■pre-conditioner

Note: The equation of this paragraph is based on the concept of Weiss et al.(Preconditioning Applied to Variable and Constant Density Flows,1995)

Governing equation is

$$\boldsymbol{\Gamma}\frac{\partial \boldsymbol{Q}}{\partial t} + \frac{\partial \boldsymbol{E}}{\partial x} + \frac{\partial \boldsymbol{F}}{\partial y} = 0 \tag{1}$$

while

$$\boldsymbol{Q} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}, \qquad \boldsymbol{E} = \begin{pmatrix} \rho_1 u \\ \rho_2 u \\ \vdots \\ \rho_n u \\ \rho u^2 + p \\ \rho v u \\ \rho u H \end{pmatrix}, \qquad \boldsymbol{F} = \begin{pmatrix} \rho_1 v \\ \rho_2 v \\ \vdots \\ \rho_n v \\ \rho u v \\ \rho v^2 + p \\ \rho v H \end{pmatrix} \tag{2}$$

and the Preconditioner matrix is

$$\boldsymbol{\Gamma} = \boldsymbol{I} + \Phi \boldsymbol{\phi} \frac{\partial p}{\partial \boldsymbol{Q}}. \tag{3}$$

$\boldsymbol{I}$ is identity matrix and

$$\boldsymbol{\phi} = \begin{pmatrix} \rho_1/\rho \\ \rho_2/\rho \\ \vdots \\ \rho_n/\rho \\ u \\ v \\ H \end{pmatrix} \tag{4}$$

and $\frac{\partial p}{\partial \boldsymbol{Q}}$ is raw vector.

Φ is calculated from the equations below.

$$\Phi = \frac{1}{U_r^2} - \frac{1}{c^2} \tag{5}$$

$$U_r = \min(\max(V_v, |V|), c), \qquad V_v = \frac{\mu}{\rho \Delta s} \tag{6}$$

$|V|$ is the absolute value of local velocity. Δs is the shorter one of lengths of cell. V_v represents reference velocity of viscosity.

The governing equation is deformated as below.

$$\boldsymbol{\Gamma}\frac{\partial \boldsymbol{Q}^{n+1}}{\partial t} + \frac{\partial \boldsymbol{E}^{n+1}}{\partial x} + \frac{\partial \boldsymbol{F}^{n+1}}{\partial y} = 0 \tag{7}$$

$$\frac{\partial \boldsymbol{Q}^{n+1}}{\partial t} + \boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{E}^{n+1}}{\partial x} + \boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{F}^{n+1}}{\partial y} = 0 \tag{8}$$

$$\int \frac{\partial \boldsymbol{Q}^{n+1}}{\partial t} dV + \int \boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{E}^{n+1}}{\partial x} dV + \int \boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{F}^{n+1}}{\partial y} dV = 0 \tag{9}$$

$$\begin{aligned}\frac{\Delta \boldsymbol{Q}_{i,j}}{\Delta t}\mathrm{Vol}_{i,j} &+ (\boldsymbol{\Gamma}^{-1}\boldsymbol{E}^{n+1})_{i+1/2,j}\Delta si_{i+1/2,j} - (\boldsymbol{\Gamma}^{-1}\boldsymbol{E}^{n+1})_{i-1/2,j}\Delta si_{i-1/2,j} \\ &+ (\boldsymbol{\Gamma}^{-1}\boldsymbol{F}^{n+1})_{i,j+1/2}\Delta sj_{i,j+1/2} - (\boldsymbol{\Gamma}^{-1}\boldsymbol{F}^{n+1})_{i,j-1/2}\Delta sj_{i,j-1/2} = 0\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\Delta \boldsymbol{Q}_{i,j}}{\Delta t}\mathrm{Vol}_{i,j} &+ (\boldsymbol{\Gamma}^{-1}\boldsymbol{E}^{n})_{i+1/2,j}\Delta si_{i+1/2,j} - (\boldsymbol{\Gamma}^{-1}\boldsymbol{E}^{n})_{i-1/2,j}\Delta si_{i-1/2,j} \\ &+ (\boldsymbol{A}\Delta\boldsymbol{Q})_{i+1/2,j}\Delta si_{i+1/2,j} - (\boldsymbol{A}\Delta\boldsymbol{Q})_{i-1/2,j}\Delta si_{i-1/2,j} \\ &+ (\boldsymbol{\Gamma}^{-1}\boldsymbol{F}^{n})_{i,j+1/2}\Delta sj_{i,j+1/2} - (\boldsymbol{\Gamma}^{-1}\boldsymbol{F}^{n})_{i,j-1/2}\Delta sj_{i,j-1/2} \\ &+ (\boldsymbol{B}\Delta\boldsymbol{Q})_{i,j+1/2}\Delta sj_{i,j+1/2} - (\boldsymbol{B}\Delta\boldsymbol{Q})_{i,j-1/2}\Delta sj_{i,j-1/2} = 0\end{aligned} \tag{11}$$

