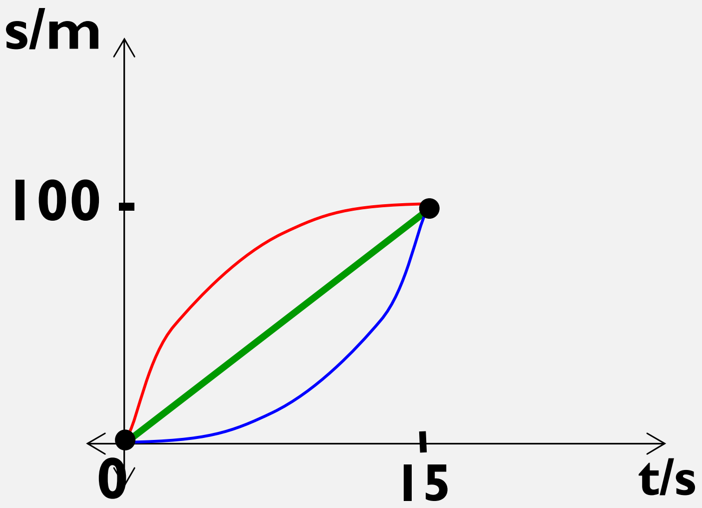
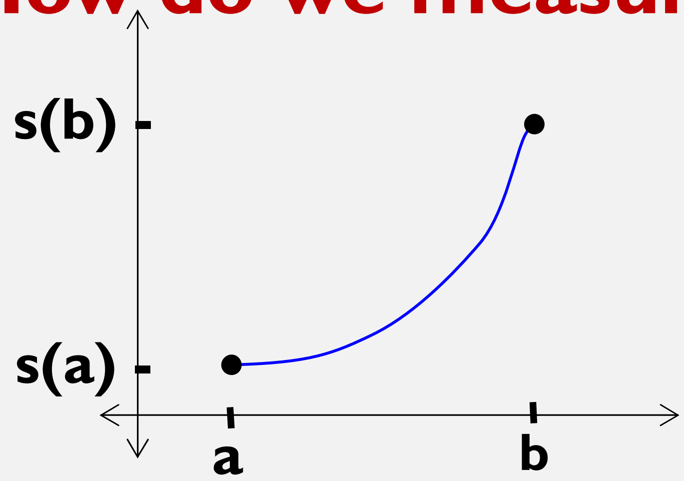
How do we measure speed?



Average velocity over the interval $[0, 15]$ $= \dfrac{100m}{15s} = \dfrac{\Delta s}{\Delta t} = \dfrac{s(15) - s(0)}{15 - 0}$

How do we measure speed?



x	3	4	5
$s(x)$	25	30	40

Average velocity over the interval $[a, b]$ $= \dfrac{\Delta s}{\Delta t} = \dfrac{s(b) - s(a)}{b - a}$

