➢ Parametric Equations

x, y 共同的自变量 t

If x and y are both given as functions of a third variable, t, then the equations $\boldsymbol{x = f(t)}$ and $\boldsymbol{y = g(t)}$ are called **parametric equations** and $\boldsymbol{t}$ is the **parameter**. 参数

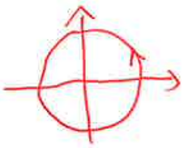
✧ $\{(x, y)\} = \{f(t), g(t)\}$ This set of points is called the **parametric curve.**

✧ **Direction of path (or motion) of the curve:**

When the points are plotted in order of increasing values of t, the curve is traced out in a specific direction.

Example: $\frac{dy}{dx} = -\frac{x}{y}$

$x^2+y^2=C$



parametric equation:

$\begin{cases} x(t)=\cos t \\ y(t)=\sin t \end{cases} \Rightarrow$ at $t=0$: $x(t)=1$, $y(t)=0$

$y'(t)=\cos 0>1$

$\therefore y(t)\uparrow$.

✧ $\boldsymbol{\frac{dy}{dx}}$ ***and*** $\boldsymbol{\frac{d^2y}{dx^2}}$

If the equation $x = f(t)$ and $y = g(t)$ define y as a differentiable function of x and $\frac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)}{\frac{dx}{dt}}$$

e.g. $\begin{cases} x=t^2 \\ y=t^3-2t \end{cases}$ $\frac{dy}{dx}=\frac{3t^2-2}{2t}=\frac{3}{2}t-t^{-1}$ $\frac{d^2y}{dx^2}=\frac{(\frac{3}{2}t-t^{-1})'}{2t}=\frac{\frac{3}{2}+t^{-2}}{2t}$

✧ **Horizontal & Vertical Tangent**

The curve represented by $x = f(t)$ and $y = g(t)$ has a

- horizontal tangent at $(f(t_0), g(t_0))$ if $g'(t)=0,\ f'(t)\neq 0$ → slope $=0$
- vertical tangent at $(f(t_0), g(t_0))$ if $f'(t)=0,\ g'(t)\neq 0$ → slope $=\pm\infty$

Practice.

1. A curve in the plane is defined parametrically by the equations $x = t^2$ and $y = t^3 - 2t$
   (a) Sketch the curve according to your **calculator** in the xy −plane for $-2 \leq t \leq 2$. Indicate the direction in which the curve is traced as t increases.
   (b) For what values of t does the curve have a vertical tangent?
   (c) For what values of t does the curve have a horizontal tangent?
   (d) Find the equation of the tangent lines to the curve at $t = \pm 2$.

(b) $\frac{dy}{dx}=\frac{3t^2-2}{2t}$

$t=0$

(c) $3t^2-2=0$

$t=\pm\frac{\sqrt{6}}{3}$

(d) $t=2$, $\frac{dy}{dx}=2.5$ passes $(4,4)$ $\Rightarrow y=4+2.5(x-4)$

$t=-2$: $\frac{dy}{dx}=-2.5$ passes $(4,-4)$ $\Rightarrow y=-4-2.5(x-4)$

2. A particle moves in the xy −plane so that its position at any time t , $0 \le t \le 4$, is given by the equations $x(t) = \cos t + t\sin t$ and $y(t) = \sin t - t\cos t$.

(a) Using the **calculator** to sketch the curve according to your calculator in the xy −plane for $0 \le t \le 4$. Indicate the direction in which the curve is traced as t increases.

(b) At what time t , $0 < t < 4$, does the line tangent to the path of the particle have a slope of -1?

(c) At what time t , $0 < t < 4$, does $x(t)$ attain its maximum value? What is the position of the particle $(x(t), y(t))$ at this time?

(d) At what time t, $0 < t < 4$, is the particle on the y-axis?

(b) $\frac{dy}{dx} = \frac{\cos t - (\cos t + t\cdot(-\sin t))}{-\sin t + (\sin t + t\cos t)} = \frac{t\sin t}{t\cos t} = \tan t = -1$

$t = \frac{3}{4}\pi$

(c) $x'(t) = -\sin t + \sin t + t\cos t = t\cos t = 0$ $(t>0)$

$\because \cos t = 0 \quad \therefore t = \frac{\pi}{2}$

t	$(0,\frac{\pi}{2})$	$\frac{\pi}{2}$	$(\frac{\pi}{2},4)$
$x'(t)$	+	max	−

$\therefore x(\frac{\pi}{2}) = \frac{\pi}{2}$ $y(\frac{\pi}{2}) = 1$ $\therefore (\frac{\pi}{2}, 1)$

(d) $x(t) = 0 = \cos t + t\sin t$

Calculator:

$t \approx 2.798$

3. (Calculator) An object moving along a curve in the xy-plane is in position $(x(t), y(t))$ at time $t \ge 0$ with $\frac{dx}{dt} = 2 - \sin(t^2)$. At time $t = 3$, the object is at position $(2,7)$. What is the x-coordinate of the position of the object at time $t = 6$?

