1. $\int \sqrt{x}\sin(x^{\frac{3}{2}})dx$

$= \int \sin(x^{\frac{3}{2}})\ \frac{2}{3}\ d(x^{\frac{3}{2}})$

$= -\frac{2}{3}\cos(x^{\frac{3}{2}}) + C$

A 2. If $\int_0^6 f(x)\,dx = 12$, what is the value of $\int_0^6 f(6-x)\,dx$?

$u = 6-x$ $x=0 \to u=6$

$du = -dx$ $x=6 \to u=0$

(A) 12 (B) 6 (C) 0 (D) −6

$= -\int_6^0 f(u)\,du = \int_0^6 f(u)\,du = 12$

B 3. If the substitution $u = 1+\sqrt{x}$ is made, $\int \frac{(1+\sqrt{x})^{3/2}}{\sqrt{x}}\,dx =$ $\int u^{\frac{3}{2}}\ 2du$

$du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$

(A) $\frac{1}{2}\int u^{3/2}\,du$ (B) $2\int u^{3/2}\,du$ (C) $\frac{1}{2}\int \sqrt{u}\,du$ (D) $2\int \sqrt{u}\,du$

C 4. If the substitution $u = \ln x$ is made, $\int_e^{e^2} \frac{1-(\ln x)^2}{x}\,dx =$ $\int_1^2 1-u^2 \cdot du$

$du = \frac{1}{x}dx$

(A) $\int_e^{e^2} (\frac{1}{u}-u^2)\,du$

$x=e \to u=1$

(B) $\int_e^{e^2} (\frac{1}{u}-u)\,du$

$x=e^2 \to u=2$

(C) $\int_1^2 (1-u^2)\,du$

(D) $\int_1^2 (1-u)\,du$

5. If f is continuous and $\int_1^8 f(x)\,dx = 15$, find the value of $\int_1^2 x^2 f(x^3)\,dx$. $= \int_1^8 \frac{1}{3} f(u)\,du$

$u = x^3$

$= \frac{15}{3} = 5$

$du = 3x^2\,dx$

$x=1 \to u=1$

$x=2 \to u=8$

Integration Tech

1. $\int_1^3 \frac{x+3}{x^2+6x}dx = \int_1^3 \frac{\frac{1}{2}}{x^2+6x}\, d(x^2+6x) = \frac{1}{2}\left[\ln|x^2+6x|\right]_1^3$

$= \frac{1}{2}(\ln 27 - \ln 7)$

2. $\int_0^1 \frac{x}{e^{x^2}}dx = \int_0^1 e^{-x^2}\cdot\left(-\frac{1}{2}\right)d(-x^2) = -\frac{1}{2}\cdot\left[e^{-x^2}\right]_0^1$

$= -\frac{1}{2}\left(e^{-1}-1\right)$

$= \frac{1}{2}\cdot\left(1-\frac{1}{e}\right)$

3. Which of the following is the antiderivative of $f(x) = \tan x$?

D

(A) $\sec x + \tan x + C$

(B) $\csc x + \cot x + C$

(C) $\ln|\csc x| + C$

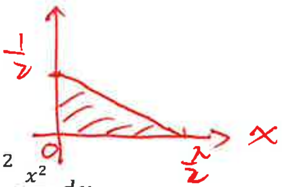
(D) $-\ln|\cos x| + C$

$\int \frac{\tan x \cdot \sec x}{\sec x} dx$

$= \int \frac{d(\sec x)}{\sec x} = \ln|\sec x| + C$

$= -\ln|\cos x| + C$

4. What is the area of the region in the first quadrant bounded by the curve $y = \frac{\cos x}{2+\sin x}$ and the vertical line $x = \frac{\pi}{2}$?



$x=0: y=\frac{1}{2}$

$x\in(0,\frac{\pi}{2}), x\uparrow: y\downarrow$

$x=\frac{\pi}{2}: y=0$

$\therefore$ Area $= \int_0^{\frac{\pi}{2}} \frac{\cos x}{2+\sin x}dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x}\, d(2+\sin x)$

