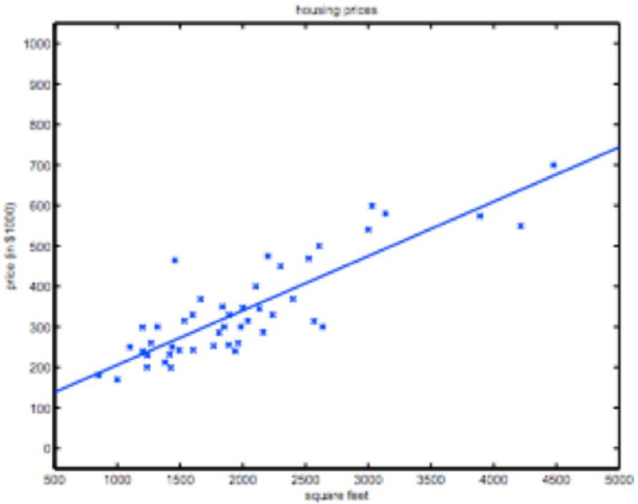
Logistic Regression

GA DAT3

Agenda

1. Linear Regression Review
2. From Linear to Logistic
3. Performance Measures
4. GLMs, Exponential Family
5. Relationship to Naive Bayes

Linear Reg Review



$$h(x) = \sum_{i=0}^{n} \theta_i x_i = \theta^T x$$

Squared Error Loss

$$J(\theta) = \frac{1}{2}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2$$

Why?

Probabilistic Interpretation

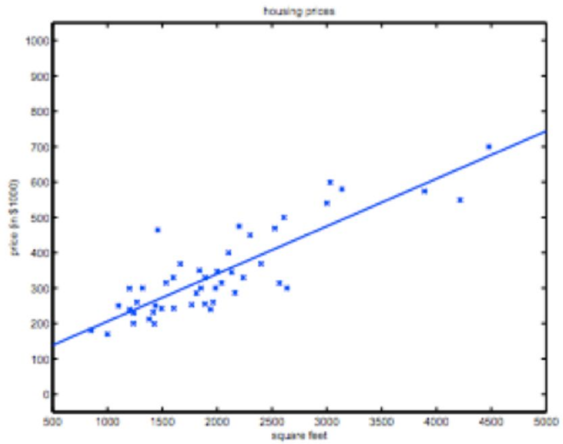
$$y^{(i)} = \theta^T x^{(i)} + \epsilon^{(i)}$$

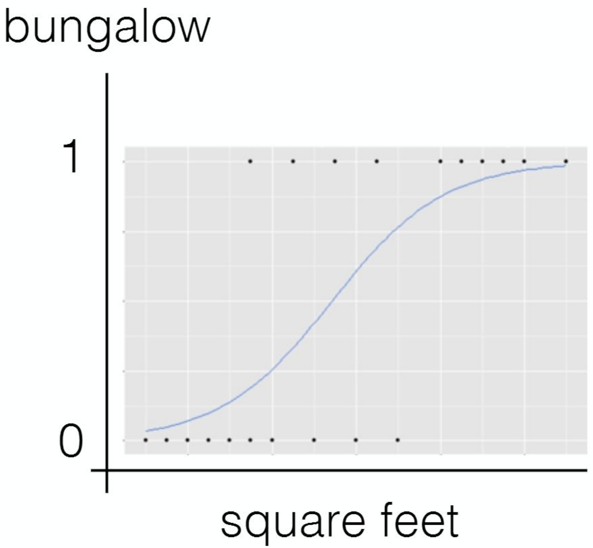
$$p(\epsilon^{(i)}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\epsilon^{(i)})^2}{2\sigma^2}\right).$$

We can minimise **cost**
or
maximise likelihood

What's the likelihood?

Linear to Logistic

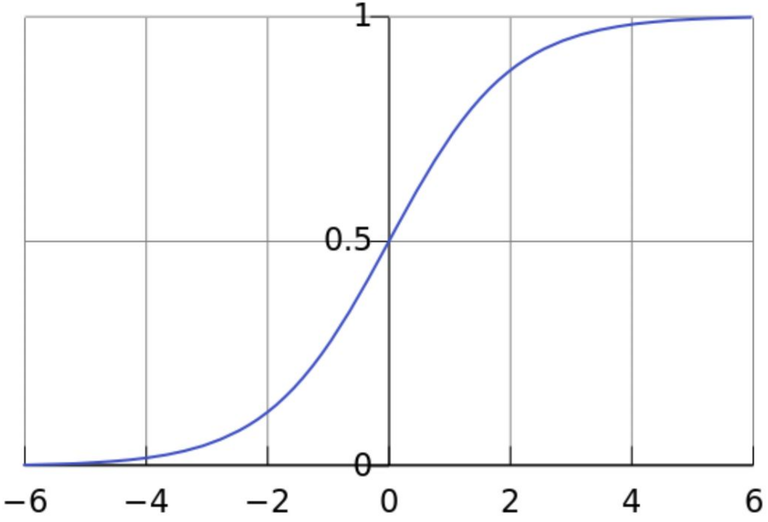




Logistic function

The logistic function always returns a value between zero and one.

$$F(t) = \frac{1}{1 + e^{-t}}$$



Problems with just using linear regression to classify?

Classification vs Clustering?

Examples of Classification?

