➢ **Stright-Line Motion**

1. A particle moves along the x-axis so that at any time $t \geq 0$, its position is given by $x(t) = -\frac{1}{2}\cos t - 3t$.

   What is the acceleration of the particle when $t = \frac{\pi}{3}$?

   $V(t) = x'(t) = \frac{1}{2}\sin t - 3$

   $a(t) = V'(t) = \frac{1}{2}\cos t$

   $a(\frac{\pi}{3}) = \frac{1}{2}\cos\frac{\pi}{3} = \frac{1}{4}$

2. A particle moves along the x-axis so that at any time t, its position is given by $x(t) = \sqrt{t}\ln t$. For what values of t is the particle at rest?

   $V(t) = x'(t) = \frac{1}{2}t^{-\frac{1}{2}}\cdot \ln t + t^{\frac{1}{2}}\cdot\frac{1}{t} = t^{-\frac{1}{2}}(\frac{1}{2}\ln t + 1)$

   "at rest" means $V(t) = 0$

   $\therefore \frac{1}{2}\ln t = -1$

   $\ln t = -2$

   $t = e^{-2}$

3. (Calculator) A particle moves along the x-axis so that at any time t, its position is given by $x(t) = 3\sin t + t^2 + 7$. What is the velocity of the particle when its acceleration is zero?

   $V(t) = x'(t) = 3\cos t + 2t$

   $a(t) = x''(t) = -3\sin t + 2 = 0$

   $\Rightarrow \sin t = \frac{2}{3}$

   $\therefore \cos t = \pm\frac{\sqrt{5}}{3}$

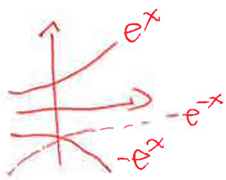
   $\therefore V(t) \approx \pm\sqrt{5} + 1.4594 = 3.695$ or -0.777

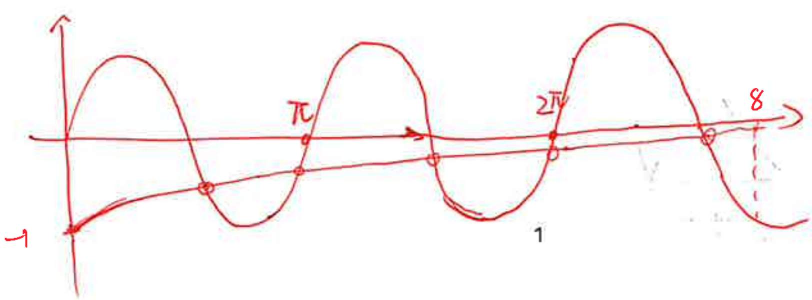
4. (Calculator) Two particles start at the origin and move along the x-axis. For $0 \leq t \leq 8$, their respective position functions are given by $x_1(t) = \sin^2 t$ and $x_2(t) = e^{-t}$. For how many values of t do the particles have the same velocity?

(A) 3
(B) 4
(C) 5
(D) 6

$V_1(t) = x_1'(t) = 2\sin t\cdot\cos t = \sin 2t$

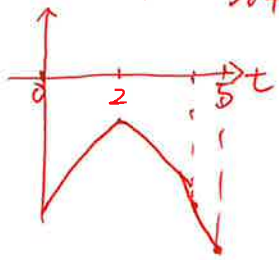
$V_2(t) = x_2'(t) = -e^{-t}$





5. A particle moves along a line so that at time t, where $0 \le t \le 5$, its velocity is given by $v(t) = -t^3 + 6t^2 - 15t + 10$. What is the minimum acceleration of the particle on the interval?

$a(t) = v'(t) = -3t^2 + 12t - 15 = -3(t^2 - 4t + 5)$



$a(t)_{min} = a(5)$

$= -30$

6. A particle moves along the x-axis so that at any time $t \ge 0$, its velocity is given by $v(t) = -t^3 e^{-t}$. At what value of t does v attain its minimum?

$v'(t) = -3t^2 e^{-t} + t^3 e^{-t}$

$= e^{-t} t^2(-3+t)$

$v'(t) = 0$: $t = 0$ or 3

0 − 3 + $v'(t)$

at time $t=3$, v attains its min.

D

7. The position of a particle moving along a line is given by $s(t) = t^3 - 12t^2 + 21t + 10$ for $t \ge 0$. For what value of t is the speed of the particle increasing?

