- **Rolle's Theorem, Mean Value Theorem**

1. Let f be the function given by $f(x) = \sin(\pi x)$. What are the values of c that satisfy Rolle's Theorem on the closed interval $[0,2]$?

$f'(x) = \cos(\pi x) \cdot \pi$

Let $f'(c) = 0$ then $\cos(\pi c) = 0$, $\pi c = \frac{\pi}{2}, \frac{3}{2}\pi \Rightarrow c = \frac{1}{2}, \frac{3}{2}$

$\therefore c = \frac{1}{2}$ or $\frac{3}{2}$

2. Let f be the function given by $f(x) = -x^3 + 3x + 2$. What are the values of c that satisfy the Mean Value Theorem on the closed interval $[0,3]$?

$AROC = \frac{f(3) - f(0)}{3-0} = -\frac{16+2}{3} = -6$

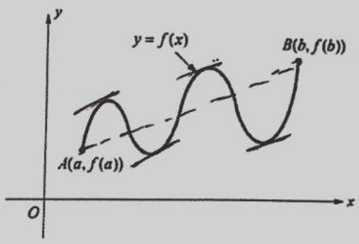
$f'(x) = -3x^2 + 3 = -3(x^2 - 1)$

let $f'(c) = -6$ then $= \pm\sqrt{3}$

$\because c \in (0,3)$

$\therefore c = \sqrt{3}$

3. The figure shows the graph of f. On the closed interval $[a, b]$, how many values of c satisfy the conclusion of the Mean Value Theorem



Four values of c.

B 4. Let f be the function given by $f(x) = \frac{x}{x+2}$. What are the values of c that satisfy the Mean Value Theorem on the closed interval $[-1,2]$?

(A) -4 only (B) 0 only (C) 0 and $\frac{3}{2}$ (D) -4 and 0

$f(x) = \frac{x+2-2}{x+2} = 1 - \frac{2}{x+2}$

$\frac{f(2) - f(-1)}{3} = \frac{1}{2}$

$f'(x) = (-2) \cdot (-1)(x+2)^{-2}$

$= 2(x+2)^{-2}$

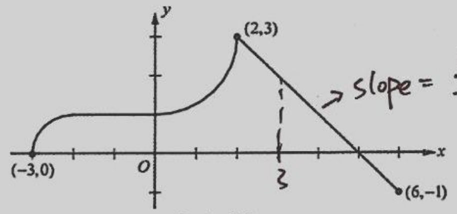
$f'(c) = \frac{1}{2}$.

$\frac{1}{(c+2)^2} = \frac{1}{4}$

$\therefore c+2 = \pm 2$

$c = 0$ or -4

$\notin (-1,2)$



slope $= \frac{4}{-4} = -1$

Graph of f

B 5. The continuous function f is defined on the interval $-3 \le x \le 6$. The graph of f consists of two quarter circles and two line segments, as shown in the figure above. Which of the following statements must be true?

I. The average rate of change of f on the interval $-3 \le x \le 6$ is $-\frac{1}{9}$. ✓

$\frac{f(6) - f(-3)}{9} = \frac{-1-0}{9} = -\frac{1}{9}$

II. There is a point c on the interval $-3 < x < 6$, for which $f'(c)$ is equal to the average rate of change of f on the interval $-3 \le x \le 6$. ×

III. If h is the function given by $h(x) = f(\frac{1}{2}x)$, then $h'(6) = -\frac{1}{2}$. ✓

$h'(x) = f'(\frac{1}{2}x) \cdot \frac{1}{2}$

$h'(6) = f'(3) \cdot \frac{1}{2}$

$= -1 \cdot \frac{1}{2}$

(A) I and II only

(B) I and III only

(C) II and III only

(D) I, II, and III

HW -- Analytical Application of Differentiation

- **Extreme Values, Critical Points, Increasing/Decreasing Intervals, First Derivative Test**

D 1. At what values of x does $f(x)=(x-1)^3(3-x)$ have the absolute maximum?

$f'(x)=3(x-1)^2(3-x)+(x-1)^3\cdot(-1)=(x-1)^2\cdot(9-3x-x+1)$
$=(x-1)^2\cdot(-4x+10)$

(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) $\frac{5}{2}$

A 2. At what values of x does $f(x)=x-2x^{2/3}$ have a relative minimum?

$f'(x)=1-2\cdot\frac{2}{3}x^{-\frac{1}{3}}=0$ undefined $x=0$

$\frac{4}{3}\frac{1}{\sqrt[3]{x}}=1$

$\sqrt[3]{x}=\frac{4}{3}$

$x=\frac{64}{27}$

(A) $\frac{64}{27}$ (B) $\frac{16}{9}$ (C) $\frac{4}{3}$ (D) 2

B 3. What is the minimum value of $f(x)=x^2\ln x$? Domain $=\{x|x>0\}$

$f'(x)=2x\ln x+x^2\cdot\frac{1}{x}$
$=2x\ln x+x$
$=x(2\ln x+1)$

$(x=0)$ or $\ln x=-\frac{1}{2}$
Omit
$x=e^{-\frac{1}{2}}$

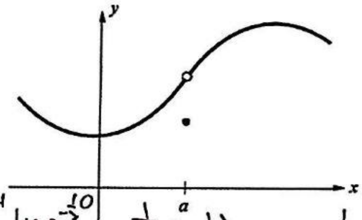
$f(e^{-\frac{1}{2}})=e^{-1}\ln e^{-\frac{1}{2}}=\frac{1}{e}\cdot(-\frac{1}{2})=-\frac{1}{2e}$

(A) $-e$

(B) $-\frac{1}{2e}$

(C) $-\frac{1}{e}$

(D) $-\frac{1}{\sqrt{e}}$



D 4. The graph of a function f is shown above. Which of the following statements about f are true?

