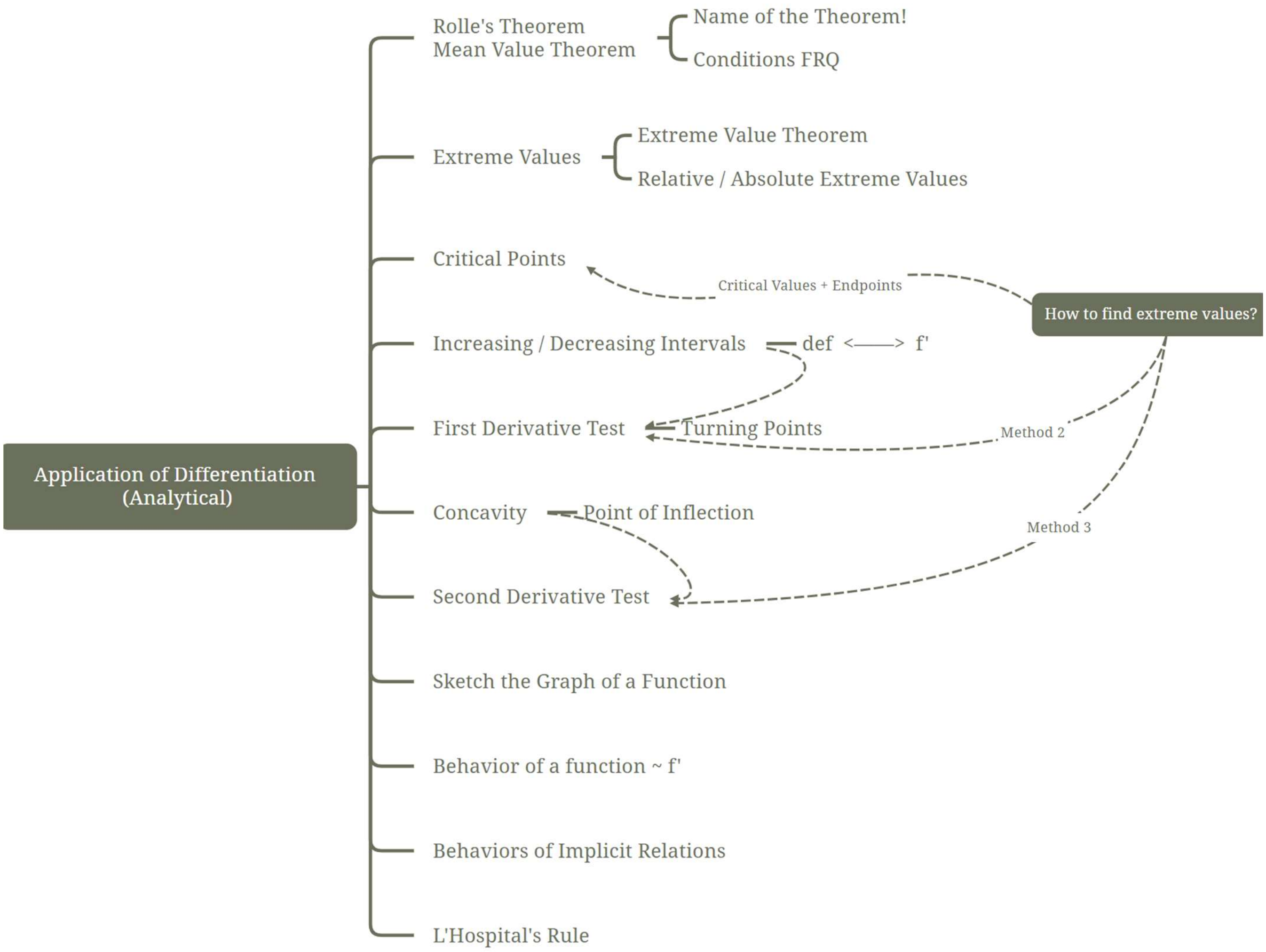
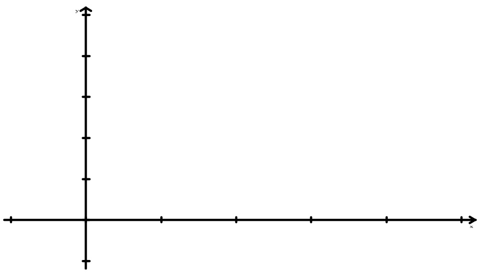

Application of Differentiation (Analytical)
Rolle's Theorem
Mean Value Theorem
Name of the Theorem!
Conditions FRQ
Extreme Values
Extreme Value Theorem
Relative / Absolute Extreme Values
Critical Points
Critical Values + Endpoints
How to find extreme values?
Increasing / Decreasing Intervals
def <——> f'
First Derivative Test
Turning Points
Method 2
Concavity
Point of Inflection
Method 3
Second Derivative Test
Sketch the Graph of a Function
Behavior of a function ~ f'
Behaviors of Implicit Relations
L'Hospital's Rule

☺ **Activity**

Step 1: Place two points anywhere on the coordinate plane below that have the <u>same y-values</u>.

