When can a Bayesian divide and conquer?
General optimality guarantees for distributed nonparametric Bayesian inference

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This talk is based on joint works with Botond Szabó and Harry van Zanten.

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  has computational complexity $O(n)$. This makes many optimization or MCMC based methods infeasible when n is very large.
- Popular methods like Gaussian process regression typically have computational complexity $O(n^3)$ and memory complexity $O(n^2)$ even in conjugate models.

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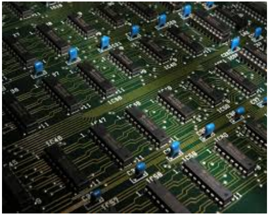

Divide and conquer - formally

Formally:

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The distribution of the average of the draws of the local posteriors give us a generalized "posterior": the m-times convolution of the local posteriors rescaled with m:

$$\tilde{\Pi}(A|X^{(1:m)}) := \mathop{\circledast}_{j=1}^{m} \Pi(m \cdot |X^{(j)})(A).$$

We perform inference on θ using $\tilde{\Pi}(\cdot|X^{(1:m)})$.

Divide and conquer - nomenclature

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- *"Consensus Monte Carlo"* in Steven L Scott et al.(2016), "Bayes and big data: The consensus Monte Carlo algorithm"

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$$O(n^3) \text{ becomes } \begin{cases} O\left(\frac{n^3}{m^3}\right) & \text{if ran in parallel,} \\ O\left(\frac{n^3}{m^2}\right) & \text{otherwise.} \end{cases}$$

  So we can even obtain linear or constant complexity in n if $m \simeq n$.
- It is super simple to implement (even parallely): no need to rewrite code for existing Bayesian methods (it is “plug-and-play”).

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Short answer: *(almost) always, but it is not always good. Inference might become highly suboptimal.*

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In this work, we provide general sufficiency conditions that guarantee optimal frequentist convergence rates for the divide and conquer strategy.

- We demonstrate that the guarantees hold for a variety of nonparametric models and priors.

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In this work, we provide general sufficiency conditions that guarantee optimal frequentist convergence rates for the divide and conquer strategy.

- We demonstrate that the guarantees hold for a variety of nonparametric models and priors.
- Similar guarantees are provided for methods that are based on general loss functions instead of likelihoods for G-Bayes and M-estimation.

Setting and notation

In this talk, I will assume iid $X_i \sim P_{\theta^\circ}$ for some $\theta^\circ \in \Theta$ and that the model $\{P_\theta : \theta \in \Theta\}$ is dominated.

As shorthand notation, let

$$\ell(\theta_1, \theta_2; X_i) := \log\left(\frac{p_{\theta_1}}{p_{\theta_2}}(X_i)\right)$$

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Recall:

$$\bar{\theta} \sim \tilde{\Pi}(\cdot | X^{(1:m)}) := \circledast_{j=1}^{m} \Pi(m \cdot | X^{(j)}).$$

Divide and conquer convergence rate

Definition: The posterior distribution $\Pi(\cdot|X^{(n)})$ is said to *contract at rate* $\varepsilon_n \to 0$ in d at θ° if

$$\Pi\left(\theta \in \Theta : d(\theta, \theta^\circ) \leq M_n \varepsilon_n | X^{(n)}\right) \to 1 \text{ in } P_{\theta^\circ}\text{-probability},$$

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Suppose the full data posterior contracts at rate ε_n, does

$$\tilde{\Pi}\left(\bar{\theta} : d(\bar{\theta}, \theta^\circ) \leq M_n \varepsilon_n \middle| X^{(1:m)}\right) \overset{P_{\theta^\circ}}{\to} 1$$

also for all $M_n \to \infty$?

Divide and conquer convergence rate Theorem

Theorem ((Szabó, V., van Zanten))

*Let ε_n be a sequence of positive numbers for which the **likelihood geometry condition** and **testing condition** hold. Assume furthermore that $\Pi^{(j)}$'s are such that the **prior mass** and **entropy conditions** are satisfied.*

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Then, ε_n is a contraction rate for the generalized posterior. That is, it holds that

$$\tilde{\Pi}\left(\bar{\theta} : d(\bar{\theta}, \theta^\circ) \leq M_n \varepsilon_n \middle| X^{(1:m)}\right) \stackrel{P_{\theta^\circ}}{\to} 1 \tag{1}$$

for any $M_n \to \infty$.

