Applications of Differentiation

-- Analytical Applications

IVT

- f is continuous on a closed interval $[a, b]$
- $f(a) \cdot f(b) < 0$

→ There are at least one zero in the closed interval.

Show that there is a solution of the equation $4x^3 - 6x^2 + 3x = 2$ on $[1,2]$

Let $\mathrm{f}(x) = 4x^3 - 6x^2 + 3x - 2$

$\mathrm{f}(1) = -1 < 0$

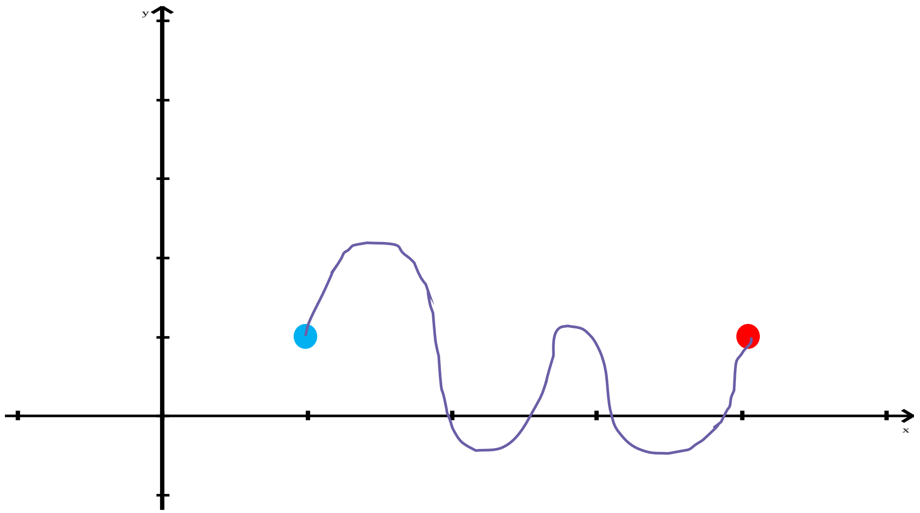
$\mathrm{f}(2) = 12 > 0$

According to the IVT, there are at least one zero in [1,2].

Activity

Step 1: Place two points anywhere on the coordinate plane below that have the **same y-values**.

Step 2: **Connect** the two points with a **continuous function** that is also **differentiable**.



There MUST be at least one point on your function where you can draw a tangent line that is horizontal. (i.e., the slope is zero)

Rolle's Theorem

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$

Q1. Determine if Rolle's Theorem applies to $f(x) = x^4 - 2x^2$ on the interval $[-2,2]$. State thoroughly the reasons why or why not the theorem applies. If the theorem does apply, find the value of c guaranteed by the theorem.

Q2. Let f be the function given by $f(x) = \mathrm{x}^3 - 9x + 1$. Find all numbers c that satisfies the conclusion of Rolle's Theorem for $f(x)$ on the closed interval $[0,3]$

The Mean Value Theorem

If f is <u>continuous</u> on the <u>closed</u> interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b), such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof:

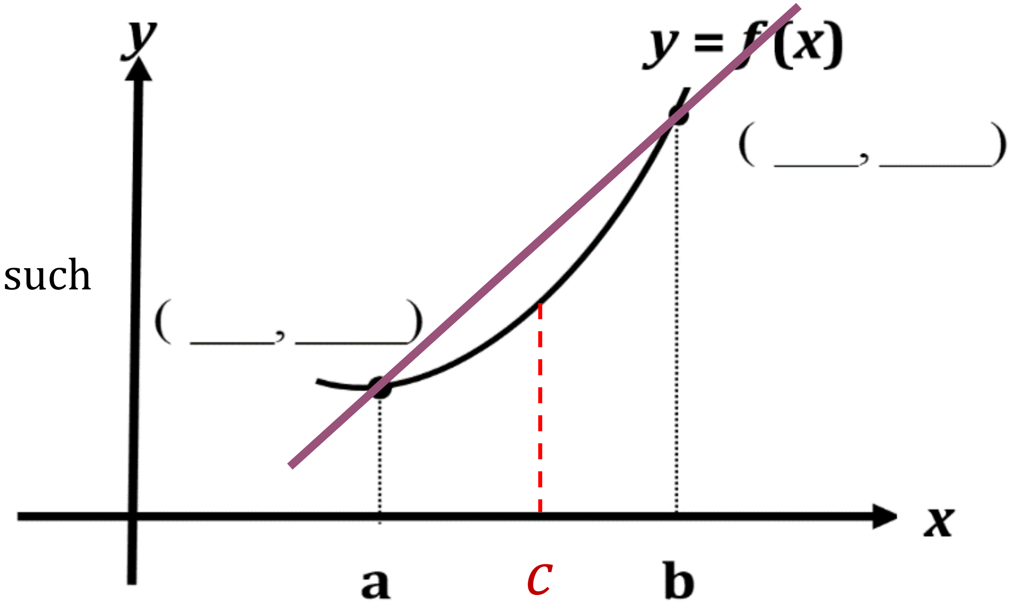
Suppose $\mathrm{F(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x - a)}$

Then we have $\mathrm{F(a) = f(a)}$ and $\mathrm{F(b) = f(a)}$

According to the Rolle's Theorem, there exist c in (a, b), such that $\mathrm{F'(c) = 0}$.

$F'(x) = f'(x) - \frac{\mathrm{f(b)-f(a)}}{\mathrm{b-a}}$

$F'(c) = f'(c) - \frac{\mathrm{f(b)-f(a)}}{\mathrm{b-a}} = 0 \rightarrow f'(c) = \frac{\mathrm{f(b)-f(a)}}{\mathrm{b-a}}$



Q3. Determine if the Mean Value Theorem applies to $f(x) = 3 - \frac{5}{x}$ on the interval $[1,5]$.

State thoroughly the reasons why or why not the theorem applies. If the theorem does apply, find the value of c guaranteed by the theorem.

Q4. Find the equation of the tangent line to the graph of $f(x) = 2x + sin\,x + 1$ on the interval $[0, \pi]$ at the point which is the solution to the Mean Value Theorem.

Q5. Explain precisely why we cannot apply the Mean Value Theorem to either of the three functions below on the provided intervals.

a. $f(x) = 3x - |x - 3|$ on $[2,5]$

b. $g(x) = \frac{2}{x+2}$ on $[-3,1]$

c. $\mathrm{h}(x) = x^{\frac{2}{3}}$ on $[-1,3]$

Extrema of a Function

A function does not have to have a maximum or a minimum on an interval.

