ISEL
INSTITUTO SUPERIOR
DE ENGENHARIA DE LISBOA

PROCESSAMENTO DE IMAGEM E BIOMETRIA

IMAGE PROCESSING AND BIOMETRICS

6. **FREQUENCY FILTERING (part 3)**

Summary (part 3)

- Image processing in the transformed domain

- Image transforms

  - The DFT and IDFT: definitions and some properties

  - The DCT and the IDCT: definitions and some properties

- Exercises

Image processing in the transformed domain

1) The input image is transformed from the *spatial domain* to the *transform domain*
   - The entire image at once
   - On a block by block basis (e.g. 8x8 blocks)

2) The DIP operations are carried out on the transform domain

3) The image is transformed back to the *spatial domain*

- Some common image transforms:
  - DFT – Discrete Fourier Transform, for spectrum analysis and frequency filtering tasks
  - DCT – Discrete Cosine Transform, for image and video analysis and lossy coding

The Discrete Fourier Transform (DFT)

- F[u,v] is the spectrum of f[m,n]
- F[u,v] is computed by the **Discrete Fourier Transform (DFT)** of the input image f[m,n]
- F[u,v] = DFT[f[m,n]]

$$F[u,v] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\exp\left(-\frac{j2\pi u}{M}m\right)\exp\left(-\frac{j2\pi v}{N}n\right)$$

- F[u,v] is the **image spectrum** (represented by complex numbers)
- MATLAB: fft2.m and fftshift.m

https://www.mathworks.com/help/matlab/ref/fft2.html

https://www.mathworks.com/help/matlab/ref/fftshift.html

The Discrete Fourier Transform (DFT)

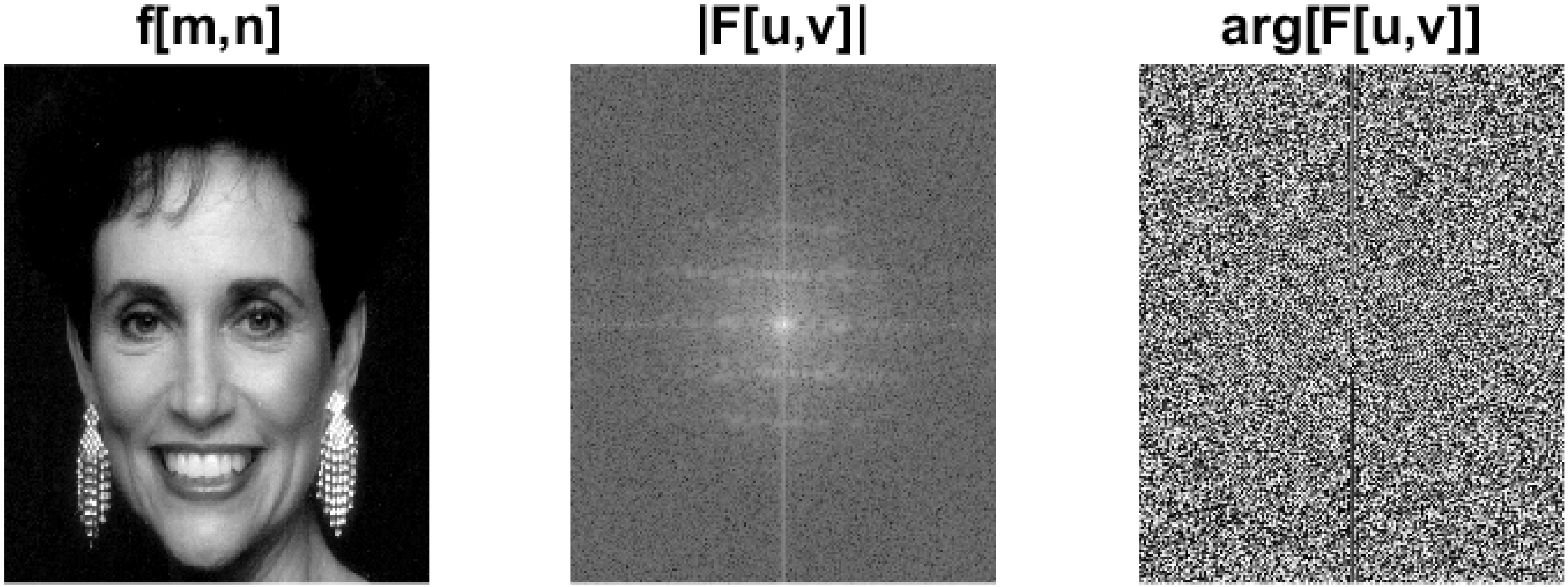
- F[u,v] = DFT[f[m,n]] is the **complex-valued** spectrum of f[m,n]

- |F[u,v]| is the **module** of the spectrum
  - Shows how the energy is spread over the (horizontal and vertical) frequency components

- arg[F[u,v]] is the **argument/phase** of the spectrum
  - Typically exhibits a rapidly changing textured pattern

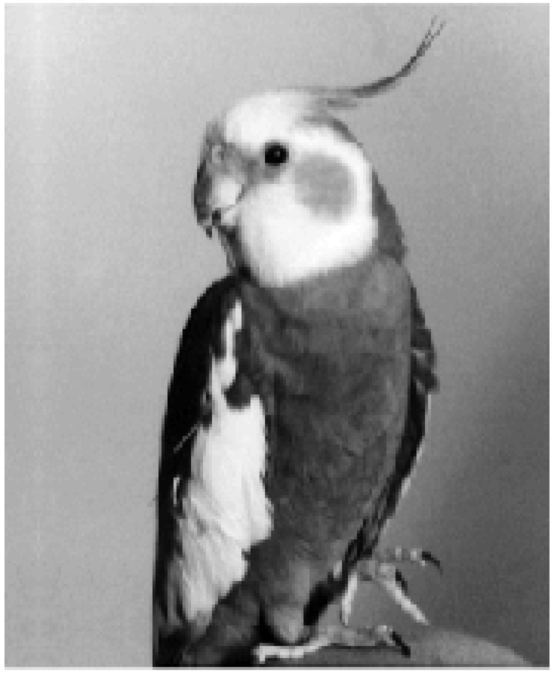
$$F[u,v] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\exp\left(-\frac{j2\pi u}{M}m\right)\exp\left(-\frac{j2\pi v}{N}n\right)$$

Example of module and argument (1)

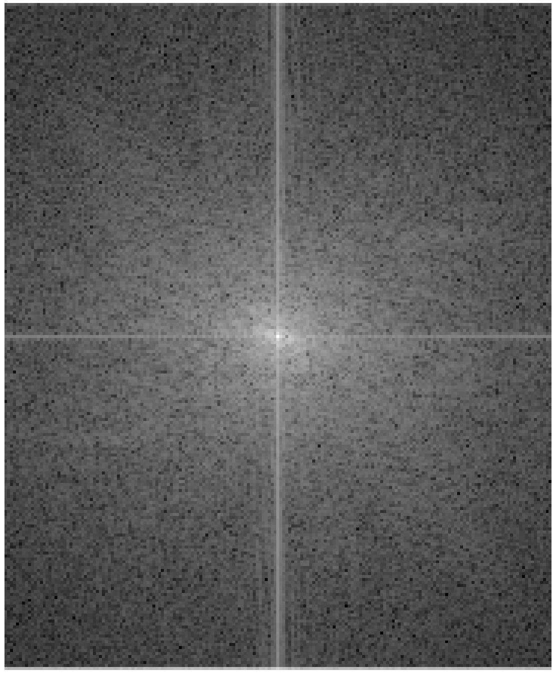


Example of module and argument (2)

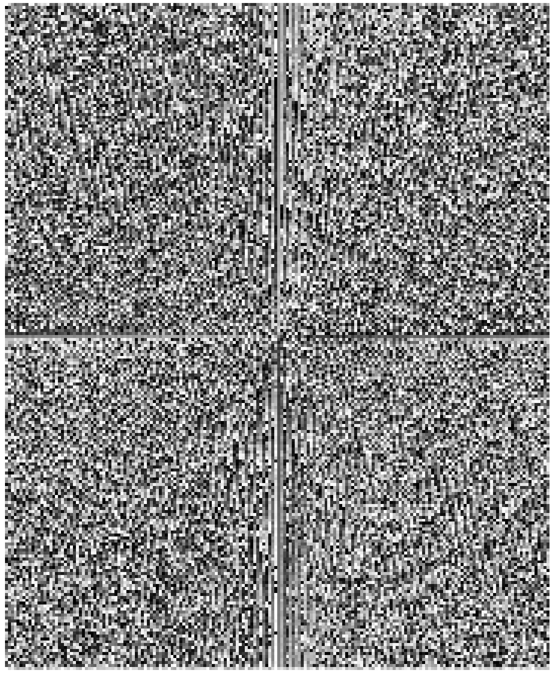
f[m,n]



|F[u,v]|

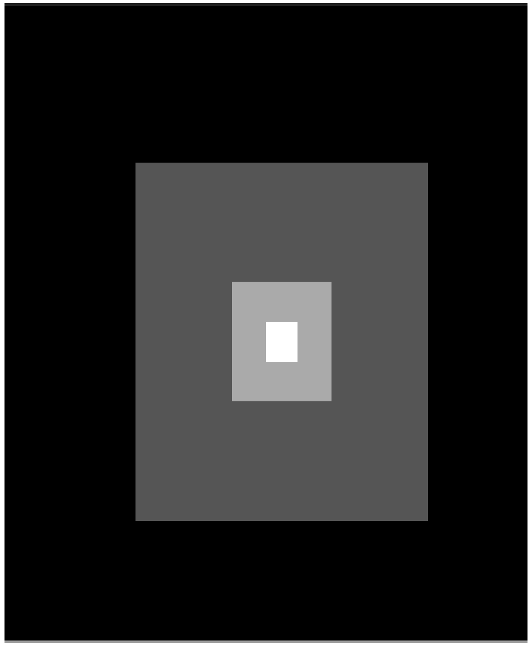


arg[F[u,v]]

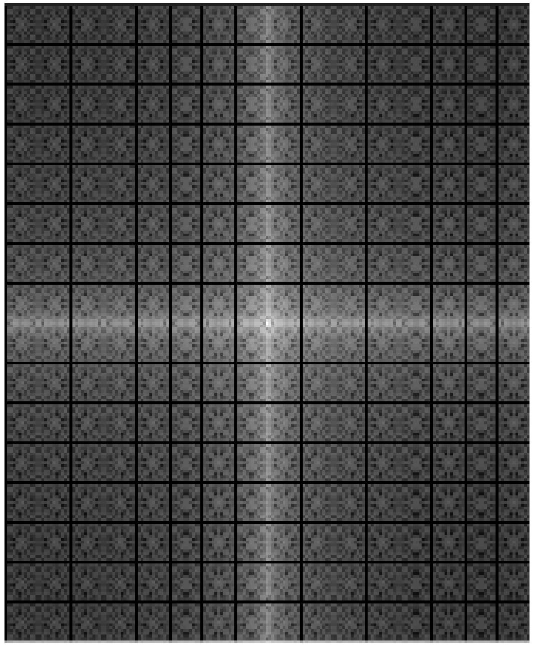


Example of module and argument (3)

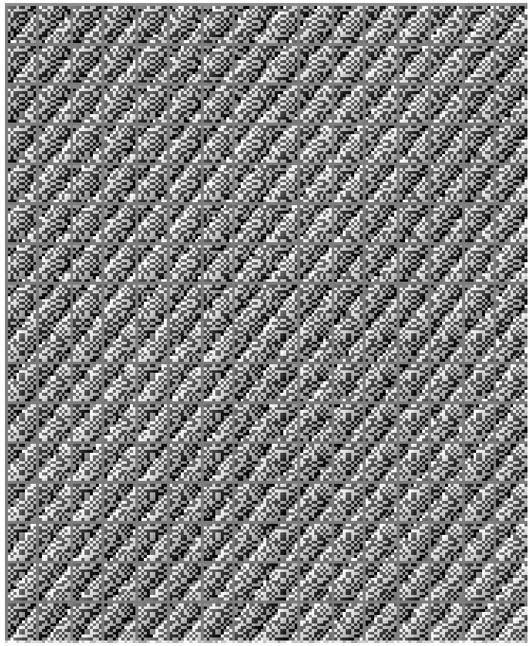
f[m,n]



|F[u,v]|

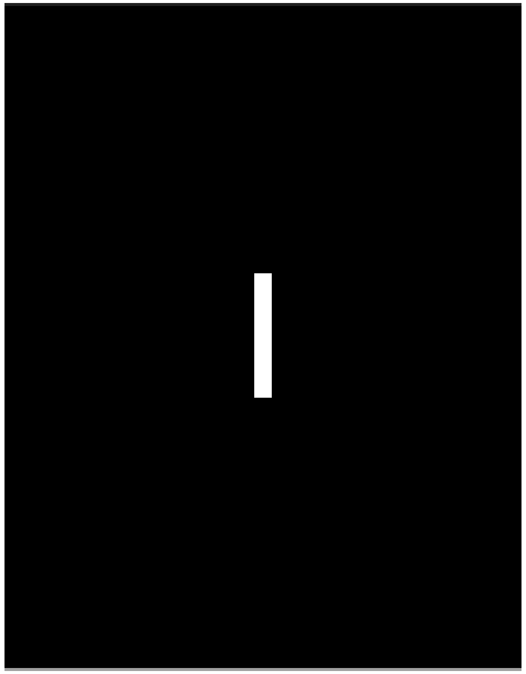


arg[F[u,v]]

