HW _ Further Applications of Integration

➢ Separable Differential Equation

1. The solution to the differential equation $\frac{dy}{dx}=\frac{3x^2}{2y}$, where $y(3)=4$, is

$\int 2y\,dy=\int 3x^2\,dx$

$y^2=x^3+c$

sub into $y(3)=4$: $c=-11$

$\therefore y^2=x^3-11$

2. If $\frac{dy}{dx}=\frac{x+\sec^2 x}{y}$ and $y(0)=2$, then $y=$

$\int y\,dy=\int (x+\sec^2 x)\,dx$

$\frac{1}{2}y^2=\frac{1}{2}x^2+\tan x+c$

$\therefore y^2=x^2+2\tan x+c$

$y=\pm\sqrt{|x^2+2\tan x+c|}$

$\because y(0)=2$

$\therefore c=4$

$\therefore y=\pm\sqrt{|x^2+2\tan x+4|}$

3. What is the value of $m+b$, if $y=mx+b$ is a solution to the differential equation $\frac{dy}{dx}=\frac{1}{4}x-y+1$?

$\frac{dy}{dx}=m$

$m=\frac{1}{4}x-(mx+b)+1$

$=(\frac{1}{4}-m)x+(1-b)$

$\therefore m=\frac{1}{4}$. $1-b=m$ $\therefore m+b=1$

4. At each point (x,y) on a certain curve, the slope of the curve is xy. If the curve contains the point $(0,-1)$, which of the following is the equation for the curve?

$\frac{dy}{dx}=xy$

$\int\frac{1}{y}\,dy=\int x\,dx$

$\ln|y|=\frac{1}{2}x^2+c$

$y=ce^{\frac{1}{2}x^2}$

$\because$ the curve passes $(0,-1)$

$\therefore -1=ce^0=c$

$\therefore y=-e^{\frac{1}{2}x^2}$

5. If $\frac{dy}{dx}=(y-4)\sec^2 x$ and $y(0)=5$, then $y=$

$\int\frac{1}{y-4}\,dy=\int\sec^2 x\,dx$

$\ln|y-4|=\tan x+c$

$|y-4|=e^{\tan x+c}$

$y-4=ce^{\tan x}$

$y=4+ce^{\tan x}$

$\because y(0)=5$

$\therefore c=1$

$\therefore y=4+e^{\tan x}$

➤ **Exponential Growth or Decay** $y = Ce^{kt}$

D 1. Bacteria in a certain culture increase at a rate proportional to the number present. If the number of bacteria doubles every four hours, in how many hours will the number of bacteria triple?

$e^{k(t_0+4)} = 2e^{kt_0}$; $\therefore k = \frac{\ln 2}{4}$; x ; $\therefore kx = \ln 3$

(A) $\ln(\frac{27}{2})$ (B) $\ln(\frac{81}{2})$ (C) $\frac{4\ln 2}{\ln 3}$ (D) $\frac{4\ln 3}{\ln 2}$

$e^{kt_0} \cdot e^{4k} = 2e^{kt_0}$; $e^{k(t_0+x)} = 3e^{kt_0}$; $x = \frac{1}{k}\ln 3$

$= 4 \cdot \frac{\ln 3}{\ln 2}$

$\therefore e^{4k} = 2$; $\therefore e^{kx} = 3$

$\therefore e^{k} = 2^{\frac{1}{4}}$; $\to y = C \cdot e^{kt}$

B 2. Population y grows according to the equation $\frac{dy}{dt} = ky$, where k is a constant and t is measured in years. If the population doubles every 15 years what is the value of k? $y(15+t_0) = 2y(t_0)$

$Ce^{(15+t_0)k} = 2Ce^{t_0 \cdot k}$

(A) 0.035 (B) 0.046 (C) 0 .069 (D) 0.078

$C \cdot e^{t_0 k} \cdot e^{15k} = C \cdot e^{t_0 k} \cdot 2$

$\therefore e^{15k} = 2$ $\therefore k = \frac{1}{15}\ln 2 \approx 0.046$

C 3. A baby weighs $\boxed{\text{6 pounds at birth}}$ and $\boxed{\text{9 pounds three months}}$ later. If the weight of baby increasing at a rate proportional to its weight, then how much will the baby weigh when she is 6 months old? $\Rightarrow W = Ce^{kt}$