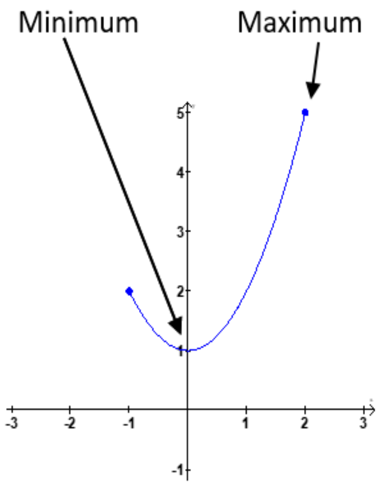


Figure 1

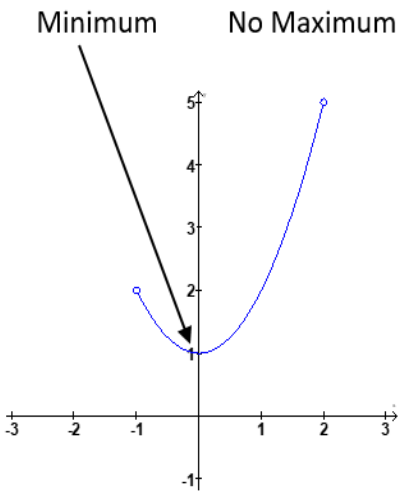


Figure 2

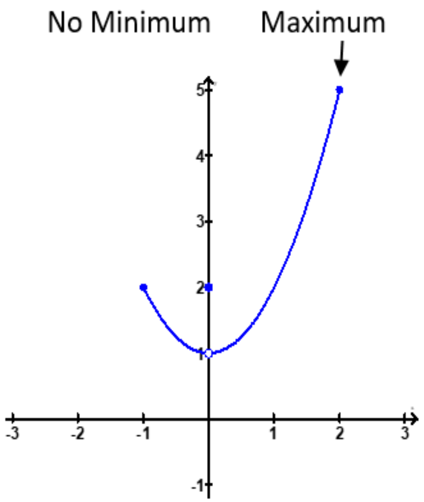


Figure 3

<u>The Extreme Value Theorem</u>

If f is **continuous** on a **closed interval**, then f has BOTH a maximum and a minimum on the interval.

Absolute and Relative Extrema

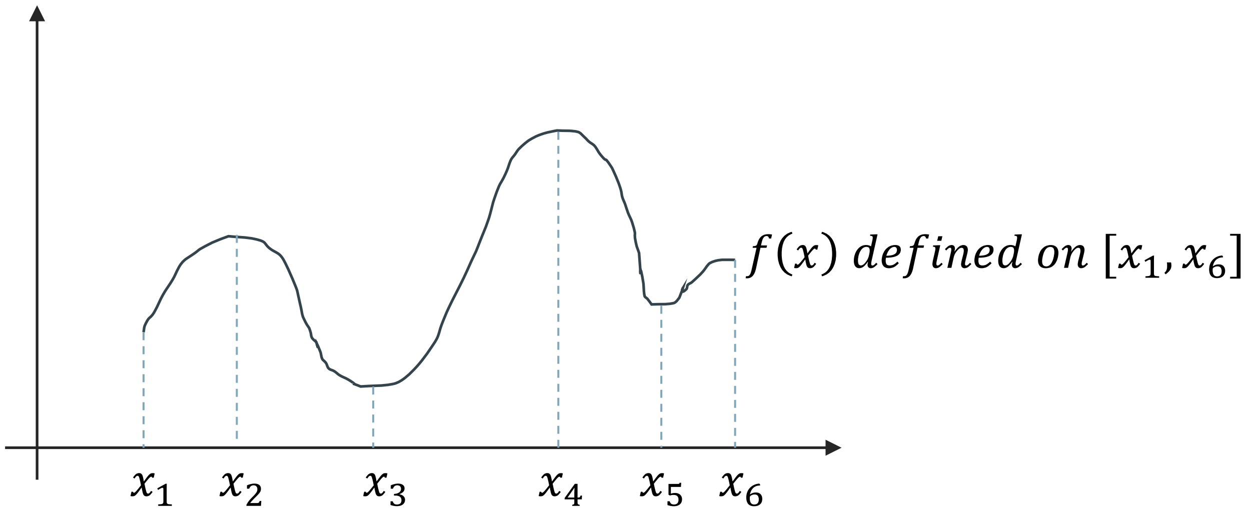
Def.

A function f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in the domain of f.

A function f has an **absolute maximum** at c if $f(c) \geq f(x)$ for all x in the domain of f.

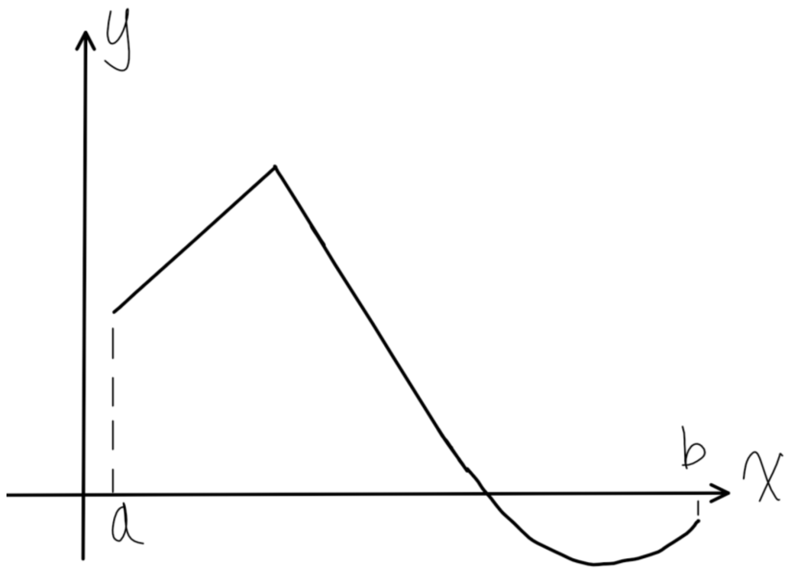
A function f has a **relative minimum** at c if $f(c) \leq f(x)$ for all x in the vicinity of c.

A function f has a **relative maximum** at c if $f(c) \geq f(x)$ for all x in the vicinity of c.



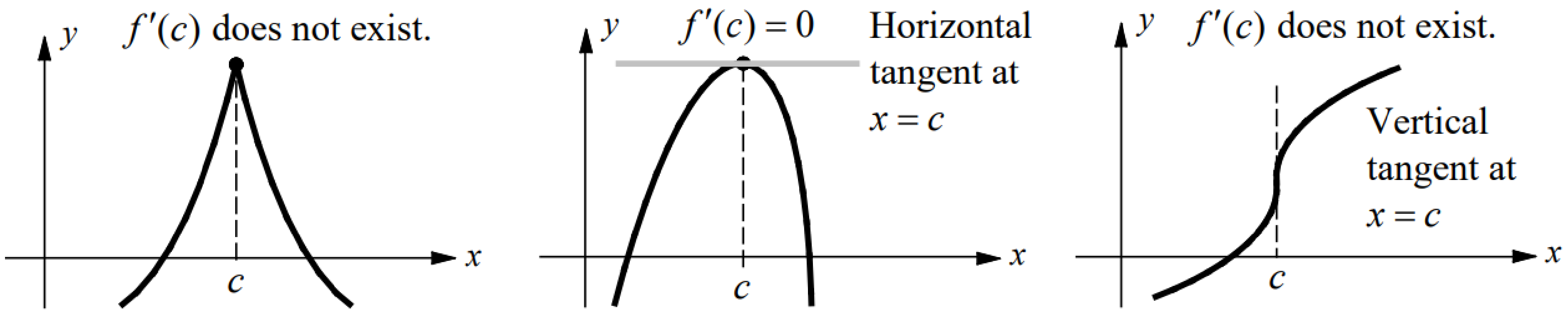
How to find extrema of a function

? How to find the relative maxima and relative minima of $f(x) = x^2 - 3x - 4$ on the interval [-4,9]



Critical Numbers

Let f be defined at c. If $f'(c) = 0$ or if $f'(c)$ does not exist, then c is a **critical number** of f.



Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at x= c, then c is a critical number of f.

Q1. If f is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$, which of the following could be false?

