

Introduction to Artificial Intelligence (INFO8006)

Exercises 5 – Learning

November 25, 2021

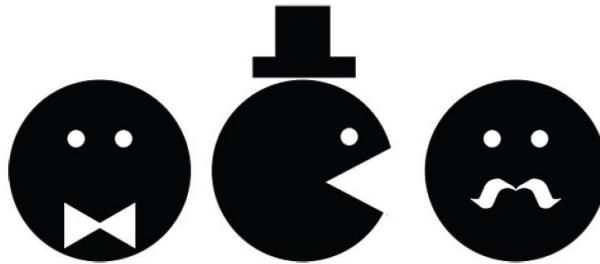
Learning outcomes

At the end of this session you should be able to

- define and apply maximum a posteriori (MAP) estimation.
- define and apply maximum likelihood estimation (MLE).
- define and apply linear regression.

Exercise 1 Pacbaby (UC Berkeley CS188, Spring 2014)

Pacman and Pacwoman have been searching for each other in the maze. Pacwoman has been pregnant with a baby, and just this morning she has given birth to Pacbaby¹. Because Pacbaby was born before Pacman and Pacwoman were reunited in the maze, he has never met his father. Naturally, Pacwoman wants to teach Pacbaby to recognize his father, using a set of pictures of Pacman. She also has several pictures of ghosts to use as negative examples.

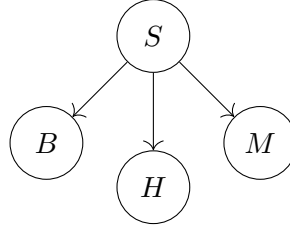


Because the pictures are black and white, and were taken from various angles, Pacwoman has decided to teach Pacbaby to identify Pacman based on salient features: the presence of a bowtie B , hat H or mustache M . The following table summarizes the content of the pictures. Each feature takes realization in $\{0,1\}$, where 0 and 1 mean the feature is respectively absent and present. The subject of the picture is described by a random variable $S \in \{0,1\}$, where 0 is a ghost and 1 is Pacman.

B	H	M	S
0	0	0	1
1	0	0	0
1	0	1	1
1	1	0	0
0	1	0	0
1	1	1	1

1. Suppose Pacbaby has a Naive Bayes based brain. Draw the Bayesian network that would represent the dependencies between S , B , H and M for Pacbaby.

¹Congratulations!



2. Write the Bayesian classification rule for this problem, *i.e.* the formula that given a data point (b, h, m) returns the most likely subject. Write the formula in terms of conditional and prior probabilities. What does the formula become under the assumptions of Pacbaby?

Given (b, h, m) , the most likely subject is given by the *maximum a posteriori* (MAP) estimation

$$\begin{aligned} s_{\text{MAP}} &= \arg \max_s P(s|b, h, m) \\ &= \arg \max_s P(b, h, m|s)P(s). \end{aligned}$$

Under the naive Bayes assumptions of Pacbaby, B , H and M become independent conditionally to S , *i.e.* $P(B, H, M|S) = P(B|S)P(H|S)P(M|S)$. Then, the formula becomes

$$s_{\text{MAP}} = \arg \max_s P(b|s)P(h|s)P(m|s)P(s).$$

3. What are the parameters of this model? Give estimates of these parameters according to the pictures provided by Pacwoman.

The parameters of the model are the elements of the prior vector $P(S)$ and the (conditional) probability matrices $P(B|S)$, $P(H|S)$ and $P(M|S)$. An (unbiased) estimation of these elements can be computed as the frequency of their respective events within the learning set (of pictures).

S	$P(S)$	$P(B = 1 S)$	$P(H = 1 S)$	$P(M = 1 S)$
0	$\frac{3}{6}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{0}{3}$
1	$\frac{3}{6}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$

4. Pacman eventually shows up wearing a bowtie, but no hat or mustache. Will Pacbaby recognize his father?

Pacbaby will recognize his father if $s_{\text{MAP}} = 1$ for $(b, h, m) = (1, 0, 0)$. Using the parameters estimated previously, we have

$$\begin{aligned} P(b|0)P(h|0)P(m|0)P(0) &= \frac{2}{3} \times \left(1 - \frac{2}{3}\right) \times \left(1 - \frac{0}{3}\right) \times \frac{3}{6} \approx 0.111 \\ P(b|1)P(h|1)P(m|1)P(1) &= \frac{2}{3} \times \left(1 - \frac{1}{3}\right) \times \left(1 - \frac{2}{3}\right) \times \frac{3}{6} \approx 0.074. \end{aligned}$$

Therefore, $s_{\text{MAP}} = 0$, meaning that Pacbaby will *not* recognize his father.

