Basics on Computer Vision

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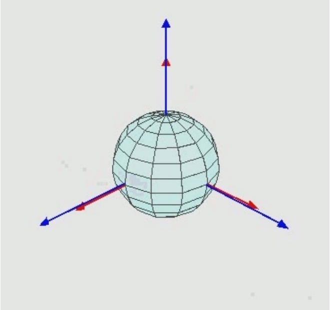
3D Transformation

3D Rotation Representations

- Rotation matrix

$$R_{3\times3} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

- Euler angles



- Axis-angle

$$\omega = \theta\hat{n}$$

- Unit quaternion

$$q = w + xi + yj + zk$$

3D Euclidean Transformation SE(3)

- 3D Rotation + 3D translation

$$x' = Rx + t$$

$$x' = [R|t]\bar{x}$$

$$\bar{x} = (x, y, z, 1)$$

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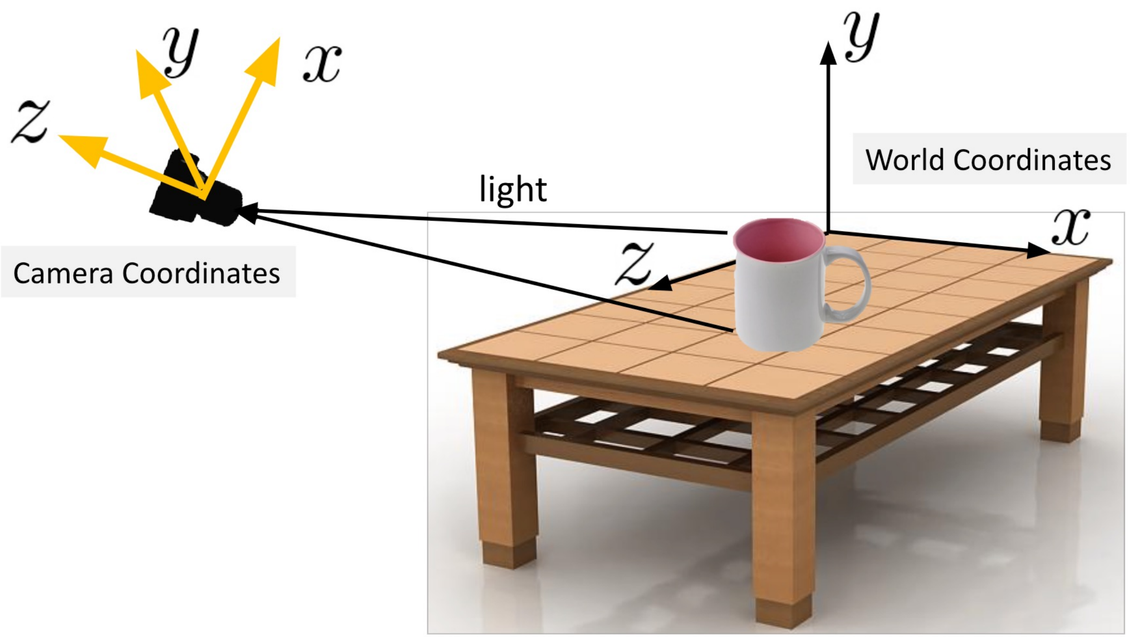
Perspective Camera Model

Perspective Camera Model



- Camera intrinsics

$$K = \begin{bmatrix} \alpha_x & s & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{bmatrix}$$

- Camera extrinsics

$$\mathbf{X}_{\mathrm{cam}} = R\mathbf{X} + \mathbf{t}$$

camera coordinates world coordinates

- Camera Projection Matrix

$$P = K[R|\mathbf{t}]$$

- World space point to image plane pixel

$$\mathbf{x} = P\mathbf{X}$$

- Camera space point to image plane pixel

$$\mathbf{x} = K[I|\mathbf{0}]\mathbf{X}_{\mathrm{cam}}$$

Back-projection to a 3D Point in Camera Coordinates

$$K = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = K^{-1} \cdot \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \cdot z$$

$$K^{-1} = \begin{bmatrix} 1/f_x & 0 & -c_x/f_x \\ 0 & 1/f_y & -c_y/f_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (u - c_x) \cdot \frac{z}{f_x} \\ (v - c_y) \cdot \frac{z}{f_y} \\ z \end{bmatrix}$$