Step 2: Connect the two points with a continuous function that is also differentiable.



There MUST be at least one point on your function where you can draw a tangent line that is horizontal. (i.e. the slope is zero)

➢ **Rolle's Theorem**

Let f be a function that is <u>continuous</u> on the <u>closed</u> interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

Q1. Determine if Rolle's Theorem applies to $f(x) = x^4 - 2x^2$ on the interval $[-2,2]$.

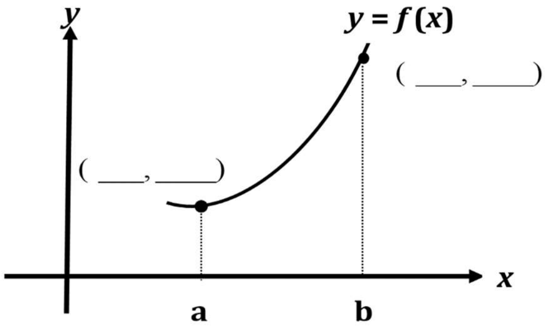
a. State thoroughly the reasons why or why not the theorem applies.

b. If the theorem does apply, find the value of c guaranteed by the theorem.

Q2. Let f be the function given by $f(x) = x^3 - 9x + 1$. Find all numbers c that satisfies the conclusion of Rolle's Theorem for f, such that $f'(c) = 0$ on the closed interval [0,3]

- **The Mean Value Theorem**

  If f is <u>continuous</u> on the <u>closed</u> interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b)such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Q3. Determine if The Mean Value Theorem applies to $f(x) = 3 - \frac{5}{x}$ on the interval $[1,5]$.

a. State thoroughly the reasons why or why not the theorem applies.

b. If the theorem does apply, find the value of c guaranteed by the theorem.

Q4. Find the equation of the tangent line to the graph of $f(x) = 2x + \sin x + 1$ on the interval $[0, \pi]$ at the point which is the solution to the Mean Value Theorem.

Q5. Explain precisely why we cannot apply the Mean Value Theorem to either of the three functions below on the provided intervals.

a. $f(x) = 3x - |x - 3|$on $[2,5]$

b. $g(x) = \frac{2}{x+2}$ on$[-3,1]$

- **Extrema of a Function**

A function does not have to have a maximum or a minimum on an interval.
For instance, in Figure 1 you can see that the function $f(x) = x^2 + 1$ has both a maximum and a minimum on the closed interval $[-1,2]$. But in Figure 2 there is no maximum on the open interval (-1,2). There is no minimum on the open interval (-1,2) in Figure 3.

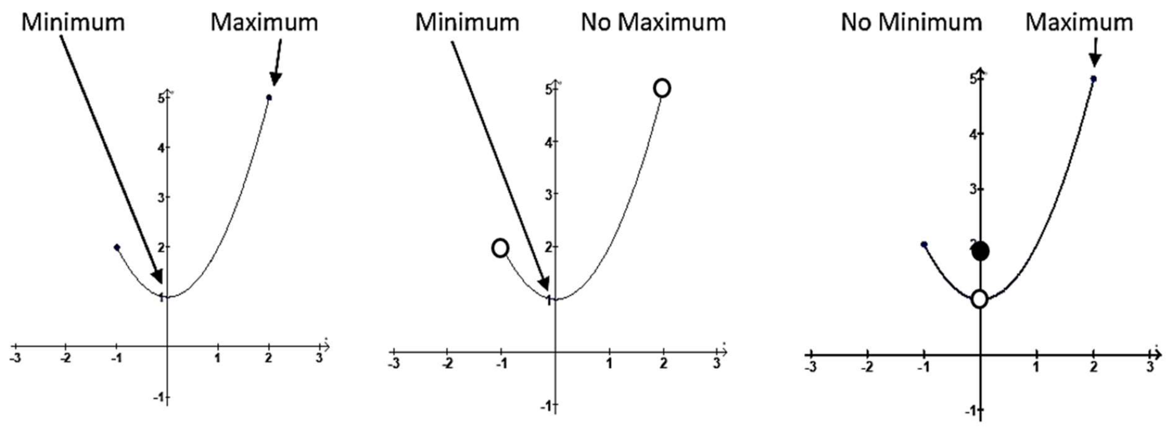


Figure 1 Figure 2 Figure 3

- **Extrema:** plural form of extreme, i.e. all maximum and minimum values
- **Minima:** plural of minimum
- **Maxima:** plural of maximum

