/ Sequence, Telescoping, Geometric Series, Def (S_n)

C 1. $\sum_{n=1}^{\infty}\frac{(3)^{n+1}}{5^n} = \sum_{n=1}^{\infty} 3\left(\frac{3}{5}\right)^n$

geometric series with $r=\frac{3}{5} \in (-1,1)$

$S_n = \frac{\frac{9}{5}\cdot(1-(\frac{3}{5})^n)}{1-\frac{3}{5}} = \frac{9}{5}\cdot\frac{5}{2}\cdot(1-(\frac{3}{5})^n) \to \frac{9}{2}$

(A) $\frac{3}{5}$ (B) $\frac{5}{2}$ (C) $\frac{9}{2}$ (D) The series diverges

A 2. If $f(x)=\sum_{n=1}^{\infty}(\tan x)^n$, then $f(1) = \sum_{n=1}^{\infty}(\tan 1)^n$

common ratio $= \tan 1 \approx 0.017 \in (-1,1)$

$S_n = \frac{\tan 1}{1-\tan 1}\cdot(1-(\tan 1)^n) \to \frac{\tan 1}{1-\tan 1} \approx -2.794$

(A) -2.794 (B) -0.61 (C) 0.177 (D) The series diverges

D 3. $\sum_{n=2}^{\infty}\frac{2}{n^2-1} = \sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right) = 1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\cdots+\frac{1}{n-3}-\frac{1}{n-1}+\frac{1}{n-2}-\frac{1}{n}+\frac{1}{n-1}-\frac{1}{n+1}$

$\left(1+\frac{1}{2}+\cancel{\frac{1}{3}}+\cancel{\frac{1}{4}}+\cdots+\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}-\cdots-\cancel{\frac{1}{n-1}}-\frac{1}{n}-\frac{1}{n+1}\right)$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{3}{2}$

$= \frac{3}{2}-\frac{1}{n}-\frac{1}{n+1} \to \frac{3}{2}$

B 4. The sum of the geometric series $\frac{2}{21}+\frac{4}{63}+\frac{8}{189}+\ldots$ is

$\frac{1}{7}\cdot\frac{2}{3}+\frac{1}{7}\cdot(\frac{2}{3})^2+\frac{1}{7}\cdot(\frac{2}{3})^3+\cdots$

$\frac{a_2}{a_1}=\frac{2}{3}$

$S_n = \frac{1}{7}\cdot\frac{2}{3}\cdot\frac{(1-(\frac{2}{3})^n)}{1-\frac{2}{3}}$

(A) $\frac{5}{21}$ (B) $\frac{2}{7}$ (C) $\frac{4}{7}$ (D) The series diverges

$\to \frac{1}{7}\cdot\frac{2}{3}\cdot 3 = \frac{2}{7}$

A 5. If $S_n=\left(\frac{3^{n-1}}{(4+n)^{20}}\right)\left(\frac{(7+n)^{20}}{3^n}\right)$, to what number does the sequence $\{S_n\}$ converge?

(A) $\frac{1}{3}$ (B) $\frac{7}{4}$ (C) $\left(\frac{7}{4}\right)^{20}$ (D) Diverges

$S_n = \frac{1}{3}\cdot\left(\frac{7+n}{4+n}\right)^{20} \to \frac{1}{3}$ as $n\to\infty$

D 6. Which of the following sequences converge?

$a_n \to \infty$ as $n\to\infty$

I. $\left\{\frac{\cos^2 n}{(1.1)^n}\right\} \to 0$ II. $\left\{\frac{e^n-3}{3^n}\right\} \to 0$ III. $\left\{\frac{n}{9+\sqrt{n}}\right\}$

(A) I only (B) II only (C) III only (D) I and II only

C 7. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n}{10(n+1)}$ × II. $\sum_{n=1}^{\infty} \arctan n \to \frac{\pi}{2} \neq 0$ as $n\to\infty$ × III. $\sum_{n=1}^{\infty} \frac{-6}{(-5)^n}$ ✓ $-\frac{1}{5} \in (-1,1)$

(A) I only (B) II only (C) III only (D) II and III only

I. $\lim_{n\to\infty} a_n = \frac{1}{10} \neq 0$

8. Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+3)} + \frac{1}{7^n}\right)$.