while

$$\boldsymbol{A} = \frac{\partial(\boldsymbol{\Gamma}^{-1}\boldsymbol{E})}{\partial \boldsymbol{Q}}, \qquad \boldsymbol{B} = \frac{\partial(\boldsymbol{\Gamma}^{-1}\boldsymbol{F})}{\partial \boldsymbol{Q}} \tag{12}$$

and the spectral radius ν_A for A and ν_B for B will be

$$\begin{cases}\nu_A = |u'_A| + c'_A \\ u'_A = u(1-\alpha) \\ c'_A = \sqrt{\alpha^2 u^2 + U_r^2}\end{cases} \tag{13}$$

$$\begin{cases}\nu_B = |u'_B| + c'_B \\ u'_B = v(1-\alpha) \\ c'_B = \sqrt{\alpha^2 v^2 + U_r^2}\end{cases} \tag{14}$$

$$\alpha = \frac{1}{2}\left(1 - \left(\frac{U_r}{c}\right)^2\right) \tag{15}$$

The time step have to be calculated from these spectral radiuses.

When LU-SGS is used, A and B is approximated as below.

$$A_{i+1/2,j} = A^{+}_{i,j} + A^{-}_{i+1,j} \tag{16}$$

$$A^{\pm}_{i,j} = \frac{1}{2}\left(A \pm \nu_A \boldsymbol{I}\right) \tag{17}$$

Therefore, (11) becomes

$$\begin{aligned}
\frac{\Delta \boldsymbol{Q}_{i,j}}{\Delta t}\mathrm{Vol}_{i,j} &+ (\boldsymbol{\Gamma}^{-1}\boldsymbol{E}^n)_{i+1/2,j}\Delta si_{i+1/2,j} - (\boldsymbol{\Gamma}^{-1}\boldsymbol{E}^n)_{i-1/2,j}\Delta si_{i-1/2,j} \\
&+ (\boldsymbol{A}^+\Delta\boldsymbol{Q})_{i,j}\Delta si_{i,j} - (\boldsymbol{A}^+\Delta\boldsymbol{Q})_{i-1,j}\Delta si_{i-1,j} \\
&+ (\boldsymbol{A}^-\Delta\boldsymbol{Q})_{i+1,j}\Delta si_{i+1,j} - (\boldsymbol{A}^-\Delta\boldsymbol{Q})_{i,j}\Delta si_{i,j} \\
&+ (\boldsymbol{\Gamma}^{-1}\boldsymbol{F}^n)_{i,j+1/2}\Delta sj_{i,j+1/2} - (\boldsymbol{\Gamma}^{-1}\boldsymbol{F}^n)_{i,j-1/2}\Delta sj_{i,j-1/2} \\
&+ (\boldsymbol{B}^+\Delta\boldsymbol{Q})_{i,j}\Delta sj_{i,j} - (\boldsymbol{B}^+\Delta\boldsymbol{Q})_{i,j-1}\Delta sj_{i,j-1} \\
&+ (\boldsymbol{B}^-\Delta\boldsymbol{Q})_{i,j+1}\Delta sj_{i,j+1} - (\boldsymbol{B}^-\Delta\boldsymbol{Q})_{i,j}\Delta sj_{i,j} = 0
\end{aligned} \tag{18}$$

