Geometric Series: $\sum_{n=1}^{\infty}(\frac{1}{2})^n = \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \cdots$

p-series: $\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

We have considered **real-number** sequences and series, such as the arithmetic sequence, the p-series, and the geometric series. In this section, we consider sequences and series whose terms are <u>**functions**</u>.

Sequence of functions, Infinite series of functions

■ **Sequence of functions** $\{f_n(x)\}_{n=1}^{\infty} = \{f_1(x), f_2(x), \ldots, f_n(x), \ldots\}$

If $\{f_n(x_0)\}_{n=1}^{\infty} = \{f_1(x_0), f_2(x_0), \ldots, f_n(x_0), \ldots\}$ is convergent, then we say the sequence of functions is convergent at point $x = x_0$.

✎ Example. $f_n(x) = x^n$, find the interval of convergence.

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & |x|<1 \\ \infty\ (DNE), & |x|\geq 1 \end{cases}$$

$\therefore$ convergence interval is $(-1, 1)$

✎ $f_n(x) = \frac{2xn+(-1)^n x^2}{n}$, find the interval of convergence. $\mathbb{R}$

$\lim_{n\to\infty} f_n(x) = 2x$

■ **Series of functions** $\sum_{n=1}^{\infty} f_n(x)$

We can view series of functions as a **function** where the domain is the interval where the series converges.

✎ $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ is a (geometric) power series. Find the interval where the series converges.

geometric series with common ratio $= x$

$S_n = \frac{1\cdot(1-x^n)}{1-x}$

$$\lim_{n\to\infty} S_n = \begin{cases} \frac{1}{1-x}, & |x|<1 \\ \infty, & |x|\geq 1 \end{cases}$$

➢ **Power Series** linear combination of power functions

A power series about $x = 0$ is $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ c_i : coefficients

More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-c)^n = c_0 + c_1(x-c) + c_2(x-c)^2 + \cdots$ is called a power series centered at $x = c$

■ Convergence of a Power Series

In most cases, the convergent interval can be found by using the **Ratio Test**.

Example. Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{\sqrt{n+3}}$

Let $\lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+1} x^{n+1}}{\sqrt{n+4}}}{\frac{(-2)^n x^n}{\sqrt{n+3}}} \right| = 2|x| < 1$

When $|x| < \frac{1}{2}$, the series converges.

Endpoints: ① $x = \frac{1}{2}$, $\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

(i) $\frac{1}{\sqrt{n+3}} > 0$ (ii) $\lim_{n\to\infty} \frac{1}{\sqrt{n+3}} = 0$ (iii) $\frac{1}{\sqrt{n+4}} \le \frac{1}{\sqrt{n+3}}$

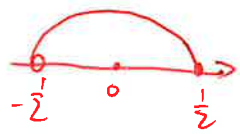
So the series converges by the Alternating Series Test

② $x = -\frac{1}{2}$, $\sum_{n=0}^{\infty} \frac{(-2)^n (-\frac{1}{2})^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$ — a p-series with $p = \frac{1}{2} < 1$

$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+3}}}{\frac{1}{\sqrt{n}}} = 1$, $\sum \frac{1}{\sqrt{n}}$ is

∴ divergent by the Limit Comparison Test.

Therefore, the series is convergent on $\left(-\frac{1}{2}, \frac{1}{2}\right]$



✎ Practice

1. Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} n!(2x)^n$.

$\lim_{n\to\infty} \left| \frac{(n+1)!\,(2x)^{n+1}}{n!\,(2x)^n} \right| = \lim_{n\to\infty} |(n+1)\cdot 2x|$

When $x = 0$, $\sum_{n=0}^{\infty} n!(2x)^n = 0$

when $x \neq 0$, the series diverges.

So the series converges only at $x = 0$

Radius $= 0$.

2. Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{n^2 x^n}{n!}$.

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^2 x^{n+1}}{(n+1)!}}{\frac{n^2 x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 < 1$$

The series converges for all $x \in \mathbb{R}$.

