Practice Questions

B 1. What are all values of $x$ for which the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n^3}$ converges?

(A) $-1 < x < 1$ (B) $-1 \le x \le 1$ (C) $-1 < x \le 1$ (D) $-1 \le x < 1$

$$\left| \frac{\frac{(-1)^{n+2} x^{n+1}}{(n+1)^3}}{\frac{(-1)^{n+1} x^n}{n^3}} \right| \to |x| < 1 \text{ as } n\to\infty$$

$x=1 : \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

$x=-1 : \sum_{n=1}^{\infty} \frac{-1}{n^3}$ $p=3$

A 2. What are all values of $x$ for which the series $\sum_{n=0}^{\infty} \frac{n(x-2)^n}{3^n}$ converges?

(A) $-1 < x < 5$ (B) $-1 < x \le 5$ (C) $-2 \le x < 4$ (D) $-2 < x \le 4$

$$\left| \frac{\frac{(n+1)(x-2)^{n+1}}{3^{n+1}}}{\frac{n(x-2)^n}{3^n}} \right| \xrightarrow{n\to\infty} \left|\frac{x-2}{3}\right| < 1 \quad \therefore |x-2| < 3$$

$-1 \quad 2 \quad 5$, $3$

$x=-1 : \sum_{n=0}^{\infty} n\cdot(-1)^n$ div

$x=5 : \sum_{n=0}^{\infty} n.$ div.

D 3. What are all values of $x$ for which the series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$ converges?

(A) $0 < x < 2$ (B) $0 \le x < 2$ (C) $-1 < x \le 2$ (D) All real $x$

$$\left| \frac{\frac{x^{n+2}}{(n+2)!}}{\frac{x^{n+1}}{(n+1)!}} \right| = \left| \frac{x}{n+2} \right| \to 0 \text{ as } n\to\infty$$

C 4. What are all values of $x$ for which the series $\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{2^n \sqrt{n}}$ converges?

(A) $-2 < x < 2$ (B) $-2 \le x < 2$ (C) $-2 < x \le 2$ (D) All real $x$

$$\left| \frac{\frac{(-1)^{n+1} x^{n+1}}{2^{n+1}\sqrt{n+1}}}{\frac{(-1)^n x^n}{2^n \sqrt{n}}} \right| \xrightarrow{n\to\infty} \left|\frac{x}{2}\right| < 1 \quad \therefore |x| < 2$$

$x=2 : \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ con

$x=-2 : \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ div

C 5. What are all values of $x$ for which the series $\sum_{n=1}^{\infty} n!(3x-2)^n$ converges?

(A) No values of $x$ (B) $(-\infty, \frac{2}{3}]$ (C) $x = \frac{2}{3}$ (D) $[\frac{2}{3}, \infty)$

$$\left| \frac{(n+1)!(3x-2)^{n+1}}{n!(3x-2)^n} \right| = \left| (n+1)\cdot(3x-2) \right| < 1$$

when $3x-2=0$

$x = \frac{2}{3}$

## Practice Questions

B

1. The power series expansion for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Which of the following is a power series expansion for $\frac{1}{1+x^3}$?

$r=-x^3$. $\sum_{n=0}^{\infty}(-x^3)^n = 1-x^3+x^6-x^9+\cdots$

(A) $1+x^2+x^4+x^6+\cdots$

(B) $1-x^3+x^6-x^9+\cdots$

(C) $1+\frac{x^3}{3}+\frac{x^6}{6}+\frac{x^9}{9}+\cdots$

(D) $1-\frac{x^3}{3}+\frac{x^6}{6}-\frac{x^9}{9}+\cdots$

C

2. The power series expansion for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Which of the following is a power series expansion for $\frac{1}{2-x}$?

$\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}} = \frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$

$=\frac{1}{2}\cdot\left(1+\frac{x}{2}+\frac{x^2}{4}+\cdots\right)$

$=\frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\cdots$

(A) $1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots$

(B) $1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\cdots$

(C) $\frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}+\cdots$

(D) $\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\cdots$

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4. A power series expansion for $f(x)=\frac{1}{1-x}$ can be obtained from the sum of the geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, if you let $a=1$ and $r=x$. Let $g(x)$ be defined as $g(x)=\frac{1}{1+x}$.

(a) $g(x)=\sum_{n=0}^{\infty}(-x)^n = 1-x+x^2-x^3+\cdots+(-x)^n+\cdots$

$|x|<1$

(a) Write the first four terms and the general term of the power series expansion of $g(x)$.

