Techniques for finding antiderivatives

Substitution

If $\boldsymbol{u = g(x)}$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f\big(g(x)\big)\cdot \boldsymbol{g'(x)\,dx} = \int f(u)\boldsymbol{du}$$

$$\boxed{\begin{gathered} \boldsymbol{u = g(x)} \\ \boldsymbol{\frac{du}{dx} = \frac{d}{dx}[g(x)] = g'(x)} \\ \boldsymbol{du = g'(x)dx} \end{gathered}}$$

$$\int \boldsymbol{\cos x} \sin x\, \boldsymbol{dx} = \int u\, du$$

$$u = sinx$$

$$= \frac{1}{2}\mathrm{u}^2 + \mathrm{C}$$

$$\frac{du}{dx} = cos\ x$$

$$= \frac{1}{2}(\sin x)^2 + \mathrm{C}$$

$$du = \cos x\, dx$$

Substitution

$$\int \cos(\boldsymbol{5\theta - 3})\, d\theta = \int cos\, u\, \frac{1}{5} d\mathrm{u}$$

$$\boldsymbol{u = 5\theta - 3}$$

$$\frac{\boldsymbol{du}}{\boldsymbol{d\theta}} = 5$$

$$\boldsymbol{du = 5d\theta}$$

$$\boldsymbol{d\theta = \frac{1}{5} du}$$

$$= \frac{1}{5}\sin u + C$$

$$= \frac{1}{5}\sin(5\theta - 3) + C$$

Substitution

$$\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x =$$

Substitution

$$\int_0^{\frac{\pi}{2}} \frac{3\cos x}{\sqrt{1+3\sin x}}\,\mathrm{d}x =$$

Substitution

If $\int_{-1}^{3} f(x+k)\,dx = 8$, where k is a constant, then $\int_{k-1}^{k+3} f(x)\,dx =$

(A) $8-k$ (B) $8+k$ (C) 8 (D) $k-8$

Substitution

$$\int_0^{\frac{\pi}{4}} \frac{\tan x}{\sqrt{\sec x}}\,\mathrm{d}x =$$

Substitution

Evaluate $\int_{e}^{e^2} \frac{(\ln x)^2}{x}\,dx$.

Substitution

Evaluate $\int_0^{\pi/4} (e^{\tan x} + 2)\sec^2 x \, dx$.

Substitution

$$\int_0^{\pi/2} \cos x \, e^{\sin x} \, dx =$$

Rational Functions

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \quad \tan^{-1} x + C$$

$$\int \frac{x}{1+x^2}\,\mathrm{d}x = \quad \frac{1}{2}\int \frac{1}{1+x^2}\,\mathrm{d}(1+x^2) = \frac{1}{2}\ln|1+x^2| + C = \frac{1}{2}\ln(1+x^2) + C$$

$$\int \frac{1+x}{1+x^2}\,\mathrm{d}x = \quad \int \frac{1}{1+x^2}\,\mathrm{d}x + \int \frac{x}{1+x^2}\,\mathrm{d}x \quad = \tan^{-1} x + \frac{1}{2}\ln(1+x^2) + C$$

Basic Rules

1. Separate numerator $\frac{1+x}{1+x^2} = \frac{1}{1+x^2} + \frac{x}{1+x^2}$

Rational Functions

$$= \int \frac{\mathbf{1}}{\boldsymbol{x^2 - 1}}\, \mathbf{d}(\boldsymbol{x^2} - \mathbf{1})$$

$$\int \frac{x^3 - 3x}{x^2 - 1}\, \mathrm{d}x \quad = \int x - \frac{2x}{x^2 - 1}\, \mathrm{d}x = \int x\, \mathrm{d}x - \int \frac{\mathbf{2}\boldsymbol{x}}{\boldsymbol{x^2} - \mathbf{1}}\, \mathbf{d}\boldsymbol{x}$$

$$= \frac{1}{2}x^2 - \ln|x^2 - 1| + C$$

Basic Rules

2. Divide improper fractions

$$\frac{x^3 - 3x}{x^2 - 1} = \frac{(x^2 - 1)x - 2x}{x^2 - 1} = x - \frac{2x}{x^2 - 1}$$

