Calibrated Model Criticism Using Split Predictive Checks

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Boston University

with Jiawei Li

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  - provide asymptotic theory showing good **calibration** & **power**

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- p-values provide measure of fit only if they are well-calibrated!

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double use of data

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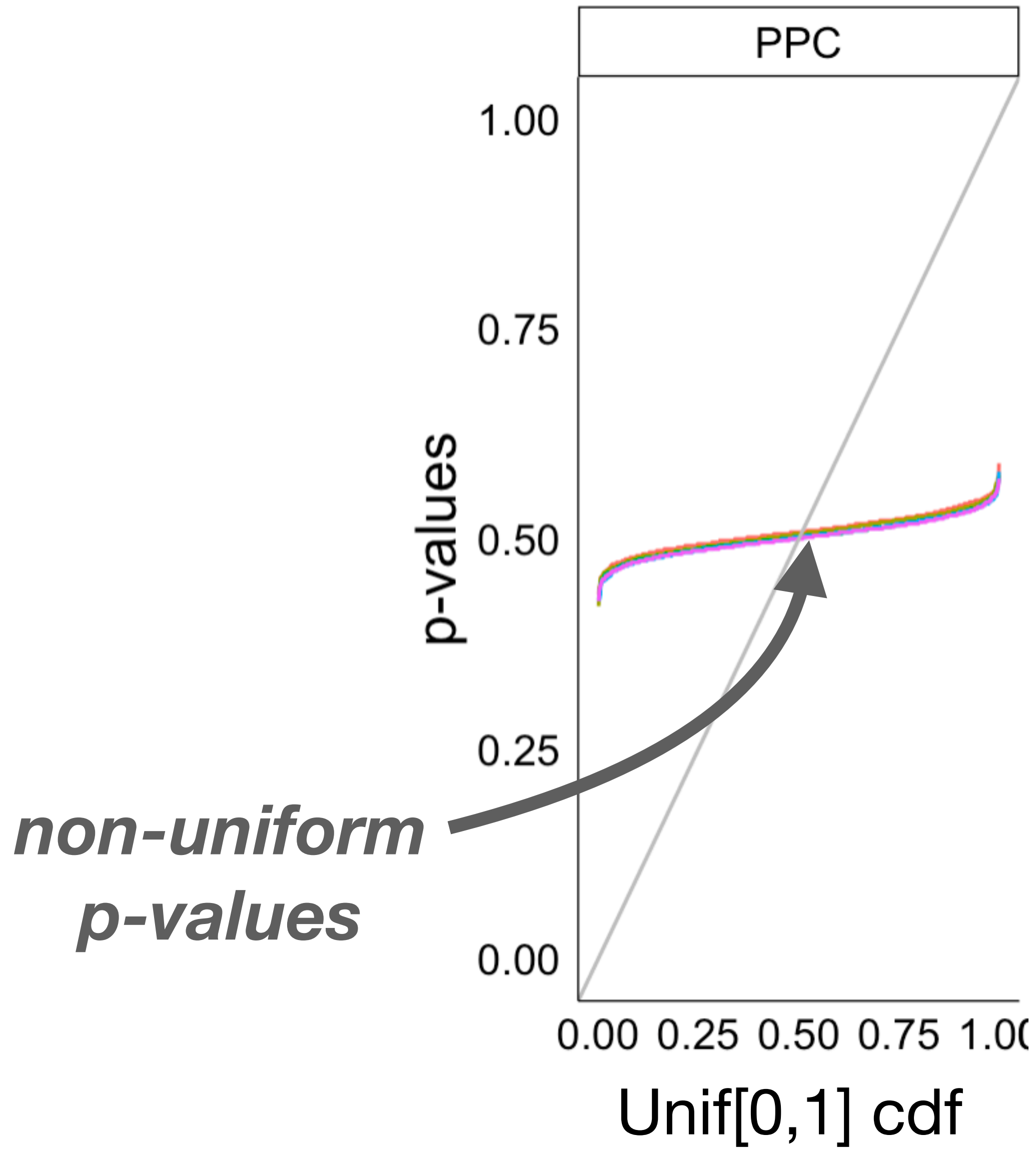

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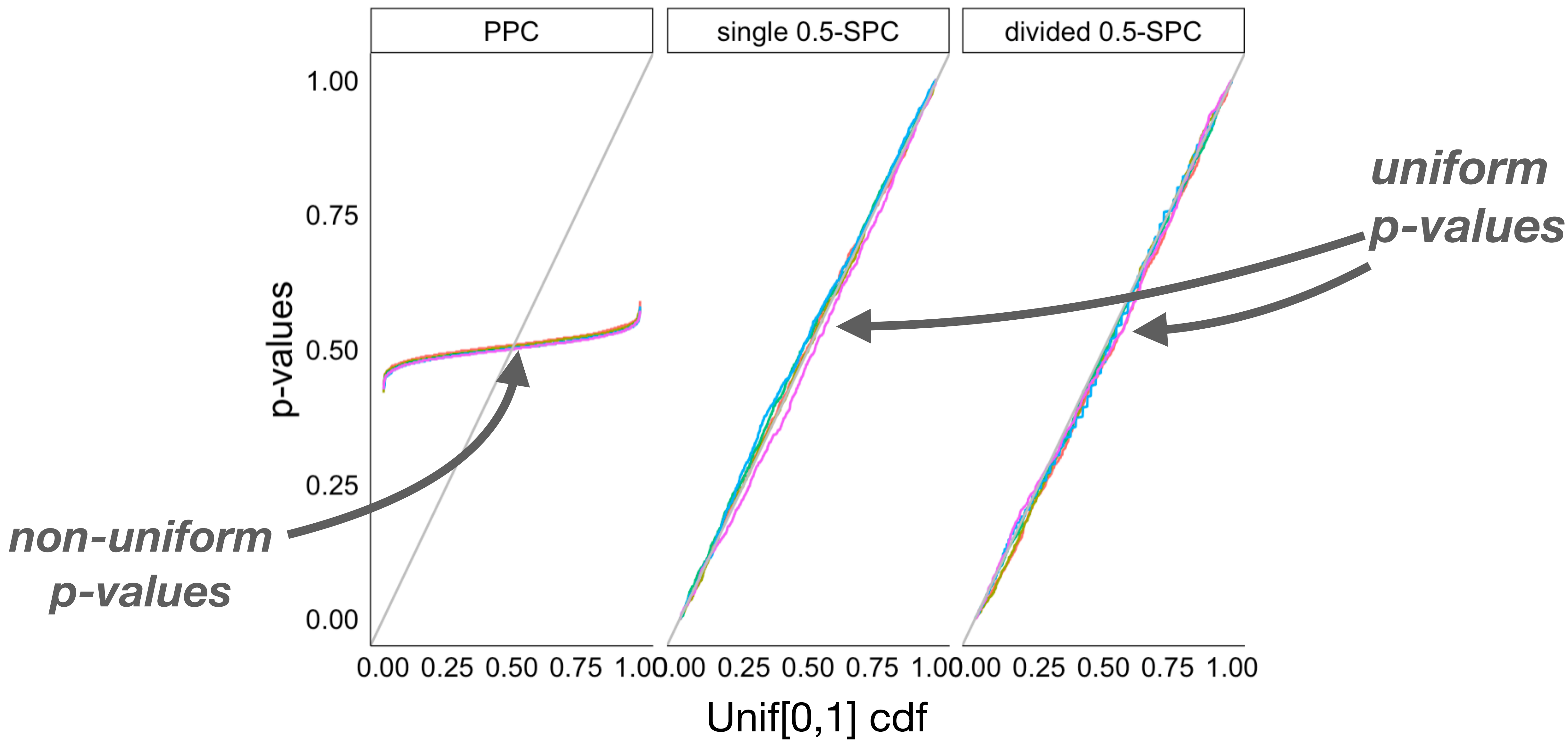

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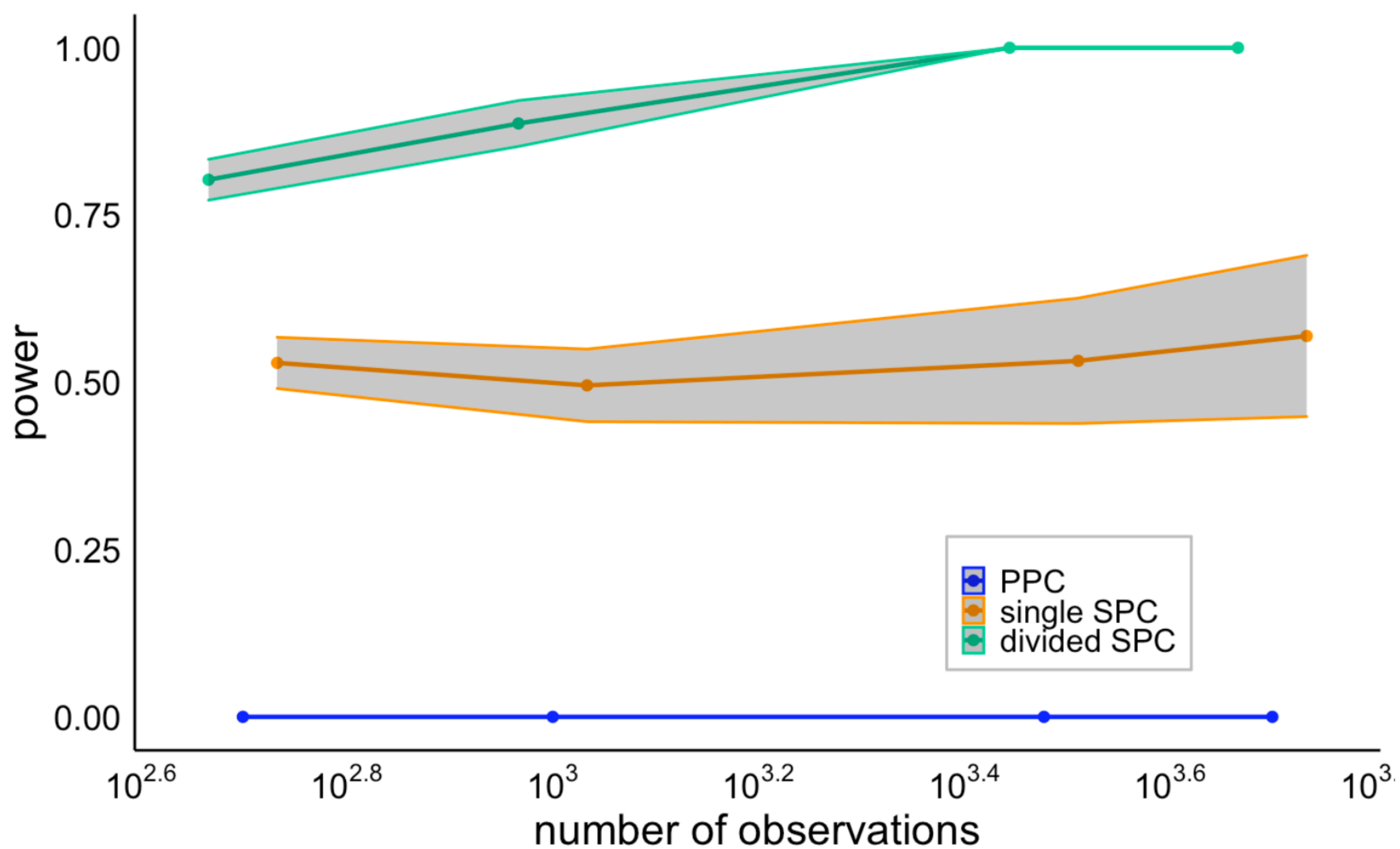