(b) Write the first four terms and the general term of the power series expansion of $g(x^2)$.

(c) Write the first four terms and the general term of the power series expansion of $h$, where $h(x)=\int g(x^2)\,dx$ and $h(0)=0$.

(b) $g(x^2)=\sum_{n=0}^{\infty}(-1)^n x^{2n} = 1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+\cdots$

(d) Find the value of $h(1)$.

$|x|<1$

(c) $h(x)=\int\sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n\int x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n+1}+C$

$h(0)=C=0$

$\therefore h(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n+1} = x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots+(-1)^n\frac{1}{2n+1}x^{2n+1}+\cdots$

$|x|<1$

(d) $h(x)=\int\frac{1}{1+x^2}dx=\tan^{-1}x+C$; $\therefore h(x)=\tan^{-1}x$

$h(0)=0\Rightarrow C=0$ $h(1)=\frac{\pi}{4}$

(13 Handout)

Practice Questions

7. Let $P(x) = 3-2(x-2)+5(x-2)^2-12(x-2)^3+3(x-2)^4$ be the fourth-degree Taylor polynomial for the function $f$ about $x=2$. Assume $f$ has derivatives of all orders for all real numbers.

(a) Find $f(2)$ and $f'''(2)$. $f(2)=3$ $f'''(2) = -12\times 3! = -72$

(b) Write the third-degree Taylor polynomial for $f'$ about 2 and use it to approximate $f'(2.1)$.

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_2^x f(t)\,dt$ about 2.

(d) Can $f(1)$ be determined from the information given? Justify your answer.

8. Let $f$ be the function given by $f(x) = \sin(2x)+\cos(2x)$, and let $P(x)$ be the third-degree Taylor polynomial for $f$ about $x=0$.

(a) Find $P(x)$.

(b) Find the coefficient of $x^{19}$ in the Taylor series for $f$ about $x=0$.

(c) Use the Lagrange error bound to show that $\left|f(\frac{1}{5})-P(\frac{1}{5})\right| < \frac{1}{100}$

(d) Let $h$ be the function given by $h(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for $h$ about $x=0$.

$f'(x)=2\cos 2x - 2\sin 2x$
$f''(x)= -4\sin 2x - 4\cos 2x$
$f'''(x)= -8\cos 2x + 8\sin 2x$
$f^{(4)}(x)= 16\sin 2x + 16\cos 2x$
$= 16\cdot\sqrt{2}\cdot\left(\frac{\sqrt{2}}{2}\sin 2x+\frac{\sqrt{2}}{2}\cos 2x\right)$
$= 16\sqrt{2}\cdot \sin\left(2x+\frac{\pi}{4}\right)$

(a) $\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(2x)^{2n+1}}{(2n+1)!}(-1)^n = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+1}\cdot(-1)^n$

$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!}(-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$

$P(x) = 1 + 2x - 2x^2 - \frac{4}{3}x^3$

(b) power $= 19 = 2n+1 \Rightarrow 2n=18 \Rightarrow n=9$

$\therefore$ coefficient $= \frac{2^{19}}{19!}(-1)^9 = -\frac{2^{19}}{19!}$

(c) $\left|f(\frac{1}{5})-P(\frac{1}{5})\right| = \left|f^{(4)}(\xi)\cdot\frac{(\frac{1}{5})^4}{4!}\right|$ $f^{(4)}(x) \le 16\sqrt{2}$

$\le 16\sqrt{2}\cdot\frac{1}{4!\,5^4} < \frac{1}{100}$

(d) $P_3(x) = x + x^2 - \frac{2}{3}x^3$

Practice Questions

B

1. A series expansion of $\frac{\arctan x}{x}$ is

$$\frac{\tan^{-1}x}{x} = \frac{x}{x} - \frac{\frac{x^3}{3}}{x} + \frac{\frac{x^5}{5}}{x} - \frac{\frac{x^7}{7}}{x} + \cdots$$

(A) $1-\frac{x}{3}+\frac{x^3}{5}-\frac{x^5}{7}+\cdots$

(B) $1-\frac{x^2}{3}+\frac{x^4}{5}-\frac{x^6}{7}+\cdots$

(C) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$

(D) $x-\frac{x^3}{3}+\frac{x^4}{5}-\frac{x^6}{7}+\cdots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

