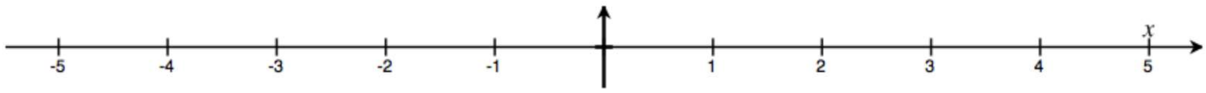
- Application of Differentiation (Contextual)
  - Straight-Line Motion
    - Position, Distance, Velocity, Acceleration, Speed...
    - Velocity & Speed
      - Speed up, Slow down
    - Velocity & Acceleration
  - Interpretation of Derivative
  - Rates of Change in Other Contexts
  - Related Rates
  - Linear Approximation of a Function

Considering the **straight-line motion** where an object moves along a straight line:

- **Position function:**

Notation for a position function with respect to time t is usually $\boldsymbol{s(t)}$ or $x(t)$ if the object is moving along the *x*-axis and $y(t)$ if the object is moving along the *y*-axis.

<u>Example 1</u>: For $s(t) = t^2 - 2t - 3$, show its position on the number line for $t = 0,1,2,3,4$.



- **Displacement =**

- **Total Distance =**

- **Average velocity** $= \dfrac{\boldsymbol{Final\ position - initial\ position}}{\boldsymbol{time\ inerval}} = \dfrac{\boldsymbol{s(t_1) - s(t_0)}}{\boldsymbol{t_1 - t_0}}$

- **Velocity Function** $\boldsymbol{v(t)}$=

  ✧ Moving forward (to the right) when $v(t)$ ___ 0

  Moving backward (to the left) when $v(t)$ ___ 0

  object **stopped** when $v(t)$ ___ 0

- **Speed =**

  The speed of an object must either be positive or <u>zero (meaning the object has stopped)</u>.

  ✧ Speed up when ________________

  Slow down when ______________

- **Acceleration** $\boldsymbol{a(t)}$ =

  ✧ a(t) > 0: object accelerating to the right, v(t) ______

  a(t) < 0: object accelerating to the left, v(t) ______

  a(t) =0: v(t) ____________________

Example 2: A particle moves along the x-axis with position function $s(t) = t^2 - 4t + 2$.

$v(t) =$ ____________________ $a(t) =$ ____________________

Complete the chart for the first 5 seconds and show where the particle is on the number line.

t	s(t)	v(t)	$\lvert v(t) \rvert$	a(t)	what direction the particle is moving	speeding up or slowing down
0						
2						
4						
6						

✧ Speed up when a(t) and v(t) have the ____________ sign(s)

Slow down when a(t) and v(t) have the ____________ sign(s)

➢ **The Relationship Between Velocity and Acceleration**

Fill in each box with either of the phrases: "speeding up," "slowing down," "constant speed," or "stopped."

How are we moving?	a(t) > 0	a(t) < 0	a(t) = 0
v(t) > 0			
v(t) < 0			
v(t) = 0			

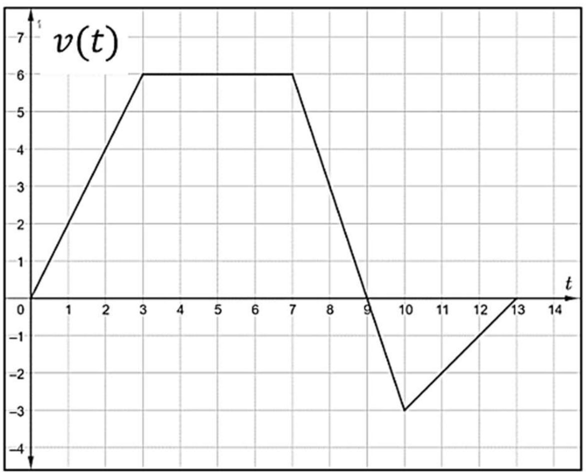
Example 3: The graph below models the velocity of a bug on the interval $0 \le t \le 13$.

a. Find $v(3)$ and $v(11)$.

b. Find $a(1), a(5)$, and $a(9)$.

c. At what time does the bug turn around?

d. On what interval does the bug have a negative acceleration?



Example 4:

A particle starts moving at time $t = 0$ and moves along the x-axis so that its position at time $t \geq 0$ is given by $x(t) = t^3 - \frac{9}{2}t^2 + 7$.

(a) Find the velocity of the particle at any time $t \geq 0$.

