➢ **Def. A differential equation** in x and y is an equation that involves the derivatives of y. $y', y'', \ldots$

☺ **Examples of Differential Equations:**

$y''+2y'=3y$ $\qquad$ $f''(x)+2f'(x)=3f(x)$ $\qquad$ $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3y$ ← second order D.E. 最高阶数

B ☺ **Example.** A particle moves along a straight line. Its velocity, v, is inversely proportional to the square of the distance, s, it has traveled. Which equation describes this relationship?

(A) $v(t)=\frac{k}{t^2}$ $\qquad$ (B) $v(t)=\frac{k}{s^2}$ $\qquad$ (C) $\frac{dv}{dt}=\frac{k}{t^2}$ $\qquad$ (D) $\frac{dv}{dt}=\frac{k}{s^2}$

代入后使等式成立

✧ A **solution to a differential equation** is a function that satisfies the differential equation when the function and its derivatives are substituted into the equation.

eg. $\frac{dy}{dx}=2x$

- general solution: contains all possible solutions with arbitrary constants

general: $y=x^2+C$

- particular solution: obtained by fixing constants using initial conditions or boundary conditions

Initial condition $y(0)=1$

$\therefore C=1$

$\therefore$ particular: $y=x^2+1$

➢ **Separable Differential Equations**

The equation $y'=f(x,y)$ is a separable equation if all x terms can be collected with dx and all y terms with dy.

The differential equation then has the form $\frac{dy}{dx}=f(x)g(y)$, then the equation can be solved.

$\int \frac{1}{g(y)}\,dy=\int f(x)\,dx$

Separation of Variables:

Step 1. Collect all x terms on one side with dx and all y terms on the other side with dy

Step 2. Integrate each side separately.

✧ **Practice**

1. Find the general solution of $f'(x)=5f(x)$

$\frac{dy}{dx}=5y$

$\frac{1}{y}\,dy=5\,dx$

$\ln|y|=5x+c$

$y=\pm e^{5x+c}=\pm e^{c}\cdot e^{5x}$

$\therefore y=C\cdot e^{5x}$

2. Find the general solution of $\frac{dy}{dx}=-\frac{x}{y}$

$y\,dy=-x\,dx$

$\frac{1}{2}y^2=-\frac{1}{2}x^2+c$

$\therefore x^2+y^2=C$

3. Find the general solution of $(x+3)y' = 2y$

$$\frac{1}{2y}\,dy = \frac{1}{x+3}\,dx$$

$$\frac{1}{2}\cdot \ln|y| = \ln|x+3| + C$$

$$y = e^{2\ln|x+3|+C}$$

$$= e^{\ln(x+3)^2 + C}$$

$$\therefore\ y = C\cdot(x+3)^2$$

4. Consider the differential equation $\frac{dy}{dx} = \frac{2x+3}{e^y}$.

(a) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $y(0) = 2$. Write an equation for the line tangent to the graph of f at $(0,2)$.

(b) Find $f''(0)$ with the initial condition $y(0) = 2$.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = \frac{2x+3}{e^y}$ with the initial condition $y(0) = 2$.

(a) slope $= \left.\frac{dy}{dx}\right|_{(0,2)} = 3e^{-2}$

tangent line: $y = 3e^{-2}x + 2$

(b) $f''(x) = 2\cdot e^{-y} + (2x+3)(-e^{-y})\frac{dy}{dx} = 2e^{-y} - e^{-2y}(2x+3)^2$

$= e^{-2y}\left[2e^{y} - (2x+3)^2\right]$

$f''(0) = (2e^2 - 9)\,e^{-4}$

(c) $\int e^y\,dy = \int 2x+3\,dx$

$e^y = x^2 + 3x + C$

Sub $y(0)=2$ into: $e^2 = C$

$\therefore\ y = \ln(x^2 + 3x + e^2)$

➢ Exponential Growth and Decay

In many real-word situations, a quantity y increases or decreases at a rate (k) proportional to its size at a given time t. If y is a function of time t, then $\frac{dy}{dt} =$ ky

