Introduction to Artificial Intelligence (INFO8006)

Exercises 4 – Reasoning over time

November 19, 2021

Learning outcomes

At the end of this session you should be able to

- formulate a Markov model for discrete-time reasoning problems.
- define the simplifying assumptions of Markov processes.
- define and apply prediction, filtering and smoothing in Markov processes.
- apply the simplified matrix algorithm(s) to hidden Markov models.
- define the Kalman filter assumptions and manipulate multivariate Gaussian distributions.

Exercise 1 Umbrella World (AIMA, Section 15.1.1)

You are a security guard stationed at a secret underground installation. You want to know whether it is raining today, but your only access to the outside world occurs each morning when you see the director coming in with, or without, an umbrella. For each day t, the evidence is a single variable $Umbrella_t \in \{1, 0\}$, *i.e.* whether the umbrella appears or not, and the (hidden) state is a single variable $Rain_t \in \{1, 0\}$, *i.e.* whether it is raining or not.

You believe that from one day $t-1$ to the next t, the chances that the weather stays the same are 70 %. You also believe that the director brings his umbrella 90 % of the time when it is raining, and 20 % of the time otherwise.

1. You would like to represent your umbrella world as a Markov model. What formal assumptions correspond to your beliefs ?

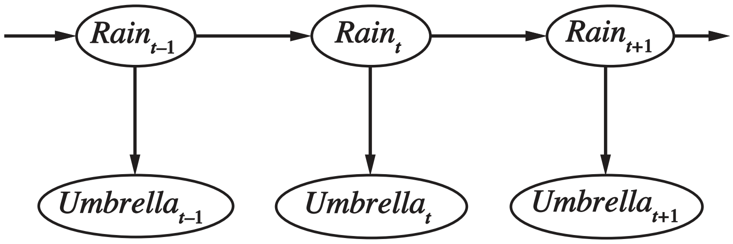
   The first belief describes the umbrella world as a *first-order Markov process*, such that

   $$P(R_t \mid R_{0:t-1}) = P(R_t \mid R_{t-1}),$$

   i.e. R_t is independent from $R_{0:t-2}$ given R_{t-1}. The second belief is equivalent to a *first-order sensor Markov assumption*, implying that

   $$P(U_t \mid R_{0:t}, U_{0:t-1}) = P(U_t \mid R_t).$$

2. Sketch a Bayesian network structure describing the umbrella world and provide the transition and sensor models.



R_{t-1}	$P(R_t = 1 \mid R_{t-1})$
1	0.7
0	0.3

R_t	$P(U_t = 1 \mid R_t)$
1	0.9
0	0.2

3. Express the distributions $P(R_{t+1} \mid R_{t-1})$, $P(U_t \mid R_{t-1})$ and $P(R_t \mid R_{t-1}, U_t)$ in terms of the transition and sensor models.

$$
\begin{aligned}
P(R_{t+1} \mid R_{t-1}) &= \sum_{r_t} P(R_{t+1} \mid r_t) P(r_t \mid R_{t-1}) \\
&= \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.58 & 0.42 \\ 0.42 & 0.58 \end{pmatrix} \\
P(U_t \mid R_{t-1}) &= \sum_{r_t} P(U_t \mid R_{t-1}, r_t) P(r_t \mid R_{t-1}) \\
&= \sum_{r_t} P(U_t \mid r_t) P(r_t \mid R_{t-1}) \\
&= \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.69 & 0.41 \\ 0.31 & 0.59 \end{pmatrix} \\
P(R_t \mid R_{t-1}, U_t) &= \frac{P(U_t \mid R_{t-1}, R_t) P(R_t \mid R_{t-1})}{P(U_t \mid R_{t-1})} \\
&= \frac{P(U_t \mid R_t) P(R_t \mid R_{t-1})}{P(U_t \mid R_{t-1})}
\end{aligned}
$$

4. Suppose you observe an unending sequence of days on which the umbrella appears. Show that, as the days go by, the probability of rain on the current day tends monotonically towards a fixed point. Calculate this fixed point.