- **The Extreme Value Theorem**

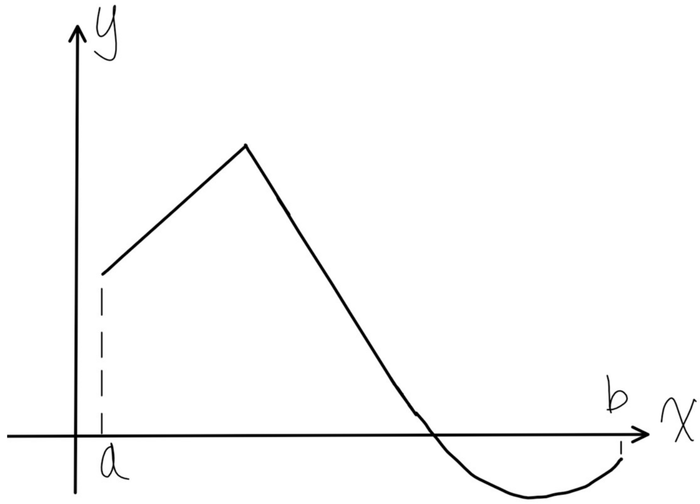
If f is ______________ on a ___________ **interval**, then f has BOTH a maximum and a minimum on the interval.

- **Absolute and Relative Extrema**

- **Def.** A function f has an **absolute minimum** at c if $f(c) \le f(x)$ for all x in the domain of f.
  A function f has an **absolute maximum** at c if $f(c) \ge f(x)$ for all x in the domain of f.

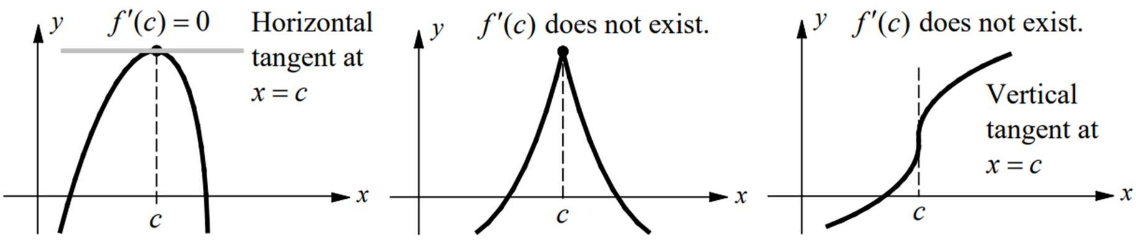
- **Def.** A function f has a **relative minimum** at c if $f(c) \le f(x)$ for all x in the vicinity of c.
  A function f has a **relative maximum** at c if $f(c) \ge f(x)$ for all x in the vicinity of c.

? How to find the relative maxima and relative minima of $f(x) = x^2 - 3x - 4$ on the interval [-4,9]



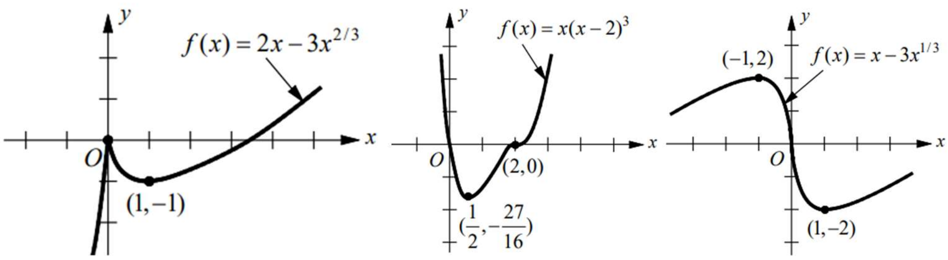
- **Critical Numbers**

  Let f be defined at c. If $f'(c) = \mathbf{0}$ or if $f'(c)$ **does not exist**, then c is a **critical number** of f.



- **Relative Extrema Occur Only at Critical Numbers**

  If f has a relative minimum or relative maximum at x= c, then c is a critical number of f.



Q1. If f is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$, which of the following **could** be false?

(A) $f'(c) = 0$ for some c such that $a < c < b$.

(B) f has a minimum value on $a \leq x \leq b$.

(C) f has a maximum value on $a \leq x \leq b$.

Q2. The function f is defined on the closed interval $[0,1]$ and satisfies $f(0) = f\left(\frac{1}{2}\right) = f(1)$. On the open interval $(0,1)$, f is continuous and strictly increasing. Which of the following statements is true?

(A) f attains both a minimum value and a maximum value on the closed interval [0,1].

(B) f attains a minimum value but not a maximum value on the closed interval [0,1].

(C) f attains a maximum value but not a minimum value on the closed interval [0,1].

(D) f attains neither a minimum value nor a maximum value on the closed interval [0,1].

Q3. Let g be a function given by $g(x) = x^2e^{kx}$, where k is a constant. For what value of k does g have a critical point at $x = \frac{3}{2}$?

