- **Parametric Equations**

If x and y are both given as functions of a third variable, t, then the equations $\boldsymbol{x = f(t)}$ and $\boldsymbol{y = g(t)}$ are called **parametric equations** and $\boldsymbol{t}$ is the **parameter.**

✧ $\{(x, y)\} = \{f(t), g(t)\}$ This set of points is called the **parametric curve.**

✧ **Direction of path (or motion) of the curve**:

When the points are plotted in order of increasing values of t, the curve is traced out in a specific direction.

Example: $\frac{dy}{dx} = -\frac{x}{y}$

✧ $\boldsymbol{\frac{dy}{dx}}$ ***and*** $\boldsymbol{\frac{d^2y}{dx^2}}$

If the equation $x = f(t)$ and $y = g(t)$ define y as a differentiable function of x and $\frac{dx}{dt} \neq 0$, then

$\frac{dy}{dx} =$ ______________________________

$\frac{d^2y}{dx^2} =$ ______________________________

✧ **Horizontal & Vertical Tangent**

The curve represented by $x = f(t)$ and $y = g(t)$ has a

- horizontal tangent at $(f(t_0), g(t_0))$ if ______________________________
- vertical tangent at $(f(t_0), g(t_0))$ if ______________________________

Practice.

1. A curve in the plane is defined parametrically by the equations $x = t^2$ and $y = t^3 - 2t$
   (a) Sketch the curve according to your **calculator** in the xy −plane for $-2 \leq t \leq 2$. Indicate the direction in which the curve is traced as t increases.
   (b) For what values of t does the curve have a vertical tangent?
   (c) For what values of t does the curve have a horizontal tangent?
   (d) Find the equation of the tangent lines to the curve at $t = \pm 2$.

2. A particle moves in the xy $-$plane so that its position at any time t, $0 \le t \le 4$, is given by the equations $x(t) = \cos t + t\sin t$ and $y(t) = \sin t - t\cos t$.

(a) Using the **calculator** to sketch the curve according to your calculator in the xy $-$plane for $0 \le t \le 4$. Indicate the direction in which the curve is traced as t increases.

(b) At what time t, $0 < t < 4$, does the line tangent to the path of the particle have a slope of -1?

(c) At what time t, $0 < t < 4$, does $x(t)$ attain its maximum value? What is the position of the particle $(x(t), y(t))$ at this time?

(d) At what time t, $0 < t < 4$, is the particle on the y-axis?

3. (Calculator) An object moving along a curve in the *xy*-plane is in position $(x(t),\ y(t))$ at time $t \ge 0$ with $\frac{dx}{dt} = 2 - \sin(t^2)$. At time $t = 3$, the object is at position $(2, 7)$. What is the *x*-coordinate of the position of the object at time $t = 6$?

(A) 8.135 (B) 9.762 (C) 10.375 (D) 11.308

4. A particle moving along the curve is defined by the equation $y = x^3 - 4x^2 + 4$. The *x*-coordinate of the particle, $x(t)$, satisfies the equation $\frac{dx}{dt} = \frac{t}{\sqrt{t^2+9}}$, for $t \ge 0$ with initial condition $x(0) = 1$.

(a) Find $x(t)$ in terms of t.

(b) Find $\frac{dy}{dt}$ in terms of t.

(c) Find the location of the particle at time $t = 4$.

(d) Write an equation for the line tangent to the curve at time $t = 4$.

➢ **Arc Length in Parametric Form**

✧ **Review.**

- If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is $L =$ ________
- If g' is continuous on $[c, d]$, then the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is $L =$ ________

✧ **Parametric Form**

If a curve C is given by the parametric equations $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$, then the arc length or the distance traveled by a particle along the curve is given by:

$L =$ ______________________________________

The magnitude of displacement of a particle is the distance between its initial and final positions. The displacement of a particle between time $t = a$ and $t = b$ is given by:

$|Displacement| =$ ______________________________________

Practice.

1. **(Calculator)**

A particle moves in the xy-plane so that its position at any time t, for $0 \leq t$, is given by $x(t) = e^t$ and $y(t) = 2\cos(t)$.

