Integration

- ✓ We know how to calculate velocity v(t) from position function s(t). This helped us to understand the idea of the derivative or rate of change of a function.

Now we consider the reverse problem:

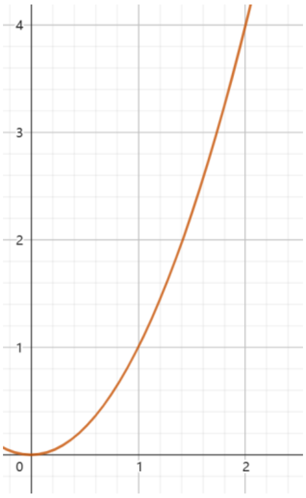
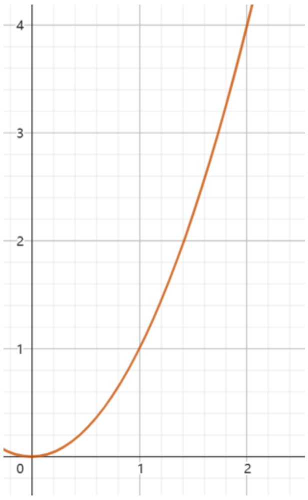
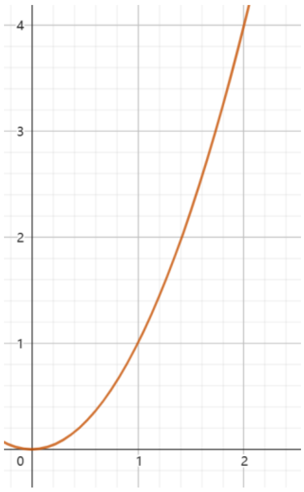
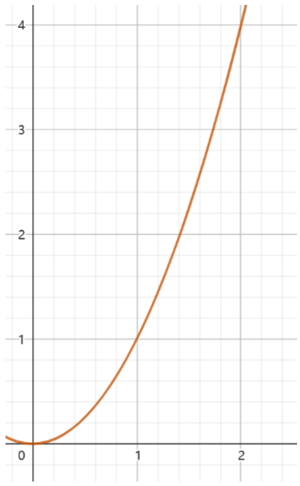
Given velocity, how do we calculate the distance the car has traveled?

This will give us the idea of **definite integration**.

- ➢ **Approximating the area under a curve using rectangles**

Consider the function of $v(t) = t^2$ for $t \in [0,2]$. Use the rectangle(s) to estimate the area under the curve.

Time Interval	0~1	1~2	
Velocity is **at most**			$[s(2) - s(0)] =$
Velocity is **at least**			$[s(2) - s(0)] =$

- ➢ **Def. Riemann Sum**

Let f be a continuous function defined on the closed interval $[a,b]$, and let Δ be a partition of $[a,b]$ given by $a = x_0 < x_1 < \cdots < x_n = b$, where Δx_i is the width of the ith interval. If c_i is <u>**any**</u> point in the ith interval, then the sum $\sum_{i=1}^{n} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n$ is called a **Riemann Sum** for f on the interval $[a,b]$.

- If every subinterval is of equal width, then $\Delta x =$ ________________

- ➢ **Left, Right, and Midpoint Riemann Sum Approximation**

If c_i is the left endpoint of each subinterval then $\sum_{i=1}^{n} f(c_i)\Delta x$ is called a Left Riemann Sum.

If c_i is the right endpoint of each subinterval then $\sum_{i=1}^{n} f(c_i)\Delta x$ is called a Right Riemann Sum.

If c_i is the midpoint of each subinterval then $\sum_{i=1}^{n} f(c_i)\Delta x$ is called a Midpoint Riemann Sum.

- The general form of Riemann sum is $\sum_{i=1}^{n} f(c_i)\Delta x =$ ____________________, where $c_i =$ ____________

Integration

Q1. Approximate the area of the region bounded by the graph of $f(x) = -x^2 + x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$

(1) by using a left Riemann sum with four subintervals

(2) by using a right Riemann sum with four subintervals

(3) by using a midpoint Riemann sum with four subintervals

t (hours)	0	2	4	5	6	9	12
$P'(t)$ people/hour	41	30	54	26	21	44	11

Q2. Tiffani posts a picture of her posing with Sir Isaac from the pop group Sir Isaac and the Newtones on her Instagram at 9 AM. The rate that her followers view her picture is modeled with selected values shown in the table above where $t = 0$ represents 9 AM and the rate is measured in people per hour. Use the data in the table above to approximate the following.

(1) Use a Right Riemann Sum with 3 subintervals to approximate the area between $P'(t)$ and the t-axis from $t = 0$ to $t = 5$. Include units of measure with your answer.