$= \left[\ln|2+\sin x|\right]_0^{\frac{\pi}{2}}$

$= \ln 3 - \ln 2$

$= \ln\frac{3}{2}$

5. $\int_0^2 \frac{x^2}{x+1}dx$

$= \int_0^2 \frac{(x+1)x-(x+1)+1}{x+1}dx$

$= \int_0^2 x-1+\frac{1}{x+1}dx$

$= \left[\frac{1}{2}x^2-x\right]_0^2 + \left[\ln|x+1|\right]_0^2$

$= (2-2)+\ln 3$

$= \ln 3$

6. $\int_1^e \frac{\cos(\ln x)}{x}dx$

$= \int_1^e \cos(\ln x)\, d(\ln x)$

$= \left[\sin(\ln x)\right]_1^e$

$= \sin(\ln e) - \sin(\ln 1)$

$= \sin 1 - \sin 0$

$= \sin 1$

1. $\int_0^{\pi} 4\sin^4\theta \, d\theta = \int_0^{\pi} 4\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta = \int_0^{\pi} 1 - 2\cos 2\theta + \cos^2 2\theta \; d\theta$

$$= \left[\theta - \sin 2\theta\right]_0^{\pi} + \underline{\underline{\int_0^{\pi} \frac{1+\cos 4\theta}{2} d\theta}}$$

$$= \left[\frac{\theta}{2} + \frac{\sin 4\theta}{8}\right]_0^{\pi}$$

$$= \frac{\pi}{2}$$

$$I = \pi + \frac{\pi}{2} = \frac{3}{2}\pi$$

2. $\int_0^{\frac{\pi}{4}} 4\tan^2\theta \, d\theta = \int_0^{\frac{\pi}{4}} 4(\sec^2\theta - 1)\, d\theta$

$$= 4 \cdot \int_0^{\frac{\pi}{4}} d(\tan\theta) - [4\theta]_0^{\frac{\pi}{4}}$$

$$= [4\tan\theta]_0^{\frac{\pi}{4}} - \pi$$

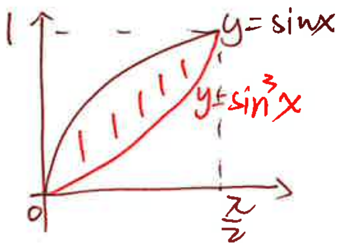
$$= 4 - \pi$$

3. $\int \sec^4 x \, dx = \int \sec^2 x \cdot d(\tan x)$

$$= \int (\tan^2 x + 1)\; d(\tan x)$$

$$= \frac{1}{3}\tan^3 x + \tan x + C$$

4. Find the area bounded by the curves $\underline{y = \sin x \text{ and } y = \sin^3 x \text{ between } x = 0 \text{ and } x}$ $= \frac{\pi}{2}$



$$\underline{\underline{\int_0^{\frac{\pi}{2}} \sin x \, dx}} - \underline{\underline{\int_0^{\frac{\pi}{2}} \sin^3 x \, dx}} = \frac{1}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin x \, dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$$

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = -\int_0^{\frac{\pi}{2}} 1 - \cos^2 x \; d(\cos x)$$

$$= \left[-\left(\cos x - \frac{1}{3}\cos^3 x\right)\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{1}{3}\cos^3 x - \cos x\right]_0^{\frac{\pi}{2}}$$

$$= -\left(\frac{1}{3} - 1\right) = \frac{2}{3}$$

$\sec\theta > 0$

D 1. If the substitution $x = 2\tan\theta$ is made in $\int \frac{x^3}{\sqrt{x^2+4}}\,dx$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the resulting integral is

$dx = 2\sec^2\theta\, d\theta$

$x^2+4 = 4\sec^2\theta$

$$= \int \frac{8\tan^3\theta}{2\sec\theta}\cdot 2\sec^2\theta\, d\theta$$

$$= \int 8\tan^3\theta\sec\theta\, d\theta$$

(A) $4\int \tan^2\theta\sec\theta\, d\theta$

(B) $4\int \tan^2\theta\sec^2\theta\, d\theta$

(C) $8\int \tan^3\theta\, d\theta$

(D) $8\int \tan^3\theta\sec\theta\, d\theta$

2. $\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx =$ $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec\theta\cdot\tan\theta}\sec\theta\tan\theta\, d\theta = [\theta]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{12}$

$x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$x=\sqrt{2} \rightarrow \theta = \frac{\pi}{4}$

$x=2 \rightarrow \theta = \frac{\pi}{3}$

3. If $0 < \theta < \frac{\pi}{2}$, then $\int \frac{\sqrt{x^2-1}}{x^4}\,dx =$ $\frac{1}{3}\cdot\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C$

