Sparse Linear Mixed Model Selection via Streamlined Variational Bayes

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joint work with

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Outlook

Introduction

- linear mixed models in general form
- two- and three-level models
- variational approximations

Efficient sparse linear mixed model selection

- streamlined variational inference
- global-local shrinkage priors

Simulated and real data experiments

- simulated data involving three-level models
- Perinatal data (two-level model)

Motivation

- A variety of applications (e.g. longitudinal or grouped data) can be studied through linear mixed-effects models having fixed and random effects in the linear predictor
- In some cases, datasets include a large number of predictors, but only a few are effectively relevant
- These predictors are often introduced as fixed effects in the model to guarantee parsimony and validity of the inferential conclusions, especially in sparse model settings
- A proper fixed effects selection procedure is recommended to identify relevant predictors
- Other approaches are based on
  - random effect selection induced by their covariance matrix decomposition (Chen and Dunson, 2003; Yang, 2013),
  - joint fixed and random effects selection (Kinney and Dunson, 2007; Yang *et al.*, 2020)

General Model Formulation

We study linear mixed models having the general form

$$\boldsymbol{y}|\boldsymbol{\beta}, \boldsymbol{u}, \sigma^2 \sim N(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{u}, \sigma^2\boldsymbol{I}), \quad \boldsymbol{u}|\boldsymbol{G} \sim N(\boldsymbol{0}, \boldsymbol{G}), \quad \boldsymbol{\beta} \sim N(\boldsymbol{\mu}_{\boldsymbol{\beta}}, \boldsymbol{\Sigma}_{\boldsymbol{\beta}}),$$
$$\sigma^2|a_{\sigma^2} \sim \text{Inverse-}\chi^2(\nu_{\sigma^2}, 1/a_{\sigma^2}), \quad a_{\sigma^2} \sim \text{Inverse-}\chi^2(1, 1/(\nu_{\sigma^2}s^2_{\sigma^2})),$$
$$\boldsymbol{G}|\boldsymbol{A}_{\boldsymbol{G}} \sim p(\boldsymbol{G}|\boldsymbol{A}_{\boldsymbol{G}}), \quad \boldsymbol{A}_{\boldsymbol{G}} \sim p(\boldsymbol{A}_{\boldsymbol{G}}),$$

where the auxiliary variable a_{σ^2} and matrix $\boldsymbol{A}_{\boldsymbol{G}}$ facilitate the implementation of variational inference (Maestrini and Wand, 2021)

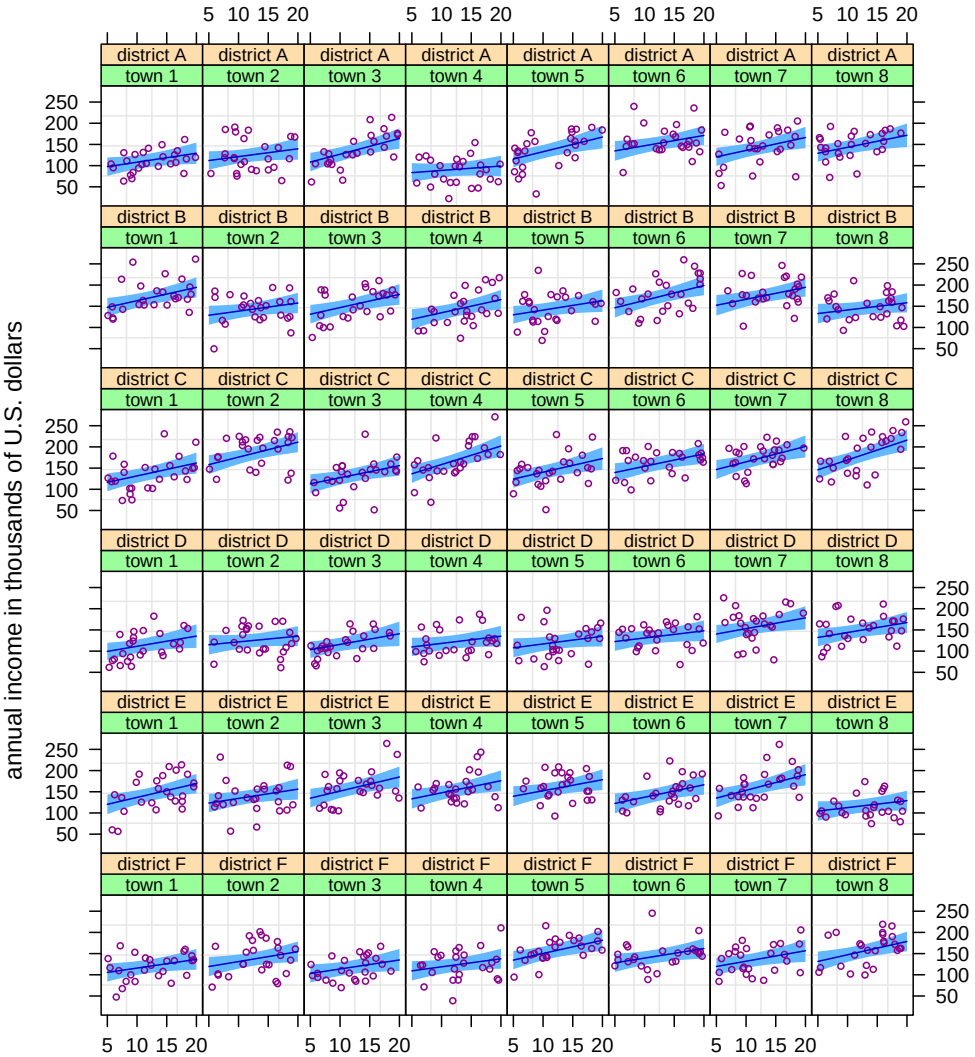
These are Gaussian response models where

- $\boldsymbol{\beta}$ is a vector of fixed effects
- $\boldsymbol{u}$ is a vector of random effects
- $\boldsymbol{X}$ is a fixed effects design matrix
- $\boldsymbol{Z}$ is a random effects design matrix
- $\boldsymbol{G}$ is a random effects covariance matrix

