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The cubic interpolation spline for functions with boundary layer on a Bakhvalov mesh

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Abstract. The problem of cubic spline interpolation on the Bakhvalov mesh of functions with region of large gradients is considered. Asymptotically accurate error estimates $O(N^{-4})$ are obtained for a class of functions with an exponential boundary layer in case $1/N \leq \varepsilon$, where N is number of nodes, ε is small parameter. In case $\varepsilon \leq 1/N$ we have experimentally shown that the error estimates of traditional spline interpolation are not uniform in a small parameter, and the error itself can increase indefinitely when the small parameter tends to zero at a fixed number of nodes N . A modified cubic spline is proposed for which uniform estimates of the order $O(N^{-4})$ have been experimentally confirmed.

1. Introduction

Cubic splines are widely used for smooth interpolation of functions [1], [2]. When using difference methods to solve singularly perturbed problems strongly nonuniform grids are used. In this case, there is a need restore function for all values of the independent variable. In case of a piecewise uniform grid of G. I. Shishkin [3], in [4] error estimates are obtained. It is shown that the convergence of the interpolation process is nonuniform in a small parameter. In this paper, we study the cubic spline interpolation [2] on the mesh of N. S. Bakhvalov [5] which densifies in the boundary layer. In case $1/N \leq \varepsilon$ uniform for these N and ε error estimates of order $O(N^{-4})$ are obtained and experimentally confirmed. In case $\varepsilon \leq 1/N$ it is experimentally shown that these estimation are not uniform in a small parameter ε . It is shown that for $\varepsilon \rightarrow 0$ the interpolation error has unlimited growth, and the development of special interpolation methods for this class of problems is necessary. There is offered a modified interpolation spline for which experimentally established uniform in ε convergence.

Introduce the notations. Set the mesh of interval $[0, 1]$:

$$\Omega = \{x_n : x_n = x_{n-1} + h_n, n = 1, 2, \dots, N, x_0 = 0, x_N = 1\}.$$

Denote by $S(\Omega, k, 1)$ the space of polynomial splines of degree k of defect 1 [2] on the mesh Ω . If necessary, we consider the partition Ω extended to the left of the point 0 with the step $h_1 = x_1 - x_0$ and to the right of the point 1 with the step $h_N = x_N - x_{N-1}$. We set $h = 1/N$. By C and C_j we mean positive constants independent of the parameter ε and the number of grid nodes. In this case, the same symbol C_j can denote different constants. Will write $f = O(g)$



if the estimate $|f| \leq C|g|$ and $f = O^*(g)$ if $f = O(g)$ and $g = O(f)$. $C[a, b], L_2[a, b]$ – spaces of continuous and quadratically summable on $[a, b]$ functions with the norms $\|\cdot\|_{C[a,b]}$ and $\|\cdot\|_{L_2[a,b]}$ accordingly, (\cdot, \cdot) is the scalar product in $L_2[0, 1]$. We say that the square matrix $A = \{a_{nk}\}$ has a diagonal predominance in rows with a measure predominance of $r > 0$ if

$$\min_n (|a_{nn}| - \sum_{k \neq n} |a_{nk}|) \geq r.$$

2. Problem statement

Let us a function $u(x)$ be presented in the form:

$$u(x) = q(x) + \Phi(x), \quad x \in [0, 1], \quad (2.1)$$

where

$$|q^{(j)}(x)| \leq C_1, \quad |\Phi^{(j)}(x)| \leq \frac{C_1}{\varepsilon^j} e^{-\alpha x/\varepsilon}, \quad 0 \leq j \leq 4, \quad (2.2)$$

where the function $q(x)$ and $\Phi(x)$ do not explicitly given, $\alpha > 0, \varepsilon > 0$.

We set the grid of the interval $[0, 1]$ based on [5].

Let us $\sigma = \min \left\{ \frac{1}{2}, \frac{4\varepsilon}{\alpha} \ln \frac{1}{\varepsilon} \right\}$. We define the function

$$g(t) = \begin{cases} -\frac{4\varepsilon}{\alpha} \ln [1 - 2(1 - \varepsilon)t], & 0 \leq t \leq \frac{1}{2}, \\ \sigma + (2t - 1)(1 - \sigma), & 1/2 \leq t \leq 1. \end{cases} \quad (2.3)$$

In case $\sigma = \frac{4\varepsilon}{\alpha} \ln \frac{1}{\varepsilon} < \frac{1}{2}$ we define the mesh Δ with nodes x_n , $n = 0, 1, \dots, N$, and steps $h_n = x_n - x_{n-1}$ using the formula

$$x_n = g(n/N). \quad (2.4)$$

In case $\sigma = \frac{1}{2}$ we define the mesh Ω as uniform mesh with step $h = 1/N$.