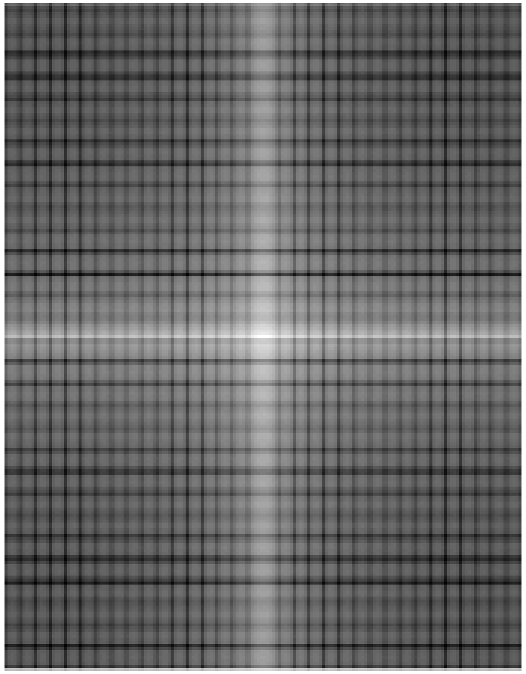


Example of module and argument (4)

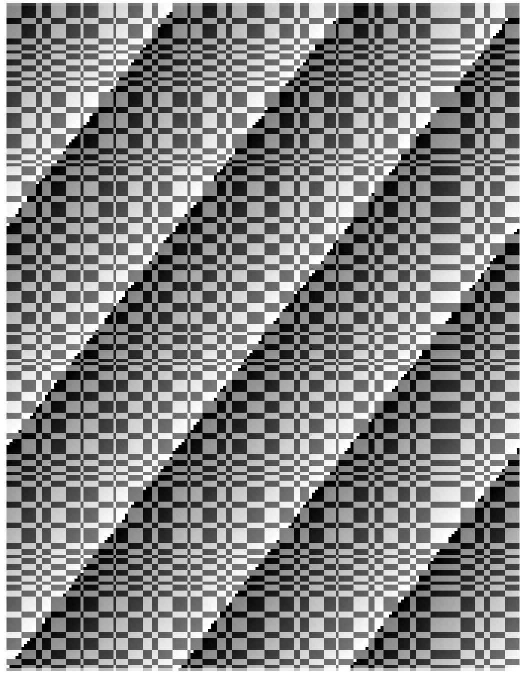
f[m,n]



|F[u,v]|



arg[F[u,v]]



The Inverse Discrete Fourier Transform (IDFT)

- F[u,v] is the spectrum of f[m,n]
- f[m,n] is computed by the **Inverse Discrete Fourier Transform (IDFT)** of the spectrum F[u,v]
- f[m,n] = IDFT[F[u,v]]

$$f[m,n] = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F[u,v]\exp\left(\frac{j2\pi m}{M}u\right)\exp\left(\frac{j2\pi n}{N}v\right)$$

- MATLAB: ifft2.m and ifftshift.m

https://www.mathworks.com/help/matlab/ref/ifft2.html

https://www.mathworks.com/help/matlab/ref/ifftshift.html

DFT/IDFT Properties (1)

$$F[u,v] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n] \exp\left(-\frac{j2\pi u}{M}m\right)\exp\left(-\frac{j2\pi v}{N}n\right)$$

- F[0,0] is the DC coefficient
- F[0,0] corresponds to the sum of all the image pixels

$$F[0,0] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]$$

- Thus, the average intensity value of f[m,n] can be computed as

$$m_f = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n] = \frac{1}{MN} F[0,0]$$

DFT/IDFT Properties (2)

$$F[u,v] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\exp\left(-\frac{j2\pi u}{M}m\right)\exp\left(-\frac{j2\pi v}{N}n\right)$$

- The energy preserving property

$$E_f = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^2[m,n] = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} |F[m,n]|^2$$

- Or, equivalently, the power preserving property

$$P_f = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^2[m,n] = \frac{1}{(MN)^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} |F[m,n]|^2$$

DCT and IDCT

- Discrete Cosine Transform (DCT)

$$F[u,v] = C[u]C[v]\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\cos\left(\frac{(2m+1)u\pi}{2M}\right)\cos\left(\frac{(2n+1)v\pi}{2N}\right)$$

MATLAB: dct2.m and idct2.m
https://www.mathworks.com/help/images/ref/dct2.html
https://www.mathworks.com/help/images/ref/idct2.html

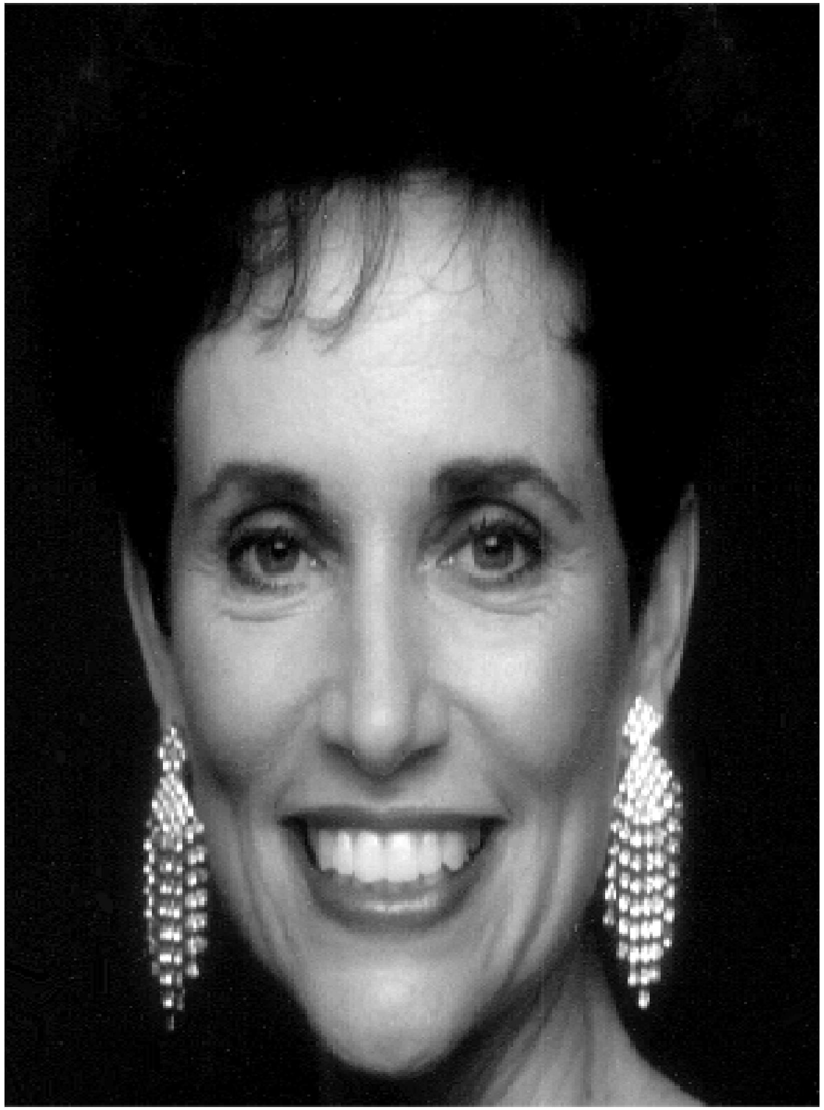
- Inverse Discrete Cosine Transform (IDCT)

$$f[m,n] = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} C[u]C[v]F[u,v]\cos\left(\frac{(2m+1)u\pi}{2N}\right)\cos\left(\frac{(2n+1)v\pi}{2N}\right)$$

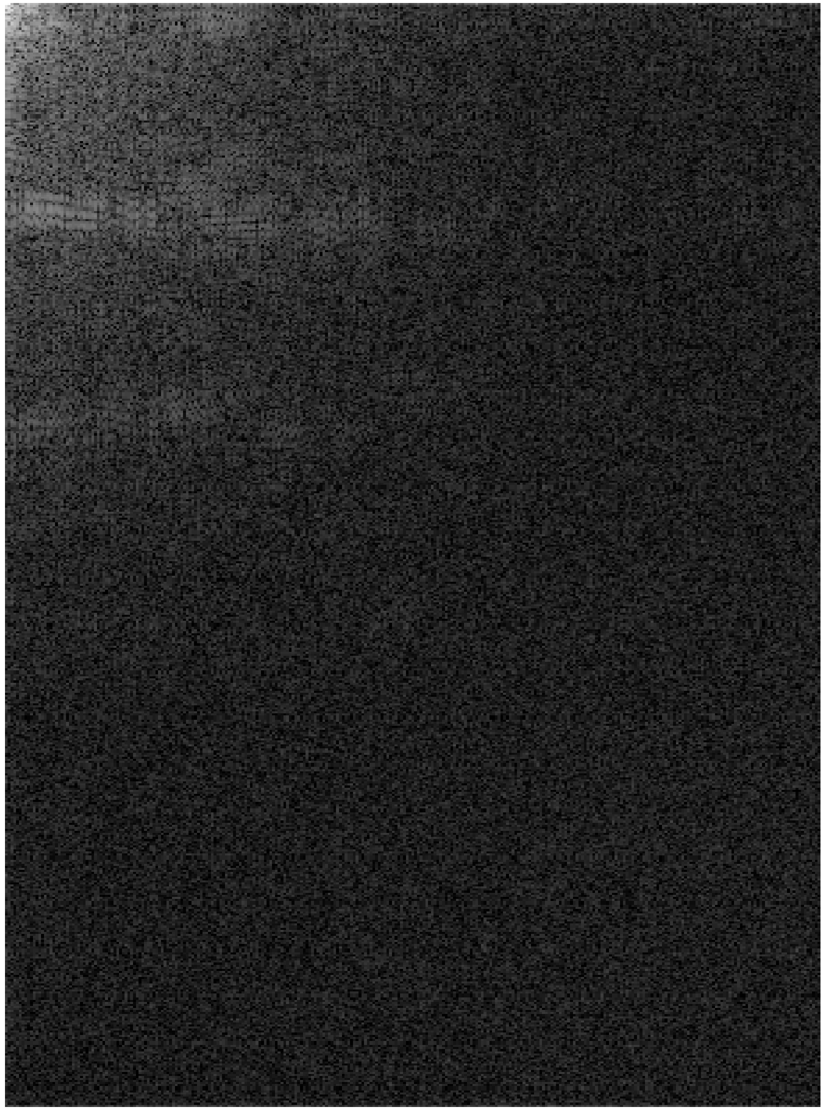
$$C[u] = \begin{cases} \frac{1}{\sqrt{M}}, & u = 0 \\ \sqrt{\frac{2}{M}}, & u \neq 0 \end{cases} \qquad C[v] = \begin{cases} \frac{1}{\sqrt{N}}, & v = 0 \\ \sqrt{\frac{2}{N}}, & v \neq 0 \end{cases}$$

Example of DCT (1)

f[m,n]



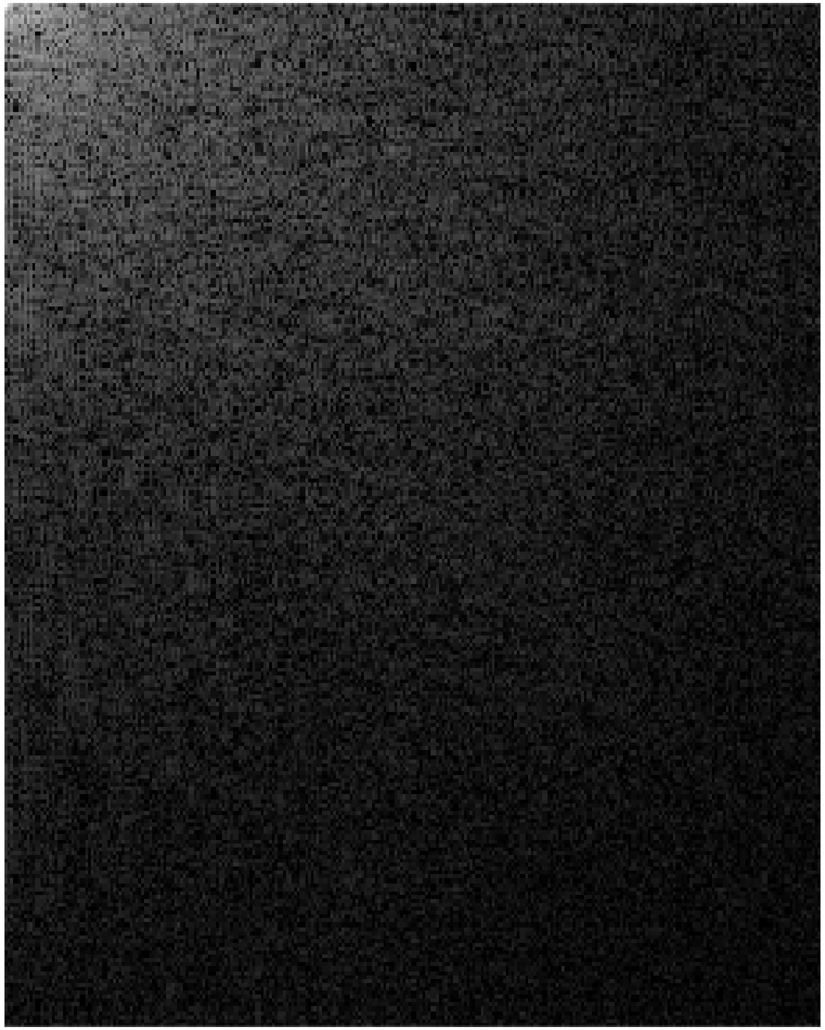
F[u,v] (DCT)



