A simple introduction to stochastic processes

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Abstract

Real world data most of the times are noisy, in the sense that they are distorted, and exhibit behaviour that cannot be described by a deterministic model. A method to analyse these behaviours and model the uncertainty (patterns not described by a deterministic model) is the use of stochastic processes. The theory of stochastic processes is considered to be an important contribution to mathematics and it continues to be an active topic of research for both theoretical reasons and applications. The task of this paper is to give a simple introduction to stochastic processes. The paper covers different matters like: a background to introduce the topic, the main properties of the process, the most common and used stochastic processes, a practical example through a C# application and a view of the main use cases.

Contents

Chapter 1

Introduction

Sometimes we are not interested in one aspect, but in the evolution of a phenomenon. Something that evolves in time or space. We want to model the object, like for example the price change of a stock. This is where stochastic processes come to our aid.

First, let us begin by giving a little historical background to the birth of the stochastic process. The first mathematical definition of a stochastic process was given by Khinchin in the early 1930s during a probability seminar. He defined it as a set of random variables indexed by the real line. After, in 1933, Andrei Kolmogorov published his book on the foundations of probability theory titled Grundbegriffe der Wahrscheinlichkeitsrechnung, where he used measure theory to develop an axiomatic framework for probability theory. The publication of this book is now widely considered to be the birth of modern probability theory, when the theories of probability and stochastic processes became parts of mathematics.

Before discussing the common definition of stochastic process, (from Greek "stochastikos": speculative, based on guesswork) [11], let's start with an example carried out during our lesson.
An homework, that we have done, consisted in the generation of a sequence of trials in which we had to compute the ratio of success to the number of trials (up to the moment), we had to calculate the so called relative frequency. We studied this quantity to see what happen asymptotically to n that tends to infinity. We saw that thanks to the **law of large numbers** this relative frequency tends to p. So the distribution becomes more and more concentrate to the p value, in the limit all the probability concentrate on p and we get a degenerate variable which takes only the variable p with probability 1. This is a first and simple example of a stochastic process. So, in this case, we have a sequence of random variables (f_1, f_2, ..., f_t, ..., f_n) which are associated with values of an index that identifies the trials (t_1, t_2, ..., t_t, ..., t_n). It is possible to view the implementation on Github at the following link[1].

[1] https://github.com/IReallyLikeYourPants/ireallylikeyourpants.github.io/tree/main/Homework/Homework%204/homework4

In general, we can define stochastic process as a family of random variables indexed through the element of another set. In our case the set T (trials) has the indexes (1, 2, ..., t, ..., n) and for each index we have associated a random variable which in our case is the relative frequency of success.

Giving the definition in this way, it is difficult to fully understand the subject. For this reason, we need first to explain the arguments concerning this definition. With this paper, we try to explain this definition in a simplistic manner and look at the most important processes.

The remaining part of the paper is subdivided as follows: Chapter 2 covers former definitions, main properties and useful information to understand the stochastic process. Chapter 3 deals with one of the simplest stochastic process, the random walk. Chapter 4 describes the most important stochastic processes like: Levy process, Brownian motion and Geometric Brownian motion. Chapter 5 analyse an implementation of the random walk in C# and in Chapter 6 we describe the uses cases and summarise our paper.

Chapter 2

Definitions and Properties

To understand and formally describe a stochastic process, we need first to start with the definition of probability space $(\Omega, \mathcal{F}, P)$ and random variable (X).

2.1 Probability space

A probability space $(\Omega, \mathcal{F}, P)$ consist of:

- *A sample space* (Ω): set of all the outcomes that we want to consider
- *The event space* $(\mathcal{F})$: set of all subset of interest from Ω such that the usual properties of σ algebra holds (for example must be close respect complementation, countable, union, must contain the sample set and the empty set, etc...)
- $\mathcal{P}$: probability measure. A measure which follows the axioms of Kolmogorov (usual definition of measure with the condition that the probability of the sample space is equal to 1)

On this probability space, which is a particular case of the measure space (where as measure we use probability), to talk about stochastic processes, we need to define random variables.

2.2 Random variable

We can define a **random variable**, usually denoted by X, as a measurable function from Ω into another measurable space (most of the time this would be the real line (R)) X: Ω -> R [18].
We consider the measurable space R with the σ algebra usually denoted with B(R) the Borel algebra (the measurable space).
Measurable means that for each set that we take from Borel algebra it is possible to specify its probability (the probability of the inverse image of the set

that we consider). For example if we take any b from the Borel algebra, the probability of b is equal to the probability of σ of those Ω belonging to the sample space such that they are mapped by x into b (definition of measurable function). In this way for each element of Borel algebra we have a probability. This would be the probability induced on this measurable space so that we get a new measurable space which is given by (R, B(R), P_b).