$\frac{1}{y}\,dy = k\,dt$ $\quad\therefore\ y = Ce^{kt}$

$\ln|y| = kt + c$

If the initial value $y(0) = y_0$, then $y =$ $y_0 e^{kt}$

$C = y_0$

Applications

① saving money (continuously compounding interest)

$P(t+\Delta t) - P(t) = r * P(t) * \Delta t$

↑ principal ↑ interest rate

when $\Delta t \to 0$: $\frac{dP}{dt} = rP$

Applications

② population size

人口变化率 ≈(出生-死亡) $\frac{\Delta P}{\Delta t} = k \cdot P$ 人口基数越大，人口变化越大.

k: ≈出生-死亡率

Example.

1. The number of bacteria in a culture increases at a rate proportional to the number present. If the number of bacteria was 600 after 3 hours and 19,200 after 8 hours, when will the population reach 120,000?

$y(t)$

$\frac{d\,y(t)}{dt} = ky$

$\therefore y = C\cdot e^{kt}$

$600 = C\cdot e^{3k}$

$19200 = C\cdot e^{8k}$ $\Rightarrow \frac{192}{6} = e^{5k}$

$32 = (e^k)^5$

$\therefore e^k = 2$

$\therefore k = \ln 2$

$\therefore C = 600\cdot e^{-\ln 8} = 75$

$\therefore y(t) = 75e^{t\cdot\ln 2} = 75e^{\ln 2^t} = 75\times 2^t$

when $y = 120{,}000$, $2^t = 1600$

$\therefore t \approx 10.644$ hours.

2. The rate at which the amount of coffee in a coffeepot changes with time is given by the differential equation $\frac{dV}{dt} = kV$, where V is the amount of coffee left in the coffeepot at any time t seconds. At time $t = 0$ there were 16 ounces of coffee in the coffeepot and at time $t = 80$ there were 8 ounces of coffee remaining in the pot. $V(0) = 16$ $\quad V(80) = 8$

(a) Write an equation for V, the amount of coffee remaining in the pot at any time t.

(b) At what rate is the amount of coffee in the pot decreasing when there are 4 ounces of coffee remaining?

(c) At what time t will the pot have 2 ounces of coffee remaining?

(a) $V(t) = V_0\cdot e^{kt}$

$\begin{cases} V(0) = 16 \\ V(80) = 8 \end{cases} \Rightarrow \begin{cases} V_0 = 16 \\ V_0 e^{80k} = 8 \end{cases} \quad \therefore \quad V_0 = 16,\ k = -\frac{\ln 2}{80}$

$\therefore V(t) = 16e^{-\frac{\ln 2}{80}t}$

(b) when $V(t) = 4$, $\frac{dV}{dt} = kV(t) = -\frac{\ln 2}{80}\cdot 4 = -\frac{1}{20}\ln 2$

(c) When $V(t) = 2$, $2 = 16e^{-\frac{\ln 2}{80}t}$ $\quad\therefore \ln\frac{1}{8} = -\frac{t}{80}\ln 2$ $\quad\therefore t = 240$

$-3\ln 2 =$

$\frac{dP}{dt}=kP(A-P)\cdot\frac{1}{A}$; $\int\frac{1}{P(A-P)}dP=\int\frac{k}{A}dt$; $\int\frac{1}{P}+\frac{1}{A-P}dP=\int k\,dt$

$\therefore \ln|P|-\ln|A-P|=kt+C$; $\left|\frac{P}{A-P}\right|=e^{kt+C}$

$\therefore \frac{P}{A-P}=C\cdot e^{kt}$; $P=A\cdot C\cdot e^{kt}-P\cdot C\cdot e^{kt}$; $P=\frac{A C\cdot e^{kt}}{1+C\cdot e^{kt}}$

times
㊟(小于等于1的参数) ★ 当P↑，资源↓，人口增长速度变慢.