Note: **final - initial**

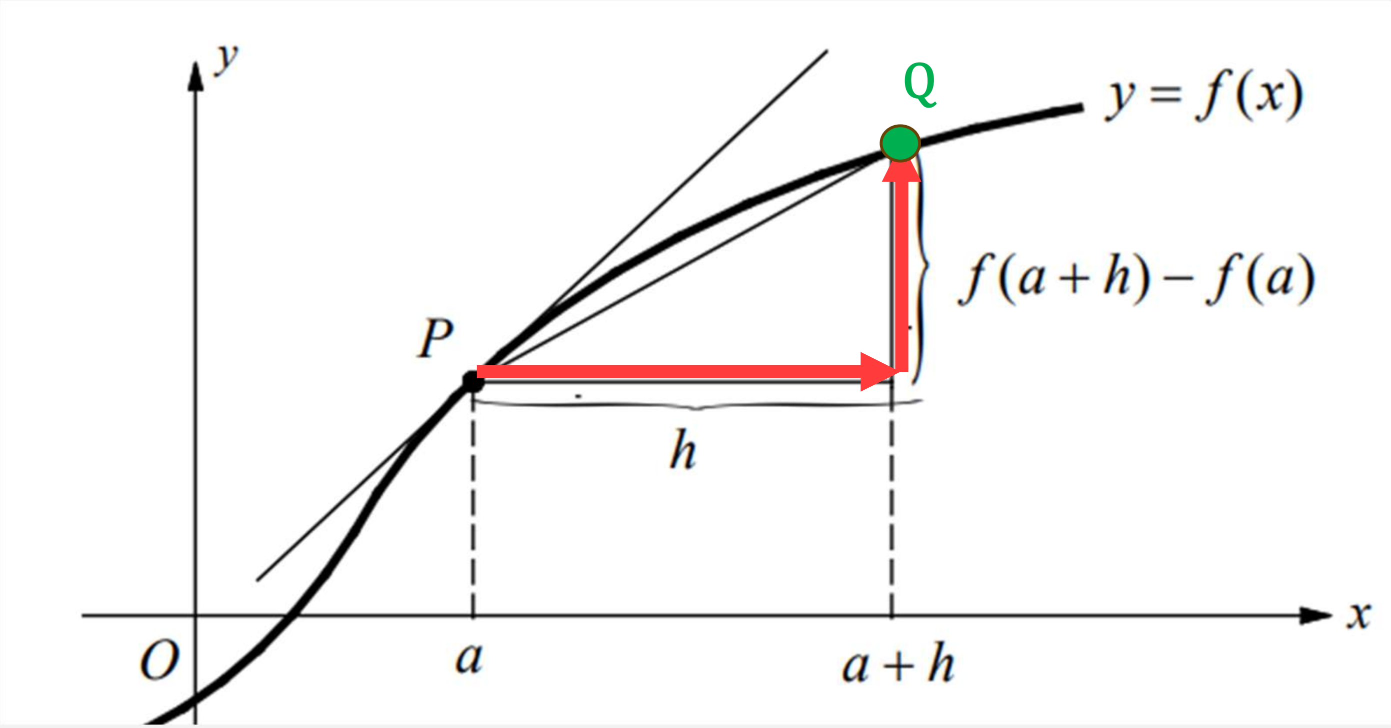
Average rate of change

Find the ***average velocity*** on the interval [4, 5]

At $t = 4$???

velocity

instantaneous rate of change



Average rate of change $= \dfrac{\Delta y}{\Delta x}$ quotient of differences

$=$ Slope of secant line over the interval $(a, a + \mathrm{h})$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{\mathrm{x} - \mathrm{a}}$$

Instantaneous rate of change

When $\boldsymbol{Q(a+h, f(a+h))}$ approaches to $\boldsymbol{P}$: secant line → tangent line

the slope of secant line → the slope of tangent line

Notation of the Derivative

Derivative of a function: $f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}\left(f(x)\right)$

At point x = a: $f'(a) = \mathrm{y}'(\mathrm{a}) = \left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{\mathrm{x=a}} = \lim_{x \to a} \frac{f(x)-f(a)}{\mathrm{x-a}}$

Practice

Find a function f and a number a such that $f'(a)=\lim_{h\to 0}\dfrac{\sin\left(\dfrac{\pi}{6}+h\right)-\dfrac{1}{2}}{h}$

The Existence of a Derivative

Recall:

- $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{\text{x} - \text{a}}$ ~ a limit of the difference quotient
- The existence of a limit ~ left-hand limit = right-hand limit = L

$f'(a)$ is defined $\leftrightarrow \lim_{x \to a} \frac{f(x) - f(a)}{\text{x} - \text{a}}$ exists $\leftrightarrow \boxed{\lim_{x \to a^-} \frac{f(x) - f(a)}{\text{x} - \text{a}}} = \boxed{\lim_{x \to a^+} \frac{f(x) - f(a)}{\text{x} - \text{a}}} = \text{L}$

left-hand derivative right-hand derivative

Differentiable & Continuous

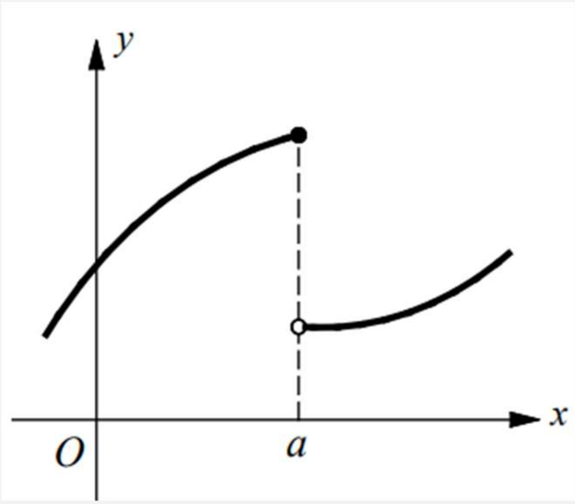
$$\begin{tikzcd} f \text{ is differentiable at } \mathrm{x{=}a} \arrow[d, shift right=3, "?"'] \\ f \text{ is continuous at } \mathrm{x{=}a} \arrow[u, shift right=3, "?"'] \end{tikzcd}$$