I. $\lim_{x\to a} f(x)$ exists. ✓

II. $x=a$ is in the domain of f. ✓

III. f has a relative minimum at $x=a$. ✓

(A) I only

(B) I and II only

(C) I and III only

(D) I, II, and III

B 5. A polynomial $f(x)$ has a relative minimum at $(-4,2)$, a relative maximum at $(-1,5)$, a relative minimum at $(3,-3)$ and no other critical points. How many zeros does $f(x)$ have?

(A) one (B) two (C) three (D) four

A 6. At $x=2$, which of the following is true of the function f defined by $f(x)=x^2e^{-x}$?

$f'(x)=2xe^{-x}+x^2e^{-x}(-1)$
$=xe^{-x}(2-x)$

$x=0, 2$

(A) f has a relative maximum.

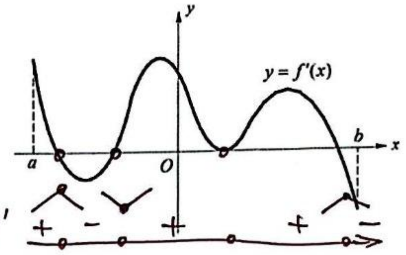
(B) f has a relative minimum.

(C) f is increasing.

(D) f is decreasing.

7.

B



The graph of f', the derivative of f, is shown in the figure above. Which of the following describes all relative extrema of f on the open interval (a,b)?

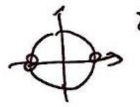
(A) One relative maximum and two relative minima
(B) Two relative maxima and one relative minimum
(C) Two relative maxima and two relative minima
(D) Three relative maxima and two relative minim

~~D~~ C 8. The first derivative of a function f is given by $f'(x) = \dfrac{3\sin(2x)}{x^2}$. How many critical values does f have on the open interval $(0,10)$? $2x \in (0, 20)$

undefined: $x=0$

$f'(x)=0$: $\sin 2x = 0$

$2x = \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$

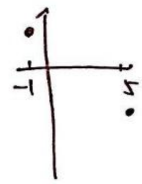


(A) four (B) five (C) six (D) seven

C 9. The function f is continuous on the closed interval $[-1,5]$ and differentiable on the open interval $(-1,5)$. If $f(-1)=4$ and $f(5)=-2$, which of the following statements could be false?

← global min

(A) There exist c, on $[-1,5]$, such that $f(c) \le f(x)$ for all x on the closed interval $[-1,5]$.
(B) There exist c, on $(-1,5)$, such that $f(c)=0$.
(C) There exist c, on $(-1,5)$, such that $f'(c)=0$.
(D) There exist c, on $(-1,5)$, such that $f(c)=2$.



x	−4	−3	−2	−1	0	1	2	3	4	5
$f'(x)$	−1	−2	0	1	2	1	0	−2	−3	−1

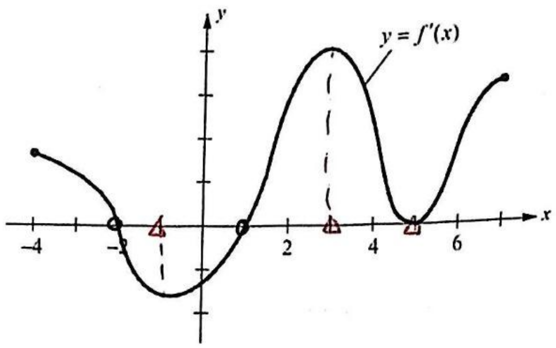
10.

D

The derivative, f', of a function f is continuous and has exactly two zeros on $[-4,5]$. Selected values of $f'(x)$ are given in the table above. On which of the following intervals is f increasing?

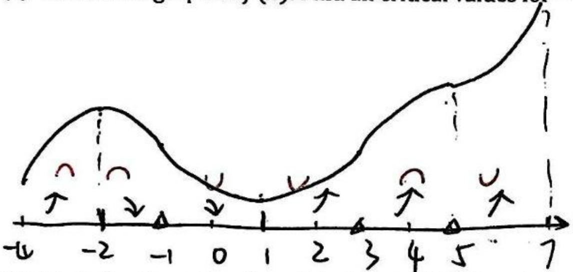
(A) $-3 \le x \le 0$ or $4 \le x \le 5$
(B) $-2 \le x \le 0$ or $4 \le x \le 5$
(C) $-3 \le x \le 2$ only
(D) $-2 \le x \le 2$ only

11.



The figure above shows the graph of f', the derivative of the function f, for $-4 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -1$, $x = 3$, and $x = 5$.

(a) Sketch the graph of $f(x)$. Find all critical values for $-4<x<7$.



critical values:

$x = -2, 1, 5$

(b) Find all values of x, for $-4 \le x \le 7$, at which f attains a relative minimum.

$x = 1$

$x = -4$

(c) Find all values of x, for $-4 \le x \le 7$, at which f attains a relative maximum.

$x = -2$

$x = 7$

(d) For $-4 \le x \le 7$, what is the absolute maximum value of f(x).

$\max\{f(-2), f(7)\}$

After learning the Integral, the answer will be more accurate: $f(7)$

➤ Second Derivative Test, Concavity, P.O.I

1. The graph of $y = x^4 - 2x^3$ has a point of inflection at

B

$y' = 4x^3 - 6x^2$

$y'' = 12x^2 - 12x = 12x(x-1) = 0 \Rightarrow x = 0, 1$

(A) $(0,0)$ only

(B) $(0,0)$ and $(1,-1)$

(C) $(1,-1)$ only

(D) $(0,0)$ and $(\frac{3}{2}, -\frac{27}{16})$

0 1

f'' + − +

f ∪ ∩ ∪

D 2. If the graph of $y = ax^3 - 6x^2 + bx - 4$ has a point of inflection at $(2,-2)$, what is the value of $a+b$?