Likelihood geometry condition

The likelihood geometry condition is satisfied if

$$\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}_{\theta^\circ}\ell\left(\theta^{(j)},\theta^\circ;X_i^{(j)}\right) \leq c_\ell \mathbb{E}_{\theta^\circ}\ell\left(m^{-1}\sum_{j=1}^{m}\theta^{(j)},\theta^\circ;X_i^{(j)}\right) \tag{2}$$

whenever $-\mathbb{E}_{\theta^\circ}\ell\left(\bar{\theta},\theta^\circ;X_i^{(j)}\right) \leq C_\ell \varepsilon_n{}^2$ for some constants $c_\ell, C_\ell > 0$.

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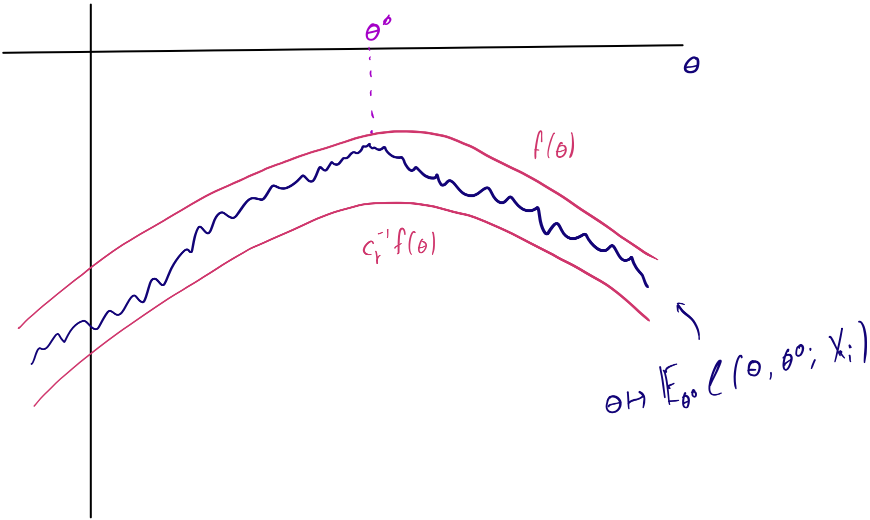
Likelihood geometry condition – a cartoon



Divide and conquer Ghoshal & van der Vaart conditions

The following **prior mass conditions** and **entropy** and **testing conditions** respectively:

$$\bigotimes_{j=1}^{m} \Pi^{(j)} \left(\theta^{(1:m)} \in \Theta^m : -\mathbb{E}_{\theta^\circ} \ell \left(\bar{\theta}, \theta^\circ ; X_i^{(j)} \right) \leq {\varepsilon_n}^2 \right) \geq e^{-C_{KL} n {\varepsilon_n}^2}, \tag{3}$$

and that there exists a $C_0 > 0$ such that for all $C \geq C_0$ there exists $\mathscr{P}_n \subset \mathscr{P}$ such that

$$\bigotimes_{j=1}^{m} \Pi^{(j)} (\theta^{(1:m)} \in \Theta^m \, : \, \bar{\theta} \in {\mathscr{P}_n}^c) \leq e^{-C n {\varepsilon_n}^2} \tag{4}$$

$$\log N(\varepsilon_n, d, \mathscr{P}_n) \leq C n {\varepsilon_n}^2. \tag{5}$$

Furthermore, writing $\bar{\theta} = m^{-1} \sum_{j=1}^{m} \theta^{(j)}$, assume that there exists a test T such that

$$P_{\theta^\circ} T \overset{n \to \infty}{\to} 0 \quad \text{and} \quad \bigotimes_{j=1}^{m} P_{\theta^{(j)}} (1 - T) \leq e^{-C_{test} M_n n {\varepsilon_n}^2} \tag{6}$$

whenever $\bar{\theta} \in \mathscr{P}_n$ and $d(\bar{\theta}, \theta^\circ) \geq M_n \varepsilon_n$.

Divide and conquer - Gaussian processes

Let $\Theta = (\Theta, \|\cdot\|)$ be a separable Banach space and let W be a centered Gaussian process on it.
Suppose that $-\mathbb{E}_{\theta}\ell(\theta', \theta; X_i) \leq \|\theta - \theta'\|^2$ and that W's *concentration function* satisfies[1]

$$\phi_{\theta^\circ}^{W}(\varepsilon_n) = \inf_{h \in \mathbb{H}: \|\theta^\circ - h\| < \varepsilon_n} \|h\|_{\mathbb{H}}^2 - \log \Pr(\|W\| \leq \varepsilon_n) \lesssim n\varepsilon_n{}^2.$$

[1] If the full data posterior corresponding to W is to contract at rate ε_n, this should be the case; see e.g. Ismaël Castillo(2008), "Lower bounds for posterior rates with Gaussian process priors" and A. W. van der Vaart and J. H. van Zanten(June 2008), "Rates of contraction of posterior distributions based on Gaussian process priors".

