The Definite Integral

- ✓ We know how to calculate velocity from position function. This helped us to understand the idea of the derivative or rate of change of a function.

Now we consider the reverse problem:

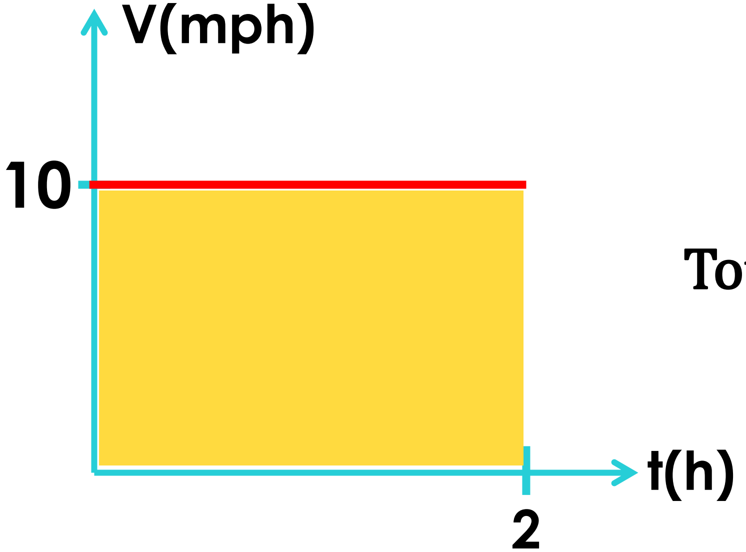
Given velocity, how do we calculate the distance the car has traveled?

This will give us the idea of the definite integral.

How do we measure distance traveled?

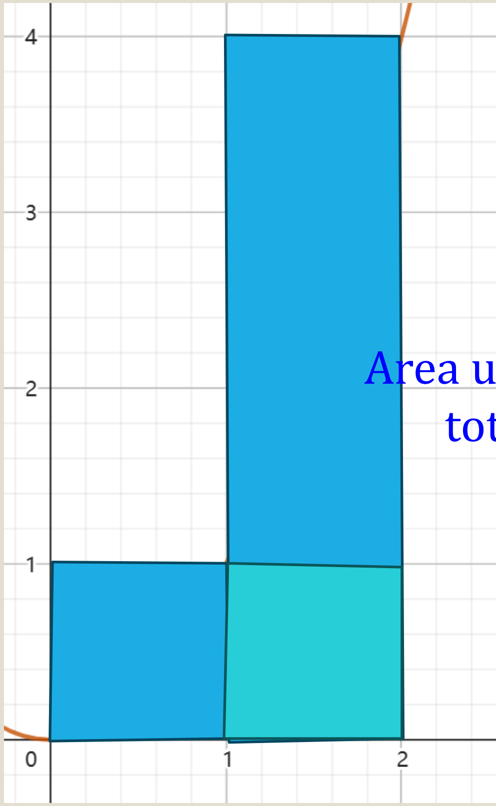
- ✓ With constant velocity we know:

$$Total\ Distance = Velocity\ \times Time$$



Total distance ~ The area under the curve

For changing velocity?



Time Interval	0~1	1~2
Velocity is **at most:**	1	4
Velocity is **at least:**	0	1

Area under the velocity curve:
total distance traveled

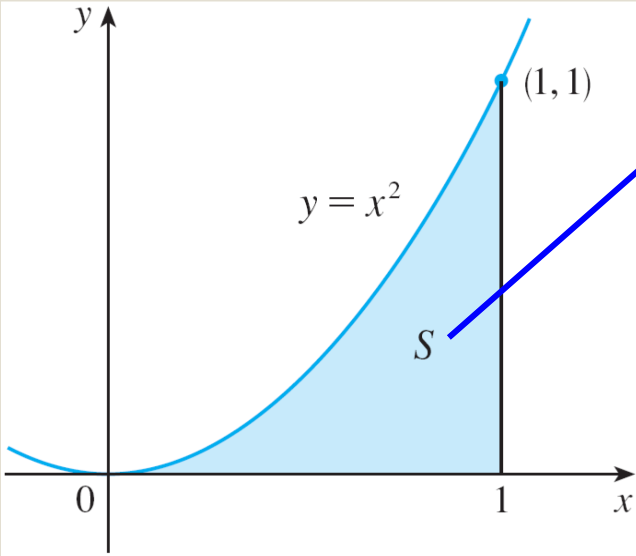
In two seconds, the car has traveled **at most:**
1*1+4*1= **5 ft**

This number is an "overestimate"

In two seconds, the car has traveled **at least:**
0*1 + 1*1 =**1 ft**

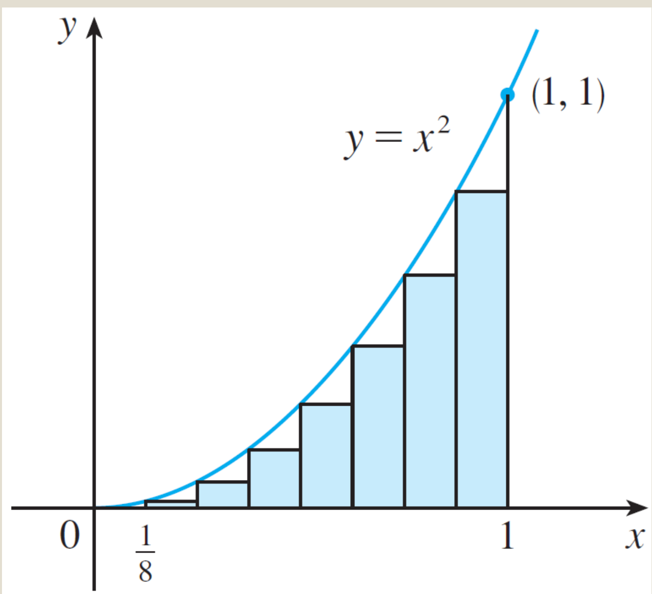
This number is an "underestimate"

