➢ The substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int \underline{f(g(x))} \cdot \underline{g'(x)\,dx} = \int f(u)du$$

$\frac{du}{dx} = g'(x)$ $\therefore du = g'(x)dx$ $\therefore$ $f(u)$ du

If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

Definite Integral :

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$x=a \longrightarrow u=g(a)$

Example 1. $\int \sin x \cos x\, dx$

$= \int \sin x \, du \Leftarrow$ $\boxed{\begin{matrix} u=\sin x \\ du=\cos x\, dx \end{matrix}}$

$= \int u\, du$

$= \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$

Example 2. $\int_0^{\frac{\pi}{2}} \sin x \cos x\, dx$

$x=0 \rightarrow u=\sin 0 = 0$

$x=\frac{\pi}{2} \rightarrow u=1$

$\therefore I = \int_0^1 u\, du = \left[\frac{1}{2}u^2\right]_0^1 = \frac{1}{2}$

Q1. $\int \cos(5\theta - 3)\, d\theta$ $= \int \cos u \cdot \frac{1}{5} du = \frac{1}{5} \cdot \sin u + C$

$u = 5\theta - 3$
$du = 5\, d\theta$

$= \frac{1}{5}\sin(5\theta - 3) + C$

Q2. $\int \frac{x}{\sqrt{1-x^2}}\, dx =$ $\int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\right) du = -u^{\frac{1}{2}} + C$

$u = 1 - x^2$
$du = -2x\, dx$

$= -\sqrt{1-x^2} + C$

Q3. $\int_0^{\frac{\pi}{2}} \frac{3\cos x}{\sqrt{1+3\sin x}}\, dx =$ $\int_1^4 \frac{1}{\sqrt{u}}\, du = \int_1^4 u^{-\frac{1}{2}} du = \left[2u^{\frac{1}{2}}\right]_1^4$

$u = 1 + 3\sin x$
$du = 3\cos x\, dx$

$= 4 - 2 = 2$

$x=0 \rightarrow u=1$

$x=\frac{\pi}{2} \rightarrow u=4$

Q4. If $\int_{-1}^{3} f(x+k)\,dx = 8$, where k is a constant, then $\int_{k-1}^{k+3} f(x)\,dx =$ $\underline{\quad 8 \quad}$

$u = x+k$

$x=-1 \rightarrow u = -1+k$

$x = 3 \rightarrow u = 3+k$

$\therefore I = \int_{k-1}^{k+3} f(u)\,du = 8$

Q5. $\int_0^{\frac{\pi}{4}} \frac{\tan x}{\sqrt{\sec x}} dx = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec x}{\sec x^{\frac{3}{2}}} dx = \int_1^{\sqrt{2}} u^{-\frac{3}{2}} du = \left[-2u^{-\frac{1}{2}}\right]_1^{\sqrt{2}}$

$u = \sec x$

$du = \tan x \sec x\, dx$

$x = 0 \rightarrow u = 1$

$x = \frac{\pi}{4} \rightarrow u = \sqrt{2}$

$= -2\cdot(2^{-\frac{1}{4}} - 1)$

$= 2 - 2^{\frac{3}{4}}$

Q6. $\int_e^{e^2} \frac{(\ln x)^2}{x} dx = \int_1^2 u^2\,du = \left[\frac{1}{3}u^3\right]_1^2 = \frac{1}{3}\cdot(7) = \frac{7}{3}$

$u = \ln x$

$du = \frac{1}{x}dx$

$x = e \rightarrow u = 1$

$x = e^2 \rightarrow u = 2$

Q7. $\int_0^{\frac{\pi}{4}} (e^{\tan x} + 2)\sec^2 x\,dx = \int_0^1 (e^u + 2)\,du = \left[e^u + 2u\right]_0^1$

$u = \tan x$

$du = \sec^2 x\,dx$

$x = 0 \rightarrow u = 0$

$x = \frac{\pi}{4} \rightarrow u = 1$

$= e + 2 - 1$

$= e + 1$

Q8. $\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x\,dx = \int_0^1 e^u\,du = \left[e^u\right]_0^1 = e - 1$

$u = \sin x$

$du = \cos x\,dx$

$x = 0 \rightarrow u = 0$

$x = \frac{\pi}{2} \rightarrow u = 1$

Basic Rules

0. Expand

$(1+e^x)^2 =$ $1+2e^x+e^{2x}$

➢ **Rational Functions**

$\int \frac{1}{1+x^2} dx =$ $\tan^{-1}x + C$

$\int \frac{x}{1+x^2} dx =$ $\int \frac{\frac{1}{2}}{1+x^2} d(1+x^2) = \frac{1}{2}\ln(1+x^2)+C$ OR $\frac{1}{2}\ln|1+x^2|+C$

Basic Rules

1. Separate numerator

$\int \frac{1+x}{1+x^2} dx =$ $\int \frac{1}{1+x^2}dx + \int \frac{x}{1+x^2}dx = \tan^{-1}x + \frac{1}{2}\ln(1+x^2)+C$

✧ **If the greatest power of the numerator is larger than or equal to that of the denominator:**