The focus of our work is on two- and three-level linear mixed models

Multilevel Data: Example

Three-level data with 6 districts, each having 8 towns where 25 residents are randomly selected (Nolan *et al.*, 2020)



Two- and Three-Level Models

A Two-level multilevel model for units clustered in m groups:

$$\boldsymbol{y}_i|\boldsymbol{\beta}, \boldsymbol{u}_i, \sigma^2 \overset{\text{ind}}{\sim} \mathrm{N}(\boldsymbol{X}_i\boldsymbol{\beta} + \boldsymbol{Z}_i\boldsymbol{u}_i, \sigma^2\boldsymbol{I}), \quad \boldsymbol{u}_i|\boldsymbol{\Sigma} \overset{\text{ind}}{\sim} \mathrm{N}(\boldsymbol{0}, \boldsymbol{\Sigma}), \quad 1 \leq i \leq m, \quad + \text{priors}$$

A three-level linear mixed model can be defined in terms of observations from the ith group and its jth subgroup as follows:

$$\boldsymbol{y}_{ij}|\boldsymbol{\beta}, \boldsymbol{u}_i^{\mathrm{L1}}, \boldsymbol{u}_{ij}^{\mathrm{L2}}, \sigma^2 \overset{\text{ind}}{\sim} \mathrm{N}(\boldsymbol{X}_{ij}\boldsymbol{\beta} + \boldsymbol{Z}_{ij}^{\mathrm{L1}}\boldsymbol{u}_i^{\mathrm{L1}} + \boldsymbol{Z}_{ij}^{\mathrm{L2}}\boldsymbol{u}_{ij}^{\mathrm{L2}}, \sigma^2\boldsymbol{I}),$$

$$\begin{bmatrix} \boldsymbol{u}_i^{\mathrm{L1}} \\ \boldsymbol{u}_{ij}^{\mathrm{L2}} \end{bmatrix} \Big| \boldsymbol{\Sigma}^{\mathrm{L1}}, \boldsymbol{\Sigma}^{\mathrm{L2}} \overset{\text{ind}}{\sim} \mathrm{N}\left(\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}^{\mathrm{L1}} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Sigma}^{\mathrm{L2}} \end{bmatrix} \right), \quad + \text{priors}$$

In this model

- $\boldsymbol{\Sigma}^{\mathrm{L1}}$ is the covariance matrix for the group-specific random effect vectors $\boldsymbol{u}_i^{\mathrm{L1}}$, $1 \leq i \leq m$
- $\boldsymbol{\Sigma}^{\mathrm{L2}}$ is the covariance matrix for the subgroup-specific random effect vectors $\boldsymbol{u}_{ij}^{\mathrm{L2}}$, $1 \leq i \leq m$, $1 \leq j \leq n_i$

Three-level Random Effects Models

The three-level model is a special case of the general specification

$$\boldsymbol{y}|\boldsymbol{\beta}, \boldsymbol{u}, \sigma^2 \sim N(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{u}, \sigma^2\boldsymbol{I}), \quad \boldsymbol{u}|\boldsymbol{G} \sim N(\boldsymbol{0}, \boldsymbol{G}), \quad + \text{priors}$$

with

$$\boldsymbol{y} = \underset{1\leq i\leq m}{\text{stack}} \left(\underset{1\leq j\leq n_i}{\text{stack}} (\boldsymbol{y}_{ij}) \right), \quad \boldsymbol{X} = \underset{1\leq i\leq m}{\text{stack}} \left(\underset{1\leq j\leq n_i}{\text{stack}} (\boldsymbol{X}_{ij}) \right),$$

$$\boldsymbol{Z} = \underset{1\leq i\leq m}{\text{blockdiag}} \left(\left[\begin{array}{c|c} \underset{1\leq j\leq n_i}{\text{stack}} (\boldsymbol{Z}_{ij}^{\text{L1}}) & \underset{1\leq j\leq n_i}{\text{blockdiag}} (\boldsymbol{Z}_{ij}^{\text{L2}}) \end{array} \right] \right),$$

$$\boldsymbol{u} = \underset{1\leq i\leq m}{\text{stack}} \left(\left[\begin{array}{c|c} (\boldsymbol{u}_{i}^{\text{L1}})^T & \left(\underset{1\leq j\leq n_i}{\text{stack}} (\boldsymbol{u}_{ij}^{\text{L2}}) \right)^T \end{array} \right]^T \right),$$

$$\boldsymbol{G} = \underset{1\leq i\leq m}{\text{blockdiag}} \left(\left[\begin{array}{cc} \boldsymbol{\Sigma}^{\text{L1}} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{n_i} \otimes \boldsymbol{\Sigma}^{\text{L2}} \end{array} \right] \right)$$

Three-level Random Effects Models (3)

If, for example, the data is divided into $m = 3$ groups each having $n_1 = 2$, $n_2 = 3$, $n_3 = 2$ subgroups, then

$$
\boldsymbol{Z} = \begin{bmatrix}
\boldsymbol{Z}_{11}^{\mathrm{L1}} & \boldsymbol{Z}_{11}^{\mathrm{L2}} & & & & & & & & & & \\
\boldsymbol{Z}_{12}^{\mathrm{L1}} & & \boldsymbol{Z}_{12}^{\mathrm{L2}} & & & & & & & & & \\
& & & \boldsymbol{Z}_{21}^{\mathrm{L1}} & \boldsymbol{Z}_{21}^{\mathrm{L2}} & & & & & & & \\
& & & \boldsymbol{Z}_{22}^{\mathrm{L1}} & & \boldsymbol{Z}_{22}^{\mathrm{L2}} & & & & & & \\
& & & \boldsymbol{Z}_{23}^{\mathrm{L1}} & & & \boldsymbol{Z}_{23}^{\mathrm{L2}} & & & & & \\
& & & & & & & \boldsymbol{Z}_{31}^{\mathrm{L1}} & \boldsymbol{Z}_{31}^{\mathrm{L2}} & & \\
& & & & & & & \boldsymbol{Z}_{32}^{\mathrm{L1}} & & \boldsymbol{Z}_{32}^{\mathrm{L2}}
\end{bmatrix}
$$

Variational Approximations

Variational approximations are a class of techniques for making approximate model fitting and inference

Idea: Approximate a complex density function p with a simpler q

How to choose q?

