Topic 2: Linear Models for Financial Time Series
Introduction

Zongye Huang

International School of Economics and Management
Capital University of Economics and Business

Feb 28th, 2017

Outline

Linear Models for Financial Time Series I

- Introduce methods and linear models useful in modeling and forecasting financial time series

  1. Simple autoregressive (AR) models,
  2. Simple moving average (MA) models,
  3. Mixed autoregressive moving average (ARMA) models,
  4. Unit-root models including unit-root tests,
  5. Exponential smoothing,
  6. Seasonal models,
  7. Regression models with time series errors,
  8. fractionally differenced models for long-range dependence.

  - For each class of models, we study their fundamental properties, introduce methods for model selection, consider ways to produce prediction, and discuss their applications.

Linear Models for Financial Time Series II

- We use real examples to introduce important statistical concepts, illustrate step-by-step data analysis, and discuss financial applications.
- The chapter also discusses methods for comparing different models, for example, backtesting and model averaging in prediction.
- Textbooks
  - *Chapter 2, "An Introduction to Analysis of Financial Data with R"*, 1st Edition, by Ruey S. Tsay.

Examples of Financial Time Series I

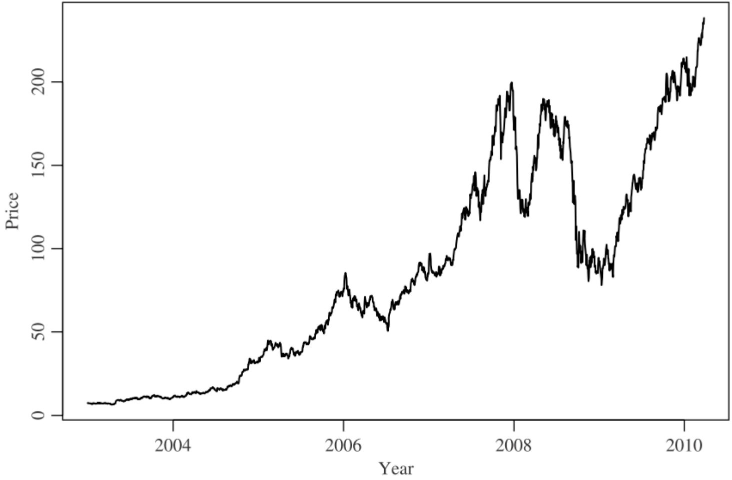

Figure 2.1. Daily closing prices of Apple stock from January 3, 2003 to April 5, 2010.

Examples of Financial Time Series II

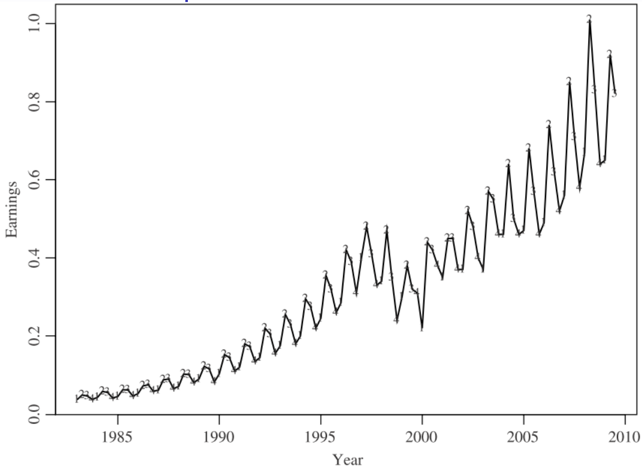

Figure 2.2. Quarterly earnings per share of Coca-Cola Company from the first quarter of 1983 to the third quarter of 2009.

Examples of Financial Time Series III

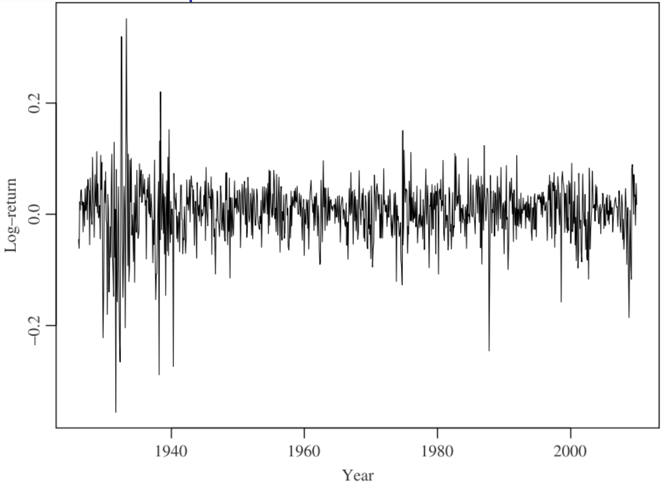

Figure 2.3. Monthly log returns of S&P 500 index from January 1926 to December 2009.

