Limit and Continuity

? What is the value of $\frac{1}{2^n}$ when n tends to infinity? (Hint: Try to graph the function of $y = \frac{1}{2^x}$)

❖ **Notation.** $\lim_{x \to c} f(x) = L$ The limit of $f(x)$ as x approaches c is L.

? $\lim_{x \to 2} \frac{1}{2^x} =$ ____________

❖ **Basic Limits**

Constant function $f(x) = k$: $\lim_{x \to 0} 4 =$ $\lim_{x \to c} k =$

Exponential function $f(x) = e^x$: $\lim_{x \to 0} e^x =$ $\lim_{x \to c} e^x =$

Polynomial function $f(x) =$______________________: $\lim_{x \to 0} f(x) =$

Limit and Continuity

❖ **Finding Limits Graphically**

Consider the graph of the function $f(x) = \frac{x^2-9}{x-3}$

Sketch the graph of $f(x)$ and find the limit of $f(x)$ as x approaches to 3:

✧ Even though $f(3)$ is not defined, the limit of $f(x)$ is _______ as x approaches to 3.

The limit of a function is where we consider values of x that are _______ to c, but not _________ to c.

❖ **Def.** If $f(x)$ becomes arbitrarily close to a single number L as x approaches c _______________, the **limit** of $f(x)$, as x approaches c, is L.

$$\lim_{x \to c} f(x) = L$$

❖ **Def. One-sided Limits**

The **right-hand limit** means that x approaches c from values greater than c.

$$\lim_{x \to c^+} f(x) = L$$

The **left-hand limit** means that x approaches c from values ___________ than c.

$$\lim_{\underline{\hspace{1.5em}}} f(x) = L$$

❖ **The existence of a Limit**

The limit of $f(x)$ as x approaches c is L iff ________________________________

Limit and Continuity

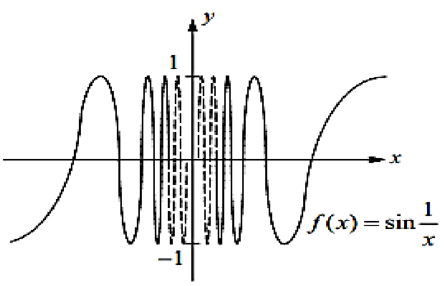
- **Limits that fails to exist**

- $f(x) = \frac{|x|}{x}$

- $f(x) = tanx$

- $f(x) = \frac{1}{x}$

- $f(x) = \sin\frac{1}{x}$



Limit and Continuity

❖ Def. Continuity

A function f is continuous at c if the following three conditions are met.

1. ______________________________

2. ______________________________

3. ______________________________

✧ Continuity over an interval I

A function f is continuous on an interval if the function is continuous at each point in the interval.

✧ Discontinuities:

1. ____________________

2. ____________________

3. ____________________

Practice

For what values of a is $f(x) = \begin{cases} x^2, x \leq 1 \\ ax + 2, 1 < x \leq 3 \end{cases}$ continuous at x=1?

Limit and Continuity

❖ Techniques for Evaluating Limits by Hand

1) Direct Sub

2) Remove the hole by cancellation

a) $\lim\limits_{x \to 2} \frac{x^2-4}{x-2}$

b) $\lim\limits_{x \to 2} \frac{x^2+3x-10}{x-2}$

3) Rationalizing

a) $\lim\limits_{x \to 0} \frac{\sqrt{x+1}-1}{x}$

b) $\lim\limits_{x \to 0} \frac{\sqrt{x+3}-2}{x-1}$

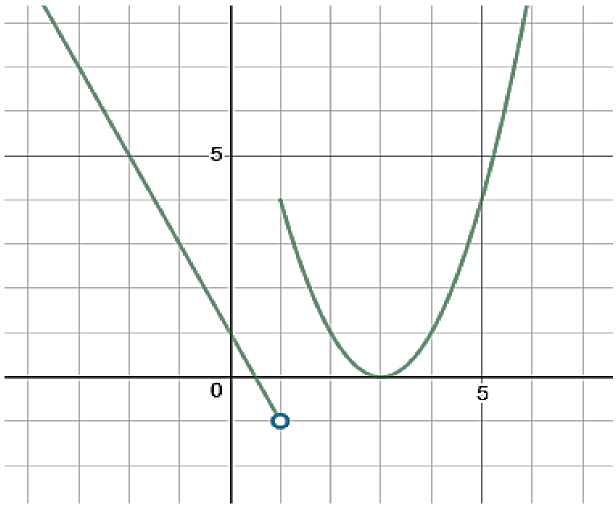
4) Setting up two cases

$$\lim_{x \to 4} \frac{|x-4|}{x-4}$$

Limit and Continuity

Practice

1. Evaluate the limits by looking at the graph of $f(x)$.



a) $\lim\limits_{x\to 3^-} f(x)$

b) $\lim\limits_{x\to 3} f(x)$

c) $\lim\limits_{x\to 1^-} f(x)$

d) $\lim\limits_{x\to 1^+} f(x)$

e) $\lim\limits_{x\to 1} f(x)$

f) $f(1)$

Limit and Continuity

❖ Limit Laws

Let c and k be real numbers and the limits $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ **exist**. Then

a) $\lim_{x \to c} f(x) \pm g(x) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

b) $\lim_{x \to c} f(x) \cdot g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

c) $\lim_{x \to c} kf(x) = k \cdot \lim_{x \to c} f(x)$

d) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, *provided* $\lim_{x \to c} g(x) \neq 0$

e) $\lim_{x \to c} [f(x)]^n = [\lim_{x \to c} f(x)]^n$

f) If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then $\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)$

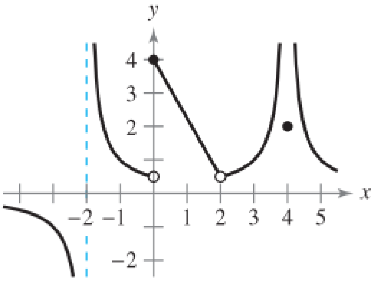
✍ **Practice:** Find the limits.