$$\begin{aligned}
\frac{\Delta \boldsymbol{Q}_{i,j}}{\Delta t}\mathrm{Vol}_{i,j} &+ (\boldsymbol{\Gamma}^{-1}\boldsymbol{E}^n)_{i+1/2,j}\Delta si_{i+1/2,j} - (\boldsymbol{\Gamma}^{-1}\boldsymbol{E}^n)_{i-1/2,j}\Delta si_{i-1/2,j} \\
&+ (\nu_A\Delta\boldsymbol{Q})_{i,j}\Delta si_{i,j} + (\boldsymbol{A}^-\Delta\boldsymbol{Q})_{i+1,j}\Delta si_{i+1,j} - (\boldsymbol{A}^+\Delta\boldsymbol{Q})_{i-1,j}\Delta si_{i-1,j} \\
&+ (\boldsymbol{\Gamma}^{-1}\boldsymbol{F}^n)_{i,j+1/2}\Delta sj_{i,j+1/2} - (\boldsymbol{\Gamma}^{-1}\boldsymbol{F}^n)_{i,j-1/2}\Delta sj_{i,j-1/2} \\
&+ (\nu_B\Delta\boldsymbol{Q})_{i,j}\Delta sj_{i,j} + (\boldsymbol{B}^-\Delta\boldsymbol{Q})_{i,j+1}\Delta sj_{i,j+1} - (\boldsymbol{B}^+\Delta\boldsymbol{Q})_{i,j-1}\Delta sj_{i,j-1} = 0
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\left(\frac{\mathrm{Vol}_{i,j}}{\Delta t} + \nu_{A_{i,j}}\Delta si_{i,j} + \nu_{B_{i,j}}\Delta sj_{i,j}\right)\Delta\boldsymbol{Q}_{i,j} \\
&+ (\boldsymbol{A}^-\Delta\boldsymbol{Q})_{i+1,j}\Delta si_{i+1,j} - (\boldsymbol{A}^+\Delta\boldsymbol{Q})_{i-1,j}\Delta si_{i-1,j} \\
&+ (\boldsymbol{B}^-\Delta\boldsymbol{Q})_{i,j+1}\Delta sj_{i,j+1} - (\boldsymbol{B}^+\Delta\boldsymbol{Q})_{i,j-1}\Delta sj_{i,j-1} \\
&+ (\boldsymbol{\Gamma}^{-1}\boldsymbol{E}^n)_{i+1/2,j}\Delta si_{i+1/2,j} - (\boldsymbol{\Gamma}^{-1}\boldsymbol{E}^n)_{i-1/2,j}\Delta si_{i-1/2,j} \\
&+ (\boldsymbol{\Gamma}^{-1}\boldsymbol{F}^n)_{i,j+1/2}\Delta sj_{i,j+1/2} - (\boldsymbol{\Gamma}^{-1}\boldsymbol{F}^n)_{i,j-1/2}\Delta sj_{i,j-1/2} = 0
\end{aligned} \tag{20}$$

and for convinience, this equation is approximated as follows:

$$\begin{aligned}
&\left(\frac{\mathrm{Vol}_{i,j}}{\Delta t} + \nu_{A_{i,j}}\Delta si_{i,j} + \nu_{B_{i,j}}\Delta sj_{i,j}\right)\Delta\boldsymbol{Q}_{i,j} \\
&+ (\boldsymbol{A}^-\Delta\boldsymbol{Q})_{i+1,j}\Delta si_{i+1,j} - (\boldsymbol{A}^+\Delta\boldsymbol{Q})_{i-1,j}\Delta si_{i-1,j} \\
&+ (\boldsymbol{B}^-\Delta\boldsymbol{Q})_{i,j+1}\Delta sj_{i,j+1} - (\boldsymbol{B}^+\Delta\boldsymbol{Q})_{i,j-1}\Delta sj_{i,j-1} \\
&+ \boldsymbol{\Gamma}^{-1}_{i,j}\left((\boldsymbol{E}\Delta si)_{i+1/2,j} - (\boldsymbol{E}\Delta si)_{i-1/2,j} + (\boldsymbol{F}\Delta sj)_{i,j+1/2} - (\boldsymbol{F}\Delta sj)_{i,j-1/2}\right) = 0.
\end{aligned} \tag{21}$$

Still more, the expressions below are used.

$$1/\alpha_{i,j} = \frac{\mathrm{Vol}_{i,j}}{\Delta t} + \nu_{A_{i,j}}\Delta si_{i,j} + \nu_{B_{i,j}}\Delta sj_{i,j} \tag{22}$$

$$D_i^+\Delta\boldsymbol{Q}_{i,j} = \Delta\boldsymbol{Q}_{i+1,j} \tag{23}$$

$$D_i^-\Delta\boldsymbol{Q}_{i,j} = \Delta\boldsymbol{Q}_{i-1,j} \tag{24}$$

$$D_j^+\Delta\boldsymbol{Q}_{i,j} = \Delta\boldsymbol{Q}_{i,j+1} \tag{25}$$

$$D_j^-\Delta\boldsymbol{Q}_{i,j} = \Delta\boldsymbol{Q}_{i,j-1} \tag{26}$$

Then (27) becomes

$$\begin{aligned}
&(\boldsymbol{I} + \alpha_{i,j}(\boldsymbol{A}^-\Delta si)_{i+1,j}D_i^+ - \alpha_{i,j}(\boldsymbol{A}^+\Delta si)_{i-1,j}D_i^- \\
&\quad + \alpha_{i,j}(\boldsymbol{B}^-\Delta sj)_{i,j+1}D_j^+ - \alpha_{i,j}(\boldsymbol{B}^+\Delta sj)_{i,j-1}D_j^-)\Delta Q_{i,j} \\
&\quad + \alpha_{i,j}\boldsymbol{\Gamma}^{-1}_{i,j}\left((\boldsymbol{E}\Delta si)_{i+1/2,j} - (\boldsymbol{E}\Delta si)_{i-1/2,j} + (\boldsymbol{F}\Delta sj)_{i,j+1/2} - (\boldsymbol{F}\Delta sj)_{i,j-1/2}\right) = 0
\end{aligned} \tag{27}$$

and moreover because $\alpha_{i,j}$ becomes small generally, the deformation below is used.