(a) Find the distance traveled by the particle from $t = 0$ to $t = 2$.

(b) Find the magnitude of the displacement of the particle between time $t = 0$ and $t = 2$.

2. **(Calculator)**

A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at any time t, where $\frac{dx}{dt} = 2\sin(t^2)$ and $\frac{dy}{dt} = \cos(t^3)$. At time $t = 1$, the object is at position $(3, 2)$.

(a) Write an equation for the line tangent to the curve at $(3, 2)$.

(b) Find the total distance traveled by the particle from $t = 1$ to $t = 3$.

(c) Find the position of the particle at time $t = 3$.

(d) Find the magnitude of the displacement of the particle between $t = 1$ and $t = 3$.

➢ Vector Valued Functions

If a particle moves in the xy −plane so that at time $t > 0$ its position vector is given by $r(t) =< x(t), y(t) >$, then the velocity vector, acceleration vector, and speed at time t are:

- ✧ **Velocity** $= v(t) = r'(t) =$ ____________
- ✧ ***Acceleration*** $= a(t) = r''(t) =$ ____________
- ✧ **Speed** $= |v(t)| =$ __________________

- ✧ $x(t)$ is increasing when ____________________

  $y(t)$ is increasing when ____________________

Practice.

1. A particle moving in the xy-plane is defined by the vector-valued function $f(t) =< t - \sin t, 1 - \cos t >$, for $0 \le t \le \pi$.

   (a) Find the velocity vector for the particle at any time t.

   (b) Find the speed of the particle when $t = \dfrac{\pi}{3}$.

   (c) Find the acceleration vector for the particle at any time t.

   (d) Find the average speed of the particle from time $t = 0$ to time $t = \pi$.

2. In the xy- plane, a particle moves along the curve defined by the equation $y = 2x^4 - x$ with a constant speed of 20 units per second. If $\dfrac{dy}{dt} > 0$, what is the value of $\dfrac{dx}{dt}$ when the particle is at the point $(1,\ 1)$

   (A) $\sqrt{2}$ (B) 2 (C) $2\sqrt{2}$ (D) 4

3. An object moving along a curve in the xy-plane is at position $(x(t),\ y(t))$ at time t, where $\frac{dx}{dt} = 1 + \cos(e^{t})$.

and $\frac{dy}{dt} = e^{(2-t^2)}$ for $t \geq 0$.

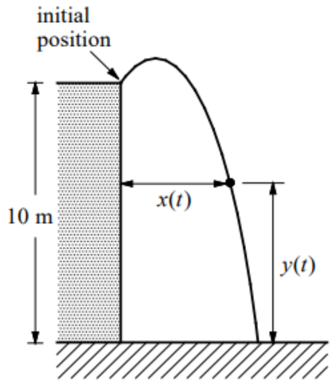
(a) At what time t is the speed of the object 3 units per second?

(b) Find the acceleration vector at time $t = 2$.

(c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 4$.

(d) Find the magnitude of the displacement of the object over the time interval $1 \leq t \leq 4$.

4.



Note: Figure not drawn to scale.

An object is thrown upward into the air 10 meters above the ground. The figure above shows the initial position of the object and the position at a later time. At time t seconds after the object is thrown upward, the horizontal distance from the initial position is given by $x(t)$ meters, and the vertical distance from the ground is given by $y(t)$ meters, where $\frac{dx}{dt} = 1.4$ and $\frac{dy}{dt} = 4.2 - 9.8t$, for $t \geq 0$.

(a) Find the time t when the object reaches its maximum height.

(b) Find the maximum vertical distance from the ground to the object.

(c) Find the time t when the object hit the ground.

(d) Find the total distance traveled by the object from time $t = 0$ until the object hit the ground.

(e) Find the magnitude of the displacement of the object from time $t = 0$ until the object hit the ground.

(f) Find the angle θ, $0 < \theta < \frac{\pi}{2}$, between the path of the object and the ground at the instance the object hit the ground.