   We are asked to prove that $x_t = P(R_t = 1 \mid U_{1:t} = 1)$ tends monotonically (with respect to t) towards a fixed point x^*. To do so, we need to find the value of x_t, which corresponds to apply *filtering* on the Markov model defined by the umbrella world. On this basis, we have

$$
\begin{aligned}
P(R_t \mid U_{1:t} = 1) &= \alpha P(U_t = 1 \mid R_t) \sum_{r_{t-1}} P(R_t \mid r_{t-1}) P(r_{t-1} \mid U_{1:t-1} = 1) \\
&= \alpha \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} P(R_{t-1} \mid U_{1:t-1} = 1) \\
\Leftrightarrow \qquad \begin{pmatrix} x_t \\ 1 - x_t \end{pmatrix} &= \alpha \begin{pmatrix} 0.63 & 0.27 \\ 0.06 & 0.14 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ 1 - x_{t-1} \end{pmatrix} \\
&= \alpha \begin{pmatrix} 0.27 + (0.63 - 0.27)\, x_{t-1} \\ 0.14 + (0.06 - 0.14)\, x_{t-1} \end{pmatrix} \\
&= \frac{1}{0.41 + 0.28\, x_{t-1}} \begin{pmatrix} 0.27 + 0.36\, x_{t-1} \\ 0.14 - 0.08\, x_{t-1} \end{pmatrix}.
\end{aligned}
$$

   Then, if there is a fixed point $x^* \in [0, 1]$, it satisfies

$$
\begin{aligned}
x^* &= \frac{0.27 + 0.36\, x^*}{0.41 + 0.28\, x^*} \\
\Leftrightarrow \quad 0.41\, x^* + 0.28\, {x^*}^2 &= 0.27 + 0.36\, x^* \\
\Leftrightarrow \quad 0 &= 0.28\, {x^*}^2 + 0.05\, x^* - 0.27 \\
\Rightarrow \quad x^* &= \frac{-0.05 + \sqrt{0.05^2 + 4 \times 0.27 \times 0.28}}{2 \times 0.28} \approx 0.897.
\end{aligned}
$$

   To show that x_t tends monotonically towards x^*, it is sufficient to prove that x_t is always between x^* and x_{t-1}, *i.e.*

$$
(x^* - x_t)(x_t - x_{t-1}) > 0,
$$

when $x_{t-1} \neq x^*$, which is left as an exercise to the motivated reader.

5. Now consider forecasting further and further into the future, given t umbrella observations. Is there a fixed point ? If yes, compute its exact value.

   We are asked to forecast the rain probability $x_k = P(R_{t+k} = 1 \mid U_{1:t} = 1)$, *i.e.* to perform *prediction.* We have

$$
\begin{aligned}
P(R_{t+k} \mid U_{1:t} = 1) &= \sum_{r_{t+k-1}} P(R_{t+k} \mid r_{t+k-1}) P(r_{t+k-1} \mid U_{1:t} = 1) \\
&= \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} P(R_{t+k-1} \mid U_{1:t} = 1) \\
\Leftrightarrow \qquad \begin{pmatrix} x_k \\ 1 - x_k \end{pmatrix} &= \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} x_{k-1} \\ 1 - x_{k-1} \end{pmatrix}.
\end{aligned}
$$

   By symmetry, we find that the fixed point $x^* = 0.5$, which highlights the fact that, without new evidences, uncertainty about the states accumulates.

Exercise 2 The coins

You are in a room containing 3 precious biased coins a, b and c. You inspect the coins and notice that the coins a, b and c have a head probability of 80 %, 50 % and 20 %, respectively.

Another person enters the room, takes the coins and put them into a bag. They draw a coin from the bag and tell you that they will repeat 4 times the same routine: hide their hand in the bag and either keep the current coin with probability $\frac{2}{3}$ or replace it by another, then toss it and show you the result. They proceed and the sequence of results are heads, heads, tail, heads. If you answer right to the following questions they will give you the coins.