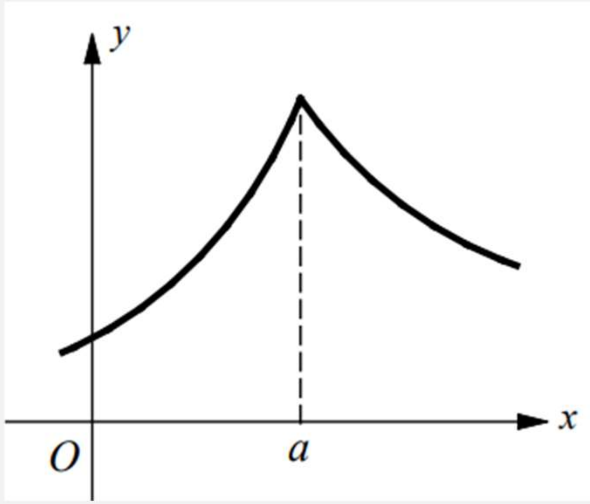
When the function is not differentiable at $x = a$...

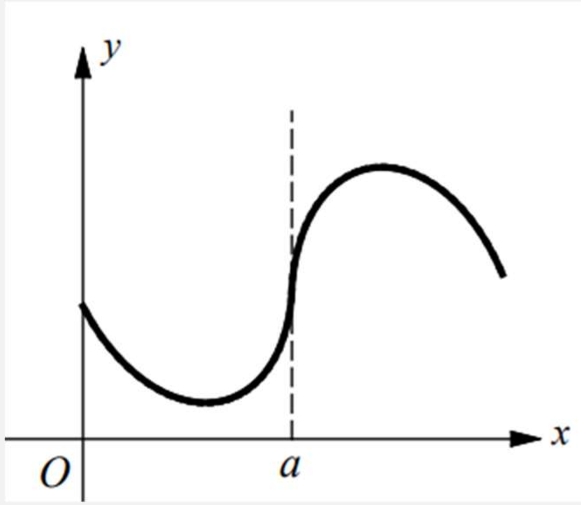
$f'(x)$ DNE: Infinity

$f'(x)$ DNE: Jump
"corner"

"The graph has a vertical tangent."