(A) $1 < t < 7$ only

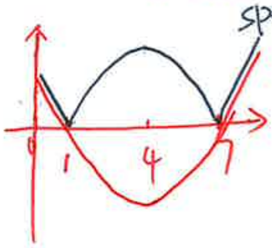
(B) $4 < t < 7$ only

(C) $0 < t < 1$ and $4 < t < 7$

(D) $1 < t < 4$ and $t > 7$

$v(t) = 3t^2 - 24t + 21$

$= 3(t^2 - 8t + 7)$



speed up when t:

$1 < t < 4$, $t > 7$

8. A particle moves along the x-axis so that its position at any time $t \ge 0$ is given by $x(t) = (t-2)^3(t-6)$.

(a) Find the velocity and acceleration of the particle at any time $t \ge 0$.

(b) Find the value of t when the particle is moving and the acceleration is zero. $v(t) \ne 0$

(c) When is the particle moving to the right?

(d) When is the velocity of the particle decreasing?

(e) When is the speed of the particle increasing?

(a) $v(t) = x'(t) = (t-2)^2 4 \cdot (t-5)$

$a(t) = v'(t) = 12(t-2)(t-4)$

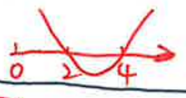
(b) $a(t) = 0$: $t = 2$ or $t = 4$

$v(2) = 0$ $v(4) \ne 0$

$\therefore t = 4$

(c) $v(t) > 0$ when $t > 5$

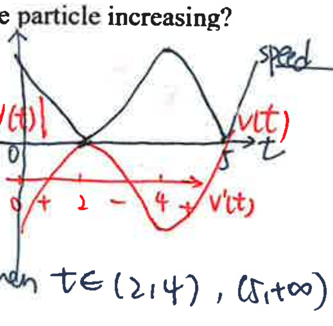
(d) $v(t) \downarrow$ when $a(t) < 0$



i.e. $t \in (2,4)$

(e) $\boxed{\text{method 1}}$ speed $= |v(t)|$

According to $a(t)$:



$\therefore$ speed up when $t \in (2,4)$, $(5, +\infty)$

method 2. $\boxed{a(t) \cdot v(t) > 0}$

0 2 4 5 t

$a(t)$: + − + +

$v(t)$: − − − +

$\therefore 2 < t < 4$,

$t > 5$

➢ Optimization Problems

1\. C

Domain $x \in (0, +\infty)$. Target $f(x) = x \cdot \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right) = x^{\frac{1}{2}} - x^{\frac{3}{2}}$

If $y = \dfrac{1}{\sqrt{x}} - \sqrt{x}$, what is the maximum value of the product of xy ?

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$

(A) $\dfrac{1}{9}$ (B) $\dfrac{\sqrt{3}}{9}$ (C) $\dfrac{2\sqrt{3}}{9}$ (D) $\dfrac{2}{3}$

$= \frac{1}{2}x^{-\frac{1}{2}} \cdot (1 - 3x)$

$f'(x) = 0$ when $x = \frac{1}{3}$

$f'(x)$ is defined on $(0, +\infty)$

$\therefore$ 0 + $\frac{1}{3}$ − $f'(x)$

↗ ↘ $f(x)$

$\therefore f_{max} = f\left(\frac{1}{3}\right) = \frac{2}{9}\sqrt{3}$

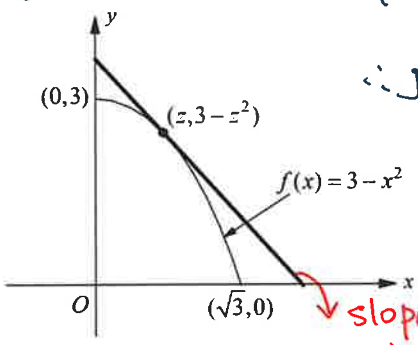
2\. B

If the maximum value of the function $y = \dfrac{\cos x - m}{\sin x}$ is at $x = \dfrac{\pi}{4}$, what the value of m ?

(A) $-\sqrt{2}$ (B) $\sqrt{2}$ (C) -1 (D) 1

$y = \cot x - m \csc x$

$y' = -\csc^2 x - m \cdot (-\cot x \csc x) = \csc x \cdot (-\csc x + m \cot x) = 0$ when $x = \frac{\pi}{4}$



$\therefore \sqrt{2} \cdot (-\sqrt{2} + m) = 0$

$\therefore m = \sqrt{2}$

x-intercept $= \frac{z^2+3}{2z}$

y-intercept $= z^2 + 3$

slope $= f'(z) = -2z$

equation: $y - (3 - z^2) = -2z(x - z)$

$\therefore$ area is

$A(z) = \frac{1}{2} \cdot (z^2+3) \cdot \frac{3+z^2}{2z}$

$= \frac{(z^2+3)^2}{4z}$

The figure above shows the graph of the function $f(x) = 3 - x^2$. For $0 < z < \sqrt{3}$, let $A(z)$ be the area of the triangle formed by the coordinate axes and the line tangent to the graph of f at the point $(z, 3 - z^2)$.