t(sec)	0	0.5	1	2
$v(t)$(ft/sec)	0	0.25	1	4

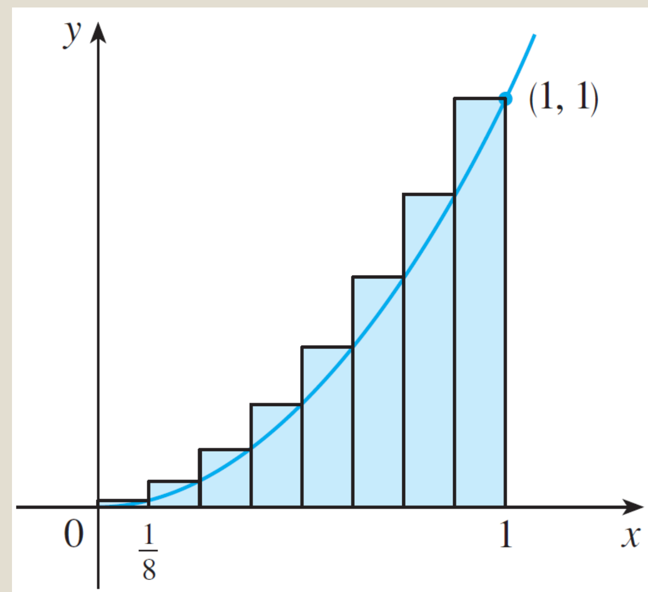


Area S: total distance traveled in the first second

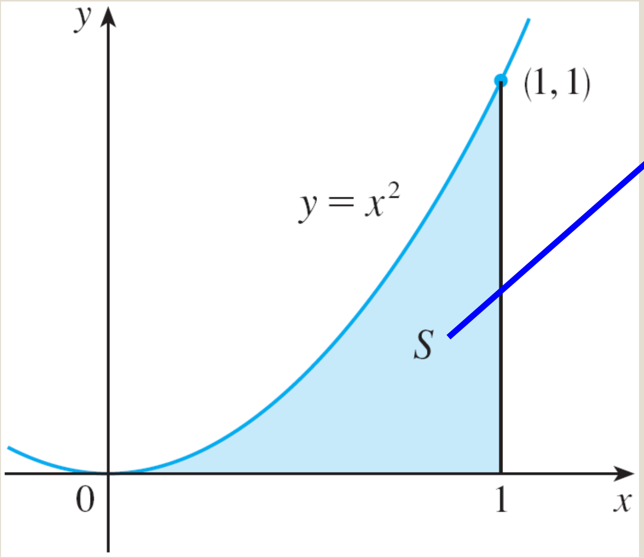
Approximating S with eight rectangles



(a) Using left endpoints



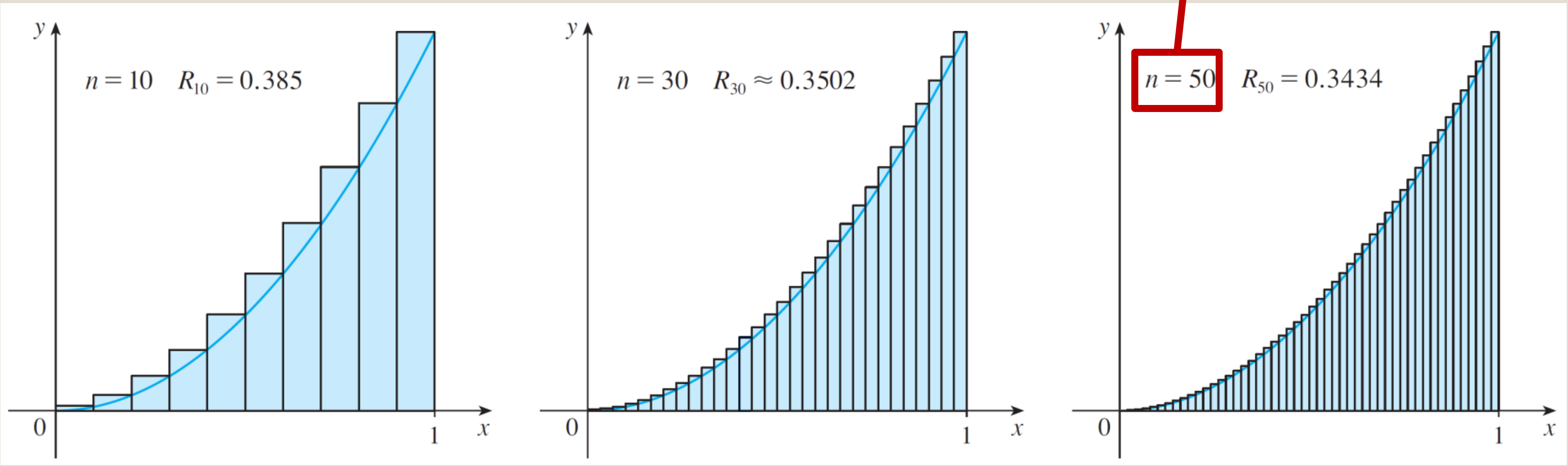
(b) Using right endpoints



Area S: total distance traveled in the first second

$n \to \infty$,

area of rectangles → area under the curve



Riemann Sum

Let f be a continuous function defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by $a = x_0 < x_1 < \cdots < x_n = b$, where Δx_i is the width of the ith interval. If c_i is any point in the ith interval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n$$

is called a **Riemann Sum** for f on the interval $[a, b]$.

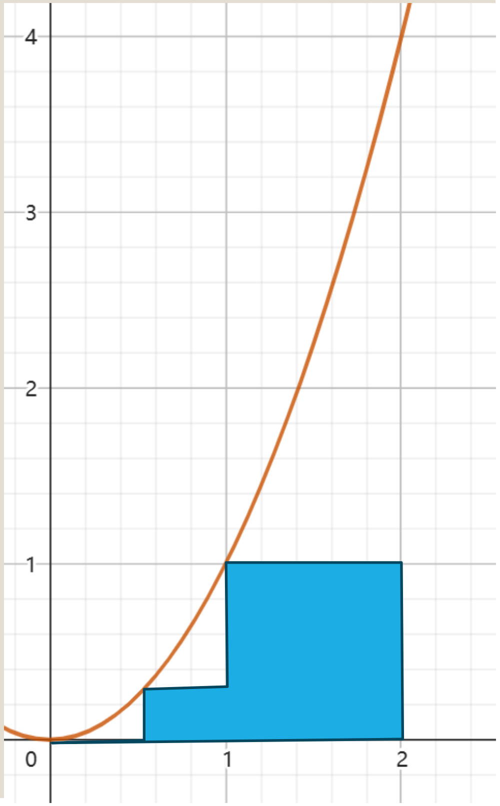
t(sec)	0	0.5	1	2
$v(t)$(ft/sec)	0	0.25	1	4

The Riemann Sum for $f(x) = x^2$ on [0,2] with partition:

$$0 = x_0 < x_1 = 0.5 < \mathrm{x}_2 = 1 < x_3 = 2$$

$$\Delta x_1 = 0.5, \Delta x_2 = 0.5, \Delta x_3 = 1$$

$$0 * 0.5 + 0.25 * 0.5 + 1 * 1 = 1.125$$



Riemann Sum

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$$\sum_{i=1}^{n} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n$$

is called a **Riemann Sum** for f on the interval $[a, b]$. $c_i = a + \ (i - \lambda)\ \Delta x, \lambda \in [0,1]$

- If every subinterval is of equal width, then $\Delta x = \dfrac{b-a}{n}$

- **Left, Right, and Midpoint Riemann Sum**

$c_i = a + \Delta x(i-1)$

If c_i is **the left endpoint** of each subinterval, then $\sum_{i=1}^{n} f(c_i)\Delta x$ is called a **Left Riemann Sum**.

