Geometric Series: $\sum_{n=1}^{\infty} (\frac{1}{2})^n = \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \cdots$

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

We have considered **real-number** sequences and series, such as the arithmetic sequence, the p-series, and the geometric series. In this section, we consider sequences and series whose terms are **functions**.

Sequence of functions, Infinite series of functions

- **Sequence of functions** $\{f_n(x)\}_{n=1}^{\infty} = \{f_1(x), f_2(x), \dots, f_n(x), \dots\}$

  If $\{f_n(x_0)\}_{n=1}^{\infty} = \{f_1(x_0), f_2(x_0), \dots, f_n(x_0), \dots\}$ is convergent, then we say the sequence of functions is convergent at point $x = x_0$.

✍ Example. $f_n(x) = x^n$, find the interval of convergence.

✍ $f_n(x) = \frac{2xn+(-1)^n x^2}{n}$, find the interval of convergence.

- **Series of functions** $\sum_{n=1}^{\infty} f_n(x)$

We can view series of functions as a **function** where the domain is the interval where the series converges.

✍ $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ is a (geometric) power series. Find the interval where the series converges.

- **Power Series**

  A power series about $x = 0$ is $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$

  More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-c)^n = c_0 + c_1(x-c) + c_2(x-c)^2 + \cdots$ is called a power series centered at ______.

  - Convergence of a Power Series

    In most cases, the convergent interval can be found by using the **Ratio Test.**

Example. Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{\sqrt{n+3}}$

- Practice

1. Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} n!(2x)^n$.

2. Find the radius of convergence and interval of convergence of the series
$$\sum_{n=0}^{\infty} \frac{n^2 x^n}{n!}.$$

3. Find the radius of convergence and the interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}\, x^{n+1}.$$

For a power series centered at 0, there are only three possibilities:

1. Converges only at 0
2. Converges on $\mathbb{R}$ (for all x)
3. Converges for $|x| < R$, and diverges for $|x| > R$, where R is a positive real number.

R: radius of convergence

Geometric Power Series

$$\_\_\_\_\_\_\_\_\_\_\_\_ = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

? Find the power series expansion of the following functions.

1. $f(x) = \frac{2}{1+x^2}$

2. $f(x) = \frac{x^2}{1+x}$

3. $f(x) = \frac{1}{1+x}$

➢ **Have a try! How to write a function into the form of power series?**

We have known the convergent interval of the series $\sum_{n=0}^{\infty} x^n$, so we can write the function $f(x) =$________________________ into the form of a series when the series converges to $f(x)$.

? Write the function $f(x) = ln\ (x + 1)$ into the form of the series and find its domain.

? Write the function $f(x) = \frac{1}{(x+1)^2}$ into the form of series and find its domain.

➢ **Operations**

- **Substitution:** New series can be generated by making an appropriate substitution in a known series.
- **Differentiation & Integration**

If the function given by $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ is differentiable, then

- $f'(x) =$ ______________________

- $\int f(x)dx =$ ____________________

Convergency:

Practice

1. If $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-2)^n}{n!} = (x-2) - \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} - \frac{(x-2)^4}{4!} + \cdots$, which of the following represents $f'(x)$?

(A) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-2)^{n-1}}{n!}$

(B) $\sum_{n=1}^{\infty} (-1)^{n} \frac{(x-2)^{n-1}}{(n+1)!}$

(C) $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-2)^{n-1}}{n!}$

(D) $\sum_{n=0}^{\infty} (-1)^{n} \frac{(x-2)^{n}}{n!}$

2. Let f be a function given by $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n}$. Find the interval of convergence for each of the following.

(1) $f(x)$

(2) $f'(x)$

(3) $\int f(x)\,dx$

3. The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n)!} = 1 - \frac{3x^2}{2!} + \frac{5x^4}{4!} - \frac{7x^6}{6!} + \cdots + (-1)^n \frac{(2n+1)x^{2n}}{(2n)!} + \cdots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that $1 - \frac{3}{2!} + \frac{5}{4!}$ approximates $f(1)$ with an error less than $\frac{1}{100}$.

(c) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Write the first four terms and the general term of the power series expansion of $\frac{g(x)}{x}$.

- **Taylor Polynomial and Maclaurin Polynomial**

? Linear Approximation of a function $f(x)$ at point $x = a$

■ How to expand a function $f(x)$ into the form of power series?

How to find the coefficients?

✧ If a function f has n derivatives at a, then the polynomial

$$P_n(x) = \_\_\_\_ + \_\_\_\_\_\_\_\_(x-a) + \_\_\_\_\_\_\_\_(x-a)^2 + \_\_\_\_\_\_\_\_(x-a)^3 + \ldots + \_\_\_\_\_\_\_\_(x-a)^n$$

is called the **nth Taylor Polynomial for f** at a.