(A) $-\frac{4}{3}$ (B) $-\frac{2}{3}$ (C) $\frac{2}{3}$ (D) 0

➢ **Guidelines for Finding Absolute Extrema on a Closed Interval**

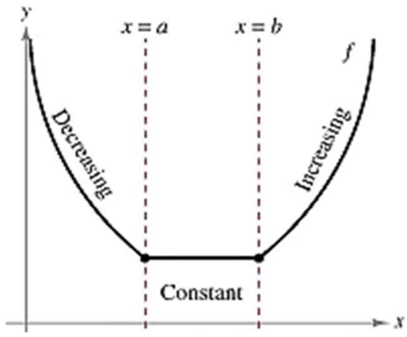
To find extrema of a continuous function f on a closed interval $[a,b]$, use the following steps.
1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of [a, b].
4. The least of these f values is the absolute minimum. The greatest is the absolute maximum.

Note: The actual maximum or minimum value is a <u>y value</u>. Where the maximum or minimum occurs would be an x value.

Q4. Find the absolute extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1,2]$.
Use your calculator to sketch the graph and verify.

Q5. Find the extrema of $f(x) = 2\sin x - \cos 2x$ on the interval $[0,2\pi]$.
Use your calculator to sketch the graph and verify.

➢ **Increasing and Decreasing Functions**



A function f is **increasing** <u>**(decreasing)**</u> on an interval <u>**if**</u> for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$ (______________)

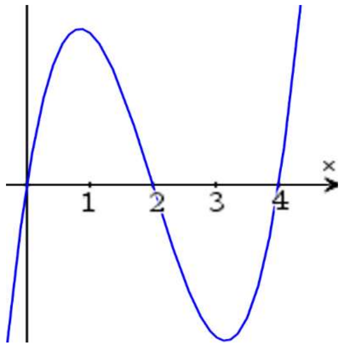
- A function is **strictly monotonic** on an interval if it is either increasing or decreasing on the entire interval.

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is ___________ on $[a, b]$.
3. If $f'(x)$ _____ 0 for all x in (a, b), then f is constant on $[a, b]$.

Q1. Intervals on Which f is Increasing or Decreasing

The graph shown to the right is of $f'(x)$, the derivative of $f(x)$. Find the critical values of $f(x)$, and state the intervals over which$f(x)$ is increasing and decreasing.



Q2. Find the intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

➢ **Guidelines for Finding Intervals on Which a Function is Increasing or Decreasing**

Let f be continuous on the interval (a, b).
1. Locate the **critical numbers** of f in (a, b), and use these numbers to determine your test intervals.
2. Determine the **sign** of $f'(x)$ by picking a "test value" in each of the intervals.
3. Use the Theorem for Increasing and Decreasing Functions to determine whether the function increases or decreases.

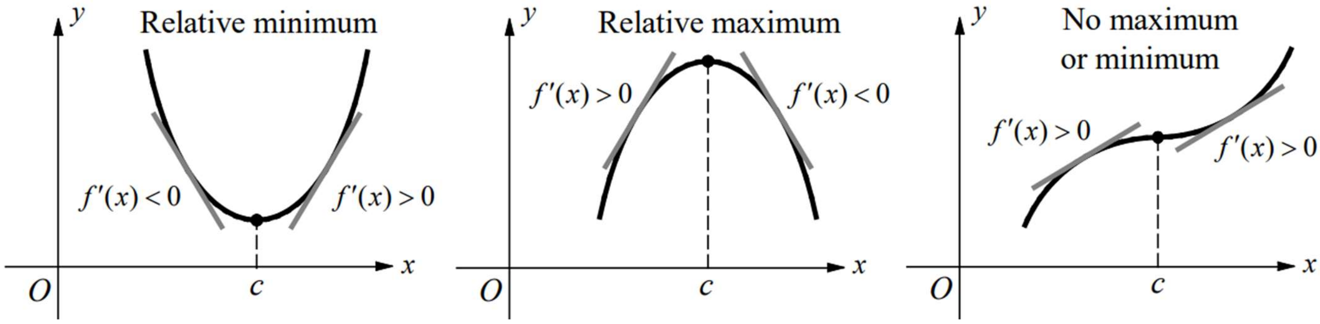
The guidelines above will also work for the interval $(-\infty, b)$, (a, ∞),or $(-\infty, \infty)$

- **Using the First Derivative Test to Determine Relative (Local) Extrema**

The First Derivative Test

Let c be a **critical number** of the function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative ___________ of f.
2. If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative ___________ of f.
3. If $f'(x)$ does not change its sign at c, then $f(c)$ is neither a relative minimum nor relative maximum.
   - Turning point: the first derivative changes its sign at the turning point.