$\boldsymbol{c_i = a + \Delta x * i}$

If c_i is **the right endpoint** of each subinterval, then $\sum_{i=1}^{n} f(c_i)\Delta x$ is called a **Right Riemann Sum**.

$\boldsymbol{c_i = a + \Delta x * (i - 0.5)}$

If c_i is **the midpoint** of each subinterval then $\sum_{i=1}^{n} f(c_i)\Delta x$ is called a **Midpoint Riemann Sum**.

Example 1

Approximate the area of the region bounded by the graph of $f(x) = -x^2 + x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$

(1) by using a left Riemann sum with four subintervals

(2) by using a right Riemann sum with four subintervals

(3) by using a midpoint Riemann sum with four subintervals

Example 2

t (hours)	0	2	4	5	6	9	12
$P'(t)$ people/hour	41	30	54	26	21	44	11

Tiffani posts a picture of her posing with Sir Isaac from the pop group Sir Isaac and the Newtones on her Instagram at 9 AM. The rate that her followers view her picture is modeled with selected values shown in the table above where $t = 0$ represents 9 AM and the rate is measured in people per hour. Use the data in the table above to approximate the following.

(1) Use a Right Riemann Sum with 3 subintervals to approximate the area between $P'(t)$ and the t-axis from $t = 0$ to $t = 5$.

(2) Use a Left Riemann Sum with 4 subintervals to approximate the area between $P'(t)$ and the t-axis from $t = 4$ to $t = 12$.

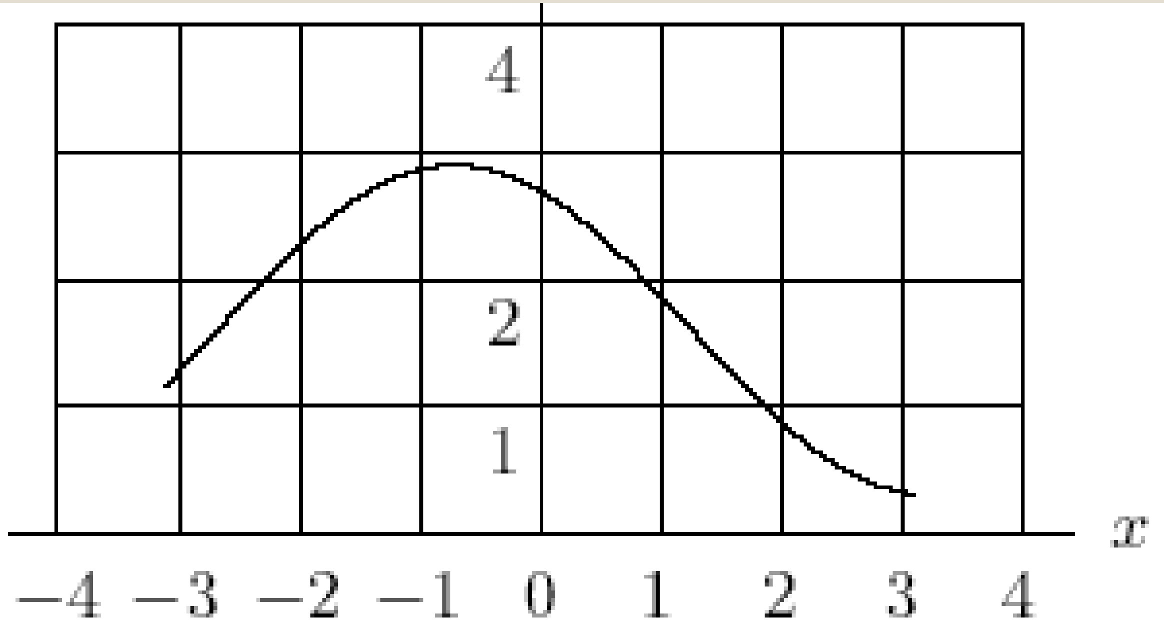
(3) Use a Midpoint Riemann Sum with 3 subintervals to approximate the area between $P'(t)$ and the t-axis from $t = 0$ to $t = 12$.

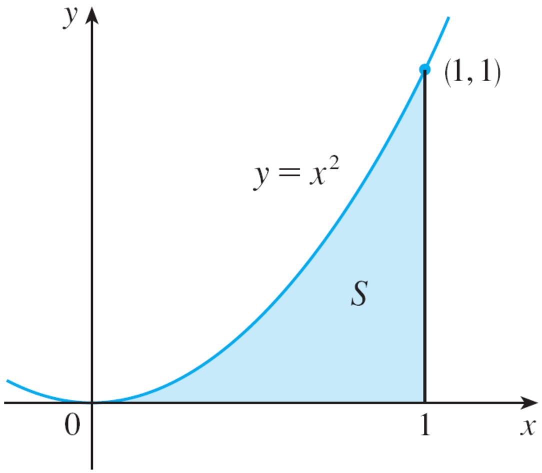
Review

Consider the graph below.

(a) Give an interval on which the left-hand sum approximation of the area under the curve on that interval is an underestimate.

(b) Give an interval on which the left-hand sum approximation of the area under the curve on that interval is an overestimate.