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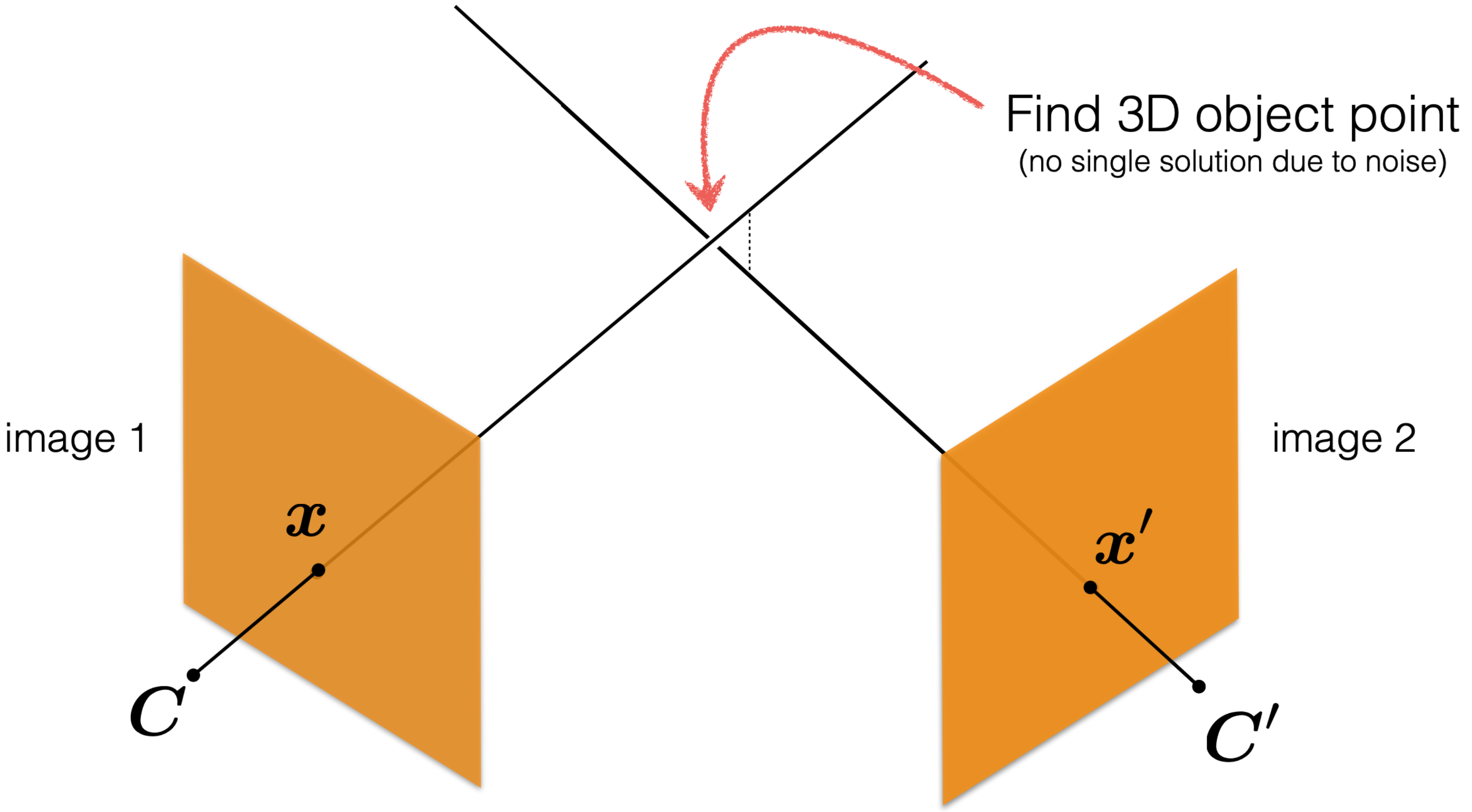
Triangulation