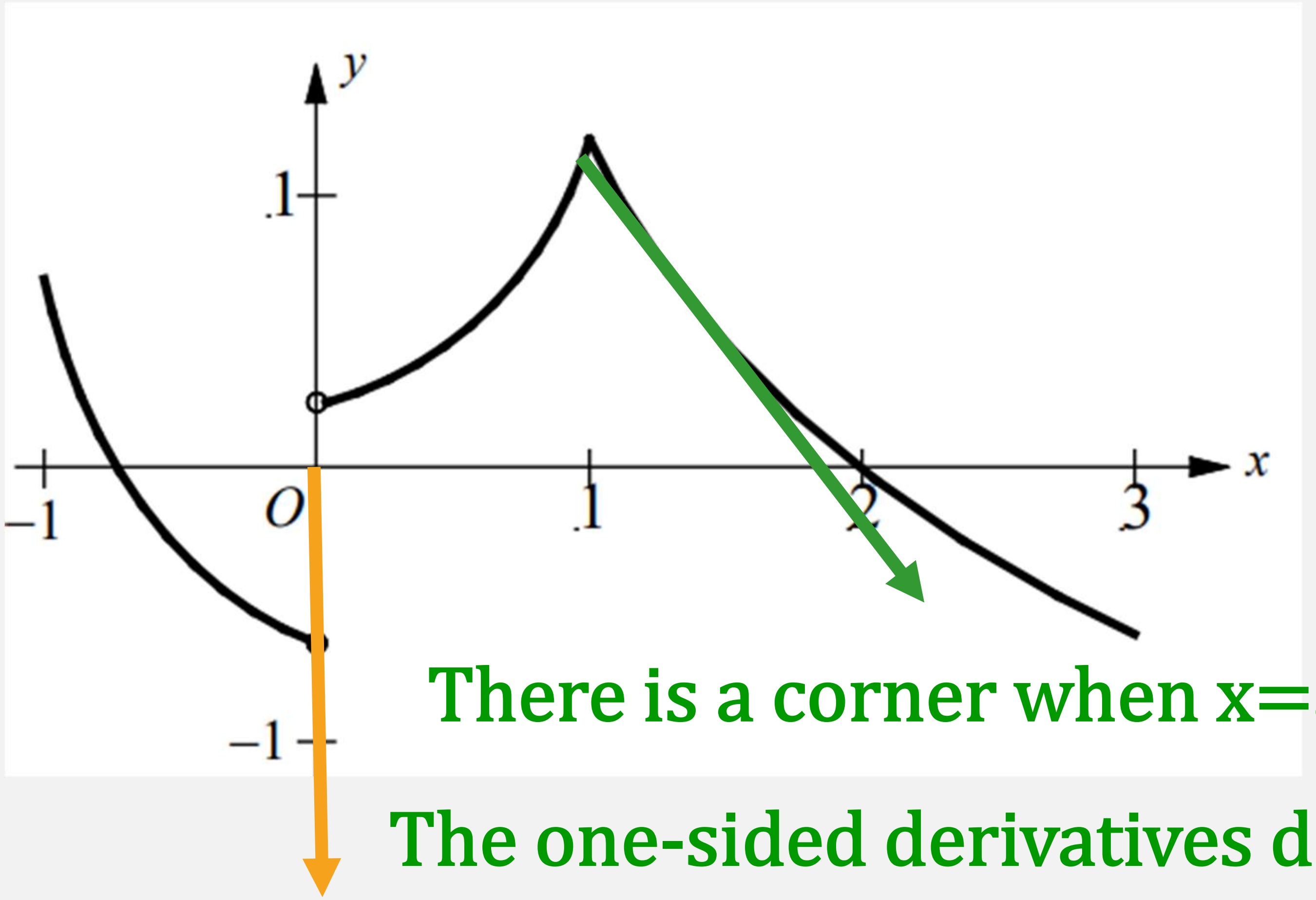




Not continuous at $x = a$

continuous at $x = a$

Practice



There is a corner when x=a. /

The one-sided derivatives differ.

Not continuous

Derivative of Power Function

- ✧ Constant Rule $\frac{d}{dx}(k) =$
- ✧ Power Rule $\frac{d}{dx}(x^n) = \quad nx^{n-1}$

Use the definition ($f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$) to find the derivative of $f(x) = x^2$

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}\frac{2x+h}{1}$$

$$= 2x$$

Practice

Q3. Find the derivative of $f(x) = x^3 - 2x + \frac{1}{x} + 5$ at $x = 2$

Q4. $\lim_{h \to 0} \frac{(3+h)^4 - 81}{h} = ?$

Hint: $\lim_{h \to 0} \frac{(3+h)^4 - 81}{h} = \mathrm{f}'(\mathrm{c})$ What is $f(x)$? $\mathrm{c} = ?$

Derivatives of piecewise functions

Example. Find the derivative of $f(x) = \begin{cases} x+3, x \leq 0 \\ 3-2x, x > 0 \end{cases}$

$$f'(x) = \begin{cases} 1, x < 0 \\ -2, x > 0 \end{cases}$$

Practice

Q5. Let f be the function defined by $f(x) = \begin{cases} mx^2 - 2, x \leq 1 \\ k\sqrt{x},\ \ x > 1 \end{cases}$

If $\boxed{f \text{ is differentiable at } \mathrm{x} = 1}$, what are the values of k and m ?