Suppose p is a posterior density function $p(\boldsymbol{\theta}|\boldsymbol{y})$. Then q is chosen minimize the Kullback–Leibler divergence (or other divergence measures):

- $KL(q(\boldsymbol{\theta})\|p(\boldsymbol{\theta}|\boldsymbol{y})) = \int q(\boldsymbol{\theta})\log\{q(\boldsymbol{\theta})/p(\boldsymbol{\theta}|\boldsymbol{y})\}\, d\boldsymbol{\theta}$
- $KL(p(\boldsymbol{\theta}|\boldsymbol{y})\|q(\boldsymbol{\theta})) = \int p(\boldsymbol{\theta}|\boldsymbol{y})\log\{p(\boldsymbol{\theta}|\boldsymbol{y})/q(\boldsymbol{\theta})\}\, d\boldsymbol{\theta}$
- etc...

Variational Bayes

The objective of variational Bayes approximations is to approximate the posterior density function $p(\boldsymbol{\theta}|\boldsymbol{y})$ with an approximating density q which is easier to obtain

- A mean field variational approximation $q^*(\boldsymbol{\theta})$ to $p(\boldsymbol{\theta}|\boldsymbol{y})$ is the minimizer of the Kullback–Leibler divergence

$$\int q(\boldsymbol{\theta}) \log\left\{\frac{q(\boldsymbol{\theta})}{p(\boldsymbol{\theta}|\boldsymbol{y})}\right\} d\boldsymbol{\theta} \quad \text{subject to} \quad q(\boldsymbol{\theta}) = \prod_{i=1}^{M} q(\boldsymbol{\theta}_i),$$

  according to a partition $\{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m\}$ of $\boldsymbol{\theta}$

- If q belongs to an exponential family of distributions,

$$q^*(\boldsymbol{\theta}_i) \propto \exp\left\{E_{q(\boldsymbol{\theta}\backslash\boldsymbol{\theta}_i)} \log p(\boldsymbol{\theta}_i|\boldsymbol{y}, \boldsymbol{\theta}\backslash\boldsymbol{\theta}_i)\right\}, \qquad i = 1, \ldots, M,$$

  where $\boldsymbol{\theta}\backslash\boldsymbol{\theta}_i$ are the entries of $\boldsymbol{\theta}$ with $\boldsymbol{\theta}_i$ omitted

MFVB Approximation

For the model in general form, a tractable variational approximation arises from

$$p(\boldsymbol{\beta}, \boldsymbol{u}, \sigma^2, a_{\sigma^2}, \boldsymbol{G}, \boldsymbol{A_G}|\boldsymbol{y}) \approx q(\boldsymbol{\beta}, \boldsymbol{u})\, q(\sigma^2) q(a_{\sigma^2})\, q(\boldsymbol{G})\, q(\boldsymbol{A_G})$$

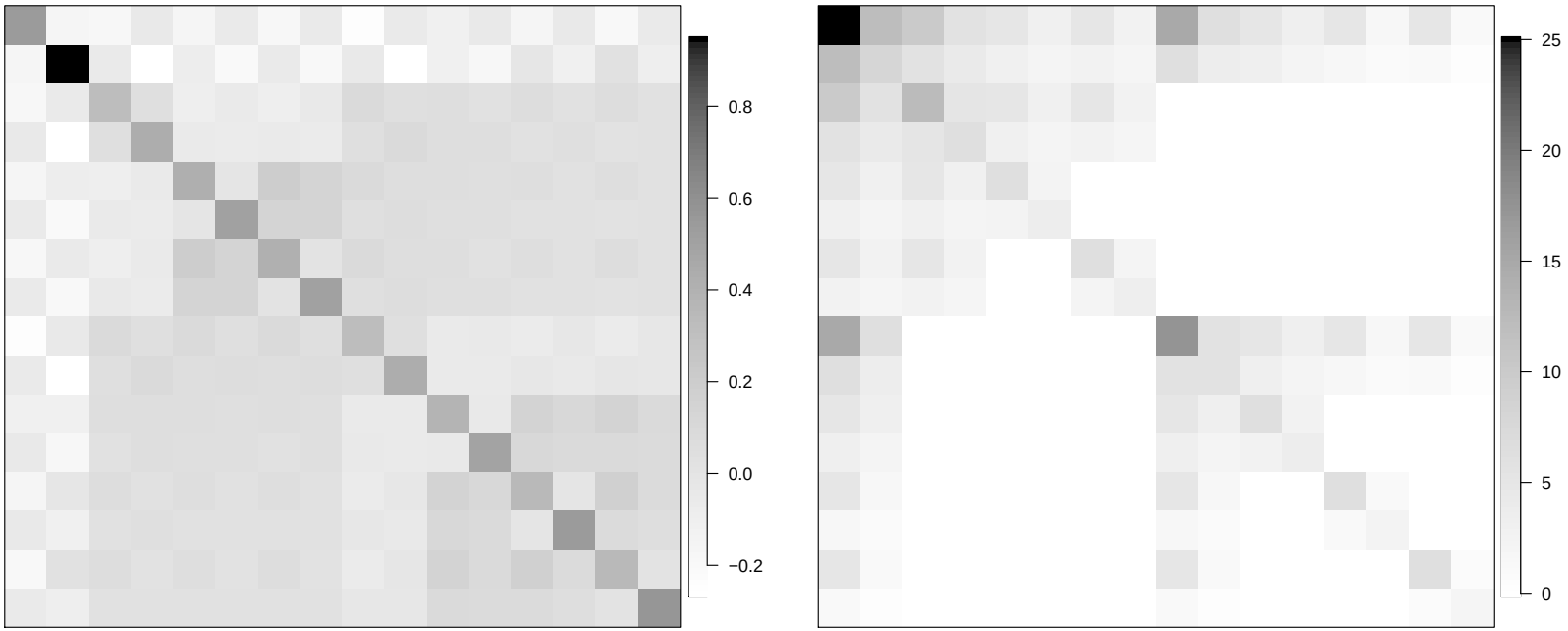
Under this restriction, the optimal MFVB approximating densities of the parameters of interest are the following:

$$q^*(\boldsymbol{\beta}, \boldsymbol{u}) \text{ is a } \mathrm{N}\left(\boldsymbol{\mu}_{q(\boldsymbol{\beta},\boldsymbol{u})}, \boldsymbol{\Sigma}_{q(\boldsymbol{\beta},\boldsymbol{u})}\right) \text{ density function,}$$
$$q^*(\sigma^2) \text{ is an Inverse-}\chi^2\left(\xi_{q(\sigma^2)}, \lambda_{q(\sigma^2)}\right) \text{ density function,}$$
$$q^*(a_{\sigma^2}) \text{ is an Inverse-}\chi^2\left(\xi_{q(a_{\sigma^2})}, \lambda_{q(a_{\sigma^2})}\right) \text{ density function,}$$
$$q^*(\boldsymbol{G}) \text{ and } q^*(\boldsymbol{A_G}) \text{ have form depending on the model specification}$$

Streamlined Variational Inference

- Note that $q^*(\boldsymbol{\beta}, \boldsymbol{u})$ is $N(\boldsymbol{\mu}_{q(\boldsymbol{\beta},\boldsymbol{u})}, \boldsymbol{\Sigma}_{q(\boldsymbol{\beta},\boldsymbol{u})})$
- At each iteration of the variational algorithm, $\boldsymbol{\Sigma}_{q(\boldsymbol{\beta},\boldsymbol{u})}$ has to be computed from a given $\boldsymbol{\Sigma}^{-1}_{q(\boldsymbol{\beta},\boldsymbol{u})}$

$\boldsymbol{\Sigma}_{q(\boldsymbol{\beta},\boldsymbol{u})}$ $\qquad$ $\boldsymbol{\Sigma}^{-1}_{q(\boldsymbol{\beta},\boldsymbol{u})}$



Linear mixed model with three-level random effects, $m = 2$, $n_1 = 2$, $n_2 = 3$, $o_{ij} = 5$, $(i = 1, 2;\ 1 \leq j \leq n_1, n_2)$ and $p = q_1 = q_2 = 2$

Naïve MFVB

Let $\boldsymbol{C} = [\,\boldsymbol{X} \;\; \boldsymbol{Z}\,]$. The naïve MFVB algorithm updates for the parameters of $q(\boldsymbol{\beta}, \boldsymbol{u})$ and $q(\sigma^2)$ are:

$$\boldsymbol{\Sigma}_{q(\boldsymbol{\beta},\boldsymbol{u})} \longleftarrow \left(\mu_{q(1/\sigma^2)}\, \boldsymbol{C}^T \boldsymbol{C} + \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{\beta}}^{-1} & \boldsymbol{O} \\ \boldsymbol{O} & E_q(\boldsymbol{G}^{-1}) \end{bmatrix} \right)^{-1},$$

$$\boldsymbol{\mu}_{q(\boldsymbol{\beta},\boldsymbol{u})} \longleftarrow \boldsymbol{\Sigma}_{q(\boldsymbol{\beta},\boldsymbol{u})} \left(\mu_{q(1/\sigma^2)}\, \boldsymbol{C}^T \boldsymbol{y} + \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{\beta}}^{-1} \boldsymbol{\mu}_{\boldsymbol{\beta}} \\ \boldsymbol{0} \end{bmatrix} \right),$$

$$\xi_{q(\sigma^2)} \longleftarrow \nu_{\sigma^2} + n,$$

$$\lambda_{q(\sigma^2)} \longleftarrow \mu_{q(1/a_{\sigma^2})} + \|\boldsymbol{y} - \boldsymbol{C}\boldsymbol{\mu}_{q(\boldsymbol{\beta},\boldsymbol{u})}\|^2 + \operatorname{tr}\left\{ \boldsymbol{\Sigma}_{q(\boldsymbol{\beta},\boldsymbol{u})} \boldsymbol{C}^T \boldsymbol{C} \right\}$$

The issues in large dimensions are:

- $\boldsymbol{Z}$ has a lot of zero elements that do not need to be stored
- $E_q(\boldsymbol{G}^{-1})$ becomes extremely sparse and inversion of $\boldsymbol{\Sigma}_{q(\boldsymbol{\beta},\boldsymbol{u})}^{-1}$ is unfeasible
- only few sub-blocks of $\boldsymbol{\Sigma}_{q(\boldsymbol{\beta},\boldsymbol{u})}$ are effectively useful for computing the other MFVB updates
- updating the expression for $\lambda_{q(\sigma^2)}$ requires operations involving huge sparse matrices

Streamlined MFVB

Streamlined MFVB:

- The mean field variational Bayes updates of $\boldsymbol{\mu}_{q(\boldsymbol{\beta},\boldsymbol{u})}$ and each of the sub-blocks of $\boldsymbol{\Sigma}_{q(\boldsymbol{\beta},\boldsymbol{u})}$ that are relevant for variational inference are expressible as a sparse matrix problem of the form:

$$\boldsymbol{A}\boldsymbol{\mu}_{q(\boldsymbol{\beta},\boldsymbol{u})} = \boldsymbol{a}$$

- The non-zero sub-blocks of $\boldsymbol{A}$ and $\boldsymbol{a}$ have close (but involved!) forms; see Nolan *et al.* (2020)
- Efficient matrix inversion and operations arise exploiting the sparse block-arrowhead matrix structure of $\boldsymbol{\Sigma}^{-1}_{q(\boldsymbol{\beta},\boldsymbol{u})}$ that occurs when two- and three-level models are considered