$$(\boldsymbol{I} - \alpha_{i,j}(\boldsymbol{A}^{+}\Delta si)_{i-1,j}D_i^{-} - \alpha_{i,j}(\boldsymbol{B}^{+}\Delta sj)_{i,j-1}D_j^{-})(\boldsymbol{I} + \alpha_{i,j}(\boldsymbol{A}^{-}\Delta si)_{i+1,j}D_i^{+} + \alpha_{i,j}(\boldsymbol{B}^{-}\Delta sj)_{i,j+1}D_j^{+})\Delta Q_{i,j} \\ + \alpha_{i,j}\boldsymbol{\Gamma}_{i,j}^{-1}\big((\boldsymbol{E}\Delta si)_{i+1/2,j} - (\boldsymbol{E}\Delta si)_{i-1/2,j} + (\boldsymbol{F}\Delta sj)_{i,j+1/2} - (\boldsymbol{F}\Delta sj)_{i,j-1/2}\big) = 0. \tag{28}$$

This equation is solved in two steps. First step is

$$\begin{aligned}&(\boldsymbol{I} - \alpha_{i,j}(\boldsymbol{A}^{+}\Delta si)_{i-1,j}D_i^{-} - \alpha_{i,j}(\boldsymbol{B}^{+}\Delta sj)_{i,j-1}D_j^{-})\Delta Q_{i,j}^{*} \\ &+ \alpha_{i,j}\boldsymbol{\Gamma}_{i,j}^{-1}\big((\boldsymbol{E}\Delta si)_{i+1/2,j} - (\boldsymbol{E}\Delta si)_{i-1/2,j} + (\boldsymbol{F}\Delta sj)_{i,j+1/2} - (\boldsymbol{F}\Delta sj)_{i,j-1/2}\big) = 0\end{aligned} \tag{29}$$

i.e.

$$\begin{aligned}&\Delta Q_{i,j}^{*} - \alpha_{i,j}(\boldsymbol{A}^{+}\Delta si)_{i-1,j}\Delta Q_{i-1,j}^{*} - \alpha_{i,j}(\boldsymbol{B}^{+}\Delta sj)_{i,j-1}\Delta Q_{i,j-1}^{*} \\ &+ \alpha_{i,j}\boldsymbol{\Gamma}_{i,j}^{-1}\big((\boldsymbol{E}\Delta si)_{i+1/2,j} - (\boldsymbol{E}\Delta si)_{i-1/2,j} + (\boldsymbol{F}\Delta sj)_{i,j+1/2} - (\boldsymbol{F}\Delta sj)_{i,j-1/2}\big) = 0\end{aligned} \tag{30}$$

$$\therefore \begin{aligned}\Delta Q_{i,j}^{*} &= \alpha_{i,j}((\boldsymbol{A}^{+}\Delta si)_{i-1,j}\Delta Q_{i-1,j}^{*} + (\boldsymbol{B}^{+}\Delta sj)_{i,j-1}\Delta Q_{i,j-1}^{*} \\ &\quad - \boldsymbol{\Gamma}_{i,j}^{-1}\big((\boldsymbol{E}\Delta si)_{i+1/2,j} - (\boldsymbol{E}\Delta si)_{i-1/2,j} + (\boldsymbol{F}\Delta sj)_{i,j+1/2} - (\boldsymbol{F}\Delta sj)_{i,j-1/2}\big)).\end{aligned} \tag{31}$$

The second one is

$$(\boldsymbol{I} + \alpha_{i,j}(\boldsymbol{A}^{-}\Delta si)_{i+1,j}D_i^{+} + \alpha_{i,j}(\boldsymbol{B}^{-}\Delta sj)_{i,j+1}D_j^{+})\Delta Q_{i,j} = \Delta Q_{i,j}^{*} \tag{32}$$

i.e.

