Sequences and Series

- **Sequence** is and ordered list of numbers

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \dots, a_n, \dots\}$$

  - ✓ Example. Arithmetic Sequence $\{n\}_{n=1}^{\infty} = \{1,2,\dots,n,\dots\}$

✧ **Convergent and Divergent**

- If a sequence has the limit L, where L is a finite real number, ($\lim_{n\to\infty} a_n = L$),

we say the sequence **converges** to L.

- If the limit does not exist, the sequence **diverges**.

Practice

1. Is the sequence $\{2n+1\}_{n=1}^{\infty}$ convergent or divergent?

   $\lim_{n\to\infty} 2n+1 = \infty$ DNE So it is divergent.

2. $\{\frac{n^2+1}{2n^2-3n+5}\}_{n=1}^{\infty}$ convergent or divergent?

   $\lim_{n\to\infty} a_n = \frac{1}{2}$ convergent

3. $\{(-1)^n\}_{n=1}^{\infty}$ convergent or divergent?

   $\lim_{n\to\infty} a_n = \begin{cases} 1 \\ -1 \end{cases}$ divergent (oscillate too much)

4. $\{\frac{1}{n} * (-1)^n\}_{n=1}^{\infty}$ convergent or divergent?

   $\lim_{n\to\infty} a_n = 0$ convergent

- **Infinite series:** Given a sequence $\{a_n\}$, $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n + \dots$ is an **infinite series.** sum of a sequence

✧ **True Sum** $S = \sum_{n=1}^{\infty} a_n$

✧ **Partial Sum** of a sequence: $S_n = \sum_{i=1}^{n} a_i$ "前n项和"

  - ✓ Example. What is the partial sum of the sequence $\{n\}_{n=1}^{\infty} = \{1,2,\dots,n,\dots\}$?

$$S_n = \frac{(1+n)n}{2}$$

$S_1 = \frac{2\times 1}{2} = 1$ $S_2 = 3$ $S_3 = \dots$

  - ■ $\{S_n\}_{n=1}^{\infty} = \{S_1, S_2, \dots, S_n, \dots\}$ is a sequence.

+

Recall: convergency of a sequence.

✧ **Convergent and Divergent**

The series is convergent ⇔ $\lim_{n\to\infty} S_n = L$

Otherwise, the series diverges.

Practice

1. Is the series $\sum_{n=1}^{\infty} n$ convergent or divergent?

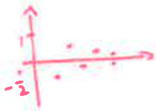
$S_n = 1+2+\cdots+n = \frac{(1+n)n}{2}$

$\lim_{n\to\infty} S_n = \infty$ (DNE) Divergent.

2. Is the series $\sum_{n=1}^{\infty}(-1)^n$ convergent or divergent?

$S_n = \begin{cases} -1 \\ 0 \end{cases}$ $\lim_{n\to\infty} S_n$ DNE Divergent

3. Is the telescoping series $\sum_{n=1}^{\infty} \frac{1}{k(k+1)}$ convergent or divergent?



$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

$S_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = 1$ convergent

4. Is the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ convergent or divergent?

$S_n = \frac{\frac{1}{2}[1-(\frac{1}{2})^n]}{1-\frac{1}{2}} = 1-(\frac{1}{2})^n$

$\lim_{n\to\infty} S_n = 1$ convergent

5. Is the geometric series $\sum_{n=1}^{\infty} 2^{3n}5^{1-n}$ convergent or divergent?

$a_n = 2^{3n}5^{1-n} = \frac{8^n}{5^n} 5 = 5\cdot(\frac{8}{5})^n$

$S_n = \frac{5\cdot\frac{8}{5}\cdot[1-(\frac{8}{5})^n]}{1-\frac{8}{5}}$ $\lim_{n\to\infty} S_n = \infty$ (DNE) divergent

Summary. **Geometric series** $\sum_{n=1}^{\infty} ar^{n-1}$ r: common ratio

可直接用结论.

$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}\cdot(1-r^n)$

$\lim_{n\to\infty} S_n = \begin{cases} \frac{a}{1-r}, & |r|<1 \\ \text{DNE}, & |r|>1 \\ \text{DNE}, & r=1 \end{cases}$ $\leftarrow$ when $r=1$ $\sum_{n=1}^{\infty} a$

$S_n = a\cdot n$

$\lim_{n\to\infty} S_n = \infty$ (DNE

Summary, for geometric series, the series is convergent when $|r|<1$

➢ **Convergence of a sequence V.S. Convergence of a series**

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

✧ This theorem provides a useful test for **divergent series**!

