Distributions for parameters

Nancy Reid

University of Toronto

Background

The First Workshop on BFF Inference and Statistical Foundations
(BFF 2014)

November 10 – November 14, 2014

Program

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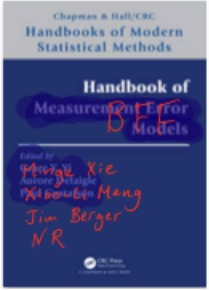

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- Abstract: "Among patients with septic shock, a resuscitation strategy targeting normalization of capillary refill time, compared with a strategy targeting serum lactate levels, did not reduce all-cause 28-day mortality."

Spiegelhalter, 2019

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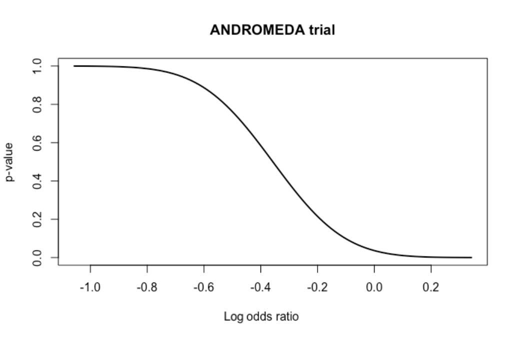

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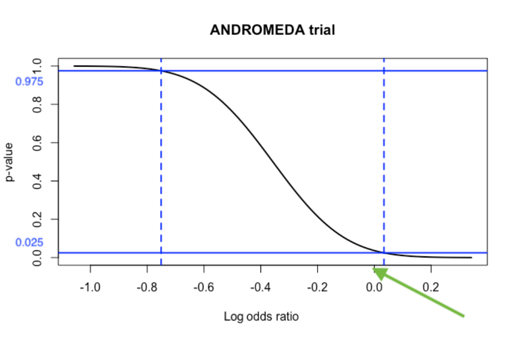

90% confidence interval: $[-0.688, -0.030]$

95% confidence interval: $[-0.751, \quad 0.034]$

99% confidence interval: $[-0.825, \quad 0.107]$

Confidence distributions

SOME PROBLEMS CONNECTED WITH STATISTICAL INFERENCE

By D. R. Cox

Birkbeck College, University of London[1]

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- "... in ... simple cases ... there seems no reason why we should not work with confidence distributions for the unknown parameter
- "These can either be defined directly, or ... introduced in terms of the set of all confidence intervals"

Biometrika (1993), **80**, 1, *pp.* 3–26
Printed in Great Britain

Bayes and likelihood calculations from confidence intervals

BY BRADLEY EFRON
Department of Statistics, Stanford University, Stanford, California 94305-4065, U.S.A.

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- "… no nuisance parameters [this] is exactly Fisher's fiducial distribution"

Seidenfeld 1992; Zabell 1992

Fiducial distribution

Inverse Probability. By R. A. FISHER, Sc.D., F.R.S., Gonville and Caius College; Statistical Dept., Rothamsted Experimental Station.

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"It is not to be lightly supposed that men of the mental calibre of Laplace and Gauss … could fall into error on a question of prime theoretical importance, without an uncommonly good reason"

… fiducial distribution

- y has density from a location model — $f(\cdot)$ known

$$f(y-\theta)$$

- equivalently — generating model

$$y=\theta+e, \quad e\sim f(\cdot)$$

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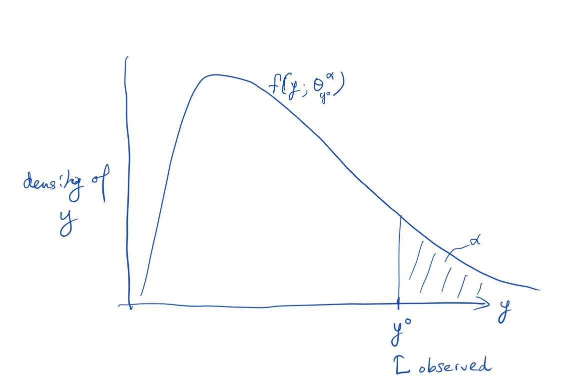
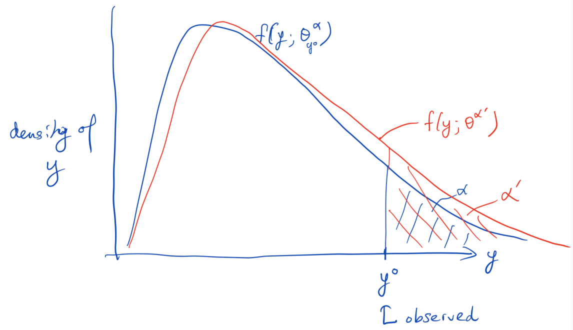
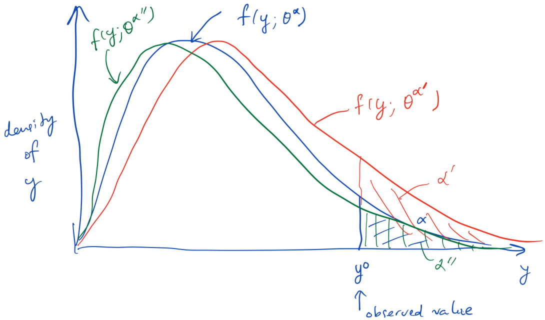
$$\frac{\partial}{\partial y}F(y,\theta)+\frac{\partial}{\partial\theta}F(y,\theta)=0 \qquad df=-\frac{\partial}{\partial\theta}F(y,\theta)d\theta$$


