Lecture 2: The ARCH Model

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Outline

1. The Volatility of Asset Returns
2. The Structure of a Volatility Model
3. The ARCH Model

Volatility is measured with the conditional variance

- Volatility here refers to the conditional variance of a time series.
- For a return series $\{r_t\}$, we are now interested in

$$\sigma_t^2 = \mathrm{Var}(r_t \mid F_{t-1})$$

where F_{t-1} is the information set at time $t-1$.

Characteristics of volatility (1)

1. There exist volatility clusters. That is, volatility may be high for certain time periods and low for other periods.

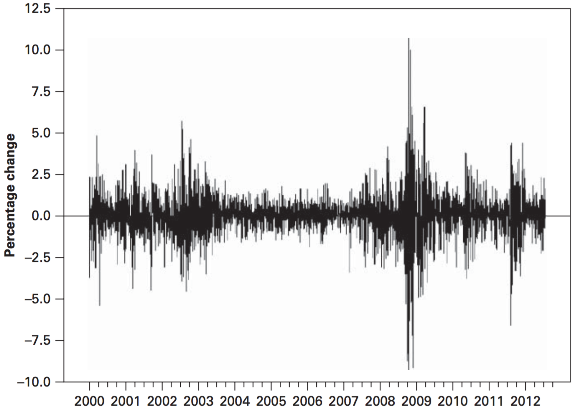

Figure: Percentage Change in the NYSE U.S.100 stock price index

Characteristics of volatility (2)

1. Volatility evolves over time in a continuous manner. That is, volatility jumps are rare.

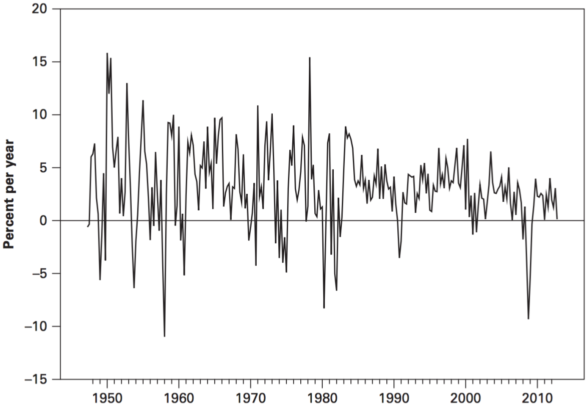

Figure: Annualized Growth Rate of Real GDP

Characteristics of volatility (3)

1. Volatility does not diverge to infinity. That is, volatility varies within some fixed range. Statistically speaking, this means that volatility is often stationary.

1. Volatility seems to react differently to a big price increase or a big price drop, referred to as the leverage effect.

The basic idea of building a volatility model

Consider the log return series $\{r_t\}$. The basic idea of a volatility model is

- $\{r_t\}$ may appear to be either serially uncorrelated or serially correlated with a minor order.
- However, $\{r_t\}$ is a dependent series and the dependence arises from its conditional variance.

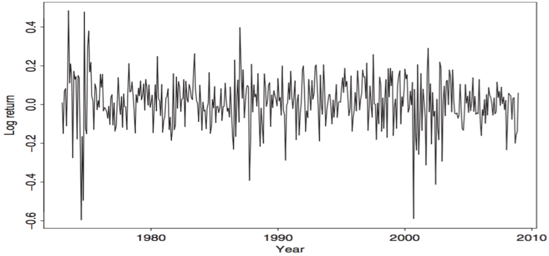

Figure: Time plot of monthly log returns of Intel stock from January 1973 to December 2008