f is continuous at $\mathrm{x} = 1$

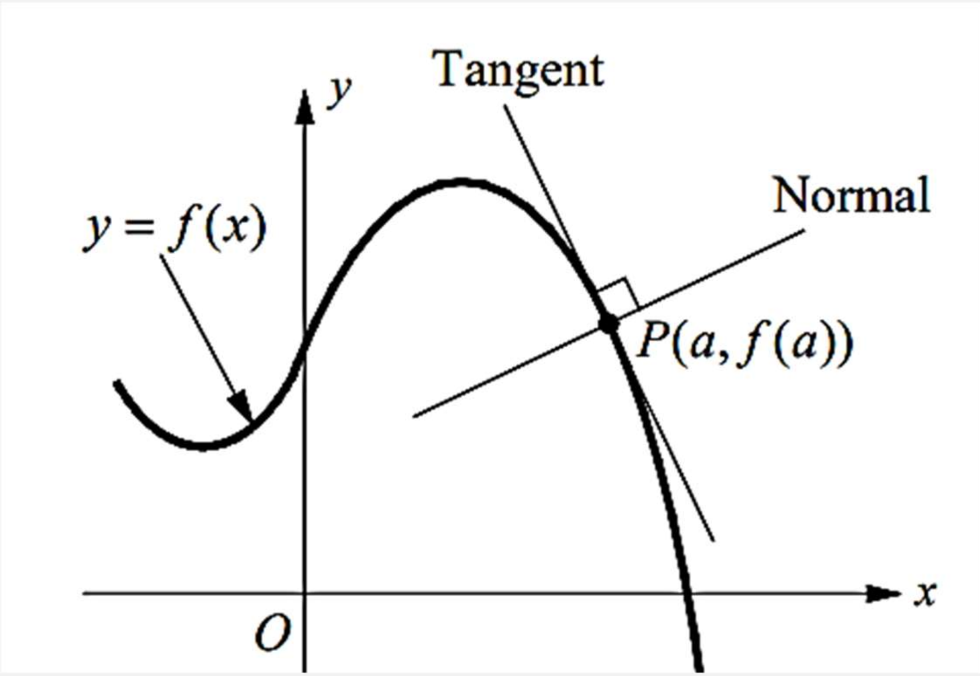
Tangent Line & Normal Line

Review. Point-slope equation

A linear equation

$$y - y_0 = m(x - x_0)$$

- slope m
- passes through the point (x_0, y_0)



$$y - f(a) = f'(a)(x - a)$$

tangent line at $\mathrm{x} = a$

- Slope: $f'(a)$
- Passes: $P(a, f(a))$

$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$

normal line at $\mathrm{x} = a$

- Slope: $-\dfrac{1}{f'(a)}$
- Passes: $P(a, f(a))$

Practice

Q6. Write the equation of the tangent line and normal line to the graph of $y = x - \frac{x^2}{10}$ at the point $(4, \frac{12}{5})$

Practice

Which of the following is an equation of the line tangent to the graph of $f(x) = x^2 - x$ at the point where $f'(x) = 3$?

(A) $y = 3x - 2$

(B) $y = 3x + 2$

(C) $y = 3x - 4$

(D) $y = 3x + 4$

Practice

If $2x+3y=4$ is an equation of the line normal to the graph of f at the point $(-1,2)$, then $f'(-1)=$

(A) $-\frac{2}{3}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{2}$ (D) $\frac{3}{2}$

Practice

If $2x - y = k$ is an equation of the line normal to the graph of $f(x) = x^4 - x$, then $k =$

(A) $\frac{23}{16}$ (B) $\frac{13}{18}$ (C) $\frac{15}{16}$ (D) $\frac{9}{8}$

Practice

A curve has slope $2x + x^{-2}$ at each point (x, y) on the curve. Which of the following is an equation for this curve if it passes through the point $(1,3)$?

(A) $y = 2x^2 + \frac{1}{x}$

(B) $y = x^2 - \frac{1}{x} + 3$

(C) $y = x^2 + \frac{1}{x} + 1$

(D) $y = x^2 - \frac{2}{x^2} + 4$

Chain Rule

If <u>y=f(u) and u=g(x) are both differentiable</u> functions,

then y=f(g(x)) is differentiable and

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)g'(x)$$

✧ If y=f(u), u=g(w), and w=h(x) are all differentiable functions, then y=f(g(h(x))) is differentiable and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx}$

$$\frac{d}{dx}\left[f\left(g(h(x))\right)\right] = f'\left(g(h(x))\right)g'\left(h(x)\right)h'(x)$$

Q11. Find y' for $y = \sqrt{1 + x^2}$

Q12. Find y' for $y = (2x^3 - 2x^2)^4$

Q13. Find y' for $y = \sqrt{x^4 - 2x + 5}$

Q14. Find $h''(x)$ for $h(x) = f(x^3)$

nth Derivative $f^n(x) = \frac{d^n y}{dx^n} = y^{(n)}(x)$

Q15. If $f(x) = \frac{1}{6}x^3 + 24\sqrt{x}$, find f'(x), f''(x), f'''(x), and f'''(9)

The Product Rule

If f and g are both differentiable, then $\frac{d}{dx}[f(x)g(x)] = f'g + g'f$

$$\lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x+h)\cdot g(x) + f(x+h)\cdot g(x) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h)\cdot g(x+h) - f(x+h)\cdot g(x)] + [f(x+h)\cdot g(x) - f(x)\cdot g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot [g(x+h) - g(x)] + g(x)\cdot [f(x+h) - f(x)]}{h} = g'f + f'g$$