(a) Find the equation of the line tangent to the graph of f at the point $(z, 3 - z^2)$.

$y - (3 - z^2) = -2z(x - z)$

(b) For what values of z does the triangle bounded by the coordinate axis and tangent line have a minimum area?

$A'(z) = \frac{4(3+z^2)(3z^2-3)}{16z^2} = 0$ then $z = 1$

$A'(z)$ is defined when $0 < z < \sqrt{3}$

0 − 1 + $\sqrt{3}$ $A'(z)$

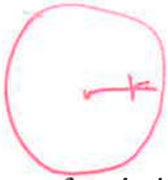
↘ ↗ $A(z)$

$\therefore$ The triangle has the min area when $z = 1$.

➢ Related Rates

B 1. The radius of a circle is changing at the rate of $1/\pi$ inches per second. At what rate, in square inches per second, is the circle's area changing when $r = 5$ in ?

(A) $\frac{5}{\pi}$ (B) 10 (C) $\frac{10}{\pi}$ (D) 15



$\frac{dr}{dt} = \frac{1}{\pi}$ in/sec

Area $= \pi r^2$

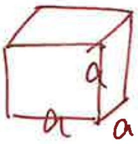
$\frac{dA}{dt} = \pi \cdot 2r \frac{dr}{dt}$

$\left.\frac{dA}{dt}\right|_{r=5} = 2\pi \cdot 5 \cdot \frac{1}{\pi} = 10$ in²/sec

D 2. The volume of a cube is increasing at the rate of $12\ \text{in}^3/\text{min}$. How fast is the surface area increasing, in square inches per minute, when the length of an edge is 20 in?

(A) 1 (B) $\frac{6}{5}$ (C) $\frac{4}{3}$ (D) $\frac{12}{5}$

$V = a^3$

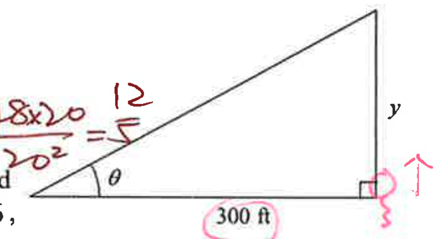


$\frac{dV}{dt} = 12\ \text{in}^3/\text{min} = 3a^2 \frac{da}{dt}$

$\frac{da}{dt} = \frac{4}{a^2}$

Surface Area $= 6a^2$.

$\frac{dA}{dt} = 12a \frac{da}{dt} = 12a \cdot \frac{4}{a^2} = \frac{48 \times 20}{20^2} = \frac{12}{5}$



C 3. In the figure shown above, a hot air balloon rising straight up from the ground is tracked camera 300 ft from the liftoff point. At the moment the camera's elevation angle is $\pi/6$, rising at the rate of 80 ft/min . At what rate is the angle of elevation changing at that moment?

(A) 0.12 radian per minute

(B) 0.16 radian per minute

(C) 0.2 radian per minute

(D) 0.4 radian per minute

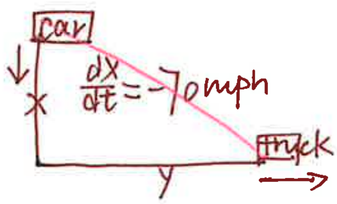
$\left.\frac{dy}{dt}\right|_{\theta=\frac{\pi}{6}} = 80$ ft/min

$\tan\theta = \frac{y}{300}$

$\therefore \sec^2\theta \frac{d\theta}{dt} = \frac{1}{300} \cdot \frac{dy}{dt}$

$\left.\frac{d\theta}{dt}\right|_{\theta=\frac{\pi}{6}} = \frac{1}{300} \cdot 80 \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{5} = 0.2$ rad/min

C 4. A car is approaching a right-angled intersection from the north at 70 mph and a truck is traveling to the east at 60 mph. When the car is 1.5 miles north of the intersection and the truck is 2 miles to the east, at what rate, in miles per hour, is the distance between the car and truck is changing?