Example of DCT (2)

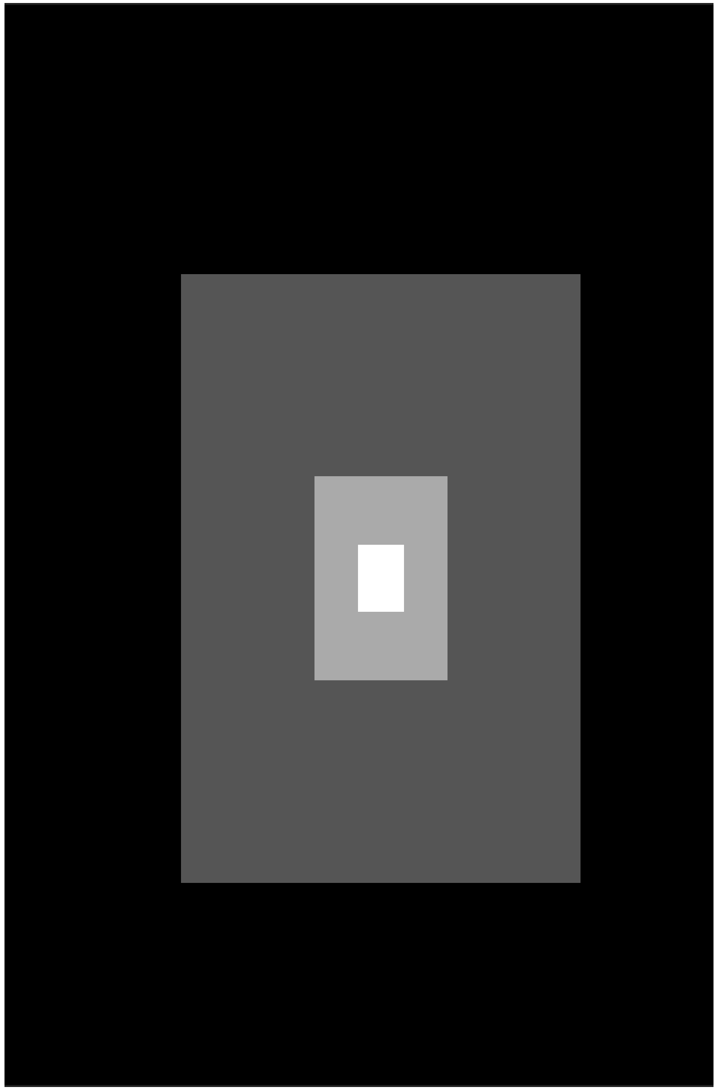
f[m,n]

F[u,v] (DCT)

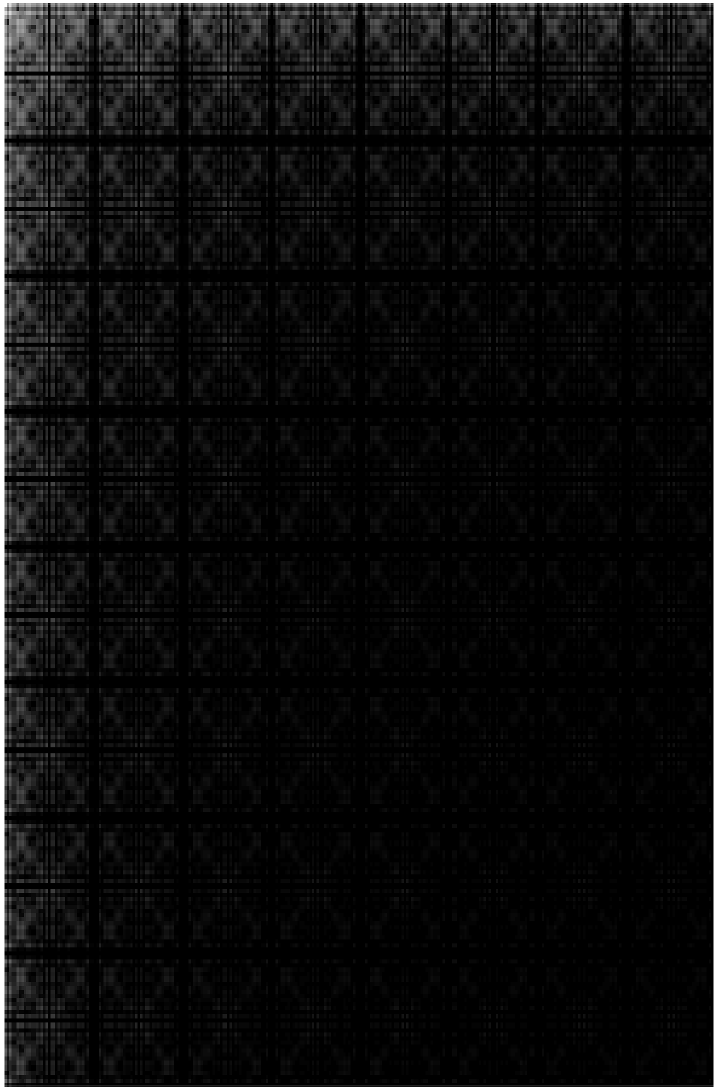


Example of DCT (3)

f[m,n]

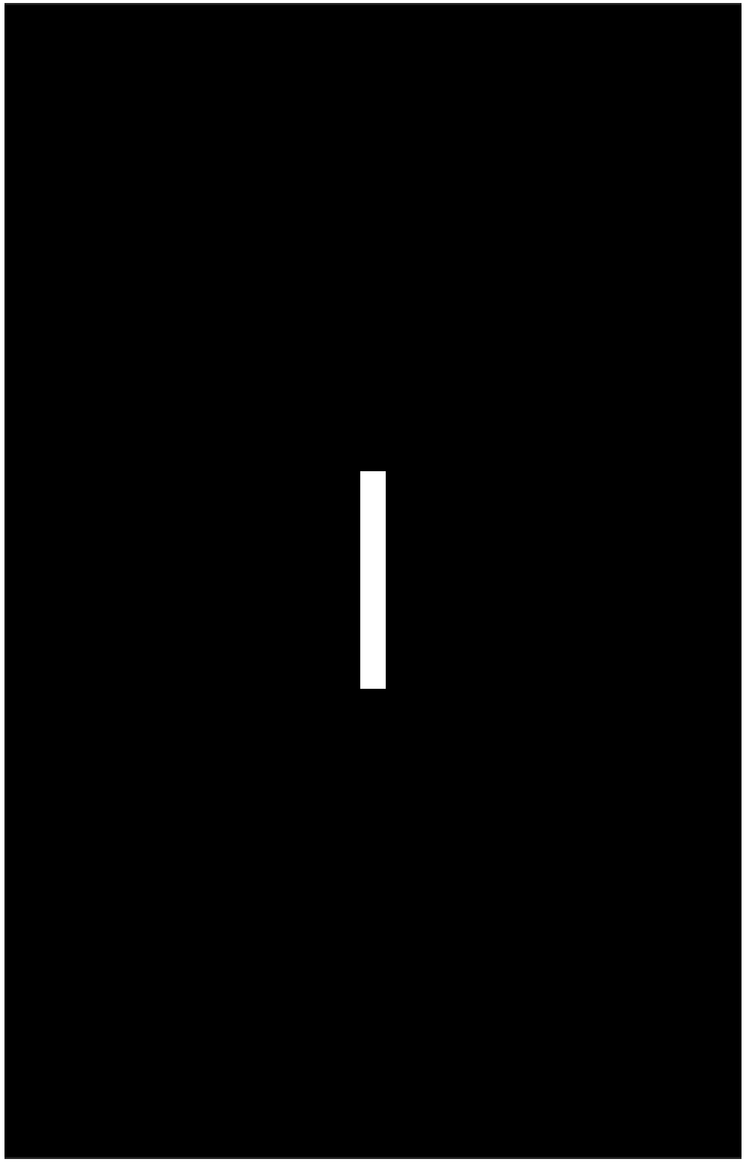


F[u,v] (DCT)

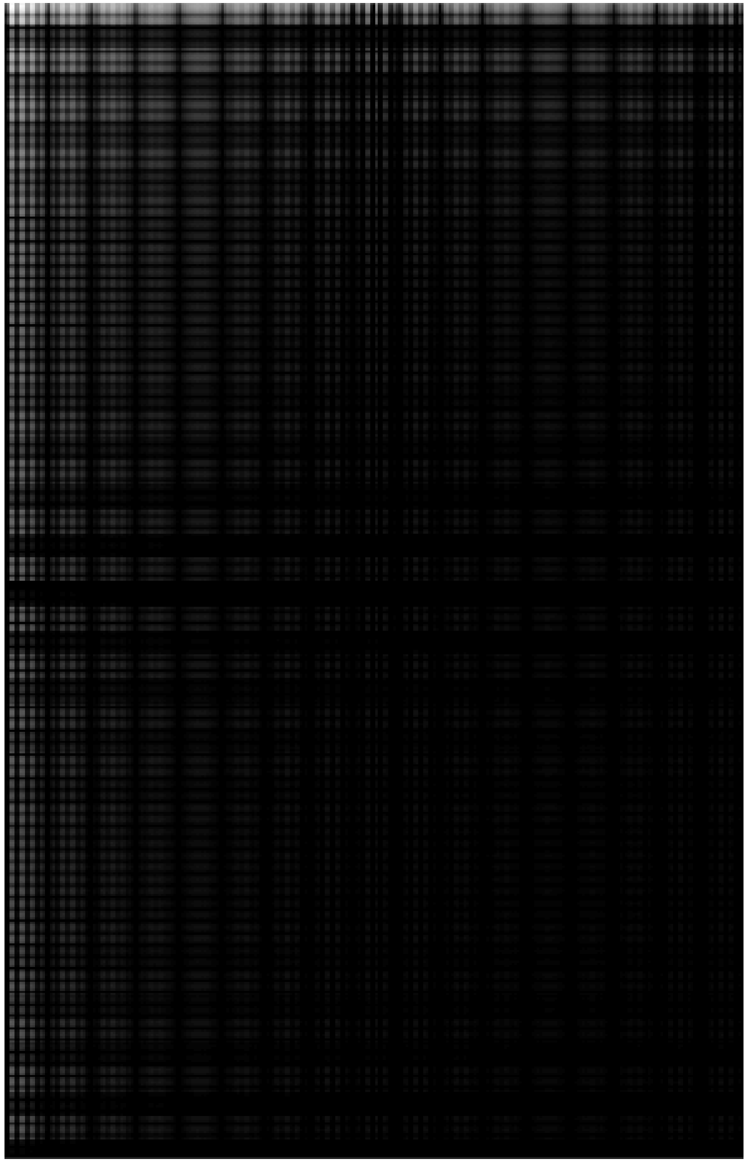


Example of DCT (4)

f[m,n]



F[u,v] (DCT)



DCT/IDCT Properties (1)

$$F[u,v] = C[u]C[v]\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\cos\left(\frac{(2m+1)u\pi}{2M}\right)\cos\left(\frac{(2n+1)v\pi}{2N}\right)$$

$$C[u] = \begin{cases} \frac{1}{\sqrt{M}}, & u = 0 \\ \sqrt{\frac{2}{M}}, & u \neq 0 \end{cases} \qquad C[v] = \begin{cases} \frac{1}{\sqrt{N}}, & v = 0 \\ \sqrt{\frac{2}{N}}, & v \neq 0 \end{cases}$$

$$F[0,0] = C[0]C[0]\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n] = \frac{1}{\sqrt{M}\sqrt{N}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]$$

- F[0,0] is the DC coefficient
- F[0,0] refers to the average intensity value (with a scaling factor)

$$m_f = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n] = \frac{1}{\sqrt{M}\sqrt{N}}F[0,0]$$

DCT/IDCT Properties (2)

$$F[u,v]=C[u]C[v]\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f[m,n]\cos\left(\frac{(2m+1)u\pi}{2M}\right)\cos\left(\frac{(2n+1)v\pi}{2N}\right)$$

$$C[u]=\begin{cases}\dfrac{1}{\sqrt{M}}, & u=0\\ \sqrt{\dfrac{2}{M}}, & u\neq 0\end{cases} \qquad C[v]=\begin{cases}\dfrac{1}{\sqrt{N}}, & v=0\\ \sqrt{\dfrac{2}{N}}, & v\neq 0\end{cases}$$

