CHAPTER 9 VECTORS

1. Vectors: Geometric View

1.1. Definition of Vectors.

Definition 1.1.1. (Vectors) Quantities has both a **magnitude** (the "how much" or "how big" part) and a **direction** in space are called vectors. Magnitude of a vector is also called **norm** or **length**.

On notation of vectors, different books or authors have different habits, you may see **A** (boldface letter), $\vec{A}$ (with an arrow above), $\vec{\mathbf{A}}$ (two ways together) etc. I will stick to $\vec{A}$, because I like using boldface letter to make readers to pay attention to those words. Formally we have:

- position vector of point A which starts at the origin: $\overrightarrow{OA}$, we denote its magnitude as $\left|\overrightarrow{OA}\right|$[1.1] or OA.
- displacement vector of B relative to A: $\overrightarrow{AB}$, its length $\left|\overrightarrow{AB}\right|$

Finally, we say **two vectors are equal** if they have the same magnitude and direction, and **the negative of a vector** $\vec{a}$ is the vector parallel to $\vec{a}$ and with the same length, but in the opposite direction. It is denoted $-\vec{a}$.

1.2. Basic Operations.

In this part, I want to briefly talking about vector addition (subtraction) and vector multiplied by a scalar. As this is just a note, I will only give you some schematic diagram[1.2]:

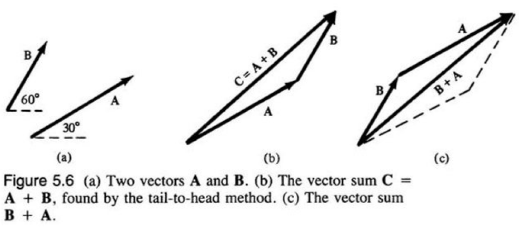

Figure 1

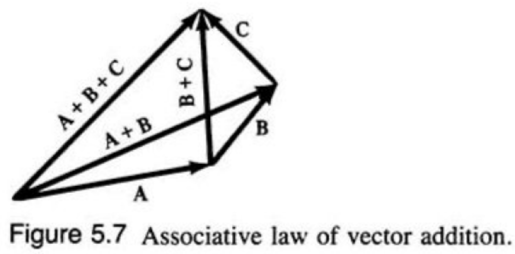

Figure 2

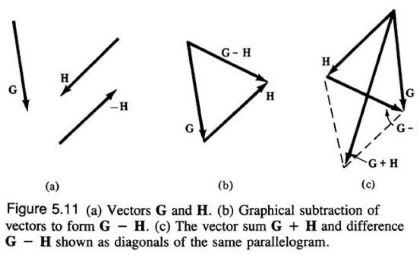

Figure 3

[1.1] Some books might use the symbol $\left\|\overrightarrow{OA}\right\|$ for the magnitude of a vector.

[1.2] The three pictures all come from the book *The Mechanical Universe: Mechanics and Heat* by Steven C. Frautschi, Richard P. Olenick, Tom M. Apostol and David L. Goodstein.

Pictures above show the way we doing vector addition or subtraction, an interesting fact is that, $\overrightarrow{BA}$, which is the sum of two vectors, is a diagonal of a parallelogram. However, as Figure 3 shows, the subtraction is another diagonal.

For a vector multiplied by a scalar, its just a stretching or compressing of the vector, omitted here. But one thing is important:

Theorem 1.2.1. (Parallel) Two non-zero vectors are parallel if and only if one is a scalar multiple of the other i.e.

$$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = \lambda \vec{b}. \tag{1.1}$$

1.3. The Scalar Product.

Definition 1.3.1. (The Scalar Product) The scalar product, also called dot product or inner product of two vectors, say $\vec{a}$, $\vec{b}$ is defined as:

$$\vec{a} \cdot \vec{b} = |\vec{a}| \left|\vec{b}\right| \cos\theta \tag{1.2}$$

Here, θ is the small angle between the two vectors (ranges from 0 to π).