The sample ACF of $\{r_t\}$ and $\{r_t^2\}$

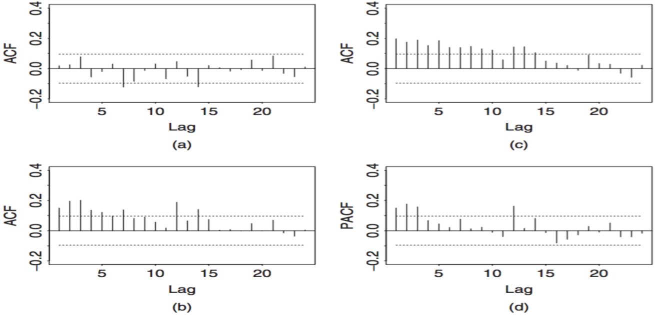

Figure: Sample ACF and PACF of various functions of monthly log stock returns of Intel Corporation from January 1973 to December 2008: (a) ACF of the log returns, (b) ACF of the squared log returns, (c) ACF of the absolute log returns, and (d) PACF of the squared log returns.

Decompose r_t into the mean and variance equations

To capture the dependence in a time series through its second moment but not the mean, we model the mean process and the variance process separately.

For a return series $\{r_t\}$, we can model it as

$$r_t = \mu_t + a_t \tag{1}$$

where μ_t represents the conditional mean and a_t is modeled to capture the conditional variance.

The mean equation

$$\mu_t = E(r_t \mid F_{t-1}) = \sum_{i=1}^{p} \phi_i y_{t-i} - \sum_{i=1}^{q} \theta_i a_{t-i} \tag{2}$$

$$y_t = r_t - \phi_0 - \sum_{i=1}^{k} \beta_i x_{it}$$

F_{t-1} is the information set at time $t-1$.

If you combine these two equations, and let $\mu_t = r_t - a_t$, you will find that it is just an ARMA(p, q) model with additional regressors x_{it}.

The variance equation

Denote the conditional variance of r_t with σ_t^2.

$$\begin{aligned}\sigma_t^2 = \mathrm{Var}(r_t \mid F_{t-1}) &= E\left((r_t - E(r_t|F_{t-1}))^2 | F_{t-1}\right) \\ &= E\left((r_t - \mu_t)^2 \mid F_{t-1}\right) \\ &= \mathrm{Var}(a_t \mid F_{t-1})\end{aligned}$$

The variance equation (cont'd)

- If we assume that $E(a_t \mid F_{t-1}) = 0$, we can see that $\sigma_t^2 = E(a_t^2 \mid F_{t-1})$.

  This result motivates us to use the series of $\{a_t^2\}$ to model the conditional variance σ_t^2.
- The simplest model is a linear model, like the following

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2$$

- Let $a_t^2 = \sigma_t^2 + \nu_t$ where ν_t is a white noise series. The above equation turns into an AR(m) model for $\{a_t^2\}$ as follows

$$a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2 + \nu_t$$

  This equation represents the essential idea of an ARCH model with just a little modification.

The procedure of building a volatility model

Building a volatility model for an asset return series consists of four steps:

1. Specify a mean equation by testing for serial dependence in the data and, if necessary, building an econometric model (e.g., an ARMA model) for the return series to remove any linear dependence.
2. Use the squared residuals of the mean equation to test for ARCH effects.
3. Specify a volatility model if ARCH effects are statistically significant, and perform a joint estimation of the mean and volatility equations.
4. Check the fitted model carefully and refine it if necessary.

Testing for the presence of ARCH effect

The Ljung-Box test for the series of a_t^2

Upon obtaining the residuals from the estimation of an adequate mean equation, we can use the squared residuals $\{\hat{a}_t^2\}$ to test the existence of autocorrelation.

- The Ljung-Box test is used to test the null hypothesis $H_0 : \rho_1 = \cdots = \rho_m = 0$.
- The $Q(m)$ statistic is calculated and compared with the critical value from $\chi^2(m)$ distribution at the desired significance level.
- The rejection of the null hypothesis implies that there is autoregressive conditional heteroskedastic (ARCH) effect.

The LM test

An auxiliary regression

We estimate a AR(m) model regarding $\{\hat{a}_t^2\}$, that is,

$$\hat{a}_t^2 = \alpha_0 + \alpha_1 \hat{a}_{t-1}^2 + \cdots + \alpha_m \hat{a}_{t-m}^2 + e_t$$

The LM test

With this model, we test the joint hypothesis

$$H_0 : \alpha_1 = \cdots = \alpha_m = 0$$

- The LM statistic is NR^2 where N is the sample size of this regression and R^2 is the coefficient of the determination of this regression.
- Given the null hypothesis is true, this statistic follows a $\chi^2(m)$ distribution.