$y' = 3ax^2 - 12x + b$

$y'' = 6ax - 12$

$y''(2) = 0 \quad \therefore a = 1$

$y(2) = -2 \Rightarrow b = 9$

(A) -2 (B) 3 (C) 6 (D) 10

C 3. At what value of x does the graph of $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ have a point of inflection? Domain: $\{x \mid x > 0\}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} + (-\frac{1}{2})x^{-\frac{3}{2}}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} + \frac{1}{2}\cdot\frac{3}{2}\cdot x^{-\frac{5}{2}}$

(A) $\frac{1}{2}$ (B) 1 (C) 3 (D) $\frac{7}{2}$

$= -\frac{1}{4}x^{-\frac{5}{2}}\cdot x + \frac{3}{4}x^{-\frac{5}{2}} = \frac{1}{4}x^{-\frac{5}{2}}(3-x) = 0 \Rightarrow x = 3$

not defined: $x = 0$

Domain

0 3

f'' + −

D 4. The graph of $y = 3x^5 - 40x^3 - 21x$ is concave up for

$y'' = 60x^3 - 240x$

$= 60x(x^2-4)$

(A) $x < 0$

(B) $x > 2$

(C) $x < 0$ or $0 < x < 2$

(D) $-2 < x < 0$ or $x > 2$

∪ ∪

−2 0 2

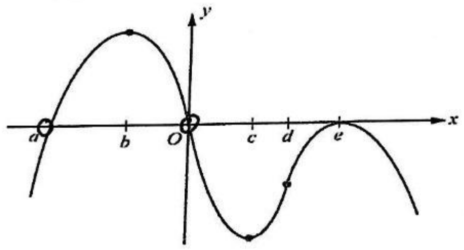
− + − + f''

A 5. Let f be a twice differentiable function such that $f(1) = 7$ and $f(3) = 12$. If $f'(x) > 0$ and $f''(x) < 0$ for all real numbers x, which of the following is a possible value for $f(5)$?

increasing ∩

increasing with a decreasing rate

(A) 16 (B) 17 (C) 18 (D) 19



x	1	3	5
f(x)	7	12	<17

5 <5

D 6. The second derivative of the function f is given by $f''(x) = x(x+a)(x-e)^2$ and the graph of f'' is shown above. For what values of x does the graph of f have a point of inflection?

(A) b and c (B) b, c and e (C) b, c and d (D) a and 0

A 7. The first derivative of the function f is given by $f'(x) = (x^3 + 2)e^x$. What is the x-coordinate of the inflection point of the graph of f?

$f''(x) = \underbrace{e^x}_{>0}(x^3 + 3x^2 + 2)$

(A) -3.196 (B) -1.260 (C) -1 (D) 0

$f''(0) > 0$

$f''(-1) > 0$

$f''(-2) > 0$

$f''(-4) < 0$

C

8. Let f be a twice differentiable function with $f'(x)>0$ and $f''(x)>0$ for all x, in the closed interval $[2,8]$. Which of the following could be a table of values for f?

(A)

x	$f(x)$
2	−1
4	3
6	6
8	8

4 3

(B)

x	$f(x)$
2	−1
4	2
6	5
8	8

3 3

(C)

x	$f(x)$
2	−1
4	1
6	4
8	8

2 3 4

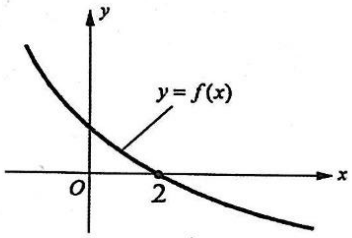
~~(D)~~

x	$f(x)$
2	8
4	4
6	1
8	−1

~~9~~. **(Calculator)** Let f be the function given by $f(x)=3\sin(\frac{2x}{3})-4\cos(\frac{3x}{4})$. For $0\le x\le 7$, f is increasing most rapidly when $x=$

$f'(x)=3\cos(\frac{2}{3}x)\cdot\frac{2}{3}+4\sin(\frac{3}{4}x)\cdot\frac{3}{4}=2\cos(\frac{2}{3}x)+3\sin(\frac{3}{4}x)$
$=0$

(A) 0.823 (B) 1.424 (C) 1.571 (D) 3.206

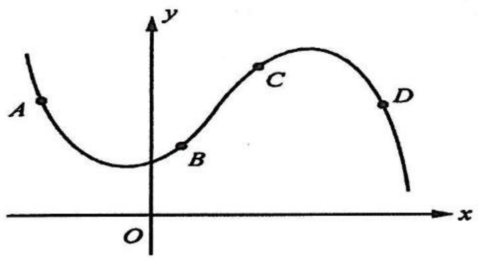


$f(2)=0$. $f'(2)<0$
$f''(2)>0$ (concave up)

C

10. The graph of a twice differentiable function f is shown in the figure above. Which of the following is true?

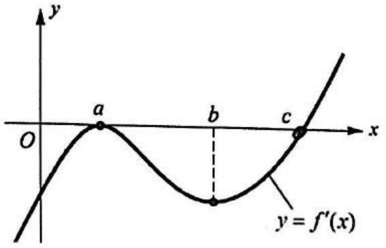
(A) $f''(2)<f(2)<f'(2)$
(B) $f'(2)<f''(2)<f(2)$
(C) $f'(2)<f(2)<f''(2)$
(D) $f(2)<f'(2)<f''(2)$



B

11. At which of the five points on the graph in the figure above is $\frac{dy}{dx}>0$ and $\frac{d^2y}{dx^2}>0$?

(A) A (B) B (C) C (D) D



D **12.** The graph of f', the derivative of function f, is shown above. If f is a twice differentiable function, which of the following statements must be true?

