Lecture 4: Alternative Volatility Models

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Outline

1. The IGARCH Model
2. The GARCH-M Model
3. Models with Asymmetry: EGARCH and TGARCH
4. The CHARMA model
5. Random Coefficient Autoregressive Model (RCA)

EGARCH(m, s): general case

- EGARCH(m, s)

$$
\begin{aligned}
a_t &= \sigma_t \epsilon_t, \; \epsilon_t \sim i.i.d.(0, 1) \\
\ln(\sigma_t^2) &= \alpha_0 + \frac{1 + \beta_1 B + \cdots + \beta_{s-1} B^{s-1}}{1 - \alpha_1 B - \cdots - \alpha_m B^m} g(\epsilon_{t-1})
\end{aligned}
$$

- Asymmetry via $g(\epsilon_t)$

$$
\begin{aligned}
g(\epsilon_t) &= \theta \epsilon_t + \gamma \left[|\epsilon_t| - E\left(|\epsilon_t|\right) \right] \\
&= \begin{cases} (\theta + \gamma)\epsilon_t - \gamma E(|\epsilon_t|), & \text{when } \epsilon_t \geq 0 \\ (\theta - \gamma)\epsilon_t - \gamma E(|\epsilon_t|), & \text{when } \epsilon_t < 0 \end{cases}
\end{aligned}
$$

Example: EGARCH(1, 1)

- EGARCH(1, 1)

$$a_t = \sigma_t \epsilon_t, \ \epsilon_t \sim i.i.d.N(0, 1) \text{ and } E(|\epsilon_t|) = \sqrt{2/\pi}$$

$$\begin{aligned}
\ln(\sigma_t^2) &= \alpha_0 + \frac{1}{1 - \alpha B} g(\epsilon_{t-1}) \\
g(\epsilon_t) &= \theta \epsilon_t + \gamma \left(|\epsilon_t| - \sqrt{2/\pi} \right)
\end{aligned}$$

- Derive σ_t^2

$$(1 - \alpha B) \ln(\sigma_t^2) = \begin{cases} \alpha^* + (\theta + \gamma)\epsilon_{t-1}, & \text{when } \epsilon_{t-1} \geq 0 \\ \alpha^* + (\theta - \gamma)\epsilon_{t-1}, & \text{when } \epsilon_{t-1} < 0 \end{cases}$$

where $\alpha^* = \alpha_0(1 - \alpha) - \gamma\sqrt{2/\pi}$. Therefore,

$$\sigma_t^2 = \sigma_{t-1}^2 \exp(\alpha^*) \begin{cases} \exp\left((\gamma + \theta) \frac{a_{t-1}}{\sigma_{t-1}} \right) \\ \exp\left((\gamma - \theta) \frac{|a_{t-1}|}{\sigma_{t-1}} \right) \end{cases}$$

What is a CHARMA model?

- CHARMA: Conditional heteroskedasticity ARMA model (Tsay, 1987)
- A general CHARMA model

$$r_t = \mu_t + a_t$$
$$a_t = \delta_{1t} a_{t-1} + \delta_{25} a_{t-2} + \cdots + \delta_{mt} a_{t-m} + \eta_t$$

- Gaussian white noise: $\eta_t \sim i.i.d. N(0, \sigma_\eta^2)$
- Random coefficient: $\boldsymbol{\delta}_t = (\delta_{1t}, \delta_{2t}, \dots, \delta_{mt})'$.
- Let $\mathrm{a}_{t-1} = (a_{t-1}, \dots, a_{t-m})$. Then CHARMA model is rewritten as

$$a_t = \mathrm{a}'_{t-1} \boldsymbol{\delta}_t + \eta_t$$

Random coefficients

- $\{\boldsymbol{\delta}_t\} = \{(\delta_{1t}, \delta_{2t}, \ldots, \delta_{mt})'\}$ is a sequence of i.i.d. random vectors.
- $\{\boldsymbol{\delta}_t\}$ is independent of $\{\eta_t\}$.
- $E(\boldsymbol{\delta}_t) = \mathbf{0}$ and $E(\boldsymbol{\delta}_t \boldsymbol{\delta}_t') = \boldsymbol{\Omega}$.

$$\boldsymbol{\Omega} = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1m} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{m1} & \omega_{m2} & \cdots & \omega_{mm} \end{pmatrix}$$

Conditional variance of a_t in CHARMA

- The conditional variance of a_t, i.e. $\mathrm{Var}_{t-1}(a_t) = \sigma_t^2$:

$$\begin{aligned}\sigma_t^2 &= \sigma_\eta^2 + \mathbf{a}'_{t-1}\boldsymbol{\Omega}\mathbf{a}_{t-1} \\ &= \sigma_\eta^2 + (a_{t-1}, \dots, a_{t-m})\boldsymbol{\Omega}(a_{t-1}, \dots, a_{t-m})'\end{aligned}$$

- Since $\boldsymbol{\Omega}$ is positive semi-definite, $\sigma_t^2 \geq \sigma_\eta^2 > 0$ is always true.

- The conditional variance is similar to ARCH but with difference.
  - When $m = 1$, $\sigma_t^2 = \sigma_\eta^2 + \omega_{11} a_{t-1}^2$.
  - When $m = 2$, $\sigma_t^2 = \sigma_\eta^2 + \omega_{11} a_{t-1}^2 + \underbrace{2\omega_{12} a_{t-1} a_{t-2}}_{\text{cross-product term}} + \omega_{22} a_{t-2}^2$.

Problem of CHARMA

- The number of cross-product terms increases with the order of m.
- The higher-order properties are hard to derive.

Why use random coefficient autoregressive model?

- Random coefficient: account for variability among different subjects.
  - Panel data; hierarchical models
- Time series model: the coefficients in the mean equation evolve over time.

RCA(p)

- The mean equation

$$r_t = \phi_0 + \sum_{i=1}^{p} (\phi_i + \delta_{it}) r_{t-i} + a_t$$

- Random coefficients

$$\{\boldsymbol{\delta}_t\} = \{(\delta_{1t}, \delta_{2t}, \ldots, \delta_{pt})'\}$$

  - Independent series;
  - $E(\boldsymbol{\delta}_t) = \mathbf{0}$ and $\mathrm{Var}\boldsymbol{\delta}_t = \boldsymbol{\Omega}_\delta$;
  - $\{\boldsymbol{\delta}_t\}$ is independent of $\{\mathbf{a}_t\}$

Conditional mean and variance

- The conditional mean

$$\mu_t = E_{t-1}(r_t) = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i}$$

- The conditional variance

$$\begin{aligned}\sigma_t^2 &= E_{t-1}\left((r_t - \mu_t)^2\right) = E_{t-1}\left(((\mathbf{r}_{t-1})'\boldsymbol{\delta}_t + a_t)^2\right) \\ &= \sigma_a^2 + (r_{t-1}, \ldots, r_{t-p})\boldsymbol{\Omega}_\delta(r_{t-1}, \ldots, r_{t-p})'\end{aligned}$$

- Similar to CHARMA but with the quadratic function of r_{t-i}.