A

$x(3) = 2$

(A) 8.135 (B) 9.762 (C) 10.375 (D) 11.308

$x(6) - x(3) = \int_3^6 2 - \sin(t^2)\, dt \approx 6.135$

4. A particle moving along the curve is defined by the equation $y = x^3 - 4x^2 + 4$. The x-coordinate of the particle, $x(t)$, satisfies the equation $\frac{dx}{dt} = \frac{t}{\sqrt{t^2+9}}$, for $t \ge 0$ with initial condition $x(0) = 1$.

(a) Find $x(t)$ in terms of t.

(b) Find $\frac{dy}{dt}$ in terms of t.

(c) Find the location of the particle at time $t = 4$.

(d) Write an equation for the line tangent to the curve at time $t = 4$.

(a) $\int \frac{dx}{dt}\, dt = \int \frac{t}{\sqrt{t^2+9}}\, dt = \int \frac{1}{2}(t^2+9)^{-\frac{1}{2}} d(t^2+9)$

$= (t^2+9)^{\frac{1}{2}} + C$

$\because x(0) = 1 \quad \therefore C = -2$

$\therefore x(t) = \sqrt{t^2+9} - 2$

(b) $\frac{dy}{dt} = 3x^2 \frac{dx}{dt} - 8x\frac{dx}{dt} = 3(\sqrt{t^2+9} - 2)^2 \cdot \frac{t}{\sqrt{t^2+9}} - 8(\sqrt{t^2+9} - 2)\cdot \frac{t}{\sqrt{t^2+9}}$

(c) $x(4) = 3$

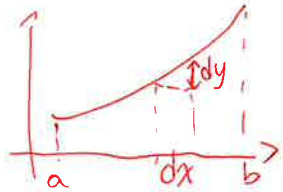
$y(4) = -5$

$\therefore (3, -5)$

(d) slope $= \frac{dy}{dx}\Big|_{t=4} = [3x^2 - 8x]_{x=3} = 3$

$y + 5 = 3(x-3)$

➢ Arc Length in Parametric Form



$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+(f'(x))^2}\ dx$

✧ Review.

- If f' is continuous on $[a,b]$, then the length of the curve $y=f(x)$ from $x=a$ to $x=b$ is $L=\int_a^b \sqrt{1+(f'(x))^2}\,dx$
- If g' is continuous on $[c,d]$, then the length of the curve $x=g(y)$ from $y=c$ to $y=d$ is $L=\int_c^d \sqrt{1+(g'(y))^2}\,dy$

✧ Parametric Form

If a curve C is given by the parametric equations $x=f(t)$ and $y=g(t)$ such that C does not intersect itself on the interval $a \le t \le b$, then the arc length or the distance traveled by a particle along the curve is given by:

$$L = \int_a^b \sqrt{(f'(t))^2+(g'(t))^2}\ dt$$

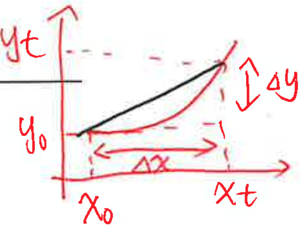
$$\sqrt{(dx)^2+(dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt = \sqrt{(f'(t))^2+(g'(t))^2}\ dt$$

The magnitude of displacement of a particle is the distance between its initial and final positions. The displacement of a particle between time $t=a$ and $t=b$ is given by:

$$|Displacement| = \sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$= \sqrt{\left(\int_a^b x'(t)dt\right)^2+\left(\int_a^b y'(t)dt\right)^2}$$

OR $x(b)-x(a)$ OR $y(b)-y(a)$



$\Delta x = \int_a^b x'(t)\,dt$

$\Delta y = \int_a^b y'(t)\,dt$

Practice.

1. **(Calculator)**

A particle moves in the xy-plane so that its position at any time t, for $0 \le t$, is given by $x(t)=e^t$ and $y(t)=2\cos(t)$.

$x'(t)=e^t \quad y'(t)=-2\sin t$

(a) Find the distance traveled by the particle from $t=0$ to $t=2$.

