Logic for Computer Science

Exercises, tutorial 3

Pascal Fontaine

November 9, 2021

1. Show that $(p \wedge q) \vee (p \wedge r)$ is logically equivalent to $p \wedge (q \vee r)$. Then assume you want to build a circuit that outputs $(a_1 \wedge a_2) \vee (a_1 \wedge a_3) \vee (a_1 \wedge a_4) \vee (a_1 \wedge a_5)$, how would you do?

Solution: You can show the logical equivalence using truth tables, or do a case analysis for the models: assume you have a model of the first (two cases) then it is a model of the second. And vice-versa.

Then the second formula is equivalent to $a_1 \wedge (a_2 \vee a_3 \vee a_4 \vee a_5)$.

2. As a reward for saving his daughter from pirates, the King has given you the opportunity to win a treasure hidden inside one of three trunks. But he is not the easy type, and filled both other trunks with dangerous and quick snakes. You must select the correct trunk. Trunks 1 and 2 are each inscribed with the message "This trunk contains no treasure", and Trunk 3 is inscribed with the message "The treasure is in Trunk 2". The Queen, who never lies, tells you that only one of these inscriptions is true, while the other two are wrong. Which trunk should you open?

Solution: Let's denote t_1, t_2, t_3 the propositional variables stating that the treasure is in trunk 1, 2, or 3, and i_1, i_2, i_3 stating that the inscriptions are true. The inscriptions provide us with the following solid information:

- $i_1 \Leftrightarrow \neg t_1$
- $i_2 \Leftrightarrow \neg t_2$
- $i_3 \Leftrightarrow t_2$

We also know that one and only one trunk holds the treasure: $t_1 \vee t_2 \vee t_3$, $\neg t_1 \vee \neg t_2$, $\neg t_1 \vee \neg t_3$, $\neg t_2 \vee \neg t_3$ and the Queen tells us that one and only one inscription is true, that is, $i_1 \vee i_2 \vee i_3$, $\neg i_1 \vee \neg i_2$, $\neg i_1 \vee \neg i_3$, $\neg i_2 \vee \neg i_3$.

Summarizing:

1. $i_1 \Leftrightarrow \neg t_1$
2. $i_2 \Leftrightarrow \neg t_2$
3. $i_3 \Leftrightarrow t_2$
4. $t_1 \vee t_2 \vee t_3$
5. $\neg t_1 \vee \neg t_2$
6. $\neg t_1 \vee \neg t_3$
7. $\neg t_2 \vee \neg t_3$
8. $i_1 \vee i_2 \vee i_3$
9. $\neg i_1 \vee \neg i_2$
10. $\neg i_1 \vee \neg i_3$
11. $\neg i_2 \vee \neg i_3$

We could easily encode this as input for a SAT solver. We can also simplify the problem, using the first three to eliminate the propositions i. Actually, $i_1 \vee i_2 \vee i_3$ is a logical consequence of 2 and 3, as well as $\neg i_2 \vee \neg i_3$.

1. $i_1 \Leftrightarrow \neg t_1$
2. $i_2 \Leftrightarrow \neg t_2$
3. $i_3 \Leftrightarrow t_2$
4. $t_1 \vee t_2 \vee t_3$
5. $\neg t_1 \vee \neg t_2$
6. $\neg t_1 \vee \neg t_3$
7. $\neg t_2 \vee \neg t_3$
8. $t_1 \vee t_2$
9. $t_1 \vee \neg t_2$

and by resolution we directly get that t_1 must be true. We used the fact that at most one inscription was true, but we do not need to know that at least one inscription is true or that there is indeed a treasure in one trunk. To deduce that there are no more treasures, we need to know that there is at most one treasure.

3. Raymond Smullyan, a master of logic puzzles, published more than a dozen books containing challenging puzzles involving logical reasoning, many about an island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. What are A and B if A says "B is a knight" and B says "The two of us are opposite types"?

Solution: Let's use a and b to respectively denote A is a knight and B is a knight. A and B respectively say that b and $a \not\Leftrightarrow b$. To transform these into truths, we just say that $a \Leftrightarrow b$, i.e., b is a knight if and only if a is, and $b \Leftrightarrow (a \not\Leftrightarrow b)$.