2. Divide improper fractions

$\frac{x^2+1}{x^2-1} =$ $\frac{x^2-1+2}{x^2-1} = 1+\frac{2}{x^2-1}$

$\frac{x^3-3x}{x^2-1} =$ $\frac{(x^2-1)x-2x}{x^2-1} = x - \frac{2x}{x^2-1}$

$I = \int \frac{x^3-3x}{x^2-1}dx = \int x - \frac{2x}{x^2-1}dx$

$= \frac{1}{2}x^2 - \int \frac{d(x^2-1)}{x^2-1}$

$= \frac{1}{2}x^2 - \ln|x^2-1|+C$

3. Add and subtract terms in numerator ~ Aim to construct the derivative of the denominator

$\frac{2x}{x^2+2x+1} =$ $\frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}$

$I = \int \frac{1}{x^2+2x+1} d(x^2+2x+1) - 2\int (x+1)^{-2}\, d(x+1)$

$= \ln|x^2+2x+1| + 2(x+1)^{-1} + C$

4. Complete the square

【Review】

$\int \frac{1}{1+x^2} dx =$ $\tan^{-1}x + C$

$\int \frac{1}{\sqrt{1-x^2}} dx =$ $\sin^{-1}x + C$

$\int \frac{1}{x\sqrt{x^2-1}} dx =$ $\sec^{-1}x + C$

Q1. $\int \frac{1}{x^2-2x+2} dx =$ $\int \frac{1}{(x-1)^2+1} d(x-1) = \tan^{-1}(x-1)+C$

Q2. $\int \frac{1}{4+x^2} dx = \int \frac{1}{4\left[1+\left(\frac{x}{2}\right)^2\right]} dx = \frac{1}{4} \cdot \int \frac{1}{1+\left(\frac{x}{2}\right)^2} 2\, d\left(\frac{x}{2}\right)$

$= \frac{1}{2} \cdot \tan^{-1}\left(\frac{x}{2}\right) + C$

Q3. $\int \frac{1}{\sqrt{9-x^2}} dx = \int \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}} 3\, d\left(\frac{x}{3}\right) = \int \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} d\left(\frac{x}{3}\right) = \sin^{-1}\left(\frac{x}{3}\right) + C$

Q4. $\int \frac{1}{x\sqrt{x^2-9}} dx = \int \frac{1}{\frac{x}{3}\cdot 3 \sqrt{9\left[\left(\frac{x}{3}\right)^2-1\right]}} dx = \int \frac{1}{9} \cdot \frac{1}{\left(\frac{x}{3}\right)\sqrt{\left(\frac{x}{3}\right)^2-1}} 3\, d\left(\frac{x}{3}\right)$

$= \frac{1}{3} \cdot \sec^{-1}\left(\frac{x}{3}\right) + C$

Summary:

$\int \frac{1}{a^2+x^2} dx = \int \frac{1}{a^2} \cdot \frac{1}{1+\left(\frac{x}{a}\right)^2} a\, d\left(\frac{x}{a}\right) = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

$\int \frac{1}{\sqrt{a^2-x^2}} dx = \int \frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}} a\, d\left(\frac{x}{a}\right) = \sin^{-1}\left(\frac{x}{a}\right) + C$

$\int \frac{1}{x\sqrt{x^2-a^2}} dx = \int \frac{1}{\frac{x}{a}\cdot a \cdot a\sqrt{\left(\frac{x}{a}\right)^2-1}} a\, d\left(\frac{x}{a}\right) = \frac{1}{a} \cdot \int \frac{1}{\left(\frac{x}{a}\right)\cdot\sqrt{\left(\frac{x}{a}\right)^2-1}} d\left(\frac{x}{a}\right)$

$= \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + C$

Q5. $\int \frac{1}{\sqrt{-x^2+4x+5}} dx = \int \frac{1}{\sqrt{-(x-2)^2+9}} dx$

$= \int \frac{1}{3\sqrt{1-\left(\frac{x-2}{3}\right)^2}} 3\, d\left(\frac{x-2}{3}\right) = \sin^{-1}\left(\frac{x-2}{3}\right) + C$

Q6. $\int \frac{1}{x^2+4x+8} dx = \int \frac{1}{(x+2)^2+4} dx$

$= \int \frac{1}{4} \cdot \frac{1}{\left(\frac{x+2}{2}\right)^2+1} 2\, d\left(\frac{x+2}{2}\right) = \frac{1}{2} \tan^{-1}\left(\frac{x+2}{2}\right) + C$

➢ **Practice**

1. $\int_2^3 \frac{1}{x^2-4x+5}dx = \int_2^3 \frac{1}{(x-2)^2+1}dx = \int_0^1 \frac{1}{u^2+1}du = [\tan^{-1}u]_0^1 = \frac{\pi}{4}$

$u=x-2$
$u: 0 \to 1$

2. $\int \frac{1}{1-e^x}dx = \int \frac{1-e^x+e^x}{1-e^x}dx = \int 1+\frac{e^x}{1-e^x}dx = x+\int\frac{-1}{1-e^x}d(1-e^x)$