(A) $f'(c) = \frac{f(b)-f(a)}{b-a}$ for some c such that $a < c < b$.

(B) $f'(c) = 0$ for some c such that $a < c < b$.

(C) f has a minimum value on $a \leq x \leq b$.

(D) f has a maximum value on $a \leq x \leq b$.

Q2. The function f is defined on the closed interval [0,1] and satisfies $f(0) = f\left(\frac{1}{2}\right) = f(1)$. On the open interval (0,1), f is continuous and strictly increasing. Which of the following statements is true?

(A) f attains both a minimum value and a maximum value on the closed interval [0,1].

(B) f attains a minimum value but not a maximum value on the closed interval [0,1].

(C) f attains a maximum value but not a minimum value on the closed interval [0,1].

(D) f attains neither a minimum value nor a maximum value on the closed interval [0,1].

Q3. Let g be a function given by $g(x) = x^2e^{kx}$, where k is a constant. For what value of k does g have a critical point at $x = \frac{3}{2}$?

(A) $-\frac{4}{3}$ (B) $-\frac{2}{3}$ (C) $\frac{2}{3}$ (D) 0

The Extreme Value Theorem

If f is **continuous** on a **closed interval**, then f has BOTH a maximum and a minimum on the interval.

Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at x= c, then c is a critical number of f.

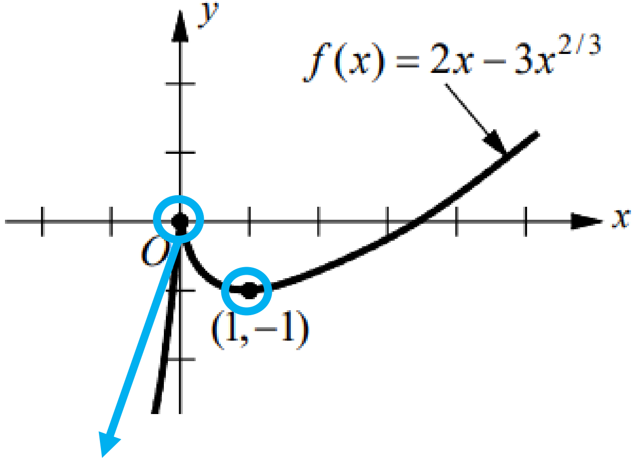
- **Guidelines for Finding Absolute Extrema on a Closed Interval**

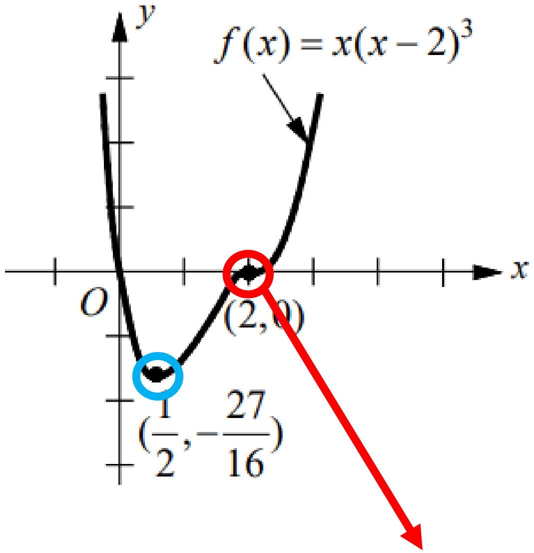
To find extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

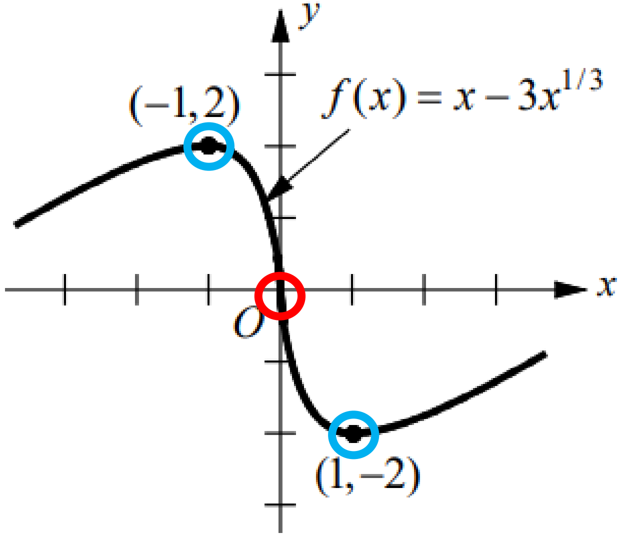
1. Find the **critical numbers** of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each **endpoint** of [a, b].
4. The least of these f values is the absolute minimum. The greatest is the absolute maximum.

Note: The actual maximum or minimum value is a y value. Where the maximum or minimum occurs would be an x value.

How to find relative minimum/maximum on $\mathbb{R}$ or on an open interval







Critical point (0,0)

$x < 0$: increasing

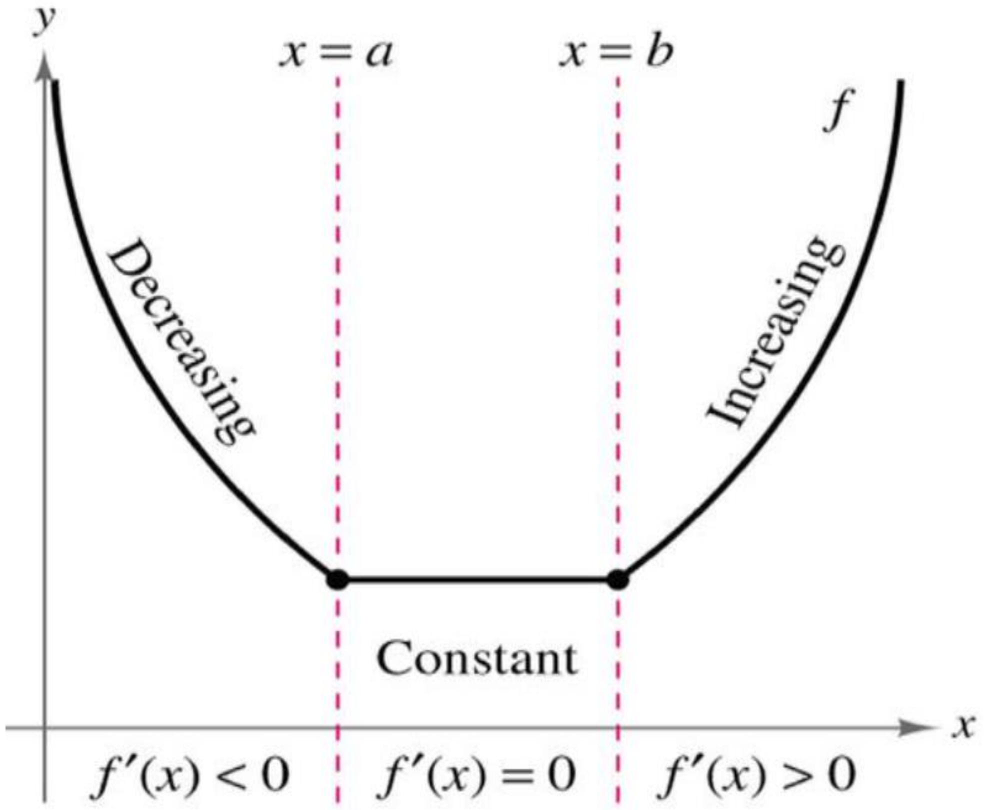
$x > 0$: decreasing

Critical point (2,0)