Now that we have recall the definition of probability space and random variable we may precede with a formal definition of **stochastic process**.

2.3 Definition of stochastic process

To define a stochastic process (random process) we need a probability space $(\Omega, \mathcal{F}, P)$ in which we have some random variables X (all from the **same** probability space).
So, from the probability space, we can start to peek some random variables (it is also possible to peek the same variable multiple times). In order to take and differentiate these variables we need to use some index as label. In general what we get is something that we can denote as $\{x_t\}$, where t is an index that belongs to the set T (set of the indexes).
This set of indexes could be finite, infinite, countable or uncountable. The object (function) thanks to which we associate indexes and random variables is called a stochastic process.
From a formal point of view, it is just a function from a set of indexes T to the set of random variables. The usual definition is: a function (which is also call sometimes an **indexed family** or collection) from a set of indexes, which could be of any type (finite, infinite, countable, uncountable) to a set of random variables (all from the same probability space) [4, 13]. In an another way it is a relationship between a set of indexes and a set of random variables along with the domain and codomain. Indexed family is defined as a mathematical function together with its domain I (in our case T) and image X. Often the elements of the set X are referred to as making up the family (in our case family of random variables) [10].
The usual notation is $\{x_t\}$ where t $\in$ T. Another important notation is $X(t, \omega) : t \in T$. It is used to reflect that it is a function of two variables $t \in T$ and $\omega \in \Omega$.

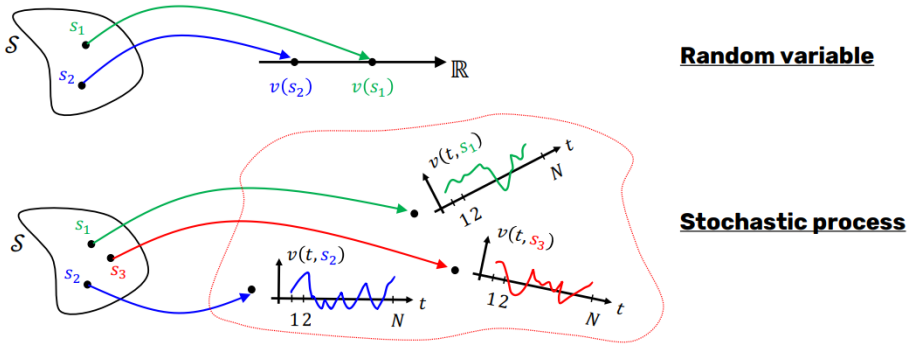


Figure 2.1: Difference between a random variable and a stochastic process[1]

2.4 General properties of a stochastic process

The index T can be defined as discrete or continuous:

- *discrete*: we have x_1, x_2, ..., x_n. In this case T is countable (a finite set or a countable infinite where there is a mapping 1 to 1 with nature numbers) [6]. Examples are the stochastic process of relative frequency of success at certain time and the random walk (with jumps at various discrete times)
- *continuous*: we might have x_t where t belongs in some interval for instance in (0, 1) or in any other one. In this case T is not countable. An example is the Poisson process, where there is a time interval (t, t + dt) and the probability of having an arrival in this interval is proportional to the size of the interval that in this case is lambda dt (a constant which increases the probability of having an arrival in an infinitesimal interval).
  Clearly when we generate a Poisson distribution on the computer we create a discrete version, instead of dt we consider 1/n as the size of the interval. As n tends to infinity, we generate infinitesimal small intervals (based on set t)