The Quotient Rule

If f and g are both differentiable, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[f(x)\cdot\frac{1}{g(x)}\right]$

$$= f \cdot \frac{d}{dx}\left[\frac{1}{g}\right] + \frac{\mathrm{df}}{\mathrm{dx}} \cdot \frac{1}{g}$$

$$= f \cdot \left(-\frac{g'}{g^2}\right) + \mathrm{f}' \cdot \frac{1}{g(x)}$$

$$= \frac{-f \cdot g' + f' \cdot g}{g^2}$$

Q16. Differentiate the function $f(x) = (x^3 - 7)(x^2 - 4x)$

Q17. Differentiate the function $f(x) = \frac{3x^2 - x}{\sqrt{x}}$

If f, g, and h are functions that is everywhere differentiable, then the derivative of $\frac{f}{g \cdot h}$ is

(A) $\frac{g\,h\,f' - f\,g'\,h'}{g\,h}$

(B) $\frac{g\,h\,f' - f\,g\,h' - f\,h\,g'}{g\,h}$

(C) $\frac{g\,h\,f' - f\,g\,h' - f\,g'h}{g^2 h^2}$

(D) $\frac{g\,h\,f' - f\,g\,h' + f\,h\,g'}{g^2 h^2}$

Q19. Differentiate the function $f(x) = (3x^3 - 2x)(2x - 1)(5x + 10)$

Derivatives of Logarithm Function

Review: Properties of logarithm

$$ln\ xy = \ln x + \ln y$$

$$ln\,\frac{x}{y} = \ln x - \ln y$$

$$ln\ x^p = p \ln x$$

$$e^{ln\ x} = x$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Derivatives of Logarithm Function

Review: Properties of logarithm

$$ln\ xy = \ln x + \ln y$$

$$ln\ x^p = p \ln x$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

$$\mathbf{\log_a x = \frac{\ln x}{\ln a} = \frac{1}{ln\ a} \cdot ln\ x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x) = \frac{1}{ln\ a} \cdot \frac{1}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$$

$$\lim_{h\to 0} \frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0} \frac{\ln\frac{x+h}{x}}{h} = \lim_{h\to 0} \frac{\ln(1+\frac{h}{x})}{\frac{h}{x}\cdot x} = \lim_{h\to 0} \frac{\frac{x}{h}\cdot \mathrm{l}\ \ (1+\frac{h}{x})}{x} = \lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)^{\frac{x}{h}}}{x} = \frac{1}{x} \lim_{u\to 0} \ln(1+u)^{\frac{1}{u}}$$

$$\lim_{u\to 0} (1+u)^{\frac{1}{u}} = e$$

Practice

Find y' if $y = \frac{\ln}{x^2}$

Derivatives of Exponential Function

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = \quad a^x \cdot \ln a$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = \quad e^x$$

$$y = a^x$$

$$\ln y = x \ln a$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln a)$$

$$\frac{1}{y} y' = \ln a$$

$$y' = \ln a \cdot y = \ln a \cdot a^x$$

Practice

Find y’ if $y = x^{\ln}$

$$\ln y = \ln x \ln x = (\ln x)^2$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left((\ln x)^2\right)$$

$$\frac{1}{y}y' = 2\ln x \cdot \frac{1}{x}$$

$$y' = 2\ln x \cdot \frac{1}{x} \cdot x^{\ln x}$$

$$y = e^{\ln} \quad = e^{\ln \quad \ln} \quad = e^{\ln x \cdot \ln x} = e^{(\ln x)^2}$$

$$y' = \frac{d}{dx}\left(e^{(\ln \quad)^2}\right) = e^{(\ln x)^2} \cdot 2\ln x \cdot \frac{1}{x}$$

Q22. Find y’ if $y = 3^{x^2 - x}$

Implicit Differentiation

✧ Explicit functions y=f(x): expresses y explicitly in terms of x

✧ Implicit functions: we are unable to solve for y as a function of x.

$$y^2 + 3x = 6xy$$

$$\frac{d}{dx}[y^2 + 3x] = \frac{d}{dx}[6xy]$$

$$2yy' + 3 = 6y + 6xy'$$

$$2yy' - 6xy' = 6y - 3$$

Guidelines:

1. Differentiate both sides of equation with respect to x
2. Collect the term $\frac{dy}{dx}$ on the left side and move all other terms to the right
3. Solve for $\frac{dy}{dx}$

$$y'(2y - 6x) = 6y - 3 \qquad y' = \frac{6y - 3}{2y - 6x}$$

Implicit Differentiation

$y^2 + 3x = 6xy$ $y' = \dfrac{6y - 3}{2y - 6x}$

- The tangent line is horizontal when $\frac{dy}{dx} =$
- The tangent line is vertical when the $\underline{\text{Denominator}}$ of $\frac{dy}{dx}$ is 0