$\boxed{\text{Aim: } \frac{d(\text{distance})}{dt} \quad x = 1.5, y = 2}$

distance $:= f(x, y) = \sqrt{x^2 + y^2}$

$\frac{dy}{dt} = 60$ mph

$f'(t) = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt}\right)$

$= 6$ mph

(A) Decreasing 15 miles per hour

(B) Decreasing 9 miles per hour

(C) Increasing 6 miles per hour

(D) Increasing 12 miles per hour

A 5. The radius r of a sphere is increasing at a constant rate. At the time when the surface area and the radius of sphere are increasing at the same numerical rate, what is the radius of the sphere? (The surface area of a sphere is $S = 4\pi r^2$.)

(A) $\frac{1}{8\pi}$ (B) $\frac{1}{4\pi}$ (C) $\frac{1}{3\pi}$ (D) $\frac{\pi}{8}$

$\frac{dS}{dt} = 4\pi 2r \frac{dr}{dt} = \frac{dr}{dt}$

$\therefore 8\pi r = 1$

$\therefore r = \frac{1}{8\pi}$

C 6. If the radius r of a cone is decreasing at a rate of 2 centimeters per minute while its height h is increasing at a rate of 4 centimeters per minute, which of the following must be true about the volume V of the cone?

(A) V is always decreasing.

(B) V is always increasing.

(C) V is increasing only when $r > h$.

(D) V is increasing only when $r < h$.

$\frac{dr}{dt} = -2\text{ cm/min}$

$\frac{dh}{dt} = 4\text{ cm/min}$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{\pi}{3}\left(2r\frac{dr}{dt}\cdot h + r^2\frac{dh}{dt}\right)$

$= \frac{\pi}{3} r\left[2\cdot(-2)h + r\cdot 4\right]$

$= \frac{4}{3}\pi r(r-h)$

>0

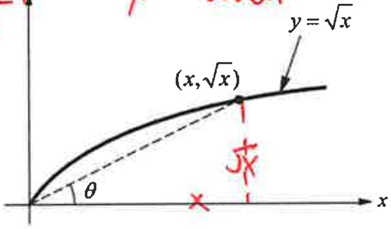
7. A particle moves along the curve $y = \sqrt{x}$. When $y = 2$, the x-component of its position is increasing at the rate of 4 units per second.

(a) The rate given by the question is $\frac{dx}{dt} = 4$ units/sec. when $y=2$

(a) What is the value of $\frac{dy}{dt}$ when $y = 2$? $\frac{dy}{dt} = \frac{1}{2}x^{-\frac{1}{2}}\frac{dx}{dt}$. When $y=2$, $x=4$.

(b) How fast is the distance from the particle to the origin changing when $y = 2$?

(c) What is the value of $\frac{d\theta}{dt}$ when $y = 2$?



So $\left.\frac{dy}{dt}\right|_{y=2} = \frac{1}{2}\cdot 4^{-\frac{1}{2}}\cdot 4$

$= 1$ unit/sec

(b) distance $= f(x) = \sqrt{(x-0)^2 + (\sqrt{x}-0)^2}$

$= \sqrt{x^2+x}$

$f'(t) = \frac{1}{2}(x^2+x)^{-\frac{1}{2}}\cdot\left[2x\cdot x'(t) + x'(t)\right]$

$\left[f'(t)\right]_{y=2} = \frac{9}{5}\sqrt{5}$ unit/sec

(c) $\tan\theta = \frac{\sqrt{x}}{x} = x^{-\frac{1}{2}}$

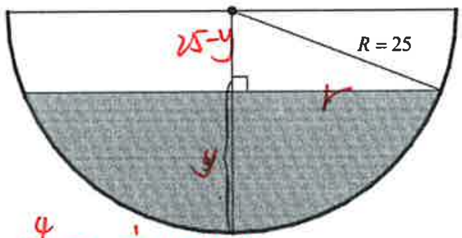
$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}(x^{-\frac{1}{2}})$

$\therefore \sec^2\theta\cdot\frac{d\theta}{dt} = -\frac{1}{2}x^{-\frac{3}{2}}\cdot\frac{dx}{dt}$

$\because$ when $y=2$, $\cos\theta = \frac{2}{\sqrt{5}}$

$\cos^2\theta = \frac{4}{5}$

$\therefore \left.\frac{d\theta}{dt}\right|_{y=2} = -\frac{1}{2}\cdot 4^{-\frac{3}{2}}\cdot 4\cdot\frac{4}{5} = -\frac{1}{5}$ rad/sec



8. As shown in the figure above, water is draining at the rate of 12 ft^3/min from a hemispherical bowl of radius 25 feet. The volume of water in a hemispherical bowl of radius R when the depth of the water is y meters is given as $V = \frac{\pi}{3}y^2(3R - y)$.

