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PS 1

① CE (AD) consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ and HH

HH capital stock

$\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ s.t.

- given prices $\{p_t, w_t, r_t\}_0^{\infty}$, the allocation of representative HH, $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves HH's problem:

$$\max_{\{c_t, i_t, x_{t+1}, k_t, l_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \sum_{t=0} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t l_t) + \pi_0 \quad (*)$$

$$x_{t+1} = (1-\delta)x_t + i_t$$

$$0 \leq k_t \leq x_t$$

$$0 \leq l_t \leq 1 \qquad \forall t \geq 0$$

$$c_t, x_{t+1} \geq 0$$

x_0 given

- given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of repr. firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ solves firm's problem.

$$\pi = \max_{\{y_t, l_t, k_t\}_{t=0}^{\infty}} \sum p_t (y_t - r_t k_t - w_t l_t)$$

$$\text{s.t. } y_t = F(k_t, l_t) \ \forall t \geq 0$$

$$y_t, k_t, l_t \geq 0$$

- markets clear in $\forall t$:

$$y_t = c_t + i_t$$

$$l_t^d = l_t^s$$

$$k_t^d = k_t^s$$

(2) SPP - Sequential formation

$$w(\bar{k}_0) = \max_{\{c_t, k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t. $F(k_t, l_t) = c_t + k_{t+1} - (1-\delta) k_t \quad \forall t$

$c_t \geq 0$

$k_t \geq 0$ $\qquad c_t = F(k_t, l_t) - k_{t+1} + (1-\delta) k_t$

$0 \leq l_t \leq 1 \quad \forall t$

$k_0 \leq \bar{k}_0$

$l_t = 1 \quad \forall t$

$\delta = 1$

also, assuming u is contin. differentiable, strictly ↑ strictly concave & bounded; F is contin. differentiable, strictly ↑, strictly concave and HD1, then define

$f(k_t) = F(k_t, 1) + (1-\delta) k_t \ ; \forall t$

output

$c_t = f(k_t) - k_{t+1}$ $\qquad f(k) = F(k, 1) + (1-\delta) k \quad \forall k$

(3)

$$w(\bar{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\underbrace{f(k_t) - k_{t+1}}_{c_t})$$

s.t.

$0 \leq k_{t+1} \leq f(k_t) \quad \forall t$ $\qquad$ ignore constraints assuming INADA conditions hold

$c_t \geq 0$

$k_0 = \bar{k}_0 > 0$, given

FOC: wrt k_{t+1}

$$-\beta^t u'(f(k_t) - k_{t+1}) + \beta^{t+1} u'(f(k_{t+1}) - k_{t+2}) \cdot f'(k_{t+1}) = 0 \quad \forall t$$

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) \cdot f'(k_{t+1}) \quad \forall t$$

Impose transversality condition:

$$\lim_{t\to\infty} \beta^t u'(f(k_t)-k_{t+1})f'(k_t)k_t = 0$$

From Th 12, Krueger, under above assumptions, allocation $\{k_{t+1}\}_{t=0}^{\infty}$ satisfying Euler equations and TVC solves sequential SPP for given k_0.

In a steady state equilibrium, which is social optimum, $c_t = c^*$, $k_{t+1} = k^* = k_0$ and

$$u'(f(k^*)-k^*) = \beta u'(f(k^*)-k^*)f'(k^*)$$

$$\beta f'(k^*) = 1$$

$$f'(k^*) = \underbrace{1+\rho}_{1/\beta} \qquad \beta = \frac{1}{1+\rho}$$

$$f'(k) = F_k(k,1) + 1 - \delta \stackrel{\delta=1}{=} \Rightarrow \underbrace{F_k(k^*,1) = 1+\rho}$$

$$f(k^*) - k^* = c^* \qquad l^* = 1$$

From CE definition, we can rewrite HH problem (*) & assuming $\pi = 0$, $l_t = 1$ & u is strictly ↑, $k_t = x_t$

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \sum_0^{\infty} p_t(c_t + \underbrace{k_{t+1} - (1-\delta)k_t}_{i_t}) = \sum_{t=0}^{\infty} p_t(r_t k_t + w_t)$$

$$c_t, k_{t+1} \geq 0 \quad \forall t$$

k_0 given.

