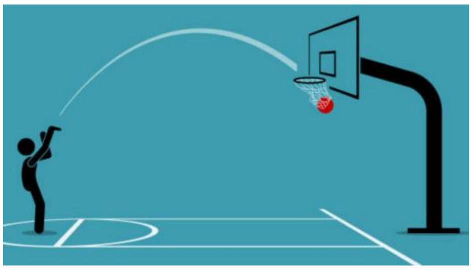
Limit and Continuity

- What is the value of $\frac{1}{2^n}$ when n tends to infinity?

- Suppose you knew that the height of the ball (in feet), t milliseconds after release could be modeled by the function $h(t) = -0.16t^2 + 2.4t + 7$. Using a **calculator,** graph the function. Trace the graph as t tends to 5.What y-value is being approached?



- **Def.** If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the **limit** of $f(x)$, as x approaches c, is L.

$$\lim_{x \to c} f(x) = L$$

- **Basic Limits**

  Constant function $f(x) = k$: $\lim_{x \to c} k =$

  Exponential function $f(x) = e^x$: $\lim_{x \to c} e^x =$

  Polynomial function $f(x) =$_______________:

- **Finding Limits Graphically**

  Consider the graph of the function $f(x) = \frac{x^3+1}{x+1}$. The domain of $f(x)$ is __________________.

  Sketch the graph of $f(x)$ and find the limit of $f(x)$ as x approaches to -1:

Limit and Continuity

✧ Even though $f(-1)$ is not defined, the limit of $f(x)$ is ________ as x approaches to -1, because the definition of a limit says that we consider values of x that are __________ to c, but not __________ to c.

❖ **Def. One-sided Limits**

The **right-hand limit** means that x approaches c from values greater than c.

$$\lim_{x \to c^+} f(x) = L$$

The **left-hand limit** means that x approaches c from values ____________ than c.

$$\lim_{\_\_\_\_} f(x) = L$$

❖ **The existence of a Limit**

The limit of $f(x)$ as x approaches x is L iff ____________________________________

❖ **Limits that fails to exists**

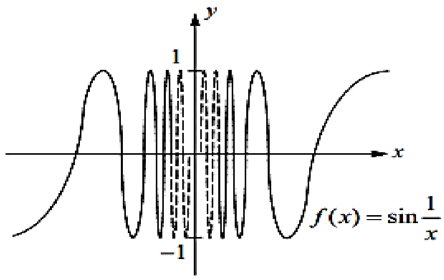
- $f(x) = \frac{|x|}{x}$

- $f(x) = tanx$

Limit and Continuity

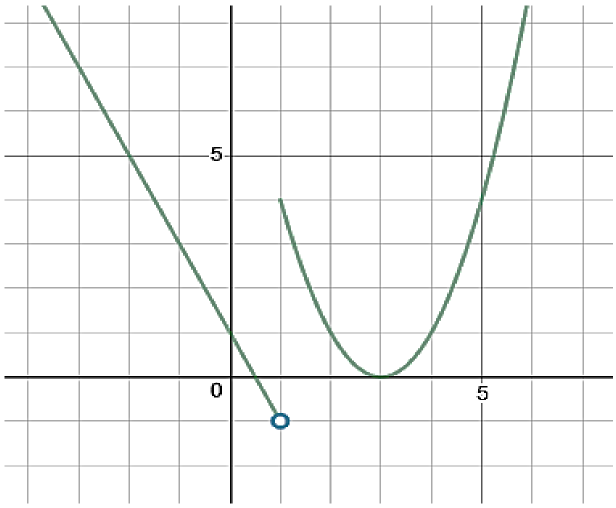
- $f(x) = \frac{1}{x}$

- $f(x) = \sin\frac{1}{x}$



- **Practice**

1. Evaluate the limits by looking at the graph of $f(x)$.



   a) $\lim_{x \to 3^-} f(x)$

   b) $\lim_{x \to 3^+} f(x)$

   c) $\lim_{x \to 3} f(x)$

   d) $f(3)$

   e) $\lim_{x \to 1^-} f(x)$

   f) $\lim_{x \to 1^+} f(x)$

g) $\lim_{x \to 1} f(x)$

h) $f(1)$

2. Sketch a function, g(x), where $\lim_{x \to 2^+} g(x) = 0$ and $\lim_{x \to 2^-} g(x) = 3$.

3. Find the limit

1) $\lim_{x \to -1} x^3 - 2x$

2) $\lim_{x \to 0} \frac{sinx}{x}$ using your calculator

Limit and Continuity

❖ **Limit Laws**

Let c and k be real numbers and the limits $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ **exist**. Then

a) $\lim_{x\to c} f(x) \pm g(x) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$

b) $\lim_{x\to c} f(x) \cdot g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

c) $\lim_{x\to c} kf(x) = k \cdot \lim_{x\to c} f(x)$

d) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, *provided* $\lim_{x\to c} g(x) \neq 0$

e) $\lim_{x\to c} [f(x)]^n = [\lim_{x\to c} f(x)]^n$

f) If f and g are functions such that $\lim_{x\to c} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$, then $\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L)$

✍ **Practice:** Find the limits.