$= x - \ln|1-e^x| + c$

3. $\int \frac{e^{2x}}{1+e^x}dx = \int \frac{(e^x+1)e^x - e^x}{1+e^x}dx = \int e^x dx - \int \frac{1}{1+e^x}d(1+e^x)$

$= e^x - \ln|1+e^x| + c$

4. $\int \frac{1-2x}{1+x^2}dx = \int \frac{1}{1+x^2}dx - \int \frac{1}{1+x^2}d(1+x^2) = \tan^{-1}x - \ln|1+x^2| + c$

5. $\int \frac{2x}{x^2+2x+1}dx = \int \frac{2x+2-2}{x^2+2x+1}dx = \int \frac{d(x^2+2x+1)}{x^2+2x+1} - \int \frac{2}{(x+1)^2}d(x+1)$

$= \ln|x^2+2x+1| + 2(x+1)^{-1} + c$

6. $\int \frac{1+\sin x}{\cos^2 x}dx = \int \sec^2 x + \frac{\sin x}{\cos^2 x}dx$

$= \tan x - \int \frac{1}{\cos^2 x}d(\cos x)$

$= \tan x + (\cos x)^{-1} + c = \tan x + \sec x + c$

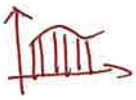
7. $\int \tan x\, dx =$ ① $\int \frac{\sin x}{\cos x} = \int \frac{-1}{\cos x}d(\cos x) = -\ln|\cos x| + c$

② $\int \frac{\tan x \sec x}{\sec x}dx = \int \frac{1}{\sec x}d(\sec x) = \ln|\sec x| + c$

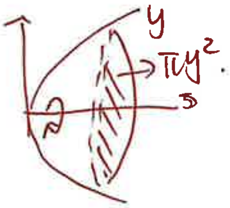
$\int \sec x\, dx \overset{①}{=} \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}dx = \int \frac{d(\sec x+\tan x)}{\sec x+\tan x} = \ln|\sec x+\tan x| + C$

8. (*) The region bounded by $y=\frac{\sin x}{\sqrt{\cos x}}$, $x=0$, $x=\frac{\pi}{4}$, and the x-axis, is revolved around the x-axis. What is the volume of the resulting solid? Hint: $\int \sec x\, dx = \ln|\sec x+\tan x| + C$

Area: "Σ lines"



Volume: "Σ area of slices"



$\text{Volume} = \int_0^{\frac{\pi}{4}} \pi y^2 dx$

$= \int_0^{\frac{\pi}{4}} \pi \frac{\sin^2 x}{\cos x}dx$

$= \pi \cdot \int_0^{\frac{\pi}{4}} \frac{1-\cos^2 x}{\cos x}dx = \pi \int_0^{\frac{\pi}{4}} \sec x - \cos x\, dx$

$\Uparrow$

$= \left(\ln(\sqrt{2}+1) - \frac{\sqrt{2}}{2}\right)\pi$

- **Trigonometric Integrals**

$\sin^2 x + \cos^2 x = 1$

$\int \sin^m x \cos^n x \, dx$

① m or n is odd: save one "sin" or "cos" to construct $d(\cos x)$ or $d(\sin x)$

② m and n are even: $\sin^2 x = \frac{1-\cos 2x}{2}$ $\quad \cos^2 x = \frac{1+\cos 2x}{2}$

1. m is odd

$\int \sin^3 x \cos^2 x \, dx = \int \underline{\sin x} \cdot \sin^2 x \cos^2 x \, \underline{dx}$ ⇒ save one sine to construct $d(\cos x)$.

$= \int \sin^2 x \quad u^2 \quad (-1)\, du$ $\quad u = \cos x \quad du = -\sin x \, dx$

Then use $\boxed{\sin^2 x = 1-\cos^2 x}$ to trans all $\sin x$ to $\cos x$. Then you will get an integral of a polynomial function which always has antiderivative.

$= -\int (1-u^2) u^2 \, du$

$= \int -u^2 + u^4 \, du = -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \boxed{-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C}$

2. n is odd similar to 1.

$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \underline{\cos x \, dx}$ $\quad u = \sin x$

$= \int u^2 (1-u^2) \quad du$ $\quad du = \cos x \, dx$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

降幂： $\sin^2 x = \frac{1-\cos 2x}{2}$ $\qquad \cos^2 x = \frac{1+\cos 2x}{2}$

3. m & n are even ⇒ Use the formula above to get the lower power. Then Use 1.2.

