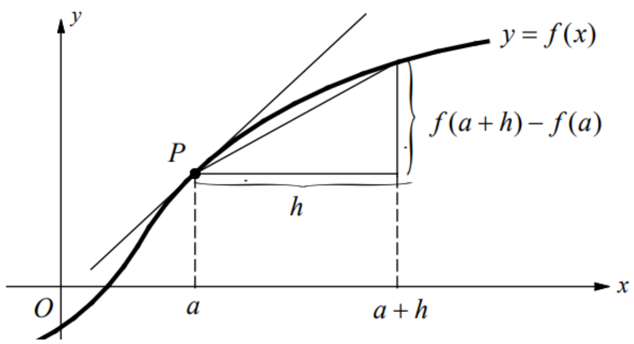
Differentiation

- **Review**



Average rate of change

The slope of secant line over an interval (a, a+h) is :

When Q(a+h, f(a+h)) approaches to P, the slope of secant line approaches to the slope of

____

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to \_\_\_\_} \frac{f(x) - f(a)}{\_\_\_\_\_\_}$$

- **Def. Derivatives**

The derivative of a function f at x, denoted as $f'(x)$ is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ if the limit exists.

Notation: _____________________________

Q1. Find a function f and a number a such that $f'(a) = \lim_{h \to 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - \frac{1}{2}}{h}$

- **Differentiability & Continuity**

If a function f is differentiable at x=a, then f is continuous at x=a.

- **One-sided Derivatives**

f(x) is differentiable at x=a iff $f'(a)$ exists iff ____

The **left-hand derivative** of f at a is: ___________________________________ (if the limit exists)

The **right-hand derivative** of f at a is: ___________________________________ (if the limit exists)

Differentiation

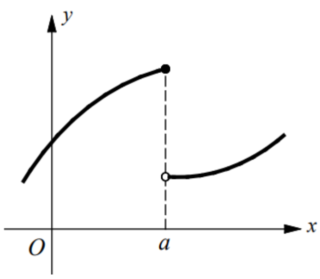
➢ **Derivatives of piecewise functions**

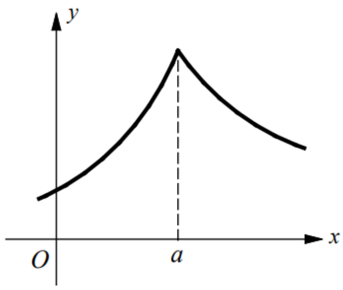
? **Example.** Find the derivative of $f(x)=\begin{cases} x+3, x\le 0 \\ 3-2x, x>0 \end{cases}$

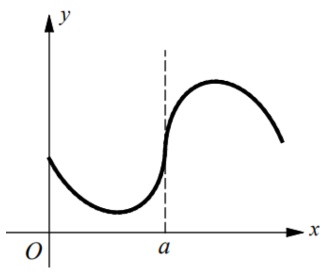
- Derivative of a linear function f(x)=ax+b is _______

Q2. Let f be the function defined by $f(x)=\begin{cases} mx^2-2, x\le 1 \\ k\sqrt{x},\ x>1 \end{cases}$. If f is differentiable at x = 1 , what are the values of k and m ?

➢ **not differentiable at x=a:**

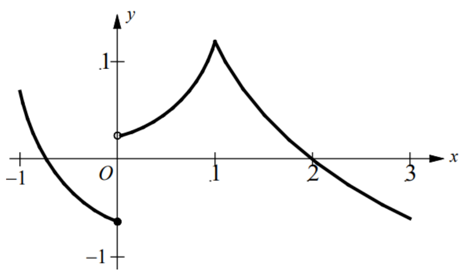






Differentiation

Q3. The graph of f is shown in the figure below. For what values of x , $-1 < x < 3$, is f not differentiable?



- **Basic Differentiable Rules**
  - ✧ **Constant Rule**
  - ✧ **Power Rule**

? Use the definition ($\boldsymbol{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$) to find the derivative of $f(x) = x^5$

Q4. Find the derivative of $f(x) = x^3 - 2x + \frac{1}{x} + 5$ at x=2

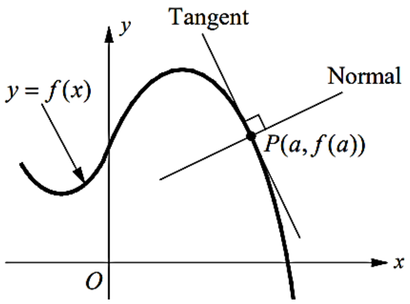
Q5. $\lim_{h\to 0}\frac{(3+h)^4-81}{h} = ?$

Differentiation

➢ **Review. Point-slope equation**

A linear equation with the slope m where the graph passes through the point (x1,y1) is _______

➢ **Tangent Line and Normal Line**



- the tangent line and the normal line both pass through the point _____________
- the slope of the tangent line of f(x) at x=a is: ________
- the slope of the normal line of f(x) at x=a is: _________
- If f'(a)=0, then the tangent line is ___________, the normal line is ___________

Point-slope equation of the tangent line is ___________________________________

Point-slope equation of the normal line is ___________________________________

Q6. Write the equation of the tangent line and normal line to the graph of $y = x - \frac{x^2}{10}$ at the point $(4, \frac{12}{5})$

Q7. Which of the following is an equation of the line tangent to the graph of $f(x) = x^2 - x$ at the point where $f'(x) = 3$?

