Bayesian Nonparametric inference for Nonlinear Hawkes processes

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Bayesian nonparametrics for nonlinear Hawkes

Joint work with

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- Vincent Rivoirard
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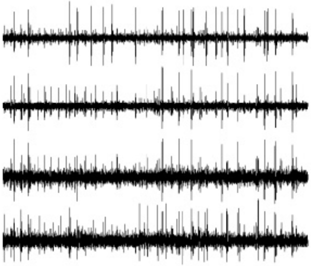
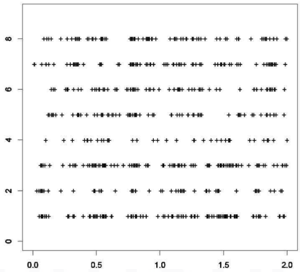
Presentation Outline

Neuronal data modelling

- Spike = electric impulse emitted by a neuron
- "Event-wise" data = times of occurrences at each location (neuron)
- *Excitation* (clustering behaviour) and *inhibition* ("cancellation")





Which neurons are **functionally connected**?
What is the **type** and **strength** of their interactions?

Point and Intensity processes

Definition 1.1 (Temporal point process)

$N = (N_t^1, \dots N_t^K)_{t>0}$ is a K-dimensional TPP if N_t^k counts the number of points until t at component k.

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Definition 1.2 (Intensity process)

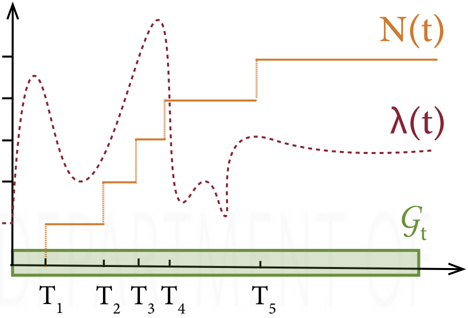
The intensity process $\lambda(t|\mathcal{G}_t) = (\lambda^1(t), \dots, \lambda^K(t))$ is the probability of observing a point at time t conditionally on the past of the process:

$$\lambda^k(t|\mathcal{G}_t)\mathrm{d}t = \mathbb{P}\left[N^k \text{ jumps in } [t, t+\mathrm{d}t] \,\middle|\, N_s, s < t \right], \quad 1 \leqslant k \leqslant K$$

Point and intensity processes

Examples:

- Poisson point process: $\lambda(t)$ is deterministic (independent of $\mathcal{G}_t$)
- **Hawkes process:** $\lambda(t)$ is a stochastic process ($\mathcal{G}_t$-predictable)



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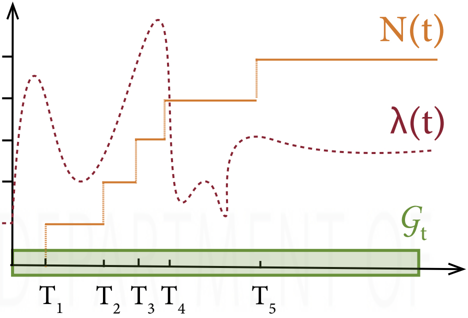

Applications: neuroscience Gerhard et al. (2017), genomics Gusto & Schbat (2005), finance Bacry & Muzy (2013), criminology Mohler (2013), epidemiology Browning et al. (2021), tweet popularity Zhao et al. 2015 ...

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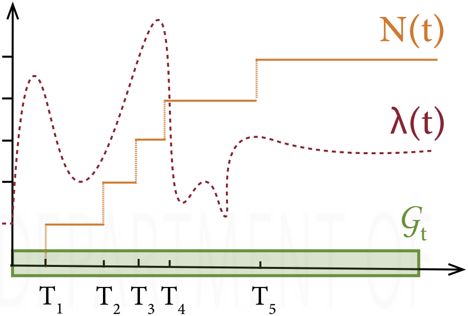

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Temporal dependencies and interactions: **excitation / contagion / diffusion, clustering / cascades, branching / causality**, **inhibition**, time decay, memory loss, nonlinear effects, state-switching, ...