$\int \sin^4 x \, dx = \int (\sin^2 x)^2 dx$

$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \int \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$

$= \frac{1}{4} \cdot \int \frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2} \, dx$

$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right) + C$

$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

$\int \sin^2 x \cos^2 x \, dx = \int \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2} \, dx$

$= \int \frac{1-\cos^2 2x}{4} \, dx$

$= \int \frac{1}{4} - \frac{1}{4}\frac{1+\cos 4x}{2} \, dx = \frac{1}{4}\int \frac{1}{2} - \frac{\cos 4x}{2} \, dx = \frac{1}{4}\left(\frac{x}{2} - \frac{\sin 4x}{8}\right) + C$

$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$

$\int \tan^m x \sec^n x \, dx$

$\tan^2 x + 1 =$ $\sec^2 x$ $\qquad$ $\sec^2 x - 1 =$ $\tan^2 x$

$d(\tan x) = \sec^2 x \, dx$

$d(\sec x) = \sec x \tan x \, dx$

1. **m is odd** ~ Save one "$\sec x \tan x$" to construct "$d(\sec x)$". Then use $\tan^2 x = \sec^2 x - 1$ to transfer all $\tan x$ to $\sec x$. → Power functions always have corresponding antiderivatives.

$\int \tan^3 x \sec^2 x \, dx = \int \tan x \cdot \tan^2 x \sec x \sec x \, dx$ $\qquad u = \sec x \qquad du = \sec x \tan x \, dx$

$= \int (u^2 - 1) \, u \, du$

$= \int u^3 - u \, du$

$= \frac{1}{4}\sec^4 x - \frac{1}{2}\sec^2 x + C$

$\int \tan^3 2x \sec^2 2x \, dx = \int \tan 2x \cdot \tan^2 2x \sec 2x \sec 2x \, dx$ $\qquad u = \sec 2x$

$du = 2\sec 2x \tan 2x \, dx$

$= \int (u^2 - 1) \quad u \quad \frac{1}{2} du$

$= \frac{1}{2}\left(\frac{1}{4}u^4 - \frac{1}{2}u^2\right) + C$

method 2. $\qquad = \frac{1}{8}\sec^4 2x - \frac{1}{4}\sec^2 2x + C$

$I = \int \tan^3 x \, d(\tan x)$ $\qquad u = \tan x$

$= \frac{1}{4}\tan^4 x + C$

⇓

2. **n is even** $\qquad d(\tan x) = \sec^2 x \, dx$

$\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x \sec^2 x \sec^2 x \, dx$ $\qquad u = \tan x \qquad \sec^2 x \, dx = du$

$= \int u^2 \cdot (u^2 + 1) \, du$

$= \frac{1}{5}u^5 + \frac{1}{3}u^3 + C$

$= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$

3. **m** is even, n=0 $\tan^2 x = \sec^2 x - 1 \Rightarrow$ Use "2."

$\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx = \int \sec^2 x\,dx - x$

$= \int d(\tan x) - x$

$= \tan x - x + c$

$\int \tan^4 x\,dx = \int (\sec^2 x - 1)^2\,dx = \int \sec^2 x \cdot \sec^2 x\,dx - 2\int \sec^2 x\,dx + \int 1\,dx$

$= \int (\tan^2 x + 1)\,d(\tan x) - 2\cdot(\tan x) + x + c$

$= \frac{1}{3}\tan^3 x + \tan x - 2\tan x + x + c$

$= \frac{1}{3}\tan^3 x - \tan x + x + c$

$\int \tan^6 x\,dx = \int (\sec^2 x - 1)^3\,dx = \int \sec^4 x \sec^2 x\,dx - 3\int \sec^2 x \sec^2 x\,dx + 3\int \sec^2 x\,dx - \int dx$

$= \int (\tan^2 x + 1)^2\,d(\tan x) - 3\int (\tan^2 x + 1)\,d(\tan x) + 3\tan x - x + c$

$= \int \tan^4 x + 2\tan^2 x + 1\,d(\tan x) - 3(\frac{1}{3}\tan^3 x + \tan x) + 3\tan x - x + c$

$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x - \tan^3 x - 3\tan x + 3\tan x - x + c$

$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + c$

➢ **Practice**

1. $\int \sin^3 nx\,dx =$

$\int \sin^2 nx \sin nx\,dx$

$= \int (1 - \cos^2 nx)\cdot(-\frac{1}{n})\,d(\cos nx)$

$= (-\frac{1}{n})\cdot(\cos nx - \frac{1}{3}\cos^3 nx) + c = -\frac{1}{n}\cos nx + \frac{1}{3n}\cos^3 nx + c$

2. $\int \sin^2 nx\,dx = \int \frac{1-\cos(2nx)}{2}\,dx = \frac{x}{2} - \frac{1}{4n}\sin(2nx) + C$

3. $\int \cos^3 x\sqrt{\sin x}\,dx = \int \cos^2 x \sqrt{\sin x}\,\cos x\,dx$ $\Leftarrow$ $\boxed{\begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array}}$

$= \int (1-u^2)u^{\frac{1}{2}}\,du$

$= \int u^{\frac{1}{2}} - u^{\frac{5}{2}}\,du$

$= \frac{2}{3}u^{\frac{3}{2}} - \frac{2}{7}u^{\frac{7}{2}} + c = \frac{2}{3}(\sin x)^{\frac{3}{2}} - \frac{2}{7}(\sin x)^{\frac{7}{2}} + c$