(A) $y = 3x - 2$

(B) $y = 3x + 2$

(C) $y = 3x - 4$

(D) $y = 3x + 4$

Q8. A curve has slope $2x + x^{-2}$ at each point (x, y) on the curve. Which of the following is an equation for this curve if it passes through the point $(1,3)$?

(A) $y = 2x^2 + \frac{1}{x}$

(B) $y = x^2 - \frac{1}{x} + 3$

(C) $y = x^2 + \frac{1}{x} + 1$

(D) $y = x^2 - \frac{2}{x^2} + 4$

Differentiation

Q9. If $2x+3y=4$ is an equation of the line normal to the graph of f at the point $(-1,2)$, then $f'(-1)=$

(A) $-\frac{2}{3}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{2}$ (D) $\frac{3}{2}$

Q10. If $2x-y=k$ is an equation of the line normal to the graph of $f(x)=x^4-x$, then $k=$

(A) $\frac{23}{16}$ (B) $\frac{13}{18}$ (C) $\frac{15}{16}$ (D) $\frac{9}{8}$

➢ **Chain Rule**

If y=f(u) and u=g(x) are both differentiable functions, then y=f(g(x)) is differentiable and $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$, or $\frac{d}{dx}[f(g(x))]=f'(g(x))g'(x)$

✧ If y=f(u), u=g(w), and w=h(x) are all differentiable functions, then y=f(g(h(x))) is differentiable and $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dw}\cdot\frac{dw}{dx}$, or

$$\frac{d}{dx}[f(g(h(x)))]=f'(g(h(x)))g'(h(x))h'(x)$$

Q11. Find y' for $y=\sqrt{1+x^2}$

Q12. Find y' for $y=(2x^3-2x^2)^4$

Differentiation

Q13. Find y' for $y = \sqrt{x^4 - 2x + 5}$

Q14. Find h''(x) if $h(x) = f(x^3)$

- **The Product Rule**

  If f and g are both differentiable, then $\frac{d}{dx}[f(x)g(x)] =$

- **The Quotient Rule**

  If f and g are both differentiable, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] =$

Q15. Differentiate the function $f(x) = (x^3 - 7)(x^2 - 4x)$

Q16. Differentiate the function $f(x) = \frac{3x^2 - x}{\sqrt{x}}$

Q17. If f, g, and h are functions that is everywhere differentiable, then the derivative of $\frac{f}{g \cdot h}$ is

(A) $\frac{g\,h\,f' - f\,g'\,h'}{g\,h}$

(B) $\frac{g\,h\,f' - f\,g\,h' - f\,h\,g'}{g\,h}$

(C) $\frac{g\,h\,f' - f\,g\,h' - f\,g'h}{g^2 h^2}$

(D) $\frac{g\,h\,f' - f\,g\,h' + f\,h\,g'}{g^2 h^2}$

Differentiation

Q18. Differentiate the function $f(x) = (3x^3 - 2x)(2x - 1)(5x + 10)$

- **Higher Derivatives**

  If f is a differentiable function, then f'(x) is also a function, so f'(x) may have a derivative of its own. The second derivative f''(x) is the derivative of f'(x). The third derivative f'''(x) is the derivative of f''(x)….

  nth derivative $f^n(x) = \frac{d^n y}{dx^n} = y^{(n)}(x)$

Q19. If $f(x) = \frac{1}{6}x^3 + 24\sqrt{x}$, find f'(x), f''(x), f'''(x), and f'''(9)

- **Review: Properties of logarithm**

$ln\ xy =$ __________________, $ln\frac{x}{y} =$ __________________

$ln\ x^p =$ __________________, $e^{ln\ x} =$ __________________

$$\log_a b = \frac{\ln b}{\ln a}$$

- **The derivatives of Logarithm Function**

$\frac{d}{dx}(\log_a x) = \frac{1}{ln\ a} \cdot \frac{1}{x}$ $\quad\quad$ $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Differentiation

Q20. Find y' if $y = \frac{\ln x}{x^2}$

Q21. Find y' if $y = x^{\ln x}$

- **The Derivative of Trigonometric Function**

$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$ $\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$