✧ When $c = 0$, then

$$P_n(x) = \_\_\_\_ + \_\_\_\_\_\_\_\_x + \_\_\_\_\_\_\_\_x^2 + \_\_\_\_\_\_\_\_x^3 + \ldots + \_\_\_\_\_\_\_\_x^n$$

is called the **nth Maclaurin Polynomial for f.**

Guidelines for Finding a Taylor Polynomial

1. Differentiate $f(x)$ several times and evaluate each derivative at c.

$$f(c),\ f'(c),\ f''(c),\ f'''(c),\ \cdot\ \cdot\ \cdot,\ f^{(n)}(c)$$

2. Use the sequence developed in the first step to form the **Taylor coefficients** $a_n = \dfrac{f^{(n)}(c)}{n!}$.

✍ **Practice**

1. Let f be the function given by $f(x) = \ln(2 - x)$. Write the third-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(1.2)$.

2. Let $P(x) = 3 - 2(x-2) + 5(x-2)^2 - 12(x-2)^3 + 3(x-2)^4$ be the fourth-degree Taylor polynomial for the function f about $x = 2$. Assume f has derivatives of all orders for all real numbers.

(a) Find $f(2)$ and $f'''(2)$.

(b) Write the third-degree Taylor polynomial for f' about 2 and use it to approximate $f'(2.1)$.

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_2^x f(t)\, dt$ about 2.

(d) Can $f(1)$ be determined from the information given? Justify your answer.

3. The second-degree Taylor polynomial of $\sec x$ about $x=\frac{\pi}{4}$ is

(A) $P_2(x)=1+\sqrt{2}(x-\frac{\pi}{4})+\sqrt{2}(x-\frac{\pi}{4})^2$

(B) $P_2(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})+\frac{3\sqrt{2}}{3!}(x-\frac{\pi}{4})^2$

(C) $P_2(x)=\sqrt{2}+\sqrt{2}(x-\frac{\pi}{4})+\frac{3\sqrt{2}}{2!}(x-\frac{\pi}{4})^2$

(D) $P_2(x)=1+\sqrt{2}(x-\frac{\pi}{4})+\frac{3\sqrt{2}}{3!}(x-\frac{\pi}{4})^2$

4. A function f has derivatives of all orders at $x=0$. Let P_n denote the nth- degree Taylor polynomial for f about $x=0$. It is known that $f(0)=\frac{1}{3}$ and $f''(0)=\frac{4}{3}$. If $P_2(\frac{1}{2})=\frac{1}{8}$, what is the value of $f'(0)$?

(A) $-\frac{3}{8}$ (B) $-\frac{3}{4}$ (C) $-\frac{5}{4}$ (D) $-\frac{3}{2}$

5. Let $P(x)=4-3x^2+\frac{13}{12}x^4-\frac{121}{360}x^6$ be the sixth-degree Taylor polynomial for the function f about $x=0$. What is the value of $f'''(0)$?

(A) $-\frac{121}{15}$ (B) $-\frac{3}{2}$ (C) 0 (D) $\frac{121}{15}$

➢ Taylor Series and Maclaurin Series

Taylor Series and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdot\cdot\cdot + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdot\cdot\cdot$$

is called the **Taylor series for $f(x)$ at c**. Moreover, if $c = 0$, then the series is called the **Maclaurin series for f**.

✍ Practice

1. Find the Maclaurin series for the function $f(x) = \ln(1+x)$.

2. Let f be a function having derivatives of all orders. The fourth degree Taylor polynomial for f about $x = 1$ is given
   $T(x) = 4 + 3(x-1) - 6(x-1)^2 + 7(x-1)^3 - 4(x-1)^4$.
   Find $f(1)$, $f'(1)$, $f''(1)$, $f'''(1)$ and $f^{(4)}(1)$.

- **Lagrange Error Bound**

- If $f(x)$ has $n+1$ derivatives on an open interval (a,b), for any $x \in [a,b]$, there exists ξ between x and x_0 such that $\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$

- If $f(x)$ has $n+1$ derivatives at c and $R_n(x)$, is the remainder term of the nth Taylor polynomial $P_n(x)$, then $f(x) =$ ________________

- The absolute value of $R_n(x)$ satisfies the inequality:

$$|R_n(x)| = |f(x) - P_n(x)| = \_\_\_\_\_\_\_\_\_\_\_\_ \leq \_\_\_\_\_\_\_\_\_\_\_\_$$

The remainder $R_n(x)$ is called the **Lagrange Error Bound.**

Practice

Let f be the function given by $f(x) = \sin(3x - \frac{\pi}{6})$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Use the Lagrange error bound to show that $\left| f(0.2) - P(0.2) \right| < \frac{1}{100}$.

- **Elementary Functions**

Function	Convergent Interval
$f(x) = \frac{1}{x} =$	
$f(x) = \frac{1}{1-x} =$	
$f(x) = \ln x =$	
$f(x) = e^x =$	
$f(x) = \sin x =$	
$f(x) = \cos x =$	
$f(x) = \tan^{-1} x =$	

- Multiplication of Power Series Power series can be multiplied the way we multiply polynomials.
- We usually find only the first few terms because the calculations for the later terms become tedious and the initial terms are the most important ones.

✍ Practice

1. Find the Maclaurin series for the function $f(x) = \cos x^2$.

2. Find the Maclaurin series for the function $f(x) = x^2 e^x - x^2$.

3. Find the first three nonzero terms in the Maclaurin series for $e^x \cos x$.