$2y - 6x$

Q23. Find $\frac{dy}{dx}$ if $y^2 = x^2 - \cos xy$

Q24. Consider the curve defined by $x^3 + y^3 = 4xy + 1$

(1) Find $\frac{dy}{dx}$

(2) Write an equation for line tangent to the curve at the point (2,1)

Q25. Consider the curve defined by $x^3 + y^3 - 6xy = 0$

(1) Find $\frac{dy}{dx}$

(2) Find the x-coordinates of each point on the curve where the tangent line is horizontal

(3) Find the y-coordinates of each point on the curve where the tangent line is vertical

Derivatives of Trigonometric Function

$\frac{d}{dx}(\sin x) =$

$\frac{d}{dx}(\cos x) =$

$$\lim_{h\to 0}\frac{sin(\mathrm{x+h})-sin\ x}{h}$$

$$=\lim_{h\to 0}\frac{sinx\cdot cosh+sinh\cdot cosx-sin\ x}{h}$$

$$=\lim_{h\to 0}\frac{sinx(cosh-1)+sinh\cdot cosx}{h}$$

$$=\lim_{h\to 0}\ sinx\frac{1-cosh}{h}+\lim_{h\to 0}\frac{sinh}{h}\cdot cosx$$

$$=sinx\cdot\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{h}+cosx=sinx\cdot 0+cosx=\cos x$$

Derivatives of Trigonometric Function

$$\frac{d}{dx}(\sin x) = \qquad \frac{d}{dx}(\cos x) =$$

$$\frac{d}{dx}(\tan x) = \qquad \frac{d}{dx}(\cot x) =$$

$$\frac{d}{dx}(\sec x) = \qquad \frac{d}{dx}(\csc x) =$$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$ Quotient Rule

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$ Quotient Rule / Chain Rule

Derivatives of Trigonometric Function

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$$

Q26. Find $\frac{dy}{dx}$ for $y = x^2 \sin x + 2x \cos x$

Q27. Find $\frac{dy}{dx}$ for $y = lnx \tan x - x^3 \sec x$

Q28. Find y’ for $y = \sin x^2$

Q29. Find y’ for $y = \csc \frac{1}{x}$

Q30. Find y' for $y = \tan^2(x^3)$

Q31. Find y' for $y = \sin^2(-3x^2 - 1)$

Q32. Differentiate $\boldsymbol{y = \ln \frac{x^2}{(x+1)^2}}$

Derivative of an inverse function

Properties of inverse function

$$f\left(f^{-1}(x)\right) = \underline{\quad x \quad}, x \in \underline{\text{range of } f}$$

$$f^{-1}\left(f(x)\right) = \underline{\quad x \quad}, x \in \underline{\text{domain of } f}$$

$$\frac{d}{dx}\left[f\left(f^{-1}(x)\right)\right] = \frac{\mathrm{d}}{\mathrm{dx}}[\mathrm{x}] = 1$$

$$f'\left(f^{-1}(x)\right) \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left[f^{-1}(x)\right] = 1$$

$$y = f^{-1}(x) \leftrightarrow x = f(y)$$

$$y' = \frac{\Delta y}{\Delta x} \leftrightarrow x' = \frac{\Delta x}{\Delta y} = f'(y)$$

$$y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

$$(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(\boldsymbol{x})\right)}$$

Q33. Let $f(2) = 5$ and $f'(2) = \frac{1}{4}$, find $(f^{-1})'(5)$

Derivative of Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1} x) =$ ____________ $\quad \frac{d}{dx}(\cos^{-1} x) =$ ____________

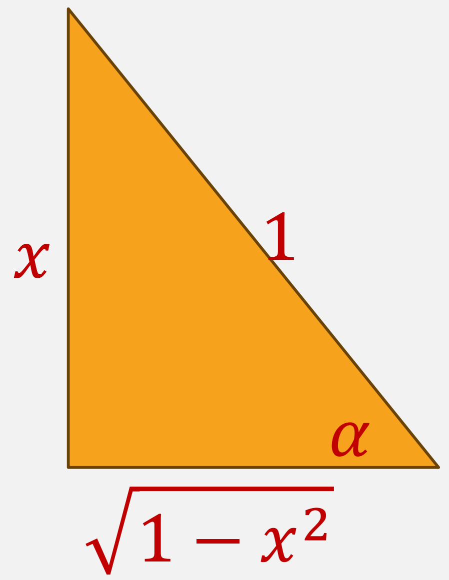
$\frac{d}{dx}(\tan^{-1} x) =$ ____________ $\quad \frac{d}{dx}(\cot^{-1} x) =$ ____________

