Logic for Computer Science

Exercises, tutorial 1

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1. Build the truth table for formula

$$G =_{\mathrm{def}} (p \Rightarrow q) \Rightarrow [(\neg p \Rightarrow q) \Rightarrow q] .$$

What can we conclude from the truth table?

Solution: The formula is valid.

2. If John hasn't met Peter the other night, this means that Peter is the murderer or that John is a liar. If Peter isn't the murderer, then John hasn't met Peter the other night and the crime happened after midnight. If the crime happened after midnight, then Peter is the murderer or John isn't a liar. Therefore Peter is the murderer.

Is the conclusion really a logical consequence of the facts?

Solution: Let us define the following propositions

r John has met Peter;

m Peter is the murderer;

l John is a liar;

c the crime happened after midnight.

The facts are expressed using the above propositions as follows:

- $G_1 =_{\mathrm{def}} \neg r \Rightarrow (m \vee l)$
- $G_2 =_{\mathrm{def}} \neg m \Rightarrow (\neg r \wedge c)$
- $G_3 =_{\mathrm{def}} c \Rightarrow (m \vee \neg l)$

Let us try to prove that $\{G_1, G_2, G_3\} \models m$. We could use a truth table to show that formula $(G_1 \wedge G_2 \wedge G_3) \Rightarrow m$ is valid (left as an exercise for the reader) or alternatively, that $\{G_1, G_2, G_3, \neg m\}$ is unsatisfiable. We will rather use a less systematic, intuitive approach.

The set $\{G_1, G_2, G_3, \neg m\}$ is unsatisfiable. Indeed an interpretation that makes true $\neg m$ and G_2 must also satisfy the literals $\neg r$ and c. An interpretation satisfying $\neg m$, $\neg r$ and c as well as formula G_3 also satisfies $\neg l$. However, an interpretation satisfying $\neg m$, $\neg r$ and c as well as formula G_1 must satisfy l. There is thus no way to find a model for all formulas in the set $\{G_1, G_2, G_3, \neg m\}$. The conclusion is indeed a logical consequence of the facts. The reasoning in this paragraph mimics a refutation proof by resolution.

3. If Robinson is elected as president, then Smith will be designated as vice-president. If Thompson is elected as president, then Smith will designated be as vice-president. Either Thompson or Robinson will be elected as president. Therefore Smith will be designated as vice-president.

Is this text correct?

4. Consider a set of five propositional variables $P =_{\text{def}} \{a, b, c, d, e\}$.

1. How many formulas, up to logical equivalence, exist that are satisfied by exactly seventeen interpretations?

2. How many formulas, up to logical equivalence, exist that are logical consequence of the formula $a \wedge b$?

Solution: First of all, notice that the number of distinct formulas on a set of propositions P equals the number of sets of subsets of P. Indeed, a formula is perfectly defined, up to a logical equivalence, by its models. And an interpretation is perfectly defined by the set of propositions that are satisfied by the interpretation. There are $2^{|P|}$ interpretations, as many as subsets of P: each proposition in P can be either true (that is, in the subset) or not, thus there are 2 choices for each proposition, and these choices multiply. And there are $2^{(2^{|P|})}$ subsets of a set with $2^{|P|}$ elements, so there are $2^{(2^{|P|})}$ ways to select, among the $2^{|P|}$ interpretations, the ones that will be models for a formula.

In the current case, P contains 5 propositions, there are thus $2^5 = 32$ subsets of P and as many distinct interpretations (that is, lines in the truth table). There exist as many formulas using propositions in P as sets of interpretations, that is $2^{(2^5)}$ distinct formulas (that is formulas that are not logically equivalent).

The number of formulas satisfied by exactly 17 interpretations is the number of sets of interpretations with exactly 17 interpretations, that is,

C_{32}^{17}.

Finally, the logical consequences of $a \wedge b$ are exactly the formulas that are true for all interpretations that make a and b true, that is, 2^3 of them (since c, d, e are free to be either true or false). The truth value for all other $2^5 - 2^3$ interpretations can either be T of F. There are thus $2^{(2^5-2^3)}$ logical consequences of $a \wedge b$.

5. Consider the formulas $p \Rightarrow r$ and $q \Rightarrow s$. Show using resolution that $(p \vee q) \Rightarrow (r \vee s)$ is a logical consequence of the two previous formulas.

Solution: Resolution works on sets of clauses. We would like to prove that $(p \vee q) \Rightarrow (r \vee s)$ is a logical consequence of $p \Rightarrow r$ and $q \Rightarrow s$, or equivalently, that it is not possible to have $(p \vee q) \Rightarrow (r \vee s)$ false when $p \Rightarrow r$ and $q \Rightarrow s$ are true. We would thus like to show that the set

$$\{p \Rightarrow r, q \Rightarrow s, \neg[(p \vee q) \Rightarrow (r \vee s)]\}$$

is unsatisfiable. Formula $p \Rightarrow r$ is actually $\neg p \vee r$ in clausal form, and $q \Rightarrow s$ is $\neg q \vee s$. Let us transform formula $\neg[(p \vee q) \Rightarrow (r \vee s)]$, by distributing the negation to first get $(p \vee q) \wedge \neg(r \vee s)$, then $(p \vee q) \wedge \neg r \wedge \neg s$ in conjunctive normal form, that is, three clauses $p \vee q$, $\neg r$, $\neg s$. We want to show that the set of clauses

$$\{\neg p \vee r, \neg q \vee s, p \vee q, \neg r, \neg s\}$$

is unsatisfiable, which is simple by unit resolution: from $\neg r$ and $\neg p \vee r$ we get $\neg p$, from $\neg s$ and $\neg q \vee s$ we get $\neg q$, and from $\neg p$ and $p \vee q$, we get q. The empty clause is obtained by resolution between $\neg q$ and q.