$x < 0$: increasing

$x > 0$: increasing

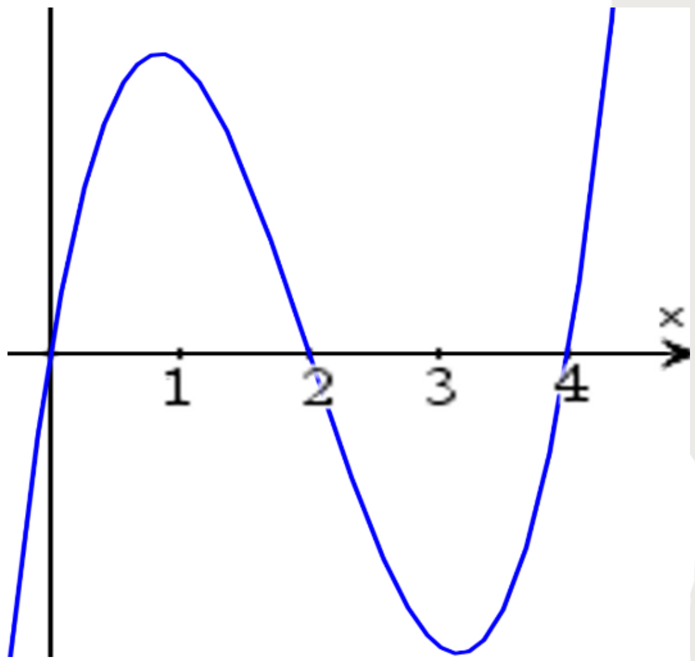
Increasing and Decreasing Functions



Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

Q1. The graph shown to the right is of $f'(x)$, the derivative of $f(x)$. Using the graph, find the critical values of $f(x)$, and state the intervals over which $f(x)$ is increasing and decreasing.



Guidelines for Finding Intervals on Which a Function is Increasing or Decreasing

1. Locate the **critical numbers** of f in (a, b), and use these numbers to determine your test intervals.

2. Determine the **sign** of $f'(x)$ by picking a "test value" in each of the intervals.

3. Use the Theorem for Increasing and Decreasing Functions to determine whether the function increases or decreases.

The guidelines above will also work for the interval $(-\infty, b), (a, \infty)$, or $(-\infty, \infty)$

Q2. Find the intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

$f'(x) = 3x(x-1)$

$f'(x) = 0 \rightarrow x = 0$ or $x = 1$

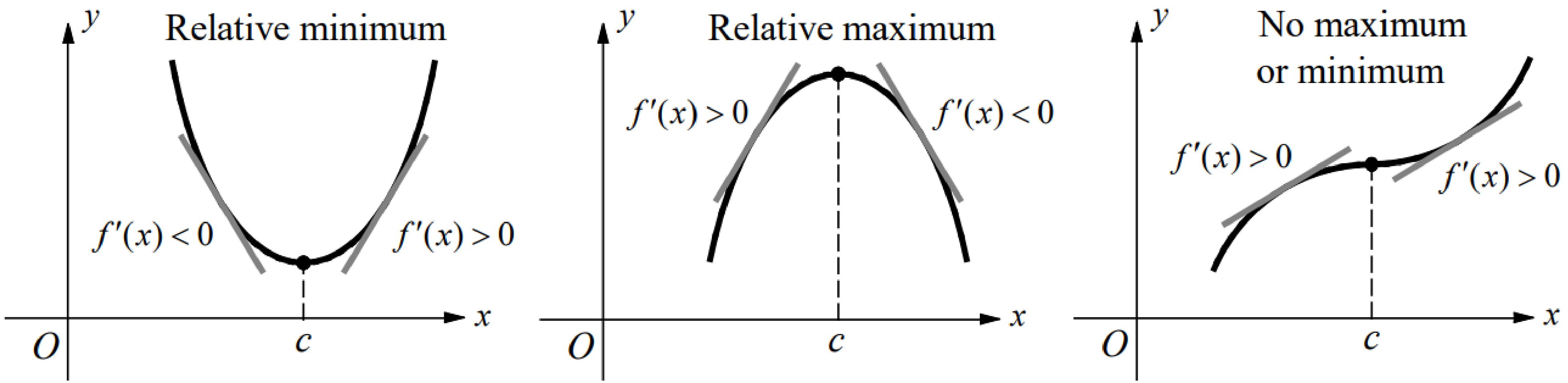
x	$(-\infty, \mathbf{0})$	0	$(\mathbf{0}, \mathbf{1})$	1	$(\mathbf{1}, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$					

The First Derivative Test

c: a critical number of the function f

Check $f'(x)$:

- If $f'(x)$ changes from negative to positive at c, then f(c) is a relative minimum of f.
- If $f'(x)$ changes from positive to negative at c, then f(c) is a relative maximum of f.
- If $f'(x)$ does not change its sign at c, then f(c) is neither a relative minimum nor relative maximum.



Q2. Find the intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

$f'(x) = 3x(x-1)$

$f'(x) = 0 \ \rightarrow x = 0 \text{ or } x = 1$

x	$(-\infty, \mathbf{0})$	0	$(\mathbf{0}, \mathbf{1})$	1	$(\mathbf{0}, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$					

Q3. Applying the First Derivative Test

a. Find the points that are relative extrema of the function $f(x) = \frac{1}{2}x - \sin x$ on the interval $(0,2\pi)$.

b. Find the points that are relative extrema of the function $f(x) = (x^2 - 4)^{\frac{2}{3}}$.

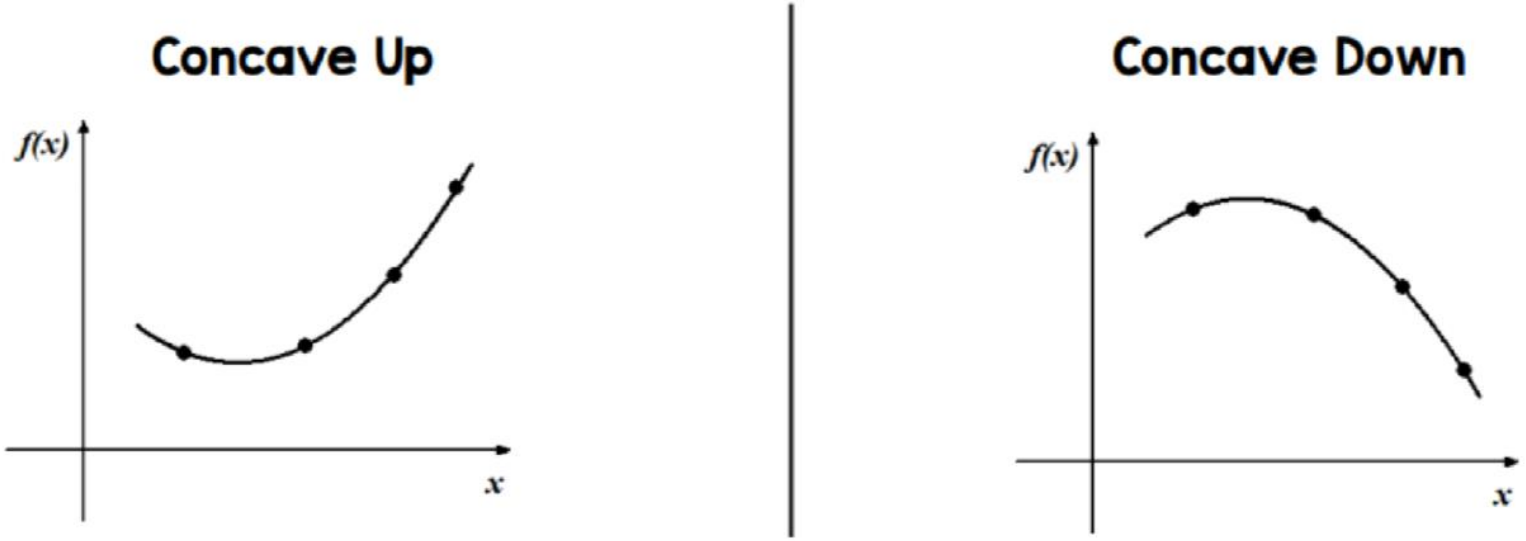
c. Find the points that are relative extrema of the function $f(x) = \frac{x^4+1}{x^2}$.