$S_n = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+3} + (\frac{1}{7})^n$

$= 1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \cdots + \frac{\frac{1}{7}(1-(\frac{1}{7})^n)}{1-\frac{1}{7}}$

$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{4} - \frac{1}{5} - \cdots - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} + \frac{1}{6}(1-(\frac{1}{7})^n)$

$= \frac{11}{6} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} + \frac{1}{6}(1-(\frac{1}{7})^n) \to \frac{11}{6} + \frac{1}{6} = 2$ as $n\to\infty$

9. Find the sum of the series $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}$

$S_n = 2 \cdot \frac{2}{3} \cdot \frac{(1-(\frac{2}{3})^n)}{1-\frac{2}{3}} \to 4$ as $n\to\infty$

$\therefore \sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = 4$

10. Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{n}{2n+3}$ (b) $\sum_{n=1}^{\infty} 2^{-n}5^n$

(a) $a_n \to \frac{1}{2}$ as $n\to\infty$

$\therefore$ divergent by the nth term test

(b) $a_n = (\frac{5}{2})^n$

Σa_n is a geometric series with common ratio $= \frac{5}{2} > 1$

$\therefore$ divergent

Integral Test, p-series

C 1. If $\int_1^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{4}$, then which of the following must be true?

I. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ diverges. ✗

II. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges. ✓

III. $\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{4}$ ✗

$\int_1^{\infty} \frac{1}{x^2+1}\, dx = [\arctan x]_1^{\infty} = \frac{\pi}{4}$

(A) none (B) I only (C) II only (D) II and III only

D 2. What are all values of p for which $\int_1^{\infty} \frac{1}{\sqrt[3]{x^p}}$ converges?

(A) $P<-3$

(B) $P<-1$

(C) $P>1$

(D) $P>3$

$x^{\frac{p}{3}}$ $\frac{p}{3} > 1$

$p > 3$

B 3. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$ ✗ II. $\sum_{n=1}^{\infty} ne^{-n^2}$ ✓ III. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ ✗

$\int_2^{\infty} \frac{1}{x \ln x}\, dx$

$= \int_2^{\infty} \frac{1}{\ln x}\, d(\ln x)$

$= [\ln(\ln x)]_2^{\infty} = \infty$

(A) I only (B) II only (C) III only (D) I and II only

I. $\int_1^{\infty} \frac{x}{2x^2+1}\, dx$

$= \int_1^{\infty} \frac{\frac{1}{4}}{2x^2+1}\, d(2x^2+1) = \frac{1}{4}\left[\ln|2x^2+1|\right]_1^{\infty} = \infty$

II. $\int_1^{\infty} x \cdot e^{-x^2}\, dx$

$= \int_1^{\infty} \frac{1}{-2} e^{-x^2}\, d(-x^2)$

$= -\frac{1}{2} \cdot [e^{-x^2}]_1^{\infty}$

$= -\frac{1}{2}(0 - e^{-1})$

$= \frac{1}{2}e^{-1}$

$f(x) = x \cdot e^{-x^2}$ positive, ↓, ctns on $[1, +\infty)$

(*)

D 4. What are all values of p for which $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^p+1}$ converges?

(A) $p>0$ (B) $p>\frac{1}{2}$ (C) $p>1$ (D) $p>\frac{3}{2}$

$\frac{\sqrt{n}}{n^p+1} \sim \frac{1}{n^{p-\frac{1}{2}}+1}$

$p - \frac{1}{2} > 1$

$p > \frac{3}{2}$

$1 + 2^{\frac{k}{2}} + 3^{\frac{k}{2}} + 4^{\frac{k}{2}} + \cdots + n^{\frac{k}{2}} + \cdots$

A 5. What are all values of k for which the series $1+(\sqrt{2})^k+(\sqrt{3})^k+(\sqrt{4})^k+\cdots+(\sqrt{n})^k+\cdots$ converges?