Basic Rules

3. Add and subtract terms in numerator

~ To construct the derivative of denominator

$$\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{x^2+2x+1}$$

$$\int \frac{2x}{x^2+2x+1}\mathrm{d}x = \int \frac{2x+2}{x^2+2x+1}\mathrm{d}x - \int \frac{2}{x^2+2x+1}\mathrm{d}x$$

$$= \int \frac{1}{x^2+2x+1} d(x^2+2x+1) - \int \frac{2}{x^2+2x+1}\mathrm{d}x$$

$$= \ln|x^2+2x+1| - \int \frac{2}{x^2+2x+1}\mathrm{d}x$$

Completing the Square

$\frac{d}{dx}[\tan^{-1} x] =$

$\frac{d}{dx}[\sin^{-1} x] =$

$\frac{d}{dx}[\cos^{-1} x] =$

$\frac{d}{dx}[\sec^{-1} x] =$

$\int \frac{1}{1+x^2} dx =$ $\mathbf{\tan^{-1} \boldsymbol{x} + \boldsymbol{C}}$

$\int \frac{1}{\sqrt{1-x^2}} dx =$ $\mathbf{\sin^{-1} \boldsymbol{x} + C = -\cos^{-1} \boldsymbol{x} + \boldsymbol{C}}$

$\int \frac{1}{x\sqrt{x^2-1}} dx =$ $\mathbf{\sec^{-1} \boldsymbol{x} + \boldsymbol{C}}$

Completing the Square

$$\int \frac{1}{x^2-2x+2}\,dx = \int \frac{1}{(x-1)^2+1}\,dx = \int \frac{1}{(x-1)^2+1}\,d(x-1)$$

$$= \tan^{-1}(x-1) + C$$

$$\int \frac{1}{4+x^2}\,dx = \int \frac{1}{4[(\frac{x}{2})^2+1]}\,dx = \int \frac{1}{4[(\frac{x}{2})^2+1]}\,2\,d(\frac{x}{2})$$

$$= \int \frac{1}{2[(\frac{x}{2})^2+1]}\,d(\frac{x}{2})$$

$$= \frac{1}{2}\tan^{-1}(\frac{x}{2}) + C$$

Completing the Square

$$\int \frac{1}{\sqrt{9-x^2}}\,dx =$$

$$\int \frac{1}{x\sqrt{x^2-9}}\,dx =$$

Completing the Square

$$\int \frac{1}{\sqrt{-x^2+4x+5}}\,dx =$$

$$\int \frac{1}{x^2+4x+8}\,dx =$$

Completing the Square

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx =$$

$$\int \frac{1}{a^2+x^2}\,dx =$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx =$$

Trigonometric Integrals $\int \sin^m x \cos^n x \, dx$

Review: $sin^2 x + \cos^2 x = 1$

Case 1.
m is odd

Save one "sine" factor to construct "d(cos x)".
Then use $\sin^2 \mathrm{x} = 1 - \cos^2\mathrm{x}$ to trans all "sin x" to "cos x".

→ Power functions always have corresponding antiderivatives.

$$\int \sin^3 x \cos^2 x \, dx \quad = \int \sin^2 x \cos^2 x \sin x \, dx$$

$$= -\int (1 - cos^2 x) \cos^2 x \, d(\cos x)$$

$$u = \cos x$$
$$du = -\sin x \, dx$$
$$I = -\int (1 - u^2) u^2 \, du$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

Trigonometric Integrals $\int \sin^m x \cos^n x \, dx$

Review: $\boldsymbol{sin^2 x + \cos^2 x = 1}$

Case 2.
n is odd

Save one "cosine" factor to construct "d(sin x)".
Then use $\cos^2 x = 1 - \text{sinx}$ to trans all "cos x" to "sin x".

→ Power functions always have corresponding antiderivatives.

$$\int \sin^2 x \cos^3 x \, dx \quad = \int \sin^2 x \cos^2 x \cos x \, dx$$

$$u = \sin x$$
$$du = \cos x \, dx$$

$$I = \int u^2 (1 - u^2) \, du$$

$$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$$

Trigonometric Integrals $\int \sin^m x \cos^n x \, dx$

Review: $\boldsymbol{\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x}$

Case 3.
m & n are even

Use $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$ to get the lower power then use the techs of case 1,2.

$$\int sin^4 x \, dx$$

$$\int sin^2 x \, cos^2 x \, dx$$

Practice. 2~4

$\int \sin^3 nx\, dx$

$\int \sin^2 nx\, dx$

$\int \cos^3 x \sqrt{\sin x}\, dx$

Trigonometric Integrals $\int \boldsymbol{tan^m x\, sec^n x\, dx}$

Review: $\boldsymbol{tan^2 x = sec^2 x - 1}$

Case 1.
m is odd

Save one "tan" factor to construct "$d(sec\, x) = tan\, x\, sec\, x\, dx$".
Then use $\boldsymbol{tan^2 x = sec^2 x - 1}$ to trans all "tan x" to "sec x".