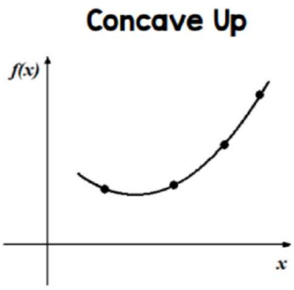


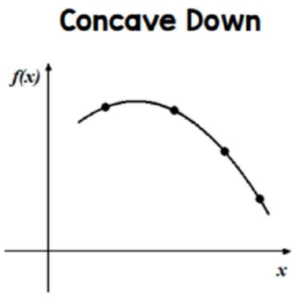
Q3. Applying the First Derivative Test

a. Find all points that are relative extrema of the function $f(x) = \frac{1}{2}x - \sin x$ on the interval $(0,2\pi)$.

b. Find all points that are relative extrema of the function $f(x) = (x^2-4)^{\frac{2}{3}}$.

c. Find all points that are relative extrema of the function $f(x) = \frac{x^4+1}{x^2}$.





- **Concavity**

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.

- **Test for Concavity**

Let f be a function whose second derivative exists on an open interval (a, b).

1. If f''_______ for all x in (a, b), then the graph of f is concave upward on (a, b).
2. If f'' _______ for all x in (a, b), then the graph of f is concave downward on (a, b).

- **Points of Inflection Theorem**

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f is not differentiable at x = c.

Q1. Determine the **open intervals** on which each graph is concave upward or downward and state any points of inflection. Justify your answer.

a. $f(x) = x^4 - 4x^3$

b. $f(x) = \frac{6}{x^2+3}$

c. $f(x) = \frac{x^2+1}{x^2-4}$

Now, we will investigate another way to find the maximum and minimum values of a function.

- **The Second Derivative Test**

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If f''________, then $f(c)$ is a relative minimum.
2. If f''________, then $f(c)$ is a relative maximum.

Q1. Find the relative extrema for $f(x) = -3x^5 + 5x^3$ using the Second Derivative Test.

Q2. Given the values below for x, $f(x)$, $f'(x)$, along with the fact that $f'(x)$ has only two zeros on the interval $(-3,10)$ and $f''(x)$ has only two zeros on the interval $(-3,10)$, answer each of the following.

x	-3	-1	1	3	5	7	10
$f(x)$	-7	1	-1	-4	3	2	-1
$f'(x)$	1	0	-1	0	2	undefined	3
$f''(x)$	-2	-1	0	2	3	0	5

a. Identify all x-values where f has a relative minimum. Justify using the First Derivative Test.

b. Identify all x-values where f has a relative maximum. Justify using the Second Derivative Test.

c. Identify all x-values where f has a point of inflection. Justify.

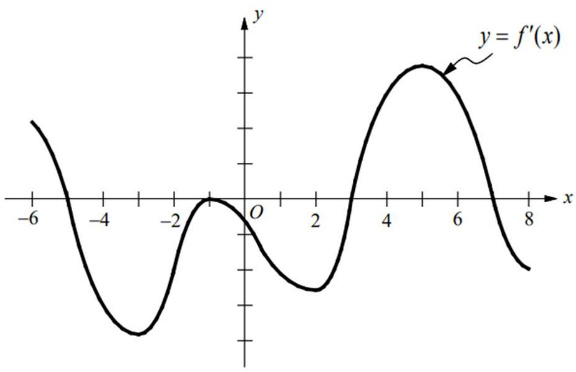
d. What is the equation of the tangent to the curve $y = f(x)$ at $x = 5$?

➢ Curve Sketching

<u>Guidelines:</u>

1. Domain
2. Intercepts
3. Symmetry
   - ✧ Even/Odd?
   - ✧ Replacing y by −y yields an equivalent equation, the curve is symmetric about the x-axis
4. Asymptotes
5. Intervals of increasing and decreasing
6. Relative Max, Relative Min
7. Point of infelection, Concavity

Q1.



<u>Note:</u> This is the graph of f', not the graph of f.

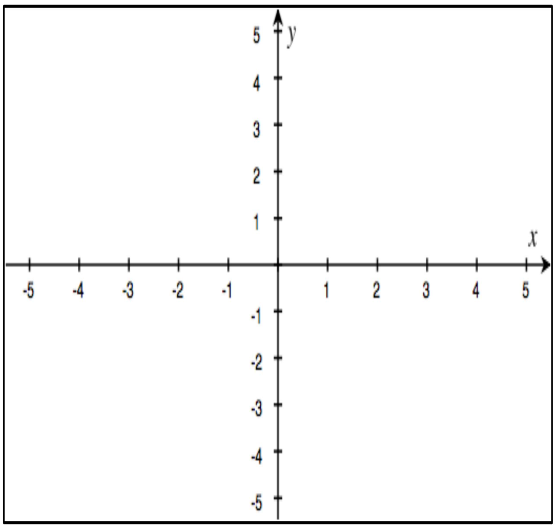
The figure above shows the graph of f'. The domain of f is the set of all real numbers x such that $-6 \le x \le 8$.