Logistic Equations

★ 当P达到A时，$\frac{dP}{dt}=0$

The differential equation $\frac{dP}{dt}=$ $kP\left(1-\frac{P}{A}\right)$ is called a **logistic equation**.

★ 当P超过A，比如一次地震，使A变小，小于P，那么$\frac{dP}{dt}<0$，人口减少.

$P(t)$: the size of the population at time t

A: the carrying capacity (the maximum population that the environment is capable of sustaining in the long run)

k: a constant

$P(t)=\frac{C\cdot Ae^{kt}}{1+C\cdot e^{kt}}$

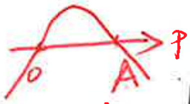
✧ **Properties:**

1. $\lim_{t\to\infty}P(t)=$ $\lim_{t\to\infty}\frac{CAe^{kt}}{C\cdot e^{kt}}=A$ ⇒ P(t)最终会稳定在A处. ⇒ $\frac{dP}{dt}$最终为0.

2. $\lim_{t\to\infty}\frac{dP}{dt}=$ 0 $\quad \frac{dP}{dt}=kP(1-\frac{P}{A})\to kP\cdot(1-\frac{A}{A})=0$ as $t\to\infty$, $P\to A$

3. The population is growing the fastest when $P=$ $\frac{A}{2}$ (max $\frac{dP}{dt}$)

$P(1-\frac{P}{A})=\frac{1}{A}P(A-P)$



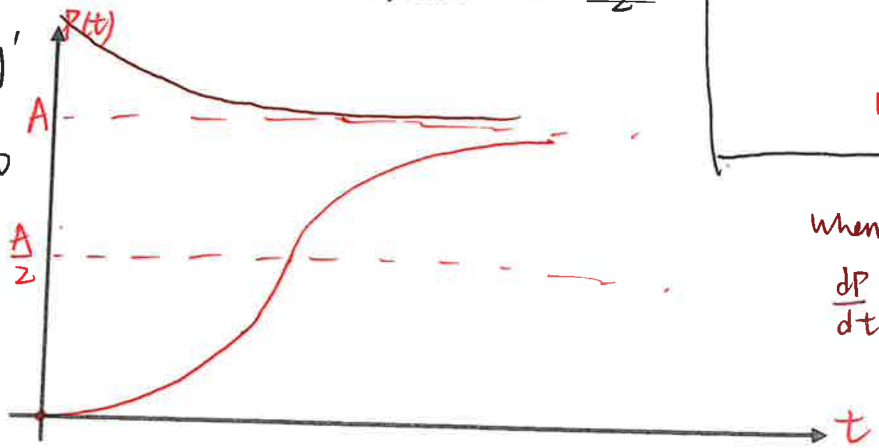
max at $P=\frac{A}{2}$.

4. The graph of $P(t)$ has a point of inflection at the point where $P=$ $\frac{A}{2}$

$\frac{d^2P}{dt^2}=\frac{k}{A}[P(A-P)]'$
$=\frac{k}{A}(A-2P)=0$
$\underline{P=\frac{A}{2}}$

when $P>\frac{A}{2}$, $\frac{d^2P}{dt^2}<0$ "∩"

when $P<\frac{A}{2}$, $\frac{d^2P}{dt^2}>0$ "∪"



Solution curves for the logistic equations with different initial conditions

when $P_0>A$
$\frac{dP}{dt}<0$ ↘
$\frac{d^2P}{dt^2}=\underbrace{\frac{k}{A}}_{<0}\underbrace{(A-2P)}_{<0}>0$
"∪"

1. A population is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt}=\frac{P}{2}\left(3-\frac{P}{20}\right)$, where the initial population $P(0)=100$ and t is the time in years. $=\frac{3}{2}P(1-\frac{P}{60})$ $\therefore A=60$ $k=\frac{3}{2}$

(a) What is $\lim_{t\to\infty}P(t)$? (a) $\therefore \lim_{t\to\infty}P(t)=60$

(b) For what values of P is the population growing the fastest? (b) when $P=\frac{A}{2}=30$.