We have another definition:(forgive my laziness) [1.3]

Definition 1.3.2. (Components):

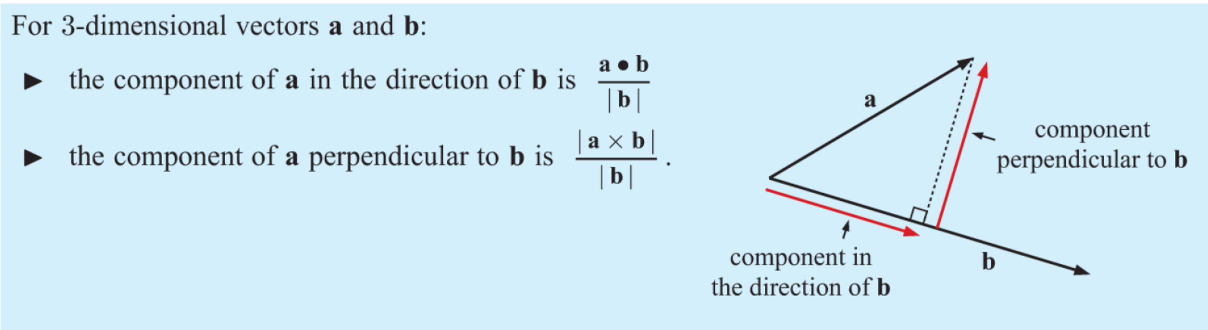

FIGURE 4

Please notice that component is a scalar! If you want the vector along the direction of $\vec{b}$ with magnitude as the component of $\vec{a}$ in the direction of $\vec{b}$, what you are searching for is a concept called projection:

Definition 1.3.3. (Projection)

$$\text{Projection } \vec{a} \text{ onto } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\left|\vec{b}\right|} \vec{b} \tag{1.3}$$

There is an useful schematic diagram for scalar product[1.4]:

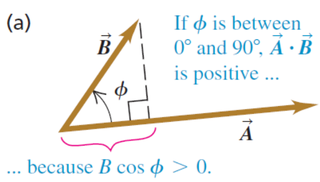

FIGURE 5

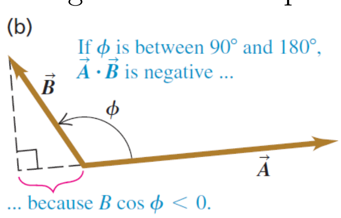

FIGURE 6

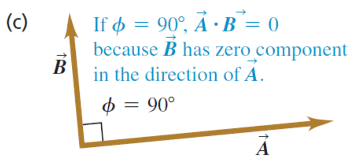

FIGURE 7

[1.3]The source of the figure is *Mathematics Application and Interpretation HL 2* by Michael Haese, Mark Humphries, Chris Sangwin and Ngoc Vo.

[1.4]These three pictures comes from *University Physics with Modern Physics* (14th ed.) by Young and Freedman

This shows that the sign of scalar product reveal the angle between two vectors, if the value of scalar product is positive, the angle will be an acute angle, negative for an obtuse angle, 0 for a right angle, actually we have:

Theorem 1.3.4. (Angle between Vectors)

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\left|\vec{b}\right|}. \tag{1.4}$$

Some special cases need us to pay more attention:

Theorem 1.3.5. (Perpendicular and Parallel)

$$\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b}=0 \tag{1.5}$$

$$\vec{a}\parallel\vec{b} \Leftrightarrow \left|\vec{a}\cdot\vec{b}\right| = |\vec{a}|\left|\vec{b}\right| \tag{1.6}$$

We will see more application of scalar product when introduction the Cartesian coordinate, just wait and see.

1.4. **The Vector Product.**

Definition 1.4.1. (The Vector Product) We denote the vector product of two vectors $\vec{a}$, $\vec{b}$, also called cross product, by $\vec{a}\times\vec{b}$. The result of it is a vector, and we define it by defining its magnitude and direction:

- magnitude:
$$\left|\vec{a}\times\vec{b}\right| = |\vec{a}|\left|\vec{b}\right|\sin\theta \tag{1.7}$$
- direction: $\vec{a}\times\vec{b}$ is perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$, and using the **right-hand rule** to choose the final direction.