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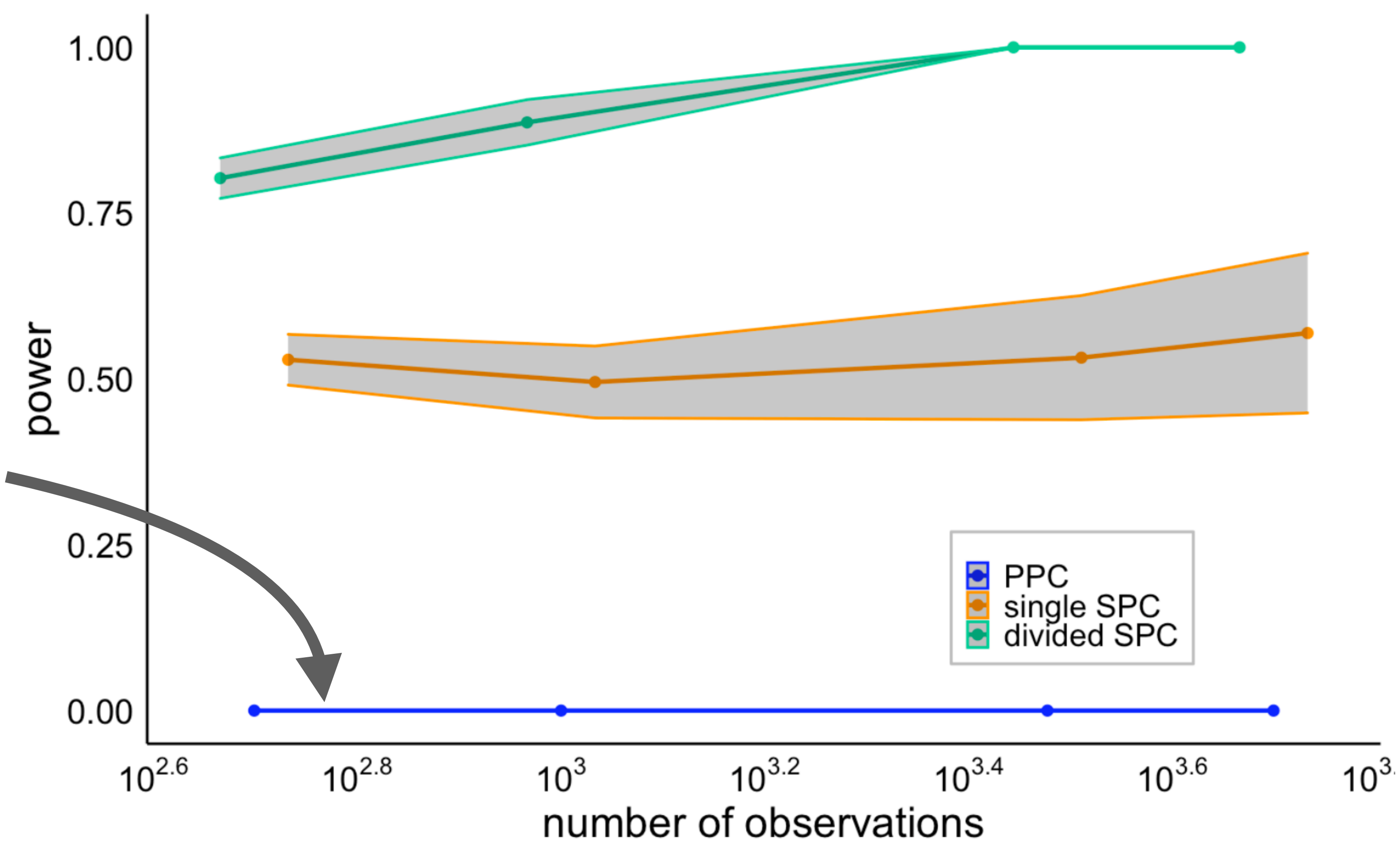

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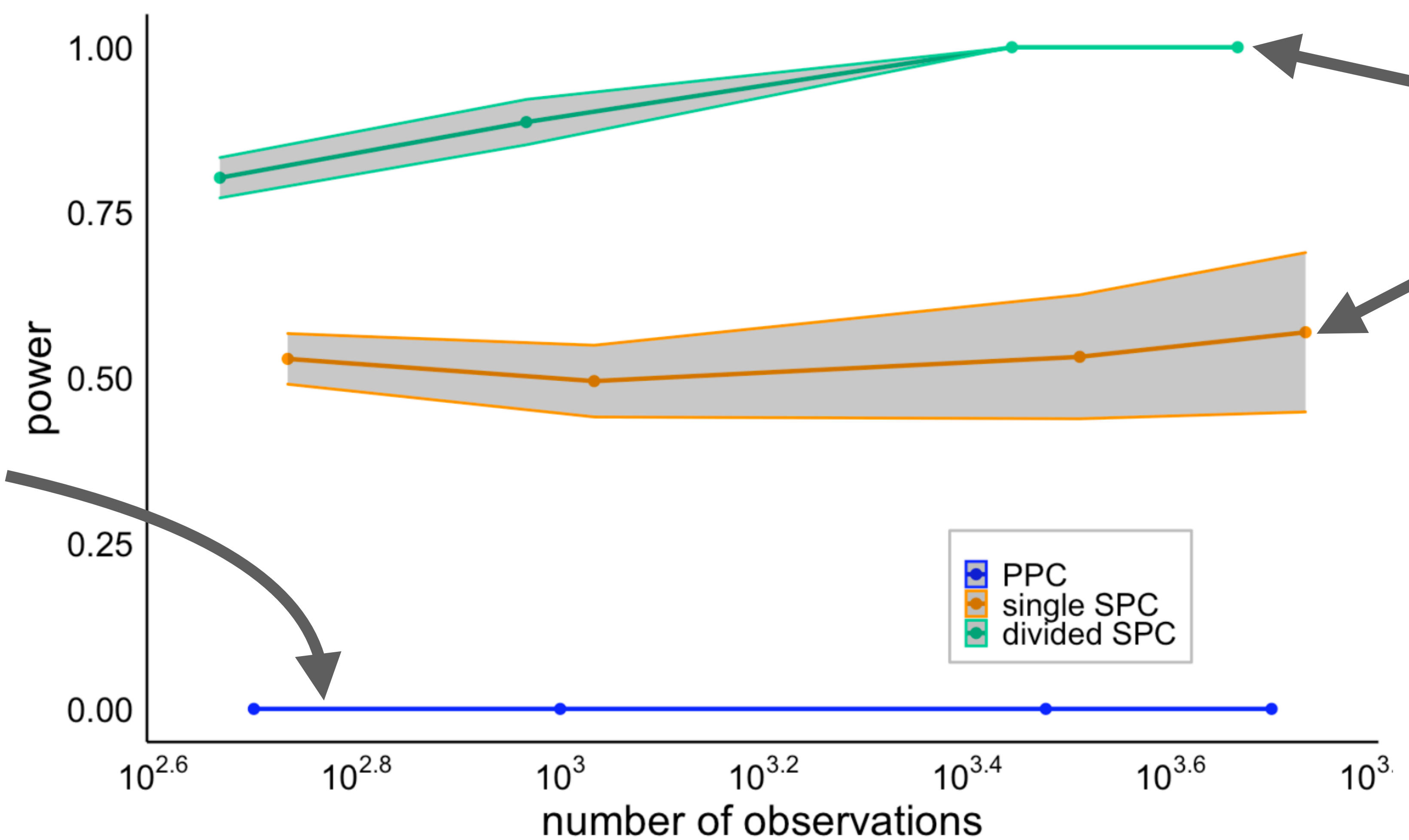

both SPCs have good power

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$$\begin{tikzcd}
 & \mathbf{x_{obs}} \\
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\end{tikzcd}$$

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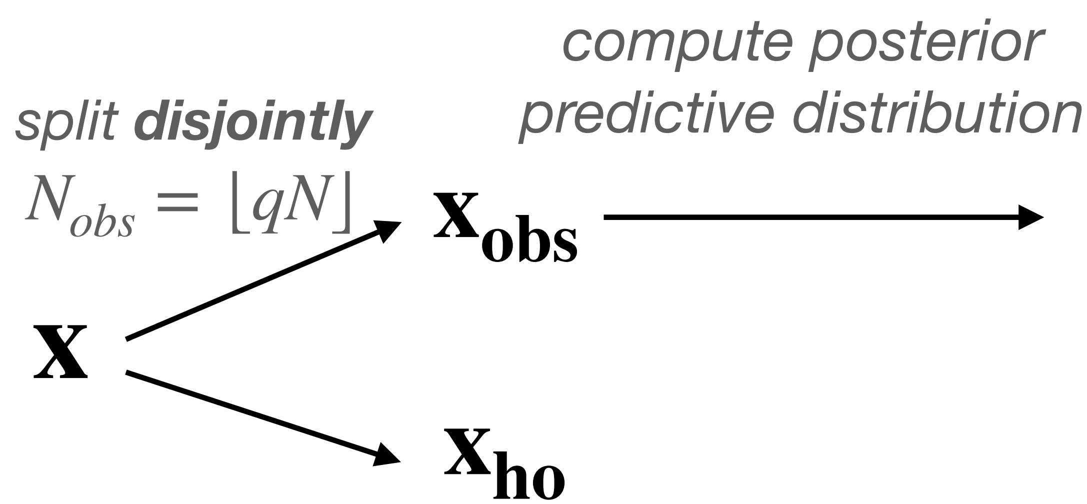

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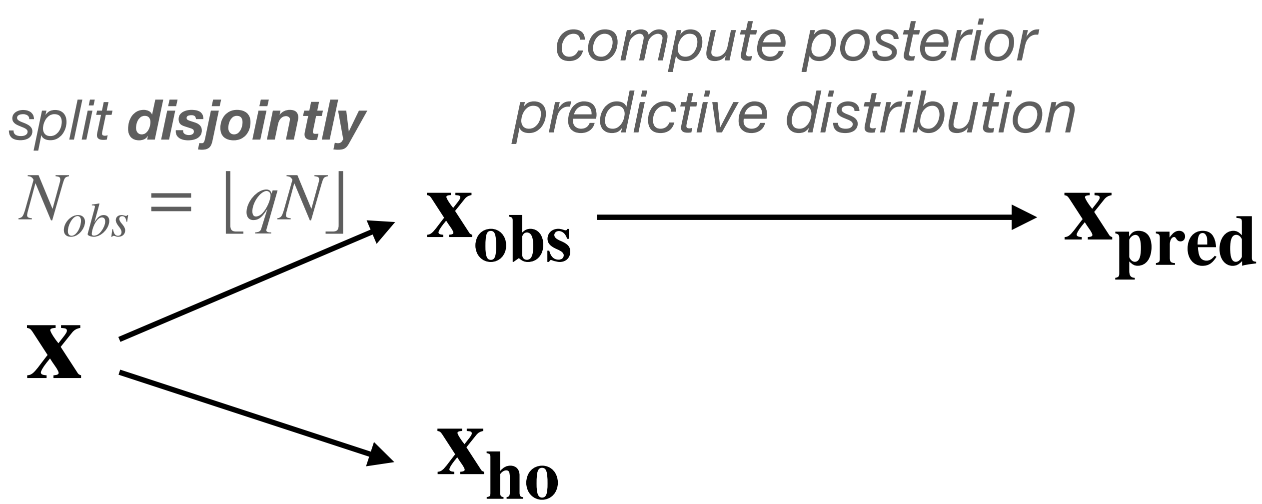

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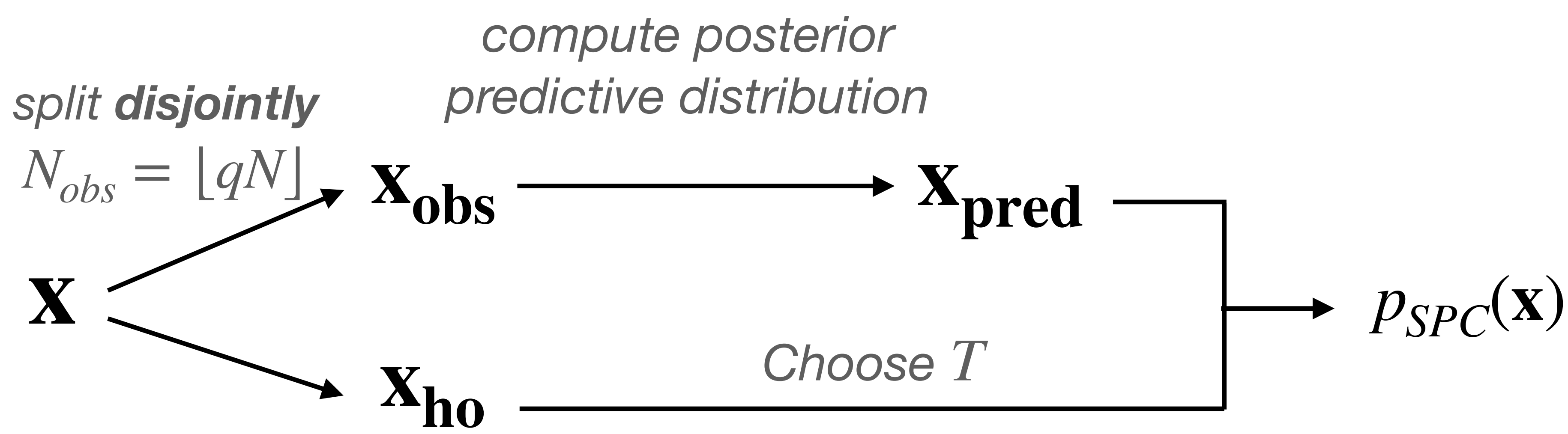

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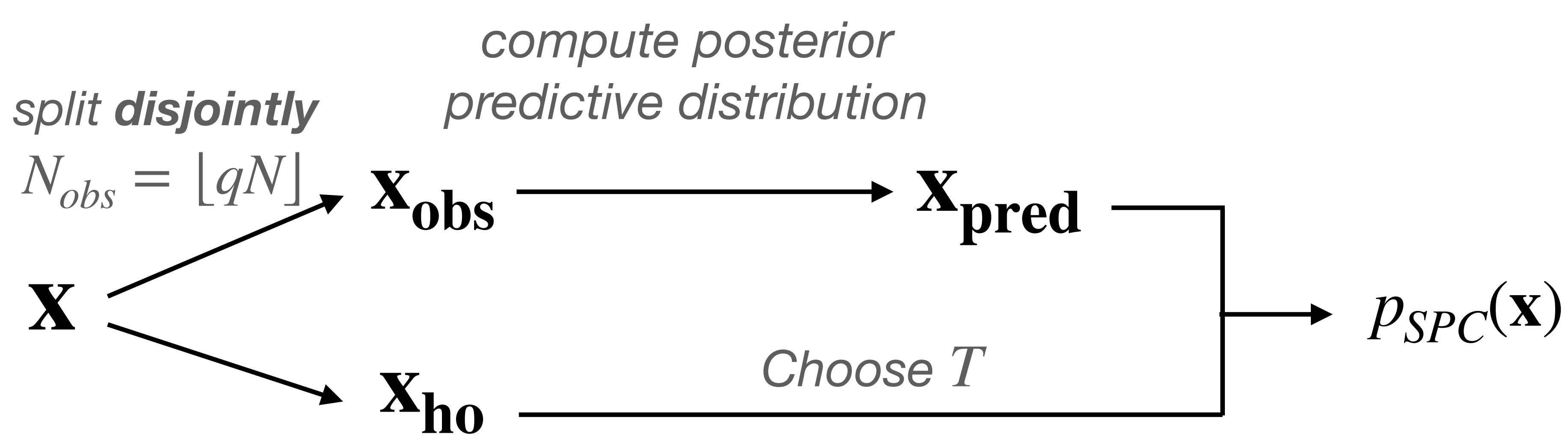

