Bayesian Sensitivity Analysis for a Missing Data Model

Bart Eggen, Stéphanie van der Pas, Aad van der Vaart

Delft University of Technology, Amsterdam University Medical Centres

1 July 2022

Co-authors

Aad van der Vaart

Stéphanie van der Pas

Outline

What is sensitivity analysis?

Observation	Outcome
1	15
2	3
3	*
4	*
5	5
⋮	⋮

What is sensitivity analysis?

Observation	Outcome
1	15
2	3
3	*
4	*
5	5
⋮	⋮

Solutions

- Assume data is **missing completely at random**
- Data imputation

What is sensitivity analysis?

What if we assume MCAR, but in reality it does not hold?

What is sensitivity analysis?

What if we assume MCAR, but in reality it does not hold?

Sensitvity Analysis

How robust are study conclusion to violations of assumptions?

How?

Statistics

Assign a parameter to the assumption

How?

Statistics

Assign a parameter to the assumption

Problems

- Sensitivity parameter is often unidentifiable
- Difficult to give intuition for practicioners

How?

Statistics

Assign a parameter to the assumption

Problems

- Sensitivity parameter is often unidentifiable
- Difficult to give intuition for practicioners

Robustness

- At what values of the sensitivity parameter do study conclusions not hold anymore?
- Size is an indicator of robustness

Bayesian sensitivity analysis

Why Bayesian?

- Put all your intuition into the prior for the sensitivity parameter
- The posterior distribution acts as a single summary of your beliefs

Bayesian sensitivity analysis

Why Bayesian?

- Put all your intuition into the prior for the sensitivity parameter
- The posterior distribution acts as a single summary of your beliefs

Problem

- Not a lot of theory available
- Wat if we are "close" to the truth with our parameter, will our conclusion converge to the truth?

Missing data model (MCAR)

Observations

$$Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} P$$

Missing data model (MCAR)

Observations

$$Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} P$$

Missingness

$$R_1, \ldots, R_n \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$$

Missing data model (MCAR)

Observations

$$Y_1, \dots, Y_n \overset{\text{iid}}{\sim} P$$

Missingness

$$R_1, \dots, R_n \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$$

Observations

$$X_i := \begin{cases} Y_i, & R_i = 1 \\ *, & R_i = 0 \end{cases}$$

Missing data model (MCAR)

Observations

$$Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} P$$

Missingness

$$R_1, \ldots, R_n \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$$

Observations

$$X_i := \begin{cases} Y_i, & R_i = 1 \\ *, & R_i = 0 \end{cases}$$

Interest

$$\mathbb{E}[g(Y)]$$

Sensitivity parameter

Conditional distributions

$$Y \mid R = 1 \sim P_1, \qquad\qquad Y \mid R = 0 \sim P_0$$

Sensitivity parameter

Conditional distributions

$$Y \mid R = 1 \sim P_1, \qquad Y \mid R = 0 \sim P_0$$

Consequence

$$P = pP_1 + (1-p)P_0$$

Sensitivity parameter [1]

$$P_0(A) = \frac{\int_A e^{q} dP_1}{\int e^{q} dP_1}$$

[1] D. O. Scharfstein, M. J. Daniels, and J. Robins (2003). "Incorporating prior beliefs about selection bias into the analysis of randomized trials with missing outcomes". In: *Biostatistics* 4.4, pp. 495–512

Sensitivity parameter [1]

$$P_0(A) = \frac{\int_A e^{q} dP_1}{\int e^{q} dP_1}$$

Example

$$q(y) = \alpha \log(y), \quad \alpha \in \mathbb{R}$$

[1] D. O. Scharfstein, M. J. Daniels, and J. Robins (2003). "Incorporating prior beliefs about selection bias into the analysis of randomized trials with missing outcomes". In: *Biostatistics* 4.4, pp. 495–512

Intuition?

$$\operatorname{logit} \Pr(R = 0 \mid Y) = \eta + q(Y)$$

Intuition?

$$\operatorname{logit} \Pr(R = 0 \mid Y) = \eta + q(Y)$$

When $q(y) = \alpha \log(y)$

- $\alpha = 0$: MCAR
- $\alpha > 0$: Higher outcomes are more likely to not be observed
- $\alpha < 0$: Lower outcomes are more likely to not be observed

Prior choices

Parametrization p, P_1

$p \sim Beta(\cdot, \cdot)$ $\qquad$ $P_1 \sim DP(a)$

Prior choices

Parametrization p, P_1

$p \sim Beta(\cdot, \cdot)$ $\qquad$ $P_1 \sim DP(a)$

Parametrization η, P

$\eta \sim$ Compact $\qquad$ $P \sim DP(a)$

Prior choices

Parametrization p, P_1

$$p \sim Beta(\cdot,\cdot) \qquad\qquad P_1 \sim DP(a)$$

Parametrization η, P

$$\eta \sim \text{Compact} \qquad\qquad P \sim DP(a)$$

We want to show

Bernstein-von Mises theorem for both parametrizations

$$\sqrt{n}\left(\chi_q(p, P_1) - \chi_q(\hat{p}, \hat{P}_1)\right) \mid R^{(n)}, X^{(n)}, q \rightsquigarrow ? \quad \text{a.s.}$$

Relation

Relation between P and P_1

$$P_1(A) = \frac{\int_A \frac{1}{1+e^{\eta+q(y)}} dP(y)}{\int \frac{1}{1+e^{\eta+q(y)}} dP(y)}$$

Relation

Relation between P and P_1

$$P_1(A) = \frac{\int_A \frac{1}{1+e^{\eta+q(y)}} dP(y)}{\int \frac{1}{1+e^{\eta+q(y)}} dP(y)}$$

Prior

$P \sim DP(a)$ is equivalent to $P_1 \sim \text{NEGP}(a, b)$

BvM for normalized extended gamma processes

Theorem

Let $X_1, \ldots, X_n \mid P \sim P$ with $P \sim \mathrm{NEGP}(a, b)$ and let $\mathcal{F}$ be a Donsker-class. Let P_0 be the true distribution of the data. Under assumption later to be specified

$$\sqrt{n}\,(P - \mathbb{P}_n) \mid X^{(n)} \rightsquigarrow B_{P_0},$$

in $\ell^\infty(\mathcal{F})$ almost surely $[P_0^\infty]$, where B_{P_0} is a P_0-Brownian bridge.

BvM for normalized extended gamma processes

Theorem

Let $X_1, \ldots, X_n \mid P \sim P$ with $P \sim \mathrm{NEGP}(a, b)$ and let $\mathcal{F}$ be a Donsker-class. Let P_0 be the true distribution of the data. Under assumption later to be specified

$$\sqrt{n}\,(P - \mathbb{P}_n) \mid X^{(n)} \rightsquigarrow B_{P_0},$$

in $\ell^\infty(\mathcal{F})$ almost surely $[P_0^\infty]$, where B_{P_0} is a P_0-Brownian bridge.

Assumptions

- b is a positive, bounded measurable function
- a is an atomless, finite measure
- $\mathcal{F}$ has envelope function F with $\int F da < \infty$
- There exist $r, q > 0$ with $1/q + 1/r < 1/2$ such that $P_0 F^r < \infty$ and $P_0 b^{-q} < \infty$

Proof strategy

- The posterior of P is a mixture (over λ) of NCRMs
- Show mixing density concentrates on big λ's
- Show the continuous part of the NCRM vanishes
- Use multiplier central limit theorem

To do

- Fix assumption on prior for η
- Show what happens when we put a prior on q (α)
- Extend model with covariates!
- ...