➢ Polar Coordinates and Slopes of Curves

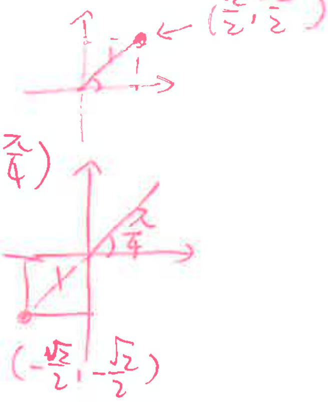
✧ If a point P has rectangular coordinates (x, y) and polar coordinates (r, θ), then

$(r, \theta) \to (x, y)$: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\qquad (x, y) \to (r, \theta)$: $\begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \frac{y}{x} \end{cases}$

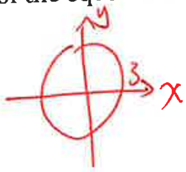
✧ Some polar curves

$(r, \theta) = (1, \frac{\pi}{4})$ $\quad (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

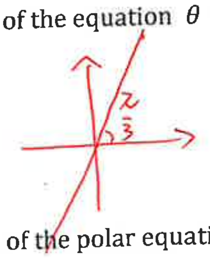
$(r, \theta) = (-1, \frac{\pi}{4})$ $\quad (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$



1. Sketch a graph of the equation $r = 3$



2. Sketch a graph of the equation $\theta = \frac{\pi}{3}$

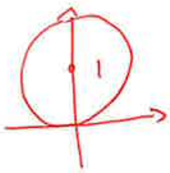


3. Sketch a graph of the polar equation $r = 2\sin\theta$

$r^2 = 2r\sin\theta$

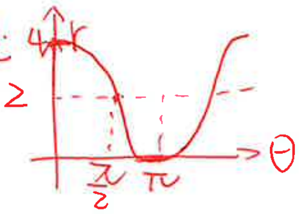
$x^2 + y^2 = 2y$

$x^2 + (y-1)^2 = 1$

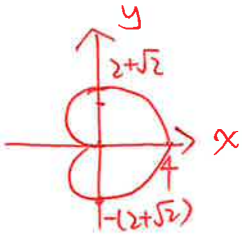


4. Sketch a graph of $r = 2 + 2\cos\theta$

Step 1.



Step 2.



$0 < \theta < \pi$: $r \downarrow$ $4 \to 0$

$\theta = \frac{\pi}{2}$: $r = 2 + \sqrt{2}$

$\pi < \theta < 2\pi$: $r \uparrow$ $0 \to 4$

✧ **Cardioid (heart-shaped):** The graph of any equation of the form $r = a(1 \pm \cos\theta)$ or $r = a(1 \pm \sin\theta)$ is a cardioid.

✧ **Limaçon**

5. Sketch a graph of the equation $r = 1 + 2\cos\theta$

midline

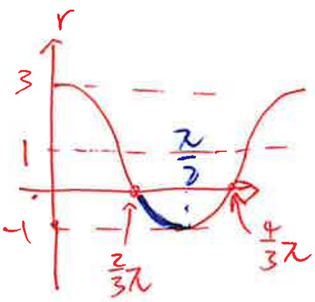
amplitude

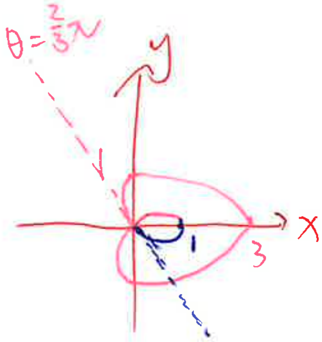
$r = 0$: $1 + 2\cos\theta = 0$

$2\cos\theta = -1$

$\cos\theta = -\frac{1}{2}$

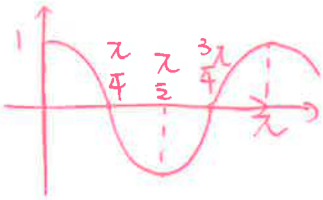
$\theta = \frac{2}{3}\pi, \frac{4}{3}\pi$