1. Provide a hidden Markov model (HMM) that describes the process.
   - The hidden state at time t is $X_t \in \{a, b, c\}$ and represents the tossed coin.
   - The evidence at time t is $E_t \in \{0, 1\}$ and represents the result (heads or tail) of the toss. We have $e_1 = e_2 = e_4 = 0$ and $e_3 = 1$.
   - The prior vector
   $$f_0 = P(X_0) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}^T$$
   - The transition matrix
   $$T = P(X_t \mid X_{t-1}) = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}$$
   - The sensor matrix
   $$B = P(E_t \mid X_t) = \begin{pmatrix} 0.8 & 0.5 & 0.2 \\ 0.2 & 0.5 & 0.8 \end{pmatrix}$$

2. What are the probabilities of the last coin given the sequence of evidences?

   We are asked to calculate the distribution $f_t = P(X_t \mid e_{1:t})$ for $t = 4$. Applying Bayes filter, we have
   $$\begin{aligned} P(X_t \mid e_{1:t}) &= \alpha P(e_t \mid X_t, e_{1:t-1}) P(X_t \mid e_{1:t-1}) \\ &= \alpha P(e_t \mid X_t, e_{1:t-1}) \sum_{x_{t-1}} P(X_t \mid x_{t-1}, e_{1:t-1}) P(x_{t-1} \mid e_{1:t-1}) \\ &= \alpha P(e_t \mid X_t) \sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1:t-1}) \\ \Leftrightarrow \qquad f_t &= \alpha\, O_t\, T\, f_{t-1} \end{aligned}$$

   where we define the observation matrices
   $$O_1 = O_2 = O_4 = \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \quad \text{and} \quad O_3 = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.8 \end{pmatrix}.$$

Then, to calculate f_4, we first need f_1, f_2 and f_3.

$$
\begin{aligned}
f_1 &= \alpha_1 \, O_1 \, T \, f_0 = \alpha_1 \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \alpha_1 \begin{pmatrix} \frac{4}{15} \\ \frac{1}{6} \\ \frac{1}{15} \end{pmatrix} = \begin{pmatrix} \frac{8}{15} \\ \frac{1}{3} \\ \frac{2}{15} \end{pmatrix} \\
f_2 &= \alpha_2 \, O_2 \, T \, f_1 = \dots \\
f_3 &= \alpha_3 \, O_3 \, T \, f_2 = \dots \\
f_4 &= \alpha_4 \, O_4 \, T \, f_3 = \dots \approx \begin{pmatrix} 0.472 & 0.374 & 0.154 \end{pmatrix}^T
\end{aligned}
$$

3. What are the probabilities of the first coin given the sequence of evidences? And of the first coin tossed?

   We are asked to calculate the distribution $P(X_k \mid e_{1:t})$ for $t = 4$ and $k = 0$. As $k < t$ this corresponds to *smoothing* our belief of the past. We have

$$
\begin{aligned}
P(X_k \mid e_{1:t}) &= \alpha P(X_k, e_{k+1:t} \mid e_{1:k}) \\
&= \alpha P(e_{k+1:t} \mid X_k, e_{1:k}) P(X_k \mid e_{1:k}) \\
&= \alpha P(e_{k+1:t} \mid X_k) P(X_k \mid e_{1:k})
\end{aligned}
$$

   As we already know $f_k = P(X_k \mid e_{1:k})$, we only need to compute $b_k = P(e_{k+1:t} \mid X_k)$. We have

$$
\begin{aligned}
P(e_{k+1:t} \mid X_k) &= \sum_{x_{k+1}} P(x_{k+1} \mid X_k) P(e_{k+1:t} \mid X_k, x_{k+1}) \\
&= \sum_{x_{k+1}} P(x_{k+1} \mid X_k) P(e_{k+1} \mid x_{k+1}) P(e_{k+2:t} \mid x_{k+1}) \\
\Leftrightarrow \qquad b_k &= T^T \, O_{k+1} \, b_{k+1}
\end{aligned}
$$

   where $b_t = b_4 = P(\textit{anything} \mid X_4) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$. Therefore, to calculate b_0, we first need b_3, b_2 and b_1.