Radius $= \infty$

The interval of convergence is $(-\infty, \infty)$

3. Find the radius of convergence and the interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)} x^{n+1}.$$

$$\lim_{n\to\infty} \left| \frac{\frac{2\cdot 4\cdot 6\cdots(2n)(2n+2)}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)} x^{n+2}}{\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)} x^{n+1}} \right| = \lim_{n\to\infty} |x| = |x| < 1$$

① $x=1$: $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)} \to 1$ as $n\to\infty$ ∴ divergent by the nth term test

② $x=-1$: $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot \cdots (2n)}{1\cdot 3\cdot \cdots (2n-1)} (-1)^{n+1} \to \pm 1$ (DNE) as $n\to\infty$ ∴ divergent by the nth term test.

So, $R=1$. Interval: $(-1,1)$

For a power series centered at 0, there are only three possibilities:

Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

1. Converges only at 0
2. Converges on $\mathbb{R}$ (for all x)
3. Converges for $|x| < R$, and diverges for $|x| > R$, where R is a positive real number.

R: radius of convergence

Endpoints: check separately.

(直接代入后，用前面 series 的知识判断).

通过 geometric series 的求和公式反推

$\sum_{n=0}^{\infty} x^n = \frac{1\cdot(1-x^n)}{1-x}$, when $|x|<1$, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

➢ **Geometric Power Series**

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \quad (|x|<1)$$

$$\Rightarrow f(x) = \frac{\text{initial term}}{1-r}$$

? Find the power series expansion of the following functions.

1. $f(x) = \frac{2}{1+x^2}$ initial term, $r = -x^2$

$$\therefore f(x) = \sum_{n=0}^{\infty} 2(-x^2)^n$$

$$|-x^2|<1$$

$$\therefore |x|<1$$

$$\left(\rightarrow S_n = \frac{2\cdot(1-(-x^2)^n)}{1-(-x^2)} = \frac{2\cdot(1-(-x^2)^n)}{1+x^2}\right)$$

2. $f(x) = \frac{x^2}{1+x}$ initial term $= x^2$, $r = -x$

$$f(x) = \sum_{n=0}^{\infty} x^2\cdot(-x)^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2}, \quad |-x|<1 \Downarrow |x|<1$$

3. $f(x) = \frac{1}{1+x}$ $r = -x$

$$f(x) = \sum_{n=0}^{\infty} (-x)^n, \quad |x|<1$$

➢ **Have a try! How to write a function into the form of power series?**

(Review)

We have known the convergent interval of the series $\sum_{n=0}^{\infty} x^n$, so we can write the function $f(x) = \frac{1}{1-x} = 1+x+x^2+\cdots$ into the form of a series when the series converges to $f(x)$.

$|x|<1$

? Write the function $f(x) = \frac{1}{1+x^2}$ into the form of series and find its domain.

$t=-x^2$

$|-x^2|<1$

$\Downarrow$

$$f(x)=\sum_{n=0}^{\infty}(-x^2)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n}, \quad |x|<1$$

? Write the function $f(x) = \ln(x+1)$ into the form of series and find its domain.

$\to$ 和 $\frac{1}{1+x}$ 有什么联系？ $\left(\because \frac{1}{1+x}=\sum_{n=0}^{\infty}(-x)^n, |x|<1\right)$

$$\int \frac{1}{1+x}dx = \ln|1+x| + C$$

$$\therefore \ln|x+1| = \int \frac{1}{1+x}dx = \int \sum_{n=0}^{\infty}(-x)^n dx = \sum_{n=0}^{\infty}\int(-x)^n dx$$

$1-x+x^2-x^3+\cdots$

$$= x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots + C$$

$|x|<1$

$f(0)=\ln 1=0 \quad \therefore C=0$ $\Leftarrow$ 为什么代入"0"？ $\because$ 易计算出 C 的取值.

$$\therefore \ln|x+1| = \ln(x+1) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}x^n \quad |x|<1$$

$(x+1)>0$

? Write the function $f(x) = \frac{1}{(x+1)^2}$ into the form of series and find its domain.

$$f(x) = -\left(\frac{1}{1+x}\right)' = -\left(\sum_{n=0}^{\infty}(-x)^n\right)' = -\sum_{n=1}^{\infty}(-1)^n n x^{n-1} = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1}$$