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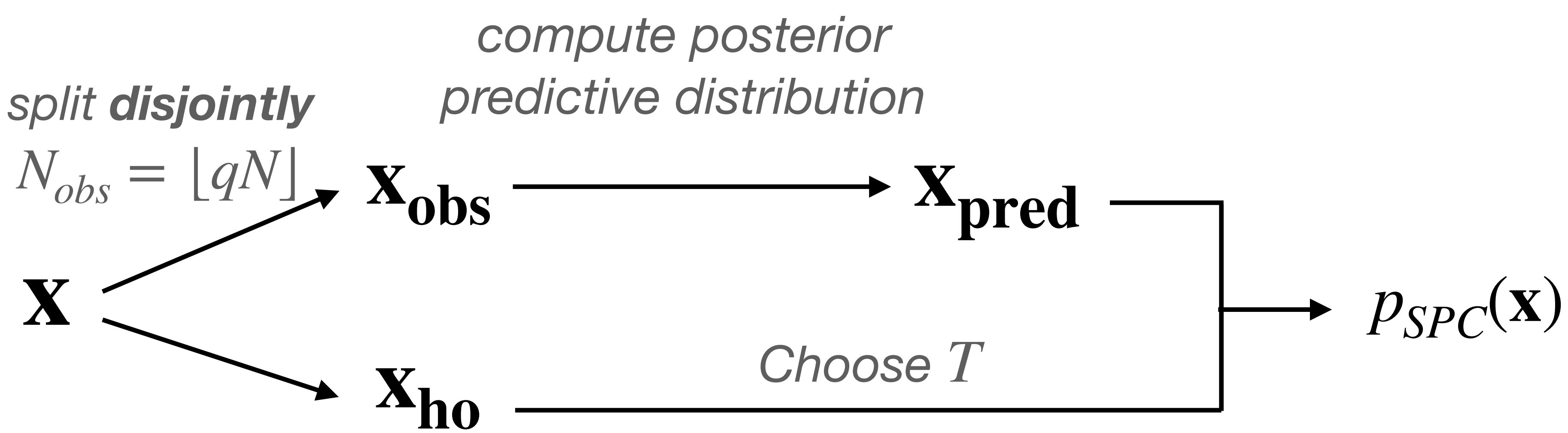

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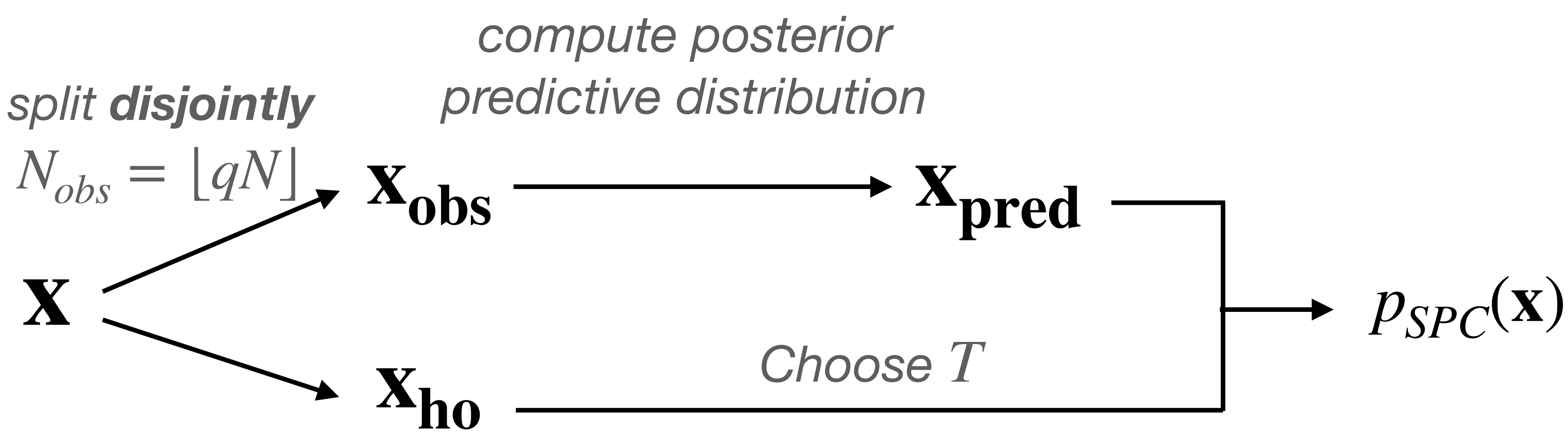

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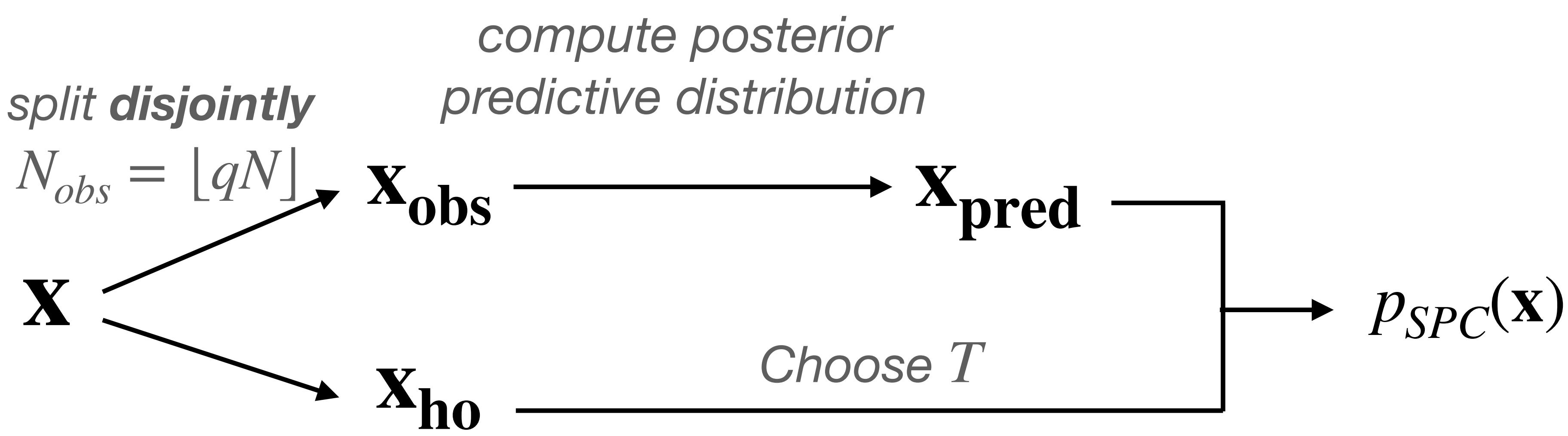

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Question 1: How is calibration and power?

Asymptotic power and calibration of single SPCs

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ρ depends on q, Fisher information matrices and asymptotic variances of T

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  - when uncertainty is over-estimated, $\rho \downarrow 0$ and power $\downarrow$ 0

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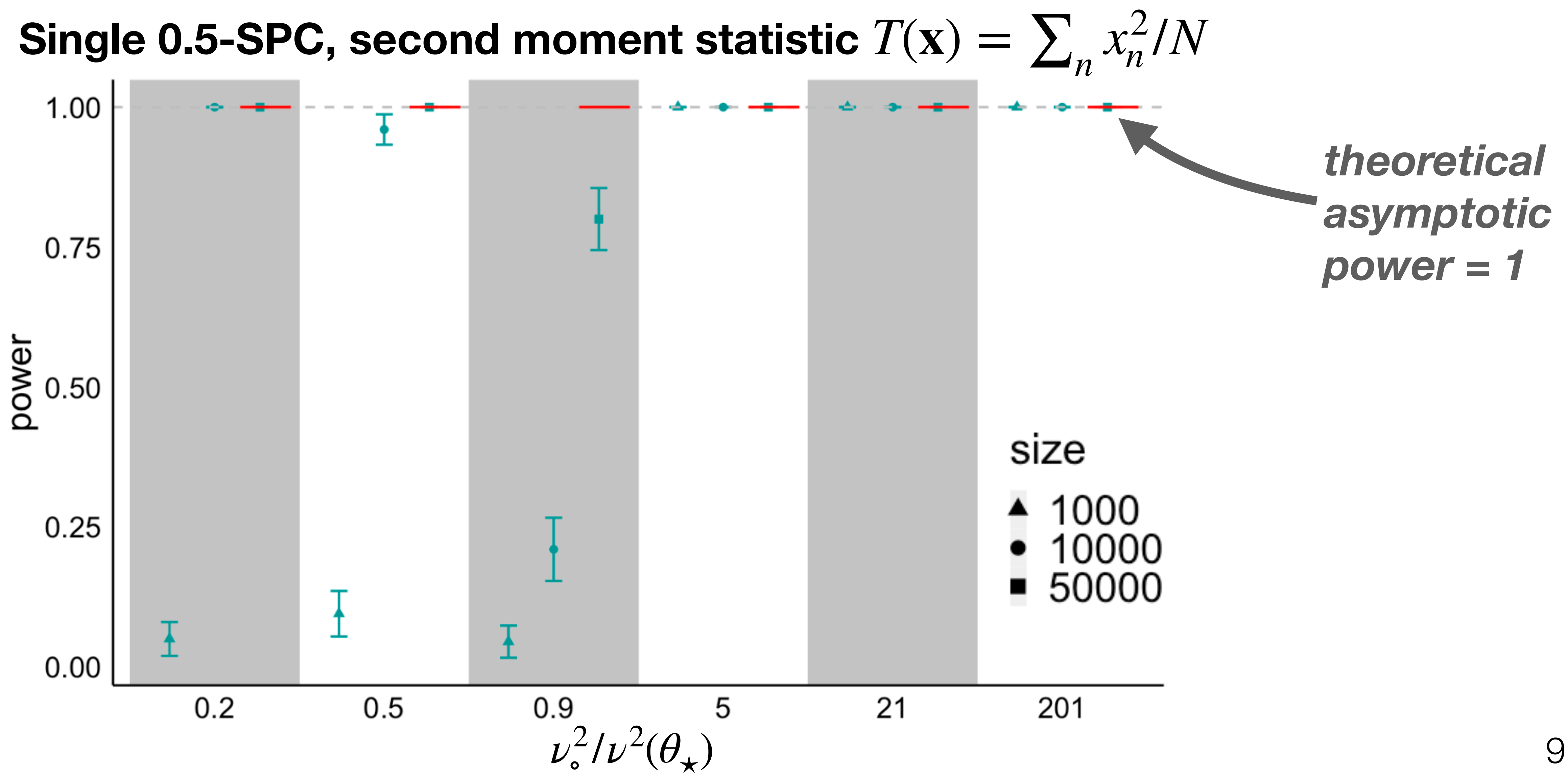