(a) For what values of x does f have a relative maximum?
(b) For what values of x does f have a relative minimum?
(c) For what values of x does the graph of f have a horizontal tangent?
(d) For what values of x is the graph of f concave upward?
(e) For what values of x is the graph of f concave downward?
(f) Suppose that $f(0) = 1$. Sketch a possible graph of f.

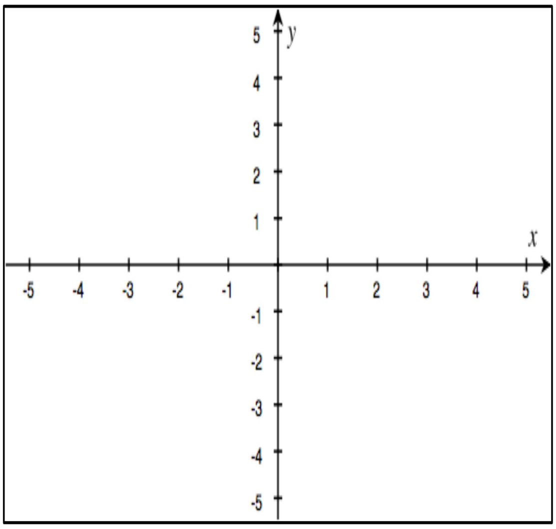
Q2. Sketch the graph of $f(x) = 2xe^{-x^2}$

➢ **Sketch a possible $f(x)$ with the following characteristics.**

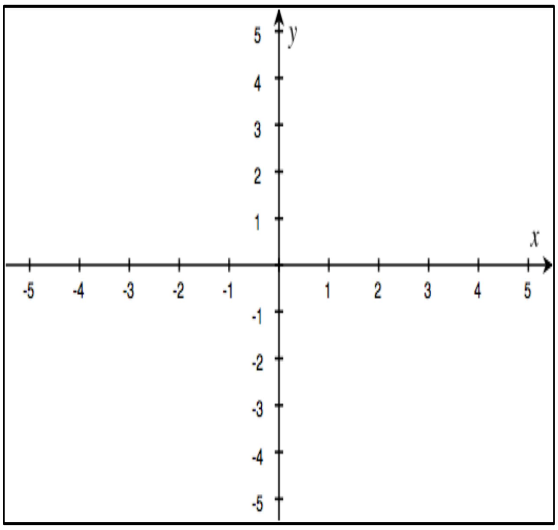
a. $f'(x) > 0$ for all $x, f'' > 0$ for all x
$f(1) = -2$



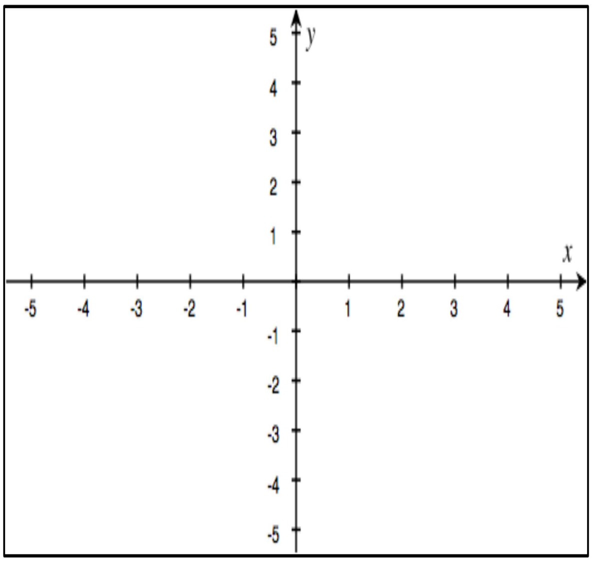
b. $f'(x) > 0$ for $x > 1, f'(x) < 0$ for $x < 1$
$f'(1) = 0, f'' > 0$ for all x, $f(1) = -1$