(A) 11.9 (B) 12.8 (C) 13.5 (D) 14.6

$\begin{cases} C = 6 \\ Ce^{3t} = 9 \end{cases}$ $\therefore e^{3t} = \frac{3}{2}$ $w(6) = C \cdot e^{6t} = 6 \cdot (e^{3t})^2 = 6 \times \frac{9}{4} = 13.5$

A 4. Temperature F changes according to the differential equation $\frac{dF}{dt} = kF$, where k is a constant and t is measured in minutes. If at time $t = 0$, $F = 180$ and at time $t = 16$, $F = 120$, what is the value of k?

(A) -0.025 (B) -0.032 (C) -0.045 (D) -0.058

$F = C \cdot e^{kt}$

$\begin{cases} 180 = C \cdot e^0 = C \\ 120 = Ce \end{cases}$

$\therefore \begin{cases} C = 180 \\ k = \frac{1}{16}\ln(\frac{2}{3}) \end{cases}$

≈ -0.0253

$120 = 180e^{16k}$

$e^{16k} = \frac{120}{180} = \frac{2}{3}$

$16k = \ln(\frac{2}{3})$

$k = \frac{1}{16}\ln(\frac{2}{3})$

- **Logistic**

$=kP(1-\frac{P}{A})=kP-\frac{k}{A}P^2$

D 1. The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt}=3P-0.0006P^2$, where the initial population is $P(0)=1000$ and t is the time in years. What is $\lim_{t\to\infty} P(t)$? $= A$

$\therefore k=3$

$\frac{k}{A}=0.0006$

$\therefore A=5000$

(A) 1000 (B) 2000 (C) 3000 (D) 5000

C 2. A healthy population $P(t)$ of animals satisfies the logistic differential equation $\frac{dP}{dt}=5P(1-\frac{P}{240})$, where the initial population is $P(0)=150$ and t is the time in years. For what value of P is the population growing the fastest?

$A=240$

$P=\frac{A}{2}=120$

(A) 48 (B) 60 (C) 120 (D) 240

B 3. A population is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt}=\frac{P}{5}\left(1-\frac{P}{150}\right)$, where the initial population is $P(0)=800$ and t is the time in years. What is the slope of the graph of P at the point of inflection?

$k=\frac{1}{5}$ $A=150$

at $P=75$

$\frac{dP}{dt}=\frac{75}{5}\cdot\frac{1}{2}=7.5$

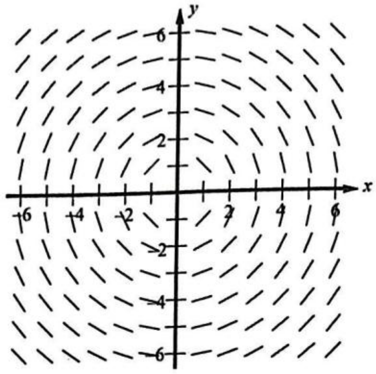
(A) 5 (B) 7.5 (C) 10 (D) 12.5

$A=\frac{1}{3}$ $\frac{A}{2}=\frac{1}{8}$

A 4. A certain rumor spreads in a small town at the rate $\frac{dy}{dt}=y(1-3y)$, where y is the fraction of the population that has heard the rumor at any time t. What fraction of the population has heard the rumor when it is spreading the fastest?

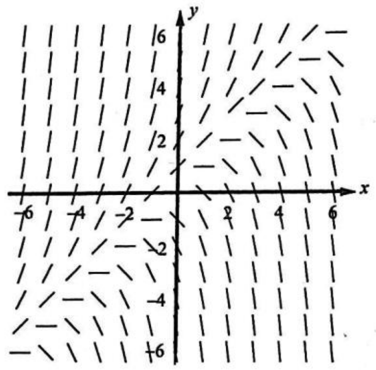
(A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$

- **Slope Field**



B 1. Shown above is a slope field for which of the following differential equations?