(c) Find the slope of the graph of P at the point of inflection.

(c) $\left.\frac{dP}{dt}\right|_{P=\frac{A}{2}}=\frac{30}{2}(3-\frac{30}{20})=22.5$

Application
养殖羊，什么时候拿去卖？
当$P=\frac{A}{2}$，$\frac{dP}{dt}$最大

2. Let f be a function with $f(2)=1$, such that all points (t, y) on the graph of f satisfy the differential equation $\frac{dy}{dt}=2y\left(1-\frac{t}{4}\right)$.

Let g be a function with $g(2)=2$, such that all points (t, y) on the graph of g satisfy the logistic differential equation $\frac{dy}{dt}=y\left(1-\frac{y}{5}\right)$.

(a) Find $y=f(t)$.

(b) For the function found in part (a), what is $\lim_{t\to\infty} f(t)$?

(c) Given that $g(2)=2$, find $\lim_{t\to\infty} g(t)$ and $\lim_{t\to\infty} g'(t)$.

(d) For what value of y does the graph of g have a point of inflection? Find the slope of the graph of g at the point of inflection.

(a) $\frac{1}{y}\,dy = 2-\frac{t}{2}\,dt$ $\quad \ln|y| = 2t-\frac{t^2}{4}+C$

$\therefore y = Ce^{2t-\frac{t^2}{4}}$

$f(2)=1$: $\quad 1 = Ce^{3}$

$\therefore C = e^{-3}$

$\therefore f(t) = e^{2t-\frac{t^2}{4}-3}$

(b) $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} e^{-\frac{t^2}{4}} = 0$

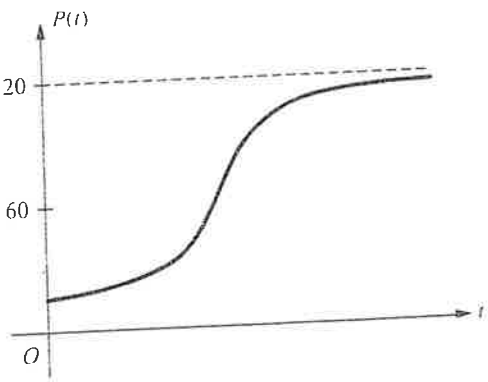
(c) $\lim_{t\to\infty} g(t) = 5$ $\quad (\because A=5,\ k=1)$

$\lim_{t\to\infty} g'(t) = 0$

(d) when $y=\frac{A}{2}=2.5$ $\qquad \left.\frac{dy}{dt}\right|_{y=2.5} = 2.5\cdot\left(1-\frac{1}{2}\right) = \frac{5}{4}$

3.

D



$A=120$

$\frac{dP}{dt} = kP\left(1-\frac{P}{A}\right) = kP - \frac{k}{120}P^2$

Which of the following differential equations for population P could model the logistic growth shown in the figure above