(2) Use a Left Riemann Sum with 4 subintervals to approximate the area between $P'(t)$ and the t-axis from $t = 4$ to $t = 12$. Include units of measure with your answer.

(3) Use a Midpoint Riemann Sum with 3 subintervals to approximate the area between $P'(t)$ and the t-axis from $t = 0$ to $t = 12$. Include units of measure with your answer.

Integration

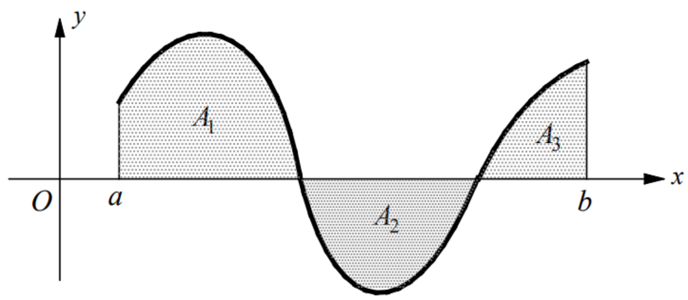
- **Def. Definite Integrals**

If f is a continuous function defined for $a \le x \le b$, then the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta x$$

If $y = f(x)$ is continuous and nonnegative over a closed interval $[a, b]$ then the area of the region bounded by the graph of f , the x-axis, and the vertical lines $x = a$ and $x = b$ is given by $Area = \int_a^b f(x)\,dx$

If $y = f(x)$ takes on both positive and negative values over a closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is obtained by adding the ____________ value of the definite integral over each subinterval where $f(x)$ does not change sign.



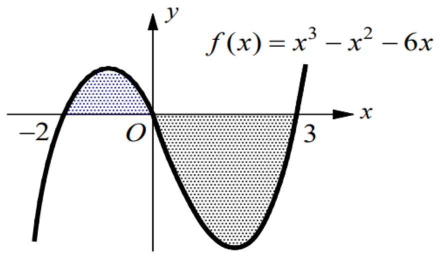
The definite integral of $f(x)$ over $[a, b]$ is $\int_a^b f(x)\,dx =$ ________________________

The total area between the curve and the x-axis over $[a, b]$ is $\int_a^b |f(x)|\,dx =$ ________________

Q3. The figure shows the graph of $f(x) = x^3 - x^2 - 6x$.

(a) Find the definite integral of $f(x)$ on $[-2,3]$ using calculator.

(b) Find the area between the graph of $f(x)$ and the x-axis on $[-2,3]$.



Q4. The expression $\frac{1}{20}[\left(\frac{1}{20}\right)^2 + \left(\frac{2}{20}\right)^2 + \cdots + \left(\frac{20}{20}\right)^2]$ is a Riemann sum approximation for ________________

Integration

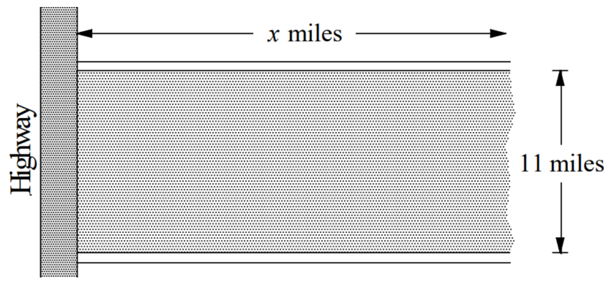
Q5. The expression $\frac{1}{10}[\frac{1}{10}+\frac{2}{10}+\cdots+\frac{20}{10}]$ is a Riemann sum approximation for ________

Q6. Which of the following limits is equal to $\int_1^3 x^3\,dx$

(A) $\lim_{n\to\infty}\sum_{i=1}^{n}(1+\frac{i}{n})^3\frac{1}{n}$ (C) $\lim_{n\to\infty}\sum_{i=1}^{n}(1+\frac{2i}{n})^3\frac{1}{n}$

(B) $\lim_{n\to\infty}\sum_{i=1}^{n}(1+\frac{i}{n})^3\frac{2}{n}$ (D) $\lim_{n\to\infty}\sum_{i=1}^{n}(1+\frac{2i}{n})^3\frac{2}{n}$

Q7. (*)(Calculator) A rectangular region located beside a highway and between two straight roads 11 miles apart are shown in the figure below. The population density of the region at a distance x miles from the highway is given by $D(x)=15x\sqrt{x}-3x^2$, where $0\le x\le 25$. How many people live between 16 to 25 miles from the highway?