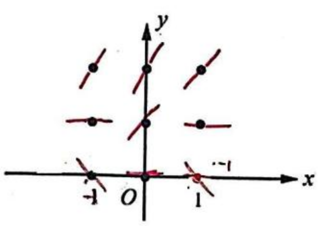
(A) $\frac{dy}{dx}=\frac{x}{y}$ (B) $\frac{dy}{dx}=-\frac{x}{y}$ (C) $\frac{dy}{dx}=\frac{x^2}{y}$ (D) $\frac{dy}{dx}=-\frac{x^2}{y}$



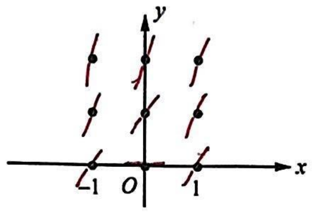
C 2. Shown above is a slope field for which of the following differential equations?

(A) $\frac{dy}{dx}=x+y$ (B) $\frac{dy}{dx}=x-y$ (C) $\frac{dy}{dx}=-x+y$ (D) $\frac{dy}{dx}=x^2-y$

3. On the axis provided, sketch a slope field for the differential equation $\frac{dy}{dx}=y-x^2$.

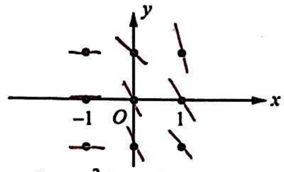


4. On the axis provided, sketch a slope field for the differential equation $\frac{dy}{dx} = x^2 + y^2$.



x	y	y'
± 1	0	1
	1	2
	2	5
0	0	0
	1	1
	2	2

5. On the axis provided, sketch a slope field for the differential equation $\frac{dy}{dx} = (x+1)(y-2)$.



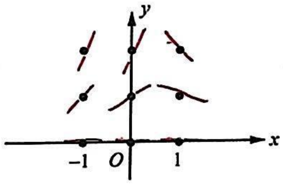
x	y	y'
-1	–	0
–	2	0
0	-1	-3
0	0	-2
0	1	-1
1	-1	-6
1	0	-4
1	1	-2

6. Consider the differential equation $\frac{dy}{dx} = \frac{y^2(1-2x)}{3}$.

(a) On the axis provided sketch a slope field for the given differential equation at the nine points indicated.

$y=0 \cdot \quad y'=0$

x	y	y'
1	1	$-\frac{1}{3}$
-1	1	1
0	1	$\frac{1}{3}$
1	2	$-\frac{4}{3}$
-1	2	4
0	2	$\frac{4}{3}$



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) $\frac{d^2y}{dx^2} = \frac{1}{3}\cdot\left[2y\cdot\frac{dy}{dx}(1-2x) + y^2\cdot(-2)\right] = \frac{1}{3}\cdot\left[\frac{2y(1-2x)\cdot y^2(1-2x)}{3} - 2y^2\right]$

$= \frac{2y^3(1-2x)^2 - 6y^2}{9}$

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $y(\frac{1}{2}) = 4$

Does f have a relative minimum, a relative maximum, or neither at $x = \frac{1}{2}$? Justify your answer.

(d) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $y(\frac{1}{2}) = 4$.

(c) $\left.\frac{dy}{dx}\right|_{(\frac{1}{2},4)} = 0$

$\left.\frac{d^2y}{dx^2}\right|_{(\frac{1}{2},4)} < 0$

$\therefore$ local max at $(\frac{1}{2}, 4)$

(d) $\int \frac{1}{y^2}\,dy = \int \frac{1}{3}\cdot(1-2x)\,dx$

$\frac{1}{y} = -\frac{1}{3}x + \frac{1}{3}x^2 + C = \frac{x^2 - x + 3C}{3}$

$y = \frac{3}{x^2 - x + C}$

$\because y(\frac{1}{2}) = 4 \quad \therefore C = 1$

$\therefore f(x) = \frac{3}{x^2 - x + 1}$