Concavity

Let f be differentiable on an open interval I.

The graph of f is **concave upward** on I if f' is increasing on the interval.

The graph of f is **concave downward** on I if f' is decreasing on the interval.

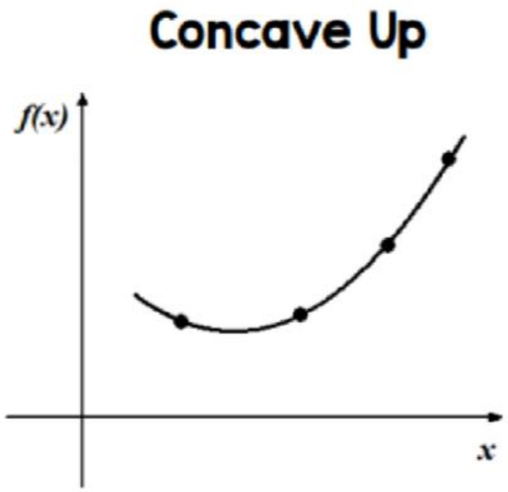


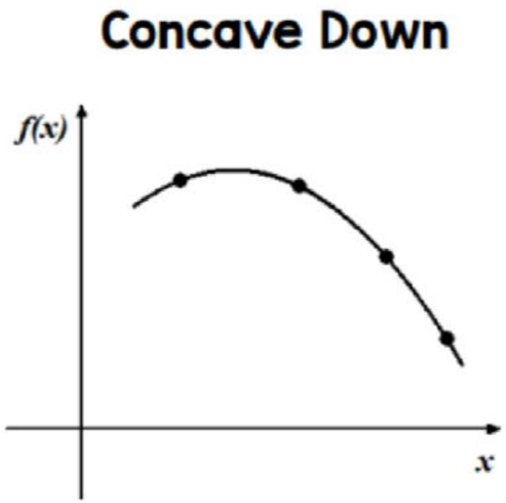
Test for Concavity

Let f be a function whose second derivative exists on an open interval (a, b).

If f''_______ for all x in (a, b), then the graph of f is concave upward on (a, b).

If f''_______ for all x in (a, b), then the graph of f is concave downward on (a, b).





Point of Inflection is the point where the concavity of the function changes.

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f is not differentiable at $\mathrm{x} = \mathrm{c}$.

Q1. Determine the open intervals on which each graph is concave upward or downward and state any points of inflection. Justify your answer.

a. $f(x) = x^4 - 4x^3$

b. $f(x) = \frac{6}{x^2+3}$

c. $f(x) = \frac{x^2+1}{x^2-4}$

The Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

If f''__________, then $f(c)$ is a relative minimum.

If f''__________, then $f(c)$ is a relative maximum.

If $f'' = 0$?

The second derivative test fails.
Use the first derivative test!

Q1. Find the relative extrema for $f(x) = -3x^5 + 5x^3$ using the <u>Second Derivative Test</u>.

Q2. Given the values below for x, $f(x)$, $f'(x)$, along with the fact that $f'(x)$ has only 2 zeros on the interval $(-3,10)$ and $f''(x)$ has only 2 zeros on the interval $(-3,10)$, answer each of the following.

x	-3	-1	1	3	5	7	10
$f(x)$	-7	1	-1	-4	3	2	-1
$f'(x)$	1	0	-1	0	2	undefined	3
$f''(x)$	-2	-1	0	2	3	0	5

a. Identify all x-values where f has a relative minimum. Justify using the First Derivative Test.

Q2. Given the values below for x, $f(x)$, $f'(x)$, along with the fact that $f'(x)$ has only 2 zeros on the interval $(-3,10)$ and $f''(x)$ has only 2 zeros on the interval $(-3,10)$, answer each of the following.

x	-3	-1	1	3	5	7	10
$f(x)$	-7	1	-1	-4	3	2	-1
$f'(x)$	1	0	-1	0	2	undefined	3
$f''(x)$	-2	-1	0	2	3	0	5

b. Identify all x-values where f has a relative maximum. Justify using the Second Derivative Test.

Q2. Given the values below for x, $f(x)$, $f'(x)$, along with the fact that $f'(x)$ has only 2 zeros on the interval $(-3,10)$ and $f''(x)$ has only 2 zeros on the interval $(-3,10)$, answer each of the following.

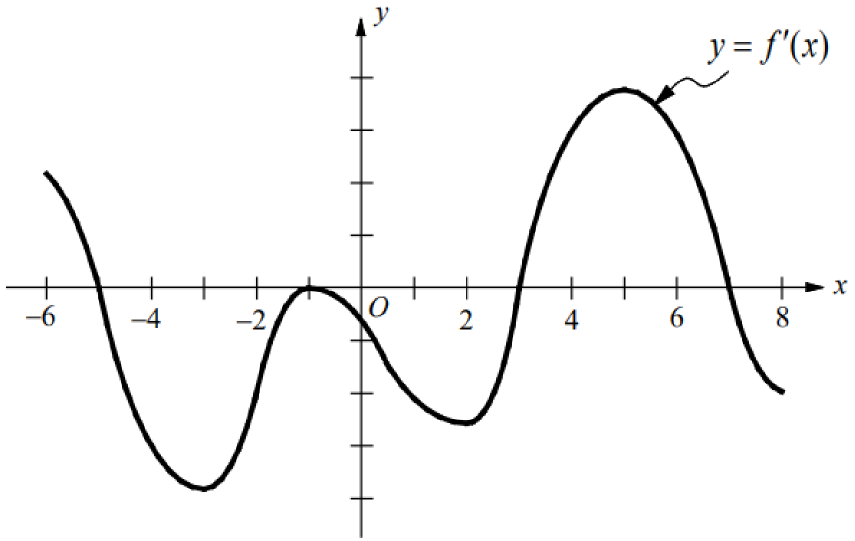
x	-3	-1	1	3	5	7	10
$f(x)$	-7	1	-1	-4	3	2	-1
$f'(x)$	1	0	-1	0	2	undefined	3
$f''(x)$	-2	-1	0	2	3	0	5

c. Identify all x-values where f has a point of inflection. Justify.

Q2. Given the values below for x, $f(x)$, $f'(x)$, along with the fact that $f'(x)$ has only 2 zeros on the interval $(-3,10)$ and $f''(x)$ has only 2 zeros on the interval $(-3,10)$, answer each of the following.

x	-3	-1	1	3	5	7	10
$f(x)$	-7	1	-1	-4	3	2	-1
$f'(x)$	1	0	-1	0	2	undefined	3
$f''(x)$	-2	-1	0	2	3	0	5

d. What is the equation of the tangent to the curve $y = f(x)$ at $x = 5$?