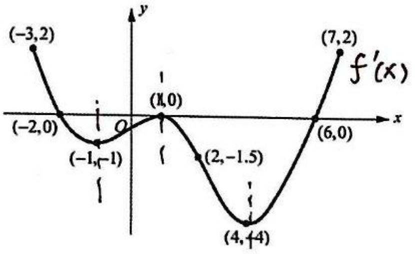
I. $f(c) \overset{<}{>} f(a)$ ✗

II. The graph of f is concave up on the interval $b < x < c$. ✓ $f'' > 0$

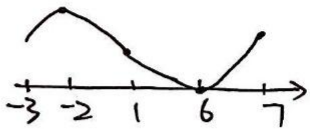
III. f has a relative minimum at $x = c$. ✓ ↘ ↗

(A) I only (B) II only (C) III only (D) II and III only

13.



The figure above shows the graph of f', the derivative of the function f, on the closed interval $[-3,7]$. The graph of f' has horizontal tangent lines at $x=-1$, $x=1$, and $x=4$. The function f is twice differentiable and $f(-2)=\frac{1}{2}$.



(a) Find the x-coordinates of each of the points of inflection of the graph of f. Justify your answer.

(a) $x = -1, 1, 4$

(b) At what value of x does f attain its absolute minimum value on the closed interval $[-3,7]$.

(b) critical: $x = -2, 1, 6$

endpoints: $x = -3, 7$

$\cancel{f(-2)=\frac{1}{2}}$ $\min\{f(-3), f(6)\}$

$\min = f(6)$

(c) Let h be the function defined by $h(x) = x^2 f(x)$. Find an equation for the line tangent to the graph of h at $x = -2$.

$h'(x) = 2x\,f(x) + x^2 f'(x)$

$h'(-2) = -4f(-2) + 4f'(-2)$

$= -4 \cdot \frac{1}{2} = -2$

$h(-2) = 4f(-2) = 2$

passes $(-2,2)$ slope $= -2$

$y - 2 = -2(x+2)$

14. Let f be a twice differentiable function with $f(1) = -1$, $f'(1) = 2$, and $f''(1) = 0$. Let g be a function whose derivative is given by $g'(x) = x^2\left[2f(x) + f'(x)\right]$ for all x.

(a) Write an equation for the line tangent to the graph of f at $x = 1$.

(a) $f'(1) = 2 \quad (1, f(1)) = (1, -1)$

$\therefore y+1 = 2(x-1)$

(b) Does the graph of f have a point of inflection when $x = 1$? Explain.

(b) We donot have enough information to draw the conclusion.

(c) Given that $g(1) = 3$, write an equation for the line tangent to the graph of g at $x = 1$.

$(1,3) \quad g'(1) = (2f(1)+f'(1)) = -2+2 = 0 \Rightarrow y = 3$

(d) Show that $g''(x) = 4xf(x) + 2x(x+1)f'(x) + x^2 f''(x)$. Does g have a local maximum or minimum at $x = 1$? Explain your reasoning.

$$g''(x) = 2x[2f+f'] + x^2[2f'+f'']$$
$$= 4xf + (2x+2x^2)f' + x^2 f''$$
$$= 4xf + 2x(x+1)f' + x^2 f''$$

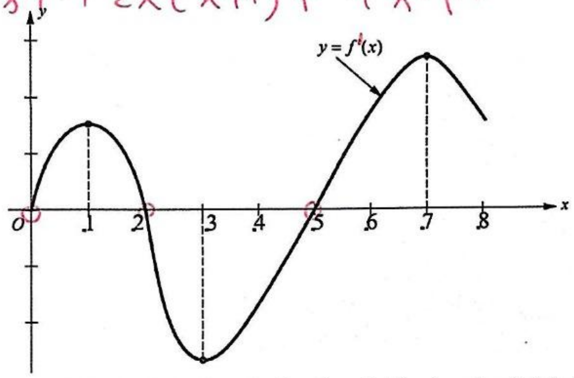
$g'(1) = 0$

$g''(1) = 4f(1) + 4f'(1) + f''(1)$

$= 4(-1) + 8 + 0$

$= 4 > 0$

$\therefore g(1)$ is a local min.



15. The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $0 \le x \le 8$.

(a) For what values of x does the graph of f have a horizontal tangent? (a) $f'(x) = 0 \ : \ x = 0, 2, 5$

(b) On what intervals is f increasing? (b) $f'(x) > 0 \Rightarrow (0,2), (5,8]$

(c) On what intervals is f concave upward? (c) $f'(x)\uparrow \Rightarrow (0,1), (3,7)$

(d) For what values of x does the graph of f have a relative maximum?

(e) Find the x-coordinate of each inflection point on the graph of f.

(d)

x	(0,2)	2	(2,5)	5	(5,8)
f'(x)	+	0	−	0	+
f	↗	max	↘	min	↗

$x = 2$

(e) $f''(x) = 0 : x = 1, 3, 7$

x	(0,1)	1	(1,3)	3	(3,7)	7	(7,8)
f''(x)	+	P.O.I	−	P.O.I	+	P.O.I	−

$x = 1, 3, 7$

16.

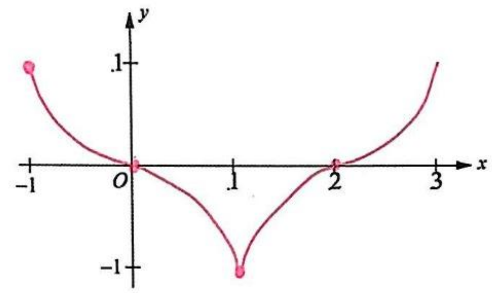
x	-1	$-1<x<0$	0	$0<x<1$	1	$1<x<2$	2	$2<x<3$
$f(x)$	1	+	0	–	–1	–	0	+
$f'(x)$	–4	– ↘	0	– ↘	DNE	+ ↑	1	+ ↑
$f''(x)$	2	+ ∪	0	– ∩	DNE	– ∩	0	+ ∪

Let f be a function that is continuous on the interval $-1 \le x < 3$. The function is twice differentiable except at $x = 1$. The function f and its derivatives have the properties indicated in the table above.

