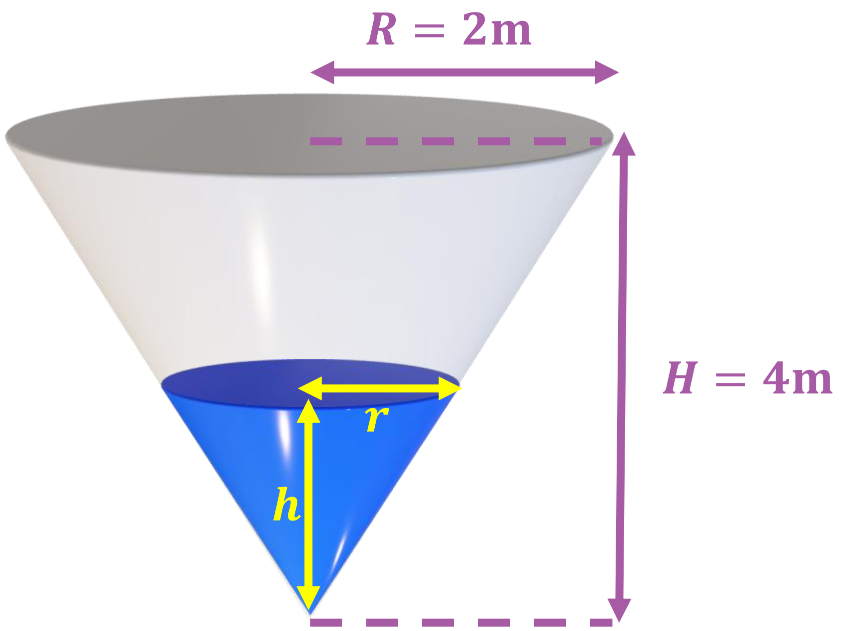
$$\frac{dV}{dt} = 0.5\ m^3/min$$

$$\left[\frac{dh}{dt}\right]_{h=2.5m} = ?$$



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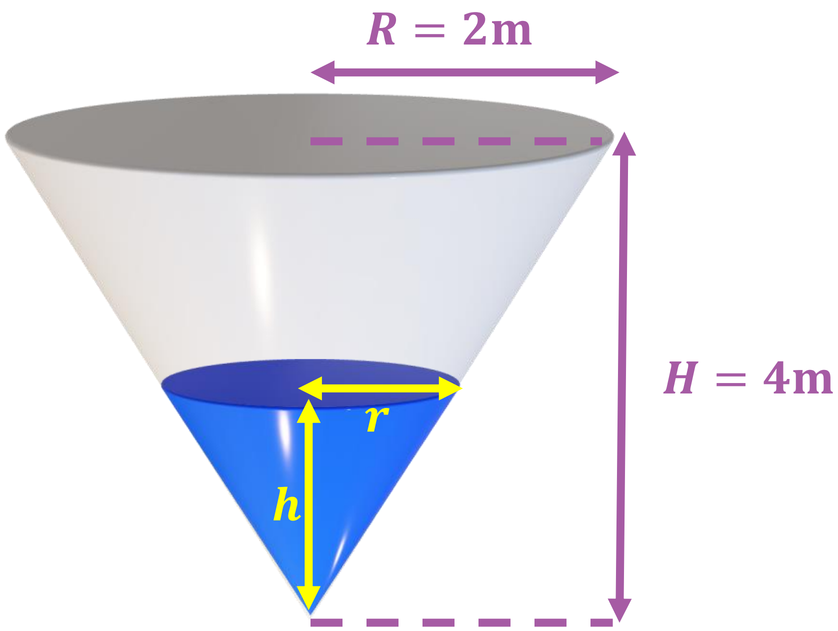

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{1}{12}\pi h^3$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{r}{h} = \frac{R}{H} = \frac{1}{2}$$

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Example 1: Water runs into a conical tank at a rate of $\boxed{0.5 \text{ m}^3/min}$. The tank stands point down and has a height of $4m$ and a base radius of $2m$. $\boxed{\text{How fast is the water level rising when the water is 2.5m deep?}}$