$$
\begin{aligned}
b_3 &= T^T \, O_4 \, b_4 = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.65 \\ 0.5 \\ 0.35 \end{pmatrix} \\
b_2 &= T^T \, O_3 \, b_3 = \dots \\
b_1 &= T^T \, O_2 \, b_2 = \dots \\
b_0 &= T^T \, O_1 \, b_1 = \dots \approx \begin{pmatrix} 0.076 & 0.055 & 0.035 \end{pmatrix}^T
\end{aligned}
$$

   Eventually,

$$
\begin{aligned}
P(X_0 \mid e_{1:4}) &= \alpha b_0 \times f_0 \approx \begin{pmatrix} 0.457 & 0.331 & 0.212 \end{pmatrix}^T \\
P(X_1 \mid e_{1:4}) &= \alpha b_1 \times f_1 \approx \begin{pmatrix} 0.580 & 0.329 & 0.091 \end{pmatrix}^T
\end{aligned}
$$

4. What is the most likely sequence of tossed coins?

   The most likely sequence $x^*_{1:t}$ given the sequence of evidences $e_{1:t}$ is the one that satisfies

$$
x^*_{1:t} = \arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})
$$

or, equivalently,

$$
\begin{aligned}
x_k^* &= \arg\max_{x_k} \left[\max_{x_{1:k-1}} P(x_{1:k}, x_{k+1:t}^* \mid e_{1:t}) \right] \\
&= \arg\max_{x_k} \left[\max_{x_{1:k-1}} \alpha P(x_{1:k}, x_{k+1:t}^*, e_{k+1:t} \mid e_{1:k}) \right] \\
&= \arg\max_{x_k} \left[\max_{x_{1:k-1}} \alpha P(e_{k+1:t} \mid x_{1:k}, x_{k+1:t}^*, e_{1:k}) P(x_{k+1:t}^* \mid x_{1:k}, e_{1:k}) P(x_{1:k} \mid e_{1:k}) \right] \\
&= \arg\max_{x_k} \left[\max_{x_{1:k-1}} \alpha P(e_{k+1:t} \mid x_{k+1:t}^*) P(x_{k+2:t}^* \mid x_{k+1}^*) P(x_{k+1:t}^* \mid x_k) P(x_{1:k} \mid e_{1:k}) \right] \\
&= \arg\max_{x_k} P(x_{k+1}^* \mid x_k) \left[\max_{x_{1:k-1}} P(x_{1:k} \mid e_{1:k}) \right] \\
&= \arg\max_{x_k} P(x_{k+1}^* \mid x_k)\, m_k(x_k)
\end{aligned}
$$

where

$$
\begin{aligned}
m_k &= \max_{x_{1:k-1}} P(x_{1:k-1}, X_k \mid e_{1:k}) \\
&= \max_{x_{1:k-1}} \alpha P(x_{1:k-1}, X_k, e_k \mid e_{1:k-1}) \\
&= \max_{x_{1:k-1}} \alpha P(e_t \mid x_{1:k-1}, X_t, e_{1:k-1}) P(X_k \mid x_{1:k-1}, e_{1:k-1}) P(x_{1:k-1} \mid e_{1:k-1}) \\
&= \max_{x_{1:k-1}} \alpha P(e_k \mid X_k) P(X_k \mid x_{k-1}) P(x_{1:k-1} \mid e_{1:k-1}) \\
&= \alpha P(e_k \mid X_k) \max_{x_{k-1}} P(X_k \mid x_{k-1}) \max_{x_{1:k-2}} P(x_{1:k-1} \mid e_{1:k-1}) \\
&= \alpha P(e_k \mid X_k) \max_{x_{k-1}} P(X_k \mid x_{k-1})\, m_{k-1}(x_{k-1})
\end{aligned}
$$

and $m_1 = P(X_1 \mid e_1) = f_1$. Therefore, we can iteratively build the vectors m_1, m_2, …, m_t to find the most likely last state

$$x_t^* = \arg\max_{x_t} m_t(x_t)$$

and, then, the most likely sequence with

$$x_k^* = \arg\max_{x_k} P(x_{k+1}^* \mid x_k)\, m_k(x_k).$$

Exercise 3 September 2019 (AIMA, Ex 15.13 and 15.14)

A professor wants to know if students are getting enough sleep. Each day, the professor observes whether the students sleep in class, and whether they have red eyes. The professor has the following hypotheses:

- The prior probability of getting enough sleep, with no observations, is 0.7.
- The probability of getting enough sleep on night t is 0.8 given that the student got enough sleep the previous night, and 0.3 otherwise.
- The probability of having red eyes is 0.2 if the student got enough sleep, and 0.7 otherwise.
- The probability of sleeping in class is 0.1 if the student got enough sleep, and 0.3 otherwise.