(a) For $-1 < x < 3$, find all values of x at which f has a relative extrema. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

critical numbers: $x=0, 1$. $f(1)$ local min

(b) On the axis provided, sketch the graph of a function that has all the given characteristics of f.



(c) Let h be the function defined by $h'(x) = f(x)$ on the open interval $-1 < x < 3$. For $-1 < x < 3$, find all values of x at which h has a relative extremum. Determine whether h has a relative maximum or a relative minimum at each of these values. Justify your answer.

(d) For the function h, find all values of x, for $-1 < x < 3$, at which h has a point of inflection. Justify your answer.

(c) $h'(x) = f(x) = 0$: $x = 0, 2$

x	$(-1,0)$	0	$(0,2)$	2	$(2,3)$
$h'(x)$	+	0	–	0	+

⇓ local max

$h''(x) = f'(x)$ $h''(0) = 0$

$h''(2) = 1 > 0 \Rightarrow$ relative min

$\therefore$ relative $\begin{cases} \text{min} & h(2) \\ \text{max} & h(0) \end{cases}$

(d) $h''(x) = f'(x) = 0$ OR DNE: $x = 0, 1$.

x	$(-1,0)$	0	$(0,1)$	1	$(1,3)$
$h''(x)$	–		–		+

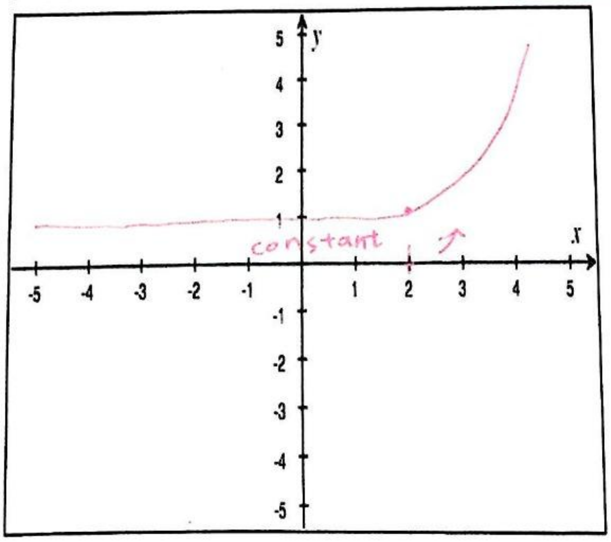
$\therefore$ P.O.I at $x=1$

➢ **Sketch the graph of a function**

1. **Sketch a possible $f(x)$ with the following characteristics.**

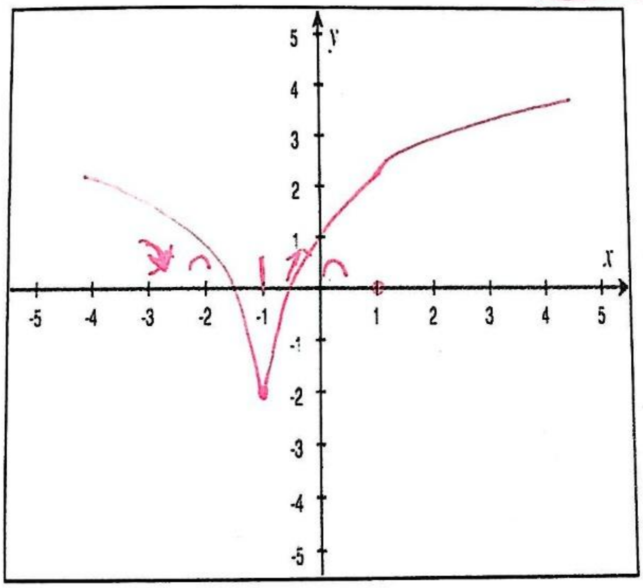
a. $f'(x) > 0$ for $x > 2$, $f'(x) = 0$ for $x \le 2$, $\underline{f'' > 0 \text{ for } x>2}$, $f(2) = 1$

∪



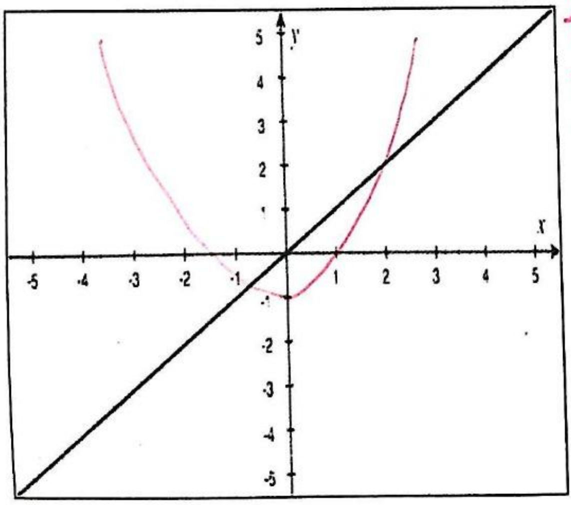
b. $f'(x) > 0$ for $x > -1, f'(x) < 0$ for $x < -1$, $\underline{f'' < 0 \text{ for } x \neq 1}$, $f(-1) = -2$

∩



2. You are given a graph of $f'(x)$. Sketch a possible graph of $f(x)$.

a.