$$\Delta Q_{i,j} + \alpha_{i,j}(\boldsymbol{A}^{-}\Delta si)_{i+1,j}\Delta Q_{i+1,j} + \alpha_{i,j}(\boldsymbol{B}^{-}\Delta sj)_{i,j+1}\Delta Q_{i,j+1} = \Delta Q_{i,j}^{*} \tag{33}$$

$$\therefore \Delta Q_{i,j} = \Delta Q_{i,j}^{*} - \alpha_{i,j}((\boldsymbol{A}^{-}\Delta si)_{i+1,j}\Delta Q_{i+1,j} + (\boldsymbol{B}^{-}\Delta sj)_{i,j+1}\Delta Q_{i,j+1}). \tag{34}$$

When calculating this equation, you have to calculate $\boldsymbol{\Gamma}^{-1}$. But this can be calculated easily. For general expression, considering the matrix next.

$$H = I + ab \tag{35}$$

while I is identity matrix, a is row vector and b is column vector. The inverse matrix is

$$H^{-1} = I - \frac{1}{1 + (b \cdot a)}ab \tag{36}$$

because

$$\begin{aligned}(I + ab)(I - \frac{1}{1 + (b \cdot a)}ab) &= I + ab - \frac{1}{1 + (b \cdot a)}ab - \frac{1}{1 + (b \cdot a)}abab \\ &= I + ab - \frac{1}{1 + (b \cdot a)}ab - \frac{1}{1 + (b \cdot a)}a(b \cdot a)b \\ &= I + ab - \frac{1}{1 + (b \cdot a)}ab - \frac{(b \cdot a)}{1 + (b \cdot a)}ab \\ &= I + (1 - \frac{1}{1 + (b \cdot a)} - \frac{(b \cdot a)}{1 + (b \cdot a)})ab \\ &= I + \frac{1 + (b \cdot a) - 1 - (b \cdot a)}{1 + (b \cdot a)}ab \\ &= I\end{aligned} \tag{37}$$

and the mathematical commutative law can be applied to this calculation. Therefore, if $a = \Phi\boldsymbol{\phi}$ and $b = \frac{\partial p}{\partial \boldsymbol{Q}}$, by using $(\boldsymbol{\phi} \cdot \frac{\partial p}{\partial \boldsymbol{Q}}) = c^2$ (I cannot explain theoretically this relation, but when calculating it, this relation is correct.)

$$\begin{aligned}\boldsymbol{\Gamma}^{-1} &= \boldsymbol{I} - \frac{1}{1 + (\Phi\boldsymbol{\phi} \cdot \frac{\partial p}{\partial \boldsymbol{Q}})} \Phi\boldsymbol{\phi} \frac{\partial p}{\partial \boldsymbol{Q}} \\ &= \boldsymbol{I} - \frac{\Phi}{1 + \Phi c^2} \boldsymbol{\phi} \frac{\partial p}{\partial \boldsymbol{Q}} \\ &= \boldsymbol{I} - \frac{1}{\frac{1}{\Phi} + c^2} \boldsymbol{\phi} \frac{\partial p}{\partial \boldsymbol{Q}}.\end{aligned} \tag{38}$$

In my program, $-\frac{1}{\frac{1}{\Phi}+c^2}$ is named as "phh".

■Dual Time Step method with pre-conditioner The governing equation is

$$\boldsymbol{\Gamma}\frac{\partial \boldsymbol{Q}}{\partial \tau} + \frac{\partial \boldsymbol{Q}}{\partial t} + \frac{\partial \boldsymbol{E}}{\partial x} + \frac{\partial \boldsymbol{F}}{\partial y} = 0 \tag{39}$$

The concept of the dual time step is when the equation is converged for τ, i.e. $\boldsymbol{\Gamma}\frac{\partial \boldsymbol{Q}}{\partial \tau} = 0$, this equation becomes

$$\frac{\partial \boldsymbol{Q}}{\partial t} + \frac{\partial \boldsymbol{E}}{\partial x} + \frac{\partial \boldsymbol{F}}{\partial y} = 0 \tag{40}$$

Therefore unsteady phenomena can be solved with pre-conditioner.

In this calculation, outer time step is counted by n, and the inner time step is counted by m. For the calculation for $n+1$ outer step, the equation at $m+1$ inner step becomes

$$\frac{\Delta\hat{\boldsymbol{Q}}}{\Delta\tau} + \boldsymbol{\Gamma}^{-1}\frac{3\boldsymbol{Q}^{m+1} - 4\boldsymbol{Q}^{n} + \boldsymbol{Q}^{n-1}}{2\Delta t} + \boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{E}^{m+1}}{\partial x} + \boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{F}^{m+1}}{\partial y} = 0 \tag{41}$$

while $\Delta\hat{\boldsymbol{Q}} = \hat{\boldsymbol{Q}}^{m+1} - \hat{\boldsymbol{Q}}^{m}$. When the inner loop is converged, that is, $\Delta\hat{\boldsymbol{Q}}$ becomes nearly zero, $\boldsymbol{Q}^{n+1}$ is set to $\boldsymbol{Q}^{m+1}$ and next n is calculated.