The professor asks you to answer the following questions:

1. Formulate the environment and hypotheses as a dynamic Bayesian network that the professor could use to detect sleep deprived students, from a sequence of observations. Provide the associated probability tables.
2. Reformulate the dynamic Bayesian network as a hidden Markov model that has only a single observation variable. Give the complete probability tables for the model.
3. For the sequence $e_{1:3}$ of observations "no red eyes, not sleeping in class", "red eyes, not sleeping in class" and "red eyes, sleeping in class", calculate the distributions $P(EnoughSleep_t \mid e_{1:t})$ and $P(EnoughSleep_t \mid e_{1:3})$ for $t \in \{1, 2, 3\}$.

Exercise 4 Hyperloop

A few years ago, ULiège decided to get into SpaceX's Hyperloop Pod competition. Briefly, what one should do to win this competition is to build the fastest and most reliable autonomous pod. One of the most important engineering problem is to be able to compute a robust estimation of the state of the pod, *i.e.* its position p (m), speed[1] $\dot{p}$ ($\mathrm{m\,s^{-1}}$) and acceleration $\ddot{p}$ ($\mathrm{m\,s^{-2}}$), given noisy sensor measurements. You received an email from students asking you what would be your solution to this estimation problem.

The email contains information about the competition and the sensors placed on the pod. The track is straight (one-dimensional) and the pod has $T = 30\,\mathrm{s}$ to go as far as possible. The initial position and speed are assumed to be centered around 0 with a standard deviation of $0.1\,\mathrm{m}$ and $0.01\,\mathrm{m\,s^{-1}}$, respectively. The time between sensor measurements is $\Delta t = 0.5\,\mathrm{s}$. The sensors placed on the pod are 1) an unbiased GPS sensor providing the pod's position with 68 % chance to be less than 100 m away from the true position and 2) an unbiased accelerometer measuring the pod's acceleration with 68 % chance to be less than $1\,\mathrm{m\,s^{-2}}$ away from the true acceleration. The acceleration $\ddot{p}$ is the result of the combination of thrust and drag. The thrust is assumed to be normally distributed around μ_a with standard deviation σ_a. The drag is proportional but opposite in direction to the speed, *i.e.* it takes the form $-\eta\,\dot{p}$.

After some research you find out that you should use a Kalman filter to solve this task.

1. Define the components of your Kalman filter in the context of the state estimation of the pod.

   The Kalman filter is a special case of the continuous Bayes filter, which assumes

   - a Gaussian prior
   $$p(x_0) = \mathcal{N}(\mu_0, \Sigma_0),$$
   - a linear Gaussian transition model
   $$p(x_t \mid x_{t-1}) = \mathcal{N}(A x_{t-1} + b, \Sigma_x)$$
   - and a linear Gaussian sensor model
   $$p(e_t \mid x_t) = \mathcal{N}(C x_t + d, \Sigma_e).$$

   In a multivariate Gaussian distribution $x = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^T \sim \mathcal{N}(\mu, \Sigma)$, the first argument is the *mean vector* and the second is the *covariance matrix*. The mean vector is simply the vector of the variables' mean, *i.e.* $\mu_i = \mathbb{E}[x_i]$. An element Σ_{ij} of the covariance matrix is the covariance between the variables x_i and x_j, *i.e.* $\Sigma_{ij} = \mathbb{E}[(x_i - \mu_i)(x_j - \mu_j)]$. Interestingly, $\Sigma_{ii} = \mathbb{E}[(x_i - \mu_i)^2] = \mathbb{V}[x_i]$, meaning that the diagonal elements of Σ are the variables' variance. Importantly, if x_i is independent from x_j, their covariance is null by definition, *i.e.* $\Sigma_{ij} = 0$.

