Continuous-Time Spline Visual-Inertial Odometry, Jacobians

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I. State

We include the state from the main paper for reference.

$$\begin{aligned}\mathbf{state} &= [f_x, f_y, c_x, c_y, s, r, p, \mathbf{kf}_1, \mathbf{kf}_2, ..., , \mathbf{kf}_N] \\ \mathbf{kf}_j &= [\mathbf{s}_{dso}, \mathbf{b}^a_j, \mathbf{b}^g_j, \mathbf{l}^r_j, \mathbf{q}^p_j, \mathbf{q}^r_j, \mathbf{c}^p_j, \mathbf{c}^r_j] \\ \mathbf{s}_{dso} &= [{}^w\mathbf{p}_{c_j}, {}^w\boldsymbol{\varphi}_{c_j}, a_j, b_j, d_{j1}, d_{j2}, ..., d_{jM}]\end{aligned}$$

II. Error State Jacobians

A. Acceleration

We synthesize acceleration from spline using the following equations (Eq. 14 & 15 in the main paper):

$$\begin{aligned}{}^i\mathbf{a}(t) &= {}^i_c\mathbf{R} \cdot {}^w_c\mathbf{R}(t)^T \cdot [s \cdot (2\mathbf{q}^p_j + 6t\mathbf{c}^p_j) + \mathbf{g}] + \mathbf{b}^a_j \qquad (1) \\ \mathbf{g} &= -9.8 \cdot [sin(p)cos(r), -sin(r), cos(p)cos(r)]^T\end{aligned}$$

We define the acceleration error to the accelerator measurement ${}^i\widetilde{\mathbf{a}}(t)$ (Eq. 22 in the main paper):

$$\mathbf{e}_{acc} = {}^i\mathbf{a}(t) - {}^i\widetilde{\mathbf{a}}(t)$$

The Jacobians with respect to the global variables in the state are:

$$\begin{aligned}\frac{\partial \mathbf{e}_{acc}}{\partial s} &= {}^i_c\mathbf{R} \cdot {}^w_c\mathbf{R}(t)^T \cdot (2\mathbf{q}^p_j + 6t\mathbf{c}^p_j) \\ \frac{\partial \mathbf{e}_{acc}}{\partial r} &= {}^i_c\mathbf{R} \cdot {}^w_c\mathbf{R}(t)^T \cdot -9.8 \cdot \begin{bmatrix} -sin(p)sin(r) \\ -cos(r) \\ -cos(p)sin(r) \end{bmatrix} \\ \frac{\partial \mathbf{e}_{acc}}{\partial p} &= {}^i_c\mathbf{R} \cdot {}^w_c\mathbf{R}(t)^T \cdot -9.8 \cdot \begin{bmatrix} cos(p)cos(r) \\ 0 \\ -sin(p)cos(r) \end{bmatrix}\end{aligned}$$

The Jacobian with respect to the acceleration bias $\mathbf{b}^a_j$ is:

$$\frac{\partial \mathbf{e}_{acc}}{\partial \mathbf{b}^a_j} = \mathbf{I}_{3\times3}$$

The Jacobians with respect to the position spline coefficients are:

$$\begin{aligned}\frac{\partial \mathbf{e}_{acc}}{\partial \mathbf{q}^p_j} &= {}^i_c\mathbf{R} \cdot {}^w_c\mathbf{R}(t)^T \cdot s \cdot 2 \\ \frac{\partial \mathbf{e}_{acc}}{\partial \mathbf{c}^p_j} &= {}^i_c\mathbf{R} \cdot {}^w_c\mathbf{R}(t)^T \cdot s \cdot 6t\end{aligned}$$

We expand ${}^w_c\mathbf{R}(t)$ in Eq. 1:

$$\begin{aligned}{}^w_c\mathbf{R}(t) &= {}^w_{c_j}\mathbf{R} \cdot exp((t\mathbf{l}^r_j + t^2\mathbf{q}^r_j + t^3\mathbf{c}^r_j)^\wedge) \\ &= exp({}^w\boldsymbol{\varphi}^\wedge_{c_j}) \cdot exp((t\mathbf{l}^r_j + t^2\mathbf{q}^r_j + t^3\mathbf{c}^r_j)^\wedge)\end{aligned}$$