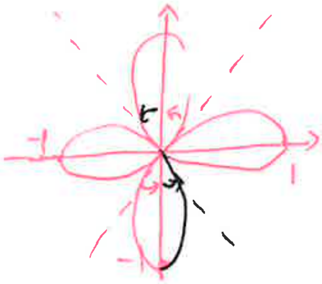




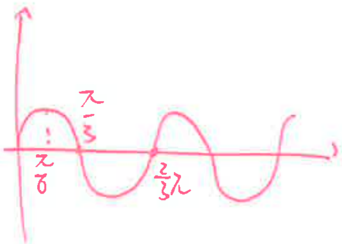
✧ **Roses**

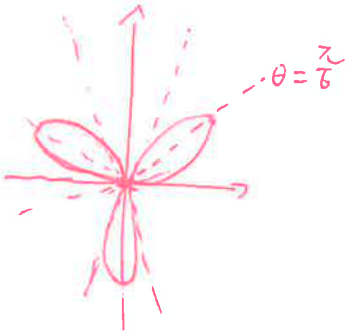
6. Sketch the curve $r = \cos 2\theta$





7. Sketch the curve $r = \sin 3\theta$



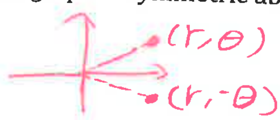


✧ In general, the graph of an equation of the form $r = a\cos n\theta$ is an 2n -leaved rose if n is even.

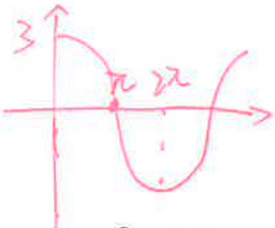
the graph of an equation of the form $r = a\sin n\theta$ is an n -leaved rose if n is odd.

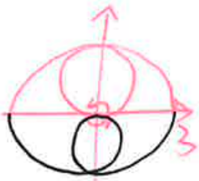
✧ **Symmetry**

- If a polar equation is unchanged when we replace θ by $-\theta$, then the graph is symmetric about the polar axis.



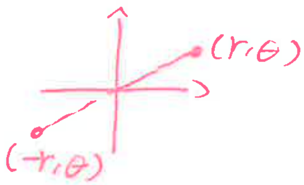
8. Sketch a graph of $r = 3\cos\frac{\theta}{2}$

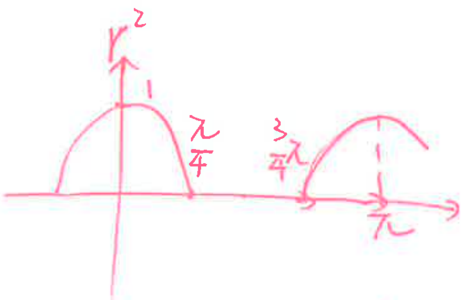


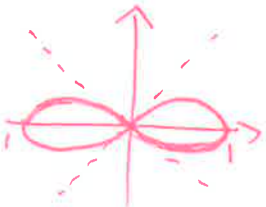


- If the equation is unchanged when we replace r by $-r$, then the graph is symmetric about the pole.

9. Sketch a graph of $r^2 = \cos 2\theta$







✧ The slope of the tangent line to the graph of $r = f(\theta)$ at point (r, θ) is:

$$\frac{dy}{dx} = \frac{\frac{d(r\sin\theta)}{d\theta}}{\frac{d(r\cos\theta)}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cdot\cos\theta}{\frac{dr}{d\theta}\cos\theta + r\cdot(-\sin\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

✧ If $r > 0$ and

(as θ increases)
w.r.t θ

- $\frac{dr}{d\theta} < 0$, then r is increasing/decreasing, which means the curve is getting closer/farther from the origin.
- $\frac{dr}{d\theta} > 0$, then r is <u>increasing</u>, which means the curve is getting <u>farther</u> from the origin.
  w.r.t. θ

Practice.

1. A curve is defined by the polar equation $r = 4\sin(2\theta)$ for $0 \le \theta \le \frac{\pi}{2}$.

   (a) Graph the curve.

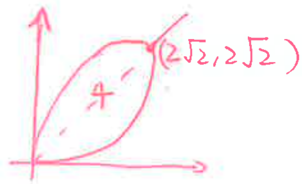
   (b) Find the slope of the curve at the point where $\theta = \pi/4$.

   (c) Find an equation in terms of x and y for the line tangent to the curve at the point where $\theta = \frac{\pi}{4}$.

   (d) Find an interval where the curve is getting closer to the origin.

   (e) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ such that the point on the curve has the greatest distance from the origin.