$-\frac{k}{2} > 1$

(A) $k<-2$ (B) $k<-1$ (C) $k>1$ (D) $k>2$

$k < -2$

6. Determine whether the following series converge or diverge.

(a) $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\cdots$

(a) $1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots+\frac{1}{n^3}+\cdots$

It's a p-series with $p=3>1$

$\therefore$ convergent.

(b) $1+\frac{1}{(\sqrt[3]{2})^2}+\frac{1}{(\sqrt[3]{3})^2}+\frac{1}{(\sqrt[3]{4})^2}+\cdots$

(b) $a_n = \frac{1}{n^{\frac{2}{3}}}$

Σa_n is a p-series with $p=\frac{2}{3}<1$

$\therefore$ divergent

7. Determine whether the series is convergent or divergent. $\sum_{n=1}^{\infty} n^{1-\pi}$

This is a p-series with $p=\pi-1>1$

$\therefore$ Convergent

D

Comparison Test

D 1. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{1}{n^2+n+3}$ $\sim \sum \frac{1}{n^2}$ ✓ II. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+2}$ $\sim \sum \frac{1}{n^2}$ ✓ III. $\sum_{n=1}^{\infty} \frac{1+4^n}{3^n}$ $\sim \sum \left(\frac{4}{3}\right)^n$ ×

(A) I only (B) II only (C) III only (D) I and II only

C 2. Which of the following series diverge?

I. $\sum_{n=1}^{\infty} \frac{1}{n!}$ $\leq \sum_{n=1}^{\infty} \frac{1}{n(n-1)}$ for $n \geq 10$ ✓ II. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ $\sim \frac{1}{n^{\frac{1}{2}}}$ $p=\frac{1}{2}<1$ div × III. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ $\sim \frac{1}{n}$ × $\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$

(A) I only (B) II only (C) II and III only (D) I, II, and III

D 3.Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n^{3/2}}{3n^3+7}$ $\sim \frac{1}{3n^{\frac{3}{2}}}$ ✓ II. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}+1}$ $\sim \frac{1}{n^{\frac{4}{3}}}$ ✓ III. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $= \frac{n(n-1)\cdots}{n\cdot n\cdots} \leq \frac{n\cdot n\cdots n\cdot 2\times 1}{n\cdot n\cdots n\cdot n\cdot n} = \frac{1}{n^2}$

(A) I only (B) I and II only (C) I and III only (D) I, II, and III

B 4. Which of the following series cannot be shown to converge using the limit comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$?

(A) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

(B) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(C) $\sum_{n=1}^{\infty} \frac{2n}{2^{n+1}\sqrt{n^2+1}}$

(D) $\sum_{n=1}^{\infty} \frac{2n^2-3n}{2^n(n^2+n-100)}$

5. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{\cos(2n)}{1+(1.6)^n}$

(b) $\sum_{n=1}^{\infty} \frac{4^n}{2^n+3^n}$

(a) $0 < \left|\frac{\cos(2n)}{1+(1.6)^n}\right| \leq \left(\frac{1}{1.6}\right)^n$ for $n \geq 1$

$\Sigma \left(\frac{1}{1.6}\right)^n$ is a geometric series with $|q| = \frac{1}{1.6} < 1$

so, it is convergent.

Therefore $\sum_{n=1}^{\infty} \frac{\cos(2n)}{1+(1.6)^n}$ is (absolutely) convergent

by the Direct Comparison Test

(b) $\frac{4^n}{2^n+3^n} \geq \frac{4^n}{3^n+3^n} = \frac{1}{2}\cdot\left(\frac{4}{3}\right)^n > 0$ for $n \geq 1$

$\Sigma \frac{1}{2}\cdot\left(\frac{4}{3}\right)^n$ is a divergent geometric series since its common ratio

is $\frac{4}{3} > 1$.