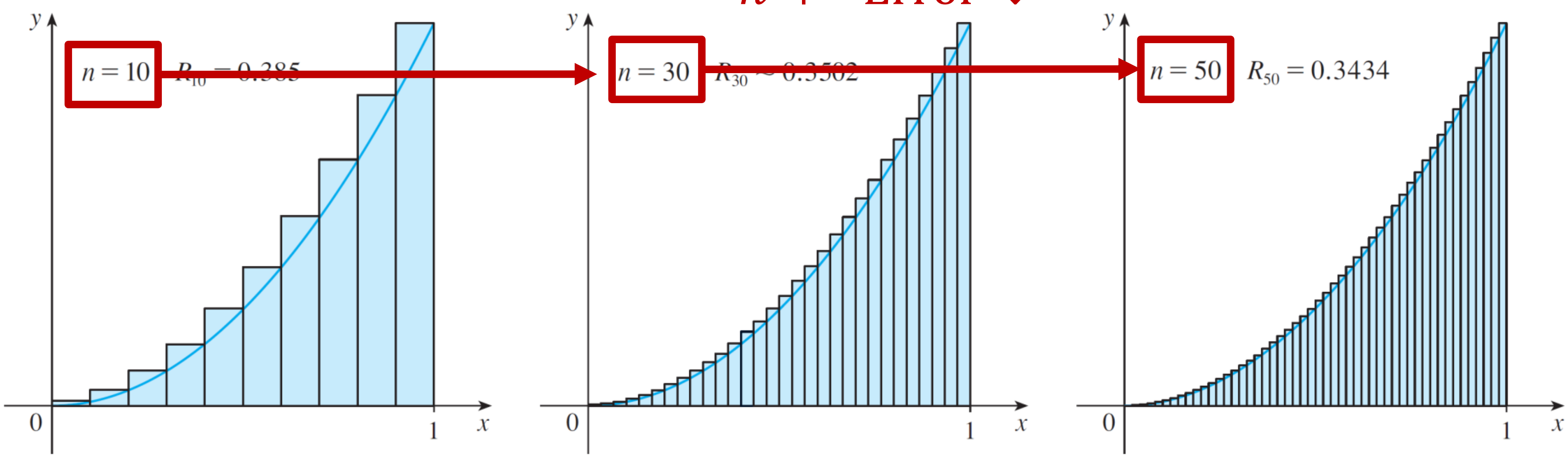




$n \to \infty$,

the sum of areas of rectangle → area under the curve

$n \uparrow$ Error $\downarrow$



Riemann Sum with equal width subinterval:

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \frac{b-a}{n}\cdot\sum_{i=1}^{n} f(c_i) = f(c_1)\Delta x + \underbrace{f(c_2)\Delta x}+f(c_3)\Delta x + \cdots + f(c_n)\Delta x$$

Each of these terms can be interpreted as the area of a rectangle.

Definite Integral

The limit of a Riemann Sum can be interpreted as a definite integral.

If f is a continuous function defined for $a \leq x \leq b$, then the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta x$$

Upper Limit of integration

Integral sign $\longrightarrow$ $\displaystyle\int_a^b f(x)\,dx$

Lower Limit of integration

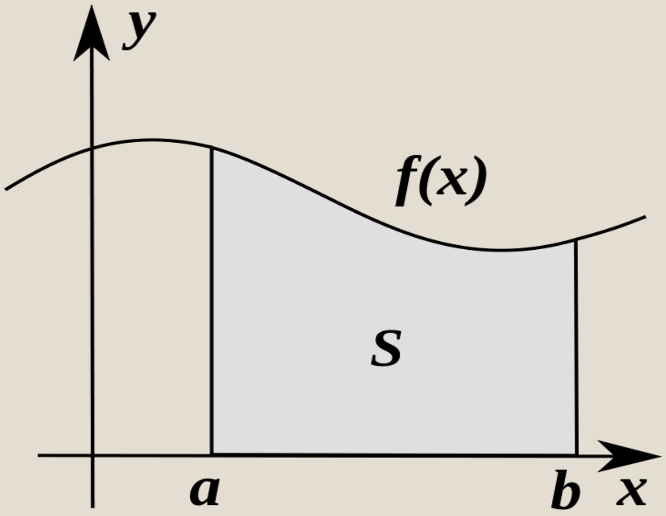
Integrand

We can think of this dx as an infinitesimally small Δx.

Definite Integral & Area

If $y = f(x)$ is continuous and **nonnegative** over a closed interval $[a, b]$ then the area of the region bounded by the graph of f , the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

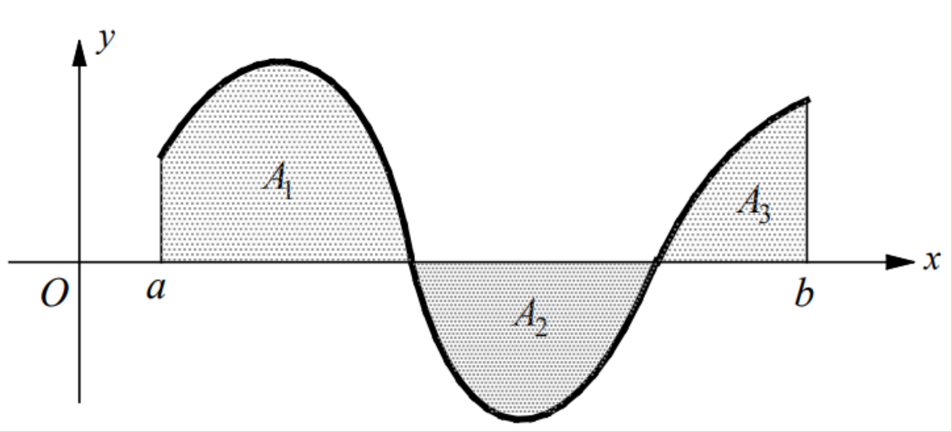
$$Area = \int_a^b f(x)\,dx \qquad \int_a^b -f(x)\,dx = ?$$



If the velocity is **positive**, the total distance traveled is exactly the area under the velocity curve!

Definite Integral & Area

If $y = f(x)$ takes on **both positive and negative values** over a closed interval $[a, b]$, then the area of the region bounded by the graph of f , the x-axis, and the vertical lines $x = a$ and $x = b$ is obtained by adding the **absolute** value of the definite integral over **each subinterval where $f(x)$ does not change sign**.