→ Power functions always have corresponding antiderivatives.

$$\int \tan^3 x \sec^2 x\, dx \quad = \int \boxed{\tan^2 x}\, sec\, x\, \boxed{\sec x \tan x\, dx}$$

$$= \int (\sec^2 x - 1)\, sec\, x\, d(\sec x)$$

$u = \sec x$

$du = \sec x \tan x\, dx$

$I = \int (u^2 - 1) u\, du$

$$= \frac{1}{4}\sec^4 x - \frac{1}{2} sec^2\, x + C$$

Trigonometric Integrals $\int \boldsymbol{tan^m x\, sec^n x\, dx}$

Review: $\boldsymbol{tan^2 x = sec^2 x - 1}$

Case 1.
m is odd

$$\int \tan^3 2x \sec^2 2x \, dx$$

Trigonometric Integrals $\int \boldsymbol{tan^m\, x\, sec^n\, x\, dx}$

Review: $\boldsymbol{tan^2 x = sec^2 x - 1}$

Case 1.
m is odd

$$\int \tan^5 2x \sec^2 2x \, dx$$

Trigonometric Integrals $\int \boldsymbol{tan^m\, x\, sec^n\, x\, dx}$

Review: $\boldsymbol{tan^2 x = sec^2 x - 1}$ $\boldsymbol{d(tanx) = sec^2 x\, dx}$

Case 2.
m is even
$n = 0$

$$\int \tan^2 x\, dx$$

$$= \int \sec^2 x - 1\, dx$$

$$= \int \sec^2 x\, dx - x$$

$$u = tanx$$
$$du = \sec^2 x\, dx$$

$$I = \int 1\, du - x$$

$$= u + C - x$$

$$= tanx - x + C$$

Trigonometric Integrals $\int \boldsymbol{tan^m\, x\, sec^n\, x\, dx}$

Review: $\boldsymbol{tan^2 x = sec^2 x - 1}$ $\boldsymbol{d(tan x) = sec^2 x\, dx}$

Case 2.
m is even
$n = 0$

$$\int \tan^4 x\, dx \quad = \int (\sec^2 x - 1)^2\, dx$$

$$\int \tan^6 x\, dx \quad = \int (\sec^2 x - 1)^3\, dx$$

Trigonometric Integrals $\int \boldsymbol{tan^m x\, sec^n x\, dx}$

Review: $\boldsymbol{tan^2 x = sec^2 x - 1}$ $\quad$ $\boldsymbol{d(tanx) = sec^2 x\, dx}$

Case 3.
n is even

Save one "$\sec^2 x$" factor to construct "$d(tan\, x) = \sec^2 x\, dx$".
Then use $\boldsymbol{sec^2 x = tan^2 x + 1}$ to trans all "sec x" to "tan x".

$$\int \tan^2 x \sec^4 x\, dx = \int \tan^2 x \sec^2 x \sec^2 x\, dx$$

$$= \int \tan^2 x\, (\tan^2 x + 1)\, d(\tan x)$$

$$= \int \tan^4 x + \tan^2 x\, d(\tan x)$$

$$= \frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C$$

Practice. 1&5

$\int \tan^2 x \sec^2 x \, dx$

$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^4 x \, dx$

Trigon Substitution

Questions with Square Root

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x$$

How to solve questions below?

$$\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x$$

$$\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$$

$$\int \sqrt{1-x^2}\,\mathrm{d}x$$

… …

Let the square root disappear !

