Logic for Computer Science

Exercises, tutorial 7

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1. What is the logical link between the following formulas?

1. $A =_{\text{def}} \forall x \exists y\, [P(x) \Rightarrow Q(x,y)]$
2. $B =_{\text{def}} \forall x\, [P(x) \Rightarrow \exists y\, Q(x,y)]$
3. $C =_{\text{def}} \forall x P(x) \Rightarrow \exists y\, Q(x,y)$
4. $D =_{\text{def}} \forall x\, [P(x) \Rightarrow \forall x \exists y\; Q(x,y)]$

Solution: It is useful to notice upfront that the first two formulas are logically equivalent. Indeed:

$$\begin{aligned}
\forall x \exists y\, [P(x) \Rightarrow Q(x,y)] &\longleftrightarrow \forall x \exists y\, [\neg P(x) \vee Q(x,y)] \\
&\longleftrightarrow \forall x\, [\neg P(x) \vee \exists y\, Q(x,y)] \\
&\longleftrightarrow \forall x\, [P(x) \Rightarrow \exists y\, Q(x,y)]
\end{aligned}$$

where the second logical equivalence stands thanks to the fact that y does not occur in $\neg P(x)$.

It is furthermore possible to show (by case analysis on the various possible interpretations) that

$$\forall x\, [P(x) \Rightarrow \exists y\, Q(x,y)]\,, \forall x P(x) \models \forall x \exists y\, Q(x,y)$$

and in particular that

$$\forall x\, [P(x) \Rightarrow \exists y\, Q(x,y)]\,, \forall x P(x) \models \exists y\, Q(x,y)$$

and finally

$$\forall x\, [P(x) \Rightarrow \exists y\, Q(x,y)] \models \forall x P(x) \Rightarrow \exists y\, Q(x,y)$$

therefore $B \models C$. However $C \not\models B$: an interpretation $\mathcal{I} = (\mathcal{D}, I_c, I_v)$ with

- $\mathcal{D} = \{a, b\}$;
- $I_c[P](a) = I_c[Q](a, a) = T$;
- $I_c[P](b) = I_c[Q](b, a) = I_c[Q](b, b) = F$;

makes C true but does not make B true.

Then,

$$D \longleftrightarrow \forall x \neg P(x) \vee \forall x \exists y\, Q(x, y)$$

and

$$\forall x\, \varphi(x) \vee \forall x\, \psi(x) \models \forall x\, [\varphi(x) \vee \psi(x)]\,,$$

whatever $\varphi(x)$ and $\psi(x)$ are (this is left as an exercise). Therefore

$$D \models \forall x\, [\neg P(x) \vee \exists y\, Q(x, y)]$$

and this last formula is logically equivalent to B. To conclude $D \models B$, and, since $B \models C$, therefore C is also a logical consequence of D.

Last, $C \not\models D$: since $D \models A$ if $C \models D$, we must also have $C \models A$ and we previously showed that it is not the case. Furthermore $A \not\models D$: an interpretation $\mathcal{I} = (\mathcal{D}, I_c, I_v)$ with

- $\mathcal{D} = \{a, b\}$;
- $I_c[P](a) = I_c[Q](a, a) = T$;
- $I_c[P](b) = I_c[Q](b, a) = I_c[Q](b, b) = F$;

makes A true but not D.

	A, B	C	D
A, B	$\longleftrightarrow$	$\models$	-
C	-	$\longleftrightarrow$	-
D	$\models$	$\models$	$\longleftrightarrow$

2. Consider the following inference rule. Is it correct? Use Herbrand theory.

$$\frac{\exists x\, [p(x) \vee q(x)]\,, \quad \forall x\, [p(x) \Rightarrow q(x)]}{\exists x\, p(x) \Rightarrow \exists x\, q(x)}$$

Solution: The rule is correct if and only if the following set of formulas is unsatisfiable:

$$\Big\{\exists x\, [p(x) \vee q(x)]\,, \forall x\, [p(x) \Rightarrow q(x)]\,, \neg[\exists x\, p(x) \Rightarrow \exists x\, q(x)]\Big\}$$

To put these formulas in Skolem form, we first need to put them in prenex form. Only the last formula is not yet in prenex form:

$$
\begin{aligned}
\neg[\exists x\, p(x) \Rightarrow \exists x\, q(x)] &\longleftrightarrow \neg[\exists x\, p(x) \Rightarrow \exists y\, q(y)] & (1)\\
&\longleftrightarrow \neg[\neg\exists x\, p(x) \vee \exists y\, q(y)] & (2)\\
&\longleftrightarrow \neg[\forall x\, \neg p(x) \vee \exists y\, q(y)] & (3)\\
&\longleftrightarrow \neg\forall x\, .\, \neg p(x) \vee \exists y\, q(y) & (4)\\
&\longleftrightarrow \exists x\, .\, \neg[\neg p(x) \vee \exists y\, q(y)] & (5)\\
&\longleftrightarrow \exists x\, .\, p(x) \wedge \neg\exists y\, q(y) & (6)\\
&\longleftrightarrow \exists x\, .\, p(x) \wedge \forall y\, \neg q(y) & (7)\\
&\longleftrightarrow \exists x \forall y\, .\, p(x) \wedge \neg q(y) & (8)
\end{aligned}
$$

where (1) comes from α-renaming (renaming quantified variables), (2) and (6) come from propositional reasoning, (3), (5) and (7) come from permutation of the negation and a quantifier ($\neg\exists x\varphi(x) \longleftrightarrow \forall x\neg\varphi(x)$ and $\neg\forall x\varphi(x) \longleftrightarrow \exists x\neg\varphi(x)$), (4) and (8) come from the fact that one can put in prefix a quantifier in a conjunction or disjunction if the rest of the formula does not contain the variable, e.g., $\forall x\, \alpha(x) \vee \beta \longleftrightarrow \forall x\, .\, \alpha(x) \vee \beta$ if x does not occur free in β.

So the above set is logically equivalent to

$$\Big\{\exists x\, [p(x) \vee q(x)]\, ,\ \forall x\, [p(x) \Rightarrow q(x)]\, ,\ \exists x \forall y\, .\, p(x) \wedge \neg q(y)\Big\}$$

After Skolemization, we obtain the equisatisfiable set of formulas

$$\Big\{p(a) \vee q(a),\ \forall x\, [p(x) \Rightarrow q(x)]\, ,\ \forall y\, .\, p(b) \wedge \neg q(y)\Big\}$$

which is obviously unsatisfiable because the set of instances

$$\Big\{p(b) \Rightarrow q(b),\ p(b) \wedge \neg q(b)\Big\}$$

is itself unsatisfiable. Actually, only the second premise is used to prove unsatisfiability, therefore the rule could be strengthened to

$$\frac{\forall x\, [p(x) \Rightarrow q(x)]}{\exists x\, p(x) \Rightarrow \exists x\, q(x)}$$

which is also a correct rule.