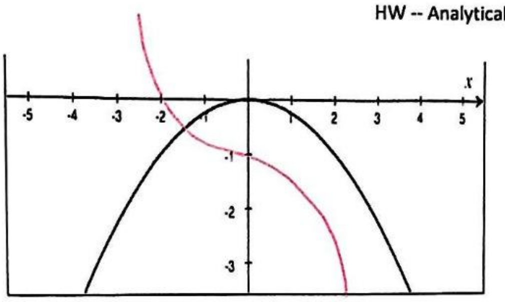


$f' = x \quad f'' = 1 > 0 \Rightarrow f: \cup$

$f \uparrow$ when $x > 0$

$f \downarrow$ when $x < 0$

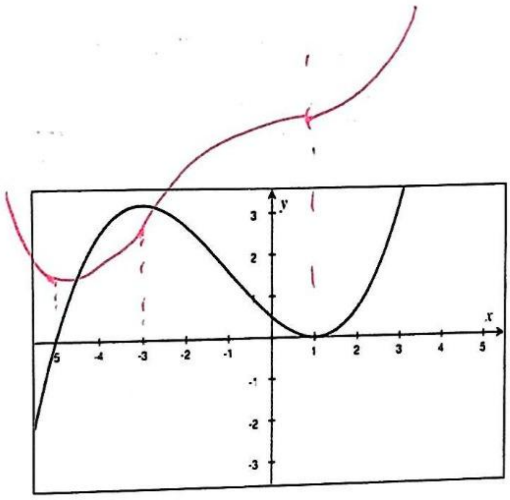
b.



$f'<0$ $f \downarrow$

f'': + 0 −

∪ ∩



c.

f: ↘ ↗

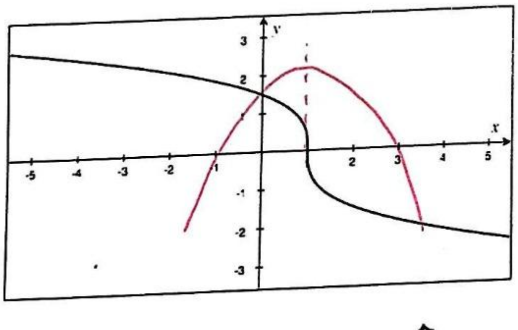
f': − + +

−5 −3 1

f'': + − +

∪ ∩ ∪

d.



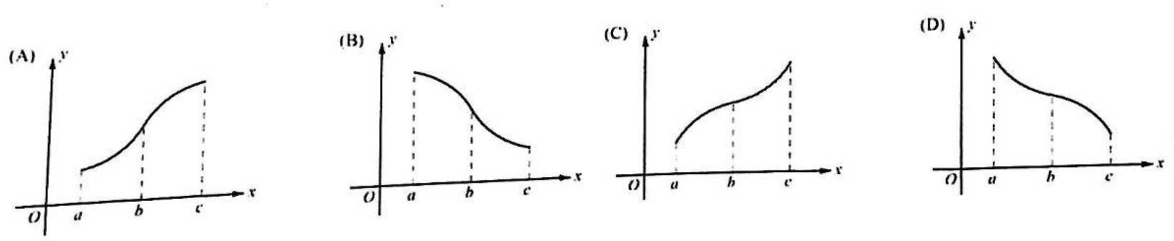
f: ↗ ↘

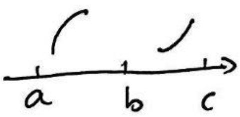
f': + 1 −

f'': − ∩

↑ ∩ ∪

C 3. If f is a function such that $f' > 0$ for $a < x < c$, $f'' < 0$ for $a < x < b$, and $f'' > 0$ for $b < x < c$ which of the following could be the graph of f?





C 4. The graph of $f(x) = xe^{-x^2}$ is symmetric about which of the following

$f(-x) = -x\,e^{-x^2} = -f(x)$ odd.

I. The x-axis
~~II~~. The y-axis
III. The origin ✓

(A) I only (B) II only (C) III only (D) II and III only

B 5. Let f be the function given by $f(x) = \dfrac{-3x^2}{\sqrt{3x^4+1}}$. Which of the following is the equation of horizontal asymptote of the graph of f?

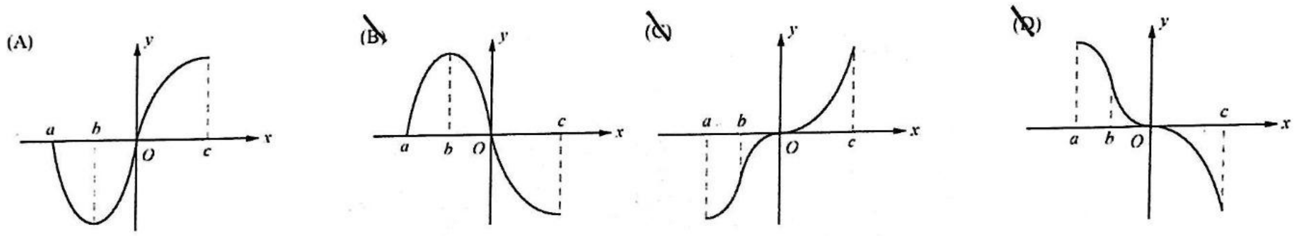
$\lim_{x\to\infty} f(x) = \frac{-3}{\sqrt{3}} = -\sqrt{3}$

$\lim_{x\to-\infty} \frac{-3x^2 \cdot \frac{1}{x^2}}{\sqrt{3x^4+1}\cdot\frac{1}{\sqrt{x^4}}} = -\sqrt{3}$

(A) $y=-3$ (B) $y=-\sqrt{3}$ (C) $y=\sqrt{3}$ (D) $y=3$

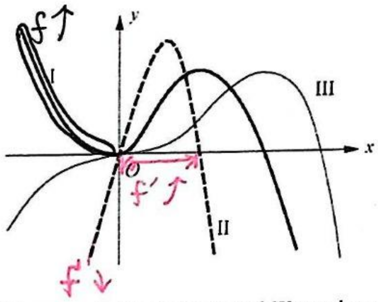
A 6. Let f be a function that is continuous on $[a,c]$, such that the derivative of function f has the properties indicated on the table below. Which of the following could be the graph of f?

x	$a<x<b$	b	$b<x<0$	0	$0<x<c$
$f'(x)$	$-$ ↘	0	$+$ ↗	3	$+$ ↗
$f''(x)$	$+$ ∪	$+$ ∪	$+$ ∪	0	$-$ ∩



7.