At first glance, the definition of vector product is kind of complex, but there are also helpful schematic diagrams[1.5]:

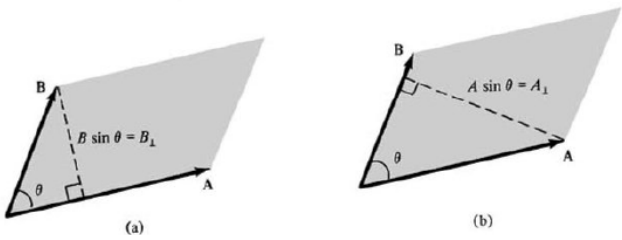

FIGURE 8

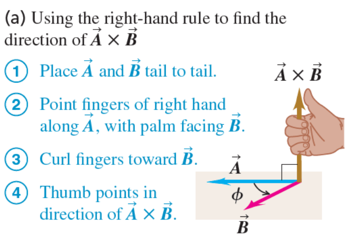

FIGURE 9

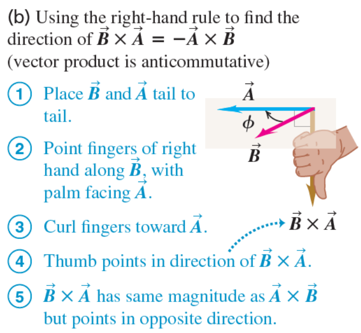

FIGURE 10

[1.5] The source of the first two figures are the same as in footnote 1.2, and the source of the last two figures are the same as in footnote 1.3.

Figure 8 shows that the magnitude of vector product is the area of the parallelogram "spaned" by the two vectors. Then Figure 9 and Figure 10 just show you how the right hand rule working. Notice that I have been ignoring the rules of operations of vectors, say like the commutative law of vector addition etc. The reason is that most of them are like the rules for numbers. However, the only difference just showed up, we have:

Theorem 1.4.2. (Anticommutative Law of Vector Product)

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \tag{1.8}$$

Vectors can be more applicable, but we have to bring something up here. Don't worry, it is Mr.Descartes comes to help.

2. Vectors with Cartesian Coordinate

2.1. Introducing Cartesian Coordinate.

I believe the readers have been quite familiar with the Cartesian coordinate system, so I just want to show you two graph of it[2.1]:

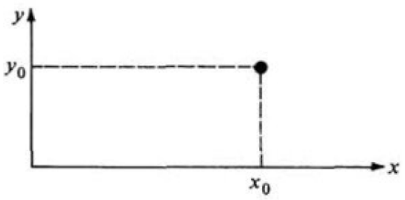

Figure 11

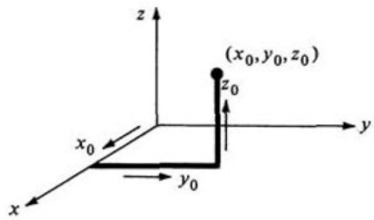

Figure 12

Here, only a little knowledge needs to be introduced:

Definition 2.1.1. (Unit Vector and $\vec{i}$, $\vec{j}$, $\vec{k}$) A unit vector is a vector that has a magnitude of 1, its only purpose is to point—that is, to describe a direction in space.

In an xy-coordinate system we can define a unit vector $\vec{i}$ **that points in the direction of the positive x-axis** and a unit vector $\vec{j}$ **that points in the direction of the positive y-axis**. If we are dealing with a xyz-coordinate, we define a unit vector $\vec{k}$ **that points in the direction of the positive z-axis**.