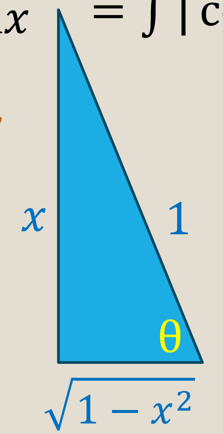
Review.

$$\boldsymbol{sin^2 x + cos^2 x = 1 \quad \rightarrow \quad sin^2 x = 1 - cos^2 x \qquad cos^2 x = 1 - sin^2 x}$$

$$\boldsymbol{sec^2 x = tan^2 x + 1 \qquad tan^2 x = sec^2 x - 1}$$

Trigon Substitution

$\int \sqrt{1-x^2}\,\mathrm{d}x \qquad = \int \sqrt{1-\sin^2\theta}\,\mathrm{d}x \quad = \int |\cos\theta|\,\mathrm{d}x$



Find the relation between $d\theta$ and dx

$\theta = \sin^{-1} x$

$\sin\theta = x$

$\sin 2\theta = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$

$\frac{d(\sin\theta)}{dx} = 1 \qquad \frac{d(\sin\theta)}{d\theta} = \cos\theta$

$\rightarrow d(\sin\theta) = dx = \cos\theta\ d\theta$

$$\mathrm{I} = \int \sqrt{1-\sin^2\theta}\ \cos\theta\,\mathrm{d}\theta = \int \cos^2\theta\,\mathrm{d}\theta = \int \frac{1+\cos 2\theta}{2}\mathrm{d}\theta \ = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

$$= \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} + C$$

Review.

$\boldsymbol{\sin^2 x + \cos^2 x = 1} \quad \rightarrow \quad \boldsymbol{\sin^2 x = 1 - \cos^2 x} \qquad \boldsymbol{\cos^2 x = 1 - \sin^2 x}$

$\boldsymbol{\tan^2 x + 1 = \sec^2 x} \qquad \boldsymbol{\sec^2 x - 1 = \tan^2 x}$

Trigon Substitution

$$\int \sqrt{9 - x^2}\, \mathrm{d}x$$

$$\boldsymbol{sin^2 x + cos^2 x = 1 \quad \rightarrow \quad sin^2 x = 1 - cos^2 x \qquad cos^2 x = 1 - sin^2 x}$$

$$\boldsymbol{sec^2 x = tan^2 x + 1 \qquad tan^2 x = sec^2 x - 1}$$

Trigon Substitution

$$\int \sqrt{9+x^2}\,\mathrm{d}x = \int \sqrt{9(1+\frac{x^2}{9})}\,\mathrm{d}x = 3\int \sqrt{(1+\left(\frac{x}{3}\right)^2)}\,\mathrm{d}x$$

$$\tan u = \frac{x}{3}$$

$$\frac{d(\tan u)}{dx} = \frac{1}{3} \qquad \frac{d(\tan u)}{du} = \sec^2 u$$

$$dx = 3d(\tan u) = 3\sec^2 u\, du$$

In the exam, it is generally a definite integral. According to the upper and lower limits of the integral, you will know the sign of the trig function and you will not need to take the absolute sign, and you even do not have to reduce u to x.

$$\mathrm{I} = 3\int \sqrt{(1+\tan^2 u)}\ 3\sec^2 \mathrm{u}\,\mathrm{du} \ = 9\int |\sec u|\sec^2 u\,\mathrm{du}$$

$$\boldsymbol{sin^2 x + cos^2 x = 1} \quad \rightarrow \quad \boldsymbol{sin^2 x = 1 - cos^2 x} \qquad \boldsymbol{cos^2 x = 1 - sin^2 x}$$

$$\boxed{\boldsymbol{sec^2 x = tan^2 x + 1}} \qquad \boldsymbol{tan^2 x = sec^2 x - 1}$$

Trigon Substitution

$$\int_0^3 \frac{1}{\sqrt{9+x^2}}\,dx = \int_0^3 \frac{1}{3\sqrt{1+\left(\frac{x}{3}\right)^2}}\,dx$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2 u}}\, 3\sec^2 u\, du$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} 3\, sec\, u\, du \;= [\ln|\sec \mathrm{u} + \tan \mathrm{u}|]_0^{\frac{\pi}{4}}$$