$R = 25$ ft

$V = \frac{\pi}{3}y^2(75-y)$

$\frac{dV}{dt} = -12\ \text{ft}^3/\text{min}$

(a) Find the rate at which the depth of water is decreasing when the water is 18 meters deep. Indicate units of measure.

$\left.\frac{dy}{dt}\right|_{y=18}$

(b) Find the radius r of the water's surface when the water is y feet deep. (b) $r = \sqrt{25^2 - (25-y)^2} = \sqrt{50y - y^2}$

(c) At what rate is the radius r changing when the water is 18 meters deep. Indicate units of measure.

(a) $\frac{dV}{dt} = \frac{dy}{dt}(50\pi y - \pi y^2)$

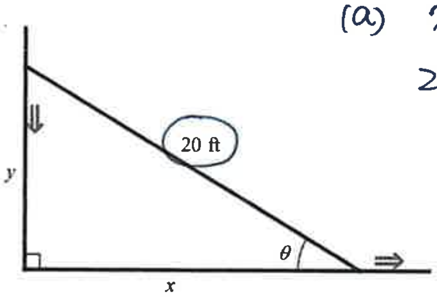
$\therefore \left.\frac{dy}{dt}\right|_{y=18} = (-12)\cdot\frac{1}{50\pi\cdot 18 - \pi 18^2} = -\frac{1}{48\pi}$ ft/min

(c) $\left.\frac{dr}{dt}\right|_{y=18} = ?$

$\frac{d}{dt}[r] = \frac{d}{dt}\left[\sqrt{50y-y^2}\right]$

$\frac{dr}{dt} = \frac{1}{2}(50y-y^2)^{-\frac{1}{2}}\cdot\left(50\frac{dy}{dt} - 2y\frac{dy}{dt}\right)$

$\left.\frac{dr}{dt}\right|_{y=18} = -\frac{7}{1152\pi}$ ft/min



(a) $x^2+y^2=20^2$

$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$

$\therefore \left.\frac{dx}{dt}\right|_{y=12}=\frac{3}{2}$ ft/sec

(b) Area $=\frac{1}{2}xy$

$\frac{dA}{dt}=\frac{1}{2}\cdot\left(\frac{dx}{dt}y+\frac{dy}{dt}x\right)$

$\left.\frac{dA}{dt}\right|_{y=12}=-7$ ft²/sec

9. In the figure shown above, the top of a 20-foot ladder is sliding down a vertical wall at a constant rate of 2 feet per second. $\frac{dy}{dt}=-2$ ft/sec

(a) When the top of the ladder is 12 feet from the ground, how fast is the bottom of the ladder moving away from the wall? when $y=12$, $x=16$ $\quad \left.\frac{dx}{dt}\right|_{y=12}=?$

(b) The triangle is formed by the wall, the ladder and the ground. At what rate is the area of the triangle is changing when the top of the ladder is 12 feet from the ground?

(c) At what rate is the angle θ between the ladder and the ground is changing when the top of the ladder is 12 feet from the ground?

(c) $\tan\theta=\frac{y}{x}$

$\sec^2\theta\,\frac{d\theta}{dt}=\frac{\frac{dy}{dt}x-\frac{dx}{dt}y}{x^2}$

when $y=12$, $x=16$ and $\cos^2\theta=\left(\frac{4}{5}\right)^2$

$\therefore \left.\frac{d\theta}{dt}\right|_{y=12}=-\frac{1}{8}$ rad/sec

10. Consider the curve given by $2y^2+3xy=1$.

(a) Find $\frac{dy}{dx}$. $\quad$ (a). $4y\frac{dy}{dx}+3\cdot y+3x\cdot\frac{dy}{dx}=0 \quad \therefore \frac{dy}{dx}=\frac{-3y}{4y+3x}$

(b) Find all points (x, y) on the curve where the line tangent to the curve has a slope of $-\frac{3}{4}$. $\quad$ (b) $\frac{dy}{dx}=-\frac{3}{4}=\frac{-3y}{4y+3x}$

(c) Let x and y be functions of time t that are related by the equation $2y^2+3xy=1$. At time $t=3$, the value of y is 2 and $\frac{dy}{dt}=-2$. Find the value of $\frac{dx}{dt}$ at time $t=3$.