1) $\lim_{x \to 0} \frac{x^2+3x-10}{x-2}$

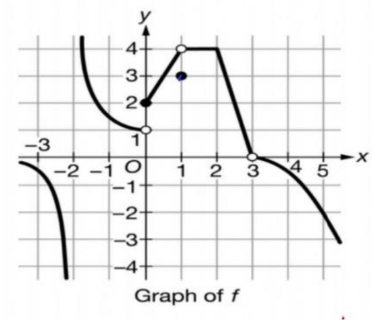
2) $\lim_{x \to 0} x^2 \sin \frac{1}{x}$

3) $\lim_{x \to 0} \sin 4x$

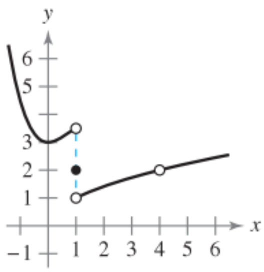
4) The graph of f is shown below. Given that $\lim_{x \to 0} h(x) = 1$, then the value of $\lim_{x \to 0} f(h(x))$ is ____________

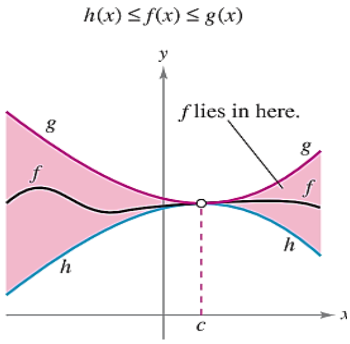


5) (Optional) The graph of f is shown on the right. $\lim_{x \to 0} f(f(x)) =$



6) (Optional) The graph of g is shown below. The value of $\lim_{x \to 0} g(1 - x^2)$ is ____________





❖ **The squeeze theorem**

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim_{x \to c} h(x) = \lim_{x \to c} g(x) = L$, then $\lim_{x \to c} f(x)$ exists and is equal to L.

Limit and Continuity

❖ **Special Limits (Evaluate** $\lim_{x \to 0} \frac{sinx}{x}$ using your calculator)

$$\lim_{x \to 0} \frac{sinx}{x} = \_\_\_\_\_\_\_\_$$

Practice: Find the limits.

a) $\lim_{x \to 0} \frac{\sin 4x}{3x}$

b) $\lim_{x \to 0} \frac{\tan x}{x}$

c) $\lim_{x \to \frac{\pi}{4}} \frac{\tan x - 1}{\sin x - \cos x}$

❖ Intermediate Value Theorem

? Given that function g is continuous, if we know that $g(10)$ is positive, while $g(15)$ is negative, what can we conclude must have occurred?

If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

Specifically, if f is continuous on [a,b] and f(a) and f(b) **differ in sign**, the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval [a,b].

✍ Practice

Let f be a function given by $f(x) = x^3 - 4x + 2$. Use the Intermediate Value Theorem to show that there is a root of the equation on [0,1]

Limit and Continuity

❖ **Asymptotes**

1. Horizontal asymptote

A line _________ is a horizontal asymptote of the graph of a function $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or

$\lim_{x\to-\infty} f(x) = L$

Practice

1. Find the horizontal asymptotes of the function.

(1) $f(x) = \frac{3x^2-2x-3}{2x^2+5x-6}$

(2) $f(x) = \frac{2x^2-2x+10}{x^4+5x^2-100}$

(3) $f(x) = \frac{\sqrt[3]{2x^3-9}}{x}$

(4) $f(x) = \frac{\sqrt{4x^2+6x}}{3x-2}$

2. Find the following limits.

(a) $\lim\limits_{x\to\infty} \frac{x^3-4x^2+7}{2x^3-3x-5}$

(b) $\lim\limits_{x\to\infty} \frac{x^{100}}{e^x}$

(c) $\lim\limits_{x\to\infty} \frac{x^{100}}{\ln x}$

(d) $\lim\limits_{x\to\infty} \frac{10-6x^2}{5+3e^x}$

(e) $\lim\limits_{x\to-\infty} \frac{10-6x^2}{5+3e^x}$

2. Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line ____ is a vertical asymptote of the graph of f.

How to find c?

The graph of rational function given by $y = \frac{f(x)}{g(x)}$ has a vertical asymptote at x=c if $g(c) = 0$?

Practice

1. Find all vertical asymptotes of the graph of each function

a) $f(x) = \frac{x}{x^2-1}$

b) $f(x) = \frac{x^2-1}{(x-1)(x-2)}$

c) $f(x) = \frac{x^2-4x-5}{x^2-x-2}$

Limit and Continuity

- **How to find the vertical asymptote x=c?**

2. Let f be the function defined by $f(x) = \frac{cx-5x^2}{2x^2+ax+b}$, where a, b, c are constants. The graph of f has a vertical asymptote at x=1, and f has a removable discontinuity at x=-2. Find the value of a, b, and c.

3. Find all asymptotes of $f(x) = \frac{\sin x}{x^2+2x}$