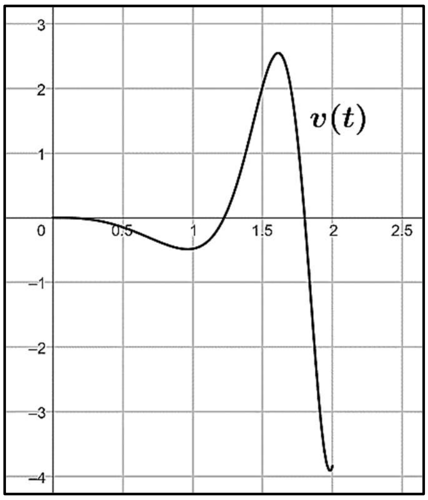
(b) For what values of t is the particle moving to the left.

(c) Find the values of t for which the particle is moving but its acceleration is zero.

(d) For what values of t is the speed of the particle decreasing?

Example 5: For $0 \leq t \leq 2$, a bug is moving along the x-axis with $v(t) = t^2 \sin(t^3 - 1.5t)$.

a. Is the bug speeding up or slowing down at $t = 0.7$?



b. Mark all points on the right figure when the bug is at rest.

c. Find the open interval(s) on $0 < t < 2$ when the bug is moving to the right.

d. Find the acceleration of the bug when $v(t) = -1$.

Example 6: A particle is moving along a horizontal line with position function $s(t) = t^3 - 9t^2 + 24t + 4$ for $t > 0$. Do an analysis of the particle's direction, acceleration, motion (speeding up or slowing down), and position.

【Summary】 To answer questions on particle motion, we will need to be able to decode questions into math equations. Consider the following phrases…

Statement	Translation
The bug is stopped…	
The bug is moving to the right	
The bug is moving to the left	
The bug turns around	
The bug is speeding up	
The bug is slowing down	

Example 7: A particle moves along the x-axis with position function, $x(t) = sin(e^{0.5t})$. Determine for which of the integer values t= 1, 2, 3, 4, 5, is the particle both to the right of the y-axis and is speeding up?

- **Interpreting the Meaning of the Derivative in Context**

Example 1: From the 2001 AP Calculus Exam BC 2

The temperature, in degrees Celsius (∘C), of the water in a pond is a differentiable function W of time t. The table to the right shows the water temperature as recorded every 3 days over a 15-day period.

t (days)	$W(t)$ ($\circ C$)
0	20
3	31
6	28
9	24
12	22
15	21

a. Use the data from the table to find an approximation for $W'(12)$.

b. Interpret the meaning of the derivative from part (a) in the context of the problem.

Scoring Guidelines from 2001

$$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer (with units)} \end{cases}$$

Example 2: From the 2013 AP Calculus Exam AB 1

On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday ($t = 0$), the plant has 500 tons of unprocessed gravel. During the hours of operation $0 \le t \le 8$, the plant processed gravel at a constant rate of 100 tons per hour.

a. Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.

Scoring Guidelines from 2013

$$2 : \begin{cases} 1 : G'(5) \\ 1 : \text{interpretation with units} \end{cases}$$

Example 3: From the 2007 AP Calculus Exam – Form B AB 3

The wind chill is the temperature, in degrees Fahrenheit ($°F$), a human feels based on the air temperature, in degrees Fahrenheit, and the wind velocity v, in miles per hour (mph). If the air temperature is $32° F$, then the wind chill is given by $W(v) = 55.6 - 22.1v^{0.16}$ and is valid for $5 \le v \le 60$.

a. Find $W'(20)$. Using correct units, explain the meaning of $W'(20)$ in terms of the wind chill.

Scoring Guidelines from 2007

$$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{explanation} \end{cases}$$

- Rates of Change in Other Contexts

Example 1: From the 2010 AP Calculus Exam AB/BC 1

There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. (t = 6). The rate $g(t)$, in cubic feet per hour, at which Janet removed snow from the driveway at time t hours after midnight is modeled by $g(t) = \begin{cases} 0, \text{ for } 0 \le t < 6 \\ 125, \text{ for } 6 \le t < 7 \\ 108, \text{ for } 7 \le t \le 9 \end{cases}$

a. Find the rate of change of the volume of snow on the driveway at 8 A.M.

Example 2: A penguin population on an island is modeled by a differentiable function P of time t, where P(t) represents the number of penguins and t is measured in years, for $0 \leq t \leq 40$. There are 100,000 penguins on the island at time $t = 0$. The birth rate for the penguins on the island is modeled by $B(t) = 1000e^{0.06}$ penguins per year and the death rate for the penguins on the island is modeled by $D(t) = 250e^{0.1t}$ penguins per year.

a. What is the rate of change of the penguin population on the island at $t = 0$?

Example 3: When a certain grocery store open, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by $f(t) = 10 + (0.8t)\,sin\left(\frac{t^3}{100}\right), 0 < t \leq 12$, where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by $g(t) = 3 + 2.4\,ln(t^2 + 2t), 3 < t \leq 12$, where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

a. Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.

Scoring Guidelines

$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{meaning} \end{cases}$

b. Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.