   In our case, the state x is a vector $\begin{pmatrix} p & \dot{p} & \ddot{p} \end{pmatrix}^T$ and according to the provided information, the prior is defined by
   $$\mu_0 = \begin{pmatrix} 0 \\ 0 \\ \mu_a \end{pmatrix} \quad \text{and} \quad \Sigma_0 = \begin{pmatrix} 0.1^2 & 0 & 0 \\ 0 & 0.01^2 & 0 \\ 0 & 0 & \sigma_a^2 \end{pmatrix}.$$

[1] In physics, the "dot" is a short-hand for the derivative with respect to time. This is called Newton's notation.

For the transition model, we want to express x_t with respect to x_{t-1}. We known that a function $f(x)$ can be rewritten as a Taylor series

$$f(a) + \frac{(x-a)}{1!}\frac{\mathrm{d}f(a)}{\mathrm{d}x} + \frac{(x-a)^2}{2!}\frac{\mathrm{d}^2 f(a)}{\mathrm{d}x^2} + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!}\frac{\mathrm{d}^i f(a)}{\mathrm{d}x^i}$$

with respect to some reference point a. In our case, the position p is a function of time and p_t is its evaluation Δt seconds after p_{t-1}. Moreover, by definition, the first and second derivatives of p at $t-1$ are $\dot{p}_{t-1}$ and $\ddot{p}_{t-1}$. Then, we have

$$p_t = p_{t-1} + \frac{\Delta t}{1}\dot{p}_{t-1} + \frac{\Delta t^2}{2}\ddot{p}_{t-1} + \mathcal{O}\left(\Delta t^3\right)$$

and, similarly,

$$\dot{p}_t = \dot{p}_{t-1} + \Delta t\, \ddot{p}_{t-1} + \mathcal{O}\left(\Delta t^2\right).$$

As mentioned in the statement, the acceleration combines thrust and drag. The thrust is normally distributed from $\mathcal{N}(\mu_a, \sigma_a^2)$ and the drag at time t is $-\eta\, \dot{p}_t$. This results in

$$\ddot{p}_t \sim \mathcal{N}\left(\mu_a - \eta\, \dot{p}_t, \sigma_a^2\right) = \mathcal{N}\left(\mu_a - \eta\,(\dot{p}_{t-1} + \Delta t\, \ddot{p}_{t-1}), \sigma_a^2\right).$$

Then, our transition model[2] is defined by

$$A = \begin{pmatrix} 1 & \Delta t & \frac{1}{2}\Delta t^2 \\ 0 & 1 & \Delta t \\ 0 & -\eta & -\eta\,\Delta t \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \mu_a \end{pmatrix} \quad \text{and} \quad \Sigma_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_a^2 \end{pmatrix}.$$

It should be noted that only the acceleration has a non-null variance in Σ_x, since the transitions of the position and speed are *deterministic*.

Concerning the sensors, there are one position sensor (e_1) and one accelerometer (e_2), both unbiased. We can assume that sensors are independent and we know that 68 % corresponds to the 1-sigma confidence interval of a one-dimensional Gaussian distribution. Therefore, we have

$$\begin{aligned} e_1 &\sim \mathcal{N}(p_t, 100^2) \\ e_2 &\sim \mathcal{N}(\ddot{p}_t, 1^2), \end{aligned}$$

which translates to

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \Sigma_e = \begin{pmatrix} 100^2 & 0 \\ 0 & 1^2 \end{pmatrix}.$$

2. While the pod is on the track, we wish to estimate at each time step t its state x_t given the evidences $e_{1:t}$ provided by the sensors until t. Determine the distribution $p(x_t \mid e_{1:t})$, with respect to the components defined previously.