$\therefore 4y+3x=4y$

$\therefore x=0$

又 $\because 2y^2+3xy=1$

$\therefore y=\pm\frac{\sqrt{2}}{2}$

$\therefore$ points are $\left(0,\pm\frac{\sqrt{2}}{2}\right)$

(c) $y(t=3)=2$

$\left.\frac{dy}{dt}\right|_{t=3}=-2$

$\left.\frac{dx}{dt}\right|_{t=3}=?$

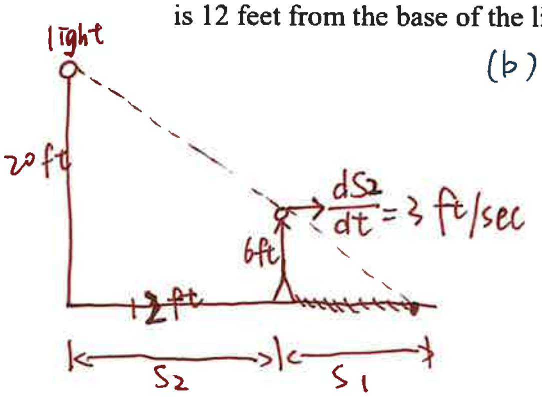
$4y\cdot\frac{dy}{dt}+3\frac{dx}{dt}y+3x\frac{dy}{dt}=0$

$\therefore \left.\frac{dx}{dt}\right|_{t=3}=\frac{3}{2}$

11. A man 6 feet tall walks at a rate of 3 feet per second away from a light that is 20 feet above the ground.

(a) At what rate is the tip of his shadow moving when he is 12 feet from the base of the light.

(b) At what rate is the length of his shadow changing when is 12 feet from the base of the light.



(a) $\left.\frac{dS}{dt}\right|_{S_2=12ft}=?$ $\quad S=S_1+S_2$

$\frac{S_1}{S_1+S_2}=\frac{6}{20} \quad \frac{S_1}{S}=\frac{3}{10} \quad \therefore \frac{S_2}{S}=\frac{7}{10}$

$\therefore \frac{dS_2}{dt}=\frac{7}{10}\frac{dS}{dt}$

$\therefore \left.\frac{dS}{dt}\right|_{S_2=12ft}=\frac{10}{7}\cdot 3=\frac{30}{7}$ ft/sec

(b) $\left.\frac{dS_1}{dt}\right|_{S_2=12}=?$

$\frac{dS_1}{dt}=\frac{3}{10}\frac{dS}{dt}$

$=\frac{3}{10}\cdot\frac{30}{7}$

$=\frac{9}{7}$ ft/sec

6

- Tangent Line Approximation

$f(x)=x^{\frac{1}{3}}$ $\quad = (8+h)^{\frac{1}{3}}$

$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}$

Linearization at $x=8$:

$L(x)=f(8)+f'(8)(x-8)$

$=2+\frac{1}{12}\cdot\underbrace{(x-8)}_{=h}$

C 1. For small values of h, the function $h(x)=\sqrt[3]{8+h}$ is best approximated by which of the following?

(A) $\frac{h}{12}$ (B) $2-\frac{h}{12}$ (C) $2+\frac{h}{12}$ (D) $3+\frac{h}{12}$

$h(x)$ can be approximated by (the tangent line at $x=8$) at $x=8+h$

$L(8+h)$

D 2. The approximate value of $y=\sqrt{1-\sin x}$ at $x=-0.1$, obtained from the line tangent to the graph at $x=0$, is

$f'(x)=\frac{1}{2}(1-\sin x)^{-\frac{1}{2}}\cdot(-\cos x) \Rightarrow f'(0)=-\frac{1}{2}$

(A) 0.9 (B) 0.95 (C) 1.01 (D) 1.05

$L(x)=f(0)+f'(0)(x-0)=1-\frac{1}{2}x$

$L(-0.1)=1+\frac{0.1}{2}=1.05$

C 3. Let $y=x^2\ln x$. When $x=e$ and $dx=0.1$, the value of dy is

$\frac{dy}{dx}=2x\cdot\ln x+x^2\cdot\frac{1}{x}$

(A) $\frac{e}{10}$ (B) $\frac{e}{5}$ (C) $\frac{3e}{10}$ (D) $\frac{2e}{5}$

$\left.\frac{dy}{dx}\right|_{x=e}=2e+e=3e$

$dy=3e\cdot 0.1=0.3e=\frac{3e}{10}$

A 4. Let f be a differentiable function such that $f(2)=\frac{5}{2}$ and $f'(2)=\frac{1}{2}$. If the line tangent to the graph of f at $x=2$ is used to find an approximation of a zero of f, that approximation is