Let $g_3(x, u) \in S(\Omega, 3, 1)$ - cubic spline interpolation on the mesh Ω , defined from conditions $g_3(x_n, u) = u(x_n)$, $0 \leq n \leq N$, $g'_3(0, u) = u'(0)$, $g'_3(1, u) = u'(1)$.

In this paper we suppose that

$$1/N \leq \varepsilon, \sigma = \frac{4\varepsilon}{\alpha} \ln \frac{1}{\varepsilon} < \frac{1}{2}. \quad (2.5)$$

The case $\varepsilon < 1/N$ is considered in the charter of numerical experiments.

3. Main result

According to [2] for cubic spline interpolation $g_3(x, u) \in S(\Omega, 3, 1)$ the error estimate hold:

$$|g_3(x, u) - u(x)| \leq \frac{5}{384} \|u^{(4)}\|_{C[0,1]} \max_n h_n^4. \quad (3.1)$$

Notice that $g_3(x, u) = g_3(x, q) + g_3(x, \Phi)$, and due the conditions (2.2) and (3.1) $\|q(x) - g_3(x, q)\|_{C[0,1]} \leq C_2 \max_n h_n^4 \leq C_2 N^{-4}$. Therefore, for the construction of cubic spline interpolation approximating $u(x)$ with order $O(N^{-4})$, it is necessary to prove the estimate

$$\|\Phi(x) - g_3(x, \Phi)\|_{C[0,1]} \leq C_2 N^{-4}. \quad (3.2)$$

In case $\sigma = 1/2$ the estimate (3.2) valid by condition (3.1). So we will assume that $\sigma < 1/2$ and $\varepsilon < e^{-1}$. Below, without loss of generality, we assume that in (2.2) $\alpha = 1$, since the general case reduces to this by replacing $\alpha x = y$ with preservation of estimates of the form (2.2).

Theorem 1. *There is constant C that is independent of ε, N such that for $1/N \leq \varepsilon$ the estimate (3.2) holds.*

4. Auxiliary results

Lemma 1. With $\sigma = \frac{4\varepsilon}{\alpha} \ln \frac{1}{\varepsilon} < \frac{1}{2}$ the sequence h_n for $n \leq N/2$ monotonically increases and the following estimates hold

$$h_n = \begin{cases} O^*\left(\frac{h}{1+(h/\varepsilon)(N/2-n)}\right), & 1 \leq n \leq N/2, \\ O^*(h), & N/2 + 1 \leq n \leq N. \end{cases} \quad (4.1)$$

Proof. From (2.3) it follows that $1 \leq n \leq N/2$

$$\begin{aligned} h_n &= -4\varepsilon \ln \left(1 - 2(1 - \varepsilon) \frac{n}{N}\right) + 4\varepsilon \ln \left(1 - 2(1 - \varepsilon) \frac{n-1}{N}\right) = 4\varepsilon \ln \left(1 + \frac{2(1 - \varepsilon)/N}{1 - 2(1 - \varepsilon)n/N}\right) = \\ &= O^*\left(\varepsilon \frac{2(1 - \varepsilon)/N}{1 - 2(1 - \varepsilon)n/N}\right) = O^*\left(\varepsilon \frac{(1 - \varepsilon)}{N/2 - (1 - \varepsilon)n}\right) = O^*\left(\varepsilon \frac{1 - \varepsilon}{\varepsilon N/2 + (1 - \varepsilon)(N/2 - n)}\right) = \\ &= O^*\left(h \frac{1 - \varepsilon}{1/2 + (h/\varepsilon)(N/2 - n)}\right) = O^*\left(\frac{h}{1/2 + (h/\varepsilon)(N/2 - n)}\right) = O^*\left(\frac{h}{1 + (h/\varepsilon)(N/2 - n)}\right), \end{aligned}$$

and the first estimate in (4.1) is proved. The second estimate is obvious, because for $N/2 + 1 \leq n \leq N$ the steps of the mesh have same lengths.

Lemma is proved.