A

2. The coefficient of $x^3$ in the Taylor series for $e^{-2x}$ about $x=0$ is

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots$$

(A) $-\frac{4}{3}$ (B) $-\frac{2}{3}$ (C) $-\frac{1}{3}$ (D) $\frac{4}{3}$

$$\frac{-8}{3!} = -\frac{4}{3}$$

D

3. A function $f$ has a Maclaurin series given by $-\frac{x^4}{3!}+\frac{x^6}{5!}-\frac{x^8}{7!}+\cdots+\frac{(-1)^n x^{2n+2}}{(2n+1)!}+\cdots$
Which of the following is an expression for $f(x)$?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x \cdot x - x^2 = -\frac{x^4}{3!} + \frac{x^6}{5!} - \cdots$$

(A) $x^3e^x - x^2$

(B) $x\ln x - x^2$

(C) $\tan^{-1}x - x$

(D) $x\sin x - x^2$

C

4. A series expansion of $\frac{x-\sin x}{x^2}$ is

$$-\sin x = -x + \frac{x^3}{3!} - \cdots$$

$$\frac{x-\sin x}{x^2} = \frac{\frac{x^3}{3!}}{x^2} - \frac{\frac{x^5}{5!}}{x^2} + \frac{\frac{x^7}{7!}}{x^2} - \cdots$$

(A) $\frac{1}{2!}-\frac{x^2}{4!}+\frac{x^4}{6!}+\cdots+\frac{(-1)^{n+1}x^{2n-2}}{(2n)!}+\cdots$

(B) $\frac{x}{2!}-\frac{x^3}{4!}+\frac{x^5}{6!}+\cdots+\frac{(-1)^{n+1}x^{2n+1}}{(2n)!}+\cdots$

(C) $\frac{x}{3!}-\frac{x^3}{5!}+\frac{x^5}{7!}+\cdots+\frac{(-1)^{n+1}x^{2n-1}}{(2n+1)!}+\cdots$

(D) $\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}+\cdots+\frac{(-1)^{n+1}x^{2n}}{(2n+1)!}+\cdots$

C

5. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{n!}$ is the Taylor series about zero for which of the following functions?

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

(A) $x\sin x$
×

(B) $x\cos x$
×

(C) $x^2e^{-x}$
$\sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n$

(D) $x\ln(x+1)$
$\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}(-1)^n$

## Practice Questions

(cal)

B 6. The graph of the function represented by the Maclaurin series $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots$ $\tan^{-1}x$

intersects the graph of $y=e^{-x}$ at $x=$

(A) 0.495 (B) 0.607 (C) 1.372 (D) 2.166

$e^{-x}=\tan^{-1}x$

$\Downarrow$

$x \approx 0.607$

D 7. What is the coefficient of $x^4$ in the Taylor series for $\cos^2 x$ about $x=0$?

(A) $\frac{1}{12}$ (B) $\frac{1}{8}$ (C) $\frac{1}{6}$ (D) $\frac{1}{3}$

$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$

$\left(1-\frac{x^2}{2}+\frac{1}{24}x^4 \cdots\right)^2$

$(-\frac{1}{2})\times(-\frac{1}{2})x^4+\frac{1}{24}x^4\times 2$

$=\frac{8}{24}x^4=\frac{1}{3}x^4$

A 8. The fifth-degree Taylor polynomial for $\tan x$ about $x=0$ is $x+\frac{1}{3}x^3+\frac{2}{15}x^5$. If $f$ is a function such that $f'(x)=\tan(x^2)$, then the coefficient of $x^7$ for $f(x)$ about $x=0$ is

(A) $\frac{1}{21}$ (B) $\frac{3}{42}$ (C) 0 (D) $\frac{1}{7}$

$\tan(x^2)=x^2+\frac{1}{3}x^6+\frac{2}{15}x^{10}+\cdots$

$f(x)=\frac{1}{3}x^3+\frac{1}{21}x^7+\frac{2}{11\times 15}x^{11}+\cdots$

C 9. The Maclaurin series for a function $f$ is given by $\sum_{n=1}^{\infty}\frac{(-2)^{n-1}}{n+1}x^n=\frac{1}{2}x-\frac{2}{3}x^2+x^3-\cdots+\frac{(-2)^{n-1}}{n+1}x^n+\cdots$.

Which of the following is the third-degree Taylor polynomial for $g(x)=\cos x\cdot f(x)$ about $x=0$?