Triangulation



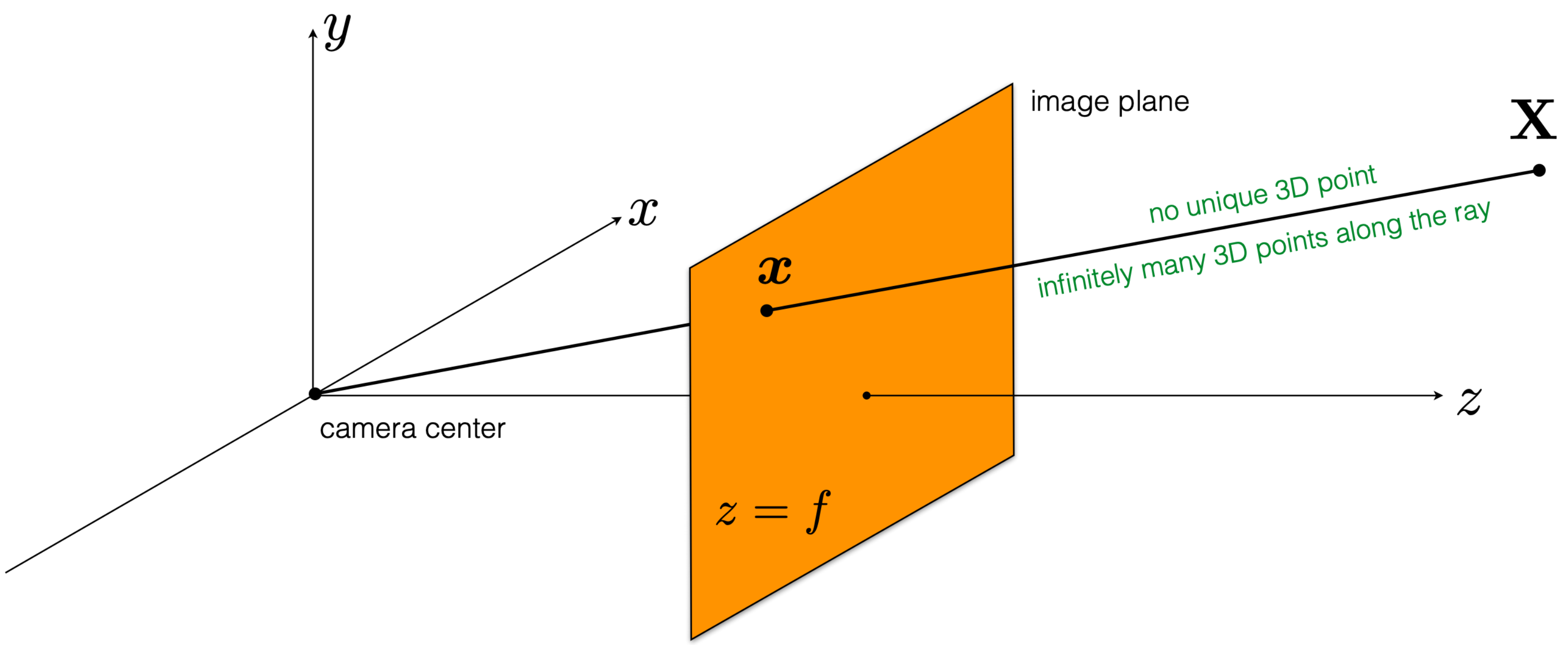
Triangulation

- Given:
  - A set of matched points $\{x_i, x_i'\}$
  - And camera matrices P, P'
- Estimate:
  - The 3D point X

Triangulation

- Q1: Can we compute X from a single correspondence x? **No!**

$$x = PX$$



Triangulation

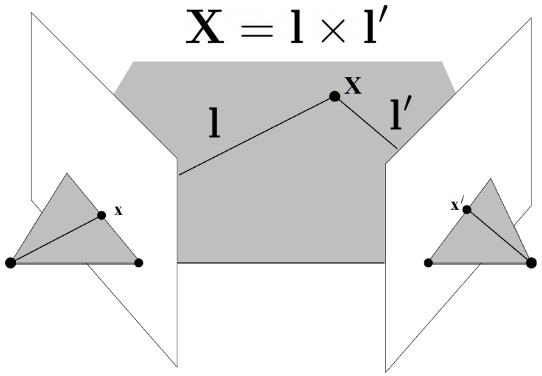
- Q2: Can we compute X from a <u>two</u> correspondences x and x'? **Maybe**

$$x = PX$$

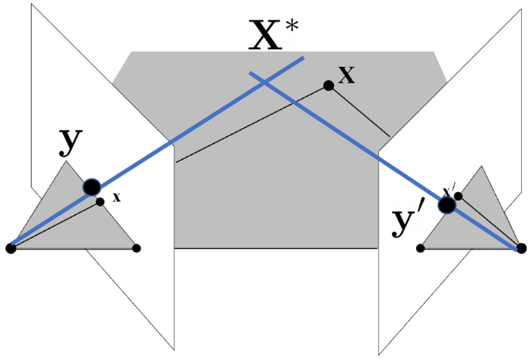
  - There will not be a point that satisfies both constraints because the measurements are usually **noisy**

$$x' = P'X \qquad x = PX$$

  - Need to find the **best fit**



Perfect measurements



Noisy measurements

- In practice, we find the correspondences

$$y, y'$$

- Find X^* that minimizes

$$d(x, PX^*) + d(x', P'X^*)$$

Triangulation

- Q2: Can we compute X from a <u>two</u> correspondences x and x'? **Maybe**

$$x = PX$$

  - There will not be a point that satisfies both constraints because the measurements are usually noisy

$$x' = P'X \qquad x = PX$$

  - Need to find the **best fit**
  - Same ray direction but differs by a scale factor

$$x = \alpha PX$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Triangulation