FOC: wrt c_t: $\beta^t u'(c_t) = \mu p_t$ wrt k_{t+1}: $\mu p_t = \mu[1-\delta + r_{t+1}]p_{t+1}$

$$\beta^{t+1} u'(c_{t+1}) = \mu p_{t+1}$$

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$$\frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{1+r_{t+1}-\delta} \quad \forall t$$

From firm's FOC: $r_t = F_k(k_t, 1)$

$f'(k_t) = F(k_t,1) + (1-\delta)k_t$, so $r_t = f'(k_t) - (1-\delta)$

same as in SPP
$$\begin{cases} c_t = f(k_t) - k_{t+1} \rightarrow \text{market clearing condition from goods market} \\ F_k(k_t, 1) = f'(k_t) \\ l_t = 1 \end{cases}$$

Substitute market clearing goods condition & firm's FOC above $\Rightarrow$

$$\frac{\beta u'(f(k_{t+1}) - k_{t+2})}{u'(f(k_t) - k_{t+1})} = \frac{1}{(1+r_{t+1}-\delta)^{-1}}$$

$$\frac{(1+r_{t+1}-\delta)\,\beta u'(f(k_{t+1}) - k_{t+2})}{u'(f(k_t) - k_{t+1})} = 1$$

$$\frac{f'(k_{t+1}) \cdot \beta u'(f(k_{t+1}) - k_{t+2})}{u'(f(k_t) - k_{t+1})} = 1 \quad \leftarrow \text{this is the same Euler equation as in SPP earlier}$$

needs transversality condition

$$\lim_{t\to\infty} \underbrace{\lambda_t}_{= p_t} k_{t+1} = 0$$

It can be shown that allocation $\{k_{t+1}\}_{t=0}^{\infty}$ are Pareto optimal and so SPP's allocations coincide with equil.'s allocations.

(4) BE:

$$v(k) = \max_{0 \le k' \le f(k)} \{ u(f(k) - k') + \beta v(k') \}$$

SP dynamic problem:

$$w(\bar{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \Rightarrow$$

s.t. $0 \le k_{t+1} \le f(k_t) \ \forall t$

k_0 given

$$\Rightarrow \max_{0 \le k_1 \le f(k_0);\, k_0 \text{ given}} u(\overbrace{f(k_0) - k_1}^{c_0}) + \beta [\max_{\substack{\{k_{t+1}\} \\ 0 \le k_{t+1} \le f(k_t)}} \sum_{t=1}^{\infty} \beta^{t-1} \cdot u(f(k_t) - k_{t+1})] \Rightarrow$$

$$w(\bar{k}_0) = \max_{\substack{0 \le k_1 \le f(k_0) \\ k_0 \text{ given}}} \{ u(f(k_0) - k_1) + \beta w(k_1) \}$$

$w(\bar{k}_0) = v(\bar{k}_0)$ under Principle of Optimality.

(5) $u(c) = \log c$

$f(k, l) = z k^{\alpha} l^{1-\alpha}$

$l = 1$

$\delta = 1$

$c_t = f(k_t) - k_{t+1}$

$f(k) = F(k, 1) = z k^{\alpha}$

Functional equation:

$$v(k) = \max_{0 \le k' \le z k^{\alpha}} \{ \log(z k^{\alpha} - k') + \beta v(k') \}$$

By guess & verify: log utility + $\delta = 1$

$v(k) = A + B \ln(k)$ ⇒ substitute the guess for

$$\max_{0 \le k' \le f(k)} \{ \log(z k^{\alpha} - k') + \beta (A + B \ln(k')) \}$$

FOC: $\frac{1}{z k^{\alpha} - k'} = \frac{\beta B}{k'} \Rightarrow k' = \frac{\beta B z k^{\alpha}}{1 + \beta B}$

Evaluate @ k' optimal

$$\log\left(\underbrace{Zk^\alpha - \beta\frac{BZk^\alpha}{1+\beta B}}_{\frac{Zk^\alpha(1+\beta B-\beta B)}{1+\beta B}}\right) + \beta A + \beta B\log\left(\beta\frac{ZBk^\alpha}{1+\beta B}\right)$$

$$= \log(Zk^\alpha) - \log(1+\beta B) + \beta A + \beta B\alpha\log(Zk) + \beta B\log\left(\frac{\beta B}{1+\beta B}\right)$$

$$= \alpha\log(Zk) - \log(1+\beta B) + \beta A +$$

$$+ \beta B\alpha\log(Zk) + \beta B\log\left(\frac{\beta B}{1+\beta B}\right) =$$

$$= \alpha\log Z + \underline{\alpha\log k} - \log(1+\beta B) + \beta A + \beta B\alpha\log Z + \beta\underline{B\alpha\log k}$$

$$+ \beta B\log\left(\frac{\beta B}{1+\beta B}\right)$$

Check:

$$A + B\ln(k) = \underbrace{\alpha\log Z - \log(1+\beta B) + \beta A + \beta B\alpha\log Z + \beta B\log\left(\frac{\beta B}{1+\beta B}\right)}_{A} +$$