a) $\lim_{x\to 0} \sin 4x$

b) $\lim_{x\to 2} \frac{x^2+3x-10}{x-2}$

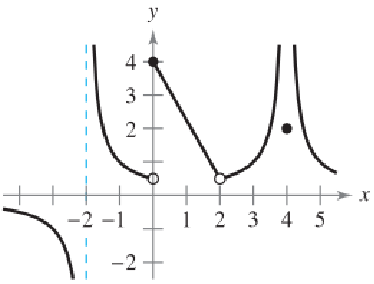
c) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

d) $\lim\limits_{x \to 0} \frac{\sqrt{x+3}-2}{x-1}$

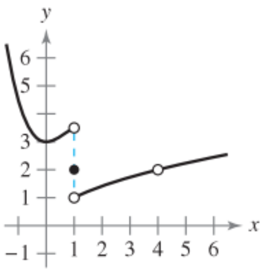
e) $\lim\limits_{x \to 0} x^2 \sin\frac{1}{x}$

f) The graph of f is shown below. Given that $\lim\limits_{x \to 0} h(x) = 1$, then the value of $\lim\limits_{x \to 0} f(h(x))$ is ____________ ,

$\lim\limits_{x \to 0} f(f(x))$ is ____________



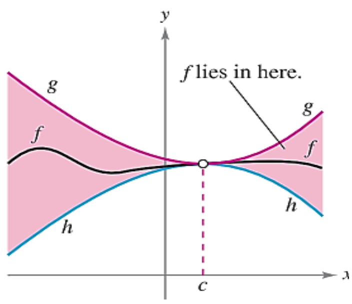
g) The graph of g is shown below. The value of $\lim\limits_{x \to 0} g(1-x^2)$ is ____________



❖ **The squeeze theorem**

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim\limits_{x \to c} h(x) = \lim\limits_{x \to c} g(x) = L$, then $\lim\limits_{x \to c} f(x)$ exists and is equal to L.

$h(x) \le f(x) \le g(x)$



❖ **Special Limits**

$$\lim_{x\to 0}\frac{sinx}{x}=1$$

✍ **Practice:** Find the limits.

a) $\lim_{x\to 0}\frac{\sin 4x}{3x}$

b) $\lim_{x\to 0}\frac{\tan x}{x}$

c) $\lim_{x\to \frac{\pi}{4}}\frac{\tan x-1}{\sin x-\cos x}$

Limit and Continuity

❖ **Def. Continuity**

A function f is continuous at c if the following three conditions are met.

1. $f(c)$ is defined.

2. $\lim_{x \to c} f(x)$ exists

3. $\lim_{x \to c} f(x) = f(c)$

✧ **Continuity over an interval I**

A function f is continuous on an interval if the function is continuous at each point in the interval.

✧ **Discontinuities:**

1. ____________________

2. ____________________

3. ____________________

Practice

For what values of a is $f(x) = \begin{cases} x^2, x \leq 1 \\ ax + 2, 1 < x \leq 3 \end{cases}$ continuous at x=1?

❖ Intermediate Value Theorem

If f is continuous on the closed interval [a,b] and k is any number between f(a) and f(b) , then there is at least one number c in [a,b] such that f(c)=k .

Specifically, if f is continuous on [a,b] and f(a) and f(b) differ in sign, the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval [a,b].

Practice

Let f be a function given by $f(x) = x^3 - 4x + 2$. Use the Intermediate Value Theorem to show that there is a root of the equation on [0,1]

❖ **Asymptotes**

1. Horizontal asymptote

A line ________ is a horizontal asymptote of the graph of a function $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or

$\lim_{x\to-\infty} f(x) = L$

Practice

1. Find the horizontal asymptotes of the function.

(1) $f(x) = \frac{3x^2-2x-3}{2x^2+5x-6}$

(2) $f(x) = \frac{2x^2-2x+10}{x^4+5x^2-100}$

(3) $f(x) = \frac{\sqrt[3]{2x^3-9}}{x}$

(4) $f(x) = \frac{\sqrt{4x^2+6x}}{3x-2}$

2. Find the following limits.

(a) $\lim_{x\to\infty} \frac{x^3-4x^2+7}{2x^3-3x-5}$

(b) $\lim_{x\to\infty} \frac{x^{100}}{e^x}$

(c) $\lim_{x\to\infty} \frac{x^{100}}{\ln x}$

(d) $\lim_{x\to\infty} \frac{10-6x^2}{5+3e^x}$

(e) $\lim_{x\to-\infty} \frac{10-6x^2}{5+3e^x}$

2. Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line _____ is a vertical asymptote of the graph of f.

How to find c?

The graph of rational function given by $y = \frac{f(x)}{g(x)}$ has a vertical asymptote at x=c if $g(c) = 0$?

Practice

1. Find all vertical asymptotes of the graph of each function

(a) $f(x) = \frac{x}{x^2-1}$

(b) $f(x) = \frac{x^2-4x-5}{x^2-x-2}$

(c) $f(x) = \frac{x-1}{x^2-5x+6}$

(d) $f(x) = \frac{\mathrm{x}^2-1}{(\mathrm{x}-1)(x-2)}$

- **How to find the vertical asymptote x=c?**

2. Let f be the function defined by $f(x) = \frac{cx-5x^2}{2x^2+ax+b}$, where a, b, c are constants. The graph of f has a vertical asymptote at x=1, and f has a removable discontinuity at x=-2. Find the value of a, b, and c.

3. Find all asymptotes of $f(x) = \frac{\sin x}{x^2+2x}$

4. $\lim_{x \to 0} f(f(x)) = ?$