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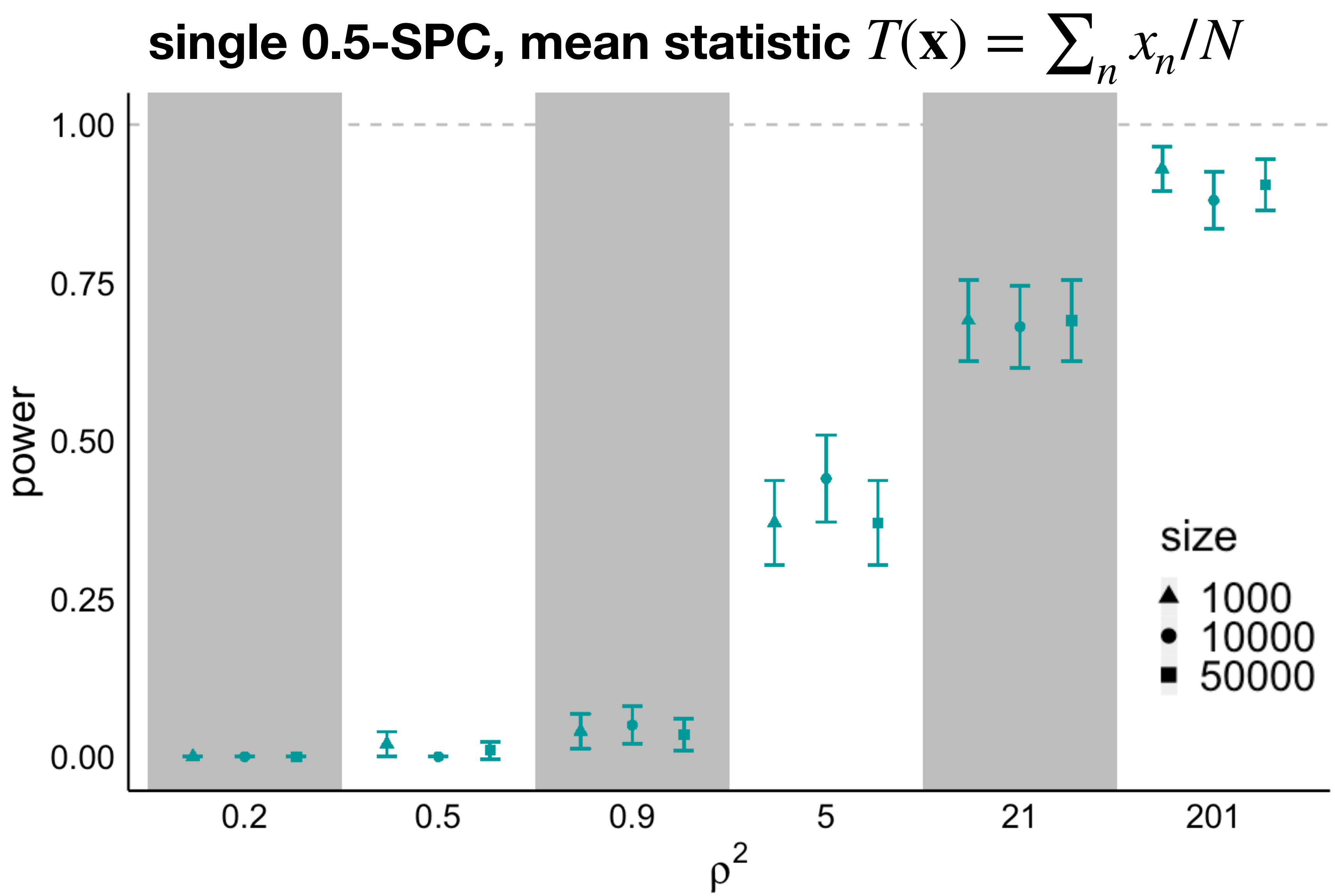

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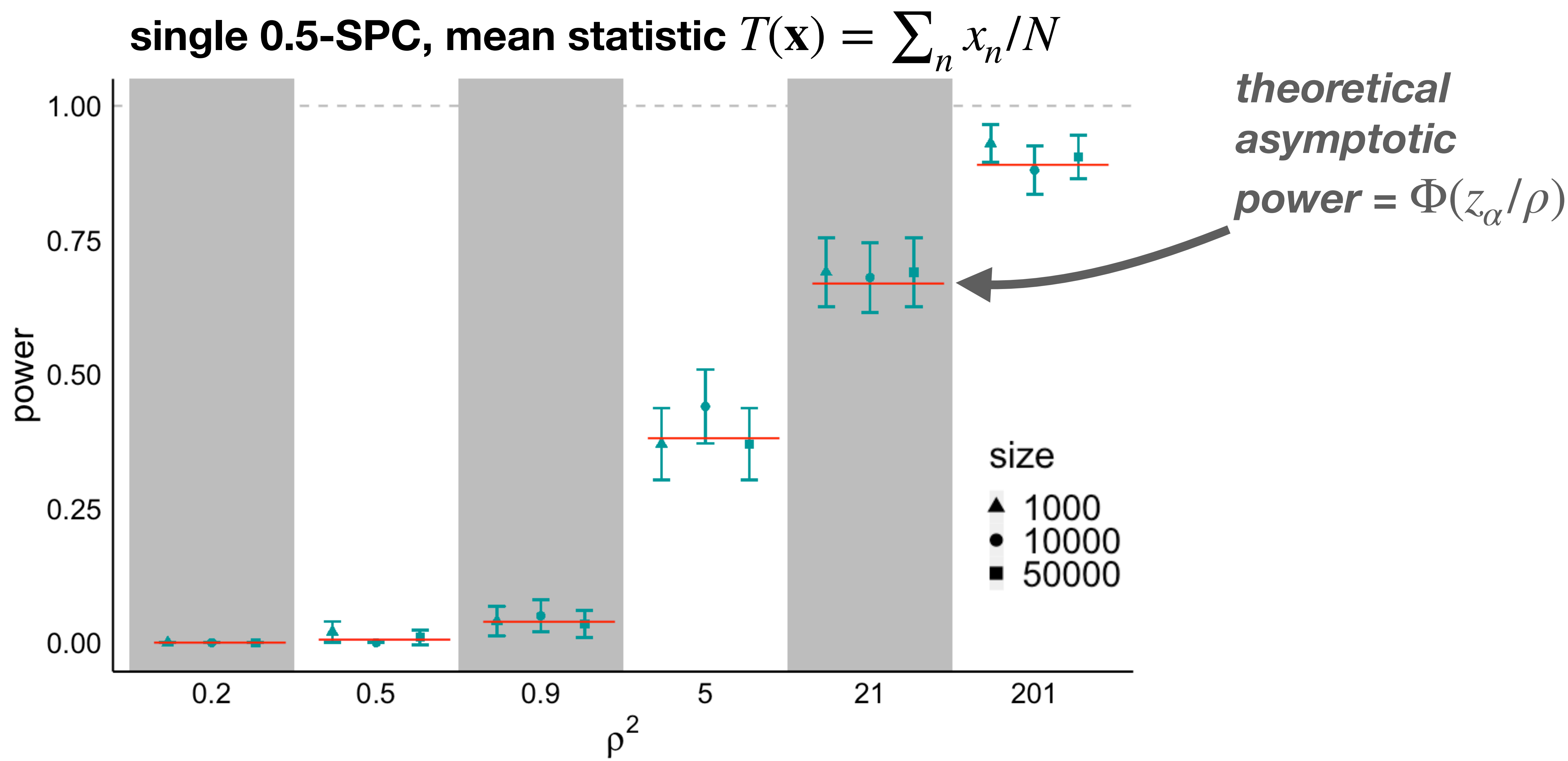

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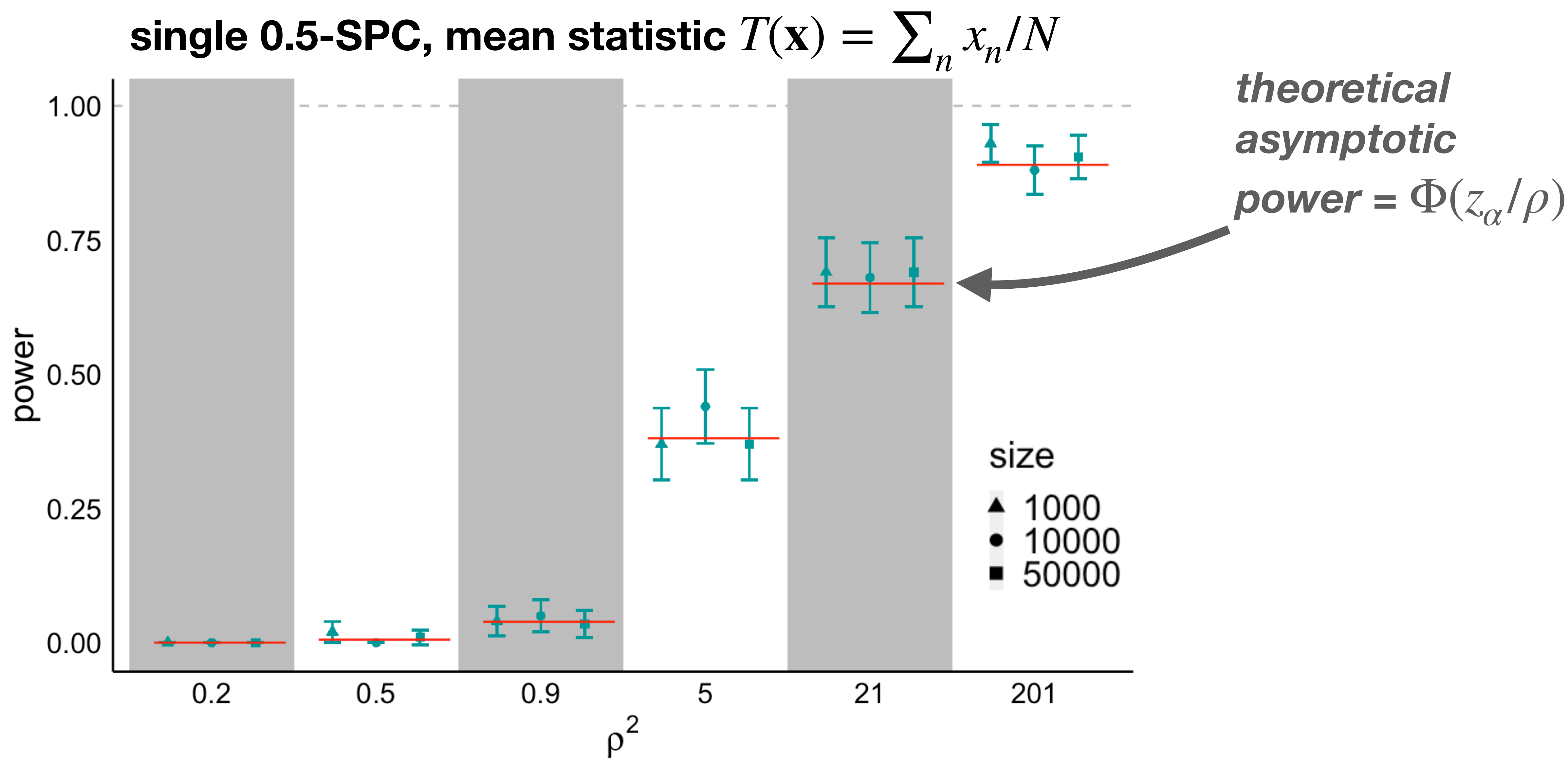

Question 2: Can we improve power to detect subtle misspecification?

An SPC with better power

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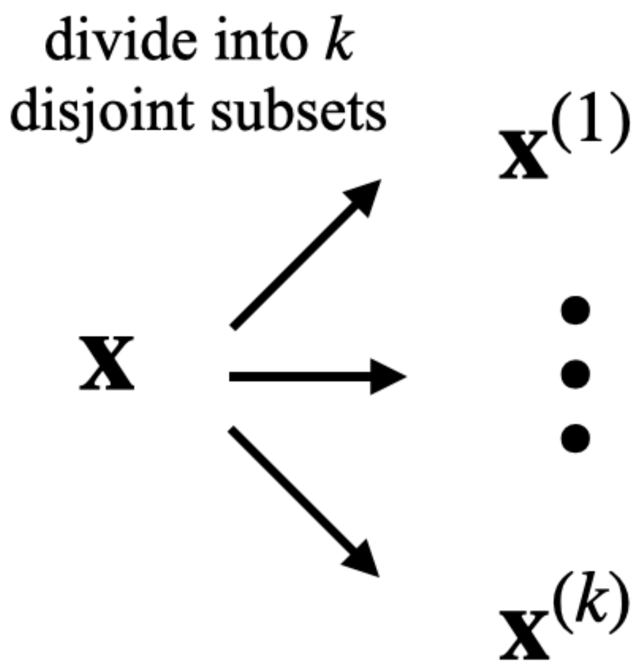
- **Idea:** Test for ***uniformity*** of single SPC p-values

Divided SPCs

Divided SPCs

$\mathbf{x}$

Divided SPCs



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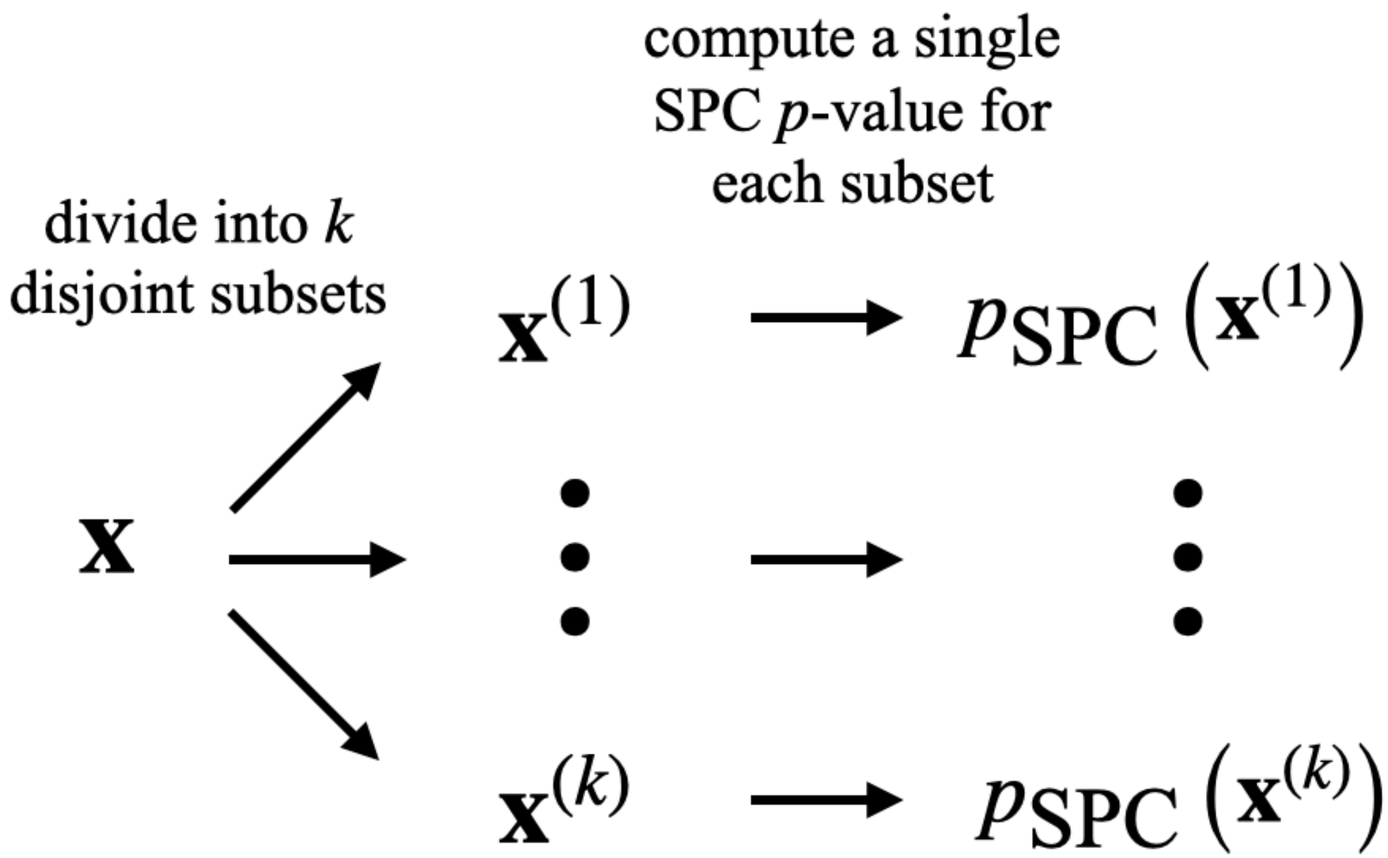