$x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$$I = \int \frac{\tan\theta}{\sec^4\theta}\sec\theta\tan\theta\, d\theta$$

$$= \int \cos^3\theta\cdot\frac{\sin^2\theta}{\cos^2\theta}\, d\theta$$

$$= \int \cos\theta\sin^2\theta\, d\theta$$

$$= \int \sin^2\theta\, d(\sin\theta)$$

$$= \frac{1}{3}\sin^3\theta + C$$

$$= \frac{1}{3}\cdot\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C$$

1. $\int \frac{dx}{x^2+x-6} = \frac{1}{5}\int \frac{1}{x-2} - \frac{1}{x+3}\, dx = \frac{1}{5}\left(\ln|x-2| - \ln|x+3|\right) + C$

$1 \quad -2$

$1 \quad 3$

$= \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right| + C$

2. $\int_4^7 \frac{5}{(x-2)(2x+1)}\, dx = \int_4^7 \frac{1}{x-2}\, d(x-2) - \int_4^7 \frac{1}{2x+1}\, d(2x+1)$

$= \frac{1}{x-2} - \frac{2}{2x+1}$

$= \left[\ln|x-2|\right]_4^7 - \left[\ln|2x+1|\right]_4^7$

$= (\ln 5 - \ln 2) - (\ln(15) - \ln 9)$

$= \ln\left(\frac{5}{2}\cdot\frac{9}{15}\right) = \ln\frac{3}{2}$

3. $\int \frac{x}{x^2+5x+6}\, dx = \int \frac{-2}{x+2}\, d(x+2) + \int \frac{3}{x+3}\, d(x+3)$

$1 \quad 2$

$1 \quad 3$

$= -2\ln|x+2| + 3\ln|x+3| + C$

$-\frac{2}{x+2} + \frac{3}{x+3}$

4. $\int \frac{2e^{2x}}{(e^x-1)(e^x+1)}\, dx = \int \frac{1}{e^x-1}\, d(e^x-1) + \int \frac{d(e^x+1)}{e^x+1}$

$\frac{e^x}{e^x-1} + \frac{e^x}{e^x+1}$

$= \ln|e^x-1| + \ln|e^x+1| + C$

$= \ln|(e^x-1)(e^x+1)| + C$

5. Let f be the function given by $f(\theta) = \int \frac{\sin\theta}{\cos\theta(\cos\theta - 1)}\, d\theta$.

$dx = d(\cos\theta) = -\sin\theta\, d\theta \quad \therefore d\theta = \frac{1}{-\sin\theta}dx = -\frac{1}{\sqrt{1-x^2}}dx$

(a) Substitute $x = \cos\theta$ and write an integral expression for f in terms of x.

$\sin\theta = \sqrt{1-x^2}$

(b) Use the method of partial fractions to find $f(\theta)$.

(a) $f(\theta) = \int \frac{\sqrt{1-x^2}}{x(x-1)} \cdot \frac{-1}{\sqrt{1-x^2}}\, dx$

$= \int \frac{-1}{x(x-1)}\, dx$

(b) $\frac{1}{\cos\theta(\cos\theta-1)} = \frac{1}{\cos\theta-1} - \frac{1}{\cos\theta}$

$= \ln|\cos\theta| - \ln|\cos\theta - 1| + C$

$\therefore I = -\int \frac{1}{\cos\theta-1} - \frac{1}{\cos\theta}\, d(\cos\theta)$

$= \int \frac{1}{\cos\theta}\, d(\cos\theta) - \int \frac{1}{\cos\theta-1}\, d(\cos\theta-1)$

1. $\int \underbrace{x}_{u}\underbrace{\sin(2x)\,dx}_{dv} =$ $x\cdot\left(-\frac{1}{2}\right)\cos(2x) + \int \frac{1}{2}\cos(2x)\,dx$

$du = dx \quad v = -\frac{1}{2}\cos(2x)$

$= -\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$

2. $\int_0^2 \underbrace{x}_{u}\underbrace{e^x\,dx}_{dv} =$ $[x\cdot e^x]_0^2 - \int_0^2 e^x\,dx$

$du = dx \quad v = e^x$

$= 2e^2 - (e^2 - e^0) = e^2 + 1$

3. If $\int \underbrace{x^2}_{u}\underbrace{\cos(3x)\,dx}_{dv} = f(x) - \frac{2}{3}\int x\sin(3x)\,dx$, then $f(x) = \frac{x^2}{3}\sin(3x)$