➢ Polar Coordinates and Slopes of Curves

✧ If a point P has rectangular coordinates (x, y) and polar coordinates (r, θ), then

$(r, \theta) \to (x, y)$: $\begin{cases} x = \_\_\_\_\_\_\_\_\_\_ \\ y = \_\_\_\_\_\_\_\_\_\_ \end{cases}$ $(x, y) \to (r, \theta)$: $\begin{cases} r = \_\_\_\_\_\_\_\_\_\_ \\ \tan\theta = \_\_\_\_\_\_\_\_\_\_ \end{cases}$

✧ Some polar curves

1. Sketch a graph of the equation $r = 3$

2. Sketch a graph of the equation $\theta = \frac{\pi}{3}$

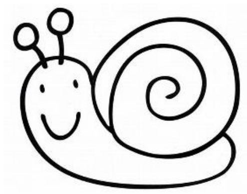
3. Sketch a graph of the polar equation $r = 2\sin\theta$

4. Sketch a graph of $r = 2 + 2\cos\theta$

✧ **Cardioid (heart-shaped)**: The graph of any equation of the form $r = a(1 \pm \cos\theta)$ or $r = a(1 \pm \sin\theta)$ is a cardioid.

✧ **Limaçon**

5. Sketch a graph of the equation $r = 1 + 2\cos\theta$



✧ **Roses**

6. Sketch the curve $r = \cos 2\theta$

7. Sketch the curve $r = \sin 3\theta$

✧ In general, the graph of an equation of the form $r = \mathrm{a} \cos \mathrm{n}\theta$ is an _____-leaved rose if n is ________________.

the graph of an equation of the form $r = \mathrm{a} \sin \mathrm{n}\theta$ is an _____-leaved rose if n is ________________.

✧ **Symmetry**

- If a polar equation is unchanged when we replace θ by $-\theta$, then the graph is symmetric about the polar axis.

8. Sketch a graph of $r = 3 \cos \frac{\theta}{2}$

- If the equation is unchanged when we replace r by $-r$, then the graph is symmetric about the pole.

9. Sketch a graph of $r^2 = \cos 2\theta$

✧ The slope of the tangent line to the graph of $r = f(\theta)$ at point (r, θ) is:

$\frac{dy}{dx} =$

✧ If $r > 0$ and

- $\frac{dr}{d\theta} < 0$, then r is increasing/decreasing, which means the curve is getting closer/farther from the origin.
- $\frac{dr}{d\theta} > 0$, then r is ________________, which means the curve is getting ______________ from the origin.

Practice.

1. A curve is defined by the polar equation $r = 4\sin(2\theta)$ for $0 \le \theta \le \frac{\pi}{2}$.

   (a) Graph the curve.

   (b) Find the slope of the curve at the point where $\theta = \pi/4$.

   (c) Find an equation in terms of x and y for the line tangent to the curve at the point where $\theta = \frac{\pi}{4}$.

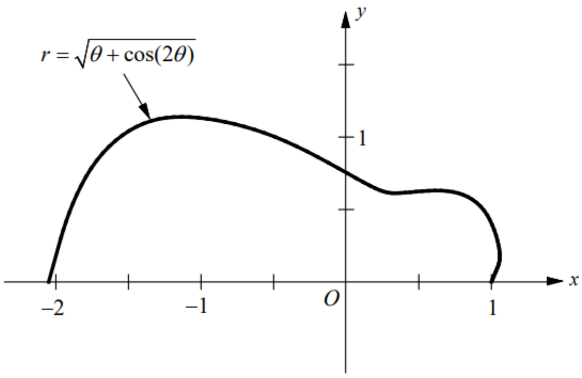
   (d) Find an interval where the curve is getting closer to the origin.

   (e) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ such that the point on the curve has the greatest distance from the origin.

2. (cal) The equation of the polar curve is given by $r = \frac{8}{1 - \cos\theta}$. What is the angle θ that corresponds to the point on the curve with x-coordinate -3?

   (A) 1.248 (B) 1.356 (C) 1.596 (D) 2.214

3.



The polar curve $r = \sqrt{\theta + \cos(2\theta)}$, for $0 \le \theta \le \pi$, is drawn in the figure above.