This equation is deformated like the section before, and finally it becomes

$$\begin{aligned}&\left(\mathrm{Vol}_{i,j}(\frac{1}{\Delta\tau} + \frac{3}{2\Delta t}\boldsymbol{\Gamma}^{-1}) + \nu_{A_{i,j}}\Delta si_{i,j} + \nu_{B_{i,j}}\Delta sj_{i,j}\right)\Delta\hat{\boldsymbol{Q}}_{i,j} \\ &+ (\boldsymbol{A}^{-}\Delta\hat{\boldsymbol{Q}})_{i+1,j}\Delta si_{i+1,j} - (\boldsymbol{A}^{+}\Delta\hat{\boldsymbol{Q}})_{i-1,j}\Delta si_{i-1,j} \\ &+ (\boldsymbol{B}^{-}\Delta\hat{\boldsymbol{Q}})_{i,j+1}\Delta sj_{i,j+1} - (\boldsymbol{B}^{+}\Delta\hat{\boldsymbol{Q}})_{i,j-1}\Delta sj_{i,j-1} \\ &+ \boldsymbol{\Gamma}_{i,j}^{-1}\left((\boldsymbol{E}\Delta si)_{i+1/2,j} - (\boldsymbol{E}\Delta si)_{i-1/2,j} + (\boldsymbol{F}\Delta sj)_{i,j+1/2} - (\boldsymbol{F}\Delta sj)_{i,j-1/2} + \frac{3\boldsymbol{Q}^{m} - 4\boldsymbol{Q}^{n} + \boldsymbol{Q}^{n-1}}{2\Delta t}\right) = 0\end{aligned} \tag{42}$$

$$\begin{aligned}&\left((\frac{\mathrm{Vol}_{i,j}}{\Delta\tau} + \nu_{A_{i,j}}\Delta si_{i,j} + \nu_{B_{i,j}}\Delta sj_{i,j}) + \frac{3\mathrm{Vol}_{i,j}}{2\Delta t}\boldsymbol{\Gamma}^{-1}\right)\Delta\hat{\boldsymbol{Q}}_{i,j} \\ &+ (\boldsymbol{A}^{-}\Delta\hat{\boldsymbol{Q}})_{i+1,j}\Delta si_{i+1,j} - (\boldsymbol{A}^{+}\Delta\hat{\boldsymbol{Q}})_{i-1,j}\Delta si_{i-1,j} \\ &+ (\boldsymbol{B}^{-}\Delta\hat{\boldsymbol{Q}})_{i,j+1}\Delta sj_{i,j+1} - (\boldsymbol{B}^{+}\Delta\hat{\boldsymbol{Q}})_{i,j-1}\Delta sj_{i,j-1} \\ &+ \boldsymbol{\Gamma}_{i,j}^{-1}\left((\boldsymbol{E}\Delta si)_{i+1/2,j} - (\boldsymbol{E}\Delta si)_{i-1/2,j} + (\boldsymbol{F}\Delta sj)_{i,j+1/2} - (\boldsymbol{F}\Delta sj)_{i,j-1/2} + \frac{3\boldsymbol{Q}^{m} - 4\boldsymbol{Q}^{n} + \boldsymbol{Q}^{n-1}}{2\Delta t}\right) = 0\end{aligned} \tag{43}$$

So if the expressions below are used,

$$1/\alpha_{i,j} = \frac{\mathrm{Vol}_{i,j}}{\Delta\tau} + \nu_{A_{i,j}}\Delta si_{i,j} + \nu_{B_{i,j}}\Delta sj_{i,j} \tag{44}$$

$$\beta_{i,j} = \frac{3\mathrm{Vol}_{i,j}}{2\Delta t}\alpha_{i,j} \tag{45}$$

$$\boldsymbol{S}_{i,j} = \boldsymbol{I} + \beta_{i,j}\boldsymbol{\Gamma}^{-1} \tag{46}$$

$$RHS_{i,j} = -\boldsymbol{\Gamma}_{i,j}^{-1}\left((\boldsymbol{E}\Delta si)_{i+1/2,j} - (\boldsymbol{E}\Delta si)_{i-1/2,j} + (\boldsymbol{F}\Delta sj)_{i,j+1/2} - (\boldsymbol{F}\Delta sj)_{i,j-1/2} + \frac{3\boldsymbol{Q}^m - 4\boldsymbol{Q}^n + \boldsymbol{Q}^{n-1}}{2\Delta t}\right) \tag{47}$$