$$+ \log k(\underbrace{\alpha + \beta B\alpha}_{B})$$

$$B = \alpha + \beta B\alpha$$

$$B - \beta B\alpha = \alpha$$

$$B(1-\beta\alpha) = \alpha$$

$$B = \frac{\alpha}{1-\beta\alpha}$$

$$A = \alpha\log Z - \log\left(1+\beta\cdot\frac{\alpha}{1-\beta\alpha}\right) + \beta A + \beta\alpha\log(Z)\cdot\frac{\alpha}{1-\beta\alpha} + \beta\cdot\frac{\alpha}{1-\beta\alpha}\log\left(\frac{\frac{\beta\alpha}{1-\beta\alpha}}{1+\beta\cdot\frac{\alpha}{1-\beta\alpha}}\right)$$

$$A(1-\beta) = \alpha\log Z\cdot\left(1+\frac{\beta\alpha}{1-\beta\alpha}\right) + \frac{\beta\alpha}{1-\beta\alpha}\log\left[\frac{\beta\alpha}{1-\beta\alpha}\times\frac{1-\beta\alpha}{1-\beta\alpha+\beta\alpha}\right] - \log\left[1+\frac{\beta\alpha}{1-\beta\alpha}\right]$$

$$A = \frac{1}{1-\beta}\left[\alpha\log Z\cdot\left[\frac{1}{1-\beta\alpha}\right] + \frac{\beta\alpha}{1-\beta\alpha}\log(\beta\alpha) - \log\left(\frac{1}{1-\beta\alpha}\right)\right]$$

$$A = \frac{1}{1-\beta}\cdot\left[\frac{\alpha}{1-\beta\alpha}\cdot\left(\beta\log Z + \beta\log(\alpha\beta)\right) - \log\left(\frac{1}{1-\beta\alpha}\right)\right]$$

policy fn: $k' = g(k) = \frac{\beta B z k^{\alpha}}{1+\beta B} = \frac{\beta \cdot \frac{\alpha}{1-\beta\alpha} \cdot z k^{\alpha}}{1+\beta \cdot \alpha/(1-\beta\alpha)} =$

$= \frac{\beta\alpha z k^{\alpha}}{1-\beta\alpha} : \left(1 + \frac{\beta\alpha}{1-\beta\alpha}\right) = \frac{\beta\alpha z k^{\alpha}}{1-\beta\alpha} \times \frac{1-\beta\alpha}{1} =$

$= \underline{\alpha\beta z k^{\alpha}}$

$k_1 = \alpha\beta z k_0^{\alpha}$
$k_2 = \alpha\beta z k_1^{\alpha}$
$k_3 = \alpha\beta z k_2^{\alpha}$
$\vdots$

6 $c_t = c^* = c_{t+1}$
$k_{t+1} = k^*$
From SPP steady state solution:

$\beta \cdot f'(k^*) = 1$ $\qquad f(k^*) = z k^{*\alpha} + (1-\delta)k$ $\quad$ "1"

$\beta \cdot \alpha z k^{*\alpha-1} = 1 \mid \wedge \frac{1}{\alpha-1}$ $\qquad f'(k^*) = \alpha z k^{*\alpha-1}$

$\alpha z k^{*\alpha-1} = \frac{1}{\beta}$

$\underbrace{k^* = \left(\frac{1}{\beta\alpha z}\right)^{\frac{1}{\alpha-1}}}$

$c^* = f(k^*) - k^* =$
$= z k^{*\alpha} - k^* =$
$= z \cdot \left(\frac{1}{\beta\alpha z}\right)^{\alpha/\alpha-1} - \left(\frac{1}{\beta\alpha z}\right)^{1/\alpha-1}$

$r^* = f'(k^*) = \alpha z k^{*\alpha-1} = \alpha z \left[\left(\frac{1}{\beta\alpha z}\right)^{\frac{1}{\alpha-1}}\right]^{\alpha-1} = \frac{\alpha z}{\beta\alpha z} = \frac{1}{\beta}$

$y^* = F(k^*, 1) = z k^{*\alpha} = z \cdot \left(\frac{1}{\beta\alpha z}\right)^{\alpha/\alpha-1}$

$w^* = F_e(k^*, 1) = (1-\alpha) z k^{*\alpha} = (1-\alpha) z \cdot \left(\frac{1}{\beta\alpha z}\right)^{\alpha/\alpha-1}$

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I attached Matlab code,
Julia code is in progress