(A) $x-\frac{1}{2}x^2-\frac{2}{3}x^3$

(B) $1-\frac{1}{2}x^2+\frac{2}{3}x^3$

(C) $\frac{1}{2}x-\frac{2}{3}x^2+\frac{3}{4}x^3$

(D) $\frac{1}{2}x-\frac{11}{12}x^2+x^3$

$\left(1-\frac{x^2}{2}+\frac{x^4}{4!}+\cdots\right)\cdot\left(\frac{1}{2}x-\frac{2}{3}x^2+x^3-\cdots\right)$

$x:\ \frac{1}{2}$

$x^2:\ -\frac{2}{3}$

$x^3:\ 1-\frac{1}{4}=\frac{3}{4}$

B 10. The Maclaurin series for the function $f$ is given by

$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots+\frac{(-1)^n x^{2n}}{(2n+1)!}+\cdots$ on its interval of convergence.

Which of the following statements about $f$ must be true?

$f'(x)=-\frac{2}{3!}x+\cdots$ $f'(0)=0$

$f''(x)=-\frac{1}{3}$ $f''(0)<0$

$\therefore$ max

(A) $f$ has a relative minimum at $x=0$.

(B) $f$ has a relative maximum at $x=0$.

(C) $f$ does not have a relative maximum or a relative minimum at $x=0$.

(D) $f$ has a point of inflection at $x=0$.

11. Let $f$ be the function given by $f(x) = e^{-x}$.

(a) Write the first four terms and the general term of the Taylor series for $f$ about $x = 0$.

(b) Use the result from part (a) to write the first four nonzero terms and the general term of the series expansion about $x = 0$ for $g(x) = \frac{1 - x - f(x)}{x}$.

(c) For the function $g$ in part (b), find $g'(-1)$ and use it to show that $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

(a) $f(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

(b) $-f(x) = -1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$

$1 - x - f(x) = -\frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$

$\frac{1-x-f(x)}{x} = -\frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} - \cdots = \sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^{n-1}}{n!}$

(c) $g'(x) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!} (n-1) x^{n-2}$

$g'(-1) = \sum_{n=2}^{\infty} (-1)^{n+1} \frac{n-1}{n!} (-1)^n$

$g'(x) = \frac{(-1+e^{-x}) \cdot x - (1 - x - e^{-x})}{x^2} = \frac{e^{-x} \cdot (x+1) - 1}{x^2}$

$= \sum_{n=2}^{\infty} (-1) \cdot \frac{n-1}{n!}$

$g'(-1) = e^1 \cdot 0 - 1 = -1$

$= \sum_{n=1}^{\infty} (-1) \frac{n}{(n+1)!}$

$\therefore \sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$

12. The Maclaurin series for $f(x)$ is given by $f(x) = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} + \cdots + \frac{(-1)^{n+1} x^{2n-1}}{(2n)!} + \cdots$.

The Maclaurin series for $g(x)$ is given by $g(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots + \frac{(-1)^n x^n}{n+1} + \cdots$.

$2n - 1 = 15 \Rightarrow n = 8$

(a) Find $f'''(0)$ and $f^{(15)}(0)$.

(a) $f'''(0) = -\frac{1}{4!} \cdot 3! = -\frac{1}{4}$

$f^{(15)}(0) = \frac{(-1)^9}{16!} \cdot 15! = -\frac{1}{16}$

(b) Find the interval of convergence of the Maclaurin series for $g(x)$.

(c) The graph of $y = f(x) + g(x)$ passes through the point $(0, 1)$. Find $y'(0)$ and $y''(0)$ and determine whether $y$ has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

(b) $\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{n+2}}{\frac{(-1)^n x^n}{n+1}} \right| = |x| < 1$

$x = 1$: $\sum \frac{(-1)^n}{n+1}$ convergent (AST)

$x = -1$: $\sum \frac{1}{n+1}$ divergent (p-series, $p=1$)

$\therefore (-1, 1]$

(c) $y(0) = f(0) + g(0) = 1$

$y'(x) = \frac{1}{2} - \frac{3}{4!}x^2 + \cdots + (-\frac{1}{2}) + \frac{2}{3}x + \cdots$

$y'(0) = 0$

$y''(x) = -\frac{3!}{4!}x + \cdots + \frac{2}{3} + \cdots$

$y''(0) = \frac{2}{3} > 0$

$\therefore$ relative minimum.