If the limit $\lim_{n\to\infty} a_n$ DNE or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ **diverges**. $\Rightarrow$ $\boxed{\text{n-term test}}$ If $\lim_{n\to\infty} a_n \neq 0$, the series is divergent.

✓ Example. $a_n = 1$ $\quad \lim_{n\to\infty} a_n \neq 0 \quad \therefore$ divergent

$a_n = \frac{1}{n}$ $\quad \lim_{n\to\infty} a_n = 0 \quad$ n-term Test fails

$$\left(\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\right.$$
$$= 1 + \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{10}\right) + \left(\frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{100}\right) + \cdots > 1 + \frac{1}{10}\cdot 9 + \frac{90}{100} + \cdots = 1 + \frac{9}{10} + \frac{9}{10} + \cdots = \infty$$

We can use the definition of convergent series to determine whether the series is convergent or divergent.

However, it is always hard to find the expression of S_n. So, we need other methods to test it.

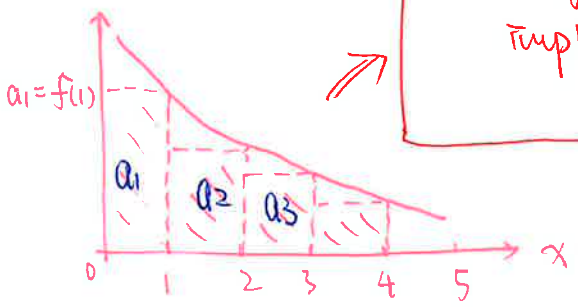
➢ $\boxed{\textbf{The Integral Test}}$ eg. $a_n = \frac{1}{n}$, $f(x) = \frac{1}{x}$ $\quad f(x)$ positive, ctns, $\downarrow$ on $[1, +\infty)$ $\quad \int_1^{\infty} \frac{1}{x}\,dx = [\ln|x|]_1^{\infty} = \infty \quad \therefore \sum_{n=1}^{\infty} a_n$ divergent.

If f is $\boxed{\text{positive}}$ continuous, and $\boxed{\text{decreasing}}$ on $[1, +\infty)$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either BOTH converge or diverge.

→ nonnegative 也可以 → 一开始不↓，但eventually decreasing 就行.

If f is negative, ctns, increasing ... : $\sum_{n=1}^{\infty} a_n = -\sum_{n=1}^{\infty}(-a_n)$, $\quad -f(x)$ positive, ctns, decreasing

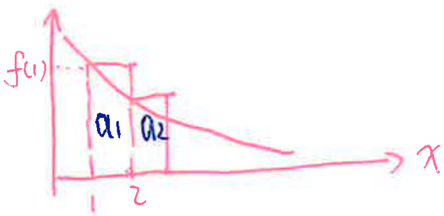
Note: The fact that "the integral converges to L" does not imply that "the series converges to L".



$$\text{Area} = a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n \leq \int_0^{\infty} f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^{\infty} f(x)\,dx$$

$$\therefore \sum_{n=1}^{\infty} a_n - \int_0^1 f(x)\,dx \leq \int_1^{\infty} f(x)\,dx$$

$\therefore$ If $\int_1^{\infty} f(x)\,dx$ is convergent, the series is convergent.



$\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} f(x)\,dx \quad \therefore$ If $\int_1^{\infty} f(x)\,dx$ is divergent, the series is divergent.

Summary: A p-series is convergent if $p > 1$

✧ **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$ $p \in \mathbb{Z}^*$

Determine whether the p-series convergent or divergent.

$f(x) = \left(\frac{1}{x}\right)^p$ positive, decreasing, ctns on $[1, \infty)$

$p > 1$: $\int_1^{\infty} \left(\frac{1}{x}\right)^p dx = \left[\frac{1}{-p+1} x^{-p+1}\right]_1^{\infty} = 0 - \frac{1}{-p+1} = \frac{1}{p-1}$ convergent!

$p = 1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent

(But do not converges to $\frac{1}{p-1}$)

✧ **Harmonic Series** $\sum_{n=1}^{\infty} \frac{1}{n}$

When $p = 1$, the p-series is called harmonic series.

✧ **General Harmonic Series** $\sum_{n=1}^{\infty} \frac{1}{an+b}$

Practice

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $f(x) = \frac{1}{x^2+1}$ is ctns, positive and decreasing on $[1, \infty)$. So we can use the Integral Test:

$\int_1^{\infty} \frac{1}{x^2+1} dx = \left[\arctan x\right]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

So the series converges.

2. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

The function $f(x)=\frac{\ln x}{x}$ is ctns. positive and decreasing on $[1,+\infty)$.

So we use the Integral test: $\int_1^{\infty} \frac{\ln x}{x} dx = \int_1^{\infty} \ln x \, d(\ln x)$

$= \left[\frac{1}{2}(\ln x)^2\right]_1^{\infty} = \infty$

$\therefore$ the series divergent

3. $\sum_{n=1}^{\infty} n^{1-\pi}$

$f(x) = x^{1-\pi}$ ctns, positive, $\searrow$ on $[1,+\infty)$

$\therefore$ Integral Test: $\int_1^{\infty} x^{1-\pi} dx = \left[\frac{1}{2-\pi} x^{2-\pi}\right]_1^{\infty} = \frac{1}{\pi-2}$

$\boxed{\text{Method 2.}}$ The p-series is convergent since $p = \pi-1 > 1$

4. $1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{25}} + \cdots$

$1 + \frac{1}{2^{\frac{2}{3}}} + \frac{1}{3^{\frac{2}{3}}} + \frac{1}{4^{\frac{2}{3}}} + \cdots$

This is a p-series with $p = \frac{2}{3} < 1$

So the series diverges.

5. $\sum_{n=1}^{\infty} (\frac{1}{2})^n$

This series is convergent since this is a geometric series

with common ratio $q = \frac{1}{2} \in (-1, 1)$

- **The Comparison Test**

  - **Direct Comparison Test**

Let $0 \le a_n \le b_n$ for all n.

1. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges
2. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

Practice

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\quad 0 \le \frac{1}{n^2+1} \le \frac{1}{n^2}$ for all n

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p=2>1$, it is convergent

So, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent. by the Direct Comparison Test.

2. $\sum_{n=1}^{\infty} \frac{n}{n^2-3}$ $\quad 0 \le \frac{n}{n^2} \le \frac{n}{n^2-3}$ for all $n \ge 2$

$\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent since it is a p-series with $p=1$.

There fore, $\sum_{n=1}^{\infty} \frac{n}{n^2-3} = -\frac{1}{2} + \sum_{n=2}^{\infty} \frac{n}{n^2-3}$ is divergent by the Direct Comparison Test.

3. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{\sqrt{n^3+1}}$ bounded "1" $\sim n^{\frac{3}{2}}$

$0 \le \frac{\sin^2 n}{\sqrt{n^3+1}} \le \frac{1}{\sqrt{n^3}} = \frac{1}{n^{\frac{3}{2}}}$ for all $n \ge 1$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a covergent p-series since $p = \frac{3}{2} > 1$.

So, $\sum_{n=1}^{\infty} \frac{\sin^2 n}{\sqrt{n^3+1}}$ is convergent by the Direct Comparison Test.

4. $\sum_{n=1}^{\infty} \frac{n^2 \cos^4 n}{n^5+1}$

$0 \le \frac{n^2 \cos^4 n}{n^5+1} \le \frac{n^2}{n^5} = \frac{1}{n^3}$ for all $n \ge 1$

? How to select b_n $\Rightarrow$ 已知收敛,发散性质的 series. eg. p-series. geometric series.

You can disregard all but the $\boxed{\text{highest powers of n in both the numerator and denominator.}}$

"leading term"

已知收敛发散性质的级数.

b_n (目标): p-series, geometric series.

✧ Limit Comparison Test

If $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is finite and positive, then both series either converge or diverge.

Practice

1. $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = 1$$

$\sum_{n=1}^{\infty} (\frac{1}{2})^n$ is convergent ($\because$ it is a geometric series with $q=\frac{1}{2} \in (-1,1)$)

$\therefore \sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is convergent by the Limit Comparison Test

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+4}}}{\frac{1}{n}} = 1$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent since it's a harmonic series.

$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$ is divergent by the Limit Comparison Test.

3. $\sum_{n=3}^{\infty} \frac{2^n}{3^n+1}$

$$\lim_{n\to\infty} \frac{\frac{2^n}{3^n+1}}{(\frac{2}{3})^n} = 1$$

$\sum_{n=3}^{\infty} (\frac{2}{3})^n$ is a geometric series with common ration $\frac{2}{3} \in (-1,1)$, so it is convergent.

So, the series $\sum_{n=3}^{\infty} \frac{2^n}{3^n+1}$ is Convergent by the Limit comparison Test

➢ Alternating Series Test

Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge if

≈ nth term test

1. $\lim_{n\to\infty} a_n = 0$ AND 2. $a_{n+1} \le a_n$ for all n greater than some integer N.