The easiest is to write a truth table (4 lines):

a	b	$a \Leftrightarrow b$	$b \Leftrightarrow (a \not\Leftrightarrow b)$
F	F	T	T
F	T	F	
T	F	F	
T	T	T	F

The only consistent explanation is that they are both knaves.

4. Write a formula φ to express that exactly two out of the three propositions p, q and r are false, and a formula ψ to express that at least two of the three same propositions are false. How do these two formulas relate?

Solution: It is simple to write φ by considering all cases:

$$\varphi = (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$$

and ψ by simply removing the constraint on the negative literal

$$\psi = (p \wedge q) \vee (p \wedge r) \vee (q \wedge r).$$

Obviously, $\varphi \models \psi$, but $\psi \not\models \varphi$ (since the interpretation that makes all propositions true is a model of ψ and not of φ). To prove this using resolution, one would need to build the conjunctive normal form of $\neg(\varphi \Rightarrow \psi)$ and show it is unsatisfiable (and thus $\varphi \Rightarrow \psi$ is valid, and thus $\varphi \models \psi$). Since $\neg(\varphi \Rightarrow \psi) \longleftrightarrow \varphi \wedge \neg\psi$, one needs to add all the clauses of the CNF of φ, and all clauses of the CNF of $\neg\psi$. Fortunately, it is trivial to build a CNF for $\neg\psi$

$$(\neg p \vee \neg q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg r)$$

but the CNF for φ requires distributing

$$(p \vee q) \wedge (p \vee r) \wedge (q \vee r) \wedge (\neg r \vee \neg q \vee \neg p)$$

1. $\neg p \vee \neg q$
2. $\neg p \vee \neg r$
3. $\neg q \vee \neg r$
4. $p \vee q$
5. $p \vee r$
6. $q \vee r$
7. $\neg r \vee \neg q \vee \neg p$
8. From (1) and (5): $\neg q \vee r$
9. From (6) and (8): r
10. From (2) and (9): $\neg p$
11. From (3) and (9): $\neg q$
12. From (4), (10), and (11): $\perp$

5. What is the relation between the following formulas?

- $p \Rightarrow [(q \Rightarrow r) \Rightarrow s]$
- $p \Rightarrow [(q \vee r) \Rightarrow s]$
- $p \Rightarrow [(q \wedge r) \Rightarrow s]$
- $p \Rightarrow [(q \Leftrightarrow r) \Rightarrow s]$
- $p \Rightarrow [(q \oplus r) \Rightarrow s]$

For each formula, provide a model, and an interpretation that makes it false.

Solution: One could study all the possible consequences using resolution, which would be incredibly tedious. One could also build truth tables, which again could prove a bit tedious. It is clear however that formulas have the same truth value whenever p is false or s is true. When it is not the case (p is true and s is false), then the formulas reduce to the negation of the subformula with q and r. It suffices to then study the consequences between those formulas, which is easily done by truth tables.

6. Simulate the DPLL algorithm (with clause copying) on the following set of clauses:

1. $p \vee q \vee r$
2. $\neg p \vee r \vee s$
3. $\neg p \vee r \vee \neg s$
4. $\neg p \vee \neg r \vee s$
5. $\neg p \vee \neg r \vee \neg s$
6. $\neg q \vee \neg r \vee s$
7. $p \vee q \vee \neg r$
8. $p \vee \neg q \vee r$

Solution: This below is not the solution, just a summary. For the complete solution, we need to provide the rule applications, with detailed consequences.

No clause is unit. No propagation can occur. Let's split on p. We obtain

a. $\{p, r \vee s, r \vee \neg s, \neg r \vee s, \neg r \vee \neg s, \neg q \vee \neg r \vee s\}$

b. $\{\neg p, q \vee r, \neg q \vee \neg r \vee s, q \vee \neg r, \neg q \vee r\}$

Again, there is no unit propagation possible. Let's split on r on the first set:

c. $\{p, r, s, \neg s, \neg q \vee s\}$ which is unsatisfiable, and

d. $\{p, \neg r, s, \neg s\}$ which is also unsatisfiable.

It remains to split set (b) using q to obtain

1. $\{\neg p, q, s, r\}$ which provides a model, and
2. $\{\neg p, \neg q, r, s, \neg r, r\}$ which is unsatisfiable.