Examples of Financial Time Series IV

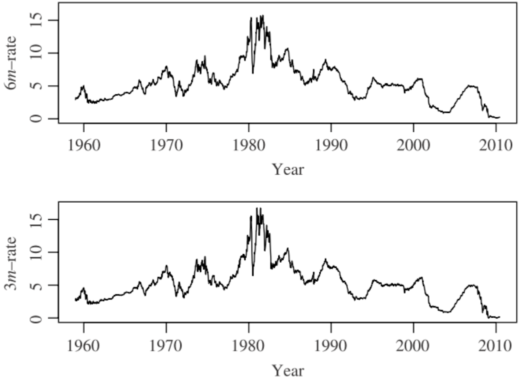

Figure 2.4. Weekly rates, from the secondary market, of the US 3-month and 6-month treasury bills from January 2, 1959 to April 16, 2010.

Examples of Financial Time Series V

- Figure 2.1 shows the daily closing price of Apple stock from January 3, 2003 to April 5, 2010. The daily prices exhibit certain degrees of variability and show an upwardmovement during the sample period.
- Figure 2.2 shows the quarterly earnings per share of Coca-Cola Company from 1983 to 2009. The quarters are marked in the plot. Besides an upward trend, the earnings also exhibit a clear annual pattern, referred to as seasonality in the time series analysis. It will be seen later that many economic and financial time series exhibit a clear seasonal pattern.
- Figure 2.3 gives the monthly log returns of the S&P 500 index from January 1926 to December 2009. From the plot, it is seen that the returns fluctuate around 0 and, except for a few extreme values, are within a fixed range.

Examples of Financial Time Series VI

- Figure 2.4 shows two time series. They are the weekly US 3-month and 6-month treasury bill rates from January 2, 1959 to April 16, 2010. The rates are from the secondary market. The upper plot is the 6-month rate and the lower plot is the 3-month rate. The two series move closely, and also exhibit certain differences. As expected, the 6-month rate was higher in general, but the 3-month rate appeared to be higher in some periods, for example, the early 1980s. This phenomenon is referred to as an inverted yield curve in the term structure of interest rates.
- In these four examples, the series x_t is observed at (roughly) equally spaced time intervals. They are the examples of financial time series that we analyze in this chapter. Our goal is to study the dynamic dependence of the series so that proper inference of the series can be made.

Stationarity I

- The foundation of statistical inference in time series analysis is the concept of weak stationarity.
  - The mean of the returns is constant over time or simply the expected return is time invariant.
    - As shown in Figure 2.3, the monthly log returns of the S&P 500 index vary around 0 over time.
    - In fact, one can divide the time span into several subperiods,and the resulting sample means of the subperiods would all be close to 0.
  - The variance of the log returns is constant over time.
    - The range of the monthly log returns is approximately $[-0.2, 0.2]$ throughout the sample span.

Let x_t be a collection of certain financial measurements over time. Putting these two time-invariant properties together, we say that the log returns x_t is *weakly stationary*.

Stationarity II

- **Weakly stationary:**
  - **A time series x_t is weakly stationary if both the mean of x_t and the covariance between x_t and x_{t-l} are time invariant, where l is an arbitrary integer.**
  - Weak stationarity implies the first two moments of x_t are time invariant and finite. $E(x_t) = \mu$ and $E(x_t - \mu)^2 = \gamma_0$.
  - The weak stationarity is important because it enables one to make inference concerning future observations (e.g., prediction).
- **Strict stationary,**
  - A time series x_t is said to be strictly stationary if the joint distribution of $(x_{t_1}, \ldots x_{t_k})$ is identical to that of $(x_{t_1+t}, \ldots x_{t_k+t})$ for all t, where k is an arbitrary positive integer and $(t_1, \ldots t_k)$ is a collection of k positive integers. In other words, strict stationarity requires that the joint distribution of $(x_{t_1}, \ldots x_{t_k})$ is invariant under time shift.

Correlation I

- The correlation coefficient between two random variables X and Y is defined as

$$\rho_{x,y} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{E\left[(X-\mu_x)(Y-\mu_y)\right]}{\sqrt{E(X-\mu_x)^2 E(Y-\mu_y)^2}}.$$

  - $-1 \leqslant \rho_{x,y} \leqslant 1$, and $\rho_{x,y} = \rho_{y,x}$.
  - If two random variables are uncorrelated, $\rho_{x,y} = 0$.