$1-x+x^2-x^3+\cdots$

$\downarrow$ derivative

$-1+2x-3x^2+\cdots$

$|x|<1$

- **Operations**
  - **Substitution:** New series can be generated by making an appropriate substitution in a known series.
  - **Differentiation & Integration**

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}\right| = \left|\frac{a_{n+1}}{a_n}(x-c)\right| < 1$$

$$\therefore |x-c| < \left|\frac{a_n}{a_{n+1}}\right|$$

If the function given by $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ $= a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$ is differentiable, then

- $f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots = \sum_{n=1}^{\infty} a_n \cdot n \cdot (x-c)^{n-1}$

- $\int f(x)dx = \sum_{n=0}^{\infty} \int a_n(x-c)^n\, dx = \sum_{n=0}^{\infty} a_n \cdot \frac{1}{n+1}(x-c)^{n+1} + C$ ← constant

$$\boxed{\left[\int f(x)\,dx\right]_{x=c}}$$

Convergency: $f'(x)$: $\lim_{n\to\infty}\left|\frac{a_{n+1}\cdot(n+1)\cdot(x-c)^{n}}{a_n \; n \; (x-c)^{n-1}}\right| = \left|\frac{a_{n+1}}{a_n}(x-c)\right| < 1$

$$\therefore |x-c| < \left|\frac{a_n}{a_{n+1}}\right|$$

$\int f(x)\,dx$: $\lim_{n\to\infty}\left|\frac{\frac{a_{n+1}}{n+2}(x-c)^{n+2}}{\frac{a_n}{n+1}(x-c)^{n+1}}\right| = \left|\frac{a_{n+1}}{a_n}(x-c)\right| < 1$

$$\therefore |x-c| < \left|\frac{a_n}{a_{n+1}}\right|$$

Radius of Convergence 保持一致 , Endpoints 单独讨论

Practice

1. If $f(x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(x-2)^n}{n!} = (x-2) - \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} - \frac{(x-2)^4}{4!} + \cdots$, which of the following represents $f'(x)$?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n(x-2)^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^{n-1}}{(n-1)!}$$

$u=n-1$

$$= \sum_{n=0}^{\infty}(-1)^{n}\frac{(x-2)^{n}}{n!}$$

(A) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^{n-1}}{n!}$

(B) $\sum_{n=1}^{\infty}(-1)^{n}\frac{(x-2)^{n-1}}{(n+1)!}$

(C) $\sum_{n=0}^{\infty}(-1)^{n-1}\frac{(x-2)^{n-1}}{n!}$

(D) $\sum_{n=0}^{\infty}(-1)^{n}\frac{(x-2)^{n}}{n!}$

D)

2. Let f be a function given by $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n}$. Find the interval of convergence for each of the following.

(1) $f(x)$

(2) $f'(x)$

(3) $\int f(x)\,dx$

(1) $\lim_{n\to\infty} \left| \dfrac{\frac{(-1)^{n+1}(x-2)^{n+1}}{n+1}}{\frac{(-1)^n (x-2)^n}{n}} \right| = |x-2| < 1$

$x-2=1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

$x-2=-1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\therefore (1, 3]$

(2) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} n (x-2)^{n-1} = \sum_{n=1}^{\infty} (-1)^n (x-2)^{n-1}$

$x=1$: $\sum_{n=1}^{\infty} -1 = -\infty$ DNE divergent

$x=3$: $\sum_{n=1}^{\infty} (-1)^n$ divergent

$\therefore (1, 3)$

(3) $\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} (x-2)^{n+1}$

$x=1$: $\sum_{n=1}^{\infty} \frac{-1}{n(n+1)}$ convergent

$x=3$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ convergent

$\therefore [1, 3]$

3. The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n)!} = 1 - \frac{3x^2}{2!} + \frac{5x^4}{4!} - \frac{7x^6}{6!} + \cdots + (-1)^n \frac{(2n+1)x^{2n}}{(2n)!} + \cdots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that $1 - \frac{3}{2!} + \frac{5}{4!}$ approximates $f(1)$ with an error less than $\frac{1}{100}$.