f(y; θ^α_{y°})
density of y
α
y°
observed
y

... fiducial/confidence distribution





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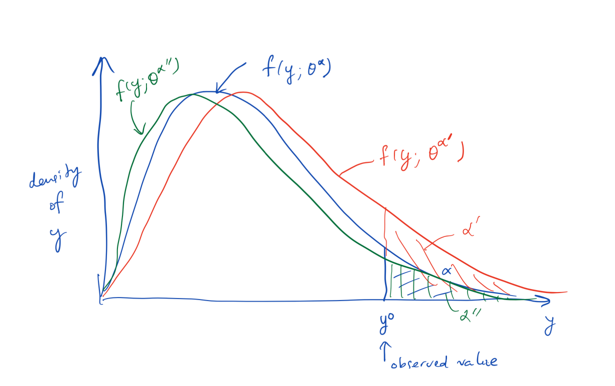

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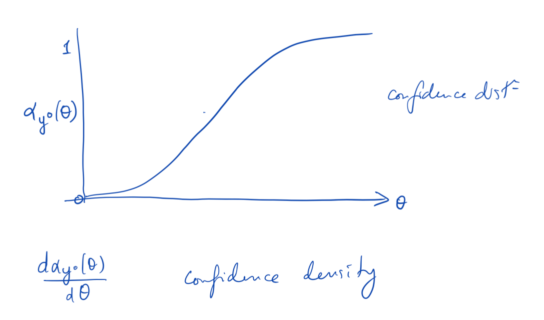

$$\alpha_{y^o}(\theta) : \theta \longrightarrow \alpha$$

inverse function

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Don’t adapt easily to multi-parameter models

Bayesian posteriors

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- consider statistical model as conditional model for data, given parameter

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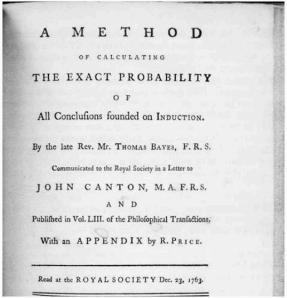

A METHOD

OF CALCULATING

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All Concluſions founded on INDUCTION.

By the late Rev. Mr. THOMAS BAYES, F. R. S.

Communicated to the Royal Society in a Letter to

JOHN CANTON, M. A. F. R. S.

AND

Publiſhed in Vol. LIII. of the Philoſophical Tranſactions,

With an APPENDIX by R. PRICE.

Read at the ROYAL SOCIETY Dec. 23, 1763.

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- added random effect for center, used default priors for covariates, change to logistic regression

ANDROMEDA, revisited

- under a range of normal priors for the log odds-ratio
- the posterior probability that the odds-ratio is less than 1

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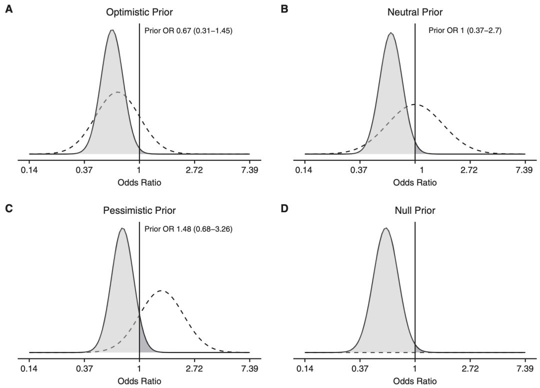

Figure 1. (A–D) Prior distributions for the odds ratio (OR) of the intervention (dashed lines). Posterior distributions of the ORs are shown by the solid lines. The light gray areas indicate the areas associated with benefit for peripheral perfusion–targeted resuscitation (i.e., OR < 1) and the dark gray areas the areas associated with harm (i.e., OR > 1). The text inside each frame reports the median and lower and upper 95% credible limits for the priors of the effect of the intervention for 28-day mortality.

