- **The substitution Rule**

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u) du$$

If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Example 1. $\int \sin x \cos x \, dx$

Example 2. $\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx$

Q1. $\int \cos(5\theta - 3)\, d\theta =$

Q2. $\int \frac{x}{\sqrt{1-x^2}} dx =$

Q3. $\int_0^{\frac{\pi}{2}} \frac{3 \cos x}{\sqrt{1+3 \sin x}} dx =$

Q4. If $\int_{-1}^{3} f(x+k)\,dx = 8$, where k is a constant, then $\int_{k-1}^{k+3} f(x)\,dx =$______________

Q5. $\int_{0}^{\frac{\pi}{4}} \frac{\tan x}{\sqrt{\sec x}}\,dx =$

Q6. $\int_{e}^{e^2} \frac{(\ln x)^2}{x}\,dx =$

Q7. $\int_{0}^{\frac{\pi}{4}} (e^{\tan x} + 2)\sec^2 x\,dx =$

Q8. $\int_{0}^{\frac{\pi}{2}} e^{\sin x}\cos x\,dx =$

Basic Rules

0. Expand

$(1+e^x)^2 =$

➢ **Rational Functions**

$\int \frac{1}{1+x^2}\,\mathrm{d}x =$

$\int \frac{x}{1+x^2}\,\mathrm{d}x =$

Basic Rules

1. Separate numerator

$\int \frac{1+x}{1+x^2}\,\mathrm{d}x =$

✧ **If the greatest power of the numerator is larger than or equal to that of the denominator:**

2. Divide improper fractions

$\frac{x^2+1}{x^2-1} =$

$\frac{x^3-3x}{x^2-1} =$

3. Add and subtract terms in numerator ~ Aim to construct the derivative of the denominator

$\frac{2x}{x^2+2x+1} =$

4. Complete the square

【Review】

$\int \frac{1}{1+x^2}\,dx =$

$\int \frac{1}{\sqrt{1-x^2}}\,dx =$

$\int \frac{1}{x\sqrt{x^2-1}}\,dx =$

Q1. $\int \frac{1}{x^2-2x+2}\,dx =$

Q2. $\int \frac{1}{4+x^2} dx =$

Q3. $\int \frac{1}{\sqrt{9-x^2}} dx =$

Q4. $\int \frac{1}{x\sqrt{x^2-9}} dx =$

Summary:

$\int \frac{1}{a^2+x^2} dx =$

$\int \frac{1}{\sqrt{a^2-x^2}} dx =$

$\int \frac{1}{x\sqrt{x^2-a^2}} dx =$

Q5. $\int \frac{1}{\sqrt{-x^2+4x+5}} dx =$

Q6. $\int \frac{1}{x^2+4x+8} dx =$

- **Practice**

1. $\int_2^3 \frac{1}{x^2-4x+5}\,dx =$

2. $\int \frac{1}{1-e^x}\,dx =$

3. $\int \frac{e^{2x}}{1+e^x}\,dx =$

4. $\int \frac{1-2x}{1+x^2}\,dx =$

5. $\int \frac{2x}{x^2+2x+1}\,dx =$

6. $\int \frac{1+\sin x}{\cos^2 x}\,dx =$

7. $\int \tan x\, dx =$

   $\int \sec x\, dx =$

8. (*) The region bounded by $y = \frac{\sin x}{\sqrt{\cos x}}, x = 0, x = \frac{\pi}{4}$, and the x-axis, is revolved around the x-axis. What is the volume of the resulting solid? Hint: $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

- **Trigonometric Integrals**

$\sin^2 x + \cos^2 x =$

$\int \boldsymbol{\sin^m x \cos^n x \, dx}$

1. m is odd

$\int \sin^3 x \cos^2 x \, dx =$

2. n is odd

$\int \sin^2 x \cos^3 x \, dx =$

降幂： $\sin^2 x =$ $\cos^2 x =$

3. m & n are even

$\int \boldsymbol{\sin^4 x \, dx} =$

$\int \boldsymbol{\sin^2 x \cos^2 x \, dx} =$

$\int \tan^m x \sec^n x \, dx$

$\tan^2 x + 1 =$ $\sec^2 x - 1 =$

1. ***m* is odd** $\sim$ Save one "$\sec x \tan x$" to construct "$d(\sec x)$". Then use $\tan^2 x = \sec^2 x - 1$ to transfer all $\tan x$ to $\sec x$. → Power functions always have corresponding antiderivatives.