$$D_i^+\Delta\hat{\boldsymbol{Q}}_{i,j} = \Delta\hat{\boldsymbol{Q}}_{i+1,j} \tag{48}$$

$$D_i^-\Delta\hat{\boldsymbol{Q}}_{i,j} = \Delta\hat{\boldsymbol{Q}}_{i-1,j} \tag{49}$$

$$D_j^+\Delta\hat{\boldsymbol{Q}}_{i,j} = \Delta\hat{\boldsymbol{Q}}_{i,j+1} \tag{50}$$

$$D_j^-\Delta\hat{\boldsymbol{Q}}_{i,j} = \Delta\hat{\boldsymbol{Q}}_{i,j-1} \tag{51}$$

it becomes

$$\begin{aligned}&\left(\boldsymbol{I} + \beta_{i,j}\boldsymbol{\Gamma}^{-1} + \alpha_{i,j}(\boldsymbol{A}^-\Delta si)_{i+1,j}D_i^+ - \alpha_{i,j}(\boldsymbol{A}^+\Delta si)_{i-1,j}D_i^- + \alpha_{i,j}(\boldsymbol{B}^-\Delta sj)_{i,j+1}D_j^+ - \alpha_{i,j}(\boldsymbol{B}^+\Delta sj)_{i,j-1}D_j^-\right)\hat{\boldsymbol{Q}}_{i,j}\\ &= \alpha_{i,j}RHS_{i,j}\end{aligned} \tag{52}$$

$$\begin{aligned}&\left(\boldsymbol{S}_{i,j} + \alpha_{i,j}(\boldsymbol{A}^-\Delta si)_{i+1,j}D_i^+ - \alpha_{i,j}(\boldsymbol{A}^+\Delta si)_{i-1,j}D_i^- + \alpha_{i,j}(\boldsymbol{B}^-\Delta sj)_{i,j+1}D_j^+ - \alpha_{i,j}(\boldsymbol{B}^+\Delta sj)_{i,j-1}D_j^-\right)\hat{\boldsymbol{Q}}_{i,j}\\ &= \alpha_{i,j}RHS_{i,j}\end{aligned} \tag{53}$$

Here we use LDU decomposition approximation such as

$$L + D + U \sim (D+U)D^{-1}(D+L). \tag{54}$$

So it becomes

$$\begin{aligned}&\left(\boldsymbol{S}_{i,j} - \alpha_{i,j}(\boldsymbol{A}^+\Delta si)_{i-1,j}D_i^- - \alpha_{i,j}(\boldsymbol{B}^+\Delta sj)_{i,j-1}D_j^-\right)\boldsymbol{S}_{i,j}^{-1}\left(\boldsymbol{S}_{i,j} + \alpha_{i,j}(\boldsymbol{A}^-\Delta si)_{i+1,j}D_i^+ + \alpha_{i,j}(\boldsymbol{B}^-\Delta sj)_{i,j+1}D_j^+\right)\hat{\boldsymbol{Q}}_{i,j}\\ &= \alpha_{i,j}RHS_{i,j}\end{aligned} \tag{55}$$

This equation is calculated three steps.

First step

$$\left(\boldsymbol{S}_{i,j} - \alpha_{i,j}(\boldsymbol{A}^+\Delta si)_{i-1,j}D_i^- - \alpha_{i,j}(\boldsymbol{B}^+\Delta sj)_{i,j-1}D_j^-\right)\hat{\boldsymbol{Q}}_{i,j}^{**} = \alpha_{i,j}RHS_{i,j} \tag{56}$$

$$\therefore \boldsymbol{S}_{i,j}\hat{\boldsymbol{Q}}_{i,j}^{**} - \alpha_{i,j}(\boldsymbol{A}^+\Delta si)_{i-1,j}\hat{\boldsymbol{Q}}_{i-1,j}^{**} - \alpha_{i,j}(\boldsymbol{B}^+\Delta sj)_{i,j-1}\hat{\boldsymbol{Q}}_{i,j-1}^{**} = \alpha_{i,j}RHS_{i,j} \tag{57}$$

$$\boldsymbol{S}_{i,j}\hat{\boldsymbol{Q}}_{i,j}^{**} = \alpha_{i,j}(RHS_{i,j} + (\boldsymbol{A}^+\Delta si)_{i-1,j}\hat{\boldsymbol{Q}}_{i-1,j}^{**} + (\boldsymbol{B}^+\Delta sj)_{i,j-1}\hat{\boldsymbol{Q}}_{i,j-1}^{**}) \tag{58}$$

$$\hat{\boldsymbol{Q}}_{i,j}^{**} = \alpha_{i,j}\boldsymbol{S}_{i,j}^{-1}(RHS_{i,j} + (\boldsymbol{A}^+\Delta si)_{i-1,j}\hat{\boldsymbol{Q}}_{i-1,j}^{**} + (\boldsymbol{B}^+\Delta sj)_{i,j-1}\hat{\boldsymbol{Q}}_{i,j-1}^{**}) \tag{59}$$