4. $\int \tan^2 x \sec^2 x \, dx = \int \tan^2 x \, d(\tan x)$

$$= \frac{1}{3}\tan^3 x + C$$

5. $\int \tan^5 x \sec^2 x \, dx = \int \tan x \sec x \cdot \tan^4 x \sec x \, dx$

$$= \int (\sec^2 x - 1)^2 \sec x \quad d(\sec x)$$

$$= \int (\sec^4 x - 2\sec^2 x + 1) \sec x \, d(\sec x)$$

$$= \frac{1}{6}\sec^6 x - \frac{1}{2}\sec^4 x + \frac{1}{2}\sec^2 x + C$$

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6. $\int_0^{\frac{\pi}{4}} \tan^2 x \sec^4 x \, dx = \int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \sec^2 x \, dx \Leftarrow$

$$\boxed{\begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \\ x = 0 \rightarrow u = 0 \\ x = \frac{\pi}{4} \rightarrow u = 1 \end{array}}$$

$$= \int_0^1 u^2(u^2+1) \, du$$

$$= \left[\frac{1}{5}u^5 + \frac{1}{3}u^3\right]_0^1$$

$$= \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$$

➢ **Trigon Substitution**

$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1-\cos^2 x \qquad \cos^2 x = 1 - \sin^2 x$

$\tan^2 x + 1 = \sec^2 x \qquad \sec^2 x - 1 = \tan^2 x$

1. $\int \sqrt{9-x^2}\,dx = \int 3\sqrt{1-\left(\frac{x}{3}\right)^2}\,dx = \int 3\,|\cos\theta|\,3\cos\theta\,d\theta = \pm 9\int \frac{1+\cos 2\theta}{2}\,d\theta$

Let $\frac{x}{3} = \sin\theta \Rightarrow d\left(\frac{x}{3}\right) = d(\sin\theta)$

$\therefore \frac{1}{3}dx = \cos\theta\,d\theta$

$\therefore dx = 3\cos\theta\,d\theta$

$= \pm\frac{9}{2}\theta \pm \frac{9}{4}\sin 2\theta + C$

$\frac{x}{3} = \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$

$\Rightarrow \cos\theta = \sqrt{1-\left(\frac{x}{3}\right)^2}$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta$

$= 2\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}$

$\therefore I = \pm\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) \pm \frac{1}{2}x\sqrt{9-x^2} + C$

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2. $\int_0^3 \frac{1}{\sqrt{9+x^2}}\,dx = \int_0^3 \frac{1}{3\sqrt{1+\left(\frac{x}{3}\right)^2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{3\cdot\sec\theta}\,3\sec^2\theta\,d\theta$

$= \left[\ln|\sec\theta+\tan\theta|\right]_0^{\frac{\pi}{4}}$

$= \ln(\sqrt{2}+1)$

let $\frac{x}{3} = \tan\theta$

$d\left(\frac{x}{3}\right) = d(\tan\theta)$

$\frac{1}{3}dx = \sec^2\theta\,d\theta$

$x=0 \to \tan\theta = 0 \to \theta = 0$

$x=3 \to \tan\theta = 1 \to \theta = \frac{\pi}{4}$

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3. $\int_3^6 \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int_3^6 \frac{1}{x^2\,3\sqrt{\left(\frac{x}{3}\right)^2-1}}\,dx = \int_3^6 \frac{1}{9\left(\frac{x}{3}\right)^2\cdot 3\sqrt{\left(\frac{x}{3}\right)^2-1}}\,dx$

Let $\frac{x}{3} = \sec\theta$

$d\left(\frac{x}{3}\right) = d(\sec\theta)$

$\frac{1}{3}dx = \sec\theta\tan\theta\,d\theta$

$x=3 \to \sec\theta = 1 \to \theta = 0$

$x=6 \to \sec\theta = 2 \to \theta = \frac{\pi}{3}$

$= \int_0^{\frac{\pi}{3}} \frac{1}{27\cdot\sec^2\theta\,\tan\theta}\,3\sec\theta\tan\theta\,d\theta$

$= \frac{1}{9}\cdot\int_0^{\frac{\pi}{3}} \cos\theta\,d\theta$

$= \frac{1}{9}\left[\sin\theta\right]_0^{\frac{\pi}{3}}$

$= \frac{\sqrt{3}}{18}$

➢ **Integration by partial fractions**

Rewriting a rational function into the sum of simpler rational functions.

$$\frac{2x+1}{(x+1)(x+2)^2} = \frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2} = \frac{A(x+2)^2+B(x+1)(x+2)+C(x+1)}{(x+1)(x+2)^2}$$

$$= \frac{(A+B)x^2+(4A+3B+C)x+(4A+2B+C)}{(x+1)(x+2)^2}$$

$$\begin{cases} A+B=0 \\ 4A+3B+C=2 \\ 4A+2B+C=1 \end{cases} \quad \therefore \begin{cases} A=-1 \\ B=1 \\ C=3 \end{cases}$$

$$\int \frac{2x+1}{(x+1)(x+2)^2}\,dx = \int \frac{-1}{x+1}\,d(x+1) + \int \frac{1}{x+2}\,d(x+2) + \int \frac{3}{(x+2)^2}\,d(x+2) = -\ln|x+1| + \ln|x+2| - \frac{3}{x+2} + C$$

$$= \ln\left|\frac{x+2}{x+1}\right| - \frac{3}{x+2} + C$$

Practice:

1. $\int \frac{x^3}{x^2-1}\,dx = \cancel{\int x + \frac{x}{x^2-1}\,dx} = \frac{1}{2}x^2 + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$

$$= \frac{(x^2-1)x+x}{x^2-1} \qquad \frac{x}{x^2-1} = \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1} \qquad \therefore \int \frac{x}{x^2-1}\,dx = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

method 2: $= \int \frac{1}{x^2-1}\,\frac{1}{2}\,d(x^2-1)$

$= \frac{1}{2}\ln|x^2-1| + C$

2. $\int \frac{5x+1}{x^2+x-2}\,dx = 2\ln|x-1| + 3\ln|x+2| + C \quad \left(= \ln|(x-1)^2(x+2)^3| + C\right)$

$$= \frac{5x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{(A+B)x+(2A-B)}{(x-1)(x+2)}$$

$$\begin{cases} A+B=5 \\ 2A-B=1 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=3 \end{cases}$$

3. $\int \frac{x+10}{(x-4)(x+3)}\,dx = 2\ln|x-4| - \ln|x+3| + C \quad \left(= \ln\left|\frac{(x-4)^2}{x+3}\right| + C\right)$

$$= \frac{A}{x-4} + \frac{B}{x+3}$$

$$= \frac{(A+B)x+(3A-4B)}{(x-4)(x+3)}$$

$$\begin{cases} A+B=1 \\ 3A-4B=10 \end{cases} \quad \therefore \begin{cases} A=2 \\ B=-1 \end{cases}$$

Extra: $\int \frac{-x+5}{1-x^2}\,dx = \int \frac{2}{1-x} + \frac{3}{1+x}\,dx = \int \frac{2}{1-x}(-1)\,d(1-x) + \int \frac{3}{1+x}\,d(1+x) = -2\ln|1-x| + 3\ln|1+x| + C$

? When should you choose the method of substitution, and when should you use partial fractions?

- **Integration by parts**

$\frac{d[f(x)g(x)]}{dx} = f'(x)g(x) + g'(x)f(x)$

$d[f(x)g(x)] = [f'(x)g(x) + g'(x)f(x)]\,dx$

$\int d[f(x)g(x)] = \int f'(x)g(x)\,dx + \int g'(x)f(x)\,dx$

$f(x)g(x) = \int g(x)\,d(f(x)) + \int f(x)\,d(g(x))$

$\overset{u=f(x)}{\underset{V=g(x)}{\Rightarrow}}$ $f(x)g(x) = u\cdot V$

$\int g(x)\,d(f(x)) = \int v\,du$

$\int f(x)\,d(g(x)) = \int u\,dv$

$\Rightarrow$ **Formula:** $\int u\,dv = \underline{uv - \int v\,du}$

- **Guidelines for Integration by parts**

After finishing all the questions below, answer: What are the methods for choosing u and dv when integrating by parts?

① u求导后会更简单 ⇒ 降幂

② $\int v\,du$ 可求. 一般$\int dv$不会升幂（不一定）

1. $\int x^2 e^{ax} dx=$

$u = x^2 \quad dv = e^{ax}dx$

$du = 2x\,dx \quad V = \frac{1}{a}e^{ax}$

易忘

$\therefore I = x^2\cdot\frac{1}{a}e^{ax} - \boxed{\int \frac{1}{a}e^{ax}\,2x\,dx}$

$\boxed{\frac{2}{a}\int x e^{ax}\,dx} = \frac{2}{a}\left[x\cdot\frac{1}{a}e^{ax} - \int\frac{1}{a}e^{ax}\,dx\right]$

$\boxed{\begin{array}{ll} u = x & dv = e^{ax}dx \\ du = dx & V = \frac{1}{a}e^{ax} \end{array}}$

$= \frac{2}{a}\cdot x\cdot\frac{1}{a}e^{ax} - \frac{2}{a^3}e^{ax} + C$

$\therefore I = \left(\frac{1}{a}x^2 - \frac{2}{a^2}x + \frac{2}{a^3}\right)e^{ax} + C$

u	dv
x^2	e^{ax}
$2x$	$\frac{1}{a}e^{ax}$
2	$\frac{1}{a^2}e^{ax}$
0	$\frac{1}{a^3}e^{ax}$

$x^2\frac{1}{a}e^{ax}$

$-2x\frac{1}{a^2}e^{ax}$

$+\frac{2}{a^3}e^{ax}$

$+C$

2. $\int x^2 \sin 2x\, dx=$

$u = x^2 \quad dV = \sin 2x\,dx$ （可不写dv，因为公式 $uv - \int v\,du$ 中，无"dv"）

$du = 2x\,dx \quad V = -\frac{1}{2}\cos 2x$

$\Rightarrow$ （把u遮住，就是V）$\Rightarrow \int \underset{u}{\underset{\shortparallel}{x^2}} \sin 2x\,dx$