   This task corresponds exactly to filtering, *i.e.* inferring the distribution

$$\begin{aligned} p(x_t \mid e_{1:t}) &= \alpha\, p(e_t \mid x_t, e_{1:t-1})\, p(x_t \mid e_{1:t-1}) \\ &= \alpha\, p(e_t \mid x_t) \int p(x_t \mid x_{t-1})\, p(x_{t-1} \mid e_{1:t-1})\, \mathrm{d}x_{t-1}. \end{aligned}$$

[2] A model only approximates the reality. It could be inaccurate. For instance, the Taylor series for p_t and $\dot{p}_t$ are only accurate for small Δt as we neglect the $\mathcal{O}(\cdot)$ terms. Additionally, at high speed, drag is usually proportional to the square of the speed, which our model does not take into account. The predictions of an inaccurate model are at best inaccurate.

We notice that the latter expression depends on the sensor model $p(e_t \mid x_t)$, the transition model $p(x_t \mid x_{t-1})$ and our previous belief $p(x_{t-1} \mid e_{1:t-1})$. When $t = 1$, this belief is the prior $p(x_0)$ and is a (multivariate) Gaussian distribution. But, from the cheat sheet for Gaussian models (slide 51, lecture 6), we know that if we have a Gaussian distribution $p(x) = \mathcal{N}(\mu_x, \Sigma_x)$ and a linear Gaussian condition distribution $p(y \mid x) = \mathcal{N}(Ax + b, \Sigma_y)$, the marginal $p(y)$ and the posterior $p(x \mid y)$ are also Gaussian and take the form

$$
\begin{aligned}
p(y) &= \int p(y \mid x)\, p(x)\, \mathrm{d}x = \mathcal{N}\Big(A\mu_x + b, \Sigma_y + A\Sigma_x A^T\Big) \\
p(x \mid y) &= \alpha\, p(y \mid x)\, p(x) = \mathcal{N}\Big(\Sigma\Big(A^T\Sigma_y^{-1}(y - b) + \Sigma_x^{-1}\mu_x\Big), \Sigma\Big)
\end{aligned}
$$

with $\Sigma = \Big(\Sigma_x^{-1} + A^T\Sigma_y^{-1}A\Big)^{-1}$. Then, by substitution for $t = 1$, we have that

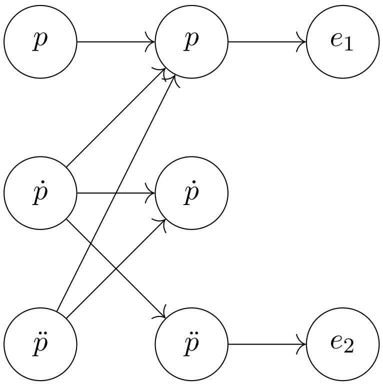
$$
\begin{aligned}
p(x_1) &= \int p(x_1 \mid x_0)\, p(x_0)\, \mathrm{d}x_0 = \mathcal{N}\Big(\underbrace{A\mu_0 + b}_{\mu_*}, \underbrace{\Sigma_x + A\Sigma_0 A^T}_{\Sigma_*}\Big) \\
p(x_1 \mid e_1) &= \alpha\, p(e_1 \mid x_1)\, p(x_1) = \mathcal{N}(\mu_1, \Sigma_1)
\end{aligned}
$$

with

$$
\begin{aligned}
\mu_1 &= \Sigma_1\Big(C^T\Sigma_e^{-1}(e_1 - d) + \Sigma_*^{-1}\mu_*\Big) \\
\Sigma_1 &= \Big(\Sigma_*^{-1} + C^T\Sigma_e^{-1}C\Big)^{-1}.
\end{aligned}
$$

Therefore, $p(x_1 \mid e_1)$ is also Gaussian and we can repeat the same process to get $p(x_2 \mid e_{1:2})$, $p(x_3 \mid e_{1:3})$ and so on until $p(x_t \mid e_{1:t})$. It should be noted that, in the preceding expressions, μ_0, Σ_0 and e_1 should be replaced respectively by μ_{t-1}, Σ_{t-1} and e_t in order to determine μ_t and Σ_t.

3. Represent the transition and sensor processes as a dynamic Bayesian network.



Supplementary materials

- Hidden Markov Models (UC Berkeley CS188, Spring 2014 Section 6).
- 68–95–99.7 rule
- Chapter 15 of the reference textbook.