(c) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Write the first four terms and the general term of the power series expansion of $\frac{g(x)}{x}$.

(a) $f'(x) = \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)2n x^{2n-1}}{(2n)!}$

$= \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{(2n-1)!} x^{2n-1}$

$f'(0) = 0$

$f''(x) = \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{(2n-1)!} (2n-1) x^{2n-2}$

$= \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{(2n-2)!} x^{2n-2}$

$f''(0) = -3 < 0$

$\therefore f$ has a local max at $x=0$

(b) $f(1) = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n)!}$

$\frac{2n+3}{(2n+1)!} \le \frac{2n+1}{(2n)!}$ for $n \ge 1$

$\left| f(1) - 1 - \left(-\frac{3}{2!} + \frac{5}{4!} \right) \right| < \frac{7}{6!}$

$\left| f(1) - \left(1 - \frac{3}{2!} + \frac{5}{4!}\right) \right| \le \frac{7}{6!} = \frac{7}{720} < \frac{1}{100}$

(c) $g(x) = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n)!} t^{2n}\,dt$

$= \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n)!} \cdot \left[\frac{1}{2n+1} t^{2n+1} \right]_0^x$

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n+1}$

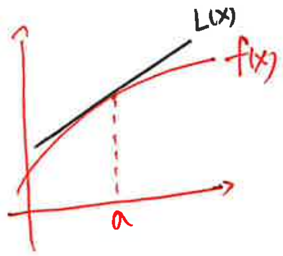
$\frac{g(x)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

➢ **Taylor Polynomial and Maclaurin Polynomial**

? Linear Approximation of a function $f(x)$ at point $x = a$

tangent Line



$$L(x) = \underbrace{f(a)}_{C_0} + \underbrace{f'(a)}_{C_1}(x-a)$$

是 power series 前 2 项.

更高幂次的 approximation:

$$P_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n$$

■ How to expand a function $f(x)$ into the form of power series?

How to find the coefficients?

conditions:

$P_n(a) = f(a) \Rightarrow C_0 = f(a)$

$P_n'(a) = f'(a) \Rightarrow C_1 = f'(a)$

$P_n''(a) = f''(a) \Rightarrow 2C_2 = f''(a) \quad \therefore C_2 = \frac{1}{2}f''(a)$

$P_n^{(3)}(a) = f'''(a) \Rightarrow 2\times 3\, C_3 = f'''(a) \quad \therefore C_3 = \frac{1}{3!}f'''(a)$

$\vdots$

$P_n^{(k)}(a) = f^{(k)}(a) \Rightarrow k!\, C_k = f^{(k)}(a)$

✧ If a function f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{1}{2!}f''(c)(x-c)^2 + \frac{1}{3!}f'''(c)(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the nth **Taylor Polynomial** for f at c.

✧ When $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

is called the nth **Maclaurin Polynomial** for f.

Guidelines for Finding a Taylor Polynomial

1. Differentiate $f(x)$ several times and evaluate each derivative at c.

   $f(c),\ f'(c),\ f''(c),\ f'''(c),\ \cdot\cdot\cdot,\ f^{(n)}(c)$

2. Use the sequence developed in the first step to form the **Taylor coefficients**

   $a_n = \dfrac{f^{(n)}(c)}{n!}$.

Practice

1. Let f be the function given by $f(x) = \ln(2-x)$. Write the third-degree Taylor polynomial for f about $x=1$ and use it to approximate $f(1.2)$.

$P_3(x) = 0 + (-1)(x-1) + (-\frac{1}{2})(x-1)^2 + (-\frac{1}{3})(x-1)^3 = -(x-1) - \frac{1}{2}(x-1) - \frac{2}{3}(x-2)^3$

$f(1) = 0$

$f'(x) = \frac{-1}{2-x}$ $\quad f'(1) = -1$

$f''(x) = \frac{-1}{(2-x)^2}$ $\quad \frac{f''(1)}{2!} = -\frac{1}{2}$

$f'''(x) = \frac{-2}{(2-x)^3}$ $\quad \frac{f'''(1)}{3!} = \frac{-2}{3!} = -\frac{1}{3}$

$\therefore f(1.2) \approx P_3(1.2) \approx -0.223$ (Calculator)

$f(2)$ $\quad f'(2)$ $\quad \frac{1}{2!}f''(2)$ $\quad \frac{1}{3!}f'''(2)$

2. Let $P(x) = 3 - 2(x-2) + 5(x-2)^2 - 12(x-2)^3 + 3(x-2)^4$ be the fourth-degree Taylor polynomial for the function f about $x = 2$. Assume f has derivatives of all orders for all real numbers.