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The Jacobian with respect to the keyframe orientation (left multiplication) is:

$$\frac{\partial \mathbf{e}_{acc}}{\partial {}^w\boldsymbol{\varphi}_{c_j}} = {}^i_c\mathbf{R} \cdot {}^w_c\mathbf{R}(t)^T \cdot [s \cdot (2\mathbf{q}^p_j + 6t\mathbf{c}^p_j) + \mathbf{g}]^\wedge$$

The Jacobians with respect to the rotation spline coefficients are:

$$\begin{aligned}\frac{\partial \mathbf{e}_{acc}}{\partial \mathbf{l}^r_j} &= {}^i_c\mathbf{R} \cdot \{{}^w_c\mathbf{R}(t)^T \cdot [s \cdot (2\mathbf{q}^p_j + 6t\mathbf{c}^p_j) + \mathbf{g}]\}^\wedge \cdot t \\ \frac{\partial \mathbf{e}_{acc}}{\partial \mathbf{q}^r_j} &= {}^i_c\mathbf{R} \cdot \{{}^w_c\mathbf{R}(t)^T \cdot [s \cdot (2\mathbf{q}^p_j + 6t\mathbf{c}^p_j) + \mathbf{g}]\}^\wedge \cdot t^2 \\ \frac{\partial \mathbf{e}_{acc}}{\partial \mathbf{c}^r_j} &= {}^i_c\mathbf{R} \cdot \{{}^w_c\mathbf{R}(t)^T \cdot [s \cdot (2\mathbf{q}^p_j + 6t\mathbf{c}^p_j) + \mathbf{g}]\}^\wedge \cdot t^3\end{aligned}$$

B. Angular Velocity

We synthesize angular velocity from spline using the following equation (Eq. 16 in the main paper):

$${}^i\boldsymbol{\omega}(t) = {}^i_c\mathbf{R} \cdot (\mathbf{l}^r_j + 2t\mathbf{q}^r_j + 3t^2\mathbf{c}^r_j) + \mathbf{b}^g_j$$

We define the angular velocity error to the gyroscope measurement ${}^i\widetilde{\boldsymbol{\omega}}(t)$ (Eq. 22 in the main paper):

$$\mathbf{e}_{gyro} = {}^i\boldsymbol{\omega}(t) - {}^i\widetilde{\boldsymbol{\omega}}(t)$$

The Jacobian with respect to the gyroscope bias $\mathbf{b}^g_j$ is:

$$\frac{\partial \mathbf{e}_{gyro}}{\partial \mathbf{b}^g_j} = \mathbf{I}_{3\times3}$$

The Jacobians with respect to the rotation spline coefficients are:

$$\begin{aligned}\frac{\partial \mathbf{e}_{gyro}}{\partial \mathbf{l}^r_j} &= {}^i_c\mathbf{R} \\ \frac{\partial \mathbf{e}_{gyro}}{\partial \mathbf{q}^r_j} &= {}^i_c\mathbf{R} \cdot 2t \\ \frac{\partial \mathbf{e}_{gyro}}{\partial \mathbf{c}^r_j} &= {}^i_c\mathbf{R} \cdot 3t^2\end{aligned}$$

III. Constraint Jacobians

A. Velocity Constraint

The velocity continuity constraint (Eq. 19) is:

$$\begin{aligned}\mathbf{c}_v(j) = &({}^w\mathbf{p}_{c_{j-1}} - {}^w\mathbf{p}_{c_j})/t_{j-1} - t_{j-1}\mathbf{q}^p_j - t^2_{j-1}\mathbf{c}^p_j \\ &- ({}^w\mathbf{p}_{c_j} - {}^w\mathbf{p}_{c_{j+1}})/t_j - t_j\mathbf{q}^p_{j+1} - 2t^2_j\mathbf{c}^p_{j+1} = \mathbf{0}\end{aligned}$$