$L(x)=f(2)+f'(2)(x-2)=\frac{5}{2}+\frac{1}{2}(x-2)$

(A) -3 (B) -2.4 (C) -1.8 (D) -1.2

Let $L(x)=0$ then $x=-3$

B 5. The approximate value of $y=\frac{1}{\sqrt{x}}$ at $x=4.1$, obtained from the line tangent to the graph at $x=4$ is

$y'(x)=-\frac{1}{2}x^{-\frac{3}{2}}$

$L(x)=y(4)+y'(4)(x-4)=\frac{1}{2}+(-\frac{1}{2})\cdot\frac{1}{8}(x-4)$

(A) $\frac{39}{80}$ (B) $\frac{79}{160}$ (C) $\frac{1}{2}$ (D) $\frac{81}{160}$

$L(4.1)=\frac{1}{2}-\frac{1}{160}=\frac{79}{160}$

C 6. Let f be the function given by $f(x)=x^2-4x+5$. If the line tangent to the graph of f at $x=1$ is used to find an approximate value of f, which of the following is the greatest value of x for which the error resulting from this tangent line approximation is less than 0.5?

$L(x)=f(1)+f'(1)(x-1)$

$=2-2(x-1)$

$=-2x+4$

$|f(x)-L(x)|\le 0.5$

(A) 1.5 (B) 1.6 (C) 1.7 (D) 1.8

$=|x^2-2x+1|=(x-1)^2<0.5$

$0\le x-1<\sqrt{0.5}$

$1\le x<\sqrt{0.5}+1\approx 1.707$

$y - y(a) = y'(a)\ (x - a)$

$y = y'(a) \cdot x + y(a) - a\,y'(a)$

D 7. The linear approximation to the function f at $x = a$ is $y = \frac{1}{2}x - 3$. What is the value of $f(a) + f'(a)$ in terms of a?

$\Rightarrow y(a) - a \cdot y'(a) = -3$

$y'(a) = \frac{1}{2}$

(A) $a - 4$

(B) $a - \frac{5}{2}$

(C) $\frac{1}{2}a - 4$

(D) $\frac{1}{2}a - \frac{5}{2}$

$\therefore y(a) = -3 + a \cdot \frac{1}{2} = \frac{a}{2} - 3$

$\therefore f(a) + f'(a) = \frac{a}{2} - 3 + \frac{1}{2} = \frac{a}{2} - \frac{5}{2}$

8. Let f be the function given by $f(x) = \dfrac{2}{e^{\sin x} + 1}$.

(a) Write an equation for the line tangent to the graph of f at $x = 0$.

(b) Using the tangent line to the graph of f at $x = 0$, approximate $f(0.1)$.

(c) Find $f^{-1}(x)$.

a) $f'(x) = 2 \cdot (-1)\,(e^{\sin x} + 1)^{-2} \cdot e^{\sin x} \cos x$

$f'(0) = -\frac{1}{2}$

$f(0) = 1$

$\therefore\ y - 1 = -\frac{1}{2}x$

$\therefore y = -\frac{1}{2}x + 1$

(b) $L(x) = -\frac{1}{2}x + 1$

$f(0.1) \approx L(0.1) = 0.95$

(c) $y = \dfrac{2}{e^{\sin x} + 1}$ $\left(x \in [-\frac{\pi}{2}, \frac{\pi}{2}] : \ e^{\sin x} \in [\frac{1}{e}, e] \ \therefore y \in [\frac{2}{e+1}, \frac{2e}{e+1}] \right)$

$\frac{2}{y} = e^{\sin x} + 1$

$\frac{2}{y} - 1 = e^{\sin x}$

$\ln(\frac{2}{y} - 1) = \sin x$

$\therefore x = \sin^{-1}[\ln(\frac{2}{y} - 1)] \Rightarrow x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\therefore f^{-1}(x) = \sin^{-1}[\ln(\frac{2}{x} - 1)]$ $\quad x \in [\frac{2}{e+1}, \frac{2e}{e+1}]$

Domain of the Inverse function is the range of function f(x)

x	-2	0	1	3	6
$f(x)$	-1	-4	-3	0	7

9. Let f be a twice differentiable function such that $f'(3) = \frac{9}{5}$. The table above gives values of f for selected points in the closed interval $-2 \le x \le 6$.

(a) Estimate $f'(0)$. Show the work that leads to your answer.

(b) Write an equation for the line tangent to the graph of f at $x = 3$.

(c) Write an equation of the secant line for the graph of f on $1 \le x \le 6$.