$$= \ln(\sqrt{2}+1)$$

$$\tan u = \frac{x}{3}$$

$$\frac{d(\tan u)}{dx} = \frac{1}{3} \qquad \frac{d(\tan u)}{du} = \sec^2 u$$

$$dx = 3d(\tan u) = 3\sec^2 u\, du$$

$$x = 0 \;\rightarrow\; \tan u = 0 \;\rightarrow\; u = 0$$

$$x = 3 \;\rightarrow\; \tan u = 1 \;\rightarrow\; u = \frac{\pi}{4}$$

Trigon Substitution

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{1}{3}\frac{1}{(\frac{x}{3})^2 9\sqrt{(\frac{x}{3})^2-1}}\,dx \quad = \int \frac{1}{27}\frac{1}{\sec^2 u\ \sqrt{\sec^2 u - 1}}\,3\tan u \sec u\,du$$

$$= \int \frac{1}{9}\frac{1}{\sec u\,|\tan\ u\,|}\tan u\,du$$

$$= \pm\frac{1}{9}\int \cos u\,du$$

$$= \pm\frac{1}{9}\sin u + C$$

$$\sec u = \frac{x}{3}$$

$$\frac{d(\sec u)}{dx} = \frac{1}{3} \qquad \frac{d(\sec u)}{du} = \tan u \sec u$$

$$dx = 3d(sec\ u) = 3\tan u \sec u\,du$$

$$\boldsymbol{sin^2 x + cos^2 x = 1 \quad \rightarrow \quad sin^2 x = 1 - cos^2 x \qquad cos^2 x = 1 - sin^2 x}$$

$$\boldsymbol{\sec^2 x = tan^2 x + 1} \qquad \boxed{\boldsymbol{tan^2 x = sec^2 x - 1}}$$

Linear Partial Fractions

$$\frac{2x+1}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$= \frac{A(x+2)^2}{(x+1)(x+2)^2} + \frac{B(x+1)(x+2)}{(x+1)(x+2)^2} + \frac{C(x+1)}{(x+1)(x+2)^2}$$

$$= \frac{A(x^2+4x+4) + B(x^2+3x+2) + C(x+1)}{(x+1)(x+2)^2}$$

$$= \frac{(A+B)x^2 + (4A+3B+C)x + (4A+2B+C)}{(x+1)(x+2)^2}$$

$A + B = 0$ $\quad$ $4A + 3B + \mathrm{C} = 2$ $\quad$ $4A + 2B + C = 1$

$A = -1$ $\quad$ $B = 1$ $\quad$ $C = 3 \rightarrow \frac{-1}{x+1} + \frac{1}{x+2} + \frac{3}{(x+2)^2}$

Linear Partial Fractions

$$\frac{2x+1}{(x+1)(x+2)^2} = \frac{-1}{x+1} + \frac{1}{x+2} + \frac{3}{(x+2)^2}$$

$$\int \frac{2x+1}{(x+1)(x+2)^2}\,dx = \int \frac{-1}{x+1} + \frac{1}{x+2} + \frac{3}{(x+2)^2}\,dx$$

$$= \int \frac{-1}{x+1}\,dx + \int \frac{1}{x+2}\,dx + \int \frac{3}{(x+2)^2}\,dx$$

$$= \int \frac{-1}{x+1}\,d(x+1) + \int \frac{1}{x+2}\,d(x+2) + \int \frac{3}{(x+2)^2}\,d(x+2)$$

$$= -\ln|x+1| + \ln|x+2| - 3(x+2)^{-1} + C$$

$$= \ln\left|\frac{x+2}{x+1}\right| - \frac{3}{x+2} + C$$

Linear Partial Fractions

$$\int \frac{x^3}{x^2-1}\, dx$$

$$\int \frac{5x+1}{x^2+x-2}\, dx$$

$$\int \frac{x+10}{(x-4)(x+3)}\, dx$$

Integration by Parts

$$\int x\,dx = ?$$ $$\frac{1}{2}x^2 + C$$

$$\int e^x\,dx = ?$$ $$e^x + C$$

$$\int \underline{xe^x}\,dx = ?$$ $$\left(\frac{1}{2}x^2 + C_1\right)\cdot\left(e^x + C_2\right)\ ??$$

Differentiate it !!!

$$\underline{x^2e^x + \frac{1}{2}x^2e^x + \cdots}$$

Integration by Parts

Review： Product Rule

If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Let us take integral on both sides !!!

$$\int d[f(x)g(x)] = \int [f(x)g'(x) + g(x)f'(x)]dx$$

Integration by Parts

$$\boxed{\int \; d[f(x)g(x)]} = \int [\, f(x)g'(x) + g(x)f'(x) \,]\ \mathrm{dx}$$

??????