Let

$$N_{n,1}(x) = \begin{cases} \frac{x-x_n}{x_{n+1}-x_n}, & x \in [x_n, x_{n+1}] \\ \frac{x_{n+2}-x}{x_{n+2}-x_{n+1}}, & x \in [x_{n+1}, x_{n+2}], \quad -1 \leq n \leq N-1 \\ 0, & x \notin [x_n, x_{n+2}] \end{cases}$$

- B -spline of first degree. Then $\|N_{n,1}\|_{L_2[0,1]} = \frac{1}{\sqrt{3}}(h_{n+1} + h_{n+2})^{1/2}$, those, in view (4.1)

$$\|N_{n,1}\|_{L_2[0,1]} = O^*(h_{n+1}^{1/2}), \quad -1 \leq n \leq N-1. \quad (4.2)$$

Let $\tilde{N}_{n,1}(x) = N_{n,1}(x) / \|N_{n,1}\|_{L_2[0,1]}$, $0 \leq n \leq N-2$. For $n = -1$ and $n = N-1$ we set $\tilde{N}_{-1,1}(x) = \tilde{N}_{0,1}(x + h_1)$, $\tilde{N}_{N-1,1}(x) = \tilde{N}_{N-2,1}(x - h_N)$. Then in view (4.2)

$$\|\tilde{N}_{n,1}\|_{C[0,1]} = O^*(h_{n+1}^{-1/2}), \quad -1 \leq n \leq N-1. \quad (4.3)$$

Let $e(x) = g_3(x, \Phi) - \Phi(x)$. We study the function $e''(x) = g_3''(x, \Phi) - \Phi''(x)$. According to ([6]), the formula $g_3''(x, \Phi) = P\Phi''(x)$ is valid, where P - orthogonal in $L_2[0, 1]$ projector on $S(\Omega, 1, 1)$. Denote by $gI(x) \in S(\Omega, 1, 1)$ the linear interpolant $\Phi''(x)$ at the nodes of the mesh. Then we have

$$e''(x) = P(\Phi''(x) - gI(x)) + (gI(x) - \Phi''(x)). \quad (4.4)$$

We represent the function $P(\Phi''(x))$ in the form

$$P(\Phi''(x) - gI(x)) = \sum_{n=-1}^{N-1} \alpha_n \tilde{N}_{n,1}(x). \quad (4.5)$$

From the conditions of orthogonality of the difference $g_3''(x, \Phi) - \Phi''(x)$ to the space $S(\Omega, 1, 1)$ we obtain a system of linear equations for coefficients $\sum_{n=-1}^{N-1} \alpha_n (\tilde{N}_{n,1}, \tilde{N}_{k,1}) = (\Phi'' - gI, \tilde{N}_{k,1})$, $-1 \leq k \leq N-1$, or in matrix form

$$\Gamma \alpha = F, \quad (4.6)$$

where $\Gamma = \{\gamma_{nk}\} = \{(\tilde{N}_{n,1}, \tilde{N}_{k,1})\}$ - Gram matrix of normalized B -splines, $F = (F_{-1}, F_0, \dots, F_{N-1})^T$, $F_j = (\Phi'', \tilde{N}_j)$. It's obvious that $0 \leq \gamma_{nk} \leq 1$.

Lemma 2. *Matrix Γ has the form*

$$\Gamma = \text{tridiag}\{a_n, c_n, b_n\}, \quad -1 \leq n \leq N-1, a_{-1} = b_{N-1} = 0, \quad (4.7)$$

$$a_{n+1} = b_n = O^*(1) > 0, \quad 0 \leq n \leq N-2, \quad (4.8)$$

$$c_n = 1, \quad 0 \leq n \leq N-2, c_{-1} = c_{N-1} = \frac{1}{\sqrt{2}}. \quad (4.9)$$

The matrix Γ has strict diagonal dominance over rows with a prevalence index $1/\sqrt{2}$.

The proof is obtained by direct calculation of the integrals taking into account (4.1)-(4.3).

Lemma 3. *The matrix Γ is invertible, and for the elements of inverse matrix $\Gamma^{-1} = \{\tilde{\gamma}_{nk}\}$ the estimates*

$$|\tilde{\gamma}_{nk}| \leq C e^{-\beta|n-k|}. \quad (4.10)$$

hold.

Proof. Invertibility of the matrix Γ and the estimates of elements follow from strict diagonal prevalence with a prevalence index of $1/\sqrt{2}$ and Demko's theorem [7]. The lemma is proved.