The energy preserving property

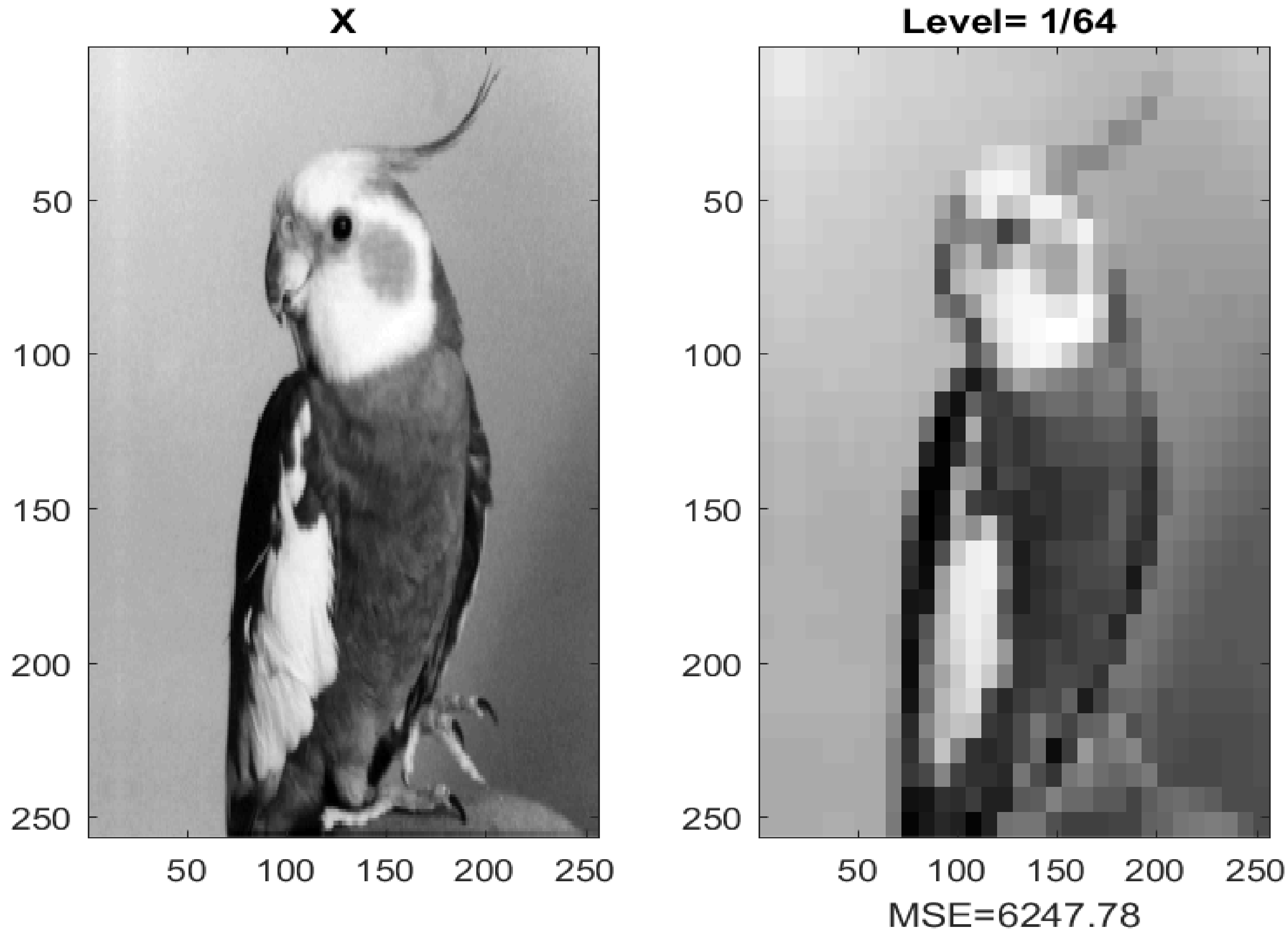
$$E_f=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f^2[m,n]=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}F^2[m,n]$$

Or, equivalently, the power preserving property

$$P_f=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f^2[m,n]=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}F^2[m,n]$$

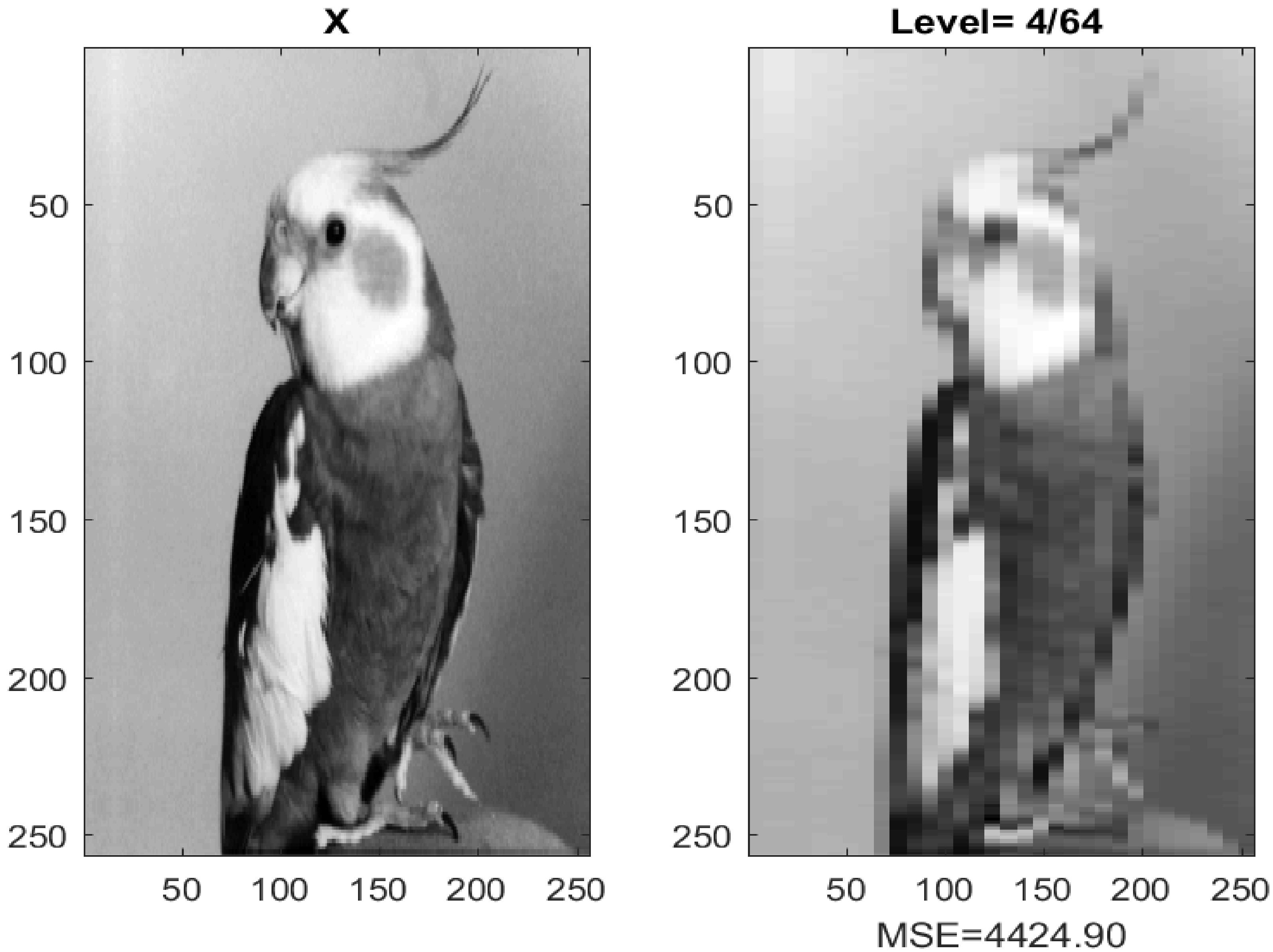
DCT/IDCT Properties (3)

Analysis of the energy preserving property for images



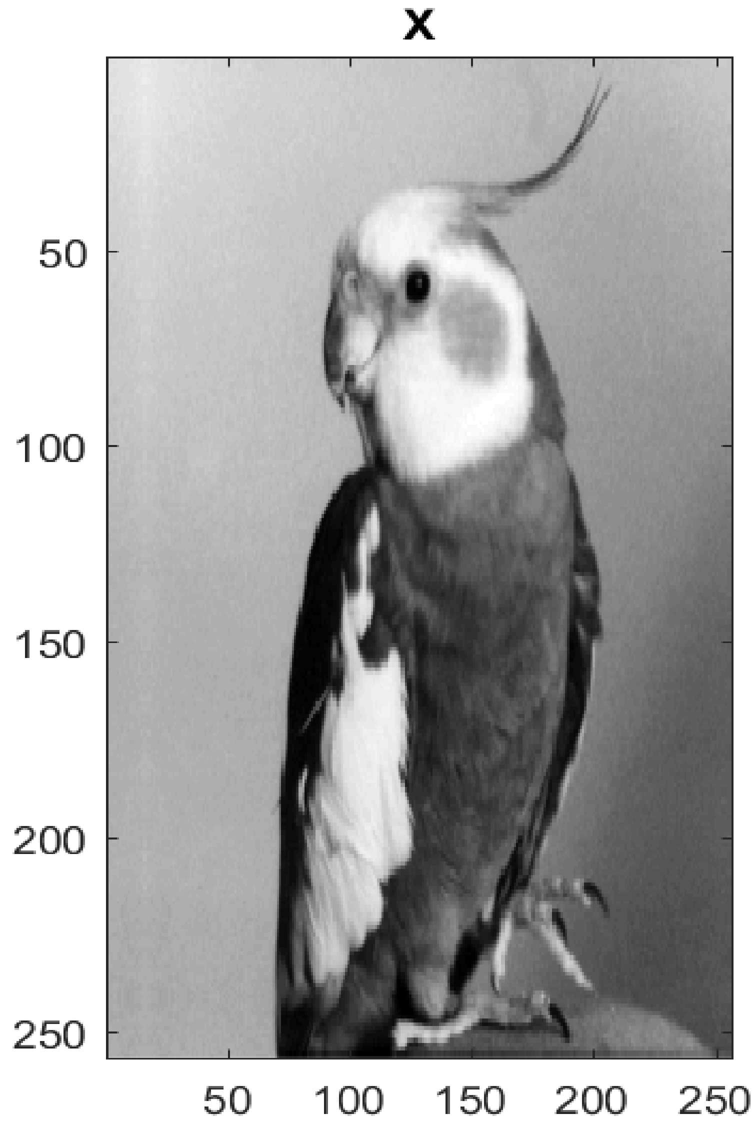
DCT/IDCT Properties (4)

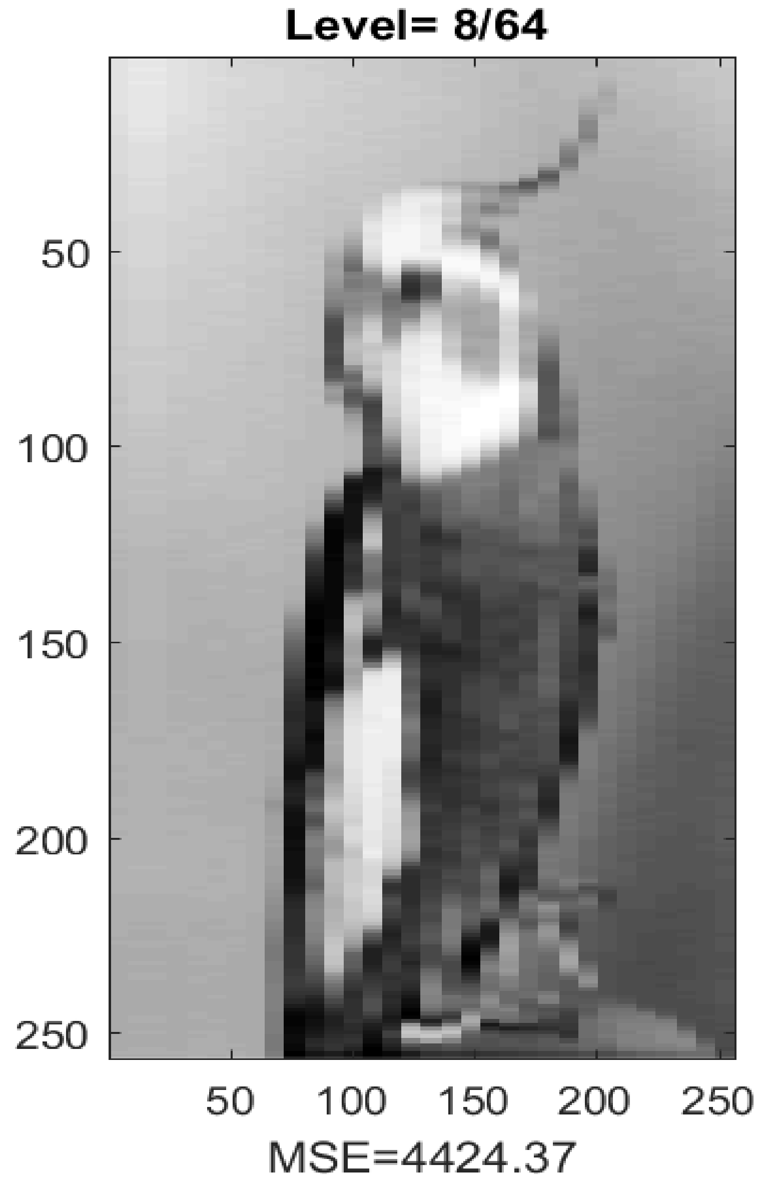
Analysis of the energy preserving property for images



DCT/IDCT Properties (5)