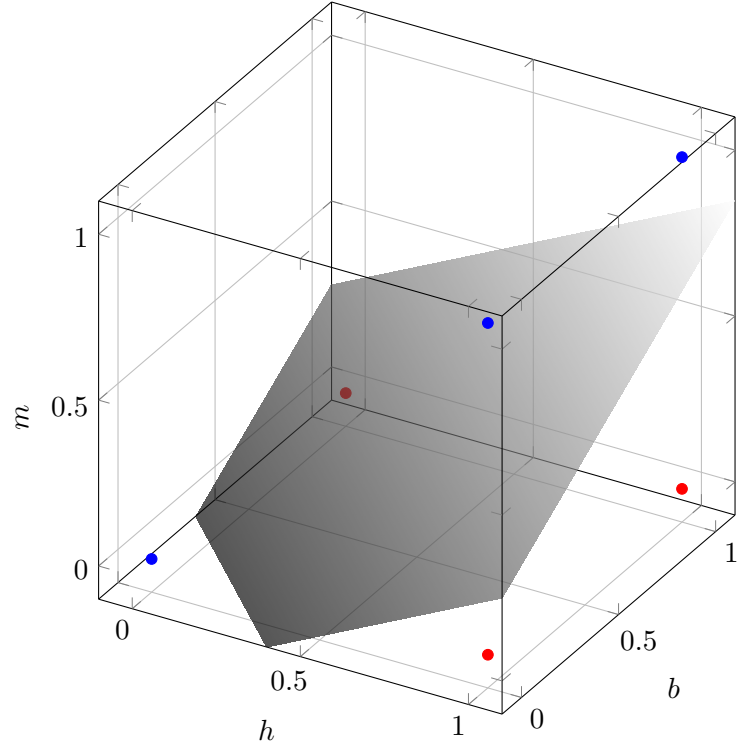
5. If Pacbaby had a perceptron based brain, meaning that he is limited to learn linear classification rules, would he be able to learn a rule that makes no mistakes on the set of pictures? In other words, is the learning set *linearly separable*?

For a training set $\{(x_i, y_i)\}$ with $x_i \in \mathbb{R}^n$ and $y_i \in \{-1, +1\}$, a linear separation consists in *any* hyperplane parameters $w \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

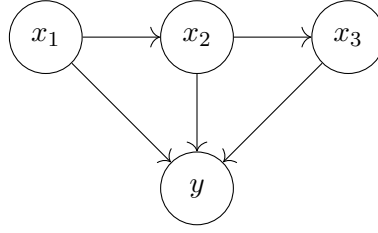
$$y_i = \text{sign}(w^T x_i + b)$$

for all (x_i, y_i) . There exists several ways of fitting these parameters to a learning set, but it is very tedious by hand. However, in low dimension (2D, 3D), it is possible to solve this problem visually. We can draw the data points on a grid with different colors for each class and try to find a plane that separates best the two classes.

In our case, we draw the $S = 0$ class in red and the $S = 1$ class in blue, and see that there is a plane that separates them perfectly. Hence, the learning set is linearly separable.



Exercise 2 Predict your grade



The hereabove Bayesian network represents how the final grade of a class is computed. In this model, x_1 , x_2 and x_3 respectively denote the grades obtained by a student at the homework, project and exam. The teaching assistant that grades the homework also grades the project and the exam, which introduces a slight bias in the corrections. In particular, $x_2 \sim \mathcal{N}(a_1x_1 + \mu_2, \sigma_2^2)$ and $x_3 \sim \mathcal{N}(a_2x_2 + \mu_3, \sigma_3^2)$. Finally, $y \sim \mathcal{N}(a_3x_1 + a_4x_2 + a_5x_3 + \mu_y, \sigma_y^2)$ stands for the final grade, which is a linear combination of the grades obtained by the student during the semester plus some Gaussian noise due to rounding errors. Answer the following questions about this model.

1. Assuming the parameters of the model are known, what is the expected value of y given x_1 and x_2 .

Our task is to find the expectation

$$\mathbb{E}_{p(y|x_1, x_2)}[y] = \int y p(y|x_1, x_2) dy.$$

We know that

$$p(y|x_1, x_2) = \int p(y|x_1, x_2, x_3) p(x_3|x_1, x_2) dx_3,$$

where $p(y|x_1, x_2, x_3)$ and $p(x_3|x_1, x_2)$ are linear Gaussian distributions given in the statement. Therefore, we have

$$p(y|x_1, x_2) = \mathcal{N}(a_3x_1 + a_4x_2 + a_5(a_2x_2 + \mu_3) + \mu_y, (a_5\sigma_3)^2 + \sigma_y^2)$$

and, by definition of a Gaussian distribution,

$$\mathbb{E}_{p(y|x_1, x_2)}[y] = a_3x_1 + (a_4 + a_5a_2)x_2 + a_5\mu_3 + \mu_y.$$

2. Suppose now that the model's parameters are unknown. Given a learning set $d = \{(x_{i,1}, x_{i,2}, y_i)\}$ of N independent and identically distributed points, determine the model that best describes d .