Performance Measures

Predict whether tumors are malignant or benign:

- **Accuracy**: fraction of instances that are classified correctly
    - does not differentiate between malignant tumors that were classified as being benign, and benign tumors that were classified as being malignant.
- In some problems, the costs associated with all types of errors may be the same
- In this problem, failing to identify malignant tumors is likely more severe than failing to identify benign tumors as malignant

Performance Measures

- **True positive**: correctly classifying a malignant tumor
- **True negative**: correctly classifying a benign tumor
- **False positive**: a benign tumor that is incorrectly classifier as being malignant
- **False negative**: a malignant tumor that is incorrectly classifier as being benign

Confusion Matrix

	p' (Predicted)	n' (Predicted)
p (Actual)	True Positive	False Negative
n (Actual)	False Positive	True Negative

Performance Measures

- ***Accuracy*** is the fraction of instances that were classified correctly

$$\mathrm{ACC} = \frac{(\mathrm{TP} + \mathrm{TN})}{(\mathrm{TP} + \mathrm{TN} + \mathrm{FP} + \mathrm{FN})}$$

Performance Measures

- ***Precision*** is the fraction of the tumors that were predicted to be malignant that are actually malignant.

$$P = TP / (TP + FP)$$

Performance Measures

- ***Recall*** (or True Positive Rate) is the fraction of malignant tumors that the system identified.

$$R = TP / (TP + FN)$$

Performance Measures

- ***Recall*** (or True Positive Rate) is the fraction of malignant tumors that the system identified.

$$R = TP / (TP + FN)$$

Performance Measures

- ***Fall-out*** *or false positive rate* ***(FPR):***

$$FPR = FP / (FP + TN)$$

Generalised Linear Models (GLMs)

We've seen

y | x ~ N(mu, sigma) ⟶ linear regression

y | x ~ Bernoulli(phi) ⟶ logistic classification

Can we find common ground?

The Exponential Family

$$p(y;\eta) = b(y)\exp(\eta^T T(y) - a(\eta))$$

The Exponential Family: Bernoulli

$$p(y;\eta) = b(y)\exp(\eta^T T(y) - a(\eta))$$

$$\begin{aligned}
p(y;\phi) &= \phi^y(1-\phi)^{1-y} \\
&= \exp(y\log\phi + (1-y)\log(1-\phi)) \\
&= \exp\left(\left(\log\left(\frac{\phi}{1-\phi}\right)\right)y + \log(1-\phi)\right)
\end{aligned}$$

What are…

$\eta =$

$$\begin{aligned}
T(y) &= \\
a(\eta) &= \\
&= \\
b(y) &=
\end{aligned}$$

The Exponential Family: Bernoulli

$$p(y;\eta) = b(y)\exp(\eta^T T(y) - a(\eta))$$

$$\begin{aligned} p(y;\phi) &= \phi^y(1-\phi)^{1-y} \\ &= \exp(y\log\phi + (1-y)\log(1-\phi)) \\ &= \exp\left(\left(\log\left(\frac{\phi}{1-\phi}\right)\right)y + \log(1-\phi)\right) \end{aligned}$$

What are…

$\eta = \log(\phi/(1 - \phi))$.

$$\begin{aligned} T(y) &= y \\ a(\eta) &= -\log(1-\phi) \\ &= \log(1+e^\eta) \\ b(y) &= 1 \end{aligned}$$

The Exponential Family: Normal

$$p(y;\eta) = b(y)\exp(\eta^T T(y) - a(\eta))$$

$$p(y;\mu) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(y-\mu)^2\right)$$

What are…

$$\begin{aligned}
\eta &= \\
T(y) &= \\
a(\eta) &= \\
&= \\
b(y) &=
\end{aligned}$$

The Exponential Family: Normal

$$p(y;\eta) = b(y)\exp(\eta^T T(y) - a(\eta))$$

$$\begin{aligned}
p(y;\mu) &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(y-\mu)^2\right) \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y^2\right)\cdot\exp\left(\mu y - \frac{1}{2}\mu^2\right)
\end{aligned}$$

What are…

$$\begin{aligned}
\eta &= \mu \\
T(y) &= y \\
a(\eta) &= \mu^2/2 \\
&= \eta^2/2 \\
b(y) &= (1/\sqrt{2\pi})\exp(-y^2/2).
\end{aligned}$$

Okay, okay…so who cares?

Constructing GLMs

1. Assume $y \mid x; \theta \sim \text{ExponentialFamily}(\eta)$

2. Given x, we want to predict $T(y)$, usually = y.
   We choose $h(x) = E[y|x]$

3. Further assume $\eta = \theta^T.x$

So we have a machinery we can crank

Constructing GLMs

Linear Regression

$$
\begin{aligned}
h_\theta(x) &= E[y|x;\theta] \\
&= \mu \\
&= \eta \\
&= \theta^T x.
\end{aligned}
$$

Logistic Classification

$$
\begin{aligned}
h_\theta(x) &= E[y|x;\theta] \\
&= \phi \\
&= 1/(1+e^{-\eta}) \\
&= 1/(1+e^{-\theta^T x})
\end{aligned}
$$

Coincidentally, this is how we get softmax regression…

Relationship to Naïve Bayes

Assuming $y \mid x \sim$ some distribution

Assuming $x \mid y \sim$ some distribution

e.g.
text classification
Gaussian Discriminant Analysis (GDA)

Q??