Linear univariate Hawkes processes

Definition 2.1 (Hawkes, 1971)

Let $\nu > 0$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $\|h\|_1 < 1$. If

$$\lambda(t) = \nu + \int_{-\infty}^{t^-} h(t-u)\mathrm{d}N_u = \nu + \sum_{T_i \in N,\, T_i < t} h(t - T_i),$$

N is a linear univariate Hawkes process with **spontaneous rate** ν and **self-exciting function** h.

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- $Im(h) \subset \mathbb{R}_+ \implies$ **(self) excitation:** events cause events
- **Branching (and causal) representation:** an event can
  - appear spontaneously at rate ν
  - be caused by a previous event at the rate $h(x)$

Multivariate nonlinear Hawkes process

Definition 2.2 (Generalized Hawkes process)

A K-dimensional continuous point process N is a Hawkes process if

- almost surely N^k and N^l never jump simultaneously
- N^k has intensity

$$\lambda^k(t) = \phi_k \left(\nu_k + \sum_{l=1}^{K} \sum_{T_i \in N^l, T_i < t} h_{lk}(t - T_i) \right) \tag{1}$$

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- $h_{lk} : \mathbb{R}^+ \to \mathbb{R}$: **interaction function $N^l \Rightarrow N^k$**
  - for x s.t. $h_{lk}(x) > 0$: **excitation**
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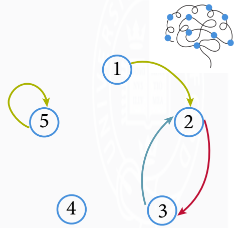
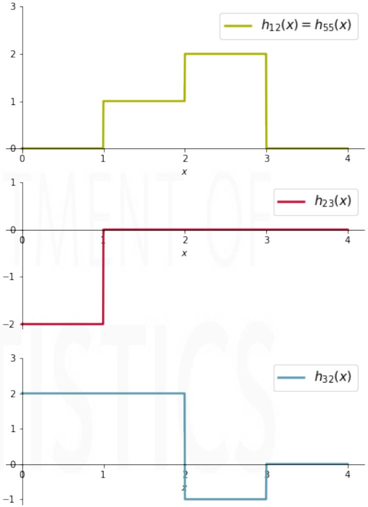
- Link functions:
  - ReLU: $\phi_k(x) = (x)_+$
  - Sigmoid: $\phi_k(x) = (1 + e^{-x})^{-1}$
  - Softplus: $\phi_k(x) = \log(1 + e^x)$
  - ...
- Existence and uniqueness of a stationary distribution and

Example with 5 neurons





- **Interaction functions:**
  - Excitating $h_{12} = h_{55}$
  - Inhibiting h_{23}
  - Mixed h_{32}
- **Connectivity graph:**
  $\Delta = (\delta_{lk})_{l,k} \in \{0,1\}^{K \times K}$,
  $\delta_{lk} = 0 \iff h_{lk} = 0$

Estimation methods

Linear parametric model $h(t) = \alpha\beta e^{-\beta t} \implies f = (\nu, \alpha, \beta)$:

- EM: Veen & Schoenberg (2008)
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Nonlinear model

- Linear approximation over a dictionary Cai et al. (2021)
- MCMC for ReLU link with $h_{lk}(t) = K_{lk} e^{-\beta_{lk} t}$ Deutsch et al. (2022)
- Gibbs sampler in the sigmoidal Hawkes with GP prior Malem-Shinitski et al. (2022) or (fixed) basis decomposition Zhou et al. (2021)

Bayesian inference problem

- Assume that we observe a Hawkes process N on $[0, T]$ with link functions $(\phi_k^0)_k$ and parameter $f_0 = (\nu_0, h_0)$ with intensity

$$\lambda^k(t; f_0, \phi_0) = \phi_k^0 \left(\nu_k^0 + \sum_{l=1}^{K} \sum_{T_i \in N^l, T_i < t} h_{lk}^0(t - T_i) \right)$$