- Sample correlation

$$\hat{\rho}_{x,y} = \frac{\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y})}{\sqrt{\sum_{t=1}^{T}(x_t - \bar{x})^2 \sum_{t=1}^{T}(y_t - \bar{y})^2}},$$

where $\bar{x}$ and $\bar{y}$ are sample means.

Autocorrelation Function (ACF) I

- Consider a weakly stationary time series x_t . The correlation coefficient between x_t and x_{t-k} is called the *lag-k autocorrelation* of x_t and is commonly denoted by ρ_k

$$\rho_k = \frac{Cov(x_t, x_{t-k})}{\sqrt{Var(x_t)Var(x_{t-k})}} = \frac{Cov(x_t, x_{t-k})}{Var(x_t)} = \frac{\gamma_k}{\gamma_0}.$$

  - Because x_t is weekly stationary, we have $Var(x_{t-k}) = Var(x_t)$.
  - $\rho_0 = 1$, $\rho_l = \rho_{-l}$, and $-1 \leqslant \rho_l \leqslant 1$.
- **Autocorrelation function (ACF)**:
  - The collection of autocorrelations, ρ_l, is called the autocorrelation function (ACF) of x_t.

Autocorrelation Function (ACF) II

Estimate Sample ACF

For a given sample $\{x_t | t = 1, ..., T\}$, let $\bar{x}$ be the sample mean, $\bar{x} = \sum_{t=1}^{T}(x_t)/T$.

- The lag-1 sample autocorrelation of x_t is

$$\hat{\rho_1} = \frac{\sum_{t=2}^{T}(x_t - \bar{x})(x_{t-1} - \bar{x})}{\sum_{t=1}^{T}(x_t - \bar{x})^2}.$$

  - Under some general conditions, $\hat{\rho_1}$ is a consistent estimate of ρ_1.
  - For example, if $\{x_t\}$ is a sequence of iid random variables and $E(x_t^2) < \infty$, then $\hat{\rho_1}$ is asymptotically normal with mean 0 and variance $1/T$.

- The lag-k ($k < T-1$) sample autocorrelation of x_t is

$$\hat{\rho_k} = \frac{\sum_{t=k+1}^{T}(x_t - \bar{x})(x_{t-k} - \bar{x})}{\sum_{t=1}^{T}(x_t - \bar{x})^2}.$$

Autocorrelation Function (ACF) III

If $\{x_t\}$ is a sequence of iid random variables satisfying $E(x_t^2) < \infty$, then $\hat{\rho}_k$ is asymptotically normal with mean 0 and variance $1/T$ for any fixed positive integer k. More generally, if x_t is a weakly stationary time series satisfying $x_t = \mu + \sum_{i=0}^{q} \psi_i a_{t-i}$, where $\psi_0 = 1$ and $\{a_j\}$ is a sequence of iid random variables with mean 0, then $\hat{\rho}_k$ is asymptotically normal with mean 0 and variance $(1 + 2\sum_{i=1}^{q} \rho_i^2)/T$ for $k > q$. This is referred to as the *Bartlett's formula* in the time series literature (Box et al., 1994).

Testing Individual ACF. For a given positive integer k, the previous result can be used to test $H_0 : \rho_k = 0$ versus $H_a : \rho_k \neq 0$. The test statistic is

$$t\text{-ratio} = \frac{\hat{\rho}_k}{\sqrt{(1 + 2\sum_{i=1}^{k-1} \hat{\rho}_i^2)/T}}.$$

For simplicity, many software packages use $1/T$ as the asymptotic variance of $\hat{\rho}_k$ for all $k \neq 0$. The t-ratio then becomes $\sqrt{T}\hat{\rho}_k$. This simplification essentially assumes that the underlying time series is a sequence of iid random variables.

In finite samples, $\hat{\rho}_k$ is a biased estimator of ρ_k. The bias is in the order of $1/T$, which can be substantial when the sample size T is small. In most financial applications, T is relatively large so that the bias is not serious.

Autocorrelation Function (ACF) IV

Portmanteau Test. The statistics $\hat{\rho}_1, \hat{\rho}_2, \ldots$ defined in Equation (2.2) is called the *sample ACF* of x_t. It plays an important role in linear time series analysis. As a matter of fact, a linear time series model can be characterized by its ACF, and linear time series modeling makes use of the sample ACF to specify a model that can capture the dynamic dependence of the data. In many financial applications, we are interested in testing jointly that several autocorrelations of x_t are 0. Box and Pierce (1970) propose the Portmanteau statistic

$$Q_*(m) = T \sum_{\ell=1}^{m} \hat{\rho}_\ell^2$$

as a test statistic for the null hypothesis $H_0 : \rho_1 = \cdots = \rho_m = 0$ against the alternative hypothesis $H_a : \rho_i \neq 0$ for some $i \in \{1, \ldots, m\}$. Under the assumption that $\{x_t\}$ is a sequence of iid random variables with certain moment conditions, $Q_*(m)$ is asymptotically a chi-squared random variable with m degrees of freedom.