Integration

- **Properties of definite integral**

1. $\int_a^a f(x)\,dx =$ ____________

2. $\int_a^b f(x)\,dx =$ ____ $\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx =$ ____________

4. $\int_a^b f(x) \pm g(x)\,dx =$ ____________

5. $\int_a^b cf(x)\,dx =$ ____________

6. $\int_a^b c\,dx =$ ____________

7. If f is even, then $\int_{-a}^a f(x)\,dx =$ ___ $\int_0^a f(x)\,dx$

   If f is odd, then $\int_{-a}^a f(x)\,dx =$ ________

8. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx$ _______

   If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx$ ____ $\int_a^b g(x)\,dx$

   If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then __________ $\leq \int_a^b f(x)\,dx \leq$ __________

Q1. Suppose $\int_0^{1.25} \cos(x^2)\,dx = 0.98$ and $\int_0^1 \cos(x^2)\,dx = 0.90$. What are the values of the following integrals?

(a) $\int_1^{1.25} \cos(x^2)\,dx$

(b) $\int_{-1}^1 \cos(x^2)\,dx$

(c) $\int_{1.25}^{-1} \cos(x^2)\,dx$

Q2. Suppose that $\int_{-3}^{4} f(x)\,dx = 5$, $\int_{-3}^{4} g(x)\,dx = -4$, and $\int_{-3}^{1} f(x)\,dx = 2$.

Find (a) $\int_{-3}^{4} [2f(x) - 3g(x)]\,dx$ (b) $\int_{1}^{4} f(x)\,dx$ (c) $\int_{-3}^{4} [g(x)+2]\,dx$.

Q3. Let f and g be continuous on the interval [1,5]. Given $\int_1^3 f(x)\,dx = -3$, $\int_1^5 f(x)\,dx = 7$, and $\int_1^5 g(x)\,dx = 9$, find the following definite integrals.

(a) $\int_3^5 f(x)\,dx$

(b) $\int_1^3 [f(x) + 3]\,dx$

(c) $\int_5^1 2g(x)\,dx$

(d) $\int_5^5 g(x)\,dx + \int_5^3 f(x)\,dx$

(e) (*) $\int_{-1}^{3} f(x + 2)\,dx$

Integration

- **Def. Antiderivative**

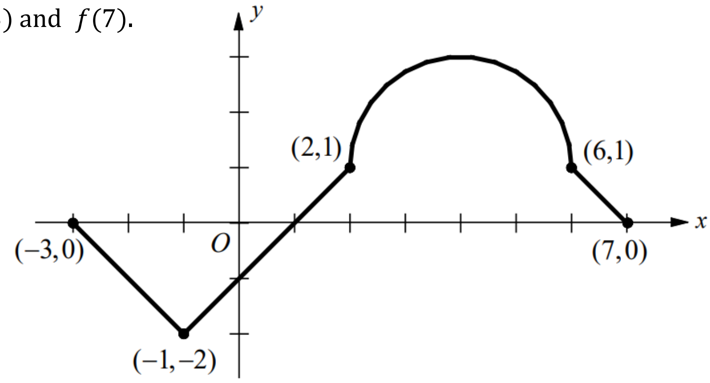
A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x on I.

If F is an antiderivative of f on I, then ________________ represents the most general antiderivative of f on I.

- **The Fundamental Theorem of Calculus (FTC)**

Let f be continuous on $[a, b]$ then $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is an antiderivative of f.

Q1. Let f be a function defined on the closed interval [-3,7] with $f(2) = 3$. The graph of f' consists of three line segments and a semicircle, as shown below. Find $f(-3)$ and $f(7)$.



graph of f'

Q2. (Calculator) If f is the antiderivative of $\dfrac{\sqrt{x}}{1+x^3}$ such that $f(1) = 2$, then $f(3) =$

Q3. (Calculator) If $f'(x) = \cos(x^2 - 1)$ and $f(-1) = 1.5$, then $f(5) =$

Q4. If f is a continuous function and $F'(x) = f(x)$ for all real numbers x, then $\int_2^{10} f(\frac{1}{2}x)\,dx =$

(A) $\frac{1}{2}[F(5) - F(1)]$

(B) $\frac{1}{2}[F(10) - F(2)]$

(C) $2[F(5) - F(1)]$

(D) $2[F(10) - F(2)]$

Integration

➢ Indefinite Integral

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x denoted by $\int f(x)\,dx$.

$$\int f(x)\,dx = F(x) + C \quad \Leftrightarrow F'(x) = f(x)$$

➢ Indefinite Integrals

$\int k\,dx =$ ________ $\int x^n\,dx =$ ________________________ $\int e^x\,dx =$ ____________