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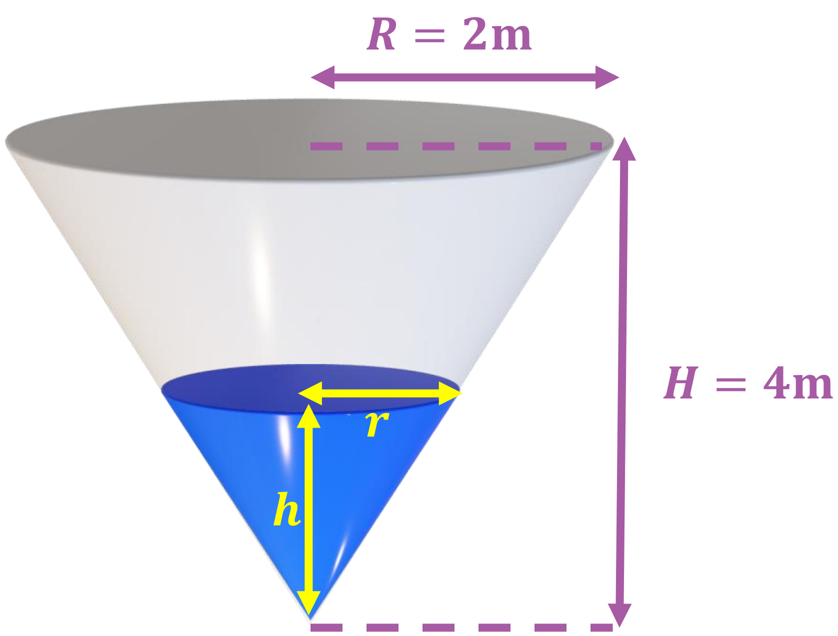

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$$V = \frac{1}{12}\pi h^3$$

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{1}{12}\pi h^3\right]$$

$$\frac{dV}{dt} = \frac{1}{12}\pi 3\text{h}^2\frac{\text{dh}}{\text{dt}} = \frac{\pi}{4}\text{h}^2\frac{\text{dh}}{\text{dt}}$$

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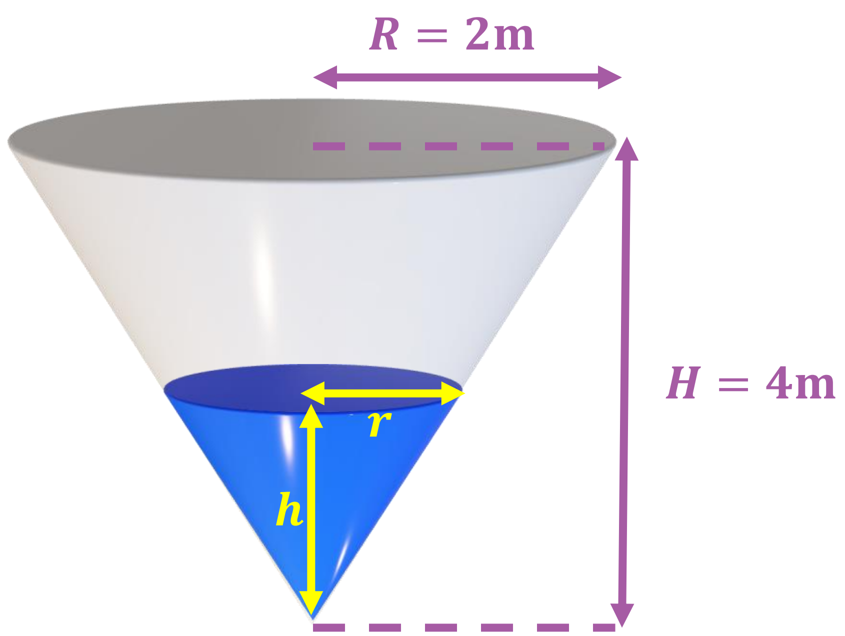
$$\frac{dV}{dt} = 0.5\ m^3/min$$

$$\left[\frac{dh}{dt}\right]_{h=2.5m} = ?$$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$



$$\left[\frac{dV}{dt}\right]_{h=2.5m} = \frac{\pi}{4} 5^2 \left[\frac{dh}{dt}\right]_{h=2.5m}$$

$$\text{So, } \left[\frac{dh}{dt}\right]_{h=2.5m} \approx 0.102 m/min$$

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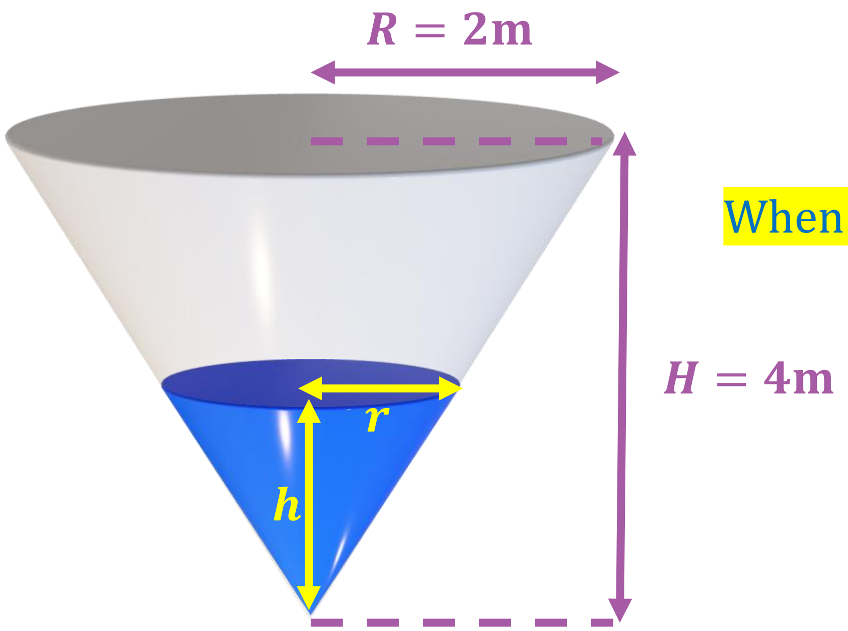