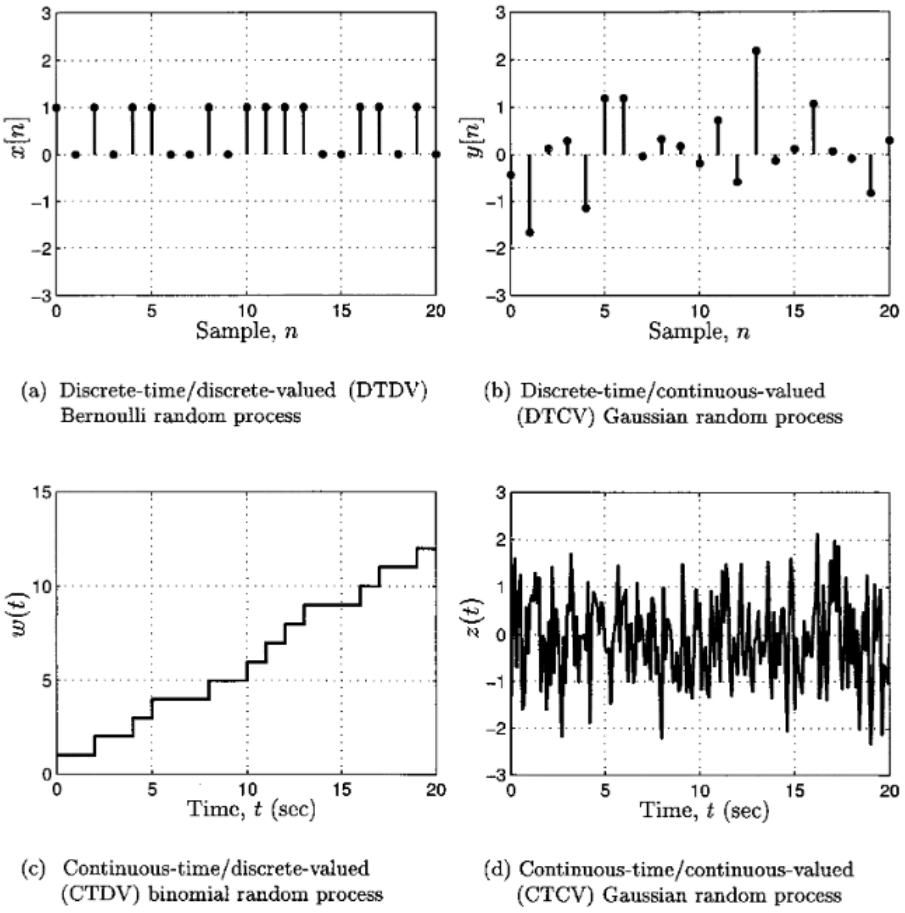
When t is a countable set, it usually represent time. In this case we talk about **discrete time process** otherwise we talk about **continuous time process**.
Most of the times the index T represents time, but it can represent anything for example also the spatial ratio.

Another classification concerns the random variables:

[1] https://cal.unibg.it/wp-content/uploads/DSI/slide/Lecture-12-Stochastic-process.pdf

- If they are discrete we talk about **discrete state process**
- Otherwise if they are continuous we talk about **continuous state process**

Combining these two kind of classification we can have 4 combinations:



(a) Discrete-time/discrete-valued (DTDV) Bernoulli random process

(b) Discrete-time/continuous-valued (DTCV) Gaussian random process

(c) Continuous-time/discrete-valued (CTDV) binomial random process

(d) Continuous-time/continuous-valued (CTCV) Gaussian random process

Figure 2.2: Typical realizations of different types of stochastic processes[2]

Other examples are:

- Discrete time and discrete state: Bernoulli trials and random walk
- Continuous time and discrete state: Poisson distribution

[2] http://guillefix.me/cosmos/static/Stochastic%2520process.html

- Discrete time and continuous state: Random walk where the increments are done instead of a Rademacher distribution through a normal distribution N(0, σ^2)
- Continuous time and continuous state: Brownian motion (price of financial instrument)

The codomain of the stochastic function is surjective but not injective. This means that each element of the index T has a value (random variable), and two different elements of T can have the same random variable.
The stochastic function is not a set, because we can have repeated elements (in a set it is not possible because there is not multiplicity). So, it can be defined as a **multi-set** (set where the multiplicity of the elements matter).
A stochastic process might also be a sequence (if we consider discrete index space) but not all sequences are stochastic processes because for the sequences, it is not required that the random variables are defined on the same probability space.

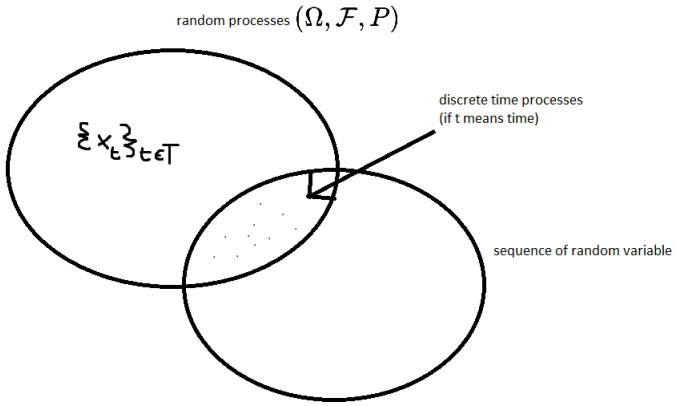


Figure 2.3: Explanation between the relationship of sequences and stochastic processes

As we said, we can consider the stochastic process as a function $X(t, \omega)$ of two variables $t \in T$ and $\omega \in \Omega$. This notation is useful because we can look at the stochastic process in a double view.

We can consider it as a function of t (where we fix t), in this case we can view it as a collection of random variables (figure 2.4).

Or we can look at this as a function of ω (where we fix ω). In this case we have different paths (trajectories), collections of random functions. This random functions are called sample paths (what we have been generating in our homework, like in the random walk represent the sequence of jumps). They are defined as a set of random functions in horizontal direction [8].