Global-Local Shrinkage Priors

- We perform selection of fixed effects through priors on $\boldsymbol{\beta}$.
- Let $\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{X}^{\mathrm{R}}\boldsymbol{\beta}^{\mathrm{R}} + \boldsymbol{X}^{\mathrm{S}}\boldsymbol{\beta}^{\mathrm{S}} + \boldsymbol{X}^{\mathrm{A}}\boldsymbol{\beta}^{\mathrm{A}}$, where
  - $\boldsymbol{\beta}^{\mathrm{R}}$ are fixed effects sharing a predictor with a random effect
  - $\boldsymbol{\beta}^{\mathrm{S}}$ are fixed effects subject to selection
  - $\boldsymbol{\beta}^{\mathrm{A}}$ are other fixed effects
- We use priors

$$\boldsymbol{\beta}^{\mathrm{R}} \sim \mathrm{N}\left(\boldsymbol{\mu}_{\boldsymbol{\beta}^{\mathrm{R}}}, \boldsymbol{\Sigma}_{\boldsymbol{\beta}^{\mathrm{R}}}\right), \quad \boldsymbol{\beta}^{\mathrm{S}} \sim p(\boldsymbol{\beta}^{\mathrm{S}}) = \prod_{h=1}^{p_{\mathrm{S}}} p(\beta_h^{\mathrm{S}}), \quad \boldsymbol{\beta}^{\mathrm{A}} \sim \mathrm{N}\left(\boldsymbol{\mu}_{\boldsymbol{\beta}^{\mathrm{A}}}, \boldsymbol{\Sigma}_{\boldsymbol{\beta}^{\mathrm{A}}}\right)$$

  where $p(\beta_h^{\mathrm{S}})$ is the pdf of a global-local shrinkage prior
- These priors admit the scale mixture representation (Polson and Scott, 2011)

$$\beta_h^{\mathrm{S}} | \tau, \zeta_h \overset{\text{ind}}{\sim} \mathrm{N}(0, \tau^2/\zeta_h), \quad \tau \sim p(\tau), \quad \zeta_h \sim p(\zeta_h),$$

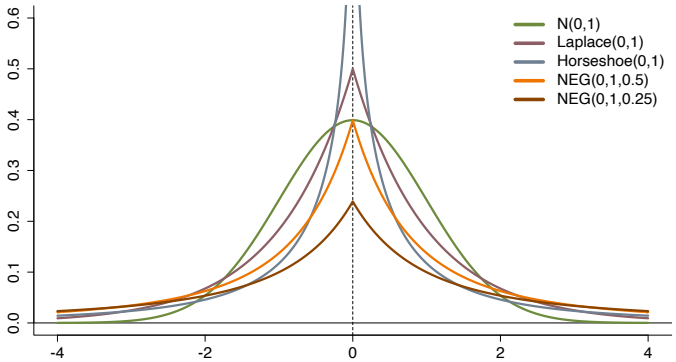
  where τ^2 is a global variance parameter, and ζ_h is a local which may induce a shrinkage effect towards zero to the hth fixed effect

Global-Local Shrinkage Priors and SAVS

- For the fixed effects subject to selection we consider the priors

$$\beta_h^S|\tau \overset{\text{iid}}{\sim} \text{Laplace}(0,\tau), \quad \beta_h^S|\tau \overset{\text{iid}}{\sim} \text{Horseshoe}(0,\tau), \quad \beta_h^S|\tau \overset{\text{iid}}{\sim} \text{NEG}(0,\tau;\lambda)$$

(NEG=Negative-Exponential-Gamma) and use the diffuse prior $\tau \sim \text{Half-Cauchy}(10^5)$



Global-local prior	$p(\beta_h^S\|\zeta_h,\tau)$	$p(\zeta_h\|a_{\zeta_h})$	$p(a_{\zeta_h})$
Laplace$(0,\tau)$	$N(0,\tau^2/\zeta_h)$	Inverse-$\chi^2(2,1)$	–
Horseshoe$(0,\tau)$	$N(0,\tau^2/\zeta_h)$	Gamma$(1/2, a_{\zeta_j})$	Gamma$(1/2,1)$
NEG$(0,\tau;\lambda)$	$N(0,\tau^2/\zeta_h)$	Inverse-$\chi^2(2, 2a_{\zeta_j})$	Gamma$(\lambda,1)$

- The posterior densities of irrelevant β_h^s's are shrunk towards zero but not sparsified
- We perform fixed-effects selection via the signal adaptive variable selector (SAVS) of Ray and Bhattacharya (2018), which is hyperparameters-tuning free:

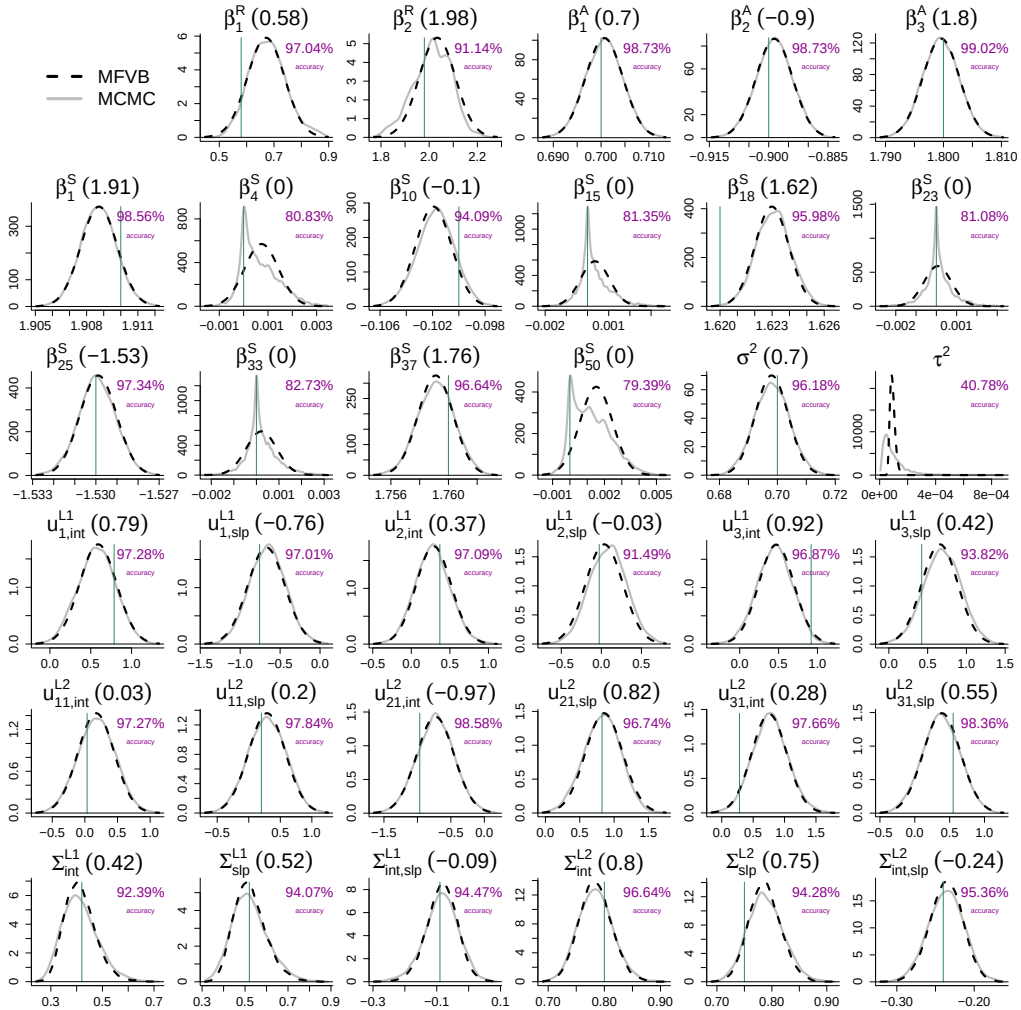
$$\text{if } \|\boldsymbol{x}_h\|^2 \leq \left|\mu^*_{q(\beta_h^S)}\right|^{-3}, \text{ then } \beta_h^S \text{ is discarded}$$

