Logic for Computer Science

Exercises, tutorial 5

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1. Transform those formulas into disjunctive normal forms

$$A = \bigwedge_{1 \leq i < n} (p_i \Rightarrow p_{i+1})$$

$$B = A \wedge (p_n \Rightarrow p_1)$$

$$C = \bigwedge_{1 \leq i, j \leq n, i \neq j} (p_i \Rightarrow \neg p_j)$$

$$D = \bigwedge_{1 \leq i \leq n} \left(\bigvee_{j \neq i} p_j \right)$$

$$E = \left[\left(\bigvee_{1 \leq i \leq n} p_i \right) \Leftrightarrow \bigwedge_{1 \leq i \leq n} \left(\bigvee_{j \neq i} p_j \right) \right]$$

$$G = \bigwedge_{1 \leq i \leq n} \left(p_i \Rightarrow \bigvee_{j \neq i} p_j \right)$$

Solution: A normalisation algorithm can be based on the following steps.

1. Eliminate all connectors but $\neg$, $\vee$, $\wedge$.
2. Propagate negations inside:

$$\neg(\varphi \wedge \psi) \longleftrightarrow \neg\varphi \vee \neg\psi \quad \text{and} \quad \neg(\varphi \vee \psi) \longleftrightarrow \neg\varphi \wedge \neg\psi$$

3. Eliminate double negations:

$$\neg\neg\varphi \longleftrightarrow \varphi$$

4. Use the distributivity rules:

$$(\varphi \vee \psi) \wedge \chi \longleftrightarrow (\varphi \wedge \chi) \vee (\psi \wedge \chi) \quad \text{and} \quad (\varphi \wedge \psi) \vee \chi \longleftrightarrow (\varphi \vee \chi) \wedge (\psi \vee \chi)$$

Considering A above:

$$A \longleftrightarrow \bigwedge_{1 \leq i < n} (\neg p_i \vee p_{i+1})$$

becomes

$$\begin{aligned}
A &\longleftrightarrow (\neg p_1 \vee p_2) \wedge (\neg p_2 \vee p_3) \wedge \ldots \wedge (\neg p_{n-1} \vee p_n) \\
&\longleftrightarrow [\neg p_1 \wedge (\neg p_2 \vee p_3) \wedge \ldots \wedge (\neg p_{n-1} \vee p_n)] \vee \\
&\qquad [p_2 \wedge (\neg p_2 \vee p_3) \wedge \ldots \wedge (\neg p_{n-1} \vee p_n)] \\
&\longleftrightarrow [\neg p_1 \wedge (\neg p_2 \vee p_3) \wedge \ldots \wedge (\neg p_{n-1} \vee p_n)] \vee \\
&\qquad [p_2 \wedge p_3 \wedge \ldots \wedge p_n] \\
&\longleftrightarrow \ldots
\end{aligned}$$

Then, in each clause:

- if p_1 is true, $\neg p_1$ is false, thus p_2 needs to be true;
- if p_2 is true, $\neg p_2$ is false, thus p_3 needs to be true;
- ...

Thus

$$A \longleftrightarrow \bigvee_{1 \leq j \leq n} \left[\left(\bigwedge_{1 \leq i < j} \neg p_i \right) \wedge \left(\bigwedge_{j < i \leq n} p_i \right) \right]$$

The solutions for the other formulas are

$$B \longleftrightarrow \left(\bigwedge_{1 \leq i \leq n} p_i \right) \vee \left(\bigwedge_{1 \leq i \leq n} \neg p_i \right)$$

$$C \longleftrightarrow \bigvee_{1 \leq i \leq n} \left(\bigwedge_{1 \leq j \leq n, j \neq i} \neg p_j \right)$$

$$D \longleftrightarrow \bigvee_{1 \leq i,j \leq n, i \neq j} (p_i \wedge p_j)$$

$$E \longleftrightarrow \left(\bigwedge_{1 \leq i \leq n} \neg p_i \right) \vee \bigvee_{1 \leq i,j \leq n, i \neq j} (p_i \wedge p_j)$$

$$G \longleftrightarrow E$$

2. Put into conjunctive normal form the formula

$$H = \bigvee_{1 \leq i < j \leq n} (p_i \wedge p_j)$$

Solution: The distributivity laws will require that each clause in the CNF contains one element of each cube. Since all pair are represented in cubes, to get a clause that will contain at least one proposition in each pair, only one proposition can be left out.

$$H \longleftrightarrow \bigwedge_{1 \leq i \leq n} \left(\bigvee_{1 \leq j \leq n, j \neq i} p_j \right)$$

3. Is the following text correct?

Some students do not work. All students want to succeed. Therefore some students want to succeed without working.

4. Consider five students: Georges, Karin, Nick, Janine, et Mike.

- Anyone who likes Georges will chose Nick in his/her team;
- Nick is a friend of no friend of Mike;
- Janine will only choose Karin's friends in his team.

Show that if Karin is a friend of Mike, then Janine doesn't like Georges.

Solution: Let's define

- notations (i.e., nullary functions, constants) g for Georges, n for Nick,...
- the predicates
  - $L(x, y)$: x likes y
  - $C(x, y)$: x chooses y
  - $F(x, y)$: x is a friend of y

It is stated:

1. $\forall x \,.\, L(x, g) \Rightarrow C(x, n)$
2. $\forall x \,\neg\, [F(n, x) \wedge F(x, m)]$
3. $\forall x \,.\, C(j, x) \Rightarrow F(x, k)$

and also $F(k, m)$ (4).

Each interpretation satisfying these formulas also satisfies

5. $\neg\, [F(n, k) \wedge F(k, m)]$ (from 2)
6. $\neg F(n, k)$ (from 4 and 5)
7. $C(j, n) \Rightarrow F(n, k)$ (from 3)
8. $\neg C(j, n)$ (from 6 and 7)
9. $L(j, g) \Rightarrow C(j, n)$ (from 1)
10. $\neg L(j, g)$ (from 8 and 9)

Hence, we can conclude that Janine doesn't like Georges!