The definite integral of $f(x)$ over $[a, b]$ is

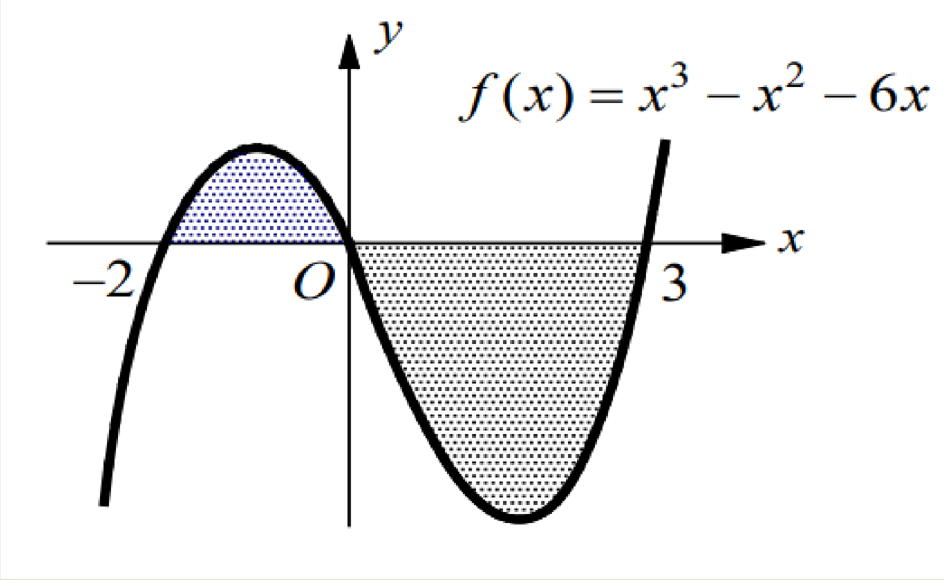
$\int_a^b f(x)\,dx = \; A_1 - A_2 + A_3$

The total area between the curve and the x-axis over $[a, b]$ is

$$\int_a^b |f(x)|\,dx = A_1 + A_2 + A_3$$

Practice -- Calculator

The figure shows the graph of $f(x) = x^3 - x^2 - 6x$.
(1) Find the definite integral of $f(x)$ on $[-2,3]$ using calculator.
(2) Find the area between the graph of $f(x)$ and the x-axis on $[-2,3]$.



Practice

$c_1 = \frac{1}{20}$ $\quad c_2 = \frac{2}{20}$ $\quad c_{20} = \frac{20}{20}$ $\qquad c_i = \frac{i}{20}$ $\qquad f(c_i) = \left(\frac{i}{20}\right)^2 = c_i^2$

The expression $\boxed{\frac{1}{20}}\underbrace{\left[\left(\boxed{\frac{1}{20}}\right)^2 + \left(\boxed{\frac{2}{20}}\right)^2 + \cdots + \left(\boxed{\frac{20}{20}}\right)^2\right]}_{n=20}$ is a Riemann sum approximation for

Equal width subinterval

$$\frac{1}{20} = \frac{b-a}{n}$$

Right Riemann Sum

$a = 0, b = 1$

$$\int_0^1 x^2\,dx$$

$f(x) = x^2$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n \to \int_a^b f(x)\,dx$$

Partition Δ : $\quad a = x_0 < x_1 < \cdots < x_n = b$

Practice

The expression $\frac{1}{10}\left[\frac{1}{10}+\frac{2}{10}+\cdots+\frac{20}{10}\right]$ is a Riemann sum approximation for ________

Practice

Which of the following limits is equal to $\int_1^3 x^3\,dx$

(A) $\lim_{n\to\infty} \sum_{i=1}^{n} (1+\frac{i}{n})^3 \frac{1}{n}$

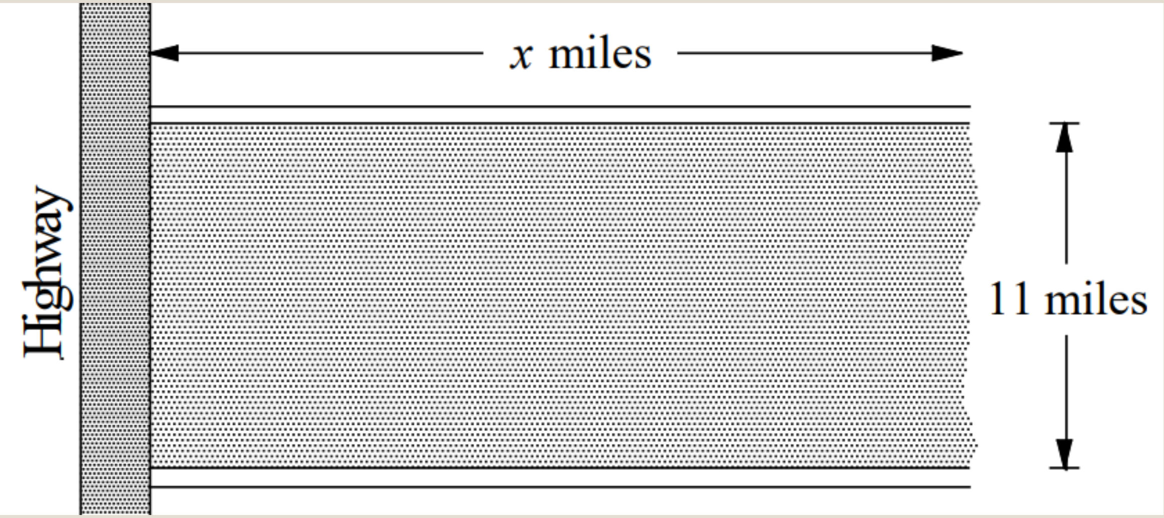
(B) $\lim_{n\to\infty} \sum_{i=1}^{n} (1+\frac{i}{n})^3 \frac{2}{n}$

(C) $\lim_{n\to\infty} \sum_{i=1}^{n} (1+\frac{2i}{n})^3 \frac{1}{n}$

(D) $\lim_{n\to\infty} \sum_{i=1}^{n} (1+\frac{2i}{n})^3 \frac{2}{n}$

Practice(*)

(Calculator) A rectangular region located beside a highway and between two straight roads 11 miles apart are shown in the figure below. The population density of the region at a distance x miles from the highway is given by $D(x) = 15x\sqrt{x} - 3x^2$, where $0 \leq x \leq 25$. How many people live between 16 to 25 miles from the highway?