Q22. Find $\frac{dy}{dx}$ for $y = x^2 \sin x + 2x \cos x$

Q23. Find $\frac{dy}{dx}$ for $y = lnx \tan x - x^3 \sec x$

Differentiation

Q24. Find y' for $y = \sin x^2$

Q25. Find y' for $y = \sin^2 x$

Q26. Find y' for $\boldsymbol{y = \csc \frac{1}{x}}$

Q27. Find y' for $y = \tan^2(x^3)$

Q28. Find y' for $y = \sin^2(-3x^2 - 1)$

Q29. Differentiate $\boldsymbol{y = \ln \frac{x^2}{(x+1)^2}}$

Differentiation

- **The Derivative of Exponential Function**

$\frac{d}{dx}(a^x) = a^x \cdot \ln a$ $\qquad$ $\frac{d}{dx}(e^x) = e^x$

Q30. Find y'' if $y = 3^{x^2-x}$

- **Implicit Differentiation**

  - Explicit functions y=f(x): expresses y explicitly in terms of x
  - Implicit functions: e.g. $xy = x^2 + 1$ we are unable to solve for y as a function of x.

    Guidelines:

    1. Differentiate both sides of equation with respect to x
    2. Collect the term $\frac{dy}{dx}$ on the left side and move all other terms to the right
    3. Solve for $\frac{dy}{dx}$

  - The tangent line is horizontal when $\frac{dy}{dx} =$ _________
  - The tangent line is vertical when the ________ of $\frac{dy}{dx}$ is 0

Q31. Find $\frac{dy}{dx}$ if $y^2 = x^2 - \cos xy$

Differentiation

Q32. Consider the curve defined by $x^3 + y^3 = 4xy + 1$

(1) Find $\frac{dy}{dx}$

(2) Write an equation for line tangent to the curve at the point (2,1)

Q33. Consider the curve defined by $x^3 + y^3 - 6xy = 0$

(1) Find $\frac{dy}{dx}$

(2) Find the x-coordinates of each point on the curve where the tangent line is horizontal

(3) Find the y-coordinates of each point on the curve where the tangent line is vertical

Differentiation

- **Derivative of an inverse function**

  - **Properties of inverse function**

$$f\left(f^{-1}(x)\right) = \_\_\_\_\_\_\_\_\_\_ \quad , x \in \_\_\_\_\_\_\_\_\_\_$$

$$f^{-1}\left(f(x)\right) = \_\_\_\_\_\_\_\_\_\_ \quad , x \in \_\_\_\_\_\_\_\_\_\_$$

Let f be a differentiable function whose inverse function f^{-1} is also differentiable. Then providing that the denominator is not zero,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Q34. Let $f(x) = x^2 - \frac{3}{x}$

(1) What is the value of $f^{-1}(8)$?

(2) What is the value of $(f^{-1})'(8)$?

Q35. Let $f(2) = 5$ and $f'(2) = \frac{1}{4}$, find $(f^{-1})'(5)$

Differentiation

➢ **Derivative of Inverse Trigonometric Functions**

$\frac{d}{dx}(\sin^{-1} x) =$ __________ $\frac{d}{dx}(\cos^{-1} x) =$ __________

$\frac{d}{dx}(\tan^{-1} x) =$ __________ $\frac{d}{dx}(\cot^{-1} x) =$ __________

$\frac{d}{dx}(\sec^{-1} x) =$ __________ $\frac{d}{dx}(\csc^{-1} x) =$ __________

Q36. Differentiate $y = x \tan^{-1} x$

Q37. Differentiate $y = \frac{1}{\cos^{-1} x}$

Q38. Differentiate $y = \arctan \sqrt{x}$

Q39. Differentiate $y = 5 \arcsin 3x$

Differentiation

- **Approximating a derivative**

If a function f is defined by a table of values, then the approximation values of its derivatives at b can be obtained from the average rate of change using values that are close to b.

x	$\cdots$	a	$\cdots$	b	$\cdots$	c	$\cdots$
$f(x)$	$\cdots$	$f(a)$	$\cdots$	$f(b)$	$\cdots$	$f(c)$	$\cdots$

For $a < b < c$,

$$f'(b) \approx \frac{f(c)-f(b)}{c-b} \text{ or}$$

$$f'(b) \approx \frac{f(b)-f(a)}{b-a} \text{ or}$$

$$f'(b) \approx \frac{f(c)-f(a)}{c-a}.$$

Q40. The temperature of the water in a coffee cup is a differentiable function F of time t. The table below shows the temperature of coffee in a cup as recorded every 3 minutes over 12minute period.

t	0	3	6	9	12
$F(t)$	205	197	192	186	181

(a) Use data from the table to find an approximation for $F'(6)$?

(b) The rate at which the water temperature decrease for $0 \le t \le 12$ is modeled by $F(t) = 120 + 85e^{-0.03t}$ degrees per minute. Find $F'(6)$ using the given model.