(a) Find $\dfrac{dr}{d\theta}$, the derivative of r with respect to θ.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate 0.5.

(c) For $\dfrac{\pi}{12} < \theta < \dfrac{5\pi}{12}$, $\dfrac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \dfrac{\pi}{2}$ that correspond to the point on the curve in the first quadrant with the least distance from the origin. Justify your answer.

➢ **Area in Polar Coordinates**

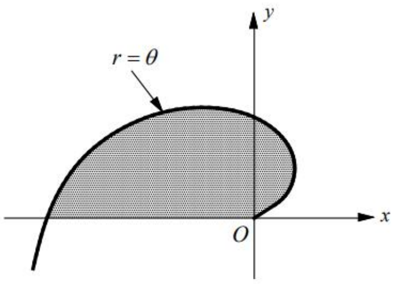
The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by:

$A =$

Practice.

1. Find the area of the region that lies inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.

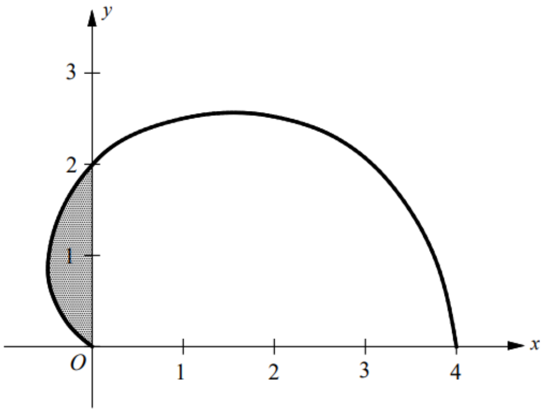
2.



The area of the shaded region bounded by the polar curve $r = \theta$ and the x-axis is

(A) $\frac{\pi^2}{4}$ (B) $\frac{\pi^3}{6}$ (C) $\frac{\pi^3}{3}$ (D) $\frac{\pi^3}{2}$

3.



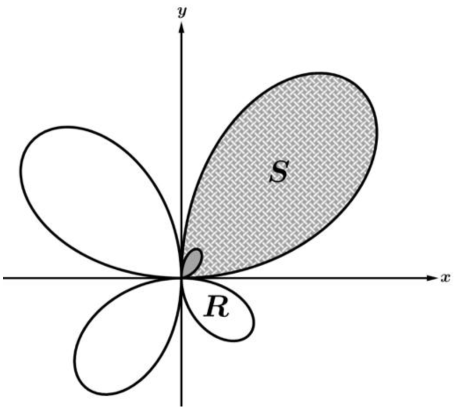
The graph of the polar curve $r = 2 + 2\cos(\theta)$ for $0 \le \theta \le \pi$ is shown above.

(a) Write an integral expression for the area of the shaded region.

(b) Write expressions for $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the curve at the point where $\theta = \dfrac{\pi}{2}$.

4.



The graph of the polar curve $r(\theta) = \theta \sin(\theta)\cos(\theta)$ is shown in the figure above for $0 \le \theta \le \frac{5\pi}{2}$. Let S be the shaded region in the outer loop of the graph of $r(\theta)$ and also outside of the inner loop of $r(\theta)$, as shown in the figure above. Let R be the region bounded by the graph of $r(\theta)$ in quadrant IV, as shown in the figure.

a. Find the area of S.

b. Find the area of R.

c. For $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$, $r(\theta) < 0$ and $\frac{dr}{d\theta} < 0$. What do these facts say about the curve relative to the origin?

Circles & Spiral	$r = 3$	$\theta = \frac{\pi}{3}$	$r = 4\cos\theta$	$r = 2\theta$
Limaçon $r = a \pm b\sin\theta$ $r = a \pm b\cos\theta$	$r = 1 + 2\sin\theta$	$r = 2 + 2\sin\theta$	$r = 2 + \sin\theta$	
Roses $r = a\sin n\theta$ $r = a\cos n\theta$	$r = a\sin 2\theta$	$r = a\sin 3\theta$	$r = a\cos 4\theta$	$r = a\cos 5\theta$