It is possible to define random (stochastic) process in a probability space $(\Omega, \mathcal{F}, P)$ in this way: X : Ω x T -> R.

It is similar to the definition of random variable Ω -> R, but with the addition of a new dimension T (the index set) [1, 2].

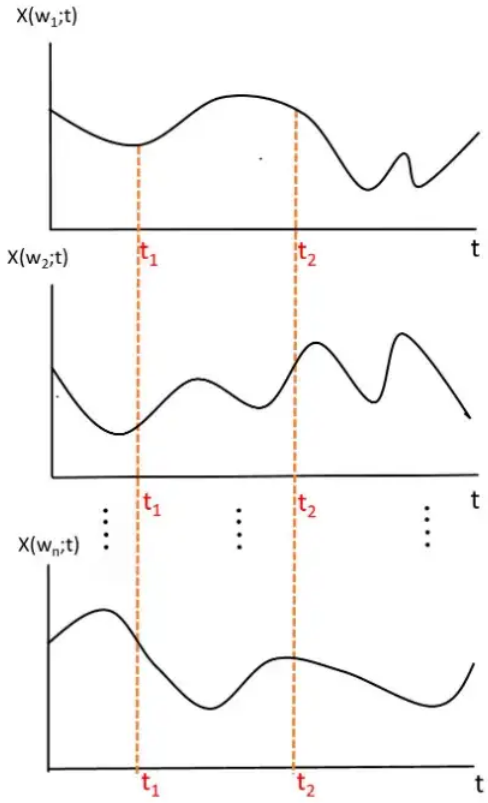


Figure 2.4: Representation of the double view[3]

[3] https://towardsdatascience.com/comprehensive-overview-of-random-variables-random-processes-and-their-properti

Chapter 3

Random walk

One of the simplest stochastic processes that we can study is the **random walk**, let's analyse it.

3.1 Definition and characteristics

A random walk [20] is a stochastic process $\{X_t\}$ where t $\in$ N (natural numbers). Each $\{X_t\}$ is a sum of Rademacher random variables defined by:

$$f(k) = \begin{cases} 1/2 & \text{if } k = -1, \\ 1/2 & \text{if } k = +1, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 3.1: The probability mass function of Rademacher distribution[1]

We assume that this random process starts from 0 and for every execution it sums 1 or substract 1 with probability 1/2. So for example in the second execution x_2 we have the sum of two Rademacher variables (so its value can be (1,1), (-1,1), (1,-1), (-1,-1)). We can notice that its execution is very similar to the Bernoulli random variable where we have 0 and 1 with probability 1-p and p. In this case we have -1 and 1 with probability 1/2. So if we execute this function we obtain a sample path. By experience we also know that the function get asymptotically to the normal distribution thanks to the central limit theorem.

We know that the random process is a sum of random variables and by the central limit theorem if we have independent and identically distributed random variables and we consider the sum of all the elements / n (mean), in our case the sum R_i (Rademacher random variable) / n, this random variable (which is the

[1] `https://en.wikipedia.org/wiki/Rademacher_distribution`

sample mean), if we standardize it, it will tend to the Normal (0,1). In order to standardize it, we need to subtract the mean and divide by the standard deviation. These random variables all have mean = 0 and variance = 1 . So if we take the mean of the sum it will still be equal to 0. The variance of each random variable, if we assume the jumps are all independent and distributed as Rademacher, is 1. The variance of the sum would be n but we are interest in the variance of the average. We know that it is σ^2/n. So in this case 1/n. In order to standardize this, we have to divide by $\sqrt{1/n}$, that is the same as multiply by $\frac{1}{\sqrt{n}}$. So we have that $\frac{1}{\sqrt{n}}$ multiplied to the sum of Rademacher distribution converges in distribution to the Normal (0,1). This normalization is necessary to keep constant the variance otherwise it will just explode, because we are summing infinite jumps and therefore we would have a divergence sum. In this way we keep the variance equal to 1 each time so we can see that it convergences to the standard normal distribution [19]. Each small ω would represent an infinite sequence of jumps.