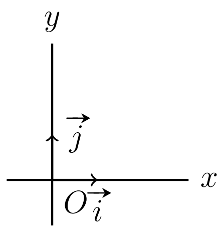

Figure 13

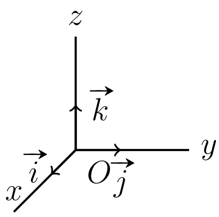

Figure 14

2.2. Converting.

Now, a critical problem, if we know a vector which means we know its magnitude and direction, how can we represent it in the Cartesian coordinate? Here suppose the meaning of the direction of a vector is denoted by the angle between the vector and x-axis which is θ. Which means we have[2.2]:

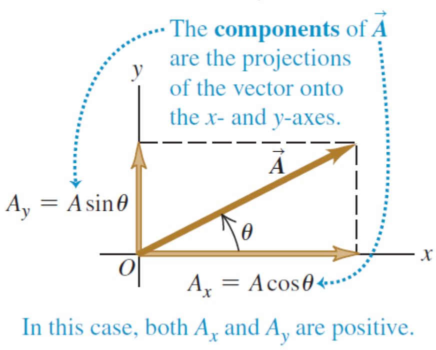

Figure 15

[2.1]The source of the two figures are the same as in footnote 1.2

[2.2]The source of the figure is the same as in footnote 1.3.

In the picture above, we have a vector $\vec{A}$ with magnitude $\left|\vec{A}\right|$, so it is easy to find its component on x-axis is $\left|\vec{A}\right|\cos\theta$ and component on y-axis is $\left|\vec{A}\right|\sin\theta$, thus, by vector addition we have:

$$\vec{A} = \left|\vec{A}\right|\cos\theta\,\vec{i} + \left|\vec{A}\right|\sin\theta\,\vec{j} \tag{2.1}$$

Then, we write[2.3]

$$\vec{A} = \begin{pmatrix} \left|\vec{A}\right|\cos\theta \\ \left|\vec{A}\right|\sin\theta \end{pmatrix} \tag{2.2}$$

Here, let $A_x = \left|\vec{A}\right|\cos\theta$, $A_y = \left|\vec{A}\right|\sin\theta$, by applying Pythagorean theorem, we have:

Theorem 2.2.1. (The Magnitude of Vectors)

$$\left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2} \tag{2.3}$$

It is not hard to see that in xyz-coordinates, the formula would be:

$$\left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{2.4}$$

A numerical example [2.4]:

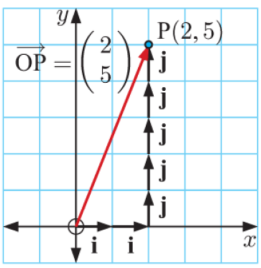

FIGURE 16

$$\text{Here } \overrightarrow{OP} = 2\,\vec{i} + 5\,\vec{j},\ \text{hence } \overrightarrow{OP} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \text{ and } \left|\overrightarrow{OP}\right| = \sqrt{2^2+5^2} = \sqrt{29}.$$

What I mean by this subsection is that, when we write a vector as things like $\vec{A} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, what we really mean is that $\vec{A} = 2\,\vec{i} + 5\,\vec{j}$.

[2.3]Some book prefer $\begin{bmatrix} \left|\vec{A}\right|\cos\theta \\ \left|\vec{A}\right|\sin\theta \end{bmatrix}$.

[2.4]The source of the figure is *Mathematics Application and Interpretation HL 2* by Michael Haese, Mark Humphries, Chris Sangwin and Ngoc Vo.

2.3. "Translating".

We finally made it here. The final jobs are to "translate" things in section 1 into coordinates.

