# Logic for Computer Science

## Exercises, tutorial 2

Pascal Fontaine

October 26, 2021

**1.** Brown, Jones and Smith are three Irish salesmen in New York. They are on trial for the fabrication and sale of alcohol (during prohibition). They swear on the Bible and declare:

- Brown: "Jones is guilty, and Smith is innocent."
- Jones: "If Brown is guilty then Smith is guilty as well."
- Smith: "I am innocent, but at least one of the two others is guilty."

Let $b$, $j$, $s$ be the three propositions "Brown is innocent", "Jones is innocent", "Smith is innocent".

1. Give a logical formula for each of the statements.
2. Is this set of statements satisfiable?
3. One statement is a logical consequence of another one. Which one?
4. If everyone is innocent, who made a false declaration?
5. As the Irish are very religious people, one could assume that they tell the truth. In this case, who is guilty?
6. If the innocents tell the truth and the guilty lie, who is guilty?

**Solution:** The statements from Brown, Jones, and Smith can be formalized respectively into $\neg j \wedge s$, $\neg b \Rightarrow \neg s$, $s \wedge (\neg b \vee \neg j)$

1. The set of statement is satisfiable. It suffices to consider an interpretation that makes propositions $s$ and $b$ true and $j$ false;

2. Smith's statement is a logical consequence of Brown's.

3. Brown and Smith lied, if everybody is innocent.

4. If all say the truth, Jones is guilty, and only him.

5. It is not possible that all are innocent (we earlier stated that if so, Brown and Smith lied). We can proceed systematically, building a set of true formulas:

   (a) $b \Rightarrow (\neg j \wedge s)$
   (b) $\neg b \Rightarrow (j \vee \neg s)$
   (c) $j \Rightarrow (\neg b \Rightarrow \neg s)$
   (d) $\neg j \Rightarrow (\neg b \wedge s)$
   (e) $s \Rightarrow (s \wedge (\neg b \vee \neg j))$
   (f) $\neg s \Rightarrow (\neg s \vee (b \wedge j))$

   The last one is a tautology, the penultimate can be simplified, and converted to a clause: $\neg s \vee \neg b \vee \neg j$ (someone is guilty). The others can be converted to clauses:

   (a) $\neg b \vee \neg j$
   (b) $\neg b \vee s$
   (c) $b \vee j \vee \neg s$
   (d) $\neg j \vee b \vee \neg s$
   (e) $j \vee \neg b$
   (f) $j \vee s$
   (g) $\neg s \vee \neg b \vee \neg j$

   By resolution, Brown is guilty ($\neg b$), then again by resolution, Smith is guilty ($\neg s$), and Jones is innocent ($j$).

**2.** Five people ($a$, $b$, $c$, $d$, $e$) have put their money into the same safe. They however have no confidence in each other and decided therefore that the safe can only be opened in the presence of $a$ and $b$, or $b$ and $c$, or $b$, $d$ and $e$. How many locks must the safe have (at least)? How many keys are needed? And who has them?

*Hint:* consider the formula

$$\varphi(p_a, p_b, p_c, p_d, p_e) =_{\text{def}} \text{"the safe can be opened"},$$

where $p_x$ is true if $x$ is present.

**Solution:** One can easily show that

$$\varphi = (p_a \wedge p_b) \vee (p_b \wedge p_c) \vee (p_b \wedge p_d \wedge p_e).$$

Now we need to find a conjunctive normal form logically equivalent to $\varphi$. The number of arguments to the conjunction will provide the number of locks, since for the safe to open, all locks must be unlocked. The clauses will indicate the number of keys and who has them.

We can systematically convert to normal form:

$$\begin{aligned} & (p_a \wedge p_b) \vee (p_b \wedge p_c) \vee (p_b \wedge p_d \wedge p_e) && (1) \\ = \quad & (p_a \wedge p_b) \vee (p_b \wedge (p_c \vee p_d) \wedge (p_c \vee p_e)) && (2) \\ = \quad & p_b \wedge (p_a \vee p_c \vee p_d) \wedge (p_a \vee p_c \vee p_e) && (3) \end{aligned}$$

We need three locks, one with one key for $b$, one with three keys for $a$, $c$, $d$, one with three keys for $a$, $c$, $e$.