Logic for Computer Science

Exercises, tutorial 4

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1. Simulate the CDCL algorithm on the following set of clauses. Take as conflict clauses the set of the negation of decision literals, and as order the order $p_i < p_j$ if $i < j$ (always choosing positive polarity):

$$
\begin{aligned}
C_1 &= \neg p_2 \vee \neg p_3 \vee \neg p_4 \vee p_5 \\
C_2 &= \neg p_1 \vee \neg p_5 \vee p_6 \\
C_3 &= \neg p_5 \vee p_7 \\
C_4 &= \neg p_1 \vee \neg p_6 \vee \neg p_7 \\
C_5 &= \neg p_1 \vee \neg p_2 \vee p_5 \\
C_6 &= \neg p_1 \vee \neg p_3 \vee p_5 \\
C_7 &= \neg p_1 \vee \neg p_4 \vee p_5 \\
C_8 &= \neg p_1 \vee p_2 \vee p_3 \vee p_4 \vee p_5 \vee \neg p_6
\end{aligned}
$$

Solution: Below are the successive queues before conflict. Boxed literals are decisions, propagations are followed by propagating clause

$$\boxed{p_1}, \boxed{p_2}, p_5(C_5), p_6(C_2), p_7(C_3), \neg p_7(C_4)$$

There is a conflict. Resolving C_4, C_3, C_2, C_5, we add

$$C_9 = \neg p_1 \vee \neg p_2$$

(alternatives are $\neg p_1 \vee \neg p_5$, which is another UIP, actually the FUIP; FUIP are usually better in practice) and backtrack to the point where this clause is propagating

$$\boxed{p_1}, \neg p_2(C_9), \boxed{p_3}, p_5(C_6), p_6(C_2), p_7(C_3), \neg p_7(C_4)$$

There is a conflict. Resolving C_4, C_3, C_2, C_6, we add

$$C_{10} \quad = \quad \neg p_1 \vee \neg p_3$$

and backtrack to the point where this clause is propagating

$$\boxed{p_1}, \neg p_2(C_9), \neg p_3(C_{10}), \boxed{p_4}, p_5(C_7), p_6(C_2), p_7(C_3), \neg p_7(C_4)$$

There is a conflict. We add

$$C_{11} \quad = \quad \neg p_1 \vee \neg p_4$$

and backtrack to the point where this clause is propagating

$$\boxed{p_1}, \neg p_2(C_9), \neg p_3(C_{10}), \neg p_4(C_{11}), \boxed{p_5}, p_6(C_2), p_7(C_3), \neg p_7(C_4)$$

There is a conflict. We add

$$C_{12} \quad = \quad \neg p_1 \vee \neg p_5$$

and backtrack to the point where this clause is propagating

$$\boxed{p_1}, \neg p_2(C_9), \neg p_3(C_{10}), \neg p_4(C_{11}), \neg p_5(C_{12}), \neg p_6(C_8), \boxed{p_7}$$

which provides a model.

2. Find the logical consequences, if any, between the formulas in each pair below:

1. $\forall x\, p(x) \vee \forall x\, q(x)$ and $\forall x\, (p(x) \vee q(x))$
2. $p(x)$ and $\exists x\, p(x)$
3. $p(x)$ and $\forall x\, p(x)$
4. $\forall x\, p(x) \wedge \forall x\, q(x)$ and $\forall x\, [p(x) \wedge q(x)]$
5. $\forall x\, \forall y\, p(x, y)$ and $\forall x\, \forall y\, p(y, x)$
6. $\forall x\, (p(x) \Rightarrow q(x))$ and $\forall x\, p(x) \Rightarrow \forall x\, q(x)$

Solution: Let's solve this using the definition of logical consequence.

1. Every interpretation that makes the formula $\forall x\, p(x) \vee \forall x\, q(x)$ true (i.e., every model) also makes either $\forall x\, p(x)$ or $\forall x\, q(x)$ true. An interpretation that makes $\forall x\, p(x)$ true, makes also $\forall x\, (p(x) \vee q(x))$ true, since each model of $p(x)$ is a model of $p(x) \vee q(x)$. The case for a model of $\forall x\, q(x)$ is similar. Thus

$$\forall x\, p(x) \vee \forall x\, q(x) \models \forall x\, (p(x) \vee q(x))\,.$$

   It remains to check if the two formulas are logically equivalent, i.e., if the reverse logical implication holds:

$$\forall x\, (p(x) \vee q(x)) \overset{?}{\models} \forall x\, p(x) \vee \forall x\, q(x)$$

   We guess that $\forall x\, (p(x) \vee q(x))$ is weaker than $\forall x\, p(x) \vee \forall x\, q(x)$. Indeed, the interpretation $\mathcal{I}\,(D, I_c, I_v)$ with D being the set of natural numbers $\mathbb{N}$, $I_c(p)$ being the predicate "is_even", and $I_c(q)$ being the predicate "is_odd", is a model of $\forall x\, (p(x) \vee q(x))$ since every natural number is either even or odd, but makes formula $\forall x\, p(x) \vee \forall x\, q(x)$ false, since it is not true that all natural numbers are even, and it it not true that all natural numbers are even.

2. Consider a model $\mathcal{I}\,(D, I_c, I_v)$ of $p(x)$. Since $d = I_v[x]$ is an element of D such that $I_c[p](d)$ is true. Thus $\mathcal{I}_{x/d}[p(x)] = T$. According to the definition of the syntax for existential quantifiers, $\mathcal{I}[\exists x\, p(x)] = T$. Thus $p(x) \models \exists x\, p(x)$ holds.

   The reverse logical consequence does not. Indeed, consider now $\mathcal{I}\,(D, I_c, I_v)$ with $D = \{d_1, d_2\}$, $\mathcal{I}[p](a) = T$, $\mathcal{I}[p](b) = F$, $\mathcal{I}[x] = b$. This interpretation satisfies $\exists x\, p(x)$ but is not a model of $p(x)$.

3. $\forall x\, p(x) \models p(x)$

4. $\forall x\, p(x) \wedge \forall x\, q(x) \longleftrightarrow \forall x\, [p(x) \wedge q(x)]$

5. $\forall x\, \forall y\, p(x, y) \longleftrightarrow \forall x\, \forall y\, p(y, x)$

6. $\forall x\, (p(x) \Rightarrow q(x)) \models \forall x\, p(x) \Rightarrow \forall x\, q(x)$

3. Express this well-know argument using logic:

- All humans are mortals
- Socrates is human

- Hence, Socrates is mortal

Solution: Let's define the predicates

- $h(x)$ true if x is human;
- $m(x)$ true if x is mortal.

The two hypotheses are

- $\forall x \,.\, h(x) \Rightarrow m(x)$
- $h(\text{Socrates})$

Hence ($\models$) $m(\text{Socrates})$.