The Jacobians with respect to the keyframe positions are:

$$\frac{\partial \mathbf{c}_v(j)}{\partial {}^w\mathbf{p}_{c_{j-1}}} = \frac{1}{t_{j-1}} \cdot \mathbf{I}_{3\times3}$$
$$\frac{\partial \mathbf{c}_v(j)}{\partial {}^w\mathbf{p}_{c_j}} = -(\frac{1}{t_{j-1}} + \frac{1}{t_j}) \cdot \mathbf{I}_{3\times3}$$
$$\frac{\partial \mathbf{c}_v(j)}{\partial {}^w\mathbf{p}_{c_{j+1}}} = \frac{1}{t_j} \cdot \mathbf{I}_{3\times3}$$

The Jacobians with respect to the position spline coefficients are:

$$\frac{\partial \mathbf{c}_v(j)}{\partial \mathbf{q}_j^p} = -t_{j-1} \cdot \mathbf{I}_{3\times3}$$
$$\frac{\partial \mathbf{c}_v(j)}{\partial \mathbf{c}_j^p} = -t_{j-1}^2 \cdot \mathbf{I}_{3\times3}$$
$$\frac{\partial \mathbf{c}_v(j)}{\partial \mathbf{q}_{j+1}^p} = -t_j \cdot \mathbf{I}_{3\times3}$$
$$\frac{\partial \mathbf{c}_v(j)}{\partial \mathbf{c}_{j+1}^p} = -2t_j^2 \cdot \mathbf{I}_{3\times3}$$

B. Rotation Constraint

The rotation continuity constraint (Eq. 17) is:

$$\mathbf{c}_r(j) = {}^w_{c_{j-1}}\mathbf{R}^T \cdot {}^w_{c_j}\mathbf{R} \cdot exp((t_{j-1}\mathbf{l}_j^r + t_{j-1}^2\mathbf{q}_j^r + t_{j-1}^3\mathbf{c}_j^r)^\wedge) = \mathbf{I}$$

The Jacobians with respect to the rotation spline coefficients are:

$$\frac{\partial \mathbf{c}_r(j)}{\partial \mathbf{l}_j^r} = {}^w_{c_{j-1}}\mathbf{R}^T \cdot {}^w_{c_j}\mathbf{R} \cdot exp(t_{j-1}\mathbf{l}_j^r + t_{j-1}^2\mathbf{q}_j^r + t_{j-1}^3\mathbf{c}_j^r) \cdot t_{j-1}$$
$$= t_{j-1} \cdot \mathbf{I}_{3\times3}$$
$$\frac{\partial \mathbf{c}_r(j)}{\partial \mathbf{q}_j^r} = {}^w_{c_{j-1}}\mathbf{R}^T \cdot {}^w_{c_j}\mathbf{R} \cdot exp(t_{j-1}\mathbf{l}_j^r + t_{j-1}^2\mathbf{q}_j^r + t_{j-1}^3\mathbf{c}_j^r) \cdot t_{j-1}^2$$
$$= t_{j-1}^2 \cdot \mathbf{I}_{3\times3}$$
$$\frac{\partial \mathbf{c}_r(j)}{\partial \mathbf{c}_j^r} = {}^w_{c_{j-1}}\mathbf{R}^T \cdot {}^w_{c_j}\mathbf{R} \cdot exp(t_{j-1}\mathbf{l}_j^r + t_{j-1}^2\mathbf{q}_j^r + t_{j-1}^3\mathbf{c}_j^r) \cdot t_{j-1}^3$$
$$= t_{j-1}^3 \cdot \mathbf{I}_{3\times3}$$

The Jacobians with respect to the keyframe orientations are:

$$\frac{\partial \mathbf{c}_r(j)}{\partial {}^w\boldsymbol{\varphi}_{c_j}} = {}^w_{c_{j-1}}\mathbf{R}^T$$
$$\frac{\partial \mathbf{c}_r(j)}{\partial {}^w\boldsymbol{\varphi}_{c_{j-1}}} = -{}^w_{c_{j-1}}\mathbf{R}^T$$