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- Given the likelihood function at $f = (\nu, h)$ and ϕ,

$$L_T(N; f, \phi) = \exp\left\{ \sum_{k=1}^{K} \left[\sum_{i=1}^{n_k} \log(\lambda^k(T_i^k; f, \phi)) - \int_0^T \lambda^k(t; f, \phi) dt \right] \right\}$$

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and a prior distribution Π on f (e.g., Gaussian processes, mixtures, splines,..), the posterior distribution is given by

$$\Pi(B|N) = \frac{\int_B L_T(N; f) d\Pi(f)}{\int_{\mathcal{F}} L_T(N; f) d\Pi(f)}, \quad B \subset \mathcal{F}$$

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  If there exists $\varepsilon > 0$ such that $\forall k$, ϕ_k injective on

$$I_k = (\nu_k - \max_{1 \leqslant l \leqslant K} \left\|h^-_{lk}\right\|_\infty - \varepsilon, \nu_k + \max_{1 \leqslant l \leqslant K} \left\|h^+_{lk}\right\|_\infty + \varepsilon)$$

Our results: posterior asymptotic properties

- Does the posterior $\Pi(.|N)$ **concentrates** around the truth f_0 when $T \to \infty$? i.e., with $\epsilon_T = o(1)$ and $d = L_1$-distance,

$$\mathbb{E}_{f_0}\left[\Pi\left(d(f, f_0) < \epsilon_T \middle| N\right)\right] \xrightarrow[T\to\infty]{} 1.$$

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Yes if

- Model assumptions:
  - ϕ_k^{-1} on $J_k = \phi_k(I_k)$ Lipschitz
  - $\inf_x \phi_k(x) > 0$ **or** $\sqrt{\phi_k}$ and $\log \phi_k$ L-Lispchitz
- Prior assumptions: standard ones for regression or density estimation (Ghosal & van der Vaart 2007) (prior mass, sieve & entropy conditions)

Our results: renewal and choice of excursions

- **Finite-memory** process: $\forall l, k,\ supp(h_{lk}) \subset [0, A]$ with $A > 0$

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- **Regeneration** times: $\tau_0 = 0, \tau_1, \tau_2, \dots$ defined as

$$\tau_j = \inf \left\{ t > \tau_{j-1};\ N|_{[t-A,t)} \neq \emptyset,\ N|_{(t-A,t]} = \emptyset \right\}, \quad j \geqslant 1$$

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$$N|_{[\tau_j,\tau_{j+1})}, \quad 0 \leqslant j \leqslant J_T - 1$$

- Concentration inequalities for the number of excursions J_T, finite-moments of the number of points per excursions, ...

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  Yes, using **Variational inference**: with $\mathcal{V}$ a mean-field variational family of distributions on $\mathcal{F}$, i.e.,

$$\mathcal{V} = \left\{ Q : dQ(f) = \prod_k dQ_{k1}(\nu_k) \prod_l dQ_{k2}(h_{lk}) \right\},$$

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where Q_{2k} includes the nonparametric prior family (Gaussian processes, basis decomposition, etc). The variational posterior is defined as

$$\hat{Q} = \arg\min_{Q \in \mathcal{V}} KL(Q || \Pi(.|N))$$

and under similar conditions $\mathbb{E}_0 \left[\hat{Q}(d(f, f_0) \lesssim \epsilon_T) \right] \xrightarrow[T\to\infty]{} 1.$

Our work in progress

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    - With data augmentation, Gibbs sampler and CAVI algorithm in the sigmoid model Zhou et al. (2021)
    - HMC in semi-parametric and low dimensional model ($K < 8$)
    - Stochastic Variational Inference?

Discussion

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- Computational bottleneck for point processes estimation

Frequentists $1 - 0$ Bayesians

  - Development of approximate and empirical Bayes methods
  - Parallelisation

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  Parametric Frequentists $2 - 0$ Nonparametric Bayesians

  - Development of approximate and empirical Bayes methods
  - Parallelisation
- Theory for
  - High-dimensional model $K \to \infty$
  - Time-varying models: time-dependent background rate X. Miscouridou (work in progress), change-points R. Browning (2021), Hidden Markov model Zhou et al. (2021)
  - Semi-parametric inference?

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Thank you for your attention!

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