Lemma 4. *For the elements F_n for any $\varepsilon \in (0, 1)$, N the estimates*

$$F_n = O(h_{n+1}^{5/2} \varepsilon^{-4} e^{-x_n/\varepsilon}), \quad -1 \leq n \leq N-1. \quad (4.11)$$

are valid.

The proof The proof is obtained by direct calculation of the integrals with taking into account (4.3) and estimates of the error of linear interpolation.

Lemma 5. *There is a constant $C_1 \in (0, 1]$ such that if $h \leq C_1 \varepsilon$ then for α_n in (4.5) the estimates*

$$\alpha_n \leq C h_{n+1}^{5/2} \varepsilon^{-4} e^{-x_n/\varepsilon}, \quad -1 \leq n \leq N-1. \quad (4.12)$$

hold.

Proof. From (4.10)-(4.11) we have

$$\begin{aligned} |\alpha_n| &\leq \left| \sum_{k=-1}^{N-1} \tilde{\gamma}_{nk} F_{nk} \right| \leq \sum_{k=-1}^{N-1} |\tilde{\gamma}_{nk}| \cdot |F_{nk}| \leq C \sum_{k=-1}^{N-1} e^{-\beta|n-k|} h_{k+1}^{5/2} \varepsilon^{-4} e^{-x_k/\varepsilon} = \\ &= C h_{n+1}^{5/2} \varepsilon^{-4} e^{-x_n/\varepsilon} \sum_{k=-1}^{N-1} e^{-\beta|n-k|} \varepsilon^{-4} e^{(x_n-x_k)/\varepsilon} \left(\frac{h_{k+1}}{h_{n+1}} \right)^{5/2}. \end{aligned} \quad (4.13)$$

Further

$$\sum_{k=-1}^{N-1} e^{-\beta|n-k|} \varepsilon^{-4} e^{(x_n-x_k)/\varepsilon} \left(\frac{h_{k+1}}{h_{n+1}} \right)^{5/2} = \sum_{k=-1}^n (\dots) + \sum_{k=n+1}^{N-1} (\dots) = \Sigma_1 + \Sigma_2. \quad (4.14)$$

From (4.1) we obtain for $n > k$

$$\frac{x_n - x_k}{\varepsilon} = \frac{h_{k+1} + \dots + h_n}{\varepsilon} \geq C_2 \frac{h}{\varepsilon} (n - k). \quad (4.15)$$

Therefore if $C_1 = \beta/2C_2$ then $(x_n - x_k)/\varepsilon \leq \beta/2(n - k)$, and given (4.1): $h_{k+1}/h_{n+1} = O(1)$. Therefore, for this choice C_1

$$\Sigma_1 \leq C_3. \quad (4.16)$$

Finally, in view of (4.1) for $n \leq k$ we have

$$\frac{h_{k+1}}{h_{n+1}} \leq C \max \left\{ \frac{|N/2 - n| + 1}{|N/2 - k| + 1}, n - k, 1 \right\} \leq C(n - k + 1). \quad (4.17)$$

Hence

$$\Sigma_2 \leq C_4 \sum_{k=n+1}^{N-1} e^{-\frac{\beta}{2}|n-k|} \leq C_5. \quad (4.18)$$

From (4.13)-(4.18) the formula (4.12) follows. Lemma is proved.

Lemma 6. *If $C_1\varepsilon \leq h \leq \varepsilon$ then the estimates hold*

$$|\alpha_n| \leq C \begin{cases} h_{n+1}^{5/2} \varepsilon^{-4} e^{-x_{n+1}/\varepsilon}, & -1 \leq n \leq N/2 - 1, \\ h^{5/2}, & N/2 \leq n \leq N - 1 \end{cases}. \quad (4.19)$$

Proof. For $-1 \leq n \leq N/2 - 1$ evaluate $|\alpha_n|$ analogously (4.13)-(4.14). From (4.1) instead (4.15) we will have

$$\frac{x_n - x_k}{\varepsilon} = \frac{h_{k+1} + \dots + h_n}{\varepsilon} \leq \left(\frac{1}{N/2 - k} + \dots + \frac{1}{N/2 - n} \right) \leq C \ln \frac{N/2 - n}{N/2 - k}. \quad (4.20)$$

Therefore, in view (4.17)

$$\Sigma_1 \leq C_6 \sum_{k=-1}^n e^{-\beta|n-k|} (n - k + 1)^{C_7} \leq C_8, \quad (4.21)$$

and (4.18) continues to be correct. So (4.19) is proved for $-1 \leq n \leq N/2 - 1$.