(A) $\frac{dP}{dt}=0.03P^2-0.0005P$

(B) $\frac{dP}{dt}=0.03P^2-0.000125P$

(C) $\frac{dP}{dt}=0.03P-0.001P^2$

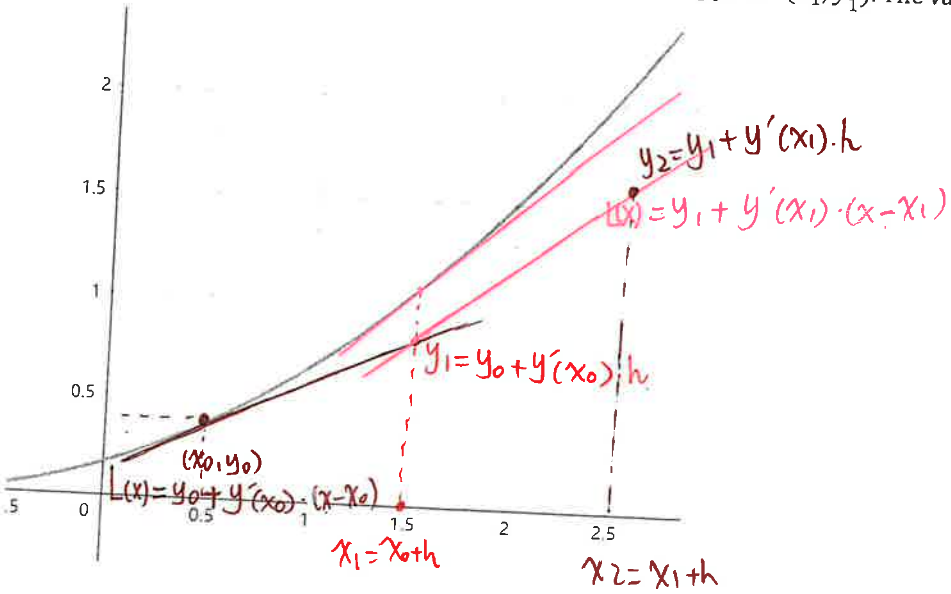
(D) $\frac{dP}{dt}=0.03P-\underline{0.00025}P^2$

$\times 120 = 0.03$

➢ Euler's Method

Euler's Method is a numerical approach to approximate the particular solution of the differential equation $y' = f(x,y)$ with an initial condition $y(x_0) = y_0$. Using a small step h and (x_0, y_0) as a starting point, move along the tangent line until you arrive at the point (x_1, y_1), where $x_1 =$ ________, $y_1 =$ ________

Repeat the process with the same step size h at a new starting point (x_1, y_1). The values of x_i and y_i are as follows.



★ We do not know $y = f(x)$

★ The only info we have is $y' = f(x, y)$, i.e. the slope so the tangent line.

1. Let f be the function whose graph goes through the point $(1, -1)$ and whose derivative is given $y' = 2 - \frac{y}{x}$. Use Euler's method starting at $x = 1$ with a step size of 0.5 to approximate $f(3)$.

$x_0 = 1$ $\quad y_0 = -1$ $\quad h = 0.5$

$x_1 = 1.5$ $\quad y'(x_0) = 3$

$y_1 = y_0 + y'(x_0) \cdot 0.5 = 0.5$

$x_2 = 2$ $\quad y'(x_1) = 2 - \frac{0.5}{1.5} = \frac{5}{3}$

$y_2 = y_1 + y'(x_1)\, 0.5 = \frac{4}{3}$

$x_3 = 2.5$ $\quad y'(x_2) = 2 - \frac{4}{3} \cdot \frac{1}{2} = \frac{4}{3}$

$y_3 = \frac{4}{3} + \frac{4}{3} \times \frac{1}{2} = 2$

$x_4 = 3$ $\quad y'(x_3) = 2 - \frac{2}{2.5} = \frac{6}{5}$

$y_4 = 2 + \frac{6}{5} \times \frac{1}{2} = \frac{13}{5} = 2.6$

$\therefore f(3) \approx 2.6$

2. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x - y + 2$ with the initial condition $f(0) = 2$. Use Euler's method starting at $x = 0$ with a step size of 0.5 to approximate $f(2)$.

$x_0 = 0$ $\quad y_0 = 2$ $\quad h = 0.5$

$x_1 = 0.5$ $\quad y_1 = 2 + \left.\frac{dy}{dx}\right|_{(x_0 = 0,\, y_0 = 2)} \times 0.5 = 2$

$x_2 = 1$ $\quad y_2 = 2 + \left.\frac{dy}{dx}\right|_{(0.5,\, 2)} \times 0.5 = 2.25$

$x_3 = 1.5$ $\quad y_3 = 2.25 + \left.\frac{dy}{dx}\right|_{(1,\, 2.25)} \times 0.5 = 2.625$

$x_4 = 2$ $\quad y_4 = 3 + \left.\frac{dy}{dx}\right|_{(1.5,\, 2.625)} = 3.0625$

$\therefore f(2) \approx 3.0625$

➢ **Slope Field**

we have learnt ways to solve some simple differential equations analytically. However, doing so can be difficult or sometimes impossible. A graphical approach to solve a differential equation is by creating **slope fields**, which show the **general shape of all solutions** to a differential equation.