Properties of definite integral

1. $\int_a^a f(x)\,dx =$

2. $\int_a^b f(x)\,dx =$ ____ $\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx =$

4. $\int_a^b f(x) \pm g(x)\,dx =$ ________________

5. $\int_a^b cf(x)\,dx =$ ________________

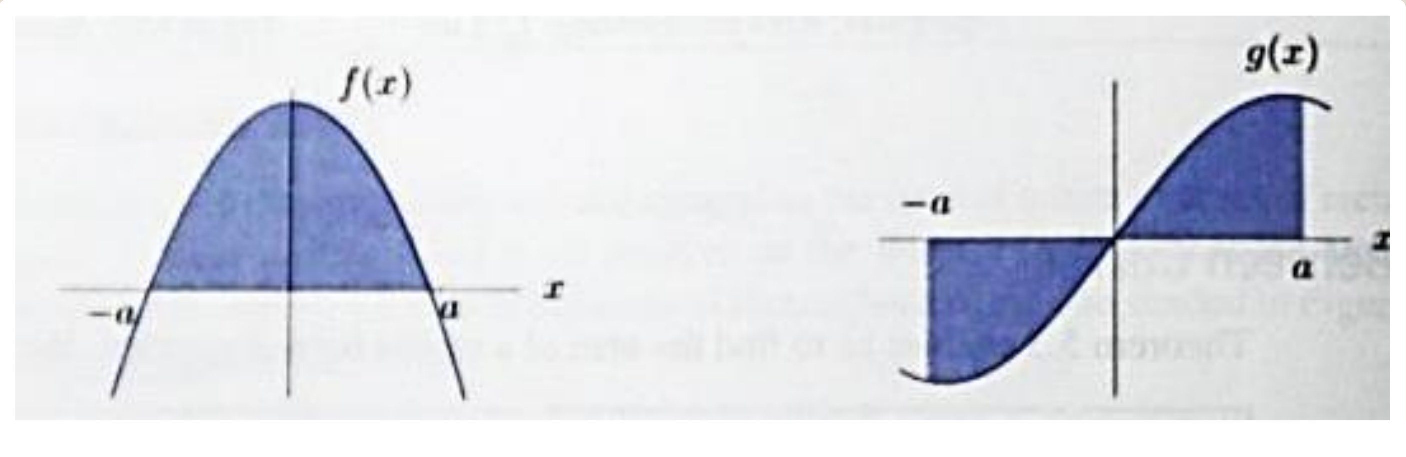
Properties of definite integral

6. $\int_a^b c \ dx =$

7. Use symmetry to evaluate integrals

If f is even, then $\int_{-a}^{a} f(x)\, dx =$

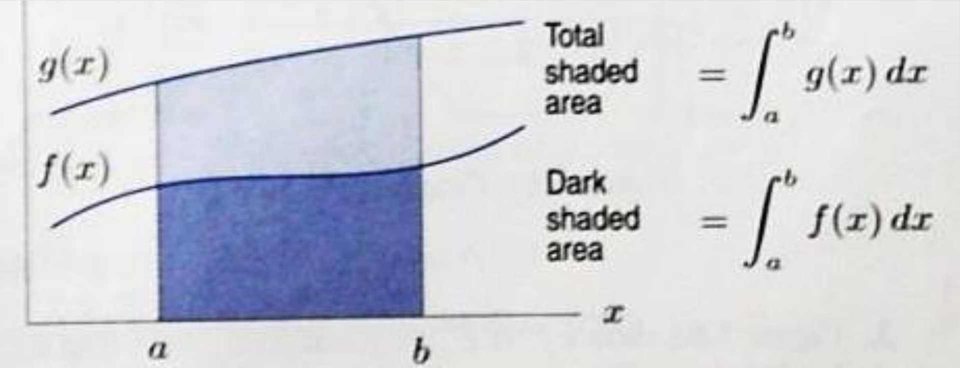
If f is odd, then $\int_{-a}^{a} f(x)\, dx =$

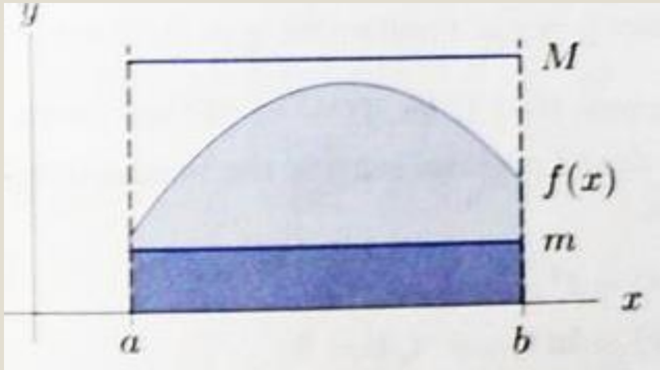


Properties of definite integral

8. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \mathbf{0}$

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx$ ____ $\int_a^b g(x)dx$





If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then ____________ $\leq \int_a^b f(x)\,dx \leq$ ____________

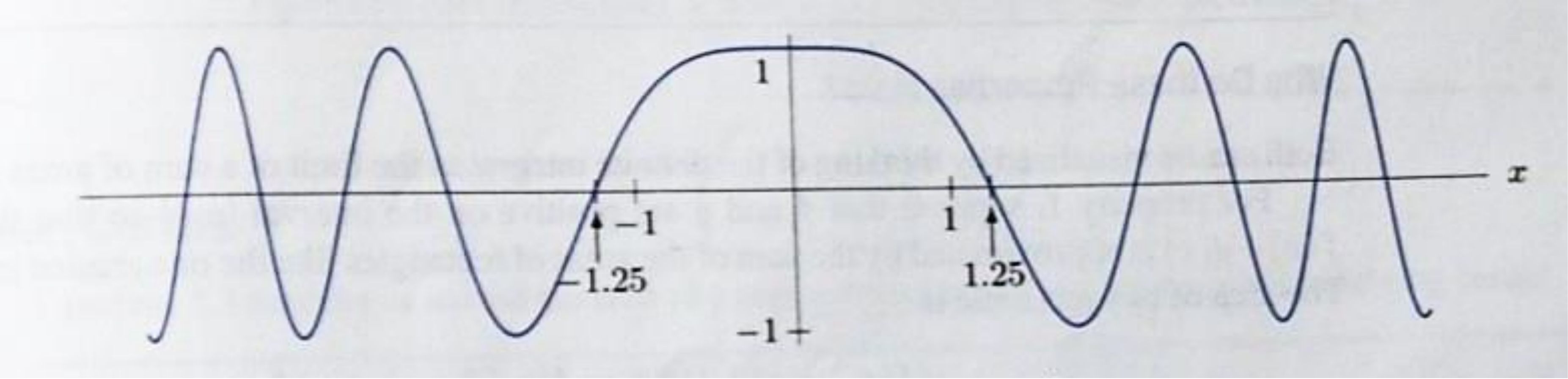
Practice

Suppose $\int_0^{1.25} \cos(x^2)\,dx = 0.98$ and $\int_0^{1} \cos(x^2)\,dx = 0.90$. The graph of f(x) is shown below. What are the values of the following integrals?