Simulated Data: Settings

- 50 simulated datasets from a three-level linear mixed model with $m = 100$ groups, each with $n_i = 15$ sub-groups of $o_{ij} = 20$ units each
- Random intercept and slope for both the group and sub-group levels ($q^{\mathrm{L1}} = q^{\mathrm{L2}} = p_{\mathrm{R}} = 2$), and $p_{\mathrm{A}} = 3$ additional fixed effects not subject to selection
- $p_{\mathrm{S}} = 50$ fixed effects subject to selection, where 40 of them are irrelevant (true value equal to zero); $\beta^{\mathrm{S}}_h|\tau \overset{\text{iid}}{\sim} \text{Horseshoe}(0, \tau)$, for $1 \leq h \leq 50$
- Hence, $\boldsymbol{X}$ has size 30,000 $\times$ 55, and $\boldsymbol{Z}$ has size 30,000 $\times$ 3,200 (96*10^6 cells, of which 99.875% are empty and do not contain any data)
- Diffuse priors for all the model parameters
- Both the naïve and streamlined MFVB approximations are obtained (200 iterations)
- Comparison with MCMC: Gibbs sampler (5,000 burn-in, 25,000 kept, thinning of size 5)
- All algorithms are implemented in C++

Simulated Data: Approximation Assessment

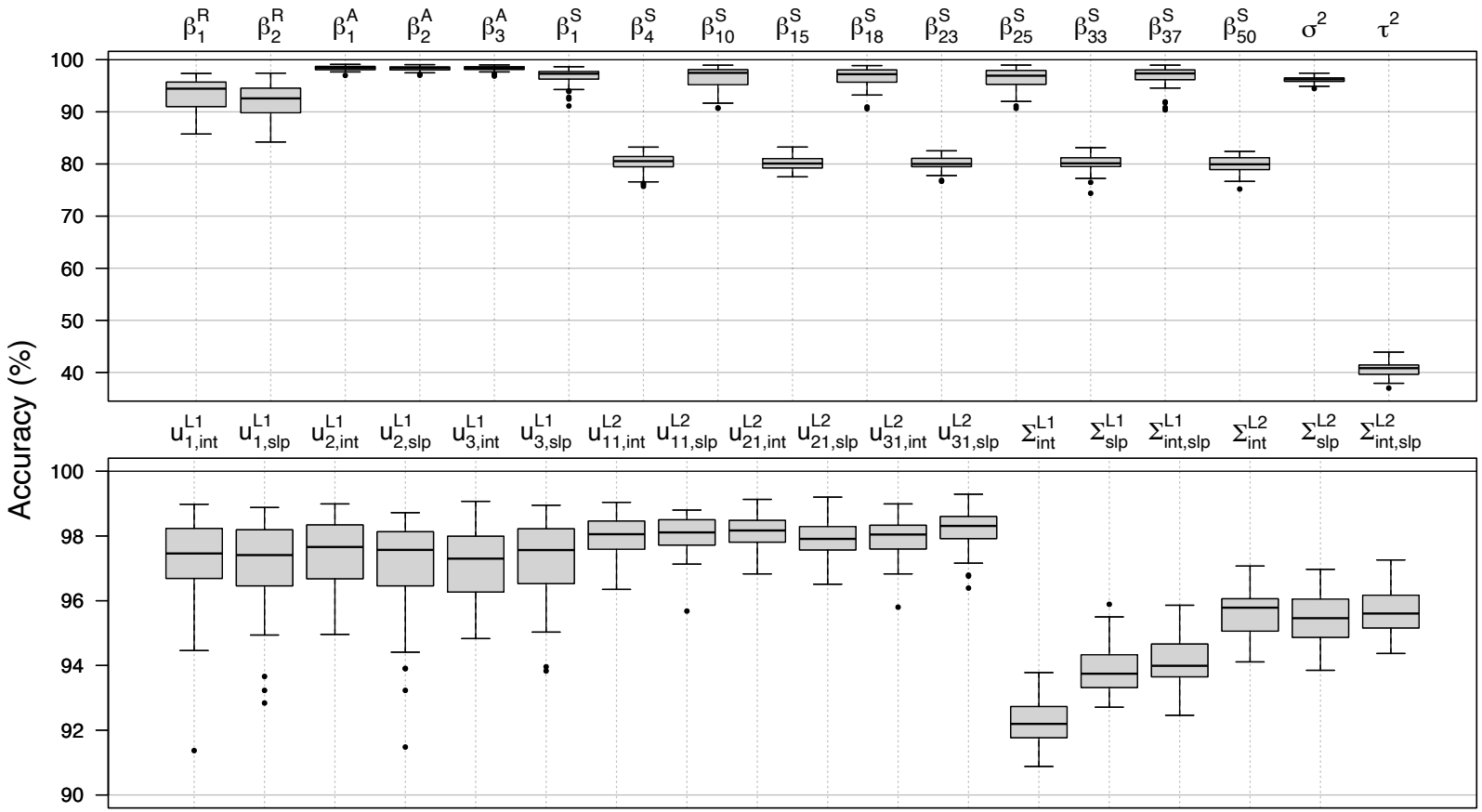
Optimal $q^*(\theta)$ densities obtained via MFVB (black dashed curves) and the corresponding MCMC-based $p(\theta|\boldsymbol{y})$ densities (grey curves). Vertical lines indicate the parameters true values.