The LM test (cont'd)

Alternatively, we can use F statistic to test the joint hypothesis.

- Let $SSR_0 = \sum_{t=m+1}^{T}(\hat{a}_t^2 - \bar{\omega})^2$, where $\bar{\omega} = (1/T)\sum_{t=1}^{T}\hat{a}_t^2$.
- Let $SSR_1 = \sum_{t=m+1}^{T}\hat{e}_t^2$ where $\hat{e}_t$ is the residuals from the regression.
- The F statistic is

$$F = \frac{(SSR_0 - SSR_1)/m}{SSR_1/(T-2m-1)} \sim F(m, T-2m-1)$$

- Rejecting the null hypothesis motivates us to model the possible ARCH effect.

An example

Go back to Figure 4. Since the return series is already stationary, we directly test the squared return series to check the ARCH effect.

- In the LM test of the ARCH effect, $F = 53.62$ and the p value is close to zero.
- The Ljung–Box statistics of the a_t^2 series also shows strong ARCH effects with $Q(12) = 89.85$, the p value of which is close to zero.
- Therefore, we can confirm that the return series of Intel stock has an ARCH effect, and next we need to model such an effect.

The basic idea of an ARCH model

Consider a series of shocks $\{a_t\}$ in a return series $\{r_t\}$. The basic idea of an Autoregressive Conditional Heteroskedasticity (ARCH) model is

1. the shock a_t of the return series is serially uncorrelated but dependent;
2. the dependence of a_t can be modeled through an autoregressive process of a_t^2.

The ARCH(m) model

An ARCH(m) model takes the following form

$$a_t = \sigma_t \epsilon_t, \ \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2 \tag{3}$$

where $\epsilon_t \sim i.i.d.(0, 1)$, $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $i = 1, \ldots, m$.

- The assumption of $\mathrm{Var}(\epsilon_t) = 1$ is to make the analysis regarding the properties of the ARCH(m) model easy;
- The assumption of $\alpha_0 > 0$ and $\alpha_i \geq 0$ is to ensure the conditional variance of a_t is positive.
- $\alpha_1, \ldots, \alpha_m$ should also satisfy some regularity conditions to ensure the unconditional variance of a_t is finite.

The Properties of an ARCH Model

- Let's take an ARCH(1) model as an example to discuss the properties of ARCH model.
- The goal is to see how such a model can capture the basic idea mentioned above and the stylized fact that highly volatile periods tend to be followed by high volatility periods.

Assume an ARCH(1) model as follows

$$a_t = \sigma_t \epsilon_t, \; \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2, \; \epsilon_t \sim i.i.d.(0,1) \tag{4}$$

where $a_0 > 0$ and $a_1 \geq 0$.

The unconditional mean and variance of a_t

The unconditional mean

$$E(a_t) = E(\sigma_t \epsilon_t) = E(\sigma_t)E(\epsilon_t) = 0$$

The second equality is ensured because σ_t and ϵ_t are independent, and the third equality comes from the assumption of $E(\epsilon_t) = 0$.

The unconditional variance

$$\begin{aligned}\mathrm{Var}(a_t) &= E(a_t^2) = E(\sigma_t^2 \epsilon_t^2) \\ &= E(\alpha_0 + \alpha_1 a_{t-1}^2) \cdot 1 = \alpha_0 + \alpha_1 \mathrm{Var}(a_{t-1})\end{aligned}$$

Assuming the unconditional mean of a_t is a constant(why?), we can have

$$\mathrm{Var}(a_t) = \frac{\alpha_0}{1 - \alpha_1}$$

Since the variance should be positive and finite, we must have $0 \leq \alpha_1 < 1$.