We know that the distribution of y given x_1 and x_2 takes the form $\mathcal{N}(w_1x_1 + w_2x_2 + b, \sigma^2)$. Then, our task is to find the parameters $h = (w_1, w_2, b, \sigma)$ that maximize the likelihood of d , i.e. the *maximum likelihood estimation* (MLE)

$$\begin{aligned} h_{\text{MLE}} &= \arg \max_w p(d|h) \\ &= \arg \max_h \prod_i p(x_i, y_i|h) \\ &= \arg \max_h \log \prod_i p(x_i, y_i|h) \\ &= \arg \max_h \sum_i \log p(x_i, y_i|h) \end{aligned}$$

$$\begin{aligned}
&= \arg \max_h \sum_i \log p(y_i|h, x_i) + \log p(x_i) \\
&= \arg \max_h \sum_i \log p(y_i|h, x_i) \\
&= \arg \max_h \sum_i \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left(-\frac{(w^T x_i - y_i)^2}{2\sigma^2} \right) \right] \\
&= \arg \max_h \sum_i -\frac{1}{2} \log(2\pi\sigma^2) - \frac{(w^T x_i - y_i)^2}{2\sigma^2} \\
&= \arg \min_h \log \sigma^2 + \frac{1}{\sigma^2} \frac{1}{N} \sum_i (w^T x_i - y_i)^2,
\end{aligned}$$

where $x_i = (x_{i,1} \ x_{i,2} \ 1)^T$ and $w = (w_1 \ w_2 \ b)^T$. In the last expression, we observe that the summation term is independent from σ . Therefore,

$$w_{\text{MLE}} = \arg \min_w \sum_i (w^T x_i - y_i)^2,$$

which exactly corresponds to a *linear regression* problem. Then, we find w_{MLE} by canceling the gradient with respect to w , *i.e.*

$$\begin{aligned}
0 &= \nabla_w \sum_i (w^T x_i - y_i)^2 \\
&= \sum_i \nabla_w (w^T x_i - y_i)^2 \\
&= \sum_i 2 (w^T x_i - y_i) x_i \\
&= \sum_i (w^T x_i - y_i) x_i.
\end{aligned}$$

This expression is a *vectorial* equality, which is equivalent to the set of element-wise equalities

$$\begin{aligned}
0 &= \sum_i (w^T x_i - y_i) x_{ij} \\
&= \sum_i \left(\sum_k w_k x_{ik} \right) x_{ij} - \sum_i y_i x_{ij} \\
&= \sum_k w_k \underbrace{\sum_i x_{ik} x_{ij}}_{(X^T X)_{jk}} - \underbrace{\sum_i y_i x_{ij}}_{(X^T Y)_j} \\
&\quad \underbrace{\hspace{1.5cm}}_{(X^T X w)_j} \\
&= (X^T X w - X^T Y)_j,
\end{aligned}$$

where $X = (x_i^T) \in \mathbb{R}^{N \times 3}$ and $Y = (y_i) \in \mathbb{R}^N$. Finally, going back to vectors, we have

$$\begin{aligned}
0 &= X^T X w_{\text{MLE}} - X^T Y \\
\Leftrightarrow w_{\text{MLE}} &= (X^T X)^{-1} X^T Y.
\end{aligned}$$

Afterwards, we find σ_{MLE} such that

$$\begin{aligned}
\sigma_{\text{MLE}} &= \arg \min_{\sigma} \log \sigma^2 + \frac{\text{MSE}}{\sigma^2} \\
&= \sqrt{\text{MSE}},
\end{aligned}$$

where MSE denotes the *mean squared error*

$$\frac{1}{N} \sum_i (w_{\text{MLE}}^T x_i - y_i)^2.$$

Exercise 3 Heteroscedastic linear regression

What becomes the expression of the weight vector w in the solution of question 2.2 if the noise is different for each sample? In particular, $y_i \sim N(w^T x, \sigma_i^2)$ and we know the values σ_i .

Exercise 4 Ridge regression

One can generalize the linear regression problem to the minimization problem

$$w^* = \arg \min_w \sum_i \ell(w^T x_i - y_i),$$

where ℓ is a *loss* function. Show that $\ell(x) = |x|$ corresponds to assuming the noise follows a Laplace distribution in contrast to $\ell(x) = x^2$, which corresponds to assuming Gaussian noise.

Exercise 5 Learning to play Pacman (August 2020)

You observe a Grandmaster agent playing Pacman. How can you use the moves you observe to train your own agent?

1. Describe formally the data you would collect, the inference problem you would consider, and how you would solve it.
2. How would you design a neural network to control your agent? Define mathematically the neural network architecture, its inputs, its outputs, its parameters, as well as the loss you would use to train it.
3. Discuss the expected performance of the resulting agent when (a) the Grandmaster agent is optimal, and (b) the Grandmaster agent is suboptimal.

Supplementary materials

- Heteroscedasticity



- Laplace distribution



- Chapter 18 of the reference textbook.