Simulated Data: Accuracy Assessment

Boxplots of the accuracy scores for a selection of model parameters and random effects

$\text{Accuracy}\{q^*(\theta)\} \equiv (1 - \frac{1}{2}\int |q^*(\boldsymbol{\theta}) - p(\theta|\boldsymbol{y})|\, \mathrm{d}\theta) \times 100\%$



Simulated Data: Selection Performance

- Get MFVB approximations for each of the 50 datasets and priors

$$\beta_h^S \overset{\text{iid}}{\sim} \text{N}(0, 10^{10}), \quad \beta_h^S|\tau \overset{\text{iid}}{\sim} \text{Laplace}(0, \tau),$$

$$\beta_h^S|\tau \overset{\text{iid}}{\sim} \text{Horseshoe}(0, \tau), \quad \beta_h^S|\tau \overset{\text{iid}}{\sim} \text{NEG}(0, \tau; 0.25)$$

- After sparsifying the $q^*(\beta_h^S)$ via the SAVS procedure, the selection performance is measured using the F_1-score:

$$F_1 = \left(\frac{2 \times \text{precision} \times \text{recall}}{\text{precision} + \text{recall}} \right) \times 100\%, \ \text{precision} = \frac{\text{TP}}{\text{TP} + \text{FP}}, \ \text{recall} = \frac{\text{TP}}{\text{TP} + \text{FN}},$$

TP $=$ # of fixed effects with true $\beta_h^S \neq 0$ and selected by SAVS,

TN $=$ # of fixed effects with true $\beta_h^S = 0$ and *not* selected by SAVS,

FP $=$ # of fixed effects with true $\beta_h^S = 0$ and selected by SAVS,

FN $=$ # of fixed effects with true $\beta_h^S \neq 0$ and *not* selected by SAVS

- Selection performance:
  - Gaussian: median F_1-score 63.5% (q1: 43.5% – q3: 94.2%)
  - Laplace: 95.24% (80%–100%)
  - Horseshoe and NEG: 100% (100%–100%, i.e. perfect fixed effects selection for each of the 50 datasets)

Simulated Data: Speed and Memory Saving

Speed and memory saving assessment for:

- three-level model
- p_S fixed effects subject to selection in $\{25, 100, 200\}$
- number of groups m in $\{10, 50, 100, 200\}$
- n_i and o_{ij} uniformly from the discrete sets $\{10, \ldots, 20\}$ and $\{20, \ldots, 30\}$

The table displays average and standard deviation (in brackets) of

- running time (seconds)
- total size of required data input (megabytes)

		Total runtime of the algorithm (seconds)				Total size of the required data inputs (megabytes)		
m	p_S	Streamlined MFVB	Naïve MFVB	$\frac{\text{Naïve MFVB}}{\text{Streamlined MFVB}}$	MCMC	Streamlined MFVB	Naïve MFVB	$\frac{\text{Naïve MFVB}}{\text{Streamlined MFVB}}$
10	25	$0.70_{(0.06)}$	$3.86_{(0.50)}$	**5.54**	$13.03_{(0.96)}$	$1.39_{(0.07)}$	$11.06_{(1.00)}$	**7.97**
	100	$3.90_{(0.23)}$	$6.51_{(0.82)}$	**1.67**	$46.99_{(1.47)}$	$3.68_{(0.24)}$	$12.73_{(1.34)}$	**3.46**
	200	$9.71_{(1.22)}$	$12.09_{(2.04)}$	**1.24**	$176.22_{(9.37)}$	$6.60_{(0.52)}$	$15.09_{(1.89)}$	**2.29**
50	25	$3.29_{(0.11)}$	$365.07_{(37.10)}$	**111.01**	$63.28_{(3.76)}$	$6.69_{(0.20)}$	$240.40_{(13.92)}$	**35.91**
	100	$19.66_{(0.87)}$	$415.29_{(51.66)}$	**21.12**	$138.42_{(5.72)}$	$18.37_{(0.81)}$	$254.04_{(20.84)}$	**13.83**
	200	$49.25_{(1.47)}$	$501.33_{(39.76)}$	**10.18**	$305.25_{(6.51)}$	$34.00_{(1.04)}$	$269.34_{(14.90)}$	**7.92**
100	25	$6.77_{(0.32)}$	$2877.33_{(177.77)}$	**424.72**	$151.05_{(10.48)}$	$13.56_{(0.34)}$	$967.34_{(42.15)}$	**71.33**
	100	$40.42_{(1.66)}$	$3172.66_{(116.83)}$	**78.50**	$266.71_{(10.06)}$	$36.77_{(1.29)}$	$1010.25_{(26.94)}$	**27.47**
	200	$97.81_{(2.07)}$	$3403.55_{(204.38)}$	**34.80**	$494.15_{(9.34)}$	$67.53_{(1.31)}$	$1013.90_{(48.67)}$	**15.01**
200	25	$13.30_{(0.30)}$	$>$ 5 hours	$>$ **1355**	$328.87_{(20.04)}$	$26.90_{(0.56)}$	$3817.06_{(113.85)}$	**141.88**
	100	$79.97_{(1.18)}$	$>$ 5 hours	$>$ **225**	$578.88_{(13.45)}$	$73.74_{(1.01)}$	$3941.15_{(97.90)}$	**53.45**
	200	$197.20_{(4.13)}$	$>$ 5 hours	$>$ **95**	$904.64_{(15.77)}$	$135.84_{(2.73)}$	$3991.66_{(102.48)}$	**29.38**