(a)



(b) $\frac{dy}{dx} = \frac{4\cos(2\theta)\cdot 2\cdot\sin\theta + 4\sin(2\theta)\cdot\cos\theta}{8\cos(2\theta)\cdot\cos\theta - 4\sin(2\theta)\sin\theta}$

$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{4\cdot\frac{\sqrt{2}}{2}}{-4\cdot\frac{\sqrt{2}}{2}} = -1$

(c) $r(\frac{\pi}{4}) = 4$

$\begin{cases} x = 2\sqrt{2} \\ y = 2\sqrt{2} \end{cases}$

$\therefore y - 2\sqrt{2} = -(x - 2\sqrt{2})$

$y = -x + 4\sqrt{2}$

(d) $r'(\theta) = 4\cos(2\theta)\cdot 2 < 0$

$\therefore \frac{\pi}{4} < \theta < \frac{\pi}{2}$

(e) $\theta = \frac{\pi}{4}$

$r'(\theta) = 8\cos 2\theta = 0$

$\therefore \theta = \frac{\pi}{4}$

$r - r\cos\theta = 8$

D

2. (cal) The equation of the polar curve is given by $r = \frac{8}{1-\cos\theta}$. What is the angle θ that corresponds to the point on the curve with x-coordinate -3?

   $r\cos\theta = -3$

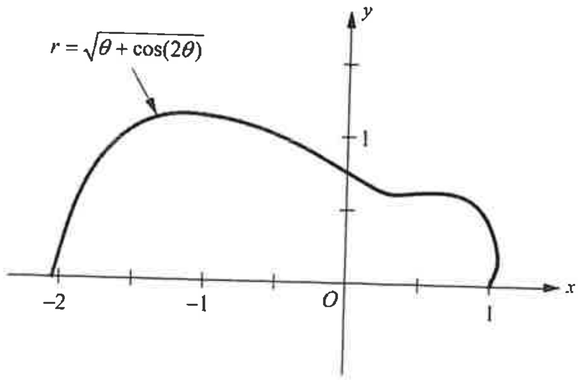
   (A) 1.248 (B) 1.356 (C) 1.596 (D) 2.214

$\therefore r = 8 + (-3) = 5$

$\therefore \cos\theta = -\frac{3}{5}$

cal: $\theta \approx 2.214$

3.



The polar curve $r = \sqrt{\theta + \cos(2\theta)}$, for $0 \le \theta \le \pi$, is drawn in the figure above.

(a) Find $\dfrac{dr}{d\theta}$, the derivative of r with respect to θ.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate 0.5.

(c) For $\dfrac{\pi}{12} < \theta < \dfrac{5\pi}{12}$, $\dfrac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \dfrac{\pi}{2}$ that correspond to the point on the curve in the first quadrant with the least distance from the origin. Justify your answer.

(a) $\dfrac{dr}{d\theta} = \dfrac{1}{2}(\theta + \cos 2\theta)^{-\frac{1}{2}} \cdot (1 + 2(-\sin 2\theta))$

$= \dfrac{1 - 2\sin(2\theta)}{2\sqrt{\theta + \cos(2\theta)}}$

(b) $x = r\cos\theta = 0.5 = \sqrt{\theta + \cos(2\theta)} \cdot \cos\theta = 0.5$

Calculator: $\theta = 0.910$

(c) r is decreasing as θ increases.

The curve is getting closer to the origin.

(d) distance $= r$

θ	$(0, \frac{\pi}{12})$	$\frac{\pi}{12}$	$(\frac{\pi}{12}, \frac{5}{12}\pi)$	$\frac{5}{12}\pi$	$(\frac{5}{12}\pi, \frac{\pi}{2})$
$r'(\theta)$	+	0	−	0	+
$r(\theta)$	↗	max	↘	min	+

$r'(\theta) = 0$: $1 - 2\sin(2\theta) = 0$

numerator of $r'(\theta)$

$\therefore 2\theta = \dfrac{\pi}{6}$, $\theta = \dfrac{\pi}{12}$

OR $2\theta = \dfrac{5}{6}\pi$, $\theta = \dfrac{5}{12}\pi$

$r(0) = 1$

$r(\frac{5}{12}\pi) = \sqrt{\frac{5}{12}\pi + \frac{1}{2}} > r(0)$

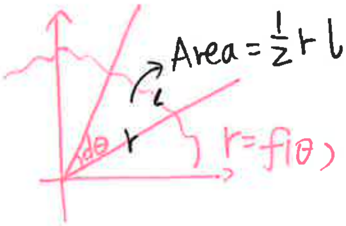
$\therefore$ when $\theta = \frac{5}{12}\pi$, the curve has the least distance from the origin.