Suppose we have three vectors:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

- $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3$;
- $-\vec{a} = -\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$
- $\vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$

Let me prove the third one to you:

$$\begin{aligned} \vec{a} + \vec{b} &= a_1\vec{i} + a_2\vec{j} + a_3\vec{k} + b_1\vec{i} + b_2\vec{j} + b_3\vec{k} \\ &= (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k} \\ &= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \end{aligned}$$

The proof of other results usually follow the same idea, so I will just list the result here now:

- $\vec{a} \parallel \vec{b} \Leftrightarrow a_1 = \lambda b_1, a_2 = \lambda b_2, a_3 = \lambda b_3$
- $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- $|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- By the way, if we want to find a unit vector whose direction is the same as $\vec{a}$, we can have $\dfrac{1}{|\vec{a}|}\vec{a}$
- $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$
- $\vec{a} \perp \vec{b} \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

The vector product or the cross product is a little bit hard:

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \\ &= a_1b_1\vec{i} \times \vec{i} + a_1b_2\vec{i} \times \vec{j} + a_1b_3\vec{i} \times \vec{k} \\ &\quad + a_2b_1\vec{j} \times \vec{i} + a_2b_2\vec{j} \times \vec{j} + a_2b_3\vec{j} \times \vec{k} \\ &\quad + a_3b_1\vec{k} \times \vec{i} + a_3b_2\vec{k} \times \vec{j} + a_3b_3\vec{k} \times \vec{k} \end{aligned}$$

Please notice that, by applying definition 1.4.1 and Anticommutative law 1.4.2, we have:

$$\vec{i} \times \vec{i} = \vec{0}, \vec{j} \times \vec{j} = 0, \vec{k} \times \vec{k} = 0$$
$$\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{i} = -\vec{k}$$
$$\vec{i} \times \vec{k} = -\vec{j}, \vec{k} \times \vec{i} = \vec{j}$$
$$\vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{j} = -\vec{i}$$

So keep our calculation, we will have:

$$\begin{aligned}\vec{a} \times \vec{b} &= a_1b_2\vec{k} - a_1b_3\vec{j} - a_2b_1\vec{k} + a_2b_3\vec{i} + a_3b_1\vec{j} - a_3b_2\vec{i} \\ &= (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k} \\ &= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}\end{aligned}$$

It is hard to remember this result, but if know determinants, then we can actually write it in another way:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{2.5}$$

2.4. **Triple Product.**

We have so called triple product. that is $\vec{c} \cdot (\vec{a} \times \vec{b})$ or $(\vec{a} \times \vec{b}) \cdot \vec{c}$ (they are the same, because dot product has commutative law):

Definition 2.4.1. (Triple Product) For the value of the triple product, we have:

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = |\vec{c}| \left|(\vec{a} \times \vec{b})\right| \cos\psi \tag{2.6}$$

Here, ψ is the angle between $\vec{c}$ and $(\vec{a} \times \vec{b})$. Or, in coordinate, using determinant, we have:

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{2.7}$$

The key thing is that there is a geometric meaning of triple product, its absolute value is the volume of the parallelepiped[2.5]:

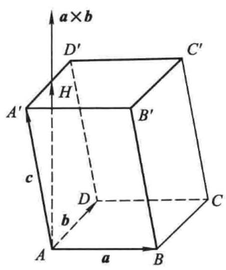

FIGURE 17

[2.5]The source of the figure is 《解析几何（第三版）》by 丘维声.

This is because $|\vec{c}|\cos\psi$ is actually the height of the parallelepiped, while $\left|(\vec{a} \times \vec{b})\right|$ is the area of the base of the parallelepiped. Hence it is obvious to see it.

3. *Epilogue

3.1. Inequalities.

Using dot product (scalar product) we can prove two famous and important inequalities:

Theorem 3.1.1. (Cauchy-Schwarz Inequalities)

$$\sum_{i=1}^{3} x_i y_i \leq \sqrt{\sum_{i=1}^{3} x_i^2}\sqrt{\sum_{i=1}^{3} y_i^2} \tag{3.1}$$

Proof. Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ We have

$$\cos\theta = \frac{x_1y_1 + x_2y_2 + x_3y_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}\sqrt{y_1^2 + y_2^2 + y_3^2}}$$

Clearly, $\cos\theta \leq 1$ Hence

$$\frac{x_1y_1 + x_2y_2 + x_3y_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}\sqrt{y_1^2 + y_2^2 + y_3^2}} \leq 1$$