<u>Note:</u> This is the graph of f', not the graph of f.

The figure above shows the graph of f'. The domain of f is the set of all real numbers x such that $-6 \le x \le 8$.

(a) For what values of x does f have a relative maximum?

(b) For what values of x does f have a relative minimum?

(c) For what values of x does the graph of f have a horizontal tangent?

(d) For what values of x is the graph of f concave upward?

(e) For what values of x is the graph of f concave downward?

(f) Suppose that $f(0) = 1$. Sketch a possible graph of f.

Curve Sketching

Sketch the graph of $f(x) = 2xe^{-x^2}$

Guidelines:

- Domain
- Intercepts
- Symmetry
    - Even/Odd?
    - Replacing y by $-$y yields an equivalent equation, the curve is symmetric about the x-axis
- Asymptotes
- Intervals of increasing and decreasing
- Relative Max, Relative Min

Sketch a possible f(x) with the following characteristics

$f'(x) > 0$ for all x

$f'' > 0$ for all x

$f(1) = -2$

Sketch a possible f(x) with the following characteristics

$f'(x) > 0$ for $x > 1$

$f'(x) < 0$ for $x < 1$

$f'(1) = 0$

$f'' > 0$ for all x

$f(1) = -1$

Sketch a possible f(x) with the following characteristics

$f'(x) > 0$ for $x < -3$,

$f'(x) < 0$ for $-3 < x < 1$

$f'(x) > 0$ for $x > 1$

$f'(-3) = f'(1) = 0$,

$f(0) = 0$

$f'' < 0$ for x < -1, $f'' > 0$ for x > -1

Sketch a possible f(x) with the following characteristics

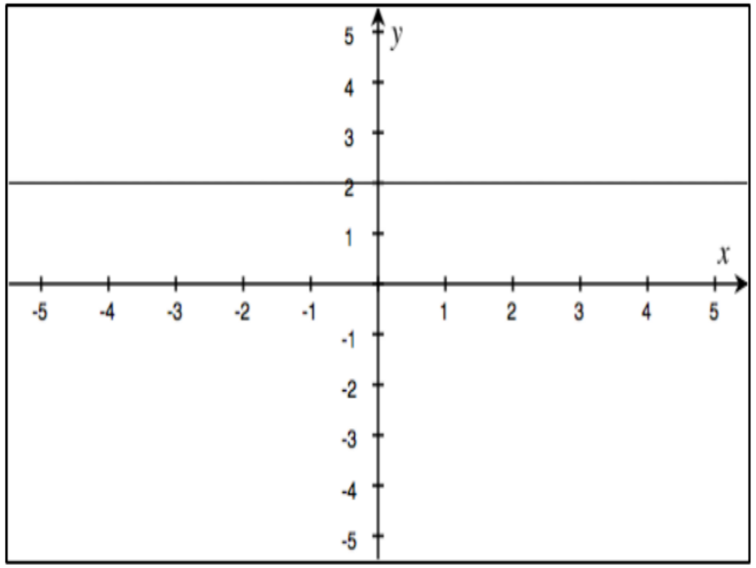
$f'(x) > 0$ for $x > 2$,

$f'(x) = -\frac{1}{2}$ for $x < 2$

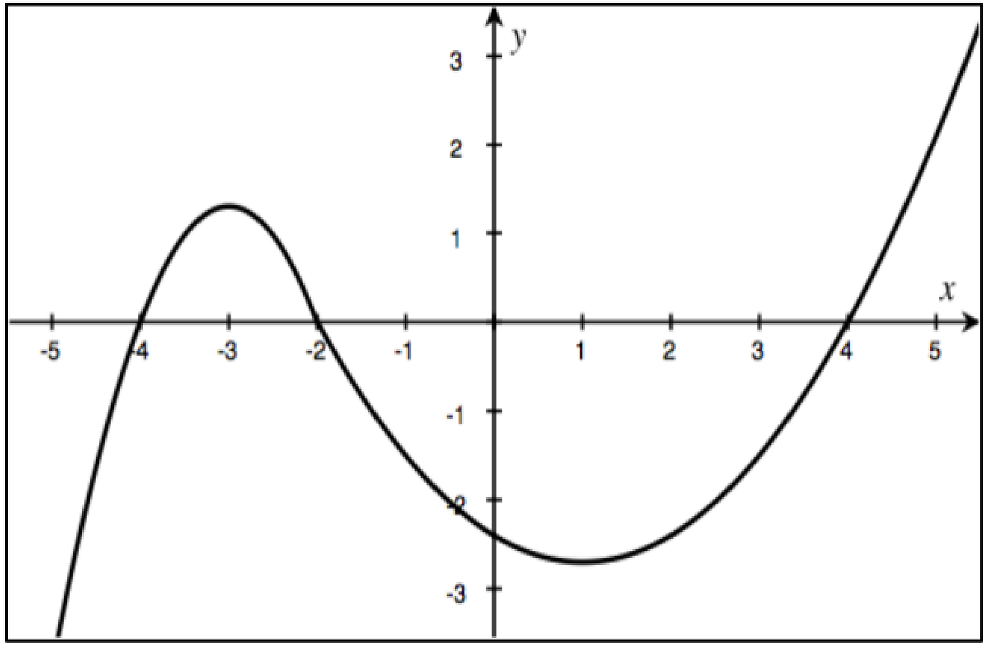
$f'(2)$ does not exist,

$f'' < 0$ for x > 2, $\quad f(2) = 0$

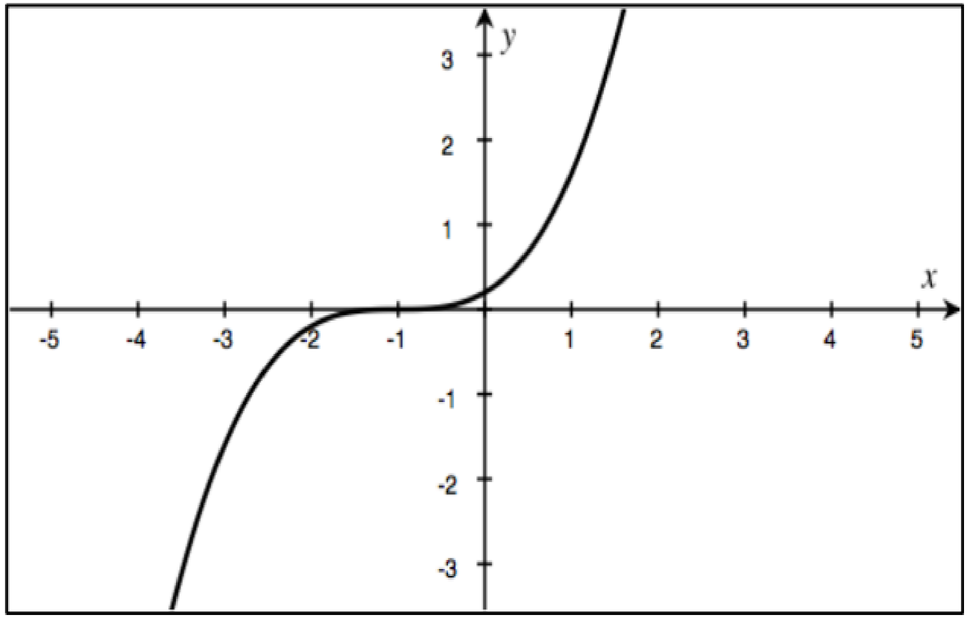
Given a graph of $f'(x)$, sketch a possible graph of $f(x)$