(d) Suppose $f''(x) > 0$ for all x in the closed interval $1 \le x \le 6$. Use the line tangent to the graph of f at $x = 3$ to show $f(5) \ge \frac{18}{5}$.

(e) Suppose $f''(x) > 0$ for all x in the closed interval $1 \le x \le 6$. Use the secant line for the graph of f on $1 \le x \le 6$ to show $f(5) \le 5$.

(a) $f'(0) \approx \frac{f(1) - f(-2)}{3} = -\frac{2}{3}$

(b) $f'(3) = \frac{9}{5}$ $\quad f(3) = 0$

$\therefore y = \frac{9}{5}(x-3)$

(c) $AROC = \frac{f(6) - f(1)}{6-1} = 2$

$\therefore y - 7 = 2(x-6)$

$\therefore y = 2x - 5$

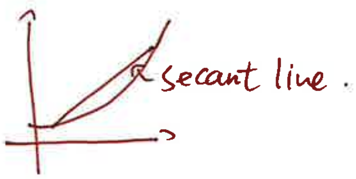
(d) $f''(x) > 0$ concave up.

Linear approximation $L(x) = \frac{9}{5}(x-3)$

$L(5) = \frac{18}{5}$

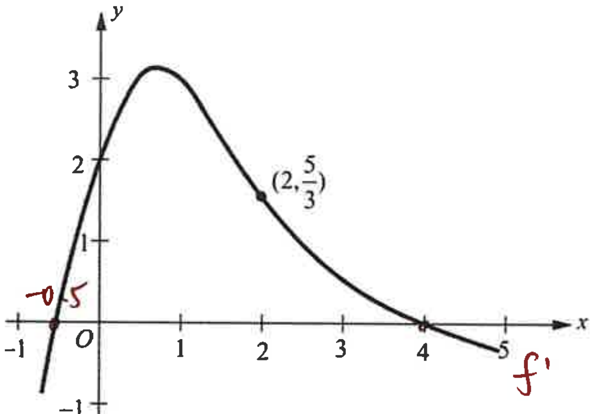
Since $f''(x) > 0$, then the tangent line approximation is smaller than the real value. Therefore, $f(5) \geq \frac{18}{5}$.

(e)



$f''(x) > 0$: the secant line lies above the curve for $[1, 6]$.

Therefore $y = 2x - 5$ at $x = 5$: $2 \times 5 - 5 = 5 \geq f(5)$



Graph of f'

10. Let f be twice differentiable function on the interval $-1 < x < 5$ with $f(1) = 0$ and $f(2) = 3$. The graph of f', the derivative of f, is shown above. The graph of f' crosses the x- axis at $x = -0.5$ and $x = 4$. Let h be the function given by $h(x) = f(\sqrt{x+1})$.

(a) Write an equation for the line tangent to the graph of h at $x = 3$.

(b) The second derivative of h is $h''(x) = \frac{1}{4}\left[\frac{\sqrt{x+1}f''(\sqrt{x+1}) - f'(\sqrt{x+1})}{(x+1)^{3/2}}\right]$. Is $h''(3)$ positive, negative, or zero? Justify your answer.

(c) Suppose $h''(x) < 0$ for all x in the closed interval $0 \le x \le 3$. Use the line tangent to the graph of h at $x = 3$ to show $h(2) \le \frac{31}{12}$. Use the secant line for the graph of h on $0 \le x \le 3$ to show $h(2) \ge 2$

(a) $h'(x) = f'(\sqrt{x+1}) \cdot \frac{1}{2}(x+1)^{-\frac{1}{2}}$

$h'(3) = \frac{5}{12}$

$h(3) = f(2) = 3$

$\therefore y - 3 = \frac{5}{12}(x-3)$

(b) numerator $(x=3) = 2\underbrace{f''(2)}_{<0} - \underbrace{f'(2)}_{\frac{5}{3}} < 0$ negative

$(x+1)^{\frac{3}{2}} > 0$ for $x=3$

$\therefore h''(x) < 0$ at $x=3$

(c) tangent line: $y = 3 + \frac{5}{12}(x-3)$

$y(2) = \frac{31}{12}$

Since $h''(x) < 0$, the tangent line lies above the curve of $h(x)$,

$h(2) \le \frac{31}{12}$.

secant line: slope $= \frac{h(3) - h(0)}{3} = 1$

$\therefore y = (x-3) + 3 = x$ $\quad y(2) = 2$

$h''(x) < 0$, then concave down, the secant line is below the curve of $h(x)$. So $h(2) \ge 2$.