$du = 2x\,dx \quad v = \frac{1}{3}\sin(3x)$

$I = x^2\cdot\frac{1}{3}\sin(3x) - \int \frac{1}{3}\sin(3x)\cdot 2x\,dx$

4. $\int x^2 \underbrace{\ln x}_{u}\,dx =$ $\ln x\cdot\frac{1}{3}x^3 - \int \frac{1}{3}x^3\,\frac{1}{x}\,dx = \frac{1}{3}x^3\ln x - \frac{1}{3}\cdot\frac{1}{3}x^3 + c$

$v = \frac{1}{3}x^3$

$du = \frac{1}{x}dx$

$= \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$

5. $\int_0^{\pi/4} \underbrace{x}_{u}\,\underbrace{\sec^2 x\,dx}_{dv} =$ $[x\tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{\tan x\sec x}{\sec x}\,dx$

$du = dx \quad v = \tan x$

$= \frac{\pi}{4} - [\ln|\sec x|]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} - \ln\sqrt{2}$

6. $\int \sec^3 x\,dx =$ $\int \sec x\,\sec^2 x\,dx = \sec x\tan x - \int \tan x\sec x\tan x\,dx$

$\boxed{u = \sec x \quad v = \tan x \quad du = \sec x\tan x\,dx}$

$= \sec x\tan x - \int \sec x(\sec^2 x - 1)\,dx$

$= \sec x\tan x - I + \int \sec x\,dx$

$\therefore I = \int \sec^3 x\,dx = \frac{1}{2}\left(\sec x\tan x + \ln|\sec x + \tan x|\right) + C$

$du = f'(x)dx \quad v = \frac{1}{n}\sin(nx)$

7. $\int \underbrace{f(x)}_{u}\underbrace{\cos(nx)\,dx}_{dv} =$ $\quad f(x)\cdot\frac{1}{n}\sin(nx) - \int \frac{1}{n}\sin(nx)\cdot f'(x)\,dx$

A

(A) $\frac{1}{n}f(x)\sin(nx) - \frac{1}{n}\int f'(x)\sin(nx)\,dx$

~~(B)~~ $\frac{1}{n}f(x)\cos(nx) - \frac{1}{n}\int f'(x)\cos(nx)\,dx$

~~(C)~~ $n\,f(x)\cos(nx) + \frac{1}{n}\int f'(x)\sin(nx)\,dx$

~~(D)~~ $n\,f(x)\cos(nx) - \frac{1}{n}\int f'(x)\cos(nx)\,dx$

D

8. If $\int \underbrace{\arccos x}_{u}\,\underbrace{dx}_{dv} = x\arccos x + \int f(x)\,dx$, then $f(x) =$

$du = -\frac{1}{\sqrt{1-x^2}}dx \quad v = x$ $\qquad = -\int x\cdot -\frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{x}{\sqrt{1-x^2}}\,dx$

(A) $-x\sqrt{1-x^2}$ (B) $x\sqrt{1-x^2}$ (C) $-\frac{1}{\sqrt{1-x^2}}$ (D) $\frac{x}{\sqrt{1-x^2}}$

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	−2	3	4	−1
3	2	−1	−3	5

A

9. The table above gives values of f, f', g, and g' for selected values of x.

If $\int_1^3 \underbrace{f(x)}_{u}\underbrace{g'(x)\,dx}_{dv} = 8$, then $\int_1^3 f'(x)g(x)\,dx =$

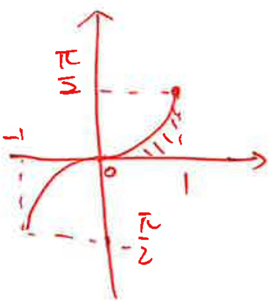
$[f(x)g(x)]_1^3 - \int_1^3 g(x)f'(x)\,dx = f(3)g(3) - f(1)g(1) - \int_1^3 f'g\,dx$

(A) −4 (B) −1 (C) 5 (D) 8

$= -2 + 6 - ?$

10. Find the area of the region bounded by $y = \arcsin x$, $y = 0$, and $x = 1$. Show the work that leads to your answer.

$\therefore ? = -4$



$\int_0^1 \arcsin x\,dx$

$\int \underbrace{\arcsin x}_{u}\,\underbrace{dx}_{dv} = \arcsin x\cdot x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$du = \frac{1}{\sqrt{1-x^2}}dx \quad v = x$

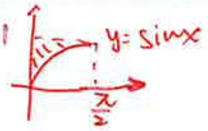
$= x\cdot\sin^{-1}x + \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}\,d(1-x^2)$

$= x\sin^{-1}x + (1-x^2)^{\frac{1}{2}} + C$

$= \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^1$

$= \frac{\pi}{2} + 0 - 1 = \frac{\pi}{2} - 1$

Method 2.