The unconditional covariance of a_t

Since ϵ_t and ϵ_{t-i} for $i \neq 0$ are independent,

$$\begin{aligned}\mathrm{Cov}(a_t, a_{t-i}) &= E(a_t a_{t-i}) = E(\sigma_t \epsilon_t \sigma_{t-i} \epsilon_{t-i}) \\ &= E(\sigma_t \sigma_{t-i}) E(\epsilon_t \epsilon_{t-i}) = 0\end{aligned}$$

- What we get now?
  - a_t has constant unconditional mean and variance,
  - a_t is serially uncorrelated.

The kurtosis of a_t

- Assume that $\epsilon \sim N(0,1)$, implying that $E(\epsilon_t^4) = 3$. Thus, we have

$$\begin{aligned} E(a_t^4) &= E(\sigma_t^4 \epsilon_t^4) = E(\sigma_t^4)E(\epsilon_t^4) = 3E(\sigma_t^4) \\ &= 3\left(\alpha_0^2 + 2\alpha_0\alpha_1 E(a_{t-1}^2) + \alpha_1^2 E(a_{t-1}^4)\right) \end{aligned}$$

- Assume that a_t is fourth-order stationary so that we can define $m_4 = E(a_t^4) = E(a_{t-1}^4)$. Then, using the fact that $E(a_t^2) = \alpha_0/(1-\alpha_1)$, we can solve m_4 from the above equation.

$$m_4 = \frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}$$

The kurtosis of a_t (cont'd)

This result regarding m_4 has two important implications:

1. Since the fourth moment of a_t is positive, we see that α_1 must also satisfy the condition $1 - 3\alpha_1^2 > 0$, that is, $0 \leq \alpha_1^2 < \frac{1}{3}$.
2. The kurtosis of a_t is

$$\text{kurtosis} = \frac{E(a_t^4)}{E(a_t^2)^2} = \frac{3(1-\alpha_1^2)}{1-3\alpha_1^2} > 3$$

Thus, the the excess kurtosis of a_t is positive and the tail distribution of a_t is heavier than that of a normal distribution.

The conditional mean

- Let's write $E_{t-1}(a_t)$ to represent the conditional mean given the information set F_{t-1}, i.e., $E(E(a_t \mid F_{t-1}))$.
- Since ϵ_t is i.i.d, we have $E_{t-1}(\epsilon_t) = E(\epsilon_t) = 0$. Thus,

$$
\begin{aligned}
E_{t-1}(a_t) = E_{t-1}(\sigma_t \epsilon_t) = E_{t-1}\left((\alpha_0 + \alpha_1 a_{t-1}^2)^{1/2}\epsilon_t\right) \\
= (\alpha_0 + \alpha_1 a_{t-1}^2)^{1/2} E_{t-1}(\epsilon_t) = 0
\end{aligned}
$$

The conditional variance

- The conditional variance of a_t is

$$\begin{aligned}\mathrm{Var}_{t-1}(a_t) &= E_{t-1}(a_t^2) = E_{t-1}\left(\sigma_t^2\epsilon_t^2\right)\\ &= E_{t-1}\left((\alpha_0 + \alpha_1 a_{t-1}^2)\epsilon_t^2\right)\\ &= E_{t-1}(\alpha_0 + \alpha_1 a_{t-1}^2)E_{t-1}(\epsilon_t^2)\\ &= (\alpha_0 + \alpha_1 a_{t-1}^2)E(\epsilon_t^2)\\ &= \alpha_0 + \alpha_1 a_{t-1}^2 = \sigma_t^2\end{aligned}$$

- How does the conditional variance capture the stylized fact?

An ARCH model is essentially an AR process

- Let $a_t^2 = \sigma_t^2 + \eta_t$, where $\eta_t \sim i.i.d.(0, \sigma_h^2)$. We can rewrite an ARCH(1) process as
$$a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \eta_t$$
Since $0 \leq \alpha_1 < 1$, we know that an ARCH(1) process is actually an AR(1) process regarding $\{a_t^2\}$.
- The properties regarding an ARCH(1) model can be easily extended to an ARCH(m) model.