True Sum $= \underbrace{S_N}_{\text{finite real number}} + R_N$, $R_N \in [-a_{N+1}, a_{N+1}]$

when $N \to \infty$, $a_N \to 0$

➢ Alternating Series Estimation Theorem (Error Bound)

If $S = \sum_{n=1}^{\infty}(-1)^n a_n$ is the sum of a convergent alternating series that satisfies the condition $a_{n+1} \le a_n$, then the remainder $R_n = S - S_n$ is smaller than a_{n+1}, $|R_n| \le a_{n+1}$

proof: $R_n = (-1)^{n+1} a_{n+1} + (-1)^{n+2} a_{n+2} + \cdots$

$= (-1)^{n+1} \cdot [a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots] \Rightarrow a_{n+1} \underbrace{- a_{n+2} + a_{n+3}}_{\le 0} \underbrace{- a_{n+4} + a_{n+5}}_{\le 0} - \cdots$ $(\because a_{n+1} \le a_n)$

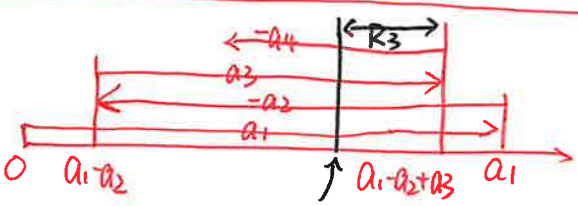
$\therefore \le a_{n+1}$

$\underbrace{a_{n+1} - a_{n+2}}_{\ge 0} + \underbrace{a_{n+3} - a_{n+4}}_{\ge 0} + \underbrace{a_{n+5} - a_{n+6}}_{\ge 0} + \cdots$

$\therefore a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots \ge a_{n+1} - a_{n+2}$

$|a_{n+1} - a_{n+2}| \le a_{n+1}$

Therefore, $|R_n| \le a_{n+1}$



the series converges to L

Apparently, $|R_3| \le a_4$

✎ Practice.

1. Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$\frac{1}{\sqrt{n}} > 0$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}}$

$\therefore$ the series is convergent by the Alternating Series Test

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n-1}$

$\lim_{n\to\infty} (-1)^n \frac{n}{2n-1} = \pm\frac{1}{2}$ (DNE)

$\therefore$ the series diverges by the nth-term test.

2. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$

$\frac{1}{n} > 0$

$\frac{1}{n+1} \le \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ the series converges by the A.S.T.

3. $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\cdots+\frac{(-1)^{n+1}}{(2n-1)!}+\cdots$

$\frac{1}{(2n-1)!} > 0$

$\lim_{n\to\infty} \frac{1}{(2n-1)!} = 0$

$\frac{1}{(2n+1)!} \le \frac{1}{(2n-1)!}$

$\therefore$ the Series converges by the A.S.T

4. Let $f(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+-\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$.

Use the alternating series error bound to show that $1-\frac{1}{2!}+\frac{1}{4!}$ approximates $f(1)$

with an error less than $\frac{1}{500}$.

$f(1) = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$

Error $\le \underbrace{\frac{1}{6!}}_{\frac{1}{720}} < \frac{1}{500}$

$\sum |a_n|$ converges?

Yes → Absolutely Convergent

No → $\sum a_n$ con?

Yes → Conditionally Convergent.

No → div

➢ **Absolute and Conditional Convergence**

✧ $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

✧ $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

✍ Practice. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sqrt[n]{e}}{n^2}$

$0 < \frac{\sqrt[n]{e}}{n^2} = \frac{e^{\frac{1}{n}}}{n^2} \le \frac{e}{n^2}$ for $n \ge 1$

$\sum \frac{e}{n^2}$ is a p-series with $p=2>1$

$\therefore \sum \frac{e}{n^2}$ is convergent

$\therefore \sum_{n=1}^{\infty} \left| \frac{(-1)^n \sqrt[n]{e}}{n^2} \right|$ is convergent by the Direct comparison test.

$\therefore$ the given series is absolutely convergent.

2. $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-\frac{2}{3}}$

$\sum \left| (-1)^{n+1} n^{-\frac{2}{3}} \right| = \sum \frac{1}{n^{\frac{2}{3}}}$ is divergent (p-series with $p=\frac{2}{3}<1$)

$\sum (-1)^{n+1} n^{-\frac{2}{3}}$: $n^{-\frac{2}{3}} > 0$, $\lim_{n\to\infty} n^{-\frac{2}{3}} = 0$, $(n+1)^{-\frac{2}{3}} \le n^{-\frac{2}{3}}$

$\therefore$ convergent by the A.S.T

$\therefore$ Conditionally convergent.

- Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = q$

理解: $\frac{a_{n+1}}{a_n} = q$

$\Downarrow$

geometric series with common ratio q. for $n > N$

N: a finite large number.

So we can use the conclusion of the geometric series. :)

- If $q < 1$, the series converges absolutely.
- If $q > 1$ or the limit DNE, the series diverges
- If $q = 1$, the ratio test fails.

Example. $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = 1$

$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right| = 1$

$\sum \frac{1}{n}$ div

$\sum \frac{1}{n^2}$ con

Practice

1. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} \right| = 0 < 1$

$\therefore$ the series is convergent by the Ratio Test

2. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{5^n}$

$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{(n+1)^3}{5^{n+1}}}{(-1)^n \frac{n^3}{5^n}} \right| = \frac{1}{5} < 1$

$\therefore$ Convergent by the ratio test.

Determine whether the series below is conditionally convergent or absolute convergent.

3. $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

$\lim_{n\to\infty} \frac{3^n}{2^n-1} = \infty$ (DNE)

$\therefore$ divergent by the nth term test

4. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+3}$

$\sum \left| (-1)^n \frac{\sqrt{n}}{n+3} \right| = \sum \frac{\sqrt{n}}{n+3}$

$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+3}}{\frac{1}{\sqrt{n}}} = 1$

$\sum \frac{1}{\sqrt{n}}$ is divergent since it is a p-series with $p = \frac{1}{2} < 1$

$\therefore \sum \frac{\sqrt{n}}{n+3}$ is divergent by the Limit Comparison Test.

$\frac{\sqrt{n}}{n+3} > 0$

$\lim_{n\to\infty} \frac{\sqrt{n}}{n+3} = 0$

$\frac{\sqrt{n+1}}{n+4} \le \frac{\sqrt{n}}{n+3}$

$\therefore \sum (-1)^n \frac{\sqrt{n}}{n+3}$ is convergent by the A.S.T.

$\Rightarrow$ Therefore, the given series is conditionally convergent.

5. $\sum_{n=1}^{\infty} (-1)^n \frac{e^n}{n!}$

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{e^n}{n!} \right|$:

$\lim_{n\to\infty} \left| \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} \right| = 0 < 1$

$\therefore$ the series is absolutely convergent by the Ratio Test.

$$\sum_{n=1}^{\infty} a_n$$

↓

n-th term test $\lim_{n\to\infty} a_n = 0$? —NO→ divergent

↓ Yes

special series ? —Yes→ / —NO→ "Test"

p-series $a_n = \frac{1}{n^p}$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$

$p>1$: convergent

$p\le 1$: divergent

geometric series $a_n = a\cdot r^{n-1}$ $\sum_{n=1}^{\infty} a\cdot r^{n-1}$

$|r|<1$: convergent

$|r|\ge 1$: divergent

Alternating Series $a_n = (-1)^n b_n$ or $(-1)^{n+1} b_n$, $b_n \ge 0$

If $\lim_{n\to\infty} b_n = 0$ AND $b_{n+1} \le b_n$

then convergent

Telescoping series (Def of convergence)

Find S_n, $\lim_{n\to\infty} S_n = L$?

The limit exists : convergent

DNE : divergent

Comparison Test

(pick $\{b_n\}$)

Direct Comparison Test

$0 \le a_n \le b_n$

$\sum_{n=1}^{\infty} b_n$ converges

$\Downarrow$

$\sum_{n=1}^{\infty} a_n$ converges

$0 \le b_n \le a_n$

$\sum_{n=1}^{\infty} b_n$ diverges

$\Downarrow$

$\sum_{n=1}^{\infty} a_n$ diverges

Limit Comparison Test

$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$

$a_n, b_n > 0$

$\sum_{n=1}^{\infty} b_n$ con, div

$\Downarrow \quad \Downarrow$

$\sum_{n=1}^{\infty} a_n$ con, div

Ratio Test

$\sum_{n=1}^{\infty} a_n$ $a_n \ne 0$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = q$

$q<1$ convergent (absolutely)

$q>1$, DNE divergent

$q=1$ the ratio test fails

Integral Test

$a_n = f(n)$

$f(x)$: cts, positive, decreasing on $[1, +\infty)$

$\int_1^{\infty} f(x)\,dx$ div, con

$\Downarrow \quad \Downarrow$

$\sum_{n=1}^{\infty} a_n$ div, con