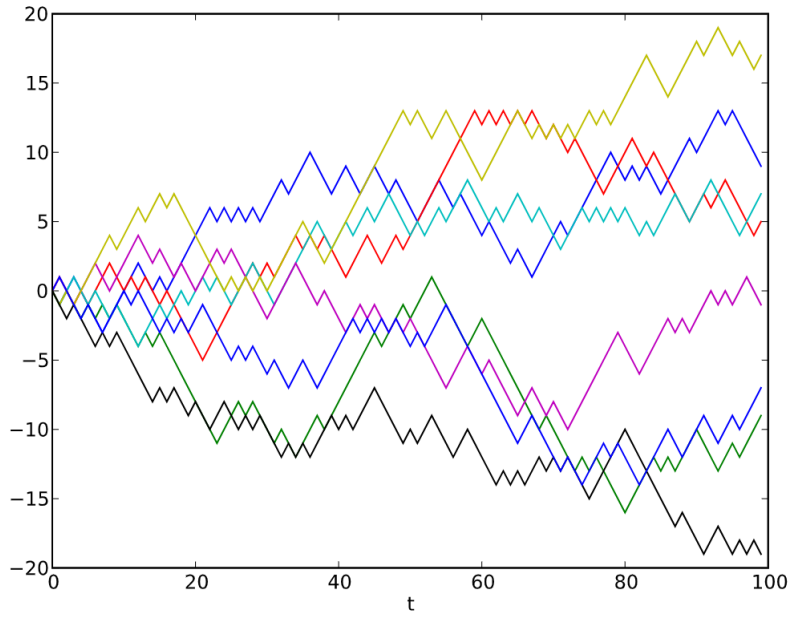


Figure 3.2: Example of eight random walks (sample paths) in one dimension starting from 0. The graph shows the position on the line (vertical axis) as time changes (number of steps taken - horizontal axis).[3]

[3] https://it.wikipedia.org/wiki/Passeggiata_aleatoria

Chapter 4

Most important stochastic processes

Before talking about the most known stochastic processes we need to introduce some new definitions.

4.1 Stationary increments

Let's consider for example a random walk. We take two points in the index set, like for example t and t+Δt, and we look at the increment that the random process had. We call this increment $I_t(\Delta t)$ and we notice that it is given by all the sum of the random variables between these two points (up or down) so it is a random variable itself.

We can define it has: $I_t(\Delta t) = W_{t+\Delta t}$ - W_t.

The mean of this jump is 0 because it is the sum of Rademacher variables. For the variance, we have sum of random variables that are independent and they all have same distribution and variance equal to 1. Since they are independent the variance of the sum is equal to the sum of the variance so in this case we have that the variance of the jump is equal to Δt, so it is proportional to the interval. The distribution of this jump is the sum of the Rademacher random variables, and it not depends on t, so it doesn't matter where t is located. Whenever is t the distribution is not affected by the position of the interval that we consider. We can tell that if we take two jumps with equal Δt they are going to have the same distribution.

When the distribution of the increment depends only on the side of the interval Δt and not on the position where the interval is located we say that the process has **stationary increments**. So we have that $I_t(\Delta t) = W_{t+\Delta t}$ - W_t has a distribution which only depends on Δt and not on t. For each given Δt, all $I_t(\Delta t)$: t $\in$ T have the same distribution.

4.2 Probabilistic continuity

In probability we have two main forms of continuity:

- **In probability** (weaker form)
- **Almost sure continuity** or continuous with probability one (stronger form)

Let's consider a random process X(t, ω) as a function T x Ω => R. Let's take a generic sample path of the random process, we will call it X(., ω) : T => R. For each t we have X(t, ω_t). We want to make sure that given another time for example s, when s gets closer to t also the value of the sample path gets closer to the value t. In practice we want that the limit for s that tends to t of X(s, ω) is equal to the value of t [5]. To summarize it, we say that given a time t $\in$ T, X is said to be **continuous in probability** at t if, for all $\epsilon > 0$, $\lim_{s \to t} \mathbf{P}\left(\left\{\omega \in \Omega \,\middle|\, \left|X_s(\omega) - X_t(\omega)\right| \geq \varepsilon\right\}\right) = 0$.

In the other case, given a time t $\in$ T, X is said to be **continuous with probability one** at t if: $\mathbf{P}\left(\left\{\omega \in \Omega \left|\lim_{s \to t} \left|X_s(\omega) - X_t(\omega)\right| = 0\right.\right\}\right) = 1$.

4.3 CADLAG process

Càdlàg is a property of the sample path. Let's see an example given a sample path X(., ω): T =>R and a point t_0. For t that tends to t_0 from the right, we have that X(t) -> X(t_0) this is defined as right continuity. The other requirement is the existence of the left limit, this value can also not be equal to the value of the function (because the function can be not continuous). So we need only that the limit of t that approaches from the left exists. In formula: x(t) -> (c < ∞). A prominent example of CADLAG function are CDF (cumulative distribution functions) just by definition (Pr{X <= x}) [7].