(7) $\alpha = 1/3$
$Z = 1$
$\delta = 1$
$\beta = 0,96$

shock to $K^* \to 0.8 K^*$
shock to $\alpha \to 1.052$

$$K^* = \left(\frac{1}{\beta \cdot 1/3 \cdot 1} \right)^{\frac{1}{1/3 - 1}}$$

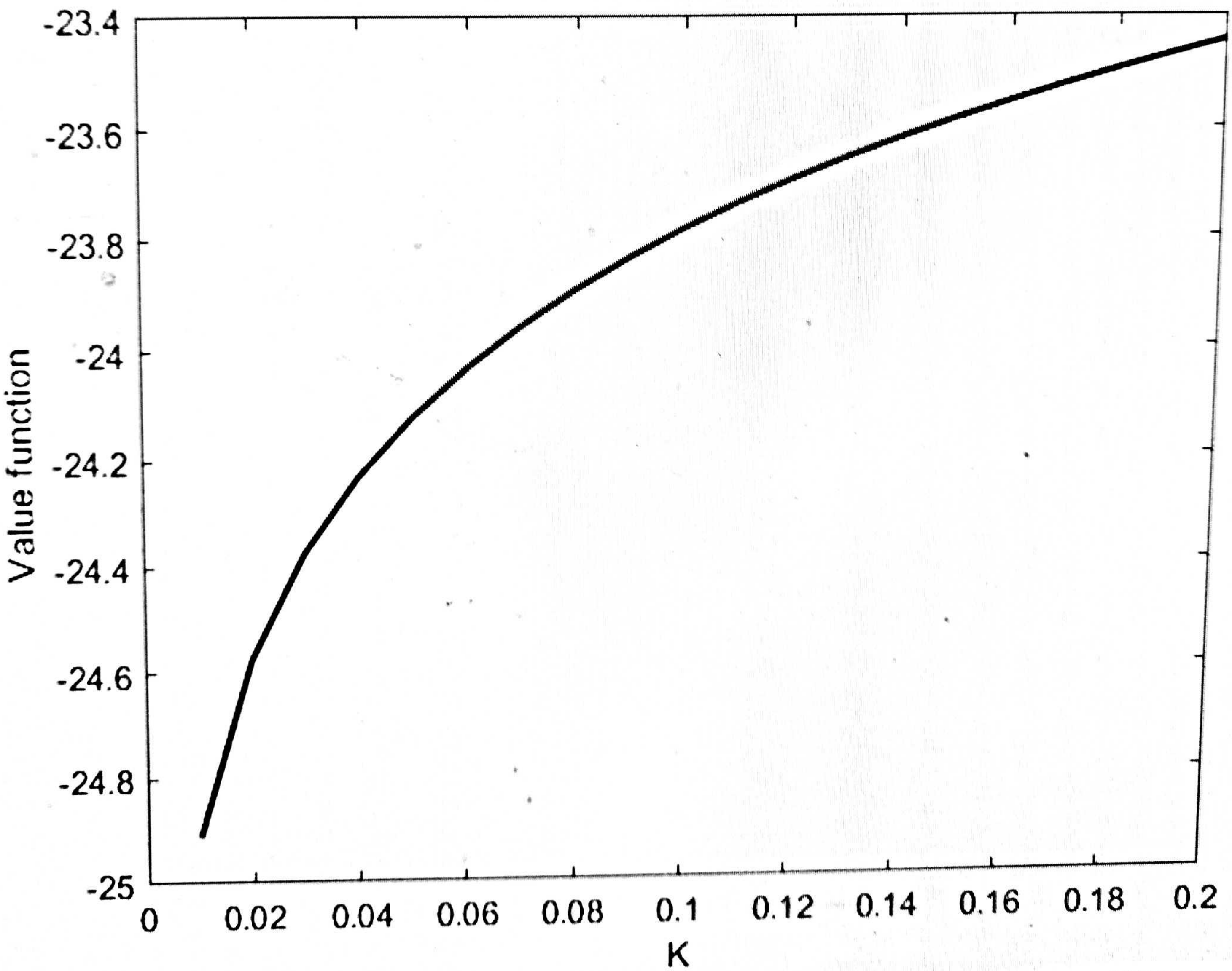
$$C^* = \left(\frac{1}{\beta \cdot 1/3 \cdot 1} \right)^{1/3/(1/3-1)} - \left(\frac{1}{1/3 \beta} \right)^{1/(1/3-1)}$$

$$r^* = \frac{1}{\beta}$$

$$y^* = \left(\frac{1}{\beta \cdot 1/3} \right)^{1/3/(1/3-1)}$$

$$w^* = (1 - 1/3)\left(\frac{1}{1/3 \beta} \right)^{1/3/(1/3-1)}$$

(8) Done


-23.4
-23.6
-23.8
-24
-24.2
-24.4
-24.6
-24.8
-25
Value function
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0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
K