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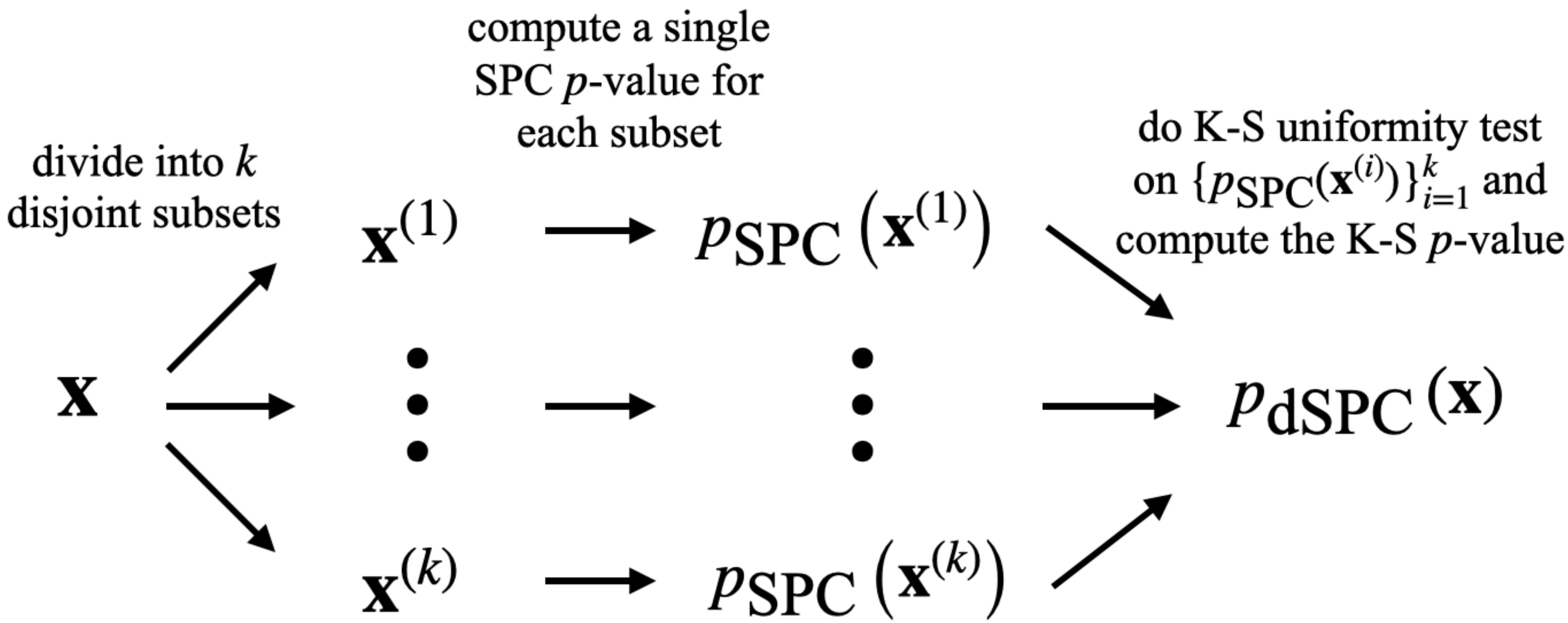

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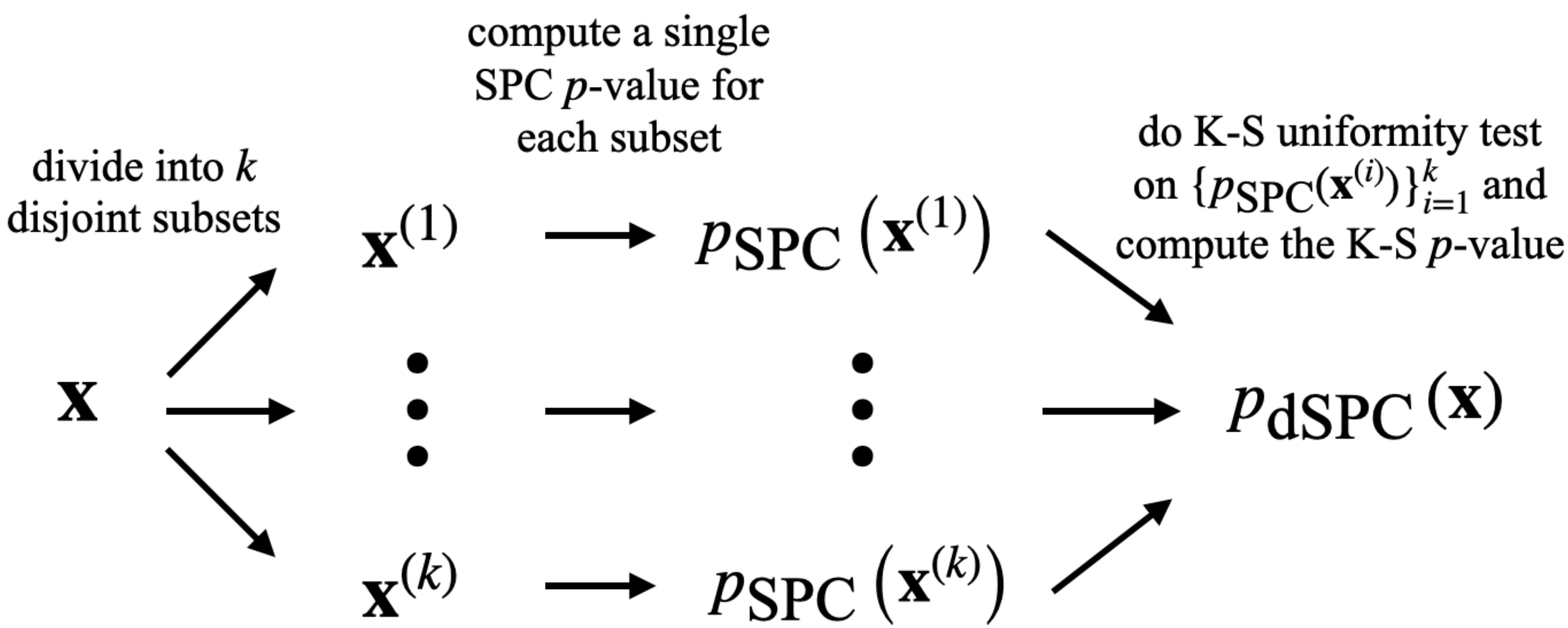

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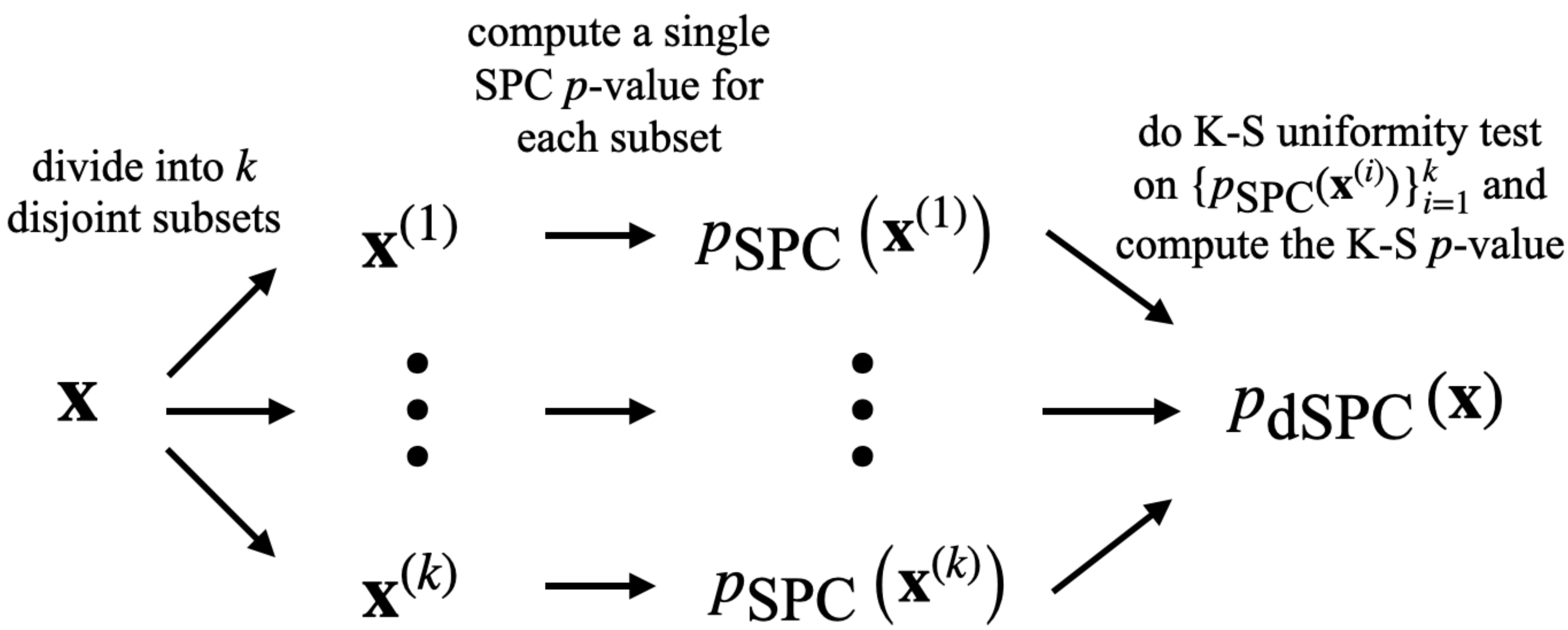

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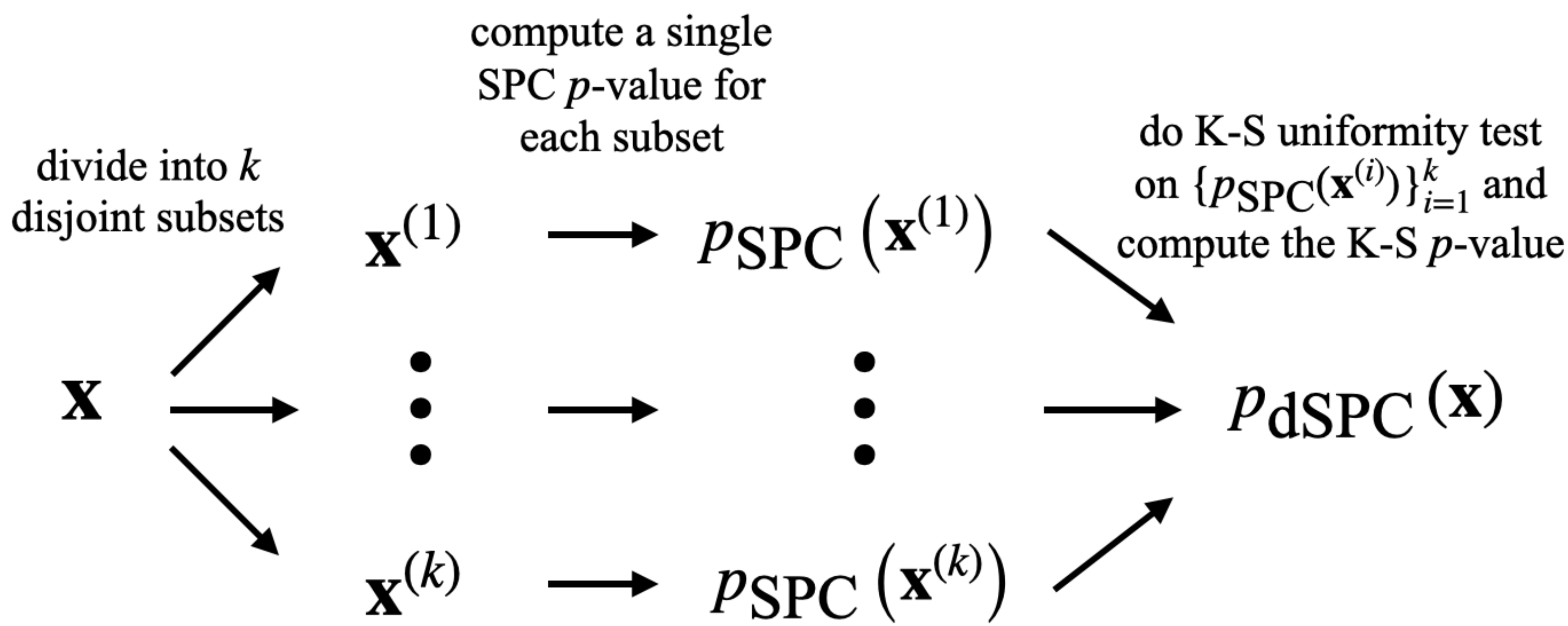

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Question 3: How is calibration and power of divided SPCs?