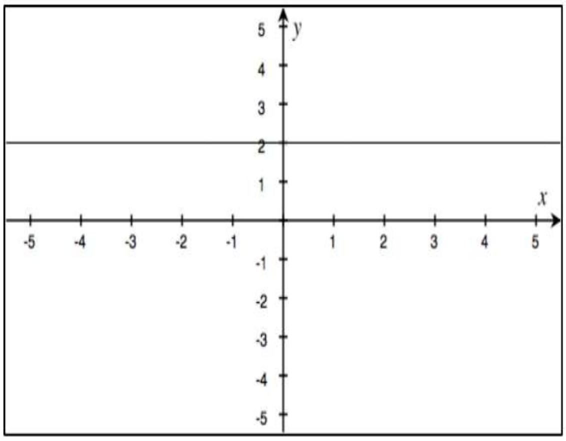
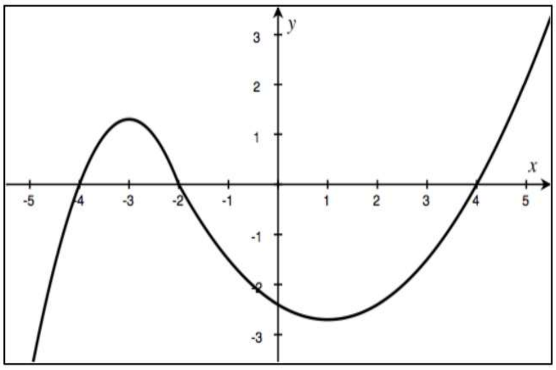
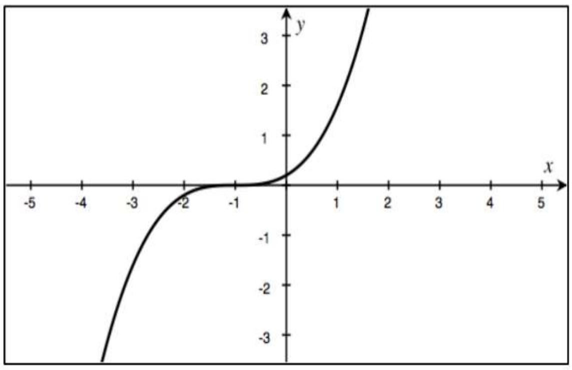
c. $f'(x) > 0$ for $x < -3, f'(x) < 0$ for $-3 < x < 1$
$f'(x) > 0$ for $x > 1, f'(-3) = f'(1) = 0, f(0) = 0$
$f'' < 0$ for x $< -1, f'' > 0$ for x > -1

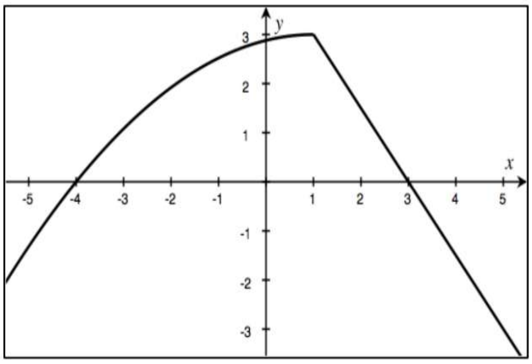
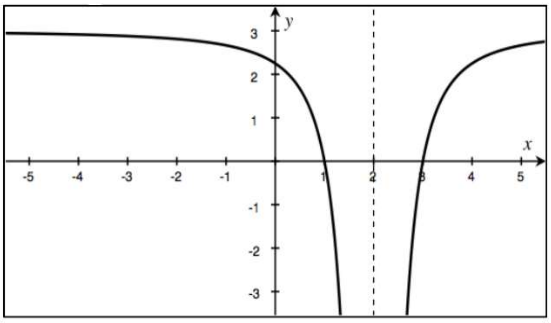
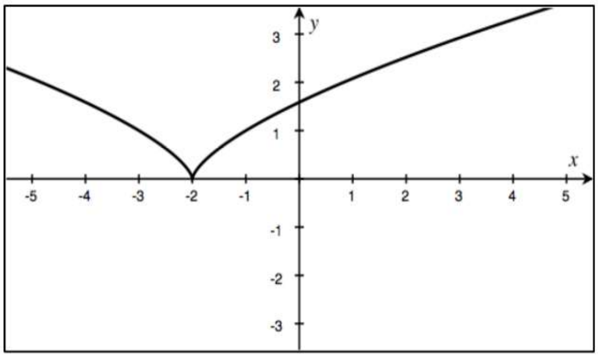


d. $f'(x) > 0$ for $x > 2, f'(x) = -\frac{1}{2}$ for $x < 2$
$f'(2)$ does not exist, $f'' < 0$ for x > 2, $f(2) = 0$



➢ You are given a graph of $f'(x)$. **Sketch a possible graph of $f(x)$.**

a. 	Possible $f(x)$
b. 	Possible $f(x)$
c. 	Possible $f(x)$

d. $f'(x)$ 	Possible $f(x)$
e. $f'(x)$ 	Possible $f(x)$
f. $f'(x)$ 	Possible $f(x)$

- **Behaviors of Implicit Relations**
- **Increasing/Decreasing Behavior**

  Consider the following relation which is an ellipse with center (0,0). $4x^2 + 9y^2 = 36$

  a. Use implicit differentiation to show that $\frac{dy}{dx} = -\frac{4x}{9y}$

  b. Find all increasing/decreasing intervals

- **Concavity**

  Consider the following relation: $3(x - y) = 4 + 3\cos y$

Q1. Let $f(x)$ be a function such that $\frac{dy}{dx} = 3x - 2y - 8$. If $f(x)$ contains the point $(2, -1)$, which of the following best describes the point $(2, -1)$ on the graph of $y = f(x)$?

