**Sequences and Series**

- **Sequence** is an ordered list of numbers

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \dots, a_n, \dots\}$$

  - ✓ Example. Arithmetic Sequence $\{n\}_{n=1}^{\infty} = \{1,2,\dots,n,\dots\}$

  - ✧ **Convergent and Divergent**

    - If a sequence has the limit $L$, where $L$ is a finite real number, (________________________________________),

    we say the sequence **converges** to L.

    - If the limit does not exist, the sequence **diverges**.

✎ Practice

1. Is the sequence $\{2n+1\}_{n=1}^{\infty}$ convergent or divergent?

2. $\{\frac{n^2+1}{2n^2-3n+5}\}_{n=1}^{\infty}$ convergent or divergent?

3. $\{(-1)^n\}_{n=1}^{\infty}$ convergent or divergent?

4. $\{\frac{1}{n} * (-1)^n\}_{n=1}^{\infty}$ convergent or divergent?

- **Infinite series**: Given a sequence $\{a_n\}$, $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ is an **infinite series.**

  - ✧ **True Sum** $S = \sum_{n=1}^{\infty} a_n$

  - ✧ **Partial Sum** of a sequence: $S_n = \sum_{i=1}^{n} a_i$

    - ✓ Example. What is the partial sum of the sequence $\{n\}_{n=1}^{\infty} = \{1,2,\dots,n,\dots\}$?

$$S_n = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

      - ■ $\{S_n\}_{n=1}^{\infty} = \{S_1, S_2, \dots, S_n, \dots\}$ is a sequence.

  - ✧ **Convergent and Divergent**

    The series is convergent $\Leftrightarrow$ ____________________________________________

    Otherwise, the series diverges.

**Sequences and Series**

✎ Practice

1. Is the series $\sum_{n=1}^{\infty} n$ convergent or divergent?

2. Is the series $\sum_{n=1}^{\infty}(-1)^n$ convergent or divergent?

3. Is the telescoping series $\sum_{n=1}^{\infty} \frac{1}{k(k+1)}$ convergent or divergent?

4. Is the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ convergent or divergent?

5. Is the geometric series $\sum_{n=1}^{\infty} 2^{3n}5^{1-n}$ convergent or divergent?

Summary. **Geometric series** $\sum_{n=1}^{\infty} ar^{n-1}$

## Sequences and Series

- **Convergence of a sequence V.S. Convergence of a series**

  If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

  - This theorem provides a useful test for **divergent series**!

    If the limit $\lim_{n\to\infty} a_n$ DNE or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ **<u>diverges</u>**.

    - ✓ Example. $a_n = 1$

$$a_n = \frac{1}{n}$$

We can use the definition of convergent series to determine whether the series is convergent or divergent.

However, it is always hard to find the expression of $S_n$. So, we need other methods to test it.

- **The Integral Test**

  If f is positive, continuous, and decreasing on $[1, +\infty)$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} \boldsymbol{a_n}$ and $\int_{1}^{\infty} \boldsymbol{f(x)\,dx}$ either **BOTH** converge or diverge.

✧ **p-series** $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{n^p}+\cdots$

Determine whether the p-series convergent or divergent.

✧ **Harmonic Series** $\sum_{n=1}^{\infty}\frac{1}{n}$

When $p=1$, the p-series is called harmonic series.

✧ **General Harmonic Series** $\sum_{n=1}^{\infty}\frac{1}{an+b}$

✍ Practice

1. $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$

2. $\sum_{n=1}^{\infty}\frac{\ln n}{n}$

3. $\sum_{n=1}^{\infty} n^{1-\pi}$

4. $1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{25}} + \cdots$

5. $\sum_{n=1}^{\infty} (\frac{1}{2})^n$

## Sequences and Series

- **The Comparison Test**

  - **Direct Comparison Test**

Let $0 \le a_n \le b_n$ for all n.

1. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ ________________
2. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ ________________

Practice

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

2. $\sum_{n=1}^{\infty} \frac{n}{n^2-3}$

3. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{\sqrt{n^3}+1}$

4. $\sum_{n=1}^{\infty} \frac{n^2 \cos^4 n}{n^5+1}$

? How to select $b_n$

### ✧ Limit Comparison Test

If $a_n > 0$, $b_n > 0$, and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$, where L is finite and positive, then both series either converge or diverge.

✍ Practice

1. $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$

3. $\sum_{n=3}^{\infty} \frac{2^n}{3^n+1}$

## Sequences and Series

- **Alternating Series Test**

Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge if

1. $\lim_{n\to\infty} a_n = 0$ **AND** 2. $a_{n+1} \le a_n$ for all n greater than some integer N.

- **Alternating Series Estimation Theorem (Error Bound)**

If $S = \sum_{n=1}^{\infty}(-1)^n a_n$ is the sum of a convergent alternating series that satisfies the condition $a_{n+1} \le a_n$, then the remainder $R_n = S - S_n$ is smaller than _________, $|R_n| \le$ _____________

✍ Practice.

1. Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n-1}$

2. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$

3. $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^{n+1}}{(2n-1)!} + \cdots$

4. Let $f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + - \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$.

   Use the alternating series error bound to show that $1 - \frac{1}{2!} + \frac{1}{4!}$ approximates $f(1)$ with an error less than $\frac{1}{500}$.

- **Absolute and Conditional Convergence**
  - $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.
  - $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Practice. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n . \sqrt[n]{e}}{n^2}$

2. $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-\frac{2}{3}}$

## Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = q$

- If $q < 1$, the series converges absolutely.
- If $q > 1$ or the limit DNE, ________________________________
- If $q = 1$, the ratio test fails.

**Example.** $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Practice

1. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

2. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{5^n}$

Sequences and Series

**Determine whether the series below is conditionally convergent or absolute convergent.**

3. $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

4. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+3}$

5. $\sum_{n=1}^{\infty} (-1)^n \frac{e^n}{n!}$

## Series Convergence/Divergence Flow Chart