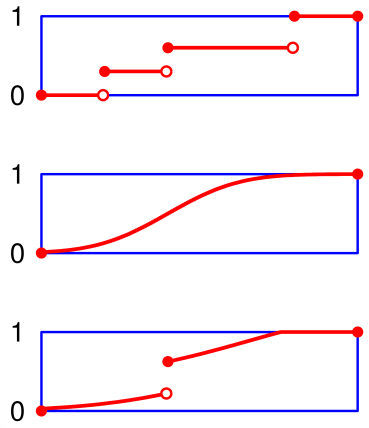


Figure 4.1: Cumulative distribution functions are examples of càdlàg functions.[2]

4.4 Lévy process

Now that we have reviewed the definitions above we are ready to introduce the most well known class of stochastic processes which are the so called Lévy processes [17]. A process belongs to this class if it satisfies certain conditions:

- Starts from 0. $X_0 = 0$
- Stationary increments ($X_{t+\Delta t}$ - X_t) this distribution depends only on Δt. It is independent of t. So for Δt fixed these random variables have all the same distribution
- Continuity in probability
- Independence of the increments. If we have a sample path and we consider an arbitrary subdivision t_1, t_2, t_3 where $t_1 < t_2 < t_3$ then for whatever choice of t the corresponding increments must be independent (P(A U B) = P(A) P(B))

One of the most prominent member of this class is the Brownian motion.

4.5 Brownian motion

We are now ready to introduce the famous Brownian Motion (Wiener process), the most important process within Lévy processes [3, 14]. Usually to denote the Brownian Motion we use these letters W, B or sometimes also X. Brownian motion is a continuous time and space process and it is characterised by the following properties:

- The increments must be independent
- The increments must be stationary
- The distribution must be a Normal(0,Δt), mean = 0 and variance = Δt, Δt (size of the interval)
- We assume to start from 0 (origin of the motion)
- The path must be continuous with probability 1 (almost surely we need to have continuous path) P($\omega \in \Omega$: X(.,ω) is continuous) = 1

So, we can see, that we have a situation very similar in the random walk, where we had increments with mean = 0 and variance = Δ (size of the interval). Therefore we can define also this process as a continuous version of the random walk where we have just replaced the discrete jumps with normal ones. It is also important to notice that thanks to the functional central limit theorem we have that the scaling limit of the random walks is given by the Brownian Motion.

[2] https://en.wikipedia.org/wiki/C%C3%A0dl%C3%A0g

If we want to standardize this process, instead of taking Δt we can take $\sigma^2 \Delta t$, to scale the variance by using this constant. And also for the mean, instead of taking 0, it is possible to add a drift and the motion of the process would drift towards the value [12].

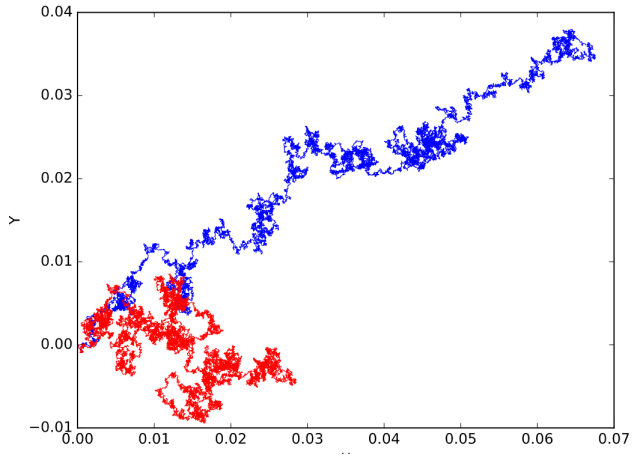


Figure 4.2: 2D Wiener processes with drift (blue) and without drift (red).[3]

4.6 Geometric Brownian motion

A geometric Brownian motion (also called GBM or exponential Brownian motion) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (Wiener process) with drift [9].

We can define that a stochastic process is a GMB when it satisfies the following stochastic differential equation:

$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$

where W_t is a Brownian motion, μ the percentage drift and σ the percentage volatility. μ and σ need to be constants.

Geometric Brownian motion is used to model population growth and also to analyse the behaviour of stock prices.

It can be used to model stock prices because:

- It has only positive values
- The calculations are easy
- Its returns are independent of the stock price, like in reality
- It shows the same pattern as in the real stock prices

[3] `https://en.wikipedia.org/wiki/Wiener_process`

But GBM also has some "flaws" because it is not a realistic model (constant mean and variance):

- Stock prices sometimes because of some unpredictable events could have strong jumps, in GBM this is not possible, the path is continuous (there is no discontinuity)
- Stock prices also have strong volatility and they change value constantly over time, in GBM the volatility is assumed to be constant

To make the process more realistic it is possible to drop the assumption that the volatility (σ) is constant. We call local volatility model a model where the volatility is a deterministic function of the stock price and time. If instead we assume that the volatility has randomness that can for example be described by a different equation through a different Brownian Motion the model is called stochastic volatility model.