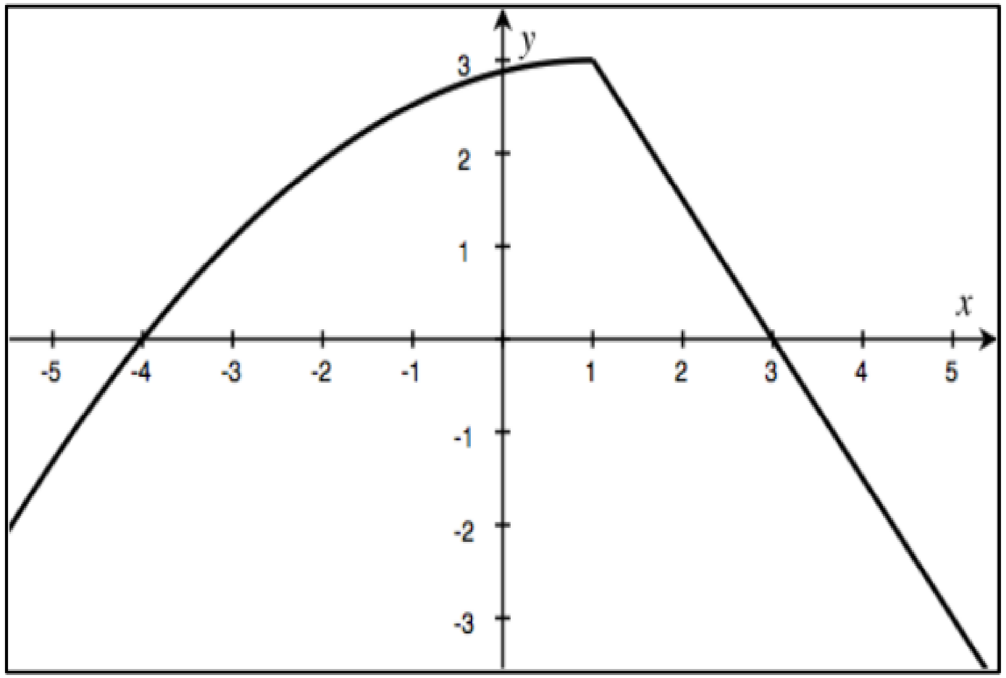
Given a graph of $f'(x)$, sketch a possible graph of $f(x)$



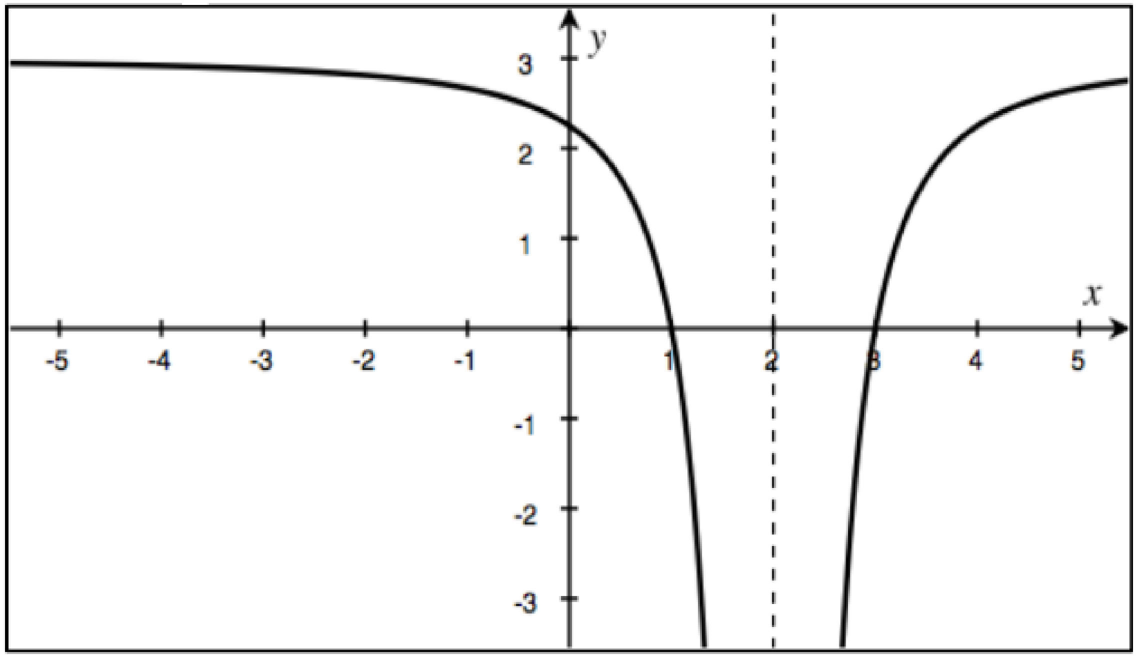
Given a graph of $f'(x)$, sketch a possible graph of $f(x)$



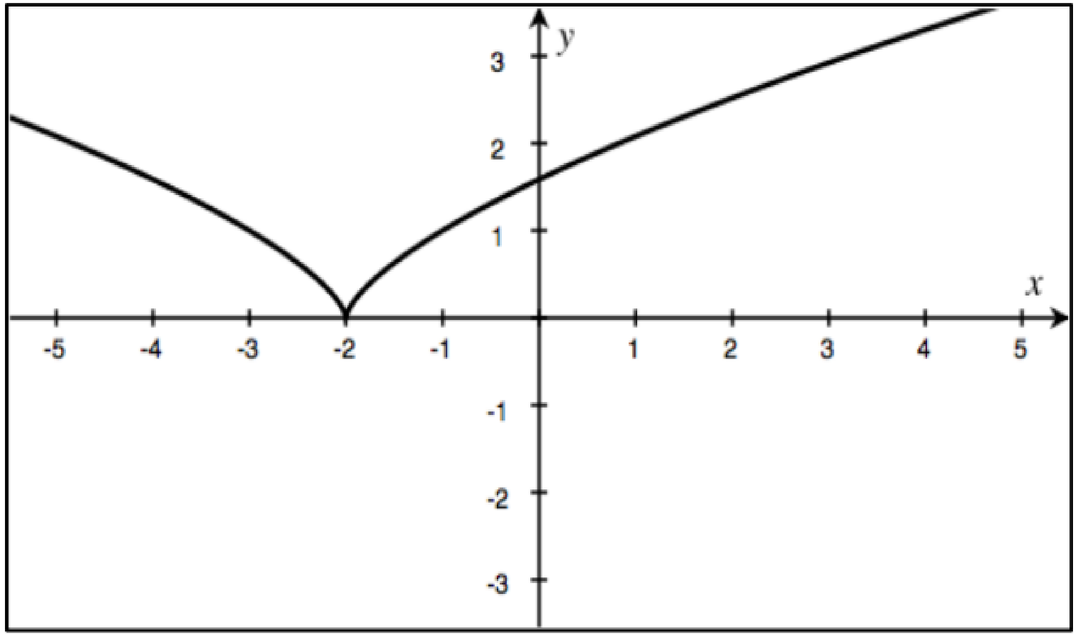
Given a graph of $f'(x)$, sketch a possible graph of $f(x)$



Given a graph of $f'(x)$, sketch a possible graph of $f(x)$



Given a graph of $f'(x)$, sketch a possible graph of $f(x)$



Behaviors of Implicit Relations

✧ **Increasing/Decreasing Behavior**

Consider the following relation which is an ellipse with center (0,0). $4x^2 + 9y^2 = 36$

a. Use implicit differentiation to show that $\frac{dy}{dx} = -\frac{4x}{9y}$

b. Find all increasing/decreasing intervals

Behaviors of Implicit Relations

- **Concavity**

  Consider the following relation: $3(x - y) = 4 + 3\cos y$

Q1. Let $f(x)$ be a function such that $\frac{dy}{dx} = 3x - 2y - 8$. If $f(x)$ contains the point $(2, -1)$, which of the following best describes the point $(2, -1)$ on the graph of $\mathrm{y} = f(x)$?