$= \int \sin 2x\,dx = V.$

$(\ln x)' = \frac{1}{x}$ 降幂 & $\ln x$ 反导数未知 $\therefore u = \ln x$

3. $\int x^3 \ln x \, dx =$

$u = \ln x \quad v = \int x^3 dx$

$du = \frac{1}{x} dx \quad v = \frac{1}{4}x^4$

$I = \ln x \, \frac{1}{4}x^4 - \int \frac{1}{4}x^4 \, \frac{1}{x} dx$

$= \frac{x^4}{4} \ln x - \frac{1}{16}x^4 + C$

$\sin^{-1}x$ 反导未知 $\Rightarrow u = \sin^{-1}x \quad (\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$ 降幂)

4. $\int x \sin^{-1} x \, dx =$

$u = \sin^{-1}x \quad v = \int x \, dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \quad v = \frac{1}{2}x^2$

$I = \sin^{-1}x \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot \frac{1}{\sqrt{1-x^2}} dx$

$= \frac{x^2}{2} \sin^{-1}x - \frac{1}{2} \underbrace{\boxed{\int \frac{x^2}{\sqrt{1-x^2}} dx}}_{= I_1} \Rightarrow$

$\sin\theta = x$

$d(\sin\theta) = dx$

$\cos\theta \, d\theta = dx$

$\therefore I_1 = \int \frac{\sin^2\theta}{|\cos\theta|} \cos\theta \, d\theta$

$= \pm \int \frac{1-\cos 2\theta}{2} d\theta$

$= \pm \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} + c \right)$

$= \frac{x^2}{2} \sin^{-1}x \mp \frac{\theta}{4} \pm \frac{\sin 2\theta}{8} + C$

$= \frac{x^2}{2} \sin^{-1}x \mp \frac{\sin^{-1}x}{4} \pm \frac{x\sqrt{1-x^2}}{4} + C$

$\tan^{-1}x$ 反导未知. $(\tan^{-1}x)' = \frac{1}{1+x^2}$ 降幂. $\therefore u = \tan^{-1}x$.

5. $\int \tan^{-1} x \, dx =$

$u = \tan^{-1}x \quad v = \int dx$

$du = \frac{1}{1+x^2} dx \quad v = x$

$I = \tan^{-1}x \cdot x - \int x \cdot \frac{1}{1+x^2} dx$

$= \tan^{-1}x \cdot x - \int \frac{\frac{1}{2}}{1+x^2} d(1+x^2)$

$= \tan^{-1}x \cdot x - \frac{1}{2}\ln(1+x^2) + C$

构造循环. 原则：u 的类型不要变!! ☆

6. $\int e^x \sin x \, dx=$

$u=\sin x \quad v=e^x$

$du=\cos x\,dx$

$I= \sin x\, e^x - \boxed{\int e^x \cos x\,dx} = \cos x \cdot e^x - \int e^x \cdot (-\sin x)\,dx$

$= \cos x\, e^x + I$

$u=\cos x \quad v=e^x$

$du=-\sin x\,dx$

$=\sin x\, e^x - \cos x\, e^x - I$

$\therefore I = \frac{\sin x - \cos x}{2} \cdot e^x$

$u=e^x \quad dv=\sin x\,dx$

$du=e^x dx \quad v=-\cos x$

$u=e^x \quad v=\sin x$

$du=e^x dx$

$I=e^x\cdot(-\cos x) - \int(-\cos x)\cdot e^x dx = -e^x\cos x + \int \cos x\, e^x dx = e^x\sin x - \int \sin x\, e^x dx$

$\therefore I = e^x(\sin x - \cos x) - I \qquad \therefore I = \frac{1}{2}e^x(\sin x - \cos x)$

Practice:

1. $\int x \sin x \, dx$

$u=x \quad v=\cos x\cdot(-1)\,dx$

$du=dx$

$I=-x\cdot\cos x - \int -\cos x\,dx$

$=-x\cos x + \sin x + C$

2. $\int x\tan^{-1}x\,dx$

$u=\tan^{-1}x$

$du=\frac{1}{1+x^2}dx \qquad V=\frac{1}{2}x^2$

$I=\tan^{-1}x\cdot\frac{1}{2}x^2-\int\frac{1}{2}x^2\cdot\frac{1}{1+x^2}dx$

$=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx \quad =\int\frac{x^2+1-1}{1+x^2}dx=\int 1-\frac{1}{1+x^2}dx$

$=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+C \qquad = x-\tan^{-1}x+C$

3. $\int e^x\cos x\,dx=$

$u=e^x \qquad V=\sin x$

$du=e^x dx$

$I=e^x\sin x-\underline{\underline{\int\sin x\,e^x dx}} \qquad u=e^x \quad V=\cos x\,(-1)$ $\quad du=e^x dx$