Scoring Guidelines

$2 : \begin{cases} 1 : \text{considers } f(5) \text{ and } g(5) \\ 1 : \text{answer with reason} \end{cases}$

- Optimization Problems

Guidelines for Solving Optimization Problems

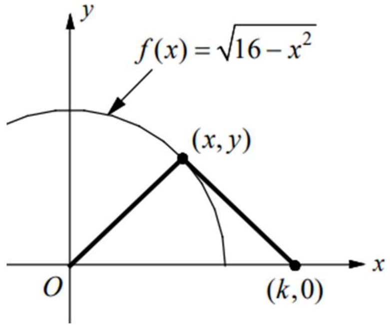
1. Read the problem carefully until you understand it.
2. In most problems it is useful to draw a picture. Label it with the quantities given in the problem.
3. Assign a variable to the unknown quantity and write an equation for the quantity that is to be maximized (or minimized), since this equation will usually involve two or more variables.
4. Use the given information to find relationships between these variables. Use these equations to eliminate all but one variable in the equation.
5. Use the first and second derivatives tests to find the critical points.

Example:

Let $f(x)=\sqrt{16-x^2}$. An isosceles triangle, whose base is the line segment from (0,0) to (k,0), where k>0, has its vertex on the graph of f as shown in the figure.

(a) Find the area of the triangle in terms of k.

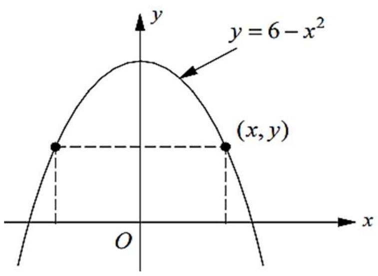
(b) For what values of k does the triangle have a maximum area?



Q1. Find the points on the curve $f(x)=\sqrt{x}$ that is nearest to the point (3,0).

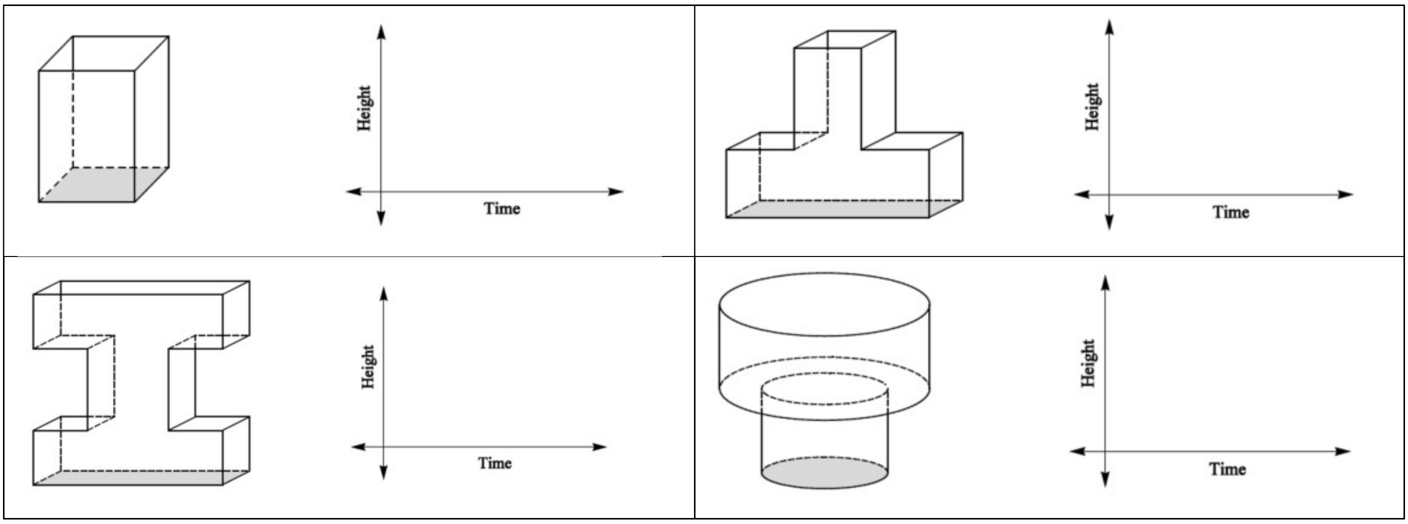
Q2. The point on the curve $y=2-x^2$ nearest to (3,2) is ________

Q3. What is the area of the largest rectangle that has its base on the x-axis and its other two vertices on the parabola $y=6-x^2$?



➢ Related Rates

Example 1: Suppose water is poured at a steady rate into containers whose shapes are shown. Sketch the graph of the height of the water versus time for each.



<u>Example 2</u>: Water runs into a conical tank at a rate of $0.5\ \mathrm{m}^3/min$. The tank stands point down and has a height of $4m$ and a base radius of $2m$. How fast is the water level rising when the water is $2.5m$ deep?