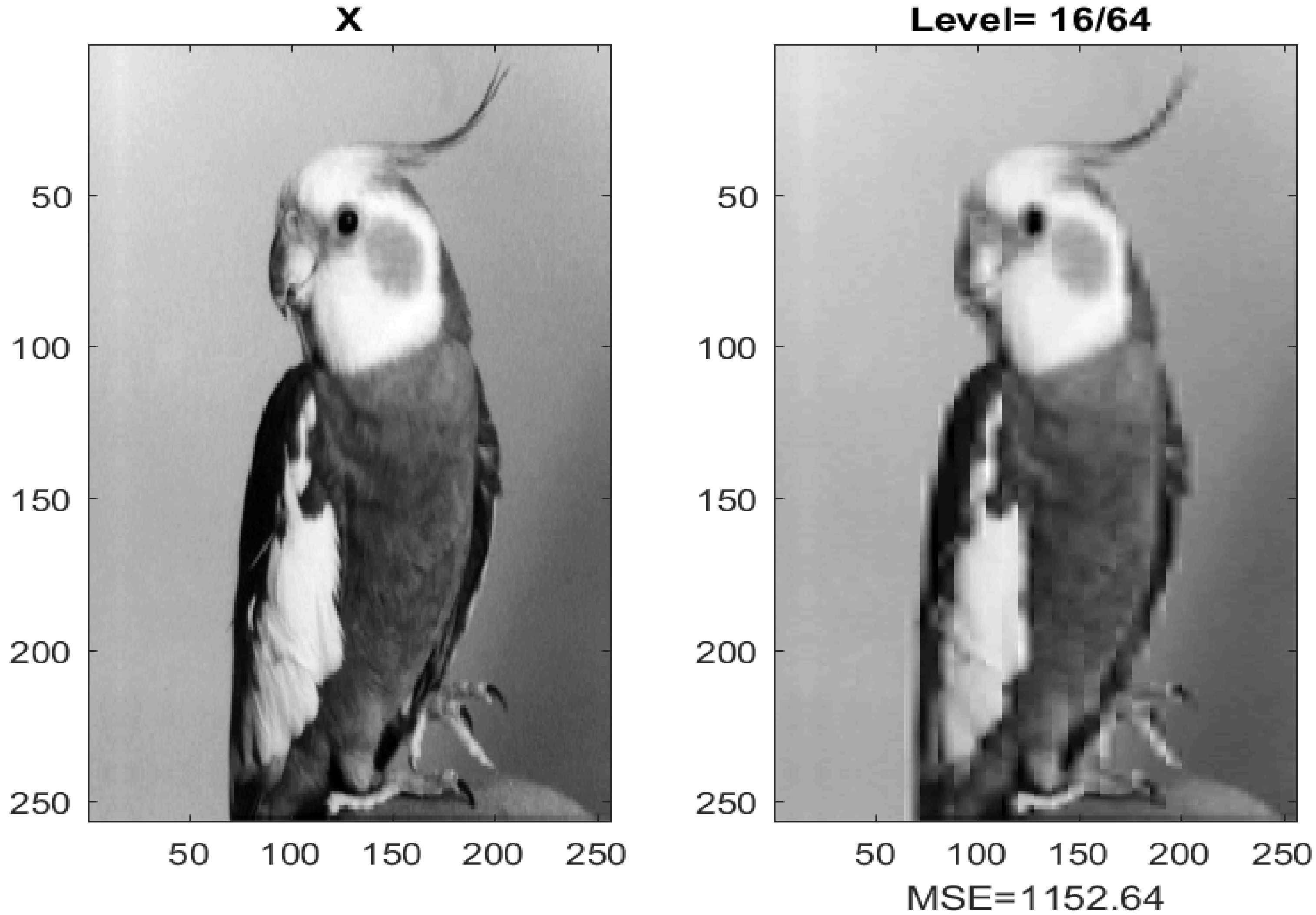
Analysis of the energy preserving property for images





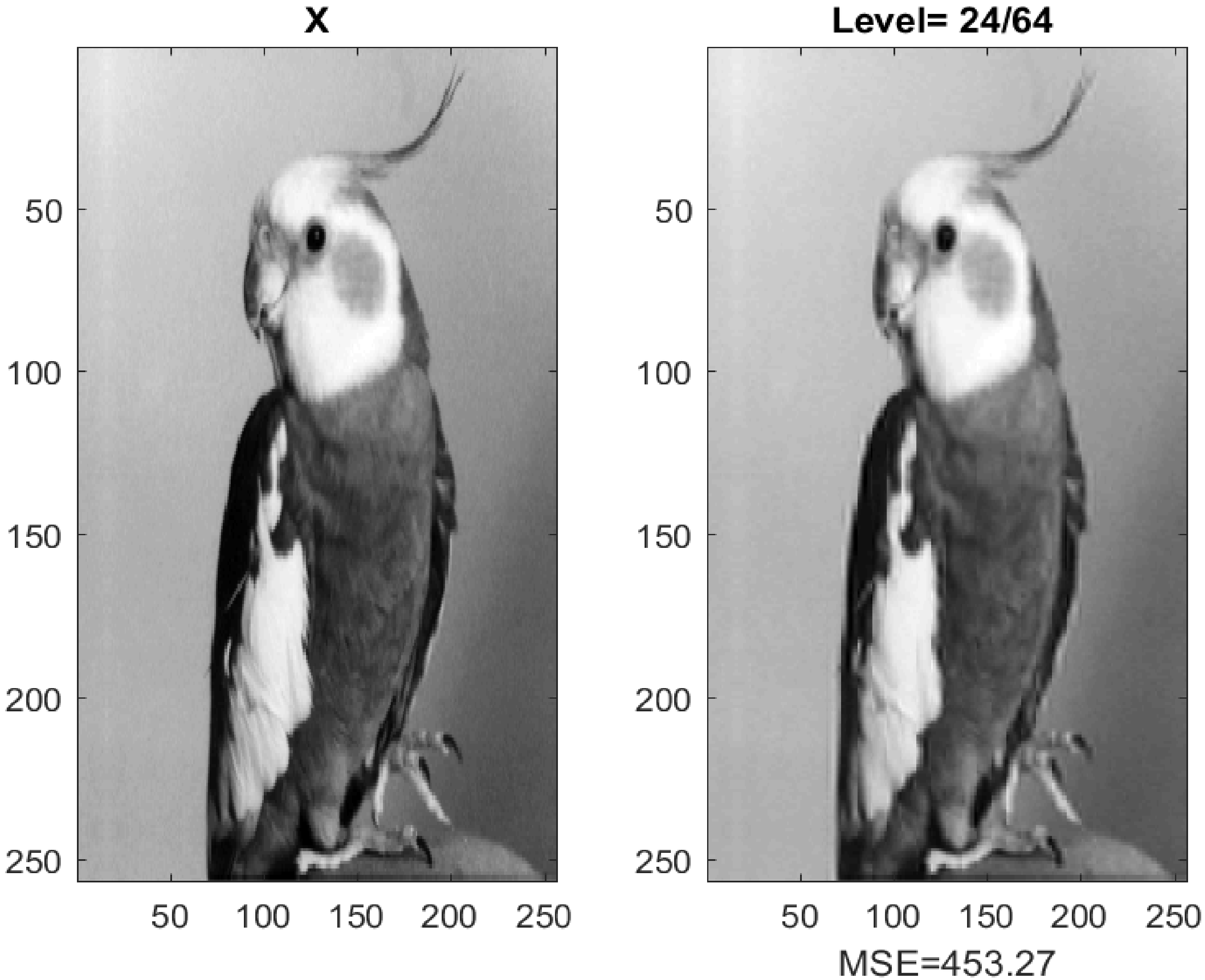
DCT/IDCT Properties (6)

Analysis of the energy preserving property for images



DCT/IDCT Properties (7)

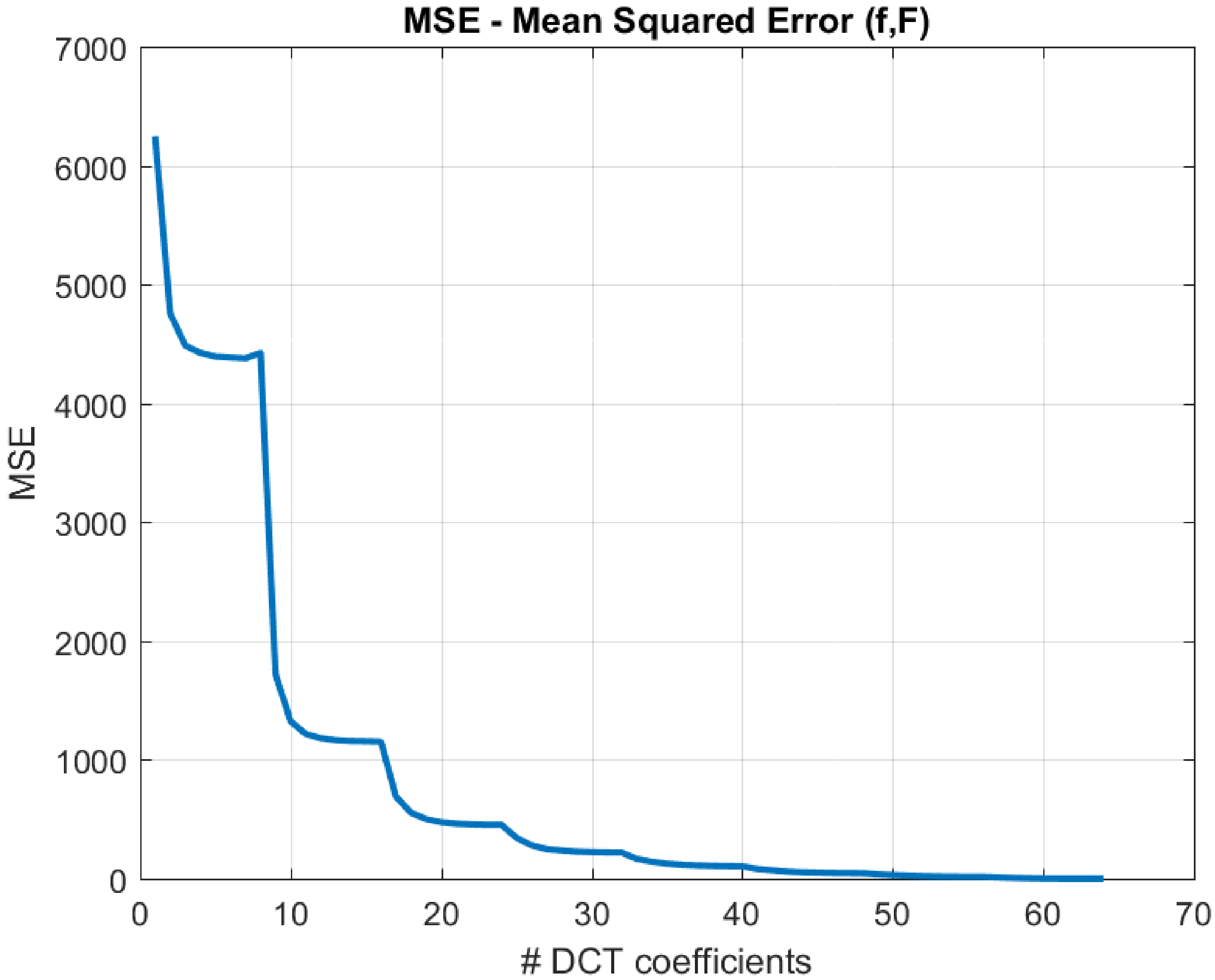
Analysis of the energy preserving property for images



DCT/IDCT Properties (8)

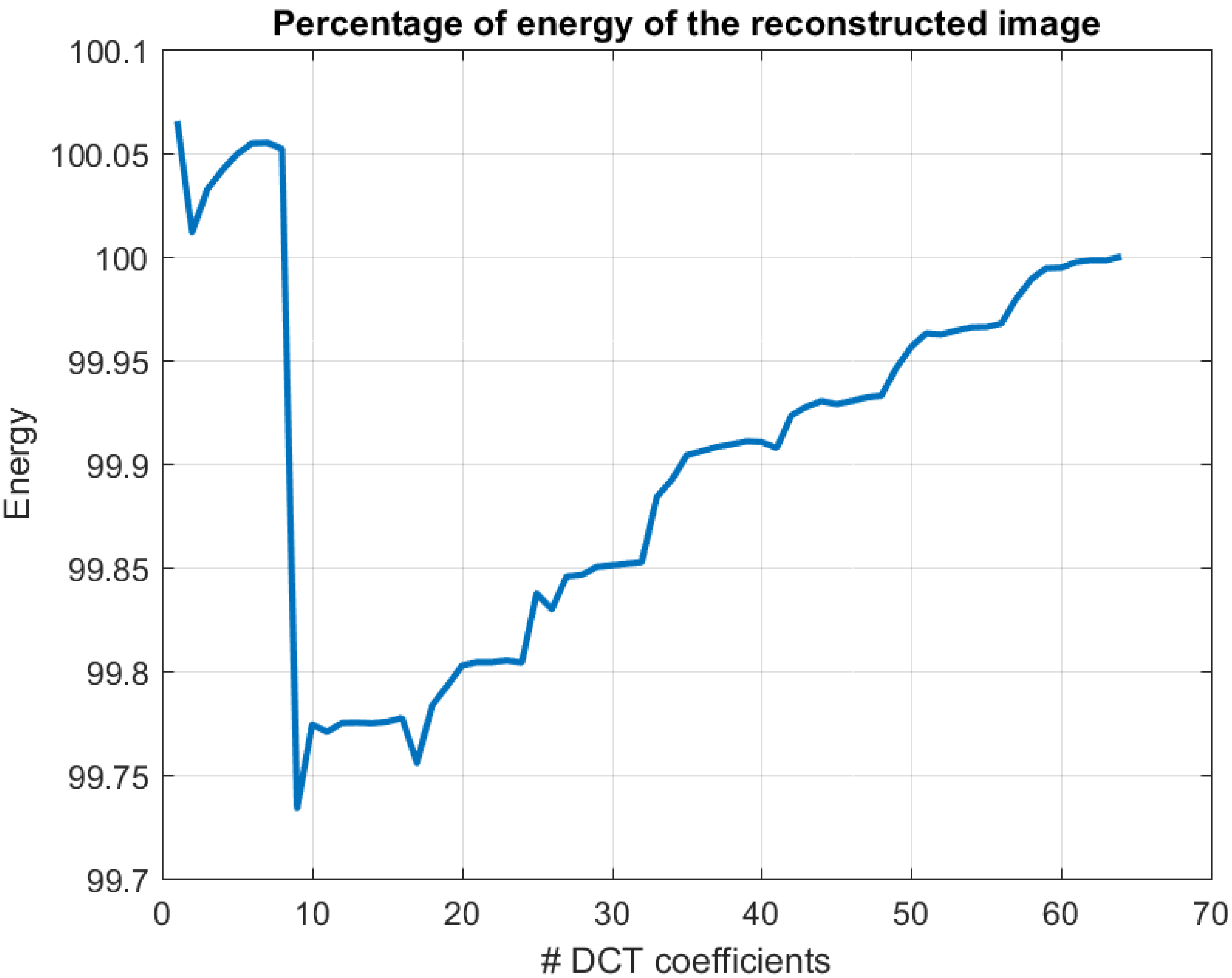
Analysis of the energy preserving property for images