(a) $\int_1^{1.25} \cos(x^2)\,dx$

(b) $\int_{-1}^{1} \cos(x^2)\,dx$

(c) $\int_{1.25}^{-1} \cos(x^2)\,dx$



Practice

Suppose that $\int_{-3}^{4} f(x)\,dx = 5$, $\int_{-3}^{4} g(x)\,dx = -4$, and $\int_{-3}^{1} f(x)\,dx = 2$.

Find (a) $\int_{-3}^{4} [2f(x) - 3g(x)]\,dx$ (b) $\int_{1}^{4} f(x)\,dx$ (c) $\int_{-3}^{4} [g(x) + 2]\,dx$.

Practice

Let f and g be a continuous function on the interval $[1,5]$. Given $\int_1^3 f(x)\,dx = -3$, $\int_1^5 f(x)\,dx = 7$, and $\int_1^5 g(x)\,dx = 9$, find the following definite integrals.

(a) $\int_3^5 f(x)\,dx$

(b) $\int_1^3 [f(x)+3]\,dx$

(c) $\int_5^1 2g(x)\,dx$

(d) $\int_5^5 g(x)\,dx + \int_5^3 f(x)\,dx$

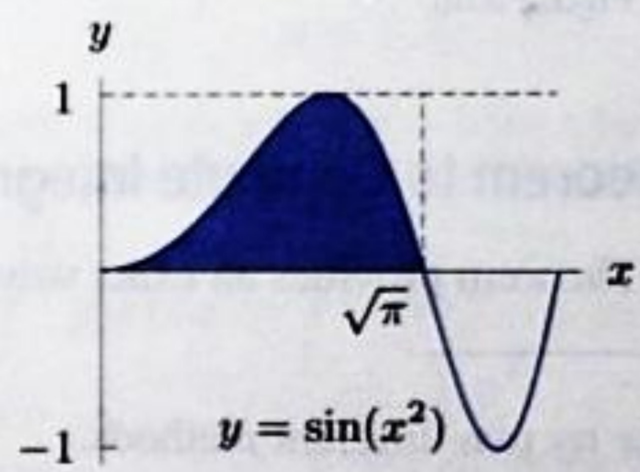
(e) $\int_{-1}^3 f(x+2)\,dx$

Practice

Example 6 Explain why $\int_0^{\sqrt{\pi}} \sin(x^2)\,dx \le \sqrt{\pi}$.

Solution Since $\sin(x^2) \le 1$ for all x (see Figure 5.60), part 2 of Theorem 5.4 gives

$$\int_0^{\sqrt{\pi}} \sin(x^2)\,dx \le \int_0^{\sqrt{\pi}} 1\,dx = \sqrt{\pi}.$$



Antiderivative

A function F is called an **antiderivative of** $\boldsymbol{f}$ on an interval I if for all x on I:

$$F'(x) = f(x)$$

If F is an antiderivative of f on I, then $\underline{\boldsymbol{F(x) + C}}$ represents the most general antiderivative of f on I.

The Fundamental Theorem of Calculus (FTC)

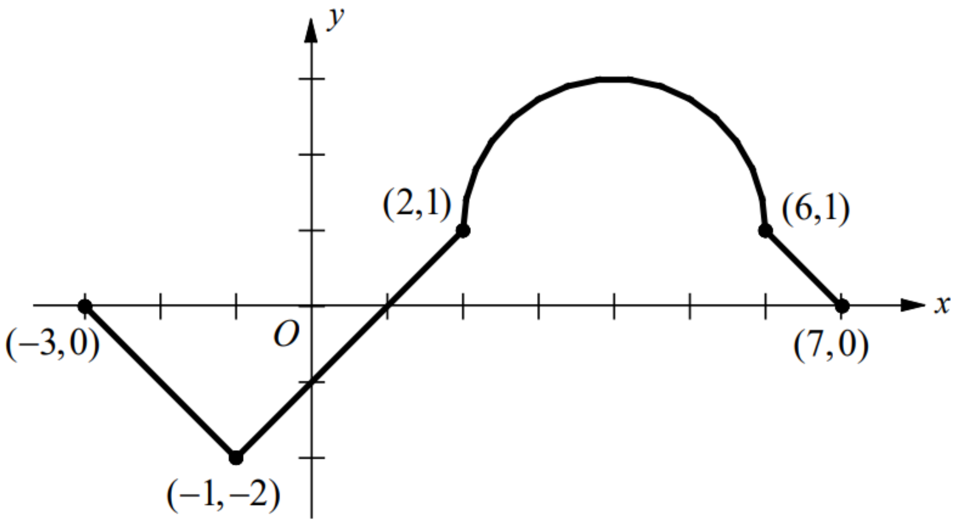
Let f be continuous on $[a, b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t)dt = F(b) - F(a)$$

If we let **f(t)** denote the **velocity function** $\boldsymbol{v(t)}$ and **F(t)** denote the **position function** $\boldsymbol{s(t)}$, then the accumulated change in position from time $t = a$ to $t = b$:

$$\int_a^b v(t)dt = s(b) - s(a)$$

Practice



graph of f'

Let f be a function defined on the closed interval $[-3,7]$ with $f(2)=3$. The graph of f' consists of three line segments and a semicircle, as shown above.

(a) Find $f(-3)$ and $f(7)$.

(b) Find an equation for the line tangent to the graph of f at $(2,3)$.

Practice

If f is the antiderivative of $\frac{\sqrt{x}}{1+x^3}$ such that $f(1)=2$, then $f(3)=$

Practice

If $f'(x) = \cos(x^2 - 1)$ and $f(-1) = 1.5$, then $f(5) =$

Practice

If f is a continuous function and $F'(x) = f(x)$ for all real numbers x, then $\int_{2}^{10} f(\frac{1}{2}x)\,dx =$

(A) $\frac{1}{2}[F(5) - F(1)]$

(B) $\frac{1}{2}[F(10) - F(2)]$

(C) $2[F(5) - F(1)]$

(D) $2[F(10) - F(2)]$

Indefinite Integral

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x denoted by $\boldsymbol{\int f(x)\, dx}$.

$$\int f(x)\, dx = F(x) + C \quad \Leftrightarrow F'(x) = f(x)$$

Formula

$\int k\,dx =$

$\int x^n\,dx =$

$\int e^x\,dx =$

$\int \sin x\,dx =$

$\int \cos x\,dx =$

$\int \sec^2 x\,dx =$

$\int \csc^2 x\,dx =$

$\int \sec x \tan x\,dx =$

$\int \csc x \cot x\,dx =$

$\int kf(x)\,dx =$

$\int [f(x) \pm g(x)]\,dx =$

Integral of Natural Logarithmic Function

$$\frac{d}{dx}[\boldsymbol{ln\ x}] =$$

$$\int_{1}^{x} \frac{1}{t}\,dt =$$

If u is a differentiable function such that $u \neq 0$, $\int \frac{1}{u}\,du =$__________

Integral of Exponential Function

$$\int \boldsymbol{e^{u}\,du} =$$

Practice

Find an antiderivative for each of the following functions.

a. $f(x) = 3x^2$

b. $g(x) = \cos x + 3$

Practice

Find the general solution of $F'(x) = \sec^2 x$.