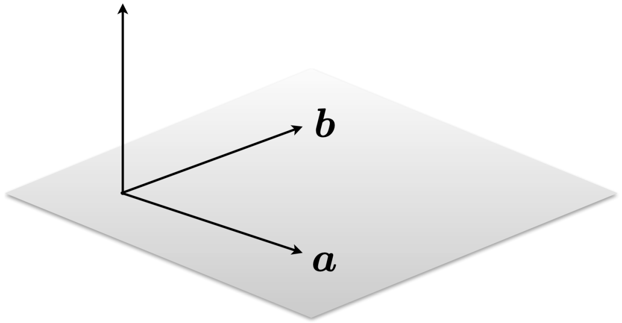
- Q3: How do we solve for unknowns in a similarity relation?

$$x = \alpha P X$$

  - Direct Linear Transform:
    - Remove scale factor, convert to linear system and solve with **SVD** (Singular Value Decomposition).
    - Recall: <u>Cross product of two vectors of same direction is zero</u>

$$x \times x = 0$$

$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$



$\boldsymbol{c} \cdot \boldsymbol{a} = 0$ $\boldsymbol{c} \cdot \boldsymbol{b} = 0$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Triangulation

- Q3: How do we solve for unknowns in a similarity relation?

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$$x \times x = x \times \alpha P X = 0$$

Triangulation - Convert to Linear System

- Rewrite x:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} - & p_1^T & - \\ - & p_2^T & - \\ - & p_3^T & - \end{bmatrix} X = \alpha \begin{bmatrix} p_1^T X \\ p_2^T X \\ p_3^T X \end{bmatrix}$$

- $x \times \alpha P X = 0$, because scaler α is **non-zero**, we **remove** α:

$$x \times PX = 0$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} p_1^T X \\ p_2^T X \\ p_3^T X \end{bmatrix} = \begin{bmatrix} y p_3^T X - p_2^T X \\ p_1^T X - x p_3^T X \\ x p_2^T X - y p_1^T X \end{bmatrix} = \boxed{\begin{bmatrix} y p_3^T - p_2^T \\ p_1^T - x p_3^T \\ x p_2^T - y p_1^T \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

Triangulation - Convert to Linear System

$$x \times PX = \begin{bmatrix} yp_3^T - p_2^T \\ p_1^T - xp_3^T \\ xp_2^T - yp_1^T \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Observation: Third line is **a linear combination** of the first and second lines.

$$xp_2^T X - yp_1^T X = -x(yp_3^T - p_2^T) - y(p_1^T - xp_3^T)$$

$$\begin{bmatrix} yp_3^T - p_2^T \\ p_1^T - xp_3^T \\ xp_2^T - yp_1^T \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} yp_3^T - p_2^T \\ p_1^T - xp_3^T \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Triangulation

- Q3: How do we solve for unknowns in a similarity relation?

$$x = \alpha P X$$

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$$x p_2^T X - y p_1^T X = -x(y p_3^T - p_2^T) - y(p_1^T - x p_3^T)$$

$$\begin{bmatrix} y p_3^T - p_2^T \\ p_1^T - x p_3^T \\ x p_2^T - y p_1^T \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \boxed{\begin{bmatrix} y p_3^T - p_2^T \\ p_1^T - x p_3^T \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

linear equations

Triangulation

- Q3: How do we solve for unknowns in a similarity relation?

$$x = \alpha PX$$

  - Direct Linear Transform:
    - Remove scale factor, **convert to linear system** and solve with **SVD**.
  - Now we can make a system of linear equations
    - Concatenate the 2D points from both images: $AX = 0$

Camera1

Camera2

$$\begin{bmatrix} y p_3^T - p_2^T \\ p_1^T - x p_3^T \\ y' p_3'^T - p_2'^T \\ p_1'^T - x' p_3'^T \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Homogeneous linear system!!!

Triangulation

- Q3: How do we solve for unknowns in a similarity relation?

$$x = \alpha PX$$

  - Direct Linear Transform:
    - Remove scale factor, convert to linear system and solve with **SVD**.
    - Recall: Cross product of two vectors of same direction is zero

$$x \times PX = 0$$

    - Observation: Third line is **a linear combination** of the first and second lines.
  - Now we can make a system of linear equations
    - Concatenate the 2D points from both images: $AX = 0$
  - To solve $AX = 0$, we apply SVD on A

$$A = UWV^T \Longrightarrow AV = UWV^TV = UW$$

$$A[v_1, \ldots, \boldsymbol{y}] = U \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{0} \end{bmatrix}$$

  - The last element $\boldsymbol{y}$ of V is the solution!!!