For $n \geq N/2$ we have analogously (4.13) taking into account $h_{n+1}/h_{N/2-1} \leq C$:

$$|\alpha_n| \leq C h_{N/2-1}^{5/2} \varepsilon^{-4} e^{-x_{N/2-1}/\varepsilon} \sum_{k=-1}^{N-1} e^{(-\beta|n-k| - (x_{N/2-1} - x_k)/\varepsilon)}. \quad (4.22)$$

Further analogously (4.14) we have

$$\sum_{k=-1}^{N-1} e^{(-\beta|n-k| - (x_{N/2-1} - x_k)/\varepsilon)} = \sum_{k=-1}^{N/2-1} (\dots) + \sum_{k=N/2}^{N-1} (\dots) = \Sigma_1 + \Sigma_2.$$

It is obvious that $\Sigma_2 \leq C$, $x_{N/2-1} - x_k < 0$. Evaluate Σ_1 . For $n \geq N/2$

$$\Sigma_1 \leq \sum_{k=-1}^{N/2-1} e^{(-\beta|N/2-k| - (x_{N/2} - x_k)/\varepsilon)},$$

and the evaluations Σ_1 are the evaluations (4.20)-(4.21) for $n = N/2 - 1$. Therefore, as by accordance with (2.3),(2.4) for $h \leq \varepsilon$ we have $\varepsilon^{-4} e^{-x_{N/2-1}} \leq C$, and from (4.1) we have $h_{N/2-1} = O(h)$, then the estimate (4.19) for $n \geq N/2$ follows from (4.22). Lemma is proved.

Lemma 7. *There are constants $C > 0, \beta > 0$ independent from ε, N such that the following estimates*

$$\|P(\Phi'' - gI)(x)\|_{C[x_n, x_{n+1}]} \leq C \begin{cases} h_{n+1}^2 \varepsilon^{-4} e^{-x_{n+1}/\varepsilon}, & -1 \leq n \leq N/2 - 1, \\ h^2, & N/2 \leq n \leq N - 1 \end{cases} \quad (4.23)$$

will be true.

Proof. Since at every node x_n only one B-spline $N_{n-1,1}$ is non-zero, then the equality $P(\Phi'' - gI)(x_n) = \alpha_{n-1} \tilde{N}_{n-1,1}(x_n)$ is valid. This, the lemma 7 and the estimates (4.3) imply the lemma.

Lemma 8. *The estimates*

$$\|e''(x)\|_{C[x_n, x_{n+1}]} \leq C \begin{cases} h_{n+1}^2 \varepsilon^{-4} e^{-x_{n+1}/\varepsilon}, & -1 \leq n \leq N/2 - 1, \\ h^2, & N/2 \leq n \leq N - 1 \end{cases}. \quad (4.24)$$

are valid.

Proof. In view of (4.4), (4.23) it is sufficiently to estimate the expression $\|gI(x) - \Phi''(x)\|_{C[x_n, x_{n+1}]}$. But the estimate of this expressions of kind (4.24) follows from the estimate of linear interpolation errors on the segment $[x_n, x_{n+1}]$.

Lemma is proved.

5. Proof of theorem

Fix $n \in [0, N - 1]$. Then, since $e(x_n) = e(x_{n+1}) = 0$, then considering $e(x)$ as the solution of the problem $e''(x) = e''(x)$ with zero boundary conditions on interval $[x_n, x_{n+1}]$, we obtain that $e(x) = \int_{x_n}^{x_{n+1}} G(x, s) e''(s) ds$, where

$$G(x, s) = \frac{1}{x_{n+1} - x_n} \begin{cases} (x - x_n)(x_{n+1} - s), & x_n \leq x \leq s, \\ (s - x_n)(x_{n+1} - x), & s < x \leq x_{n+1} \end{cases}$$

is Green function.