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- In practice, we set $k = N^{0.49}$

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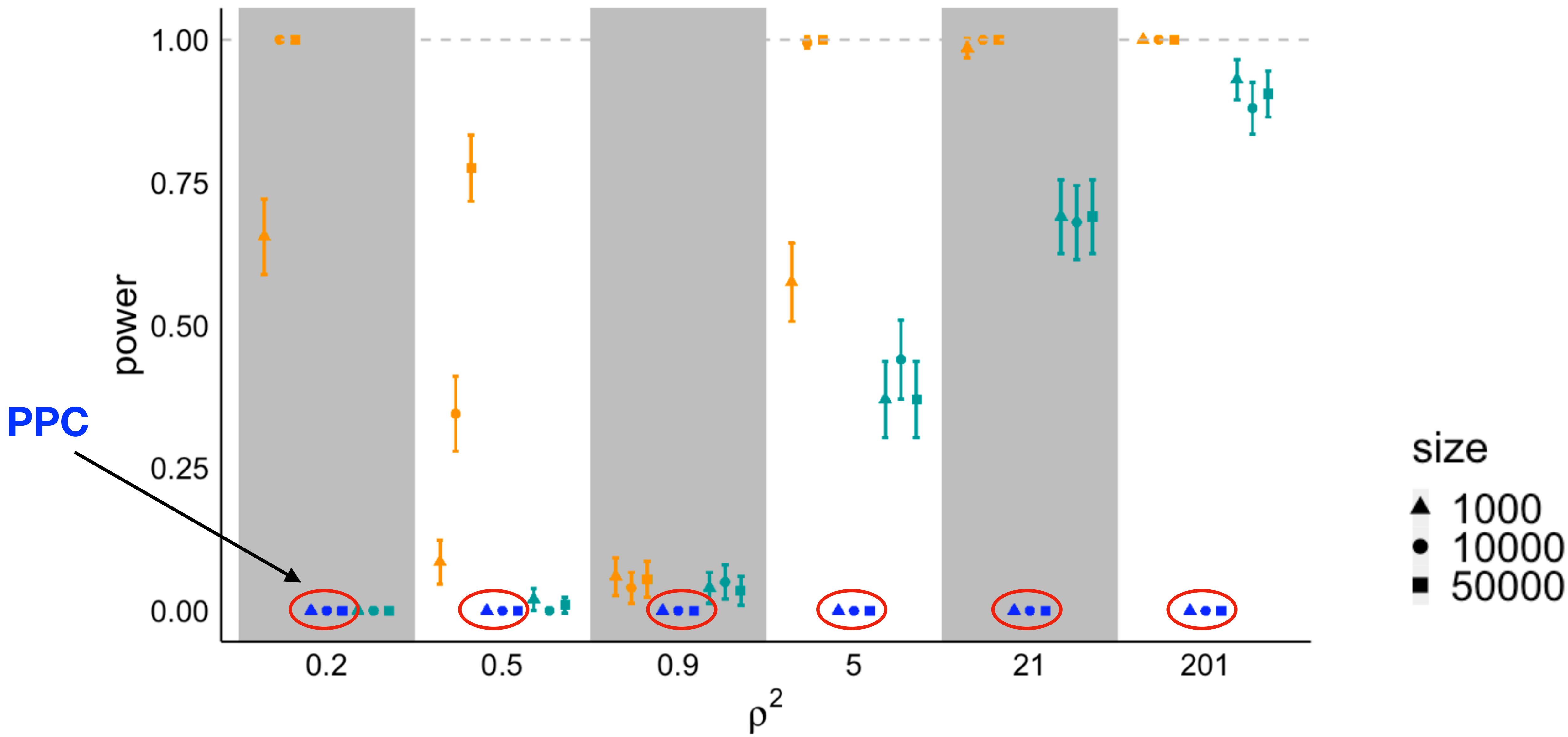
Divided SPCs have superior power



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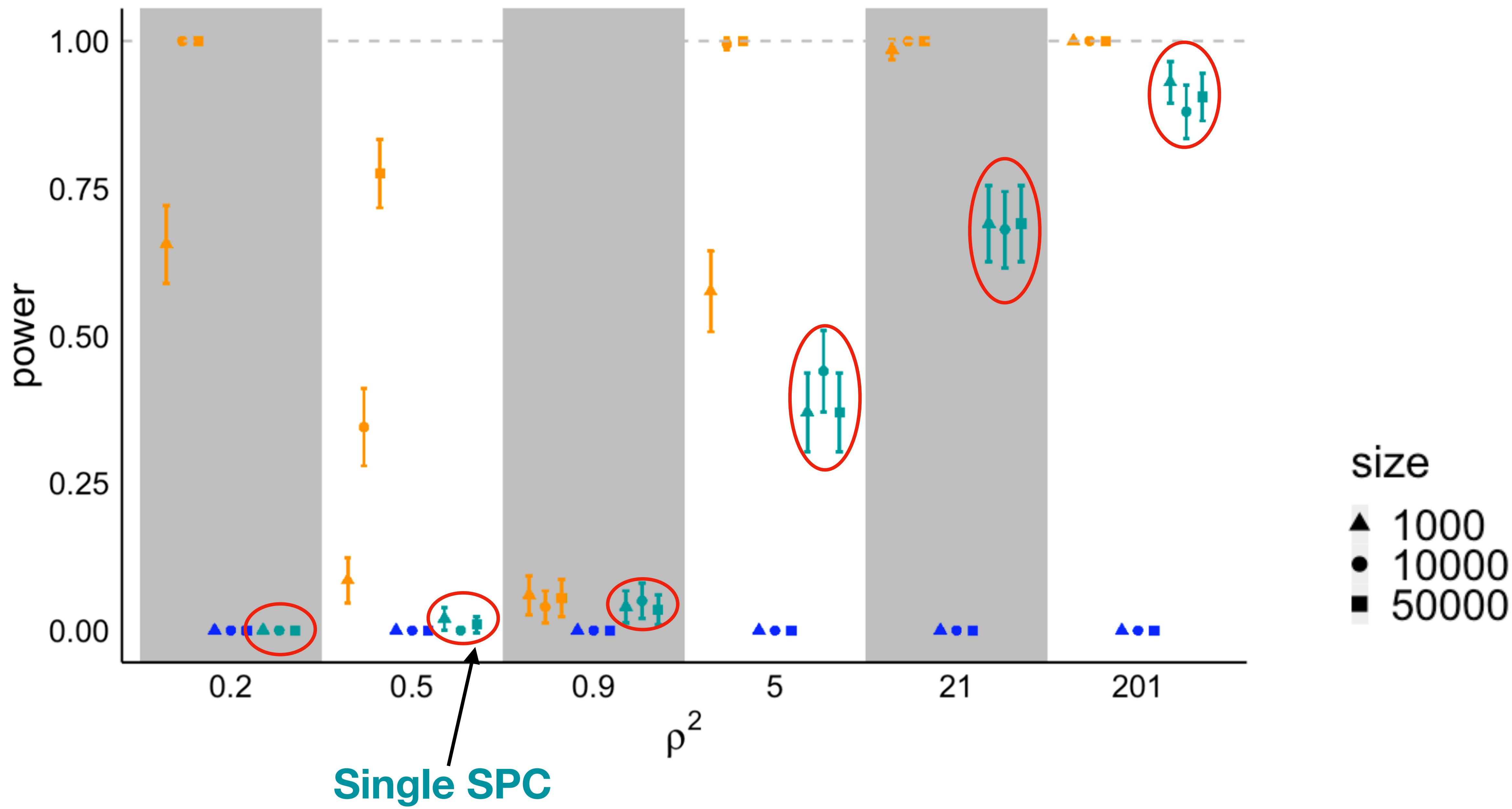

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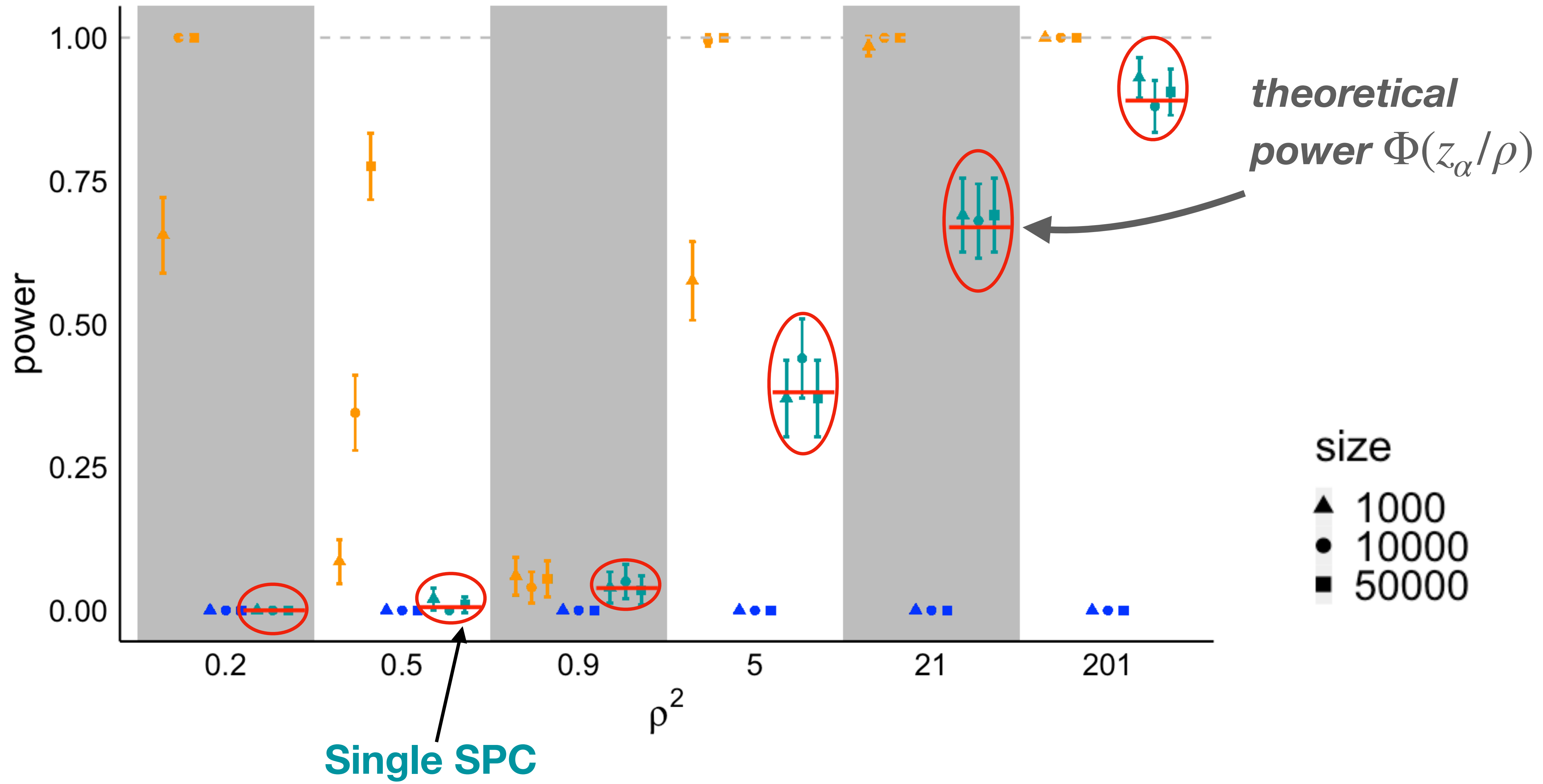