Perinatal Data (1)

National Collaborative Perinatal Project data (Klebanoff, 2009)

- A multisite prospective cohort study which took place in the United States of America between 1959 and 1974.
- Designed to identify the effects of complications during pregnancy or the perinatal period on birth and child outcomes
- Focus on predicting the height-for-age z-score (standardized measure of the WHO for the height of children after accounting for age) for 37,257 infants followed longitudinally over their first year of life

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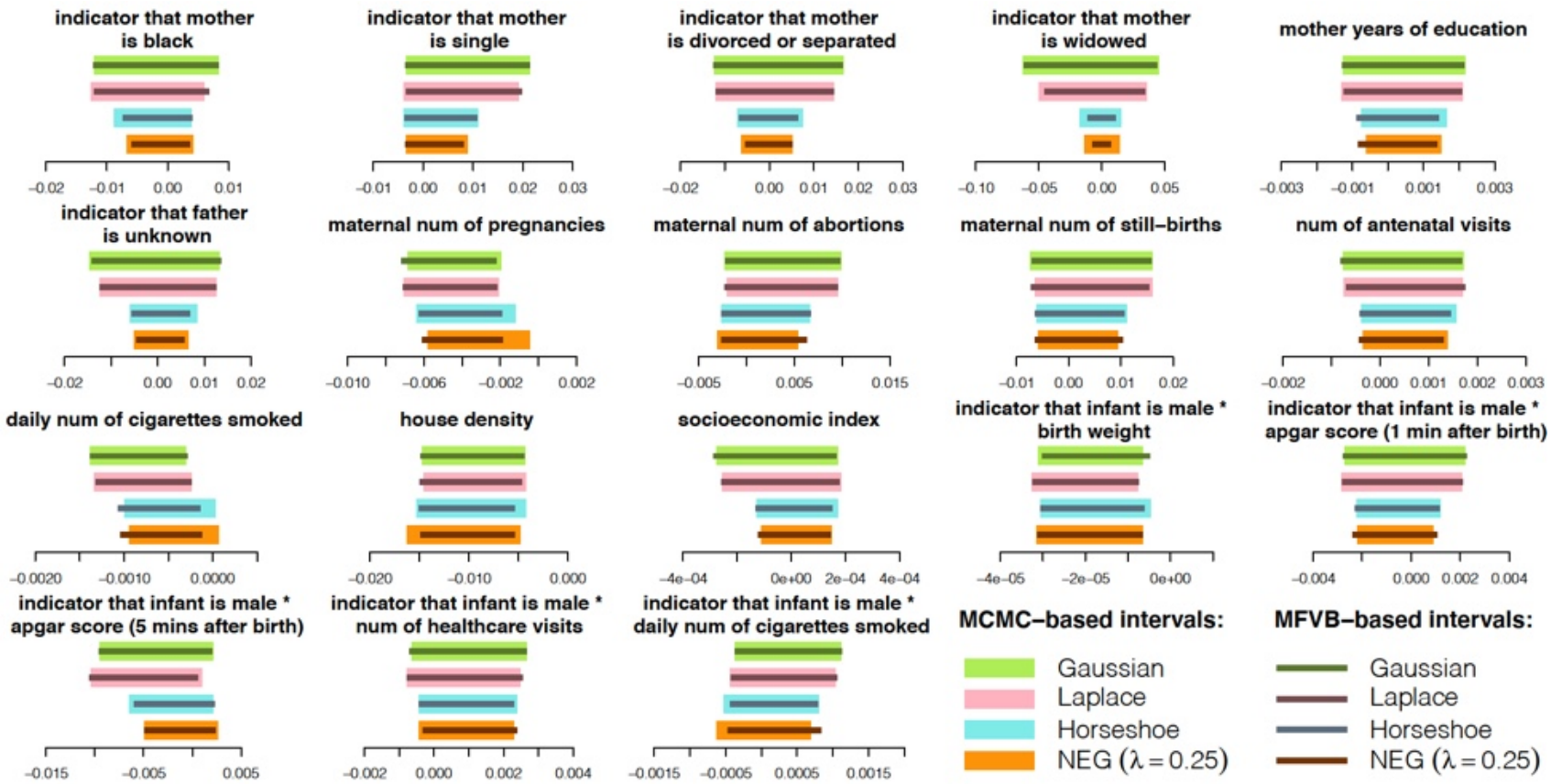
- We fit a two-level linear mixed model with
  - Intercept, age of the infant and its square, plus 38 candidate predictors subject to selection as fixed effects
  - Intercept, age of the infant and its square as random effects
  - Gaussian, Laplace, Horseshoe and Negative-Exponential-Gamma priors for fixed effects subject to selection
- MCMC (5,000 burn-in iterations followed by 25,000 iterations with thinning factor of 5) took more than 35 minutes to run
- Streamlined MFVB (200 iterations) took around 2 minutes to run

Perinatal Data (2)

The 90% high posterior density credible intervals for some of the fixed effects subject to selection



Extensions

More accurate variational approximations can be obtained at higher computational cost when using

- global-local prior specifications that are not based on auxiliary variables (see Neville *et al.*, 2014)
- less restrictive mean-field factorizations on the fixed effects, e.g. $q(\boldsymbol{\beta}, \boldsymbol{u}) = q(\boldsymbol{\beta}^{\mathrm{R}}, \boldsymbol{u})q(\boldsymbol{\beta}^{\mathrm{A}}, \boldsymbol{\beta}^{\mathrm{S}})$

With relatively little changes in the streamlined MFVB algorithms, it is possible to treat models with

- other response types (binary data with logit/probit link, Poisson, t, Skew Normal, Skew t, etc...)
- other types of priors for fixed effects selection (e.g. Bayesian lasso and spike-and-slab priors)

Other extensions include the development of streamlined variational inference for models with

- unit-specific errors, heteroskedastic covariance structures for groups and sub-groups, higher levels of nesting or crossed random effects
- priors for selecting random effects (however, this would entail formulating new streamlining strategies for the variational algorithms)

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