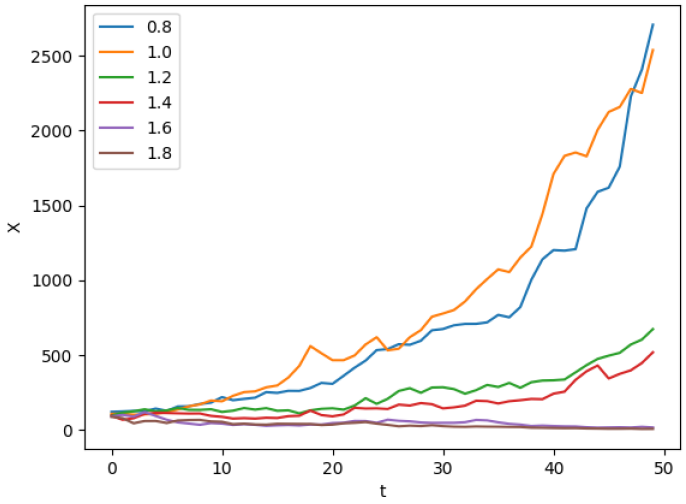


Figure 4.3: Realizations of Geometric Brownian Motion with different variances μ=1.[5]

[5] https://en.wikipedia.org/wiki/Geometric_Brownian_motion

Chapter 5

Random walk implementation

In this chapter of the paper, we analyse the trend of the random walk process thanks to an implementation in C#.

5.1 Explanation and Code

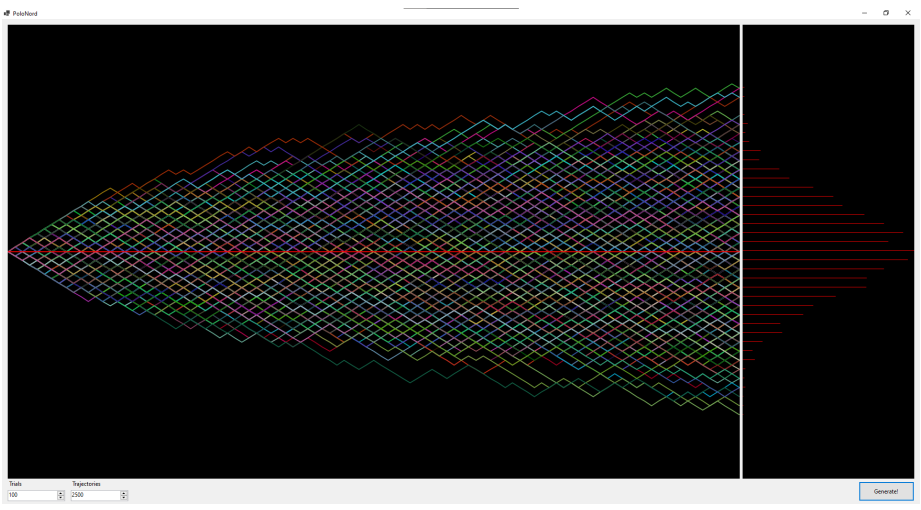


Figure 5.1: Example of the execution of the application with 100 trials and 2500 trajectories.

The application allows the generation of n trajectories with m trials. As we can see from the image 5.1, the graph on the left shows the sample paths. All the sample paths start from the middle (value 0) and for each step do 1 or -1. The value 0 is represented in the graph through the red line. On the left we can see the distribution of the sample paths. We can also notice that thanks to the functional central limit theorem we have that the scaling limit of the random walks is the Brownian Motion.

```
for (int i = 0; i < NumberOfTrajectories; i++)
{
    Random rnd = new Random();
    Color randomColor = Color.FromArgb(rnd.Next(200), rnd.Next(200), rnd.Next(200));
    Pen PenTrajectory = new Pen(randomColor, 2);
    List<Point> Punti = new List<Point>();
    double Y = maxY / 2;
    for (int x = 0; x <= Trials; x++)
    {
        double coin = r.NextDouble();
        if (coin <= Probability)
        {
            Y = Y + 1;
        }
        else
        {
            Y = Y - 1;
        }
        if (x == 0)
        {
            Y = maxY / 2;
        }
        int xDevice = FromXRealToXVirtual(x, minX, maxX, Window.Left, Window.Width);
        int yDevice = FromYRealToYVirtual(Y, minY, maxY, Window.Top, Window.Height);
        if (x == Trials)
        {
            EndY.Add(yDevice);
        }
        Punti.Add(new Point(xDevice, yDevice));
    }
    g.DrawLines(PenTrajectory, Punti.ToArray());
```

Figure 5.2: Portion of the code that shows the generation of the sample paths.

To generate the sample paths as we can see from the image 5.2 we generate a random variable with value between 0 and 1. If the value is less than 0.5 the sample path goes up otherwise down. After we transform world coordinates to device coordinates to show them on the graph. It is possible to visualise the source code on Github[1].