The Mean Squared Error (MSE) between the original and the reconstructed image



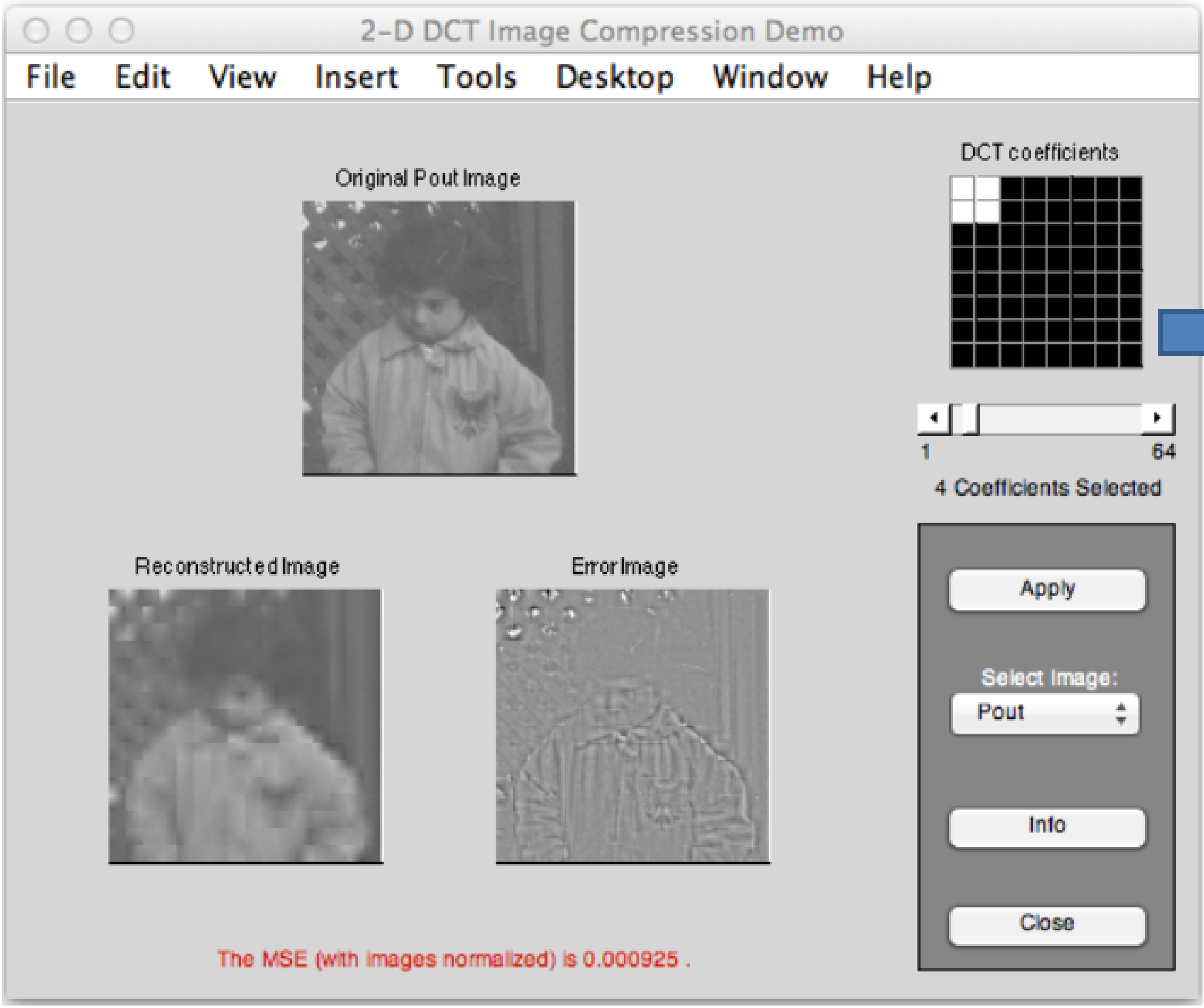
DCT/IDCT Properties (9)

Analysis of the energy preserving property for images (8x8 blocks)



DCT/IDCT Properties (10)

Analysis of the energy preserving property for images (8x8 blocks)



4 coefficients selected

DCT/IDCT Properties (11)

The DCT coefficients have the following key properties:

- **Energy Compaction** - The energy of the original pixel block is located on a few significant DCT coefficients
- **Statistical Decorrelation** – The original pixels on the spatial domain block has large correlations, the DCT coefficients are usually decorrelated. There is no strong correlation/resemblance between the values found a given neighborhood

These properties make the DCT and adequate tool **for lossy compression schemes, such as JPEG, MPEG, MP3,…**

DCT/IDCT Properties (12)

An 8×8 block from the Y image of 'Lena'

200	202	189	188	189	175	175	175
200	203	198	188	189	182	178	175
203	200	200	195	200	187	185	175
200	200	200	200	197	187	187	187
200	205	200	200	195	188	187	175
200	200	200	200	200	190	187	175
205	200	199	200	191	187	187	175
210	200	200	200	188	185	187	186

$f(i, j)$

515	65	-12	4	1	2	-8	5
-16	3	2	0	0	-11	-2	3
-12	6	11	-1	3	0	1	-2
-8	3	-4	2	-2	-3	-5	-2
0	-2	7	-5	4	0	-1	-4
0	-3	-1	0	4	1	-1	0
3	-2	-3	3	3	-1	-1	3
-2	5	-2	4	-2	2	-3	0

$F(u, v)$

DCT/IDCT Properties (13)

Another 8×8 block from the Y image of 'Lena'

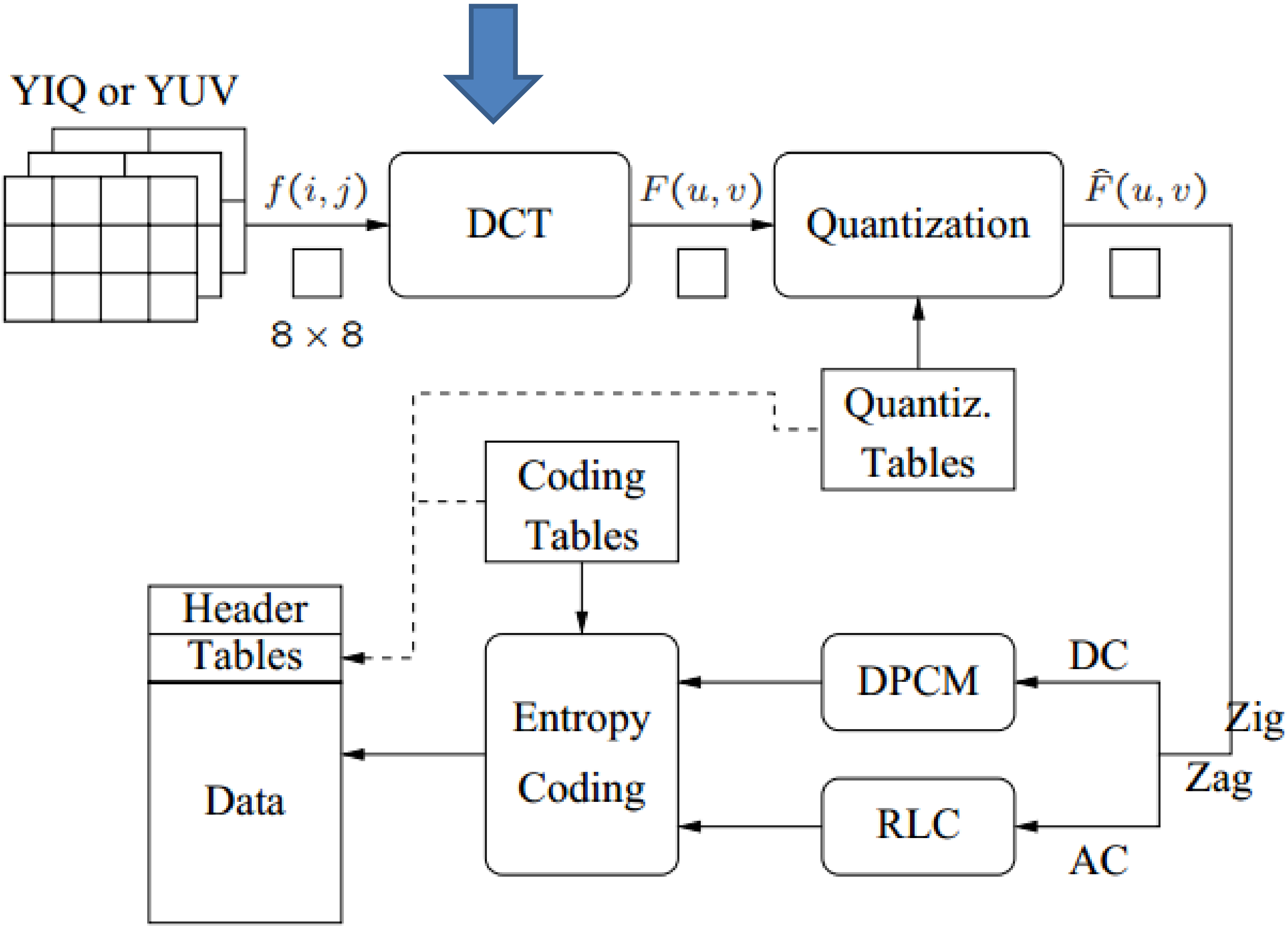
70	70	100	70	87	87	150	187
85	100	96	79	87	154	87	113
100	85	116	79	70	87	86	196
136	69	87	200	79	71	117	96
161	70	87	200	103	71	96	113
161	123	147	133	113	113	85	161
146	147	175	100	103	103	163	187
156	146	189	70	113	161	163	197

$f(i,j)$

-80	-40	89	-73	44	32	53	-3
-135	-59	-26	6	14	-3	-13	-28
47	-76	66	-3	-108	-78	33	59
-2	10	-18	0	33	11	-21	1
-1	-9	-22	8	32	65	-36	-1
5	-20	28	-46	3	24	-30	24
6	-20	37	-28	12	-35	33	17
-5	-23	33	-30	17	-5	-4	20

$F(u,v)$

DCT/IDCT Application (1)

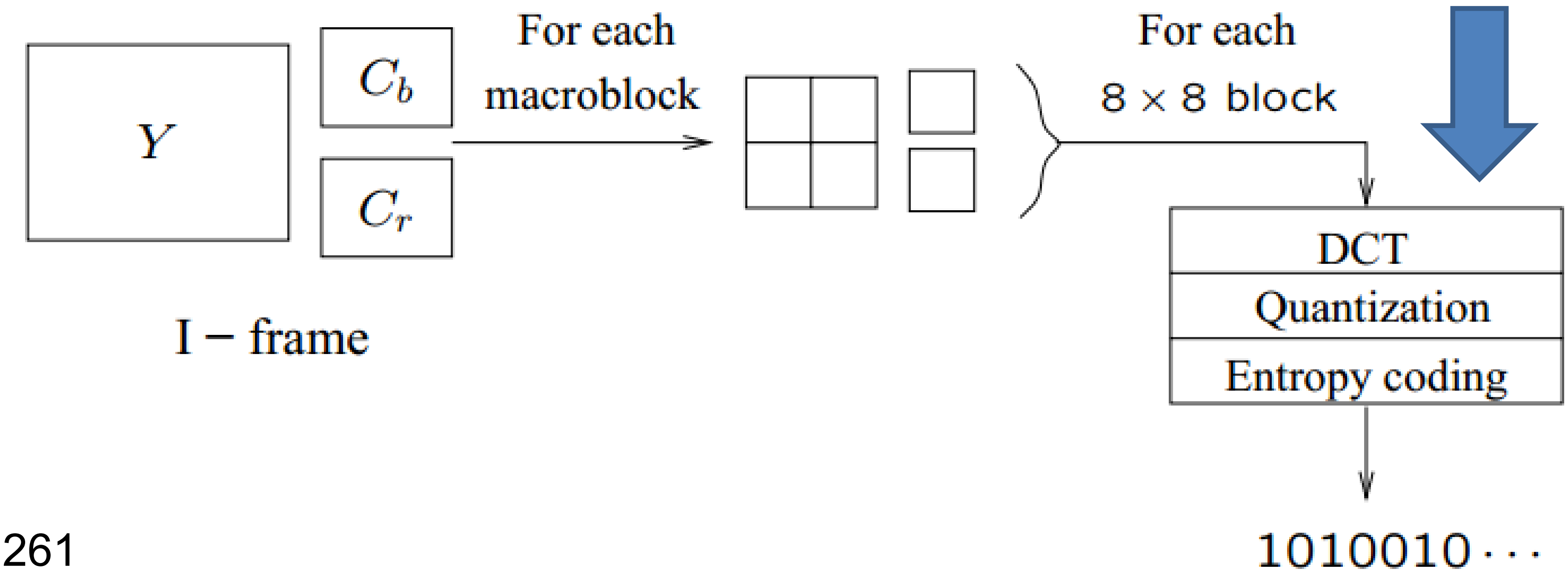


JPEG

Still
Image
Encoder

Block diagram for JPEG encoder.