(a) Find $f(2)$ and $f'''(2)$.

(b) Write the third-degree Taylor polynomial for f' about 2 and use it to approximate $f'(2.1)$.

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_2^x f(t)\,dt$ about 2.

(d) Can $f(1)$ be determined from the information given? Justify your answer.

(a) $f(2) = 3$

$f'''(2) = 3!\,(-12) = -72$

(b) $f'(x)$: $\quad P_3(x) = -2 + 10(x-2) - 36(x-2)^2 + 12(x-2)^3$

$f'(2.1) \approx P_3(2.1) \approx -1.348$

(c) $g(x)$: $\quad P_4(x) = 3(x-2) - (x-2)^2 + \frac{5}{3}(x-2)^3 - 3(x-2)^4$

(d) NO, The radius of convergence is not given, we do not know the behavior of f at $x=1$ (covergent or divergent). Also, we cannot accurately determine $f(1)$ from the given polynomial alone.

3. The second-degree Taylor polynomial of $\sec x$ about $x = \frac{\pi}{4}$ is

C

$f(\frac{\pi}{4}) = \sqrt{2}$

$f'(x) = \sec x \tan x \qquad f'(\frac{\pi}{4}) = \sqrt{2}$

$f''(x) = \sec x \tan x \cdot \tan x + \sec x \cdot \sec^2 x$

$f''(\frac{\pi}{4}) = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$

$c_2 = \frac{3}{2!}\sqrt{2}$

(A) $P_2(x) = 1 + \sqrt{2}(x - \frac{\pi}{4}) + \sqrt{2}(x - \frac{\pi}{4})^2$

(B) $P_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) + \frac{3\sqrt{2}}{3!}(x - \frac{\pi}{4})^2$

(C) $P_2(x) = \sqrt{2} + \sqrt{2}(x - \frac{\pi}{4}) + \frac{3\sqrt{2}}{2!}(x - \frac{\pi}{4})^2$

(D) $P_2(x) = 1 + \sqrt{2}(x - \frac{\pi}{4}) + \frac{3\sqrt{2}}{3!}(x - \frac{\pi}{4})^2$

B 4. A function f has derivatives of all orders at $x = 0$. Let P_n denote the nth-degree Taylor polynomial for f about $x = 0$. It is known that $f(0) = \frac{1}{3}$ and $f''(0) = \frac{4}{3}$. If $P_2(\frac{1}{2}) = \frac{1}{8}$, what is the value of $f'(0)$?

(A) $-\frac{3}{8}$ (B) $-\frac{3}{4}$ (C) $-\frac{5}{4}$ (D) $-\frac{3}{2}$

$P_2(x) = \frac{1}{3} + f'(0)\,x + \frac{1}{2!}\cdot\frac{4}{3}x^2$

$P_2(\frac{1}{2}) = \frac{1}{3} + f'(0)\cdot\frac{1}{2} + \frac{1}{6} = \frac{1}{8}$

$\therefore f'(0) = -\frac{3}{4}$

C 5. Let $P(x) = 4 - 3x^2 + \frac{13}{12}x^4 - \frac{121}{360}x^6$ be the sixth-degree Taylor polynomial for the function f about $x = 0$. What is the value of $f'''(0)$?