$=e^x\cdot(-\cos x)-\int(-\cos x)\,e^x dx$

$=-e^x\cos x+\int\cos x\,e^x dx \quad +C$

$=e^x(\sin x+\cos x)-I+C$

$\therefore I=\frac{e^x}{2}(\sin x+\cos x)+C$

4. $\int x^2\sin(2x^3)\,dx$

$=\int\sin(2x^3)\;\frac{1}{6}\;d(2x^3)$

$=\frac{1}{6}\cdot(-\cos(2x^3))+C$

$=-\frac{1}{6}\cos(2x^3)+C$

➢ Improper Integrals

✧ Improper Integrals with Infinite Integration Limits

1. If $f(x)$ is continuous on $[a,\infty)$, then $\int_a^{\infty} f(x)\,dx = \lim\limits_{t\to\infty} \int_a^t f(x)\,dx$

2. If $f(x)$ is continuous on $(-\infty,b]$, then $\int_{-\infty}^{b} f(x)\,dx = \lim\limits_{t\to-\infty} \int_t^b f(x)\,dx$

3. If $f(x)$ is continuous on R, then $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)dx + \int_a^{+\infty} f(x)\,dx$

$$= \lim_{t\to-\infty} \int_t^a f(x)dx + \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

Example.

1. $\int_0^{\infty} xe^{-x^2}\,dx = \lim\limits_{t\to\infty} \boxed{\int_0^t x e^{-x^2} dx}$

$$= \int_0^t e^{-x^2} (-\tfrac{1}{2}) d(-x^2)$$

$$= -\frac{1}{2}\cdot e^{-x^2}\Big|_0^t$$

$$= -\frac{1}{2}(e^{-t^2}-1)$$

$$I = \lim_{t\to\infty} (-\tfrac{1}{2})\cdot(e^{-t^2}-1) = \frac{1}{2}$$

思.考: $\int_1^{\infty} \frac{1}{x}\,dx = \infty$

$\int_1^{\infty} \frac{1}{x^2}\,dx = -x^{-1}\Big|_1^{\infty} = 1$

$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = 2x^{\frac{1}{2}}\Big|_1^{\infty} = \infty$

什么时候是∞，什么时候constant

"$\sum f(c_i)\,\Delta x_i$"

$f \downarrow 0$ 更快 还是 $x\uparrow$ 更快

$f: x^{-1}, x^{-0.9} \ldots$
$x^k \ (k \geq -1)$
$\Rightarrow \infty$

$f: x^k \ (k<-1)$
$\Rightarrow$ constant

2. $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_0^{+\infty} \frac{1}{1+x^2}dx$

$$= \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}dx + \lim_{t\to+\infty} \int_0^t \frac{1}{1+x^2}dx$$

$$= \lim_{t\to-\infty} \left[\tan^{-1}x\right]_t^0 + \lim_{t\to\infty} \left[\tan^{-1}x\right]_0^t$$

$$= 0-(-\tfrac{\pi}{2}) + (\tfrac{\pi}{2}-0)$$

$$= \pi$$

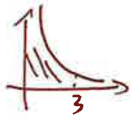
✧ Improper Integrals with **Infinite Discontinuities**

vertical asymptote at $x=b$

1. If $f(x)$ is continuous on $[a,b)$ and has an infinite discontinuity at b, then $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$
2. If $f(x)$ is continuous on $(a,b]$ and has an infinite discontinuity at a, then $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$
3. If $f(x)$ is continuous on $[a,b]$ except some number c in (a,b) at which f has an infinite discontinuity, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx$

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the value of the improper integral. If the limit fails to exists, the improper integral **diverges**.

eg. $\int_0^3 \frac{1}{x} = \lim_{t\to 0^+}\int_t^3 \frac{1}{x}\,dx = \lim_{t\to 0^+}\left[\ln|x|\right]_t^3 = \lim_{t\to 0^+}(\ln 3-\ln t) = +\infty$



Example.

1. $\int_1^5 \frac{dx}{\sqrt{x-1}} = \lim_{t\to 1^+}\int_t^5 \frac{d(x-1)}{\sqrt{x-1}} = \lim_{t\to 1^+}2\left[(x-1)^{\frac{1}{2}}\right]_t^5 = \lim_{t\to 1^+}2\left[2-(t-1)^{\frac{1}{2}}\right]$

   $x\neq 1$

   $= 4$

2. $\int_0^1 \frac{dx}{1-x} = \lim_{t\to 1^-}\int_0^t \frac{-d(1-x)}{1-x} = \lim_{t\to 1^-}\left[-\ln|1-x|\right]_0^t = \lim_{t\to 1^-}(-\ln|1-t|)$

   $x\neq 1$

   $= +\infty$

3. If $\int_0^1 \frac{ke^{-\sqrt{x}}}{\sqrt{x}}\,dx = 1$, what is the value of k?

   $\neq 0$

$= \lim_{t\to 0^+}\int_t^1 ke^{-\sqrt{x}}(-2)\,d(-\sqrt{x}) = \lim_{t\to 0^+} -2k\left[e^{-\sqrt{x}}\right]_t^1 = \lim_{t\to 0^+}\left(-2ke^{-1}+2ke^{-\sqrt{t}}\right)$

$= \frac{-2k}{e}+2k = 1$

$\therefore k = \frac{e}{2(e-1)}$