(b) Find the magnitude of the displacement of the particle between time $t=0$ and $t=2$.

(a) $\int_0^2 \sqrt{(e^t)^2+(-2\sin t)^2}\ dt \approx 7.035$

(b) $\sqrt{(x(2)-x(0))^2+(y(2)-y(0))^2} \approx 6.989$

2. **(Calculator)**

A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at any time t, where $\frac{dx}{dt}=2\sin(t^2)$ and $\frac{dy}{dt}=\cos(t^3)$. At time $t=1$, the object is at position $(3,2)$.

(a) Write an equation for the line tangent to the curve at $(3,2)$.

(b) Find the total distance traveled by the particle from $t=1$ to $t=3$.

(c) Find the position of the particle at time $t=3$.

(d) Find the magnitude of the displacement of the particle between $t=1$ and $t=3$.

(a) slope $= \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=1} = \frac{\cos 1}{2\sin 1}$

$\therefore$ tangent line: $y-2 = \frac{\cos 1}{2\sin 1}\cdot(x-3)$

(b) $\int_1^3 \sqrt{(x'(t))^2+(y'(t))^2}\ dt$

$= \int_1^3 \sqrt{(2\sin(t^2))^2+\cos^2(t^3)}\ dt$

≈ 3.166

(c) $x(3) = x(1) + \int_1^3 \frac{dx}{dt}\,dt \approx 3.927$

$y(3) = y(1) + \int_1^3 \frac{dy}{dt}\,dt \approx 1.877$

(d) $\Delta x = x(3)-x(1) \approx 0.927$

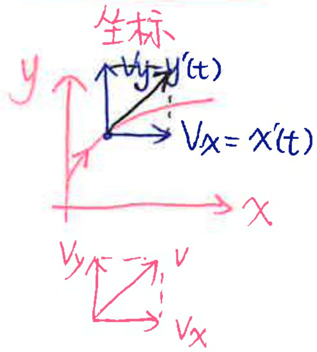
$\Delta y = y(3)-y(1) \approx 0.123$

$|displacement| \approx \sqrt{(0.927)^2+(0.123)^2} \approx 0.935$

➢ **Vector Valued Functions**

$=(x(t), y(t))$

If a particle moves in the xy –plane so that at time $t > 0$ its position vector is given by $r(t) =< x(t), y(t) >$, then the velocity vector, acceleration vector, and speed at time t are:



✧ **Velocity** $= v(t) = r'(t) =$ $\langle x'(t), y'(t) \rangle$

✧ ***Acceleration*** $= a(t) = r''(t) =$ $\langle x''(t), y''(t) \rangle$

✧ **Speed** $= |v(t)| =$ $\sqrt{(V_x)^2 + (V_y)^2} = \sqrt{(x'(t))^2 + (y'(t))^2}$

$x(t) \downarrow \Leftrightarrow x'(t) < 0$

✧ $x(t)$ is increasing when $x'(t) > 0$

$y(t)$ is increasing when $y'(t) > 0$

$y(t) \downarrow \Leftrightarrow y'(t) < 0$

Practice.

1. A particle moving in the xy-plane is defined by the vector-valued function $f(t) =< t - \sin t, 1 - \cos t >$, for $0 \le t \le \pi$.

(a) Find the velocity vector for the particle at any time t. (a) $V(t) = \langle 1 - \cos t, \sin t \rangle$

(b) Find the speed of the particle when $t = \frac{\pi}{3}$. (b) Speed $= \sqrt{(1-\cos t)^2 + \sin^2 t}$ at $t = \frac{\pi}{3}$:

(c) Find the acceleration vector for the particle at any time t.

(d) Find the average speed of the particle from time $t = 0$ to time $t = \pi$.

speed $= \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = 1$

(c) $a(t) = \langle \sin t, \cos t \rangle$

(d) $\frac{1}{\pi}\int_0^{\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt = \frac{1}{\pi}\int_0^{\pi} \sqrt{1 + \cos^2 t - 2\cos t + \sin^2 t}\, dt$

$= \sqrt{2} \cdot \sqrt{1 - \cos t}$

$I = \frac{1}{\pi}\sqrt{2}\cdot\sqrt{2}\cdot 2\left[-\cos\frac{t}{2}\right]_0^{\pi}$

$= \frac{4}{\pi}$

$\int \sqrt{1-\cos t}\, dt = \int \sqrt{2} \sin\frac{t}{2}\, dt = \sqrt{2}\, 2(-\cos\frac{t}{2}) + C$