(A) a relative minimum (B) a relative maximum (C) a point of inflection (D) none of these

Q2. The points $(-1, -1)$ and $(1, -5)$ are on the graph of a relation whose derivative is $\frac{dy}{dx} = x^2 + y$. Which of the following must be true?

(A) $(1, -5)$ is a local maximum of the relation
(B) $(1, -5)$ is a point of inflection of the relation
(C) $(-1, -1)$ is a local maximum of the relation
(D) $(-1, -1)$ is a local minimum of the relation
(E) $(-1, -1)$ is a point of inflection of the relation

Q3. Concavity of an Implicit Relation

Consider a curve whose first derivative is defined as $\frac{dy}{dx} = \frac{3}{3+cos}$. Determine the concavity of the curve at points for which $0 < y < \pi$.

- **Using L'Hospital's Rule for Determining Limits of Indeterminate Functions**

Limit Problem	Result of directly substituting in "c" for x
$\boldsymbol{\lim\limits_{x\to 2} \frac{x-2}{x^2-4}}$	
$\boldsymbol{\lim\limits_{x\to\infty} \frac{1-4x-5x^2}{3x^2-x-4}}$	

These are called **indeterminate** forms.

$$\frac{\mathbf{0}}{\mathbf{0}}, \frac{\pm\infty}{\pm\infty}$$

$$\mathbf{0}\cdot(\pm\infty)$$

$$\infty-\infty, \mathbf{0}-\mathbf{0}$$

<u>L'Hospital's Rule</u>

If we have one of the two following cases: $\lim\limits_{x\to c}\frac{f(x)}{g(x)}$ where both $\lim\limits_{x\to c} f(x)=0$ <u>and</u> $\lim\limits_{x\to c} g(x)=0$ OR where both $\lim\limits_{x\to c} f(x) \Rightarrow \pm\infty$ <u>and</u> $\lim\limits_{x\to c} g(x) \Rightarrow \pm\infty$ for any real number, c, or for c having the value of infinity or negative infinity, then $\boldsymbol{\lim\limits_{x\to c}\frac{f(x)}{g(x)} = \lim\limits_{x\to c}\frac{f'(x)}{g'(x)}}$

<u>Example 1</u>: Consider the following limits:

(1) $\lim\limits_{x\to 0}\frac{\cos x-1}{x^2-x}$

(2) $\lim\limits_{x\to\infty}\frac{e^x+x}{x^3}$

(3) $\lim\limits_{x\to\frac{\pi}{2}}\frac{\sec x+9}{\tan x}$

(4) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$

(5) $\lim\limits_{x\to\infty}(x-\sqrt{x^2+x})$

(6) $\lim\limits_{x\to\infty} x\tan\frac{1}{x}$

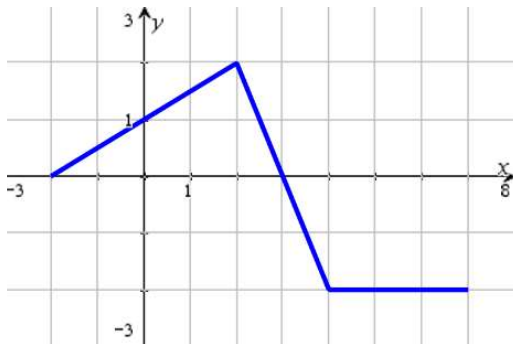
(7) $\lim\limits_{x\to 0}\frac{e^x}{x}$

(8) $\lim\limits_{x\to\infty}\left(\frac{4x^2-5x+2}{e^{5x}+\ln x}\right)$

<u>Example 2</u>: Find each of the following limits.

The graph of $f(x)$ is shown below. The functions $f(x)$ and $g(x)$ are differentiable for all x. Use the following table to help find the limit.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	-2	0	3

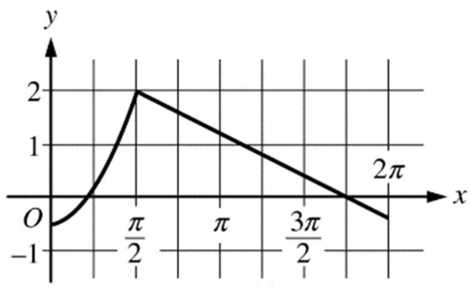


Find (1) $\lim\limits_{x\to 2} \frac{f(x)-4}{g(x)\cdot x^2}$

(2) $\lim\limits_{x\to 3} \frac{f(x)}{x^2-9}$

<u>Example 3</u>: From the 2018 AP Calculus Exam AB 5

Let f be a function defined by $f(x) = e^x \cos x$. Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g is shown below. Find the value of $\lim\limits_{x\to \pi/2} \frac{f(x)}{g(x)}$ or state that it doesn't exist.



Graph of g'