see also van Zwet et al. 2021
used empirical prior
posterior prob 0.91

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Cox 2006

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- all inference statements ~~are~~ **seem to be** probability statements about unknowns

Nature of Probability

- probability to describe physical haphazard variability

Nature of Probability

Cox 2006; R & Cox 2015

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- probability to describe the uncertainty of knowledge — epistemic
  - measures rational, supposedly impersonal, degree of belief, given relevant information — Jeffreys
  - measures a particular person's degree of belief, subject typically to some constraints of self-consistency — Ramsey, de Finetti, Savage

BFF Handbook

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Berger, Efron

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high-dimensional inference, model selection, computation

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- as with other measuring devices, within this scheme of repetition, probability is defined as a hypothetical frequency
- it is unacceptable if a procedure yielding high-probability regions in some non-frequency sense are poorly calibrated

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- such procedures, used repeatedly, give misleading conclusions

Bayesian Analysis, V1(3) 2006

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- only in very special cases can calibration be achieved for more than one parameter in the model, from the same prior

- the simplicity of a fully Bayesian approach to inference is lost

Gelman 2008; PMM LW

… objective Bayes

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- Stein's example: — sequence model

$$\begin{aligned} y_i &\sim N(\theta_i, 1/n), \quad i = 1, \ldots, k \\ \pi(\theta_i) &\propto 1 \\ \pi(\theta \mid y) &\propto N(y, I_k/n) \end{aligned}$$

Example

- $y_i \sim N(\theta_i, 1/n), \quad i = 1, \ldots, k; \quad \pi(\theta_i) \propto 1$
- posterior distribution of $a^{\mathrm{T}}\theta$ is well-calibrated
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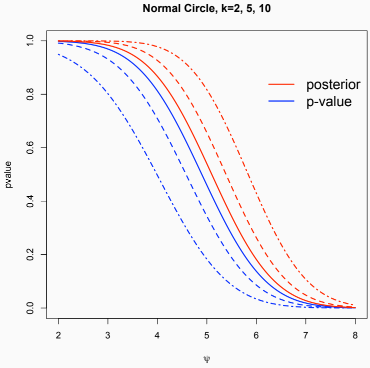

global-local shrinkage priors (horseshoe) shrink the posterior in the right direction

reference and targetted priors do the same

- calibrated posterior distributions must be targetted on the parameter of interest
- matching priors set out this requirement explicitly

defined by calibration of posterior quantiles

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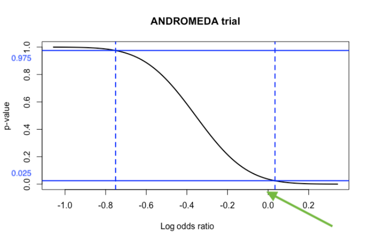
- ANDROMEDA trial used default priors on regression coefficients, except odds-ratio
  and a selection of log-normal priors on odds-ratio

Targetting on parameter

Example 1: 2×2 table

Based on conditional distribution of odds-ratio, given marginal totals



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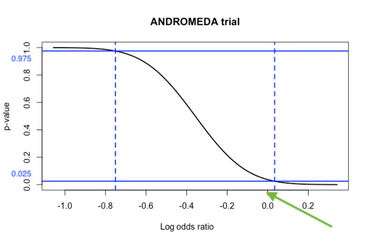
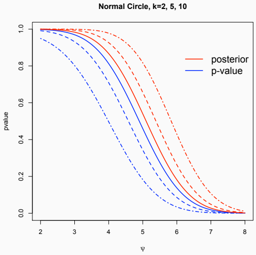


Example 2: $y_i \sim N(\theta_i, 1/n)$

Based on marginal distribution of Σy_j^2



BFF Handbook

- posterior distribution 1763
- fiducial probability 1930
- confidence distribution 1958
- structural probability 1964
- significance function 1991
- belief functions 1967

- objective Bayes, empirical Bayes
  Berger, Efron
- generalized fiducial inference
  Hannig et al., Lang, Vovk, Taraldsen & Lindqvist
- confidence distributions and confidence curves, data fusion Thornton & Xie, Hector et al.
- approximate significance functions
  Davidson & R
- inferential models Shafer, Gong, Liu & Martin

- any function $H : \mathcal{Y} \times \Theta \to (0, 1)$ which is
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- “as long as one can create confidence regions of all levels”

- avoids the problem of targetting by assuming a method for this has already been developed often likelihood ratio statistic or deviance

Generalized Fiducial

Location model:

- data-generating equation $Y_i = \theta + e_i$; with fiducial inversion $e_i = y_i^o - \theta$
- $e_i - \bar{e} = y_i^o - \bar{y}^o, \quad i = 1, \ldots, n$ is known once the data is available
- relevant distribution is conditional on $e_i - \bar{e}$
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- $y = G(u, \theta), \quad u$ has known distribution — $y_1, \ldots, y_n$
- given observed data y^o find the inverse $\theta = Q_{y^o}(U)$ — Efron 1993
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- inverse only exists if θ and Y have same dimension
- Hannig introduces an auxiliary function to adjust for dimension reduction

$$r(\theta; y^o) \propto f(y^o, \theta) J(y^o, \theta)$$

- $$r(\theta; y^o) = \frac{f(y^o, \theta) J(y^o, \theta)}{\int f(y^o, \theta) J(y^o, \theta) d\theta}$$

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- $$J(y^o, \theta) = D \left\{ \frac{\partial}{\partial \theta} G(u, \theta) \right\} \Big|_{u^o}$$

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$$J(y^o,\theta) = |\frac{dy}{d\theta}|$$

- well adapted to simulation, and applied in many interesting and complex models
- no guarantees on calibration, beyond first-order; needs to be checked in each case

Significance function

- focus on parameter of interest

many arguments point to the need for this

- $y \in \mathbb{R}^n, \quad \theta \in \mathbb{R}^p, \quad \psi \in \mathbb{R}$

- specifies a particular path to targetting, via two-stage dimension reduction

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- $n \downarrow p$ using an approximate location model

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- $p \downarrow 1$ using an approximate exponential model

- asymptotic theory applied to this exponential model gives a ?unique? pivotal quantity for inference about parameter of interest ψ
- which is a modification of the likelihood ratio statistic r^*_ψ

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- e.g. confidence distribution approach asks user to identify, from the model, a quantity that measures the parameter of interest focus parameter, Hjort & Schweder, 2016

Nature of Probability?

- Bayes / objective Bayes — epistemic / empirical
- generalized fiducial inference — empirical
- confidence distributions and confidence curves — epistemic / empirical
- approximate significance functions — empirical

ANDROMEDA

Zampieri et al 2020

Table 1. Odds Ratio, 95% Credible Interval, Probability That the Odds Ratio Is below Given Thresholds, and Absolute Difference between Groups

	28-d Outcome			90-d Outcome			
Prior	**OR (95% Credible Interval)**	**Probability OR < 1 (Probability OR < 0.8)**	**Absolute Difference (95% Credible Interval)***	**OR (95% Credible Interval)**	**Probability OR < 1 (Probability OR < 0.8)**	**Absolute Difference (95% Credible Interval)***	**Reason for Prior Use**
Optimistic	0.61 (0.41 to 0.90)	99% (92%)	−9% (−17% to −1%)	0.69 (0.47 to 1.01)	97% (79%)	−7% (−16% to 2%)	Considers an OR of 0.67 for the intervention (slightly more conservative than the effect size ANDROMEDA-SHOCK was powered to detect), while considering that there is still a 15% probability that the intervention was harmful
Neutral	0.65 (0.43 to 0.96)	98% (85%)	−7% (−16% to 1%)	0.74 (0.50 to 1.08)	94% (66%)	−5% (−14% to 4%)	Has a mean OR of 1 (i.e., absence of effect) and 50% probability of benefit and 50% of harm from the intervention
Pessimistic	0.74 (0.50 to 1.09)	94% (66%)	−5% (−13% to 3%)	0.83 (0.57 to 1.21)	83% (42%)	−3% (−11% to 6%)	Opposite values of the optimistic prior; considers a very pessimistic scenario in which the intervention is harmful but still acknowledges a 15% chance that the intervention might be beneficial
Null	0.59 (0.38 to 0.92)	98% (91%)	−8% (−17% to 1%)	0.69 (0.45 to 1.07)	95% (74%)	−6% (−15% to 4%)	No prior information is considered

Definition of abbreviation: OR = odds ratio.
*Refers to a simple model adjusted only for study arm and not for all predictors.

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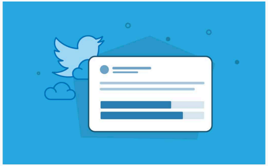

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- several current versions of fiducial inference: confidence, significance, generalized fiducial
- “making a Bayesian omelette without cracking the Bayesian eggs”

Needs in applications

- something that works
- gives ‘sensible’ answers
- not too sensitive to model assumptions
- computable in reasonable time
- provides interpretable parameters

Role of Foundations

- avoid apparent discoveries based on spurious patterns
- shed light on the structure of the problem
- obtain calibrated inferences about interpretable parameters
- provide a realistic assessment of precision
- understand when/why methods work/fail

Thank you!


PLACE BONAVENTURE


THE 2022 ISBA
WORLD MEETING
June 26th - July 1st, Hotel Bonaventure, Montreal, Canada