Since $|G(x, s)| \leq (x_{n+1} - x_n) = h_{n+1}$, then from (4.24), we get

$$\begin{aligned} \|e(x)\|_{C[x_n, x_{n+1}]} &\leq h_{n+1} \int_{x_n}^{x_{n+1}} |e''(s)| ds \leq h_{n+1}^2 \|e''(s)\|_{C[x_n, x_{n+1}]} \\ &\leq C \begin{cases} h_{n+1}^4 \varepsilon^{-4} e^{-x_{n+1}/\varepsilon}, & -1 \leq n \leq N/2 - 1, \\ h^4, & N/2 \leq n \leq N - 1 \end{cases}. \end{aligned} \quad (5.1)$$

From (4.1), (2.3), (2.4), (5.1) we get for $n \leq \frac{N}{2} - 1$

$$\|e(x)\|_{C[x_n, x_{n+1}]} \leq \left(1 - 2(1 - \varepsilon) \frac{n}{N}\right)^4 \left(\frac{1/N}{(1 + \frac{1}{\varepsilon N}(N/2 - n))}\right)^4 = \frac{16C}{N^4} \frac{(N/2 - n + \varepsilon n)^4}{(N + \frac{N/2 - n}{\varepsilon})^4} \leq \frac{C_1}{N^4}.$$

Hence we obtain (3.2) for $n \leq \frac{N}{2} - 1$.

For $N/2 \leq n \leq N - 1$ the estimate (3.2) follows from (5.1). Theorem 1 is proved.

6. Results of numerical experiments

We define the function of the form (2.1):

$$u(x) = \cos \frac{\pi x}{2} + e^{-\frac{x}{\varepsilon}}, \quad x \in [0, 1].$$

The tables show the maximum errors of spline interpolation calculated at nodes of the condensed mesh obtained from the original computational mesh by dividing each of its mesh intervals into 10 equal parts. Also the tables contain values of observed convergence rate. In the table 1 shows the errors for the traditional cubic spline $g_3(x, u)$. The errors confirm the estimates of Theorem 1. However this table shows that the error increases with decreasing ε for fixed N . This result is similar to the result obtained in [4] for the interpolation spline on the Shishkin grid.

Due to the non-uniform in ε convergence of the cubic spline $g_3(x, u)$, we construct a modified interpolation spline. We use an approach [4], where cubic spline on the Shishkin grid is considered. We define $\bar{x}_n = (x_n + x_{n+1})/2, n \in [N/2 - 1, N/2]$, $\bar{x}_n = x_n, n \in [0, N/2 - 2] \cup [N/2 + 1, N]$. Let us $gm_3(x, u) \in S(\Omega, 3, 1)$ be cubic spline determined from conditions $gm_3(\bar{x}_n, u) = u(\bar{x}_n), n \in [0, N], gm'_3(0, u) = u'(0), gm'_3(1, u) = u'(1)$.

The results of the table 2 for the modified spline $gm_3(x, u)$ show the uniform in ε error of order $O(1/N^4)$.

Table 1. The error of cubic interpolation spline $g_3(x, u)$.

εN	2^4	2^5	2^6	2^7	2^8	2^9
10^{-1}	$1.33 \cdot 10^{-4}$	$1.02 \cdot 10^{-5}$ 3.70	$6.99 \cdot 10^{-7}$ 3.88	$4.52 \cdot 10^{-8}$ 3.95	$2.89 \cdot 10^{-9}$ 3.98	$1.82 \cdot 10^{-10}$ 3.99
10^{-2}	$1.72 \cdot 10^{-4}$	$1.06 \cdot 10^{-5}$ 4.02	$6.74 \cdot 10^{-7}$ 3.98	$7.95 \cdot 10^{-8}$ 3.08	$8.80 \cdot 10^{-9}$ 3.18	$8.12 \cdot 10^{-9}$ 3.44
10^{-3}	$4.82 \cdot 10^{-4}$	$1.37 \cdot 10^{-5}$ 5.13	$7.04 \cdot 10^{-7}$ 4.29	$4.38 \cdot 10^{-8}$ 4.01	$2.71 \cdot 10^{-9}$ 4.02	$1.64 \cdot 10^{-10}$ 4.04
10^{-4}	$6.35 \cdot 10^{-3}$	$1.88 \cdot 10^{-4}$ 5.08	$5.45 \cdot 10^{-6}$ 5.10	$1.56 \cdot 10^{-7}$ 5.12	$4.45 \cdot 10^{-9}$ 5.14	$1.72 \cdot 10^{-10}$ 4.69
10^{-5}	$7.22 \cdot 10^{-2}$	$2.19 \cdot 10^{-3}$ 5.04	$6.62 \cdot 10^{-5}$ 5.05	$1.98 \cdot 10^{-6}$ 5.06	$5.86 \cdot 10^{-8}$ 5.08	$1.71 \cdot 10^{-9}$ 5.10
10^{-6}	$7.73 \cdot 10^{-1}$	$2.38 \cdot 10^{-2}$ 5.02	$7.28 \cdot 10^{-4}$ 5.03	$2.22 \cdot 10^{-5}$ 5.03	$6.76 \cdot 10^{-7}$ 5.04	$2.05 \cdot 10^{-8}$ 5.05
10^{-7}	8.06	$2.49 \cdot 10^{-1}$ 5.02	$7.70 \cdot 10^{-3}$ 5.02	$2.37 \cdot 10^{-4}$ 5.02	$7.29 \cdot 10^{-6}$ 5.02	$2.24 \cdot 10^{-7}$ 5.03
10^{-8}	83.1	2.58 5.01	$7.98 \cdot 10^{-2}$ 5.01	$2.47 \cdot 10^{-3}$ 5.01	$7.64 \cdot 10^{-5}$ 5.02	$2.36 \cdot 10^{-6}$ 5.02

