➢ **Def.** A **differential equation** in $x$ and $y$ is an equation that involves the derivatives of $y$.

☺ **Examples of Differential Equations:**

$y''+2y'=3y$ $\qquad$ $f''(x)+2f'(x)=3f(x)$ $\qquad$ $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3y$

☺ **Example.** A particle moves along a straight line. Its velocity, $v$, is inversely proportional to the square of the distance, s, it has traveled. Which equation describes this relationship?

**(A)** $v(t)=\frac{k}{t^2}$ $\qquad$ **(B)** $v(t)=\frac{k}{s^2}$ $\qquad$ **(C)** $\frac{dv}{dt}=\frac{k}{t^2}$ $\qquad$ **(D)** $\frac{dv}{dt}=\frac{k}{s^2}$

✧ A **solution to a differential equation** is a function that satisfies the differential equation when the function and its derivatives are substituted into the equation.

- general solution: contains all possible solutions with arbitrary constants
- particular solution: obtained by fixing constants using initial conditions or boundary conditions

➢ **Separable Differential Equations**

The equation $y'=f(x,y)$ is a separable equation if all x terms can be collected with dx and all y terms with dy .

The differential equation then has the form $\frac{dy}{dx}=f(x)g(y)$, then the equation can be solved.

✧ **Practice**

1. Find the general solution of $f'(x)=5f(x)$

2. Find the general solution of $\frac{dy}{dx}=-\frac{x}{y}$

3. Find the general solution of $(x+3)y' = 2y$

4. Consider the differential equation $\frac{dy}{dx} = \frac{2x+3}{e^y}$.

(a) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $y(0) = 2$. Write an equation for the line tangent to the graph of $f$ at $(0,2)$.

(b) Find $f''(0)$ with the initial condition $y(0) = 2$.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = \frac{2x+3}{e^y}$ with the initial condition $y(0) = 2$.

➢ **Exponential Growth and Decay**

In many real-word situations, a quantity $y$ increases or decreases at a rate $(k)$ proportional to its size at a given time $t$. If $y$ is a function of time $t$, then $\frac{dy}{dt} =$ ____________________

If the initial value $y(0) = y_0$, then $y =$ __________________________

**Example.**

1. The number of bacteria in a culture increases at a rate proportional to the number present. If the number of bacteria was 600 after 3 hours and 19,200 after 8 hours, when will the population reach 120,000?

2. The rate at which the amount of coffee in a coffeepot changes with time is given by the differential equation $\frac{dV}{dt} = kV$, where $V$ is the amount of coffee left in the coffeepot at any time $t$ seconds. At time $t = 0$ there were 16 ounces of coffee in the coffeepot and at time $t = 80$ there were 8 ounces of coffee remaining in the pot.

   (a) Write an equation for $V$, the amount of coffee remaining in the pot at any time $t$.

   (b) At what rate is the amount of coffee in the pot decreasing when there are 4 ounces of coffee remaining?

   (c) At what time $t$ will the pot have 2 ounces of coffee remaining?

## ➢ Logistic Equations

The differential equation $\frac{dP}{dt} =$ ______________________ is called a **logistic equation.**

$P(t)$: the size of the population at time t

$A$: the carrying capacity (the maximum population that the environment is capable of sustaining in the long run)

$k$: a constant

### ✧ Properties:

1. $\lim_{t\to\infty} P(t) =$ ____________

2. $\lim_{t\to\infty} \frac{dP}{dt} =$ ____________

3. The population is growing the fastest when $P =$ ______

4. The graph of $P(t)$ has a point of inflection at the point where $P =$ _____

**Solution curves for the logistic equations with different initial conditions**

**1.** A population is modeled by a function $P$ that satisfies the logistic differential equation $\dfrac{dP}{dt} = \dfrac{P}{2}\left(3 - \dfrac{P}{20}\right)$, where the initial population $P(0) = 100$ and $t$ is the time in years.

(a) What is $\lim_{t\to\infty} P(t)$?

(b) For what values of $P$ is the population growing the fastest?

(c) Find the slope of the graph of $P$ at the point of inflection.

**2.** Let $f$ be a function with $f(2)=1$, such that all points $(t, y)$ on the graph of $f$ satisfy the differential equation $\frac{dy}{dt}=2y\left(1-\frac{t}{4}\right)$.