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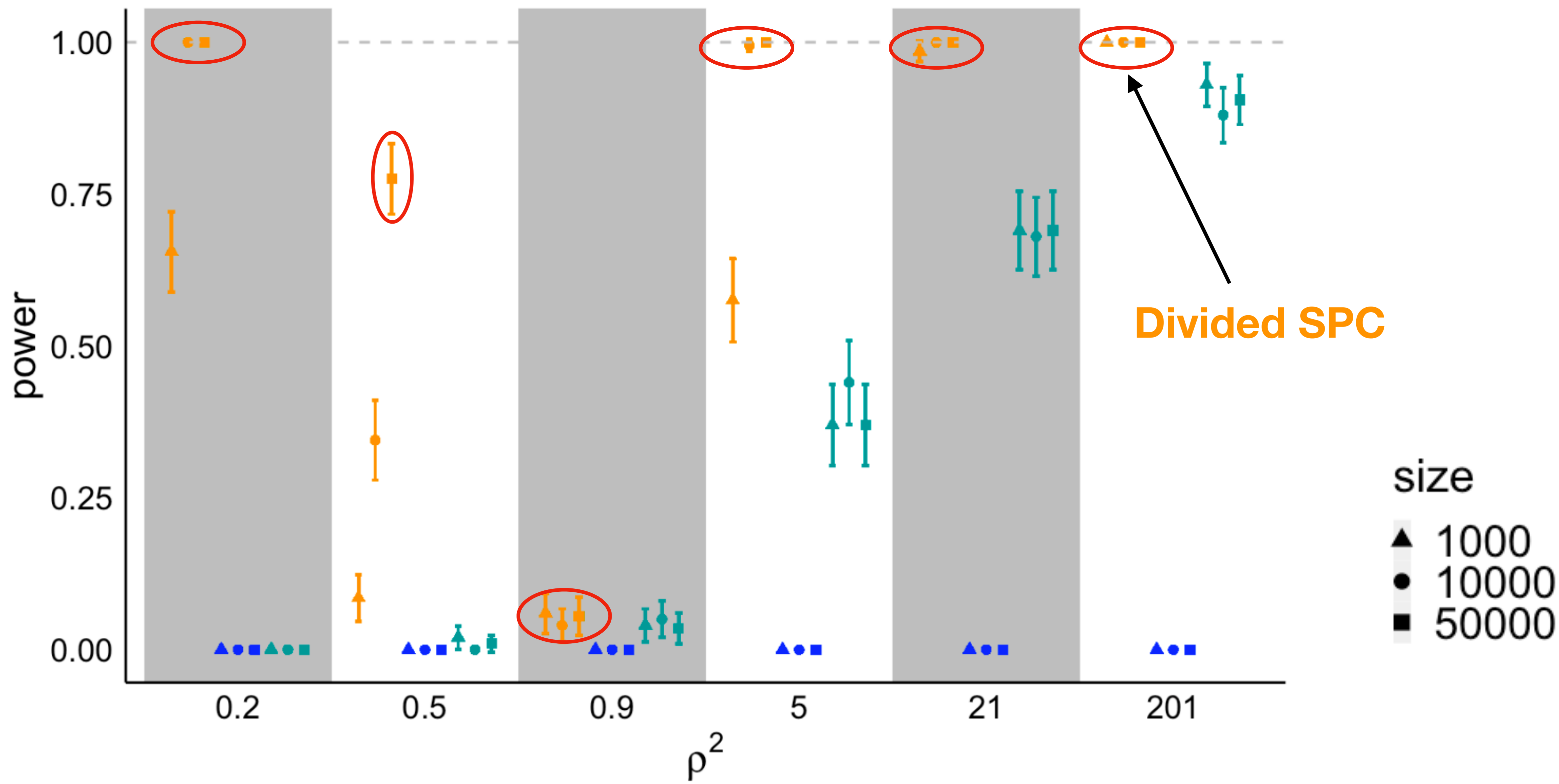

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- Consider divided SPCs if single SPCs inconclusive

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explain data well?

$$x_1 \quad x_2 \quad \ldots \quad x_i \quad x_{i+1} \quad \ldots \quad x_{N-1} \quad x_N$$

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Methods	P-values
PPC	0.132
Interpolated single 0.5-SPC	0.626
Extrapolated single 0.5-SPC	0.000
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model ***does not predict*** *future well*

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model fits data well

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- Different SPCs offer different (potentially useful!) insights

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J. Li and J. H. Huggins (2022). Calibrated Model Criticism Using Split Predictive Checks. *arXiv:2203.15897 [stat.ME]*

read the paper

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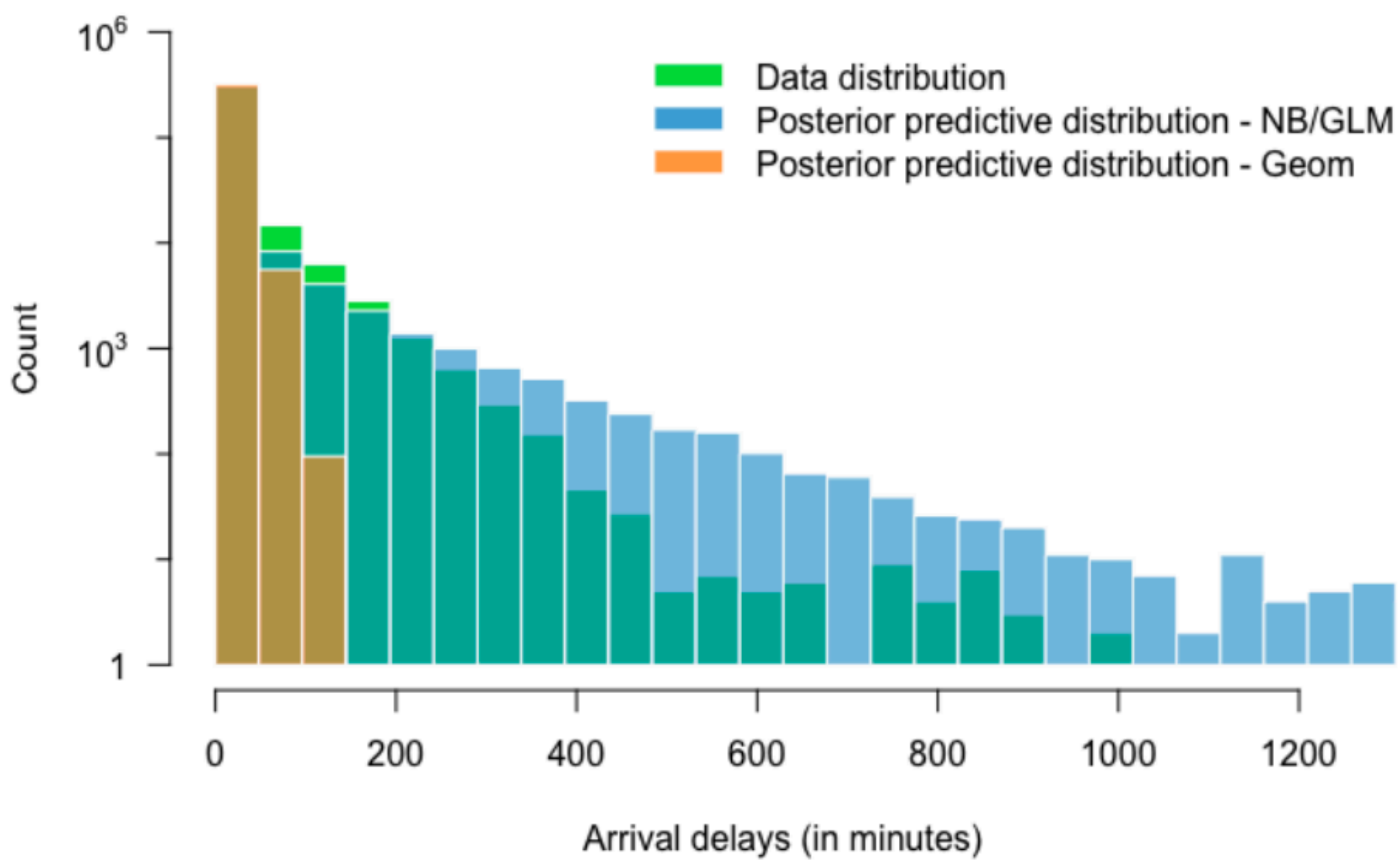

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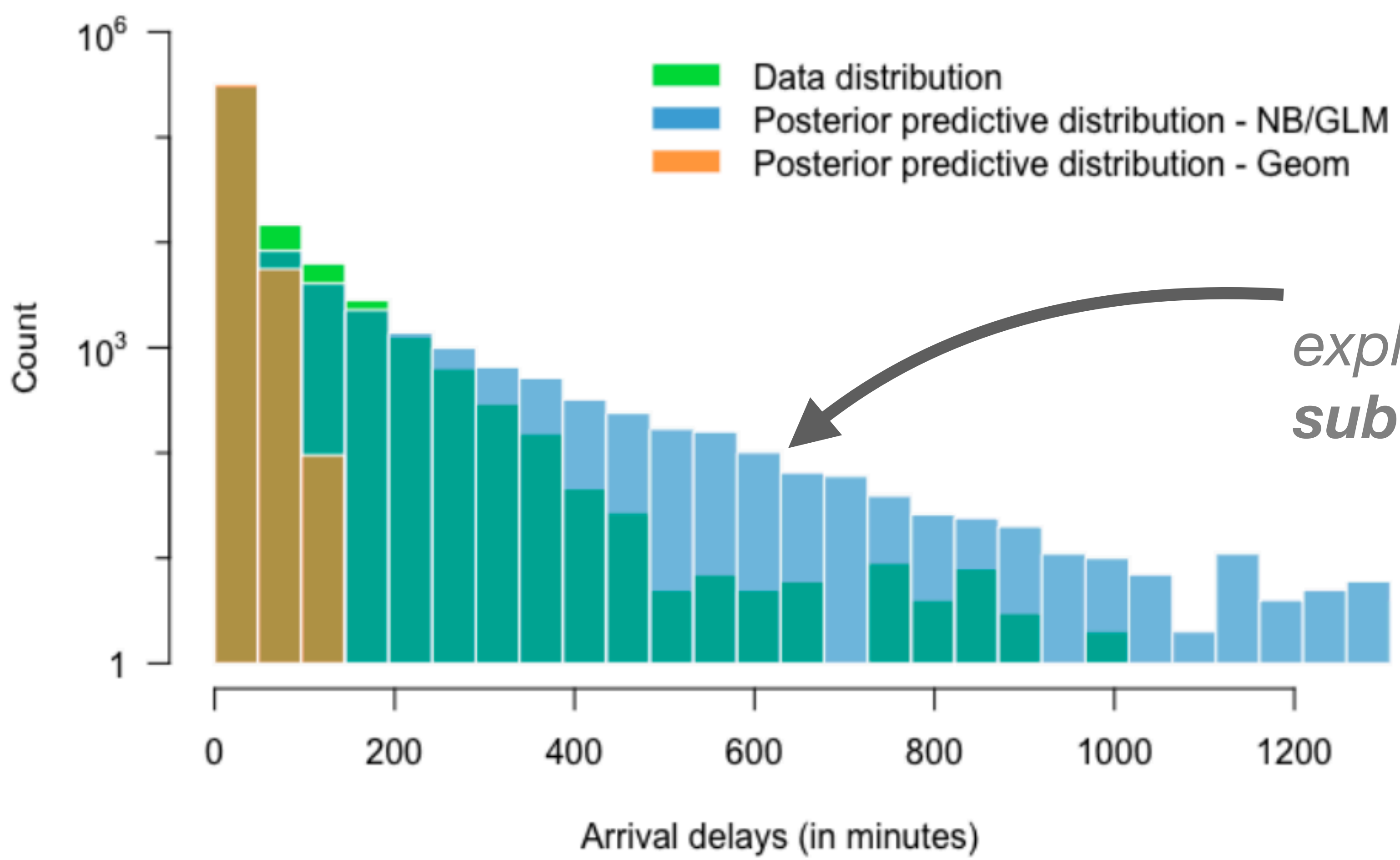

explains the data fairly well
subtle misspecification!

How to "split" in divided SPC with time-series data?