$\int \sin x\,dx =$ ____________ $\int \cos x\,dx =$ ____________ $\int \sec^2 x\,dx =$ ____________ $\int \csc^2 x\,dx =$ ____________

$\int \sec x \tan x\,dx =$ ______________________ $\int \csc x \cot x\,dx =$ _______________________

$\int kf(x)\,dx =$ ________ $\int [f(x) \pm g(x)]\,dx =$ __________________________

➢ Integral of Natural Logarithmic Function

$\frac{d}{dx}[\ln x] =$ ________

For $x \neq 0$, $\int \frac{1}{x}\,dx =$ _____________________

Q1. $\int_{\frac{\pi}{2}}^{x} \cos t\,dt = ?$

Q2. Find an antiderivative for each of the following functions.

a. $f(x) = 3x^2$

b. $g(x) = \cos x + 3$

Q3. Find the antiderivative of $x^3 - 3x + 2$.

Q4. Find the general indefinite integral $\int \sqrt{x} - \sec x \tan x\ dx$

Q5. The area of the region in the first quadrant enclosed by $f(x) = 4x - x^3$ and the x-axis is

(A) $\frac{11}{4}$ (B) $\frac{7}{2}$ (C) 4 (D) $\frac{11}{2}$

Q6. Find $\int_1^e \frac{x^2+3}{x}\,dx$

Q7.(*) $\int_0^5 \sqrt{25 - x^2}\,dx =$ ________

Integration

- **Fundamental Theorem:**
  - Let f be continuous on $[a,b]$ then $F(x)=\int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b), and $F'(x)=\frac{d}{dx}\int_a^x f(t)\,dt=$ ______________________
  - If $F(x)=\int_a^{u(x)} f(t)\,dt$, then $F'(x)=\frac{d}{dx}\int_a^{u(x)} f(t)\,dt=$ ______________________

Q1. If $F(x)=\int_1^x \frac{1}{1+u^3}\,du$, then $F'(x)=$ ______________________

Q2. If $F(x)=\int_1^{x^2+1} \sqrt{t}\;dt$, then $F'(x)=$ ______________________

Q3. For $-\frac{\pi}{2}<x<\frac{\pi}{2}$, if $F(x)=\int_0^{\sin x}\frac{dt}{\sqrt{1-t^2}}$, then $F'(x)=$

Q4. Let f be the function given by $f(x)=\int_0^x \cos(t^2+2)\,dt$ for $0\le x\le\pi$. On which of the following intervals is f increasing?

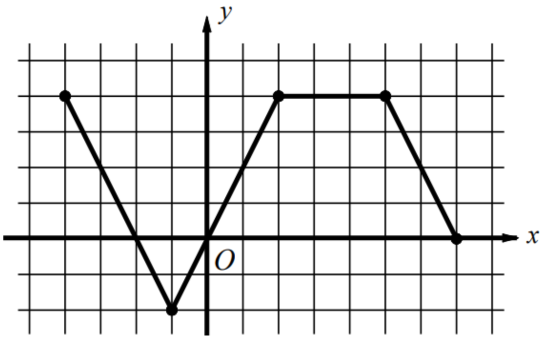
(A) $0\le x\le\frac{\pi}{2}$

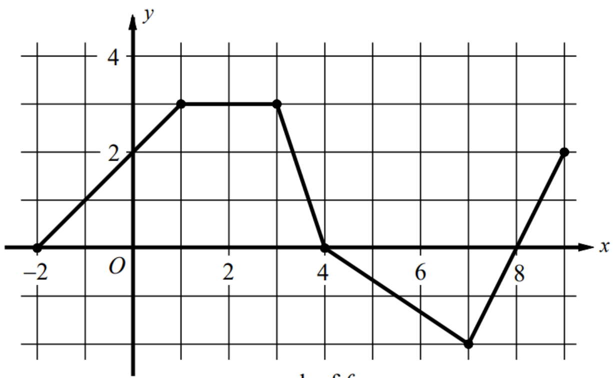
(B) $0\le x\le 1.647$

(C) $1.647\le x\le 2.419$

(D) $\frac{\pi}{2}\le x\le\pi$

Q5. The graph of the function f shown below consists of four line segments. If g is the function defined by $g(x)=\int_{-4}^{x} f(t)\,dt$, find the value of $g(6)$, $g'(6)$, and $g''(6)$.





graph of f

Q6. Let g be the function given by $g(x)=\int_{-2}^{x} f(t)\,dt$. The graph of the function f, shown above, consists of five line segments.

(a) Find $g(0)$, $g'(0)$ and $g''(0)$.