Table 2. The error of modified cubic spline $gm_3(x, u)$.

εN	2^4	2^5	2^6	2^7	2^8	2^9
10^{-1}	$2.21 \cdot 10^{-4}$	$1.85 \cdot 10^{-5}$ 3.57	$1.35 \cdot 10^{-6}$ 3.78	$9.09 \cdot 10^{-8}$ 3.89	$5.02 \cdot 10^{-9}$ 3.94	$3.78 \cdot 10^{-10}$ 3.97
10^{-2}	$1.67 \cdot 10^{-4}$	$1.09 \cdot 10^{-5}$ 3.93	$1.00 \cdot 10^{-6}$ 3.45	$1.16 \cdot 10^{-7}$ 3.11	$1.45 \cdot 10^{-8}$ 3.00	$1.47 \cdot 10^{-9}$ 3.31
10^{-3}	$1.89 \cdot 10^{-4}$	$1.10 \cdot 10^{-5}$ 4.10	$6.49 \cdot 10^{-7}$ 4.09	$4.06 \cdot 10^{-8}$ 4.00	$2.59 \cdot 10^{-9}$ 3.96	$2.16 \cdot 10^{-10}$ 3.59
10^{-4}	$2.25 \cdot 10^{-4}$	$1.35 \cdot 10^{-5}$ 4.05	$8.05 \cdot 10^{-7}$ 4.07	$4.74 \cdot 10^{-8}$ 4.09	$2.77 \cdot 10^{-9}$ 4.10	$1.61 \cdot 10^{-10}$ 4.10
10^{-5}	$2.46 \cdot 10^{-4}$	$1.51 \cdot 10^{-5}$ 4.03	$9.19 \cdot 10^{-7}$ 4.04	$5.57 \cdot 10^{-8}$ 4.04	$3.35 \cdot 10^{-9}$ 4.05	$2.00 \cdot 10^{-10}$ 4.07
10^{-6}	$2.58 \cdot 10^{-4}$	$1.60 \cdot 10^{-5}$ 4.02	$9.84 \cdot 10^{-7}$ 4.02	$6.05 \cdot 10^{-8}$ 4.02	$3.71 \cdot 10^{-9}$ 4.03	$2.26 \cdot 10^{-10}$ 4.03
10^{-7}	$2.66 \cdot 10^{-4}$	$1.65 \cdot 10^{-5}$ 4.01	$1.02 \cdot 10^{-6}$ 4.01	$6.33 \cdot 10^{-8}$ 4.01	$3.91 \cdot 10^{-9}$ 4.02	$2.41 \cdot 10^{-10}$ 4.02
10^{-8}	$2.71 \cdot 10^{-4}$	$1.68 \cdot 10^{-5}$ 4.01	$1.05 \cdot 10^{-6}$ 4.01	$6.50 \cdot 10^{-8}$ 4.01	$4.03 \cdot 10^{-9}$ 4.01	$2.5 \cdot 10^{-10}$ 4.01

7. Conclusion

The error of cubic spline interpolation on the Bakhvalov mesh in the presence of an exponential boundary layer in case $1/N \leq \varepsilon$ is estimated. The error estimates of order $O(1/N^4)$ uniform for these ε and N are proved. It is experimentally shown that for a given number of mesh nodes, the interpolation error can grow indefinitely with decreasing value of a small parameter. The results of computational experiments are presented.

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