$\int \tan^3 x \sec^2 x \, dx =$

$\int \tan^3 2x \sec^2 2x \, dx =$

2. ***n* is even**

$\int \tan^2 x \sec^4 x \, dx =$

3. **m is even**, n=0

$\int \tan^2 x \, dx =$

$\int \tan^4 x \, dx =$

$\int \tan^6 x \, dx =$

- **Practice**

1. $\int \sin^3 nx \, dx =$

2. $\int \sin^2 nx \, dx =$

3. $\int \cos^3 x \sqrt{\sin x} \, dx =$

4. $\int \tan^2 x \sec^2 x \, dx =$

5. $\int \tan^5 x \sec^2 x \, dx =$

6. $\int_0^{\frac{\pi}{4}} \tan^2 x \sec^4 x \, dx =$

- **Trigon Substitution**

$\sin^2 x + \cos^2 x =$

$\tan^2 x + 1 =$ $\qquad$ $\sec^2 x - 1 =$

1. $\int \sqrt{9 - x^2}\, dx =$

2. $\int_0^3 \frac{1}{\sqrt{9+x^2}}\, dx =$

3. $\int_3^6 \frac{1}{x^2\sqrt{x^2-9}}\, dx =$

- **Integration by partial fractions**

Rewriting a rational function into the sum of simpler rational functions.

$$\frac{2x+1}{(x+1)(x+2)^2} =$$

$$\int \frac{2x+1}{(x+1)(x+2)^2}\,dx =$$

Practice:

1. $\int \frac{x^3}{x^2-1}\,dx =$ ______________________

2. $\int \frac{5x+1}{x^2+x-2}\,dx =$ ______________________

3. $\int \frac{x+10}{(x-4)(x+3)}\,dx =$ ______________________

? When should you choose the method of substitution, and when should you use partial fractions?

- **Integration by parts**

Formula: $\int u\, dv =$ ____________________

- **Guidelines for Integration by parts**

After finishing all the questions below, answer: What are the methods for choosing u and dv when integrating by parts?

1. $\int x^2 e^{ax} dx=$

2. $\int x^2 \sin 2x\, dx=$

3. $\int x^3 \ln x \, dx=$

4. $\int x \sin^{-1} x \, dx =$

5. $\int \tan^{-1} x \, dx=$

6. $\int e^x \sin x \, dx =$

Practice:

1. $\int x \sin x \, dx$

2. $\int x \tan^{-1} x \, dx$

3. $\int e^{x} \cos x \, dx$=

4. $\int x^{2} sin(2x^{3}) \, dx$

- **Improper Integrals**

✧ Improper Integrals with **Infinite Integration Limits**

1. If $f(x)$ is continuous on $[a, \infty)$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$

2. If $f(x)$ is continuous on $(-\infty, b]$, then $\int_{-\infty}^b f(x)\,dx =$

3. If $f(x)$ is continuous on R, then $\int_{-\infty}^\infty f(x)\,dx =$

Example.

1. $\int_0^\infty xe^{-x^2}\,dx =$

2. $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2} =$

✧ Improper Integrals with **Infinite Discontinuities**

1. If $f(x)$ is continuous on $[a, b)$ and has an infinite discontinuity at b, then $\int_a^b f(x)\,dx = \lim_{\_\_\_\_} \int_a^t f(x)\,dx$

2. If $f(x)$ is continuous on $(a, b]$ and has an infinite discontinuity at a, then $\int_a^b f(x)\,dx =$

3. If $f(x)$ is continuous on $[a, b]$ except some number c in (a, b) at which f has an infinite discontinuity, then $\int_a^b f(x)\,dx =$

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the value of the improper integral. If the limit fails to exists, the improper integral **diverges**.

Example.

1. $\displaystyle\int_1^5 \frac{dx}{\sqrt{x-1}} =$

2. $\displaystyle\int_0^1 \frac{dx}{1-x} =$

3. If $\displaystyle\int_0^1 \frac{ke^{-\sqrt{x}}}{\sqrt{x}}\,dx = 1$, what is the value of k?