$\frac{d}{dx}(\sec^{-1} x) =$ ____________ $\quad \frac{d}{dx}(\csc^{-1} x) =$ ____________

$$\sin^{-1} x = \alpha \leftrightarrow \sin\alpha = x$$

$$f = \sin \qquad f' = \cos$$

$$(f^{-1})' = \frac{1}{\cos\alpha} = \frac{1}{\sqrt{1-x^2}}$$

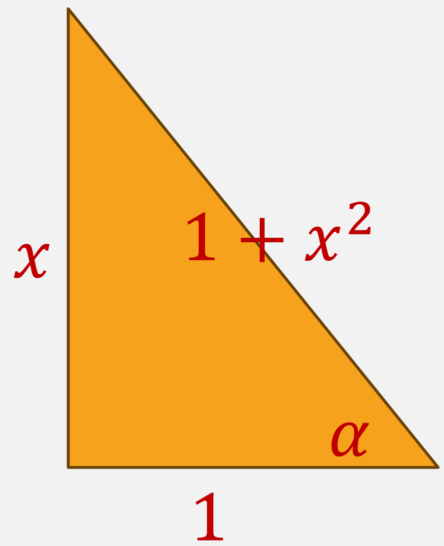


Derivative of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$



Q34. Differentiate $y = x\tan^{-1} x$

Q35. Differentiate $y = \frac{1}{\cos^{-1} x}$

Q36. Differentiate $y = \arctan\sqrt{x}$

Q37. Differentiate $y = 5\arcsin 3x$

Approximating a derivative

If a function f is defined by a table of values, then the approximation values of its derivatives at b can be obtained from the average rate of change using values that are close to b.

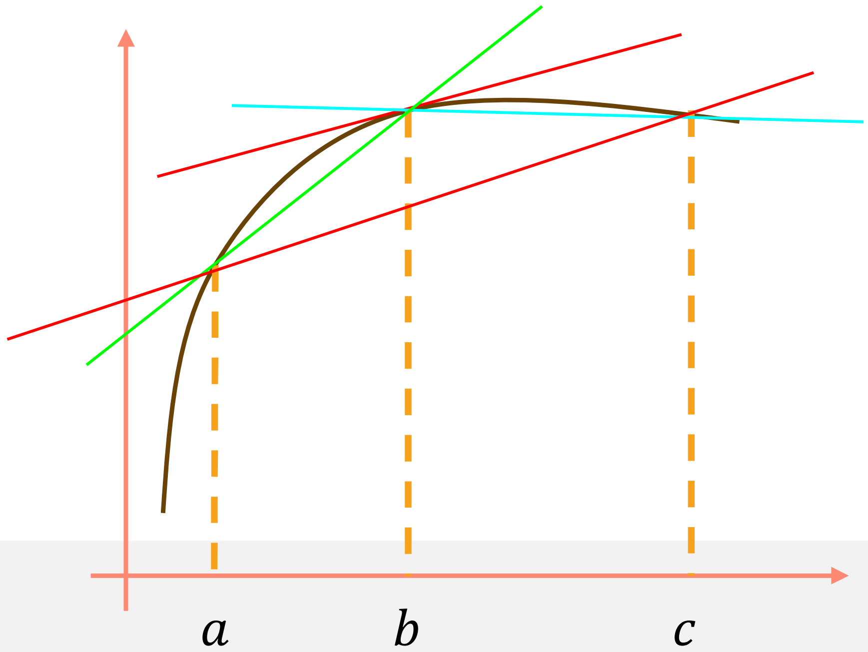
x	$\cdots$	a	$\cdots$	b	$\cdots$	c	$\cdots$
$f(x)$	$\cdots$	$f(a)$	$\cdots$	$f(b)$	$\cdots$	$f(c)$	$\cdots$

For $a<b<c$,

$$f'(b) \approx \frac{f(c)-f(b)}{c-b} \text{ or}$$

$$f'(b) \approx \frac{f(b)-f(a)}{b-a} \text{ or}$$

$$f'(b) \approx \frac{f(c)-f(a)}{c-a}.$$



The temperature of the water in a coffee cup is a differentiable function F of time t. The table below shows the temperature of coffee in a cup as recorded every 3 minutes over 12minute period.

t	0	3	6	9	12
$F(t)$	205	197	192	186	181

(a) Use data from the table to find an approximation for $F'(6)$?

(b) The rate at which the water temperature decrease for $0 \le t \le 12$ is modeled by $F(t) = 120 + 85e^{-0.03t}$ degrees per minute. Find $F'(6)$ using the given model.