Consider a differential equation $y' = f(x, y)$ in terms of x and y. For every point (x, y) in its domain, y' determines the slope of the solution function at that point. If you draw a short line segment with the slope indicated at each point on y', the **slope field (direction field)** will show the general shape of all the solution functions to that differential equation.

≈ tangent line"

$(0,0)$: $y' = 1$

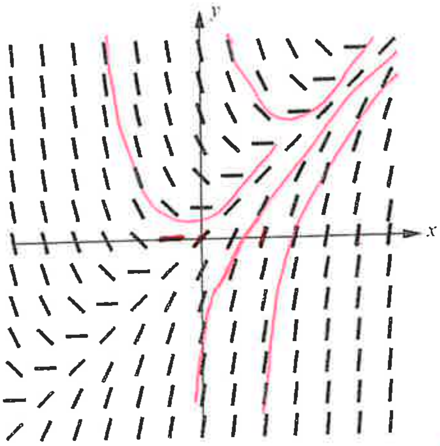
x-axis: $y = 0$

$y' = x + 1$

∴ when $x = -1$, $y' = 0$

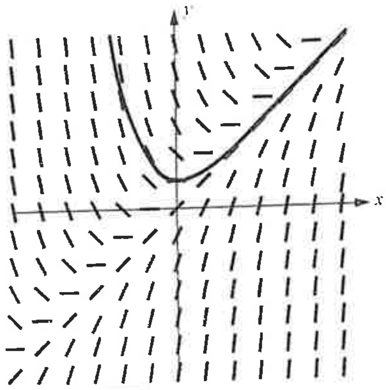
$x \uparrow$, $y' \uparrow$

$x \downarrow$, $y' \downarrow$



Slope field for $y' = x - y + 1$

⇒: $+c$ "general solution"

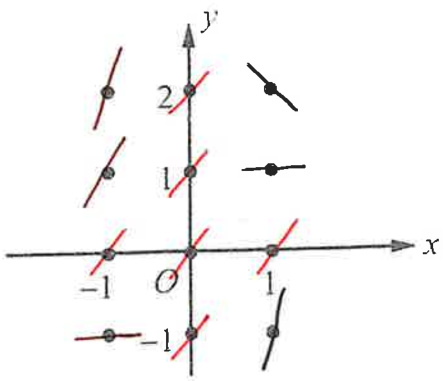


Particular solution for $y' = x - y + 1$ passing through $(0, 1)$

= initial condition

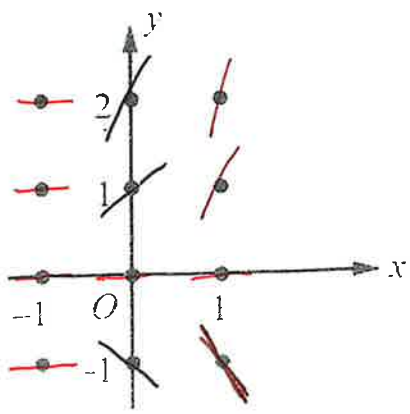
Practice.