Hence the series given is divergent by the Direct Comparison Test.

6. Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$

(b) $\sum_{n=3}^{\infty} \frac{2^n}{3^n+1}$

(a) $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+4}}}{\frac{1}{n}} = 1$

$\Sigma \frac{1}{n}$ is divergent since it's a p-series with $p=1$

Therefore, $\Sigma \frac{1}{\sqrt{n^2+4}}$ is divergent by the Limit Comparison Test

(b) $\lim_{n\to\infty} \frac{\frac{2^n}{3^n+1}}{\left(\frac{2}{3}\right)^n} = 1$

$\Sigma \left(\frac{2}{3}\right)^n$ is a geometric series with $|q| = \left|\frac{2}{3}\right| < 1$

So, it's convergent

Therefore, $\sum_{n=3}^{\infty} \frac{2^n}{3^n+1}$ is convergent by the Limit Comparison Test.

D 1. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n}$ ✓ II. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$ ✓ III. $\sum_{n=1}^{\infty} \cos(n\pi)$ $(-1)^n$ ×

(A) I only (B) II only (C) III only (D) I and II only

C 2. Which of the following series converge?

I. $\sum_{n=1}^{\infty} (-1)^n \cos(\frac{\pi}{n})$ × II. $\sum_{n=1}^{\infty} \sin(\frac{2n-1}{2})\pi$ × III. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n^2+1}$ ✓

$\lim_{n\to\infty} (-1)^n \cos(\frac{\pi}{n}) = \pm 1 \neq 0$

(A) I only (B) II only (C) III only (D) I and II only

C 3. For what integer k, $k>1$, will both $\sum_{n=1}^{\infty} \frac{(-1)^{kn}}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{n^2\sqrt{n}}{n^k+1}$ converge? $\sim \sum \frac{n^{\frac{5}{2}}}{n^k} = \sum \frac{1}{n^{k-\frac{5}{2}}}$

✓ k is odd

$k-\frac{5}{2}>1$

$k>\frac{7}{2}$

(A) 3 (B) 4 (C) 5 (D) 6

B 4. Let $s=\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ and s_n be the sum of the first n terms of the series. If $|s-s_n|<\frac{1}{500}$ what is the smallest value of n?

$a_{n+1} = \frac{1}{(n+1)^3} < \frac{1}{500}$

(A) 6 (B) 7 (C) 8 (D) 9

$n=6: \frac{1}{7^3} = \frac{1}{343}$

$n=7: \frac{1}{8^3} = \frac{1}{512}$ ✓

D 5. Which of the following series converge?

$\lim_{n\to\infty} (-1)^n 3^{\frac{1}{n}} = \pm 1 \neq 0$

I. $\sum_{n=2}^{\infty} (-1)^n \sqrt[n]{3}$ II. $\sum_{n=1}^{\infty} \frac{3^{n+1}}{\pi^n}$ ✓ $3\cdot(\frac{3}{\pi})^n$ <1 III. $\sum_{n=1}^{\infty} (\tan^{-1}(n+1) - \tan^{-1}(n))$ ✓

$3^{\frac{1}{n}}$ $n\uparrow.\ 3^{\frac{1}{n}}\downarrow$

$S_n = \tan^{-1}2 - \tan^{-1}1 + \tan^{-1}3 - \tan^{-1}2 + \cdots + \tan^{-1}(n+1) - \tan^{-1}(n)$

$= -\tan^{-1}1 + \tan^{-1}(n+1)$

$\to -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$ as $n\to\infty$

(A) I only (B) II only (C) III only (D) II and III only

A 6. Which of the following statements about the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+3)}{n^2}$ is true?

$\sim \sum (-1)^{n+1} \frac{1}{n}$

(A) The series converges conditionally.
(B) The series converges absolutely.
(C) The series converges but neither conditionally nor absolutely.
(D) The series diverges.

B 7. Which of the following series is absolutely convergent?