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$$
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\downarrow & & & \downarrow & & & \downarrow \\
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\end{array}
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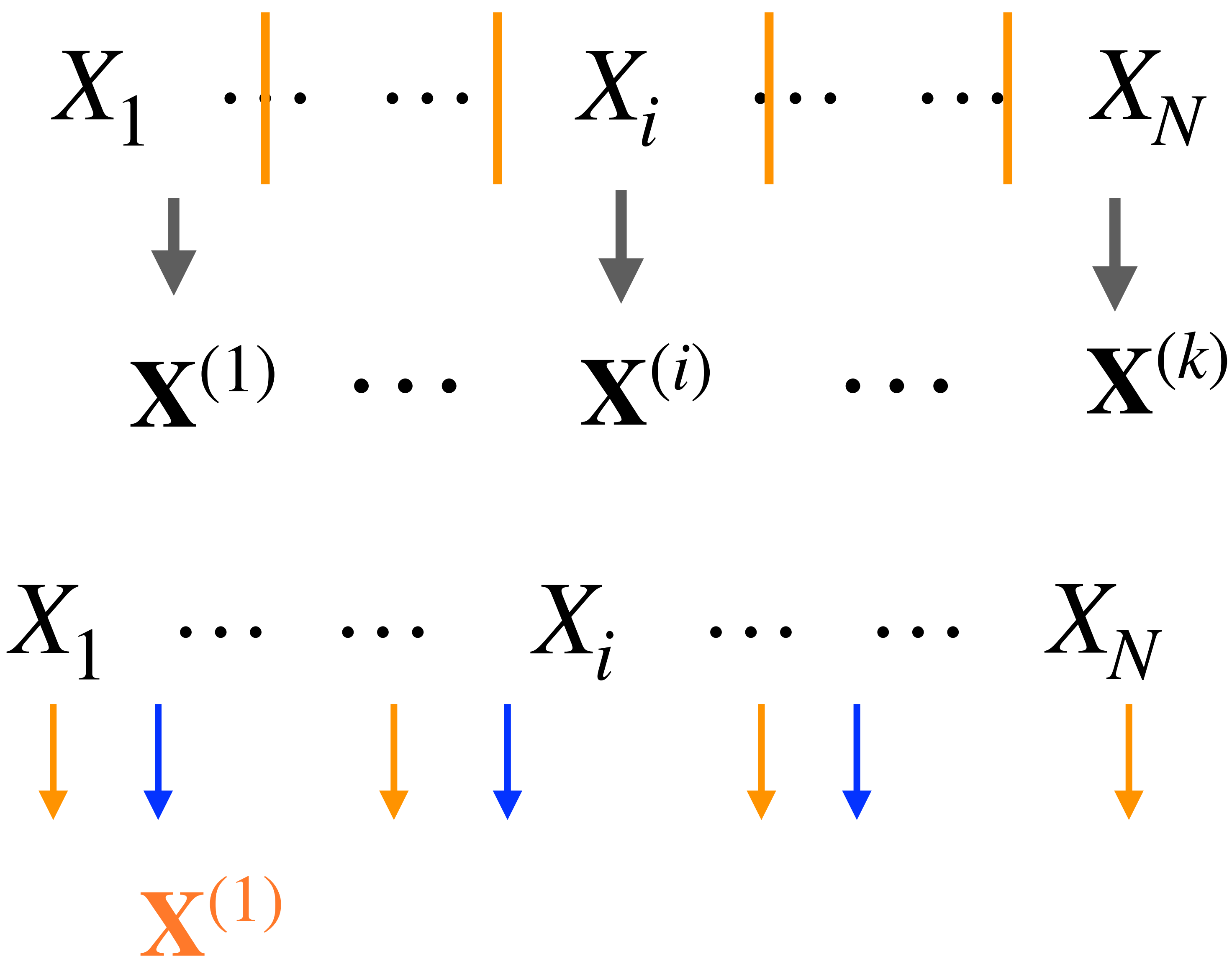

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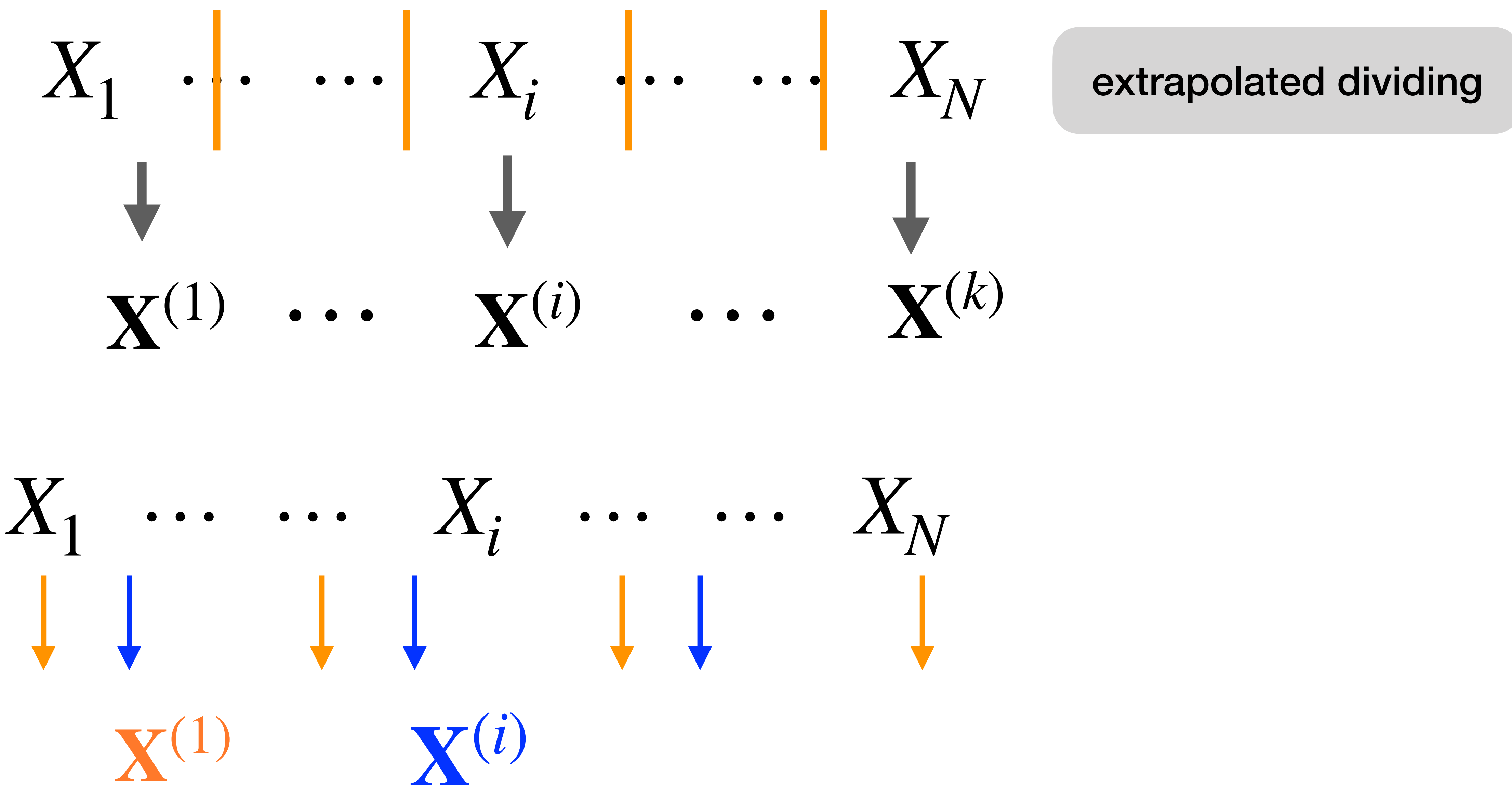

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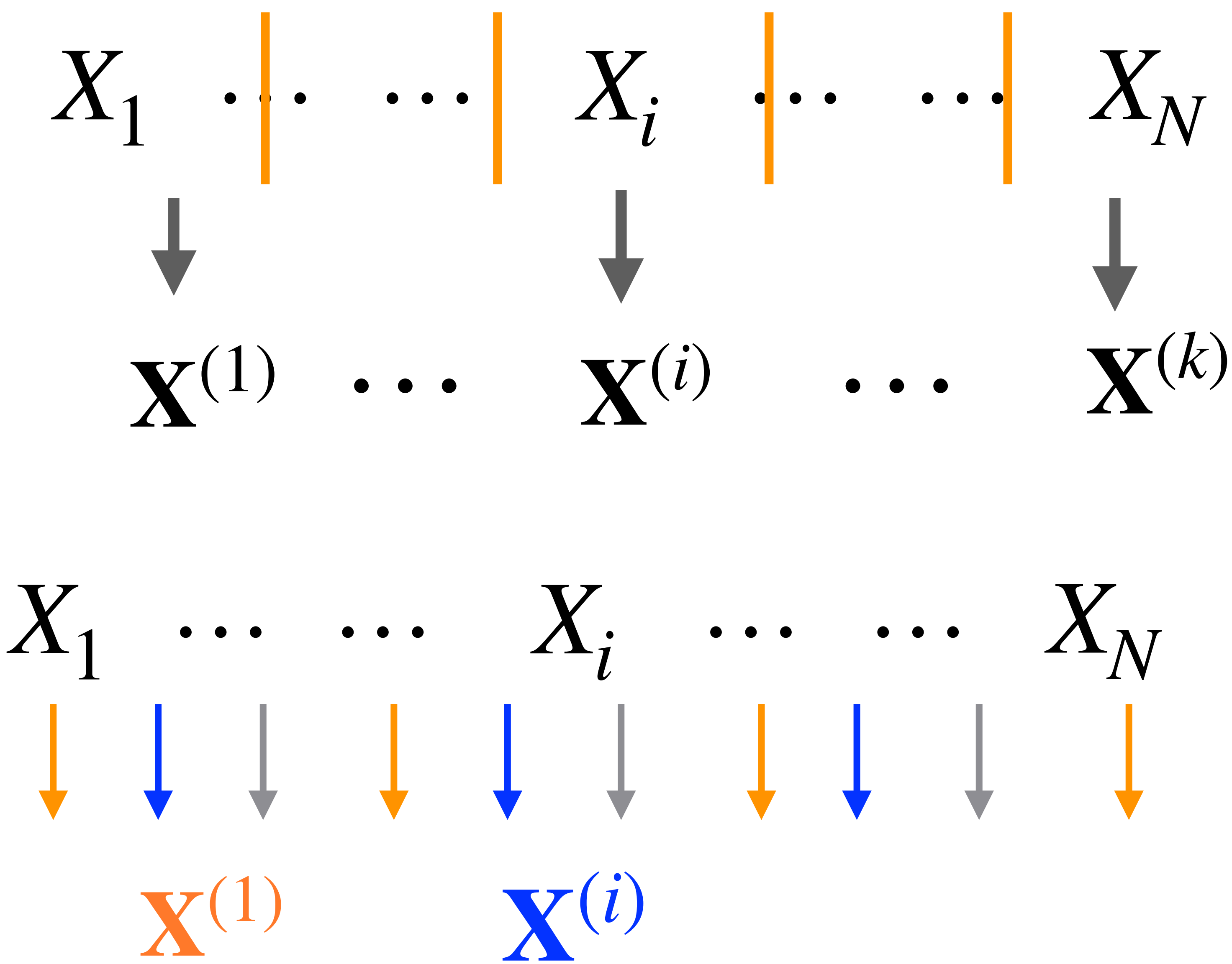

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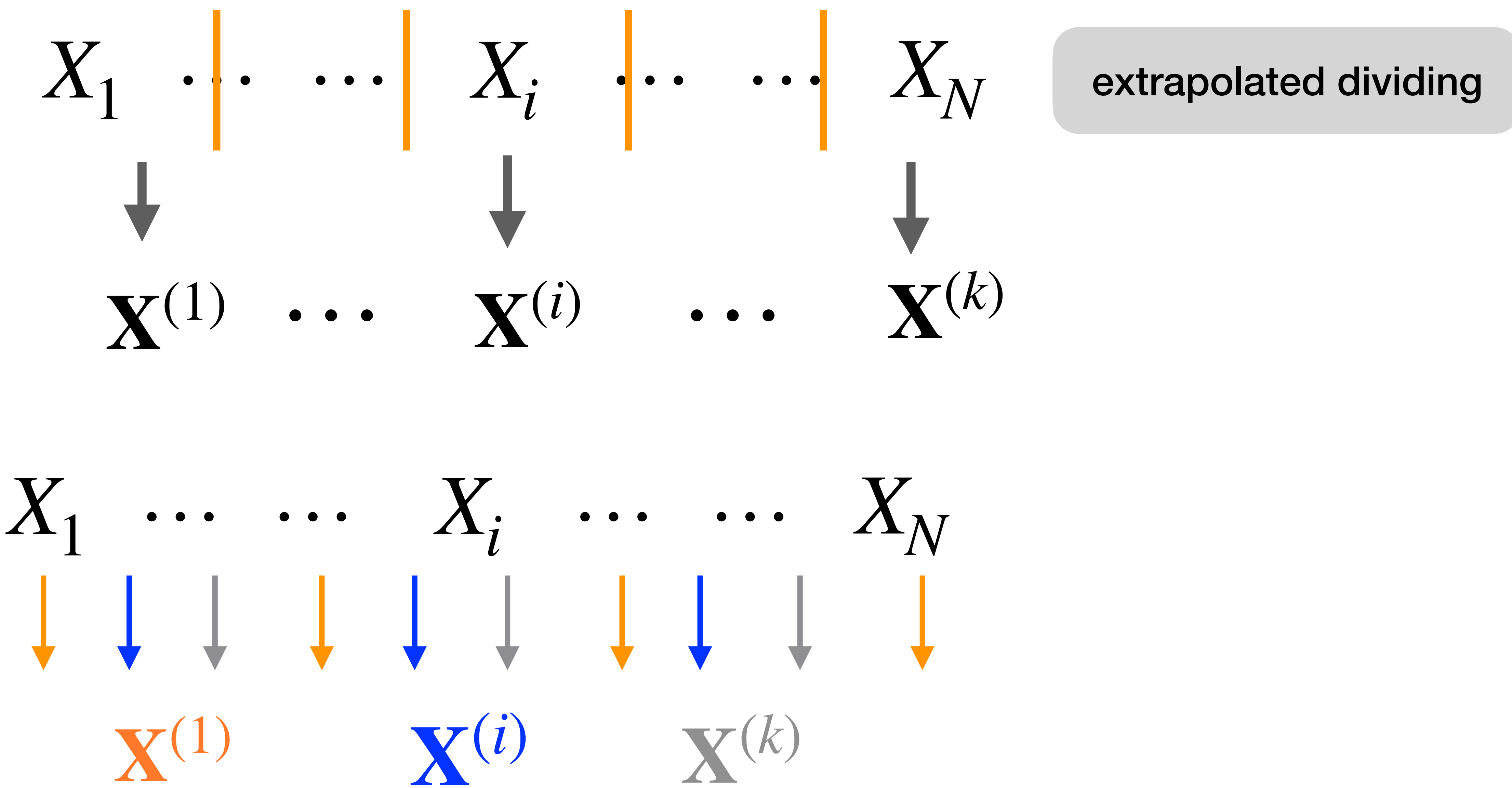

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