Review  $\text{Area} = \frac{1}{2} r l = \frac{1}{2} r \cdot (r\theta) = \frac{1}{2} r^2 \theta$

➢ Area in Polar Coordinates

The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by:

$A = \int_{\alpha}^{\beta} \frac{1}{2} f^2(\theta)\, d\theta$



$\text{Area} = \frac{1}{2} r l = \frac{1}{2} r \cdot r d\theta$

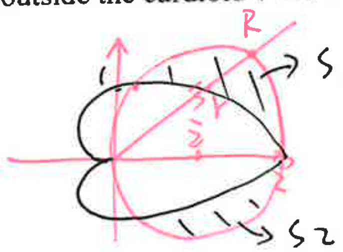
$= \frac{1}{2} (f(\theta))^2 d\theta$

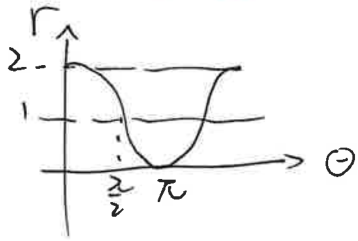
Practice.

1. Find the area of the region that lies inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.

$\to x^2 + y^2 = 3x$

$(x - \frac{3}{2})^2 + y^2 = \frac{3}{2}$





intersection : $r = 3\cos\theta = 1 + \cos\theta$

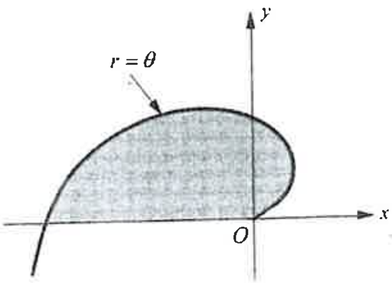
$2\cos\theta = 1, \quad \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

Area $S_1 = \frac{1}{2} \int_0^{\frac{\pi}{3}} (3\cos\theta)^2 - (1+\cos\theta)^2 \, d\theta = \frac{\pi}{2}$

$S = 2S_1 = \pi$

2.



B The area of the shaded region bounded by the polar curve $r = \theta$ and the x-axis is

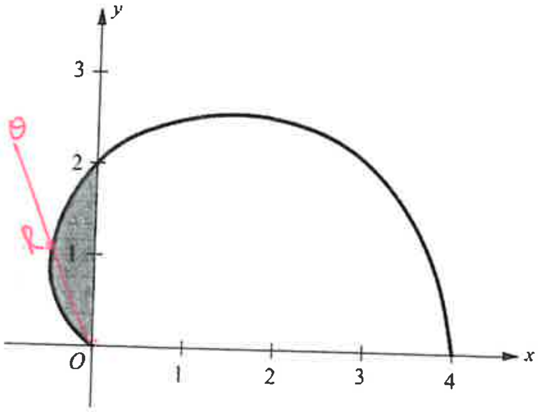
(A) $\frac{\pi^2}{4}$ (B) $\frac{\pi^3}{6}$ (C) $\frac{\pi^3}{3}$ (D) $\frac{\pi^3}{2}$

$\int_0^{\pi} \frac{1}{2} \theta^2 \, d\theta$

$= \frac{1}{2} \cdot \frac{1}{3} [\theta^3]_0^{\pi}$

$= \frac{\pi^3}{6}$

3.



The graph of the polar curve $r = 2 + 2\cos(\theta)$ for $0 \le \theta \le \pi$ is shown above.

(a) Write an integral expression for the area of the shaded region. $\int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(2+2\cos\theta)^2\, d\theta$

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the curve at the point where $\theta = \frac{\pi}{2}$.

(b) $\frac{dx}{d\theta} = \frac{d(r\cos\theta)}{d\theta} = \frac{dr}{d\theta}\cos\theta + r\cdot(-\sin\theta)$

$= 2\cdot(-\sin\theta)\cdot\cos\theta + (-\sin\theta)\cdot(2+2\cos\theta)$

$= -4\sin\theta\cos\theta - 2\sin\theta$

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cdot\cos\theta$

$= -2\sin^2\theta + \cos\theta(2+2\cos\theta)$

$= 2\cos\theta + 2(\cos^2\theta - \sin^2\theta) = 2\cos\theta + 2\cos(2\theta)$

(c) $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{\left.\frac{dy}{d\theta}\right|_{\theta=\frac{\pi}{2}}}{\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{2}}} = 1$

when $\theta = \frac{\pi}{2}$, $r = 2$

$\therefore$ passes $(0, 2)$

$\therefore$ tangent line: $y - 2 = x$

$y = x + 2$