C



Three graphs labeled I, II, and III are shown above. They are the graphs of f, f', and f''. Which of the following correctly identifies each of the three graphs?

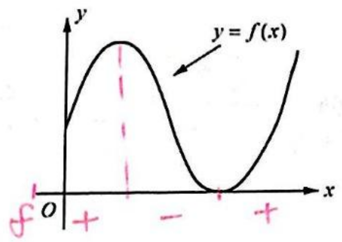
$f,\ f',\ f''$ power ↘

	f	f'	f''
(A)	I	II	III
(B)	II	I	III
(C)	III	I	II
(D)	I	III	II

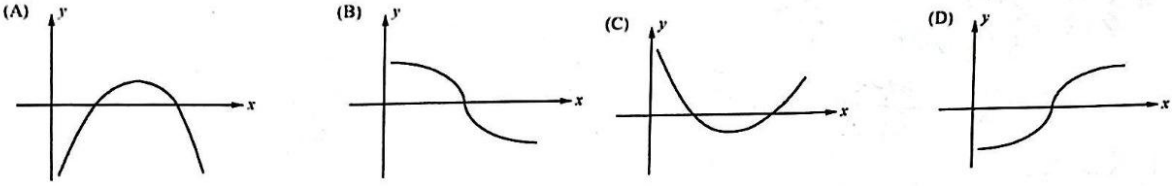
II: f''
I: f'

8.

C



The graph of f is shown in the figure above. Which of the following could be the graph of f'?



9. Sketch the graph of $y = e^x(x-2)^3$. Domain $= \mathbb{R}$

$y' = e^x (x-2)^2 (x+1) = 0 \Rightarrow x = 2, -1$

$y'' = e^x (x-2)(x^2+2x-2) = 0 \Rightarrow x = 2, -1 \pm \sqrt{3}$

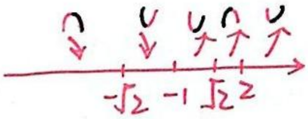
Asymptote: $\lim_{x \to -\infty} e^{x}(x-2)^3 = 0$

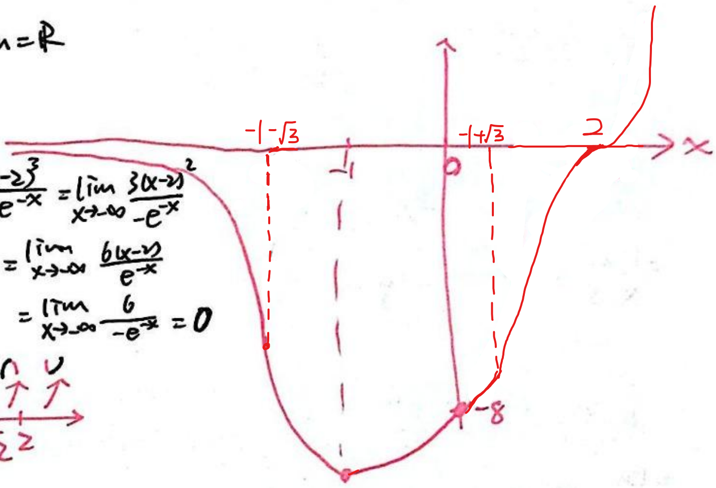
$\lim_{x \to -\infty} \frac{(x-2)^3}{e^{-x}} = \lim_{x \to -\infty} \frac{3(x-2)^2}{-e^{-x}} = \lim_{x \to -\infty} \frac{6(x-2)}{e^{-x}} = \lim_{x \to -\infty} \frac{6}{-e^{-x}} = 0$

$x \to -\infty$: $e^x (x-2)^3 < 0$

Intercepts: $x = 0, y = -8$

$y = 0, x = 2$





10. Sketch the graph of $y = -3x^5 + 5x^3$ using the Second Derivative Test.

polynomial function: smooth ⟺ ctns & differentiable

$y' = 15x^2(-x^2+1) = 0$: $x = 0, \pm 1$

$y'' = 30x(-2x^2+1)$

$y''(0) = 0$

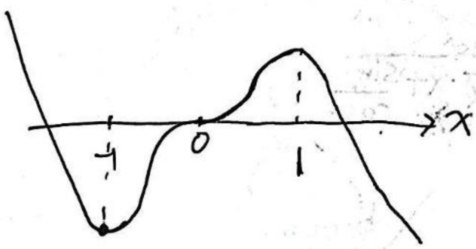
$y''(1) < 0$ max

$y''(-1) > 0$ min

End Behavior: $x \to -\infty, y \to +\infty$

$x \to +\infty, y \to -\infty$

x	(-1, 0)		(0, 1)	
y''	+	−	+	−
y	∪	∩	∪	∩



➢ **L'Hospital's Rule**

1. $\lim_{x \to -3} \left(\frac{x+3}{\sqrt{x^2-5}-2} \right)$ "$\frac{0}{0}$"

$= \lim_{x \to -3} \frac{1}{\frac{1}{2}(x^2-5)^{-\frac{1}{2}} \cdot (2x)} = \lim_{x \to -3} \frac{\sqrt{x^2-5}}{x} = -\frac{2}{3}$