Integration by Parts

$$\boxed{\int d[f(x)g(x)]} = \boxed{\int [\, f(x)g'(x) + g(x)f'(x)\,]\,\mathrm{dx}}$$

$$\boxed{\int f(x)g'(x)\,\mathrm{dx} + \int g(x)f'(x)\,\mathrm{dx}}$$

f(x)g(x)

$$\mathrm{f}(x)g(x) = \int f(x)g'(x)\,\mathrm{dx} + \boxed{\int g(x)f'(x)\,\mathrm{dx}}$$

$$\int f(x)g'(x)\,\mathrm{dx} = \mathrm{f}(x)g(x) - \int g(x)f'(x)\,\mathrm{dx}$$

Integration by Parts

$$\int \, d[f(x)g(x)] = \int [\, f(x)g'(x) + g(x)f'(x) \,]\, \mathrm{dx}$$

f(x)g(x)

$$\int f(x)g'(x)\, \mathrm{dx} + \int g(x)f'(x)\, \mathrm{dx}$$

The **formula for integration by parts**.

$$\int f(x)g'(x)\, \mathrm{dx} = \mathrm{f}(x)g(x) - \int g(x)f'(x)\, \mathrm{dx}$$

Integration by Parts

$$\int f(x)g'(x)\,\mathrm{dx} = \mathrm{f}(x)g(x) - \int g(x)f'(x)\,\mathrm{dx}$$

u = f(x) and v = g(x)

du = f′(x)dx   dv = g′(x)dx

$$\int \boldsymbol{u\,dv} = \boldsymbol{uv} - \int \boldsymbol{v\,du}$$

Integration by Parts

$$\int x\, e^x\, dx$$

$$\int \boldsymbol{u\, dv} = \boldsymbol{uv} - \int \boldsymbol{v\, du}$$

$u = x \qquad dv = e^x\, dx$

$du = 1\, dx \qquad v = e^x$

$$\int x\, e^x\, dx = uv - \int v du$$

$$= x\, e^x - \int e^x\, dx$$

$$= x\, e^x - e^x + C$$

How to choose u and v ?

- $\frac{du}{dx}$ is "better"
- The integral of v is possible, i.e.

  $\int v\, du$ is possible

Practice

$$\int x^2 \sin 2x \ \ dx$$

$\mathbf{u = x^2}$ $\quad\quad$ $\mathrm{dv} = \sin(2\mathrm{x})\, dx$

$\mathrm{du} = 2\mathrm{x}\,\mathrm{dx}$ $\quad\quad$ $\mathrm{v} = \int \sin(2\mathrm{x})\,\mathrm{dx} = -\dfrac{1}{2}\cos(2x)$

$$\int x^2 \sin(2x)\,\mathrm{d}x = -\frac{1}{2}\cos(2x) \cdot \boldsymbol{x^2} - \int \left(-\frac{1}{2}\cos(2x)\right) \cdot 2x\,\mathrm{dx}$$

$$= -\tfrac{1}{2}\cos(2x) \cdot x^2 + \textstyle\int cos(2x) \cdot x\,\mathrm{d}x$$

Practice

$$\int x^2 e^{ax}\, dx$$

Practice

Calculate $\int x^3 lnx\, dx$ using integration by parts.

Answer: $\mathrm{u} = \ln \mathrm{x}$ $\quad$ $\mathrm{v} = \int \mathrm{x}^3 \mathrm{dx}$

$\mathrm{du} = \frac{1}{\mathrm{x}}\,\mathrm{dx}$ $\quad$ $\mathrm{v} = \frac{1}{4}\mathrm{x}^4$

$$\int \mathrm{x}^3 \ln \mathrm{x}\, \mathrm{dx} = \frac{1}{4}\mathrm{x}^4 \ln \mathrm{x} - \int \frac{1}{4}\mathrm{x}^3\, \mathrm{dx}$$

$= \frac{1}{4}\mathrm{x}^4 \ln \mathrm{x} - \frac{1}{16}\mathrm{x}^4 + \mathrm{C}$, where C is a constant.