(A) a relative minimum

(B) a relative maximum

(C) a point of inflection

(D) none of these

Q2. The points $(-1,-1)$ and $(1,-5)$ are on the graph of a relation whose derivative is $\frac{dy}{dx} = x^2 + y$. Which of the following must be true?

(A) $(1,-5)$ is a local maximum of the relation

(B) $(1,-5)$ is a point of inflection of the relation

(C) $(-1,-1)$ is a local maximum of the relation

(D) $(-1,-1)$ is a local minimum of the relation

(E) $(-1,-1)$ is a point of inflection of the relation

Q3. Concavity of an Implicit Relation

Consider a curve whose first derivative is defined as $\frac{dy}{dx} = \frac{3}{3+cos}$. Determine the concavity of the curve at points for which $0 < y < \pi$.

L'Hôpital's Rule

Limit Problem	Directly substituting
$\lim_{x \to 2} \frac{x-2}{x^2-4}$	$\frac{0}{0}$
$\lim_{x \to \infty} \frac{1-4x-5x^2}{3x^2-x-4}$	$\frac{-\infty}{\infty}$

indeterminate forms

$$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$$

$$0 \cdot (\pm\infty)$$

$$\infty - \infty, 0 - 0$$

$$1^{\infty}, 0^{0}, (\pm\infty)^{0}$$

L'Hôpital's Rule

If we have one of the two following cases:

For any real number, c, or for c having the value of infinity or negative infinity

- $\lim_{x \to c} \frac{f(x)}{g(x)}$ where both $\lim_{x \to c} f(x) = 0$ <u>and</u> $\lim_{x \to c} g(x) = 0$
- OR where both $\lim_{x \to c} f(x) \Rightarrow \pm\infty$ <u>and</u> $\lim_{x \to c} g(x) \Rightarrow \pm\infty$

indeterminate forms

$$\frac{\mathbf{0}}{\mathbf{0}}, \frac{\pm\infty}{\pm\infty}$$

$$\mathbf{0} \cdot (\pm\infty)$$

$$\infty - \infty, \mathbf{0} - \mathbf{0}$$

$$\mathbf{1}^{\infty}, \mathbf{0}^{\mathbf{0}}, (\pm\infty)^{\mathbf{0}}$$

then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

Example 1: Consider the following limits:

(1) $\lim_{x \to 0} \frac{\cos x - 1}{x^2 - x}$

(2) $\lim_{x \to \infty} \frac{e^x + x}{x^3}$

(3) $\lim_{x \to \frac{\pi}{2}} \frac{\sec x + 9}{\tan x}$

(4) $\lim\limits_{x \to \infty} x \tan \frac{1}{x}$ $\mathbf{0} \cdot (\pm\infty)$

$$(5) \lim_{x \to \infty} \left(x - \sqrt{x^2 + x} \right)$$

$$\infty - \infty$$

(6) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$

$\mathbf{1}^{\infty}$

(7) $\lim\limits_{x\to 0^+}(1+\sin 4x)^{\cot x}$

(8) $\lim_{x \to 0^+} (\tan x)^x$

$\mathbf{0^0}$

(9) $\lim_{x \to \infty} (x)^{\frac{1}{x}}$

$(\pm\infty)^{\mathbf{0}}$

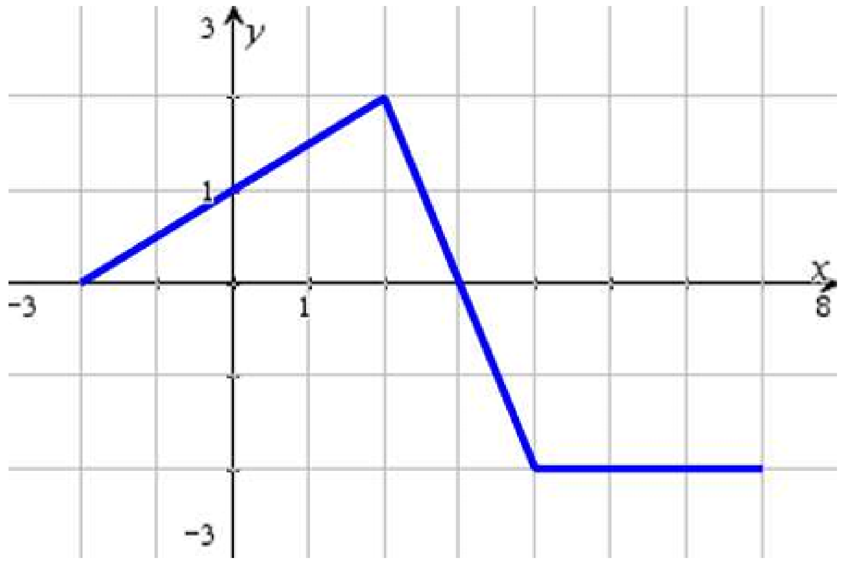
$$(10)\ \lim_{x\to 0} \frac{e^x}{x}$$

$$(11)\ \lim_{x\to\infty} \left(\frac{4x^2-5x+2}{e^{5x}+\ln x}\right)$$

Example 2: Find each of the following limits.

The graph of $f(x)$ is shown below. The functions $f(x)$ and $g(x)$ are differentiable for all x. Use the following table to help find the limit.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	-2	0	3

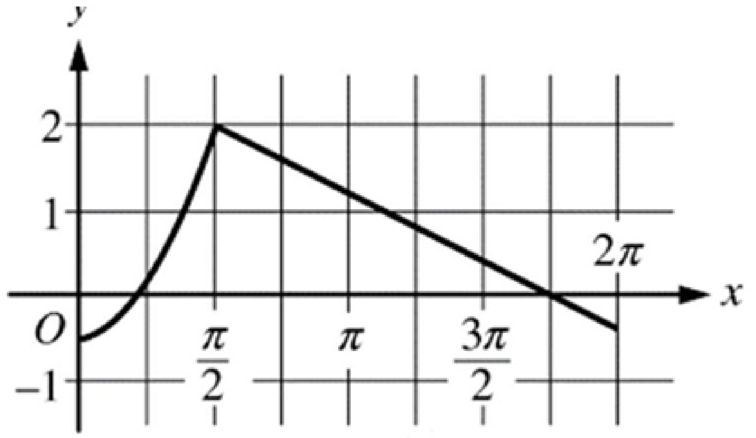


Find (1) $\lim_{x\to 2}\frac{f(x)-4}{g(x)\cdot x^2}$

(2) $\lim_{x\to 3}\frac{f(x)}{x^2-9}$

Example 3: From the 2018 AP Calculus Exam AB 5

Let f be a function defined by $f(x) = e^x \cos x$. Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g is shown below. Find the value of $\lim\limits_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it doesn't exist.



Graph of g'