Second step

$$\boldsymbol{S}_{i,j}^{-1}\hat{\boldsymbol{Q}}_{i,j}^{*} = \hat{\boldsymbol{Q}}_{i,j}^{**} \tag{60}$$

$$\therefore \hat{\boldsymbol{Q}}_{i,j}^{*} = \boldsymbol{S}_{i,j}\hat{\boldsymbol{Q}}_{i,j}^{**} \tag{61}$$

Third step

$$\left(\boldsymbol{S}_{i,j} + \alpha_{i,j}(\boldsymbol{A}^{-}\Delta si)_{i+1,j}D_i^{+} + \alpha_{i,j}(\boldsymbol{B}^{-}\Delta sj)_{i,j+1}D_j^{+}\right)\hat{\boldsymbol{Q}}_{i,j} = \hat{\boldsymbol{Q}}^{*}_{i,j} \tag{62}$$

$$\therefore \boldsymbol{S}_{i,j}\hat{\boldsymbol{Q}}_{i,j} + \alpha_{i,j}(\boldsymbol{A}^{-}\Delta si)_{i+1,j}\hat{\boldsymbol{Q}}_{i+1,j} + \alpha_{i,j}(\boldsymbol{B}^{-}\Delta sj)_{i,j+1}\hat{\boldsymbol{Q}}_{i,j+1} = \hat{\boldsymbol{Q}}^{*}_{i,j} \tag{63}$$

$$\boldsymbol{S}_{i,j}\hat{\boldsymbol{Q}}_{i,j} = \hat{\boldsymbol{Q}}^{*}_{i,j} - \alpha_{i,j}((\boldsymbol{A}^{-}\Delta si)_{i+1,j}\hat{\boldsymbol{Q}}_{i+1,j} + (\boldsymbol{B}^{-}\Delta sj)_{i,j+1}\hat{\boldsymbol{Q}}_{i,j+1}) \tag{64}$$

$$\hat{\boldsymbol{Q}}_{i,j} = \boldsymbol{S}^{-1}_{i,j}(\hat{\boldsymbol{Q}}^{*}_{i,j} - \alpha_{i,j}((\boldsymbol{A}^{-}\Delta si)_{i+1,j}\hat{\boldsymbol{Q}}_{i+1,j} + (\boldsymbol{B}^{-}\Delta sj)_{i,j+1}\hat{\boldsymbol{Q}}_{i,j+1})) \tag{65}$$

In this equation, you want $\boldsymbol{S}, \boldsymbol{S}^{-1}$. They are

$$\begin{aligned}
\boldsymbol{S}_{i,j} &= \boldsymbol{I} + \beta_{i,j}\boldsymbol{\Gamma}^{-1} \\
&= \boldsymbol{I} + \beta_{i,j}\left(\boldsymbol{I} - \frac{1}{\frac{1}{\Phi} + c^2}\boldsymbol{\phi}\frac{\partial p}{\partial \boldsymbol{Q}}\right) \\
&= (1+\beta_{i,j})\boldsymbol{I} - \frac{\beta_{i,j}}{\frac{1}{\Phi} + c^2}\boldsymbol{\phi}\frac{\partial p}{\partial \boldsymbol{Q}} \\
&= (1+\beta_{i,j})\left(\boldsymbol{I} - \frac{1}{(\frac{1}{\beta_{i,j}} + 1)(\frac{1}{\Phi} + c^2)}\boldsymbol{\phi}\frac{\partial p}{\partial \boldsymbol{Q}}\right)
\end{aligned} \tag{66}$$

$$\begin{aligned}
\boldsymbol{S}^{-1}_{i,j} &= \frac{1}{1+\beta_{i,j}}\left(\boldsymbol{I} - \frac{1}{1/(-\frac{\beta_{i,j}}{(1+\beta_{i,j})(\frac{1}{\Phi}+c^2)}) + c^2}\boldsymbol{\phi}\frac{\partial p}{\partial \boldsymbol{Q}}\right) \\
&= \frac{1}{1+\beta_{i,j}}\left(\boldsymbol{I} + \frac{\beta_{i,j}}{\frac{1+\beta_{i,j}}{\Phi} + c^2}\boldsymbol{\phi}\frac{\partial p}{\partial \boldsymbol{Q}}\right)
\end{aligned} \tag{67}$$