Which is what we want:

$$\sum_{i=1}^{3} x_i y_i \leq \sqrt{\sum_{i=1}^{3} x_i^2}\sqrt{\sum_{i=1}^{3} y_i^2}$$

□

Maybe you are more familiar with this inequality when we are in 2-D plane, but the result and proof are pretty similar. Actually, the generalized version of it should be:

$$\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2} \tag{3.2}$$

You can prove it using the similar method here as long as you think or "believe" there is a angle in n dimension space. Otherwise, you shall use some other method. I put this at here because I usually forget this important inequality, but with vector, it is quite easy to remember, just:

$$\vec{x} \cdot \vec{y} \leq |\vec{x}|\,|\vec{y}| \tag{3.3}$$

Another import inequality is:

Theorem 3.1.2. (Triangle Inequality)

$$|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}| \tag{3.4}$$

Proof. We have:

$$\begin{aligned}
(|\vec{x} + \vec{y}|)^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\
&= |\vec{x}|^2 + 2\vec{x} \cdot \vec{y} + |\vec{x}|^2 \\
&\leq |\vec{x}|^2 + 2\,|\vec{x}|\,|\vec{y}| + |\vec{x}|^2 \\
&= (|\vec{x}| + |\vec{y}|)^2
\end{aligned}$$

Hence, we have

$$(|\vec{x} + \vec{y}|)^2 \leq (|\vec{x}| + |\vec{y}|)^2$$

Notice that both $|\vec{x}+\vec{y}|$ and $|\vec{x}|+|\vec{y}|$ are non-negative numbers, and we have a trivial result that $a^2 \leq b^2$ then $a < b$ provided $a \geq 0, b \geq 0$, so we have:

$$|\vec{x}+\vec{y}| \leq |\vec{x}|+|\vec{y}|$$

□

Forget about the proof for a second, notice that this inequality just said that **the length of one side of a triangular can not be longer than the sum of the length of the other two sides** (See in Figure 1). Which you shall have learned in your young age.

3.2. **Direction Cosine.**

Some book might also introduce the concepts direction angles and direction cosine:

Definition 3.2.1. (Direction Angles and Direction Cosine) The **direction angles** of a vector $\vec{a}$ is the angle between $\vec{a}$ and $\vec{i}, \vec{j}, \vec{k}$ i.e. the angle between the vector and the three positive coordinate axes. **Commonly, they will be denoted by** α**(between x-axis)**, β**(between y-axis)**, γ**(between z-axis)**. And the **direction cosines** (or directional cosines) of a vector are the cosines of the angles between the vector and the three positive coordinate axes.

Using dot product, the direction cosines are not hard to calculate, let $\vec{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$:

$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}|\left|\vec{i}\right|} = \frac{x}{\sqrt{x^2+y^2+z^2}} \tag{3.5}$$

$$\cos\beta = \frac{\vec{a}\cdot\vec{j}}{|\vec{a}|\left|\vec{j}\right|} = \frac{y}{\sqrt{x^2+y^2+z^2}} \tag{3.6}$$

$$\cos\gamma = \frac{\vec{a}\cdot\vec{k}}{|\vec{a}|\left|\vec{k}\right|} = \frac{z}{\sqrt{x^2+y^2+z^2}} \tag{3.7}$$

Two interesting things here, first:

$$\cos\alpha^2 + \cos\beta^2 + \cos\gamma^2 = 1 \tag{3.8}$$

Second, we have said in section 2.3 that to find a unit vector whose direction is the same as $\vec{a}$, we just need $\frac{1}{|\vec{a}|}\vec{a}$, here it should be:

$$\frac{1}{|\vec{a}|}\vec{a} = \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} \end{pmatrix}$$

Do you see that, we just find that:

$$\text{The unit vector for } \vec{a}\text{: } \frac{1}{|\vec{a}|}\vec{a} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} \tag{3.9}$$