$\cos t = 1 - 2\sin^2\frac{t}{2}$

C 2. In the xy- plane, a particle moves along the curve defined by the equation $y = 2x^4 - x$ with a constant speed of 20 units per second. If $\frac{dy}{dt} > 0$, what is the value of $\frac{dx}{dt}$ when the particle is at the point (1, 1)

(A) $\sqrt{2}$ (B) 2 (C) $2\sqrt{2}$ (D) 4

speed $= 20 = \sqrt{(x'(t))^2 + (y'(t))^2}$

$\therefore (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 20^2$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (8x^3 - 1) \cdot \frac{dx}{dt}$

$\therefore (\frac{dx}{dt})^2 \cdot [1 + (8x^3 - 1)^2] = 20^2$

when $x = 1$, $(\frac{dx}{dt})^2 \cdot 50 = 20^2$

$\therefore \frac{dx}{dt} = \sqrt{\frac{400}{50}} = 2\sqrt{2}$

(Cal)

3. An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where $\frac{dx}{dt} = 1 + \cos(e^t)$.

and $\frac{dy}{dt} = e^{(2-t^2)}$ for $t \geq 0$.

(a) $(\text{speed})^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9$

(a) At what time t is the speed of the object 3 units per second?

$(1+\cos(e^t))^2 + e^{(4-2t^2)} = 9$

(b) Find the acceleration vector at time $t = 2$.

calculator: $t \approx 0.95$

(c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 4$.

(d) Find the magnitude of the displacement of the object over the time interval $1 \leq t \leq 4$.

(b) $a(t) = \langle x''(t), y''(t) \rangle$

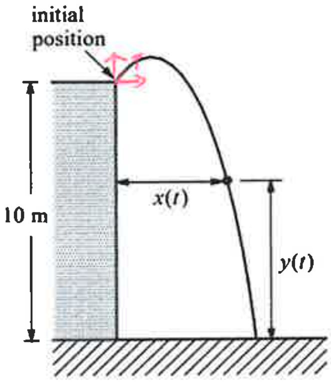
$= \langle -\sin(e^t)e^t, e^{(2-t^2)}(-2t) \rangle$

$a(2) = \langle -\sin(e^2)\cdot e^2, -\frac{4}{e^2} \rangle$

(c) $\int_1^4 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \approx 3.544$

(d) $|\text{displacement}| = \sqrt{\left(\int_1^4 x'(t)\,dt\right)^2 + \left(\int_1^4 y'(t)\,dt\right)^2}$ ≈ 2.954

4. (Cal)



Note: Figure not drawn to scale.

An object is thrown upward into the air 10 meters above the ground. The figure above shows the initial position of the object and the position at a later time. At time t seconds after the object is thrown upward, the horizontal distance from the initial position is given by $x(t)$ meters, and the vertical distance from the ground is given by $y(t)$ meters, where $\frac{dx}{dt} = 1.4$ and $\frac{dy}{dt} = 4.2 - 9.8t$, for $t \geq 0$.

(a) when $\frac{dy}{dt} = 0$, $t \approx 0.429$

(a) Find the time t when the object reaches its maximum height.

(b) Find the maximum vertical distance from the ground to the object. (b) $y(0.429) = y(0) + \int_0^{0.429} 4.2 - 9.8t\,dt$ ≈ 10.900

(c) Find the time t when the object hit the ground. (c) $y(t) = 0$ $\quad \int 4.2 - 9.8t\,dt = 4.2t - 4.9t^2 + C$

(d) Find the total distance traveled by the object from time $t = 0$ until the object hit the ground.

(e) Find the magnitude of the displacement of the object from time $t = 0$ until the object hit the ground.

$\because y(0) = 10$

$\therefore C = 10$

(f) Find the angle θ, $0 < \theta < \frac{\pi}{2}$, between the path of the object and the ground at the instance the object hit the ground.

$\therefore y(t) = 4.2t - 4.9t^2 + 10$

let $y(t) = 0$ then $t \approx 1.92$

(d) $\int_0^{1.92} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \approx 12.384$

(e) $\sqrt{\left(\int_0^{1.92} x'(t)\,dt\right)^2 + \left(\int_0^{1.92} y'(t)\,dt\right)^2} \approx 10.354$

(f) $v(1.92) = \langle x'(1.92), y'(1.92) \rangle = \langle 1.4, -14.616 \rangle$

$\tan\theta = \frac{14.616}{1.4}$ $\quad \therefore \theta \approx 1.475$