2. $\lim_{x \to -2} \left(\frac{x^3+x^2-8x-12}{x^3+8x^2+20x+16} \right)$ "$\frac{0}{0}$"

$= \lim_{x \to -2} \frac{3x^2+2x-8}{3x^2+16x+20}$ "$\frac{0}{0}$"

$= \lim_{x \to -2} \frac{6x+2}{6x+16} = -\frac{5}{2}$

3. $\lim_{x \to 1} \left(\frac{5x^4-4x^2-1}{10-x-9x^2} \right)$ "$\frac{0}{0}$"

$= \lim_{x \to 1} \frac{20x^3-8x}{-1-18x} = -\frac{12}{19}$

4. $\lim_{x \to 2} \left(\frac{3x^2-7x+2}{x-2} \right)$ "$\frac{0}{0}$"

$= \lim_{x \to 2} \frac{6x-7}{1} = 5$

5. $\lim_{x \to 0} \frac{e^x-1-x}{x^2}$ "$\frac{0}{0}$"

$= \lim_{x \to 0} \frac{e^x-1}{2x}$ "$\frac{0}{0}$"

$= \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$

6. $\lim_{x \to 0} \frac{\sin^{-1} x}{x}$ "$\frac{0}{0}$"

$= \lim_{x \to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$

7. $\lim_{x \to 0} \left(\frac{1}{\tan x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x-\tan x}{\tan x \cdot x}$ "$\frac{0}{0}$"

$= \lim_{x \to 0} \boxed{\frac{1-\sec^2 x}{\sec^2 x \cdot x + \tan x}} = \frac{-\tan^2 x}{\frac{1}{\cos^2 x} \cdot x + \frac{\sin x}{\cos x}} = \frac{-\frac{\sin^2 x}{\cos x}}{\frac{x}{\cos x} + \sin x} = \frac{-\sin^2 x}{x + \sin x \cos x}$

$= \frac{-\sin^2 x}{x + \frac{1}{2}\sin 2x}$ "$\frac{0}{0}$"

method 2

$= \lim_{x \to 0} \frac{-2\sin x \cos x}{1+\cos 2x} = 0$

method 1 $= \lim_{x \to 0} \frac{-2\sec x \cdot \sec x \tan x}{2\sec x \cdot \sec x \tan x \cdot x + \sec^2 x + \sec^2 x}$ "$\frac{0}{0}$"

$= \lim_{x \to 0} \frac{2\sec^2 x \cdot (-\tan x)}{2\sec^2 x \, (\tan x \cdot x + 1)} = 0$

8. $\lim_{x\to 1^-}\boxed{\left(\frac{2}{x^2-1}-\frac{x}{x-1}\right)} = \frac{2-x(x+1)}{x^2-1} = \frac{-x^2-x+2}{x^2-1} = \frac{(-x+1)(x+2)}{(x+1)(x-1)} = -\frac{x+2}{x+1}$

$= -\frac{3}{2}$

9. $\lim_{x\to 0^+}(\tan x)^x = y$

$= \lim_{x\to 0^+} e^{\ln y}$

$= e^{\lim_{x\to 0^+}\ln y} = e^0 = 1$

$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+} x\ln\tan x = \lim_{x\to 0^+}\frac{\ln(\tan x)}{\frac{1}{x}} \underset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^+}\boxed{\frac{\cot x\cdot\sec^2 x}{-\frac{1}{x^2}}}$

$= \lim_{x\to 0^+} -\frac{x}{\sin x}\cdot\frac{x}{\cos x}$

$= 0$

$- \frac{\frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x}}{1}x^2$

$= -x^2\cdot\frac{1}{\cos x\sin x}$

$= -\frac{x}{\sin x}\cdot\frac{x}{\cos x}$

10. $\lim_{\theta\to\pi}\frac{\sin\theta}{\theta-\pi}$ "$=\frac{0}{0}$"

$= \lim_{\theta\to\pi}\frac{\cos\theta}{1} = -1$

11. $\lim_{x\to 1}\left(\frac{\ln x-x+1}{e^x-ex}\right)$ $\quad \frac{\ln 1-1+1}{e^1-e} = \frac{0}{0}$

$= \lim_{x\to 1}\frac{\frac{1}{x}-1}{e^x-e} = \lim_{x\to 1}\frac{-\frac{1}{x^2}}{e^x} = -e^{-1}$

12. $\lim_{x\to\infty}(x)^{\frac{1}{x}}$ "∞^0"

$\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$

$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln x}{x} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0 \quad \therefore \lim_{x\to\infty}(x)^{\frac{1}{x}} = \lim_{x\to\infty}e^{\ln y} = e^{\lim_{x\to\infty}\ln y} = e^0 = 1$

13. $\lim_{x\to\infty}\left(\frac{1-4x-5x^2}{3x^2-x-4}\right)$

$= \frac{-5}{3}$

14. $\lim_{x\to 0}\frac{e^x-1}{\sin x} \underset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0}\frac{e^x}{\cos x} = \frac{1}{1} = 1$

15. Use L'Hospital's Rule to find the exact value of $\lim_{x\to\infty} x\left[\ln(x+3)-\ln x\right]$. Show the work that leads to your answer.

$\infty(\infty-\infty)$

$= x\cdot\ln\frac{x+3}{x}$

$= \frac{\ln\frac{x+3}{x}}{\frac{1}{x}} = \frac{\ln(1+\frac{3}{x})}{\frac{1}{x}}$ "$=\frac{0}{0}$"

$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{3}{x}}\cdot 3\cdot(\frac{1}{x})'}{(\frac{1}{x})'} = \lim_{x\to\infty}\frac{3}{1+\frac{3}{x}} = 3$