Area $= \frac{\pi}{2} - \int_0^1 \sin x\,dx = \frac{\pi}{2} + [\cos x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$

1. $\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{\sqrt{x-1}}\,d(x-1) = \lim_{t\to\infty} \left[2\sqrt{x-1}\right]_2^t = \lim_{t\to\infty} 2\cdot\left(\sqrt{t-1}-1\right)$
$= \infty$

2. $\int_0^\infty \frac{1}{(x+3)(x+4)}\,dx = \int_0^\infty \frac{1}{x+3} - \frac{1}{x+4}\,dx$

$\int \frac{1}{(x+3)(x+4)}\,dx = \ln|x+3| - \ln|x+4| + C$
$= \ln\left|\frac{x+3}{x+4}\right| + C$

$\therefore I = \lim_{t\to\infty}\left[\ln\left|\frac{x+3}{x+4}\right|\right]_0^t = \lim_{t\to\infty}\left(\ln\left|\frac{t+4-1}{t+4}\right| - \ln\frac{3}{4}\right)$
$= \lim_{t\to\infty}\left(\ln\left|1-\frac{1}{t+4}\right| - \ln\frac{3}{4}\right)$
$= -\ln\frac{3}{4}$

C 3. $\int_0^4 \frac{dx}{(x-1)^{2/3}} = \int_0^4 (x-1)^{-\frac{2}{3}}\,d(x-1) = \left[3(x-1)^{\frac{1}{3}}\right]_0^4 = 3(1+\sqrt[3]{3})$

(A) $3\sqrt[3]{3}$ (B) $3(1-\sqrt[3]{3})$ (C) $3(1+\sqrt[3]{3})$ (D) divergent

A 4. $\int_0^\infty x^2 e^{-x^3}\,dx = \int_0^\infty \left(-\frac{1}{3}\right)e^{-x^3}\,d(-x^3) = \lim_{t\to\infty}\left[-\frac{1}{3}e^{-x^3}\right]_0^t = \lim_{t\to\infty}\left(-\frac{1}{3}e^{-t^3}\right) - \left(-\frac{1}{3}\right) = \frac{1}{3}$

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 1 (D) divergent

B 5. $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx =$

$\int x^{-\frac{1}{2}}\ln x\,dx$ $u = \ln x$, $V = 2x^{\frac{1}{2}}$, $du = x^{-1}dx$

$= \ln x\cdot 2x^{\frac{1}{2}} - \int 2x^{-\frac{1}{2}}dx$
$= \ln x\cdot 2x^{\frac{1}{2}} - 4x^{\frac{1}{2}} + C$

(A) -6 (B) -4 (C) -2 (D) divergent

$I = \lim_{t\to 0^+}\left[\ln x\cdot 2\sqrt{x} - 4\sqrt{x}\right]_t^1$
$= -4$

$\lim_{t\to 0^+} \frac{\ln x}{x^{-\frac{1}{2}}} = \lim_{t\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}} = 0$

6. If $\int_0^1 \frac{ke^{-\sqrt{x}}}{\sqrt{x}}\,dx = 1$, what is the value of k?

$= \int_0^1 ke^{-\sqrt{x}}(-2)\,d(-\sqrt{x}) = \lim_{t\to 0^+}(-2k)\left[e^{-\sqrt{x}}\right]_t^1 = \lim_{t\to 0^+}(-2k)\left(e^{-1} - e^{t}\right)$

(A) $-\frac{1}{2}$ (B) $\frac{e}{2}$ (C) $\frac{1}{2}$ $k = \frac{1}{2(1-e)}$ (D) There is no such value of k

7. Let f be the function given by $f(x) = \frac{x}{\sqrt{x^2+1}}\,dx$.

(a) Show that the improper integral $\int_1^\infty f(x)\,dx$ is divergent.

(a) $I = \int_1^\infty \frac{\frac{1}{2}}{\sqrt{x^2+1}}\,d(x^2+1) = \lim_{t\to\infty}\left[\sqrt{x^2+1}\right]_1^t$
$= \infty$

(*) (b) Find the average value of f on the interval $[1,\infty)$.

(b) $\lim_{t\to\infty} \frac{1}{t}\int_1^t \frac{x}{\sqrt{x^2+1}}\,dx$
$= \lim_{t\to\infty}\left(\frac{\sqrt{t^2+1}}{t} - 0\right) = 1$