✧ Guideline for solving the related rate problem.

Step 1: Read the problem and make a sketch if possible.

Step 2: Write down the rates that are given.

Write down the rate you are trying to find.

Step 3: Find an equation that ties your variables together.

Step 4: Differentiate your equation with respect to time t. Remember, you are <u>implicitly</u> differentiating with respect to *t*.

Step 5: Substitute the given numerical information into the resulting equation and solve for the unknown rate.

Step 6: Label your answers in terms of the correct units (very important) and be sure you answered the question asked.

<u>Example 3</u>: A rectangle is undergoing changes to its length and width.

a. A rectangle's length is increasing at the rate of 2 inches/sec and its width is increasing at rate of 3 inches/sec. Find how fast the perimeter is changing at the moment when its length is 10 inches and its width is 6 inches.

b. Now, the rectangle's length is increasing by 3 inches/sec and its width is decreasing by 2 inches/sec. Find how fast the area is changing at the moment when its length is 10 inches and its width is 6 inches.

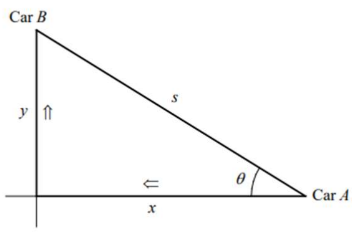
Example 4: A right triangle has sides whose lengths are changing. The short side is increasing at 3 in./sec and the long side is decreasing at 5 in/sec.
a. Find the rate of change of the area of the triangle at the moment the short side is 30 inches and the long side is 40 inches.
b. How fast is the hypotenuse changing at the moment when the short side is 30 inches and the long side is 40 inches?

Example 5: A right circular cylinder has a height and radius which are both changing.
a. The radius is growing at 2 feet/min and the height is shrinking at 3 feet/min. Find the rate of change of the volume of the cylinder at the moment the height is 10 feet and the radius is 8 feet.
b. The radius is decreasing at 4 feet/min and the height is increasing at 2 feet/min. Find the rate of change of the surface area of the cylinder at the moment the height is 10 feet and the radius is 8 feet.

Example 6: An oil tanker spills oil that spreads in a circular pattern whose radius increases at the rate of 50 feet/min. How fast are both the circumference and area of the spill increasing when the radius of the spill is 20 feet?

Example 7:

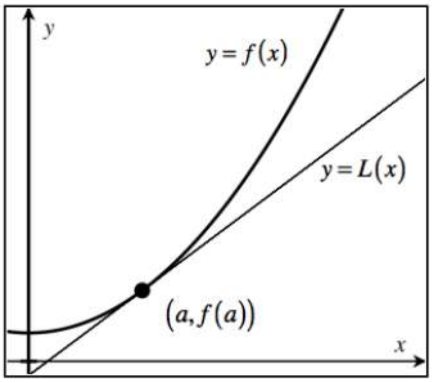
Car A is traveling due west toward the intersection at a speed of 45 miles per hour. Car B is traveling due north away from the intersection at a speed of 30 mph. Let x be the distance between Car A and the intersection at time t, and let y be the distance between Car B and the intersection at time t as shown in the figure at the right.



(a) Find the rate of change, in miles per hour, of the distance between the two cars when $x = 32$ miles and $y = 24$ miles.

(b) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 32$ miles and $y = 24$ miles.

- **Tangent Line Approximation**
- An equation for the tangent line at the point (a, f (a)) is given by :



- The linear function $L(x) = f(a) + f'(a)(x - a)$ is called the linearization of f at a. **Near** $x = a$, the function and the tangent line have nearly the same graph.

- If k were some x-value very near to a, there would be very little difference in the values of $L(k)$ and $f(k)$. In this case, we call the tangent line $f(k) \approx L(k) =$ ______________________ the **linear approximation** to the function at $x = a$.

- If the curve is concave upward, the line tangent to the graph of y=f(x) lies above/below the graph, so the tangent line approximation is greater/smaller than the real value.

Q1: Find the tangent line approximation of $f(x) = \sqrt{x-1}$ at c = 5 and approximate the number $\sqrt{3.95}$

Q2: Find the tangent line approximation of $f(x) = \tan x$ at $c = \frac{\pi}{4}$ and approximate the number $\tan 47°$

Q3: Use the linear approximation for $\boldsymbol{f(x) = \sqrt[3]{x}}$ **at** $\boldsymbol{x = 8}$ to approximate $\boldsymbol{\sqrt[3]{8.1}}$**.**

Q4: Using a calculator, find the error in using the linear approximation to $f(x) = e^x$ at $x = 0$ to approximate $\sqrt[4]{e}$.

Q5: For the tangent line to $f(x) = x^2 - 3x + 5$ at $x = 4$ to have an error of at most 0.1 when approximating $f(4+h)$, find the range of values for h.