[1] https://github.com/IReallyLikeYourPants/ireallylikeyourpants.github.io/tree/main/mini_Thesis/PortoCervo

5.2 Other examples

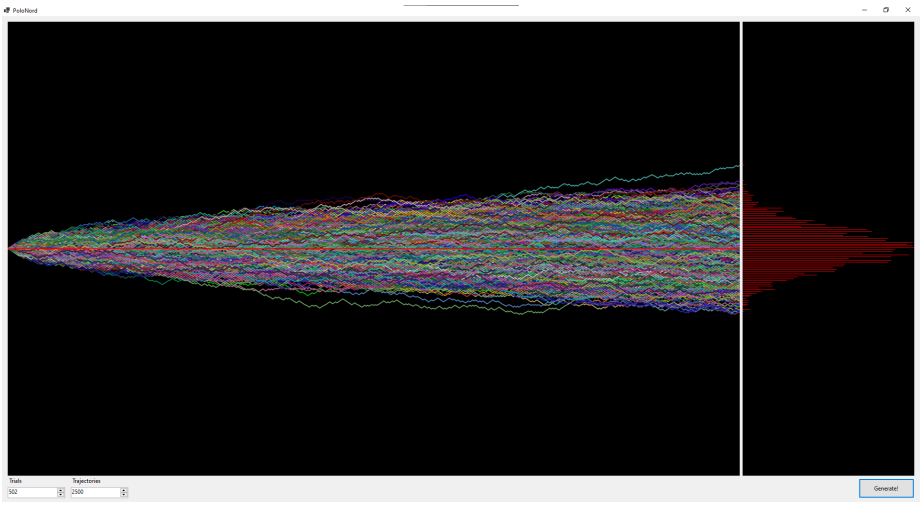


Figure 5.3: Example that shows a better view of the scaling limit.

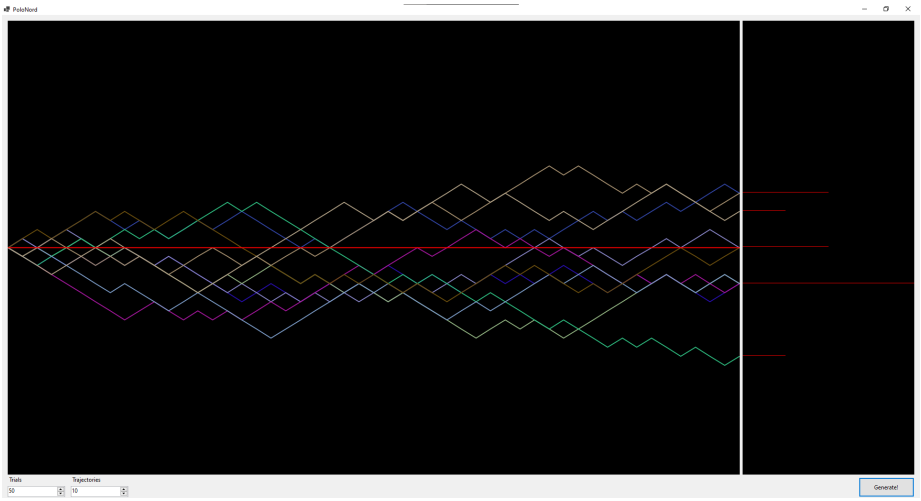


Figure 5.4: A simple execution with 10 trajectories.

Chapter 6

Use cases and Conclusions

In this final chapter we took a look at some general usage of the stochastic processes and we summarise what we covered.

6.1 Use cases

Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner. Examples include the growth of a bacterial population, electrical current fluctuating due to thermal noise, the movement of a gas molecule, molecular motors, intracellular transport, robustness in gene networks, genetic switches and oscillators, quantum fluctuations, medium access control mechanisms, data analysis, data compression, channel capacity, rate distortion, network information theory and hypothesis testing.

So as we can see, stochastic processes have applications in many disciplines such: biology, chemistry, ecology, neuroscience, physics, image processing, signal processing, control theory, information theory, computer science, cryptography and telecommunications [16, 15]. Furthermore, seemingly random changes in financial markets have motivated the extensive use of stochastic processes in finance. Examples of stochastic processes include the Wiener process (Brownian motion process), used for example by Louis Bachelier to study price changes on the Paris Bourse, and the Poisson process, used by A. K. Erlang to study the number of phone calls occurring in a certain period of time.

6.2 Conclusions

Thanks to this paper, we have analysed the most important notions concerning stochastic processes. It has served us to gain a better understanding of how they work and their properties. In the following, we analysed one of the best-known stochastic processes, the random walk, and implemented it on an application

level in order to better understand the theory. We also studied the most important stochastic processes that are still being analysed today, as they are of great use in mathematics, both in terms of theory but also on an application level, such as Brownian Motion. It is certainly possible to go into this topic in greater depth, an interesting example would be the analysis of stock simulation thanks to the Geometric Brownian motion. Due to its length, this paper is only intended to be a simple introduction to the subject.

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