Practice

$$\int x \sin^{-1} x \; dx$$

Practice

$$\int \tan^{-1} x \;\; dx$$

Practice

$$\int e^x \sin x \, dx$$

Practice

$$\int x \sin x \, dx$$

Practice

$$\int x \tan^{-1} x \, dx$$

Practice

$$\int e^x \cos x \, dx$$

Practice

Calculate $\int \mathrm{x}^2 \sin(2\mathrm{x}^3)\,\mathrm{dx}$.

Answer: $\int x^2 \sin(2x^3)\,dx = \int \frac{1}{6} \sin(2x^3)\,d(2x^3) = -\frac{1}{6} \cos(2x^3) + C$

Practice

Calculate $\int x^2 \sin(2x)\,dx$.

Answer: $$\int \boldsymbol{x^2}\,\mathbf{sin(2}\boldsymbol{x}\mathbf{)\,d}\boldsymbol{x} = -\frac{1}{2}\cos(2x)\cdot x^2 + \int \cos(2x)\cdot x\,dx$$

Calculate $\int \cos(2x)\cdot x\,dx$**:** $u = x$ $\quad v = \int \cos(2x)\,dx$

$du = 1\,dx \quad v = \frac{1}{2}\sin(2x)$

$$\int \cos(2x)\cdot x\,dx = \frac{1}{2}x\sin(2x) - \int \frac{1}{2}\sin(2x)\,dx = \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$$

$$\int x^2\sin(2x)\,dx = -\frac{1}{2}\cos(2x)\cdot x^2 + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$$

Improper Integrals -- with Infinite Integration Limits

•If $f(x)$ is continuous on $[a, \infty)$, then $\int_a^\infty f(x)\,dx = \displaystyle\lim_{t\to\infty}\int_a^t \mathbf{f(x)\,dx}$

•If $f(x)$ is continuous on $(-\infty, b]$, then $\int_{-\infty}^b f(x)\,dx = \displaystyle\lim_{t\to-\infty}\int_t^b \mathbf{f(x)\,dx}$

•If $f(x)$ is continuous on $\mathbb{R}$, then $\int_{-\infty}^\infty f(x)\,dx = \displaystyle\int_{-\infty}^a \mathbf{f(x)\,dx} + \int_a^\infty \mathbf{f(x)\,dx}$

$$= \lim_{t\to-\infty}\int_t^a \mathbf{f(x)\,dx} + \lim_{t\to\infty}\int_a^t \mathbf{f(x)\,dx}$$

Improper Integrals -- with Infinite Integration Limits

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx$$

$$\lim_{t\to\infty} -\frac{1}{2}\int_0^t e^{-x^2}\,d(-x^2)$$

$$\lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^t$$

$$\lim_{t\to\infty}\left(\frac{1}{2}-\frac{1}{2}e^{-t^2}\right)$$

$$\frac{1}{2}$$

Improper Integrals -- with Infinite Integration Limits

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} =$$

Improper Integrals -- with Infinite Discontinuities

Vertical asymptote at $x = b$

If $f(x)$ is continuous on $[a, b)$ and has an infinite discontinuity at b,

then $\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t \mathrm{f}(x)\,\mathrm{d}x$

If $f(x)$ is continuous on $[a, b)$ and has an infinite discontinuity at a,

then $\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b \mathrm{f}(x)\,\mathrm{d}x$

If $f(x)$ is continuous on $[a, b]$ except some number c in (a,b) at which f has an infinite discontinuity, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$\lim_{t \to c^-} \int_a^t \mathrm{f}(x)\,\mathrm{d}x + \lim_{t \to c^+} \int_t^b \mathrm{f}(x)\,\mathrm{d}x$$

Improper Integrals -- with Infinite Discontinuities

$$\int_0^3 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^3 \frac{1}{x}\,dx \quad = [\ln|x|]_t^3 \; = \ln 3 - \ln|t|$$

$$\lim_{t\to 0^+} (\ln 3 - \ln|t|)$$

$$\infty$$

Improper Integrals -- with Infinite Discontinuities

$$\int_1^5 \frac{dx}{\sqrt{x-1}} =$$

Improper Integrals -- with Infinite Discontinuities

$$\int_0^1 \frac{dx}{1-x} =$$

Improper Integrals -- with Infinite Discontinuities

If $\int_0^1 \frac{ke^{-\sqrt{x}}}{\sqrt{x}}\,dx = 1$, what is the value of k?