1. On the axes provided, sketch a slope field for the differential equation $y' = 1 - xy$.



x	y	y'
0	-1, 0, 1, 2	1
-1, 1	0	1
1	1	0
1	2	-1
1	-1	2
-1	1	2
-1	2	3
-1	-1	0

2. On the axes provided, sketch a slope field for the differential equation $y' = y + xy$. $= y(x+1)$



x	y	y'
—	0	0
-1	—	0
0	-1	-1
0	1	1
0	2	2
1	-1	-2
1	1	2
1	2	4

$\frac{1}{y}\,dy = (x+1)\,dx$

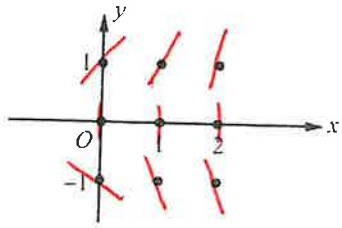
$\ln|y| = \frac{1}{2}x^2 + x + c$

$y = C \cdot e^{\frac{1}{2}x^2 + x}$

x	y	$\frac{dy}{dx}$
0 ,1,2	0	∞
0	1	1
1	1	2
2	1	3
0	-1	-1
1	-1	-2
2	-1	-3

3. Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.

(a) On the axis provided sketch a slope field for the given differential equation at the nine points indicated.



(b) $\left.\frac{dy}{dx}\right|_{(1,\sqrt{3})} = \frac{2}{3}\sqrt{3}$

$\therefore$ tangent line: $y = \sqrt{3} + \frac{2}{3}\sqrt{3}(x-1)$

(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $y(1) = \sqrt{3}$. Write an equation for the line tangent to the graph of f at $(1, \sqrt{3})$ and use it to approximate $f(1.2)$.

$f(1.2) \approx y(1.2) = 1.963$

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $y(1) = \sqrt{3}$.

(d) Use your solution from part (c) to find $f(1.2)$.

(c) $y\,dy = (x+1)\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + x + C$

Sub $y(1) = \sqrt{3}$ into: $\frac{3}{2} = \frac{3}{2} + C$

$\therefore C = 0$

$\therefore y^2 = x^2 + 2x$

$(y = \pm\sqrt{x^2+2x})$

(d) $f(1.2)$:

$y^2 = 1.2^2 + 2.4$

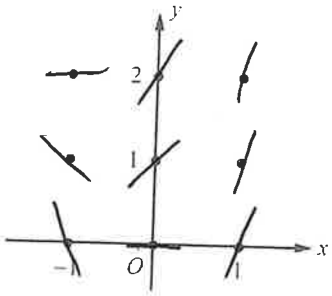
$y \approx \pm 1.96$

$\therefore f(1.2) \approx \pm 1.96$

4. Consider the differential equation $\frac{dy}{dx} = 2x + y$.

(a) On the axis provided, sketch a slope field for the given differential equation at the ~~twelve~~ Nine points indicated, and sketch the solution curve that passes through the point $(1, 1)$.

x	y	$\frac{dy}{dx}$
0	0	0
0	1	1
0	2	2
1	0	2
1	1	3
1	2	4
-1	0	-2
-1	1	-1
-1	2	0



(b) Let f be the function that satisfies the given differential equation with the initial condition $f(1) = 1$. Use Euler's method, starting at $x = 1$ with a step size of 0.1, to approximate $f(1.2)$. Show the work that leads to your answer.

(b) $x_0 = 1$ $y_0 = 1$

$x_1 = 1.1$ $y_1 = 1 + f'(1)\,0.1 = 1.3$

$x_2 = 1.2$ $y_2 = 1.3 + f'(1.1)\,0.1$

$= 1.65$

$f(1.2) \approx 1.65$

(c) Find the value of b for which $y = -2x + b$ is a solution to the given differential equation. Show the work that leads to your answer.

(d) Let g be the function that satisfies the given differential equation with the initial condition $g(1) = -2$. Does the graph of g have a local extremum at the point $(1, -2)$? If so, is the point a local maximum or a local minimum? Justify your answer.

(c) $\frac{dy}{dx} = -2$ $\quad -2 = 2x + (-2x + b)$

$\therefore b = -2$

(d) $\left.\frac{dg}{dx}\right|_{(1,-2)} = 0$

$\frac{d^2g}{dx^2} = 2 + \frac{dy}{dx} = 2 + 2x + y$

$\left.\frac{d^2g}{dx^2}\right|_{(1,-2)} = 2 > 0$

$\therefore$ local min at $(1, -2)$