So, $\left[\frac{\mathrm{dh}}{\mathrm{dt}}\right]_{h=2.5m} \approx 0.102m/min$

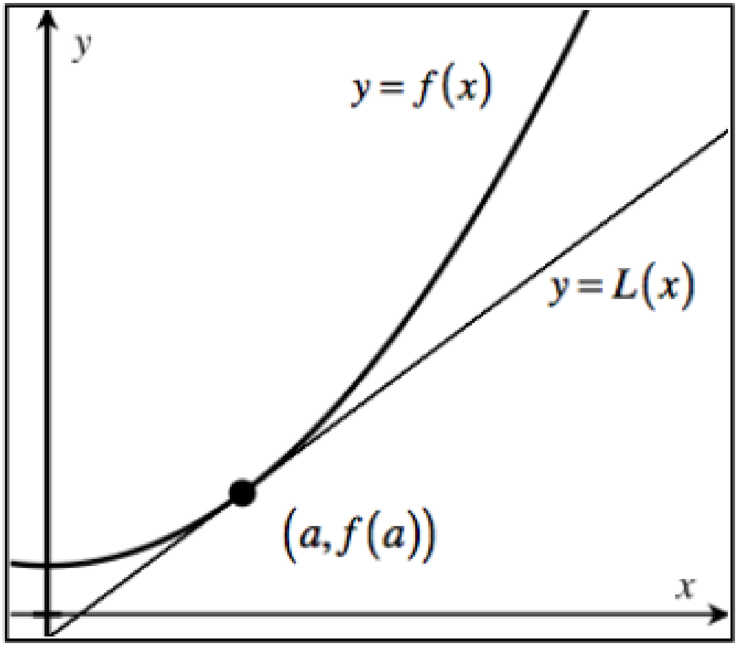
When h=2.5m, the water level is rising at a rate of 0.102 meters per minute.

Tangent Line Approximation

An equation for the tangent line of $f(x)$ at the point (a, f (a)) is given by :

$$y - f(a) = f'(x)(x - a)$$

$$\rightarrow\ y = f(a) + f'(x)(x - a)$$



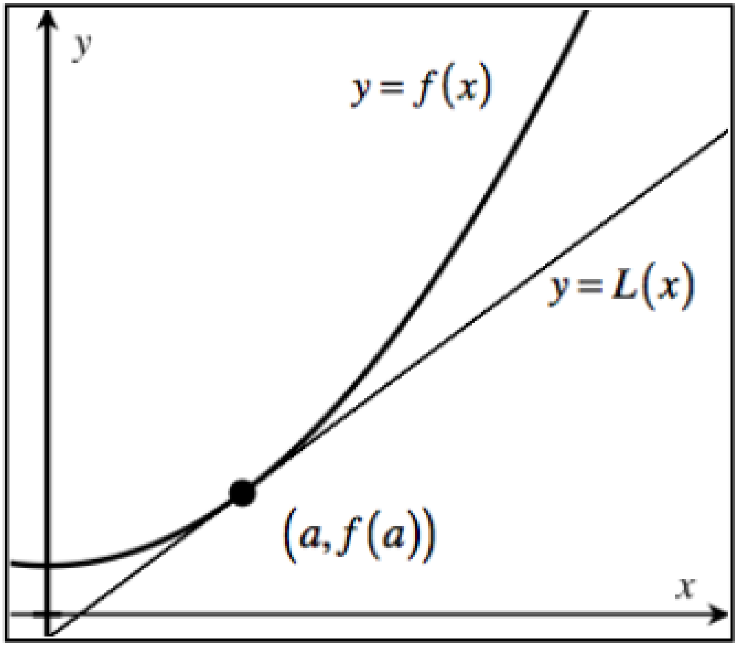
Linearization of f at $x = a$: $L(x) = f(a) + f'(a)(x - a)$

Linear approximation to the function at $x = k$:

$$f(k) \approx L(k) = f(a) + f'(a)(k - a)$$

Tangent Line Approximation

- ✧ If the curve is concave upward like the graph to the right, the line tangent to the graph of y=f(x) lies above/below the graph, so the tangent line approximation is greater/smaller than the real value.



- ✧ If the curve is concave down ~overestimate/underestimate the actual value