Let $g$ be a function with $g(2)=2$, such that all points $(t, y)$ on the graph of $g$ satisfy the logistic differential equation $\frac{dy}{dt}=y\left(1-\frac{y}{5}\right)$.

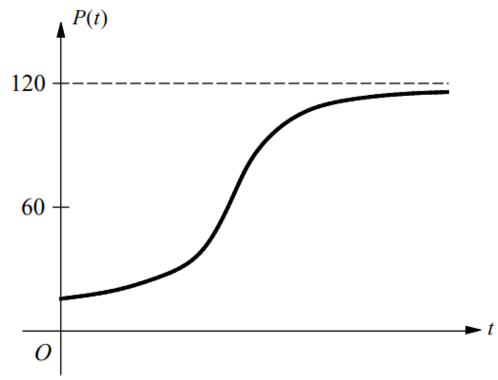
(a) Find $y=f(t)$.

(b) For the function found in part (a), what is $\lim_{t\to\infty} f(t)$?

(c) Given that $g(2)=2$, find $\lim_{t\to\infty} g(t)$ and $\lim_{t\to\infty} g'(t)$.

(d) For what value of $y$ does the graph of $g$ have a point of inflection? Find the slope of the graph of $g$ at the point of inflection.

**3.**




Which of the following differential equations for population $P$ could model the logistic growth shown in the figure above

(A) $\frac{dP}{dt}=0.03P^2-0.0005P$

(B) $\frac{dP}{dt}=0.03P^2-0.000125P$

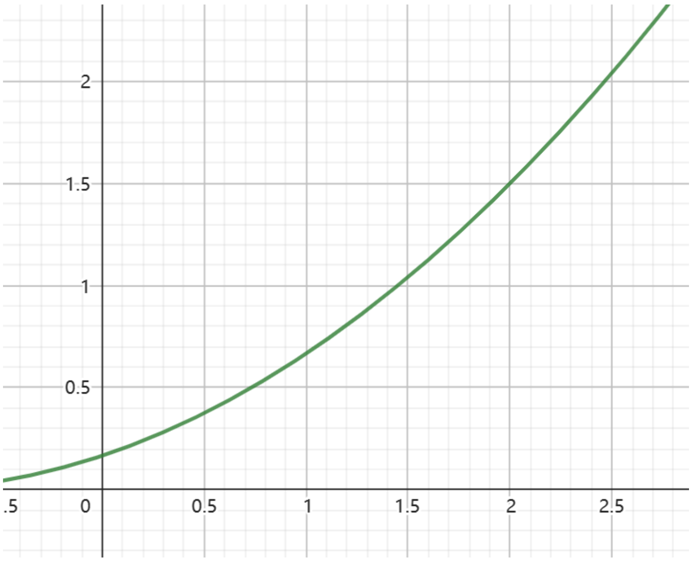
(C) $\frac{dP}{dt}=0.03P-0.001P^2$

(D) $\frac{dP}{dt}=0.03P-0.00025P^2$

## ➢ Euler's Method

Euler's Method is a numerical approach to approximate the particular solution of the differential equation $y' = f(x, y)$ with an initial condition $y(x_0) = y_0$. Using a small step $h$ and $(x_0, y_0)$ as a starting point, move along the tangent line until you arrive at the point $(x_1, y_1)$, where $x_1 =$ __________, $y_1 =$ ____________________

**Repeat** the process with the same step size h at a new starting point $(x_1, y_1)$. The values of $x_i$ and $y_i$ are as follows.




1. Let $f$ be the function whose graph goes through the point $(1, -1)$ and whose derivative is given $y' = 2 - \frac{y}{x}$.

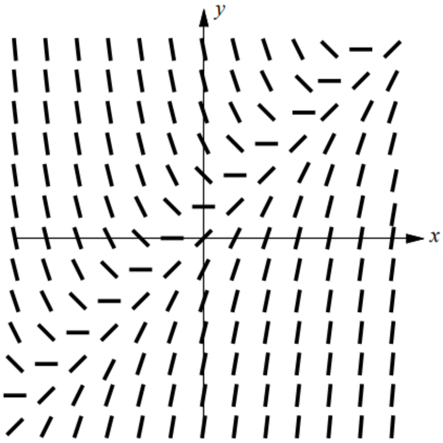
   Use Euler's method starting at $x = 1$ with a step size of $0.5$ to approximate $f(3)$.

2. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x - y + 2$ with the initial condition $f(0) = 2$. Use Euler's method starting at $x = 0$ with a step size of $0.5$ to approximate $f(2)$.

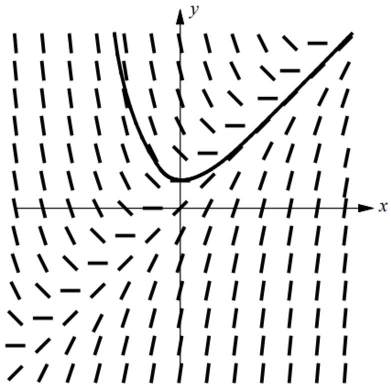
- **Slope Field**

we have learnt ways to solve some simple differential equations analytically. However, doing so can be difficult or sometimes impossible. A graphical approach to solve a differential equation is by creating **slope fields**, which show the **general shape of all solutions** to a differential equation.

Consider a differential equation $y' = f(x, y)$ in terms of $x$ and $y$. For every point $(x, y)$ in its domain, $y'$ determines the slope of the solution function at that point. If you draw a short line segment with the slope indicated at each point on $y'$, the **slope field (direction field)** will show the general shape of all the solution functions to that differential equation.



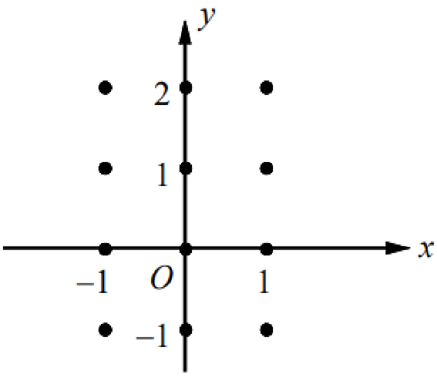

Slope field for $y' = x - y + 1$



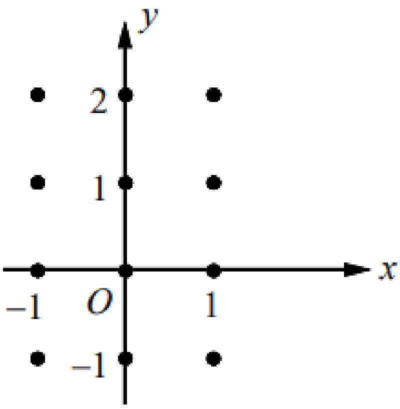

Particular solution for $y' = x - y + 1$ passing through $(0,1)$

**Practice.**

1. On the axes provided, sketch a slope field for the differential equation $y' = 1 - xy$.

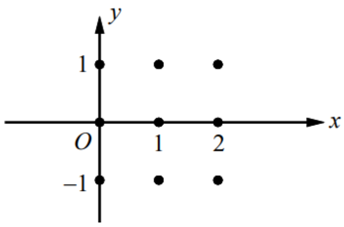



2. On the axes provided, sketch a slope field for the differential equation $y' = y + xy$.



**3.** Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.

(a) On the axis provided sketch a slope field for the given differential equation at the nine points indicated.



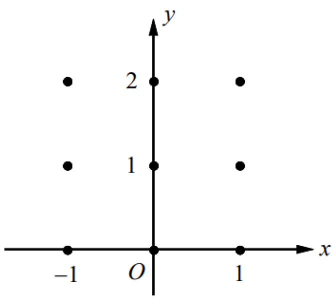

(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $y(1) = \sqrt{3}$. Write an equation for the line tangent to the graph of $f$ at $(1, \sqrt{3})$ and use it to approximate $f(1.2)$.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $y(1) = \sqrt{3}$.

(d) Use your solution from part (c) to find $f(1.2)$.

**4.** Consider the differential equation $\frac{dy}{dx} = 2x + y$.

(a) On the axis provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(1,1)$.




(b) Let $f$ be the function that satisfies the given differential equation with the initial condition $f(1) = 1$. Use Euler's method, starting at $x = 1$ with a step size of 0.1, to approximate $f(1.2)$. Show the work that leads to your answer.

(c) Find the value of $b$ for which $y = -2x + b$ is a solution to the given differential equation. Show the work that leads to your answer.

(d) Let $g$ be the function that satisfies the given differential equation with the initial condition $g(1) = -2$. Does the graph of $g$ have a local extremum at the point $(1, -2)$? If so, is the point a local maximum or a local minimum? Justify your answer.