(A) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n}$ $\sim \sum (-1)^n \cdot n^{-\frac{1}{2}}$

(B) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2\sqrt{n}}$

(C) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2-\sqrt{n}}$ $\sim \sum (-1)^n \cdot \frac{1}{n}$

(D) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n^2+1)}{n^3}$ $\sim \sum (-1)^n \cdot \frac{1}{n}$

C 8. An alternating series is given by $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+3}$. Let S_3 be the sum of the first three terms of the given alternating series. Of the following, which is the smallest number $\underline{\underline{M}}$ for which the alternating series error bound guarantees that $|S - S_3| \le M$?

$a_4 = \frac{1}{19}$

(A) $\frac{1}{4}$ (B) $\frac{1}{7}$ (C) $\frac{1}{19}$ (D) $\frac{1}{28}$

9. Let $f(x) = 1 - \frac{3x}{2!} + \frac{9x^2}{4!} - \frac{27x^3}{6!} + \cdots + \frac{(-1)^n (3x)^n}{(2n)!} + \cdots$.

Use the alternating series error bound to show that $1 - \frac{3}{2!} + \frac{9}{4!}$ approximates $f(1)$ with an error less than $\frac{1}{20}$.

$\frac{1}{(2(n+1))!} \le \frac{1}{(2n)!}$

$f(1) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}$

$R_3 \le a_4 = \frac{27}{6!} = \frac{3\times\cancel{3}\times\cancel{3}}{\cancel{6}\times 5\times 4\times\cancel{3}\times\cancel{3}\times 2\times 1} = \frac{3}{80} < \frac{1}{20}$

B 1. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ $\left|\frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}}\right| \to \infty$ ✗

II. $\sum_{n=1}^{\infty} \frac{n}{3^n}$ $\left|\frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}}\right| \to \frac{1}{3} <$ ✓

III. $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ $\left|\frac{\frac{n+1}{(\frac{2}{3})^{n+1}}}{\frac{n}{(\frac{2}{3})^n}}\right| \to \frac{3}{2} > 1$ ✗

(A) I only (B) II only (C) II and III only (D) I, II, and III

D 2. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right|$ ✓

II. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

III. $\sum_{n=1}^{\infty} \frac{n^9}{9^n}$ ✓

(A) I only (B) II only (C) I and II only (D) I, II, and III

3. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n!}{n\,2^n}$

(b) $\sum_{n=0}^{\infty} \frac{\cos^n x}{2^n}$ ✓

(c) $\sum_{k=1}^{\infty} \frac{3^k k!}{(k+3)!}$

(a) $\lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{n+1 \cdot 2^{n+1}}}{\frac{n!}{n\,2^n}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{2}\right| = \infty$ (DNE) Divergent by the ratio test

(b) $\lim_{n\to\infty}\left|\frac{\frac{\cos^{n+1} x}{2^{n+1}}}{\frac{\cos^n x}{2^n}}\right| \leq \frac{1}{2} < 1$ ∴ convergent

(c) $\lim_{k\to\infty}\left|\frac{\frac{3^{k+1}\cdot(k+1)!}{(k+4)!}}{\frac{3^k\, k!}{(k+3)!}}\right|$

$= \lim_{k\to\infty}\left|\frac{3\cdot k+1}{k+4}\right| = 3 > 1$

∴ divergent.

Q2. I. $\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\left(\frac{n}{n+1}\right)^n\right|$

$\to \frac{1}{e} < 1$

∴ convergent.

$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$

II. $\lim_{n\to\infty}\left|\frac{\frac{(n+1)!\,(n+1)!}{(2n+2)!}}{\frac{n!\,n!}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cdot(n+1)}{(2n+2)(2n+1)}\right| = \frac{1}{4} < 1$

III. $\lim_{n\to\infty}\left|\frac{\frac{(n+1)^9}{9^{n+1}}}{\frac{n^9}{9^n}}\right| = \frac{1}{9} < 1$