(b) For what values of x, in the open interval $(-2,9)$, is the graph of g concave up?

(c) For what values of x, in the open interval $(-2,9)$, is g increasing?

Integration

- **The Trapezoidal Rule**

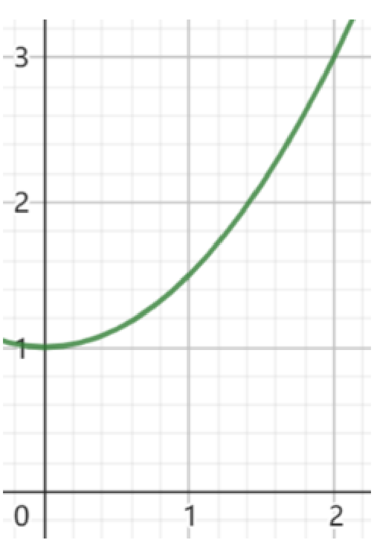
  Let f be continuous on $[a,b]$, and let Δ be a partition of $[a,b]$ given by $a = x_0 < x_1 < \cdots < x_n = b$. The Trapezoidal Rule for approximating the area under a curve is

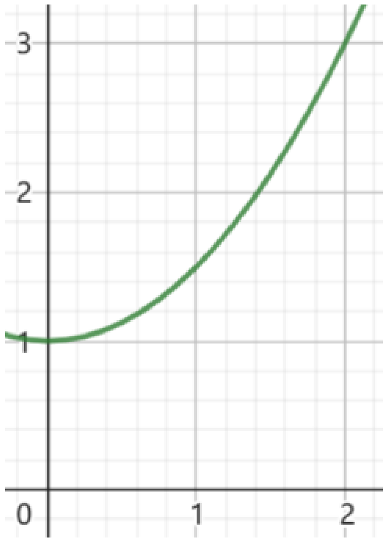
$$Area \approx \underline{\hspace{8cm}}$$

Hint: Consider the function of $y = \frac{1}{2}x^2 + 1$ for $x \in [0,2]$. Use the trapezoids to estimate the area under the curve.

(1) two equal subintervals:

(2) with four equal subintervals:





Q1. Using the Trapezoidal Rule to approximate the area under $f(x) = \sin x$ and above the x-axis on the interval $[0, \pi]$ using 4 trapezoids.

Q2. The function f is continuous on the closed interval $[-1,8]$ and has values that are given in the table below. What is the trapezoidal approximation of the area under the curve of f?

x	-1	1	4	6	8
$f(x)$	5	7	11	8	7

Q3. The following table shows the speed in miles per hour of a cyclist at various times. Use a trapezoidal approximation to find the distance (in miles) the cyclist traveled in the 12-minute time interval.

Time (min)	0	2	5	6	9	10	12
Speed (mph)	33	25	27	13	21	5	9

Q4. If four equal subdivisions on $[0,2]$ are used, what is the trapezoidal approximation of $\int_0^2 e^x\,dx$?

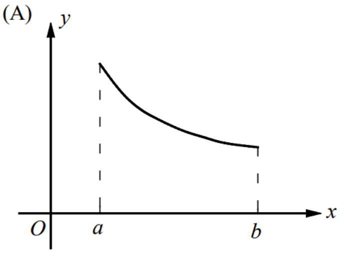
(A) $\frac{1}{4}\left[1+2\sqrt{e}+2e+2e\sqrt{e}+e^2\right]$

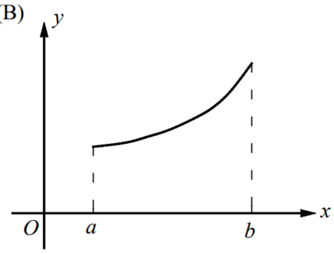
(B) $\frac{1}{2}\left[1+2\sqrt{e}+2e+2e\sqrt{e}+e^2\right]$

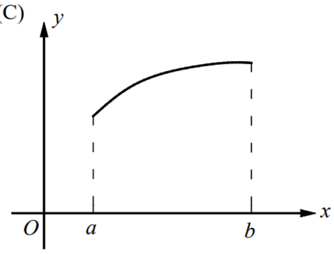
(C) $\frac{1}{4}\left[1+\sqrt{e}+e+e\sqrt{e}+e^2\right]$

(D) $\frac{1}{2}\left[1+\sqrt{e}+e+e\sqrt{e}+e^2\right]$

Q5. If a trapezoidal sum underapproximates the area under the curve on $[a,b]$, and a right Riemann sum overestimates the area, which of the following could be the graph of

(A)


(B)


(C)


(D)