Practice

Find the antiderivative of $x^3 - 3x + 2$.

Practice

Find the general indefinite integral $\int \sqrt{x} - \sec x \tan x \; dx$

Practice

The area of the region in the first quadrant enclosed by $f(x) = 4x - x^3$ and the x-axis is

(A) $\frac{11}{4}$ (B) $\frac{7}{2}$ (C) 4 (D) $\frac{11}{2}$

Practice（*）

$\int_0^5 \sqrt{25 - x^2}\, dx$=__________

Fundamental Theorem

Let f be continuous on $[a, b]$ then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b), and

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = \underline{\quad \boldsymbol{f(x)} \qquad}$$

If $F(x) = \int_a^{u(x)} f(t)\,dt$, then $F'(x) = \frac{d}{dx}\int_a^{u(x)} f(t)\,dt =$ $\boldsymbol{f(u(x)) \cdot u'(x)}$

Practice

$$\int_{\frac{\pi}{2}}^{x} \cos t \, dt = ?$$

Practice

If $F(x) = \int_1^x \frac{1}{1+u^3}\, du$, then $F'(x) =$ ______________________

Practice

If $F(x) = \int_1^{x^2+1} \sqrt{t}\ dt$, then $F'(x) =$ ________________________________

Practice

For $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, if $F(x) = \displaystyle\int_0^{\sin x} \dfrac{dt}{\sqrt{1-t^2}}$, then $F'(x) =$

Practice

Let f be the function given by $f(x)=\int_0^x \cos(t^2+2)\,dt$ for $0\le x\le\pi$. On which of the following intervals is f increasing?

(A) $0\le x\le\frac{\pi}{2}$

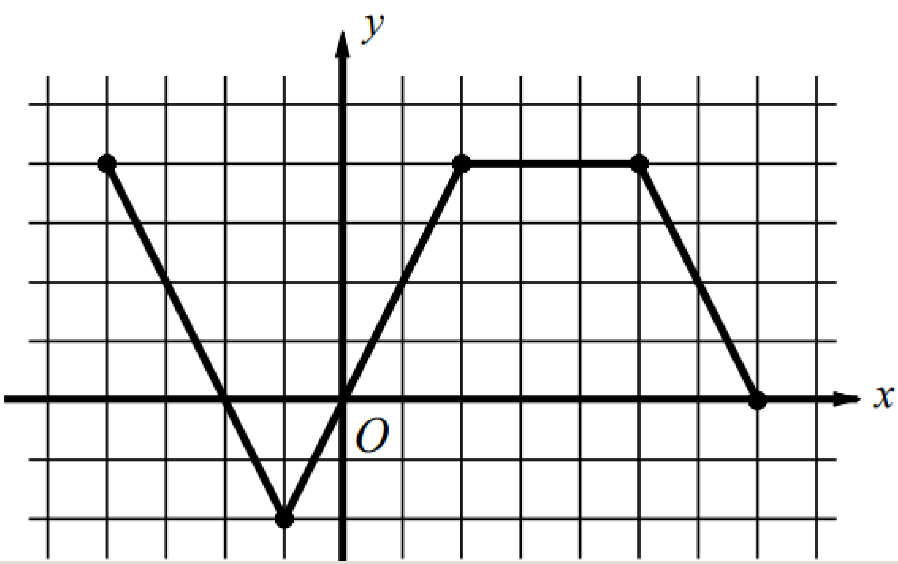
(B) $0\le x\le 1.647$

(C) $1.647\le x\le 2.419$

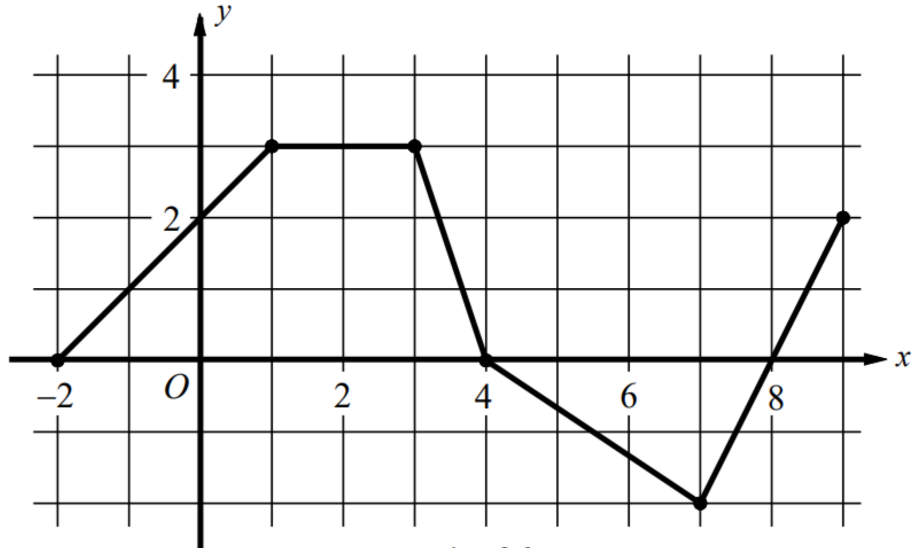
(D) $\frac{\pi}{2}\le x\le\pi$

Practice

The graph of the function f shown below consists of four line segments. If g is the function defined by $g(x) = \int_{-4}^{x} f(t)\, dt$, find the value of $g(6)$, $g'(6)$, and $g''(6)$.



Practice



graph of f

Let g be the function given by $g(x) = \int_{-2}^{x} f(t)\,dt$. The graph of the function f, shown above, consists of five line segments.

(a) Find $g(0)$, $g'(0)$ and $g''(0)$.

(b) For what values of x, in the open interval $(-2,9)$, is the graph of g concave up?

(c) For what values of x, in the open interval $(-2,9)$, is g increasing?