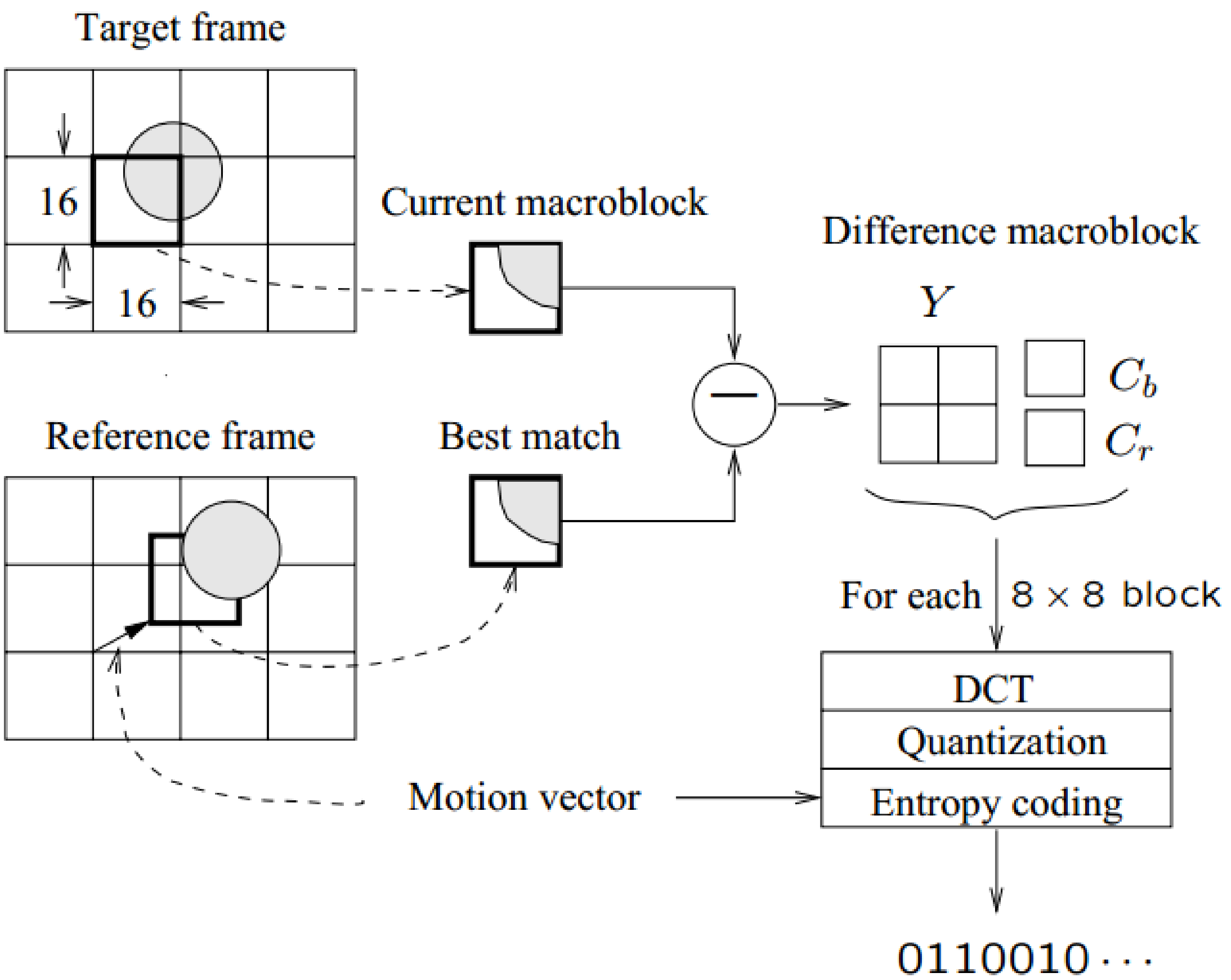
DCT/IDCT Application (2)



H.261
I-frame

Video
Encoder

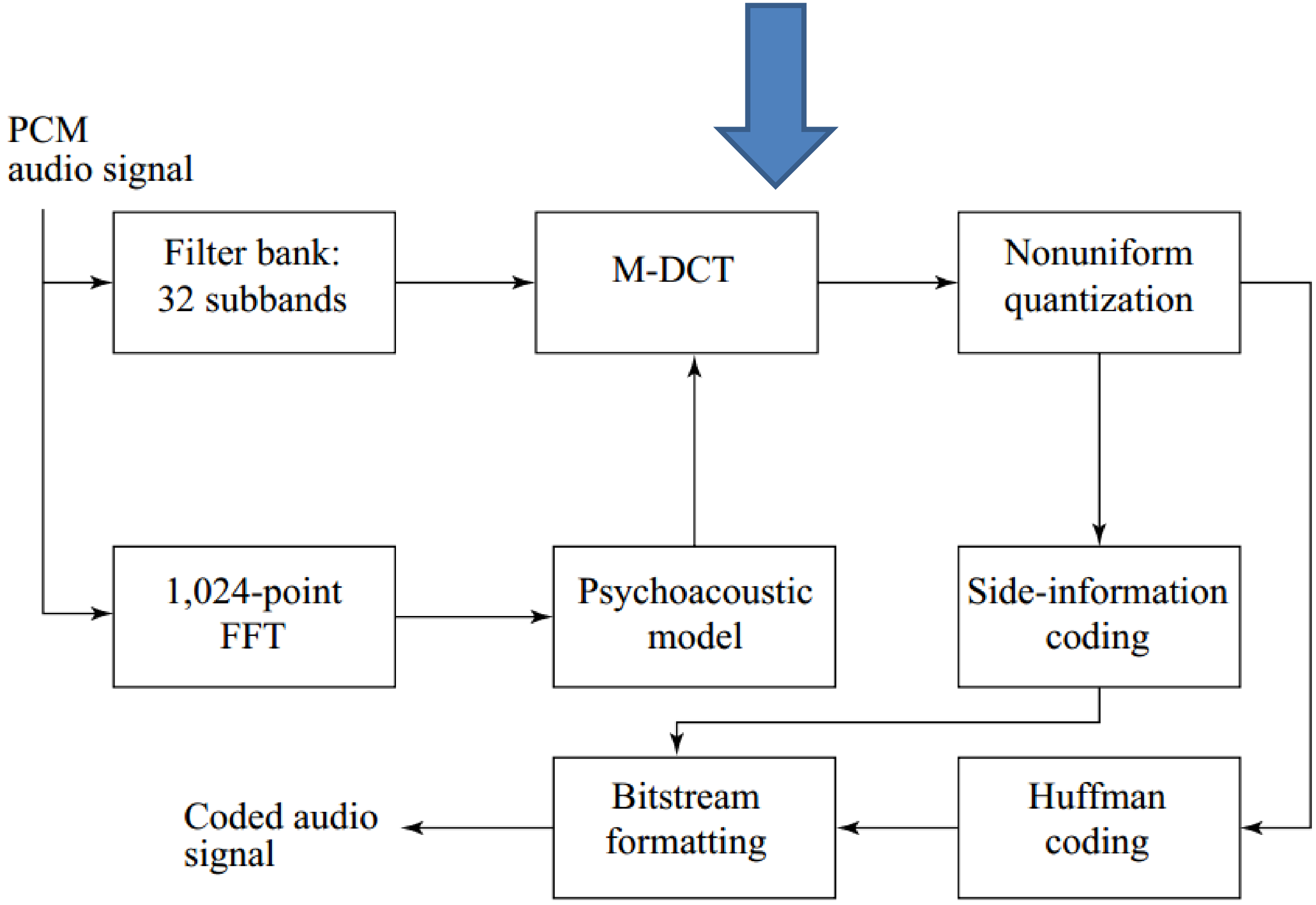
DCT/IDCT Application (3)



H.261
P-frame

Video
Encoder

DCT/IDCT Application (4)



MPEG-1
Layer 3
(MP3)

Audio
Encoder

MPEG-Audio Layer 3 Coding.

DCT/IDCT Application (5)

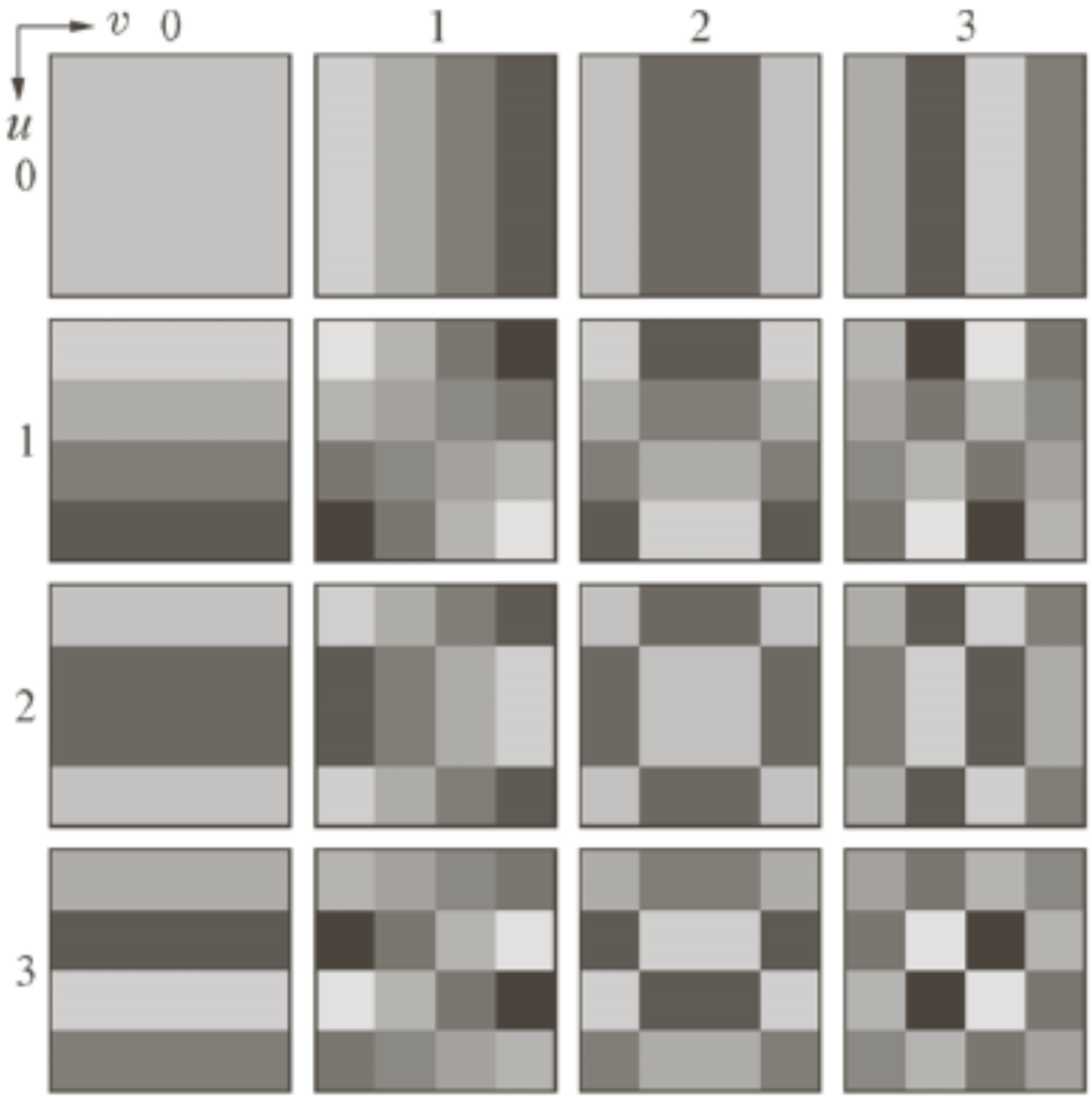


FIGURE 8.23
Discrete-cosine basis functions for $n = 4$. The origin of each block is at its top left.

DCT/IDCT Application (6)

a b c
d e f

FIGURE 8.24 Approximations of Fig. 8.9(a) using the (a) Fourier, (b) Walsh-Hadamard, and (c) cosine transforms, together with the corresponding scaled error images in (d)–(f).