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The prior blow-up of $\sqrt{m}$ turns out to be strictly necessary. We have a theorem that stating that taking $\Pi^{(j)}$ equal to the distribution of W results in a convergence rate that corresponds to throwing away $\frac{n(m-1)}{m}$ observations in typical settings.

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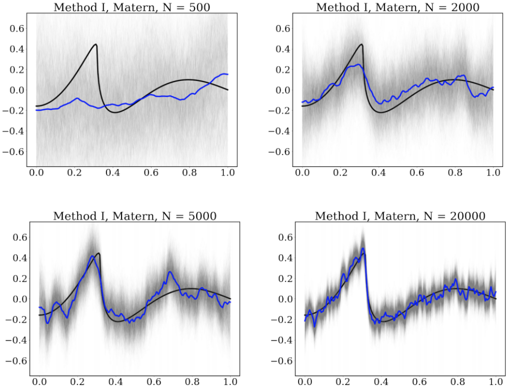
W centered Gaussian with squared exponential kernel. Local prior is with $\sqrt{m}$ blow-up.



Picture is courtesy of Amine Hadji, Tammo Hesslink, and Botond Szabó(2022), "Optimal recovery and uncertainty quantification for distributed Gaussian process regression".

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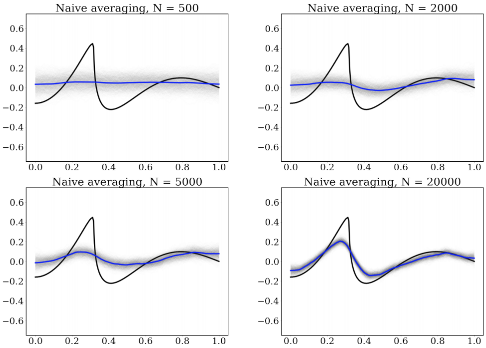

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Divide and conquer convergence rate Theorem

Theorem ((Szabó, V., van Zanten))

Let ε_n be a sequence of nonnegative numbers and let W be a Gaussian process such that $\phi^W_{\theta^\circ}(\varepsilon_n) \lesssim n{\varepsilon_n}^2$ holds.

For $j = 1, \ldots, m$, let the local priors $\Pi^{(j)}$ be equal to the distribution of $\sqrt{m}W$. Then,

$$\tilde{\Pi}(\bar{\theta} : \|\bar{\theta} - \theta^\circ\| \leq M_n\varepsilon_n | X^{(1:m)}) \overset{P_{\theta^\circ}}{\to} 1 \tag{7}$$

as long as the ***testing condition*** *and* ***likelihood geometry condition*** *are satisfied.*

Examples: for which models does all of this hold?

- **Density estimation**: Suppose we observe iid X_i's in $[0,1]^d$ from some Hölder smooth density p°. Take as local priors

$$p_{\sqrt{m}W}(X_i) = \frac{e^{\sqrt{m}W(X_i)}}{\int e^{\sqrt{m}W(x)} d\mu(x)}$$

  where W is a zero mean Gaussian process in $\ell_\infty([0,1]^d)$ satisfying (**??**) with ε_n the optimal contraction rate and suppose that p° lies in the support of W. Then, the generalized posterior contracts at rate ε_n also.

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  with $Z_i \sim N(0,\sigma^2)$ and θ belonging to e.g. a Sobolev space.
- **Classification:** Consider iid observations $(Y_1, X_1), \ldots, (Y_n, X_n)$ satisfying

$$\Psi(\theta(x)) = \Pr(Y_i = 1 | X_i = x),$$

  conditionally on P_θ for some link function Ψ and some other assumptions, a divide and conquer approach (using e.g. Gaussian process priors) attains the optimal contraction rate for θ°.

Thank you for listening!

References:

- Botond Szabó and Harry Van Zanten(2019), "An asymptotic analysis of distributed nonparametric methods."
- Amine Hadji, Tammo Hesslink, and Botond Szabó(2022), "Optimal recovery and uncertainty quantification for distributed Gaussian process regression"