(A) $-\frac{121}{15}$ (B) $-\frac{3}{2}$ (C) 0 (D) $\frac{121}{15}$

➢ **Lagrange Error Bound**

✧ If $f(x)$ has $n+1$ derivatives on an open interval (a,b), for any $x \in [a,b]$, there exists ξ between x and x_0 such that $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$

（证明略）

类似 Mean Value theorem: For continuous function $f(x)$ on $[a,b]$, $f(x)$ is differentiable on (a,b): $\exists\, c \in (a,b)$, s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$

Let $x=b \Rightarrow f'(c)\cdot(x-a) + f(a) = f(x)$

✧ If $f(x)$ has $n+1$ derivatives at c and $R_n(x)$, is the remainder term of the nth Taylor polynomial $P_n(x)$, then $f(x) =$ $\underline{P_n(x) + R_n(x)}$

✧ The absolute value of $R_n(x)$ satisfies the inequality:

$$|R_n(x)| = |f(x) - P_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}\right| \le \max_{\xi}\left|f^{(n+1)}(\xi)\right| \cdot \frac{|x-x_0|^{n+1}}{(n+1)!}$$

The remainder $R_n(x)$ is called the **Lagrange Error Bound.**

✎ Practice

Let f be the function given by $f(x) = \sin(3x - \frac{\pi}{6})$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Use the Lagrange error bound to show that $|f(0.2) - P(0.2)| < \frac{1}{100}$.

(a) $f(0) = -\frac{1}{2}$

$f'(x) = 3\cos(3x - \frac{\pi}{6})$ $\quad f'(0) = \frac{3}{2}\sqrt{3}$

$f''(x) = -9\sin(3x - \frac{\pi}{6})$ $\quad f''(0) = \frac{9}{2}$ $\quad \frac{f''(0)}{2!} = \frac{9}{4}$

$f'''(x) = -27\cos(3x - \frac{\pi}{6})$ $\quad f'''(0) = -\frac{27}{2}\sqrt{3}$ $\quad \frac{f'''(0)}{3!} = -\frac{27}{12}\sqrt{3} = -\frac{9}{4}\sqrt{3}$

$P(x) = -\frac{1}{2} + \frac{3}{2}\sqrt{3}\,x + \frac{9}{4}x^2 - \frac{9}{4}\sqrt{3}\,x^3$

(b)

$|f(x) - P(x)| = \left|\frac{f^{(4)}(\xi)}{4!}x^4\right| \le \max_{\xi}|f^{(4)}(\xi)| \cdot \frac{|x^4|}{4!}$

$f^{(4)}(x) = 81\sin(3x - \frac{\pi}{6}) \le 81$

$\therefore |f(0.2) - P(0.2)| \le 81 \cdot \frac{(0.2)^4}{4!} = 0.0054 < \frac{1}{100}$

➢ Taylor Series and Maclaurin Series

Taylor Series and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the **Taylor series for** $f(x)$ **at** c. Moreover, if $c = 0$, then the series is called the **Maclaurin series for** f.

Practice

Method 2. $\int \frac{1}{1+x}\,dx$ $\quad \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

1. Find the Maclaurin series for the function $f(x) = \ln(1+x)$.

$f(0) = 0$

$f'(x) = \frac{1}{1+x}, \quad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}, \quad \frac{f''(0)}{2!} = -\frac{1}{2}$

$f'''(x) = \frac{2}{(1+x)^3}, \quad \frac{f'''(0)}{3!} = \frac{2}{3!} = \frac{1}{3}$

$\vdots$

$f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{(1+x)^n}, \quad \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}}{n}$

$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

$+ \frac{(-1)^{n+1}}{n} x^n + \cdots$

2. Let f be a function having derivatives of all orders. The fourth degree Taylor polynomial for f about $x = 1$ is given

$T(x) = 4 + 3(x-1) - 6(x-1)^2 + 7(x-1)^3 - 4(x-1)^4$.

Find $f(1)$, $f'(1)$, $f''(1)$, $f'''(1)$ and $f^{(4)}(1)$.