DCT/IDCT Application (7)

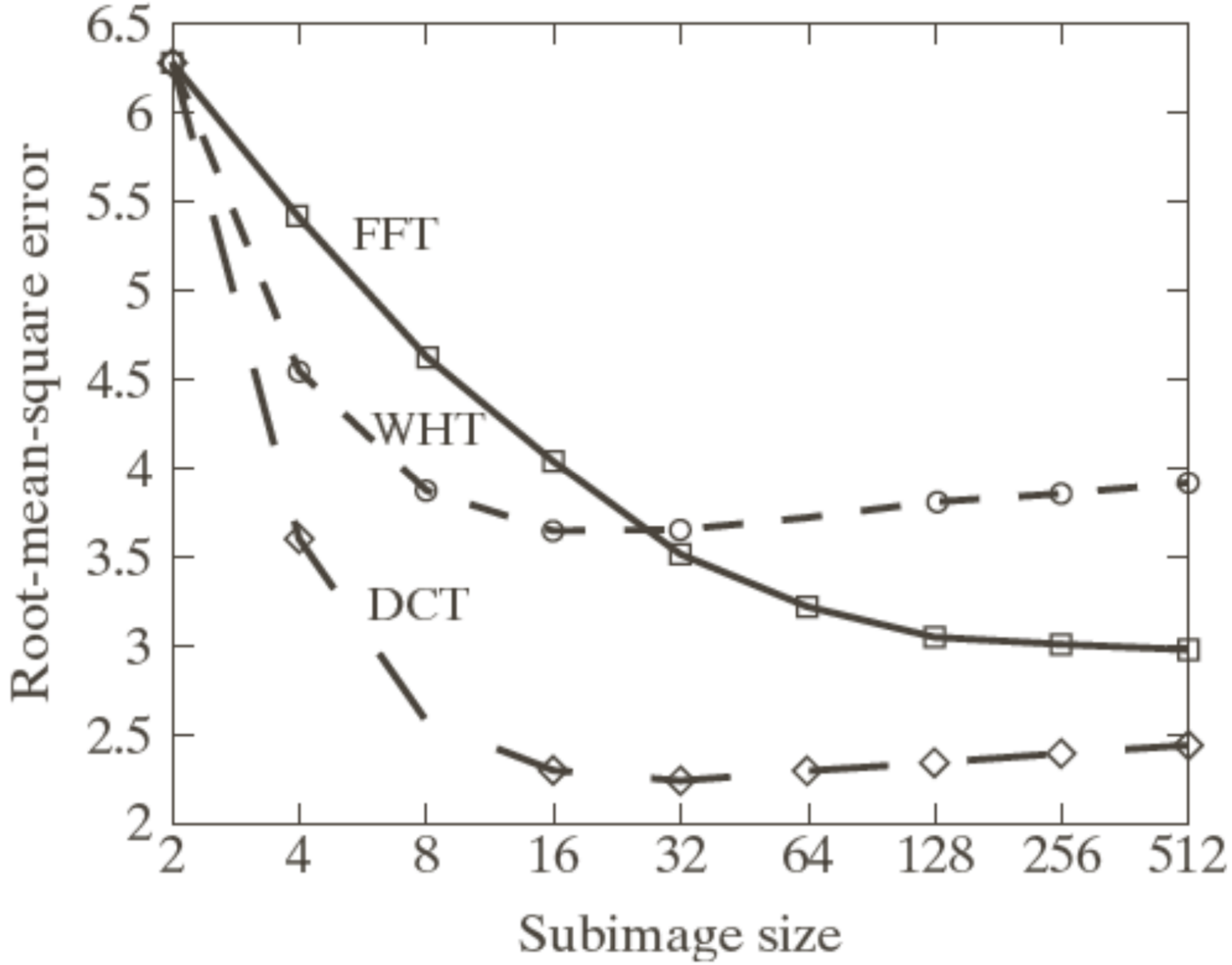
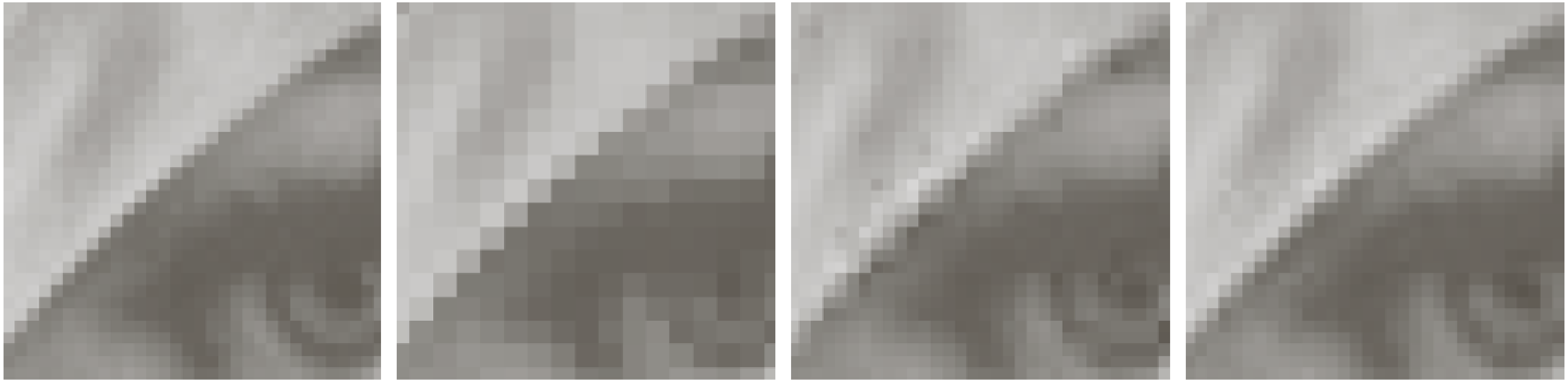


FIGURE 8.26
Reconstruction error versus subimage size.

DCT/IDCT Application (8)



a b c d

FIGURE 8.27 Approximations of Fig. 8.27(a) using 25% of the DCT coefficients and (b) 2×2 subimages, (c) 4×4 subimages, and (d) 8×8 subimages. The original image in (a) is a zoomed section of Fig. 8.9(a).

DCT/IDCT Application (9)

FIGURE 8.32 Two JPEG approximations of Fig. 8.9(a). Each row contains a result after compression and reconstruction, the scaled difference between the result and the original image, and a zoomed portion of the reconstructed image.

Exercises (1)

1. Considere a representação de imagem no domínio da frequência e as técnicas de processamento de imagem nesse domínio.

   a) {1,0} O cálculo do espetro $F[u,v]$ de uma imagem retangular $f[m,n]$ de resolução $M \times N$ é dado por

   $$F[u,v] = \mathrm{DFT}[f[m,n]] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] \times \exp\left(-j\frac{2\pi um}{M}\right) \times \exp\left(-j\frac{2\pi vn}{N}\right).$$

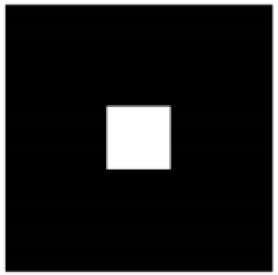
   Indique o número total de multiplicações reais, assinaladas pelo símbolo $\times$, necessárias para calcular a totalidade do espetro $F[u,v]$, no pior caso.

   b) {1,0} As imagens $f_1[m,n]$ e $f_2[m,n]$ apresentam o mesmo objeto em localizações diferentes.

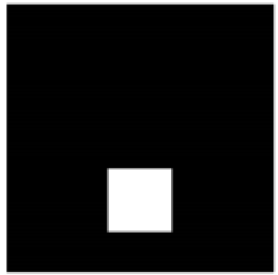
   Relativamente ao módulo e fase dos espetros destas duas imagens, qual das seguintes afirmações é verdadeira?

   i) $|F_1[u,v]| = |F_2[u,v]|$ *e* $arg[F_1[u,v]] = arg[F_2[u,v]]$.
   ii) $|F_1[u,v]| = |F_2[u,v]|$ *e* $arg[F_1[u,v]] \neq arg[F_2[u,v]]$.
   iii) $|F_1[u,v]| \neq |F_2[u,v]|$ *e* $arg[F_1[u,v]] = arg[F_2[u,v]]$.
   iv) $|F_1[u,v]| \neq |F_2[u,v]|$ *e* $arg[F_1[u,v]] \neq arg[F_2[u,v]]$.

f_1 [m,n]



f_2 [m,n]



Exercises (2)

6. {R2} A *Discrete Cosine Transform* (DCT) para imagens de resolução $M \times N$ é definida da forma que se apresenta de seguida.

$$F[u,v] = \mathrm{DCT}[f[m,n]] = C[u]C[v]\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n] \times \cos\left(\frac{(2m+1)u\pi}{2M}\right) \times \cos\left(\frac{(2n+1)v\pi}{2N}\right),$$

em que $C[u] = \begin{cases} \frac{1}{\sqrt{M}}, & u = 0 \\ \sqrt{\frac{2}{M}}, & u \in \{1,\ldots,M-1\} \end{cases}$ e $C[v] = \begin{cases} \frac{1}{\sqrt{N}}, & v = 0 \\ \sqrt{\frac{2}{N}}, & v \in \{1,\ldots,N-1\} \end{cases}$.

   a) {1,5} Indique o número total de multiplicações reais (assinaladas pelo símbolo $\times$) e de somas, necessárias para calcular a totalidade de $F[u,v]$, no pior caso.

   b) {1,5} Realizou-se o cálculo da DCT da imagem $g[m,n]$ de resolução 3×3 e obteve-se $G[u,v]$. Verificou-se que todos os coeficientes de $G[u,v]$ são nulos à exceção de $G[0,0]$ e $G[1,0]$, os quais tomam os valores 9 e -2, respetivamente. Relativamente a $g[m,n]$ indique: a sua energia; o valor da soma de todos os *pixels* que a constituem.

Exercises (3)

6. A *Discrete Cosine Transform* (DCT) para imagens de resolução $M \times N$ é definida da forma que se apresenta de seguida.

$$F[u,v] = \text{DCT}[f[m,n]] = C[u]C[v] \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] \times \cos\left(\frac{(2m+1)u\pi}{2M}\right) \times \cos\left(\frac{(2n+1)v\pi}{2N}\right),$$

em que $C[u] = \begin{cases} \frac{1}{\sqrt{M}}, & u = 0 \\ \sqrt{\frac{2}{M}}, & u \in \{1,\ldots,M-1\} \end{cases}$ e $C[v] = \begin{cases} \frac{1}{\sqrt{N}}, & v = 0 \\ \sqrt{\frac{2}{N}}, & v \in \{1,\ldots,N-1\} \end{cases}$.

a) {1,0} Seja $F[u,v] = \text{DCT}[f[m,n]]$, com $f[m,n] = \begin{bmatrix} 10 & 10 \\ 20 & 20 \end{bmatrix}$. Determine $F[0,0]$ e $F[1,0]$.

b) {1,0} Seja $g[m,n] = 5 + 2f[m,n]$. Sem realizar cálculos de DCT, indique a relação entre $G[u,v] = \text{DCT}[g[m,n]]$ e $F[u,v]$.

Exercises (4)

1. The *discrete Fourier transform* (DFT), for images with $M \times N$ resolution is defined as

$$F[u,v] = \text{DFT}[f[m,n]] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n] \exp\left(-j\frac{2\pi um}{M}\right) \exp\left(-j\frac{2\pi vn}{N}\right).$$

a) Let $F_1[u,v] = \text{DFT}[f_1[m,n]] = \begin{bmatrix} 12, & -3+j\sqrt{3}, & -3-j\sqrt{3} \\ -2, & 1-j\sqrt{3}, & 1+j\sqrt{3} \end{bmatrix}$.

i) $\{1.25\}$ Compute the module and the phase of the spectrum $F_1[u,v]$, named as $|F_1[u,v]|$ and $\arg[F_1[u,v]]$.

ii) $\{1.25\}$ For the $f_1[m,n]$ image, state: its spatial resolution; its average intensity; its energy.

b) Let $G_1[u,v] = F_1[u,v] \times \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ and $g_1[m,n] = \text{IDFT}[G_1[u,v]]$.

i) $\{1.25\}$ Compute the quotient between the energy of images $g_1[m,n]$ and $f_1[m,n]$, defined as $R = E_{g1}/E_{f1}$.

ii) $\{1.25\}$ Compute the phase spectrum $\arg[G_1[u,v]]$ and compare it with the phase spectrum $\arg[F_1[u,v]]$. Comment on the results.

Exercises (5)

6. {R2||GE} The $f[m,n]$ image has energy $E_f = 166$ J and (not centered) module and phase spectra, defined as

$$|F[u,v]| = \begin{bmatrix} 36 & 3.4641 & 3.4641 \\ C & 1.7321 & 1.7321 \\ 9 & 1.7321 & 1.7321 \end{bmatrix} \quad \text{and} \quad \arg[F[u,v]] = \begin{bmatrix} 0 & 2.6180 & -2.6180 \\ \pi & -0.5236 & 0.5236 \\ -\pi & -0.5236 & 0.5236 \end{bmatrix}.$$

a) {2.0||1.5} Compute the value of C as well as the average intensity of $f[m,n]$.

b) {1.5||1.0} Compute the spectrum of the image, $F[u,v]$.

c) {1.5||1.0} In generic terms, what is the information contained in: i) the module of the spectrum of an image? ii) the phase spectrum of an image?

Bibliography

- The images displayed in these slides are from:
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