Autocorrelation Function (ACF) V

Ljung and Box (1978) modify the $Q_*(m)$ statistic as below to increase the power of the test in finite samples,

$$Q(m) = T(T+2)\sum_{\ell=1}^{m}\frac{\hat{\rho}_\ell^2}{T-\ell}. \tag{2.3}$$

The decision rule is to reject H_0 if $Q(m) > \chi^2_\alpha$, where χ^2_α denotes the $100(1-\alpha)$th percentile of a chi-squared distribution with m degrees of freedom. Most software packages will provide the p-value of $Q(m)$. The decision rule then is to reject H_0 if the p-value is less than α, the type I error or significance level.

White Noise

White Noise. A time series x_t is called a *white noise* if $\{x_t\}$ is a sequence of iid random variables with finite mean and variance. In particular, if x_t is normally distributed with mean 0 and variance σ^2, the series is called a *Gaussian white noise*.

- For a white noise series, all the ACFs are 0.
- In practice, if all sample ACFs are close to 0, then the series is a white noise series.

Linear Time Series I

Linear Time Series. A time series x_t is said to be linear if it can be written as

$$x_t = \mu + \sum_{i=0}^{\infty} \psi_i a_{t-i}, \tag{2.4}$$

where μ is the mean of x_t, $\psi_0 = 1$, and $\{a_t\}$ is a sequence of iid random variables with mean 0 and a well-defined distribution (i.e., $\{a_t\}$ is a white noise series). It will be seen later that a_t denotes the new information at time t of the time series and is often referred to as the *innovation* or *shock* at time t. Thus, a time series is linear if it can be written as a linear combination of past innovations. In this book, we are mainly concerned with the case where the innovation a_t is a continuous random variable. Not all financial time series are linear, but linear models can often provide accurate approximations in real applications.

For a linear time series in Equation (2.4), the dynamic structure of x_t is governed by the coefficients ψ_i, which are called the ψ*-weights* of x_t in the time series literature. If x_t is weakly stationary, we can obtain its mean and variance easily by using properties of $\{a_t\}$ as

$$E(x_t) = \mu, \qquad \mathrm{Var}(x_t) = \sigma_a^2 \sum_{i=0}^{\infty} \psi_i^2, \tag{2.5}$$

where σ_a^2 is the variance of a_t. Because $\mathrm{Var}(x_t) < \infty, \{\psi_i^2\}$ must be a convergent sequence, implying that $\psi_i^2 \to 0$ as $i \to \infty$. Consequently, for a stationary series, impact of the remote shock a_{t-i} on the return x_t vanishes as i increases.

Linear Time Series II

The lag-ℓ autocovariance of x_t is

$$\begin{aligned}\gamma_\ell = \mathrm{Cov}(x_t, x_{t-\ell}) &= E\left[\left(\sum_{i=0}^{\infty}\psi_i a_{t-i}\right)\left(\sum_{j=0}^{\infty}\psi_j a_{t-\ell-j}\right)\right] \\ &= E\left(\sum_{i,j=0}^{\infty}\psi_i\psi_j a_{t-i}a_{t-\ell-j}\right) = \sum_{j=0}^{\infty}\psi_{j+\ell}\psi_j E(a_{t-\ell-j}^2) \\ &= \sigma_a^2\sum_{j=0}^{\infty}\psi_j\psi_{j+\ell}.\end{aligned}$$

Consequently, the ψ-weights are related to the autocorrelations of x_t as follows:

$$\rho_\ell = \frac{\gamma_\ell}{\gamma_0} = \frac{\sum_{i=0}^{\infty}\psi_i\psi_{i+\ell}}{1+\sum_{i=1}^{\infty}\psi_i^2}, \quad \ell \geq 0,$$

where $\psi_0 = 1$. Linear time series models are econometric and statistical models employed to describe the pattern of the ψ-weights of x_t. For a weakly stationary time series, $\psi_i \to 0$ as $i \to \infty$ and, hence, ρ_ℓ converges to 0 as ℓ increases. For asset returns, this means that, as expected, the linear dependence of the current return x_t on the remote past return $x_{t-\ell}$ diminishes for large ℓ.