$f(1) = 4$

$f'(1) = 3$

$\frac{f''(1)}{2!} = -6 \quad\quad f''(1) = -12$

$\frac{f'''(1)}{3!} = 7 \quad\quad f'''(1) = 42$

$\frac{f^{(4)}(1)}{4!} = -4 \quad\quad f^{(4)}(1) = -96$

- Elementary Functions

Function	Convergent Interval
$f(x)=\frac{1}{x}=\frac{1}{1-(1-x)}=1+(1-x)+(1-x)^2+\cdots=\sum_{n=0}^{\infty}(1-x)^n=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$ substitution	$\lvert 1-x\rvert<1$, $-1<1-x<1$; $0<x<2$, $(0,2)$
$f(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+\cdots+x^n+\cdots$	$\lvert x\rvert<1$
$f(x)=\ln x=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}(x-1)^{n+1}$	$(0,2]$
$f(x)=e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots=\sum_{n=0}^{\infty}x^n\cdot\frac{1}{n!}$	$\mathbb{R}$
$f(x)=\sin x=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$	$\mathbb{R}$
$f(x)=\cos x=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$	$\mathbb{R}$
$f(x)=\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n+1}$	$[-1,1]$

integral

derivative

example: $e^x\cdot x-x=\sum_{n=0}^{\infty}x^{n+1}\cdot\frac{1}{n!}-x=\sum_{n=1}^{\infty}x^{n+1}\frac{1}{n!}$

- Multiplication of Power Series ~~Power series~~ can be multiplied the way we multiply polynomials.
- We usually find only the first few terms because the calculations for the later terms become tedious and the initial terms are the most important ones.

$\ln x$: $\ln(1+(x-1))$ OR $\ln x=\int\frac{1}{x}dx=\sum_{n=0}^{\infty}\int(-1)^n(x-1)^n dx$

$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}(x-1)^{n+1}$

$x=0$: $x-1=-1$, $\sum_{n=0}^{\infty}(-1)\cdot\frac{1}{n+1}$ divergent

$x=2$: $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}$ convergent

$\therefore (0,2]$

e^x : $f^{(n)}(x)=e^x$ $\quad f^{(n)}(0)=1$

$\therefore e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=0<1$

$\therefore x\in\mathbb{R}$.

$\tan^{-1}x$:

$f'(x)=\frac{1}{1+x^2}$

$\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-x^2)^n$

$=\sum_{n=0}^{\infty}(-1)^n x^{2n}$

$\lvert x\rvert<1$

$\therefore f(x)=\int\sum_{n=0}^{\infty}(-1)^n x^{2n}dx$

$=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{2n+1}x^{2n+1}+C$

$f(0)=0$ $\quad\therefore C=0$

$\therefore \tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n+1}$

$x=1$: convergent $\quad x=-1$: convergent

$[-1,1]$

$\sin x$:

$f(0)=0$

$f'(x)=\cos x$ $\quad f'(0)=1$ ✓

$f''(x)=-\sin x$ $\quad \frac{f''(0)}{2!}=0$

$f'''(x)=-\cos x$ $\quad \frac{f'''(0)}{3!}=\frac{-1}{3!}$ ✓

$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}$

convergency: $\lim_{n\to\infty}\left|(-1)^{n+1}\cdot x^{2n+3}\cdot\frac{1}{(2n+3)!}\cdot\frac{1}{(-1)^n}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|=\lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right|=0$ $\quad\therefore x\in\mathbb{R}$

Practice

1. Find the Maclaurin series for the function $f(x) = \cos x^2$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!}(-1)^n$$

$$f(x) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots + \frac{(-1)^n}{(2n)!}x^{4n} + \cdots$$

$|x^2| \in \mathbb{R}$ $\therefore x \in \mathbb{R}$

2. Find the Maclaurin series for the function $f(x) = x^2 e^x - x^2$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$x^2 e^x = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} = x^2 + x^3 + \frac{x^5}{2!} + \cdots$$

$$x^2 e^x - x^2 = x^3 + \frac{x^5}{2!} + \frac{x^6}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{x^{n+2}}{n!} \quad (x \in \mathbb{R})$$

3. Find the first three nonzero terms in the Maclaurin series for $e^x \cos x$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x \cos x = 1 + x - \left(\frac{x^2}{2} - \frac{x^2}{2}\right) + \left(-\frac{x^3}{2} + \frac{x^3}{6}\right) + \cdots$$

$$= 1 + x - \frac{1}{3}x^3 + \cdots$$