Topic 5: Unit-Root

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Outline

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Unit-root Nonstationary Time Series

- So far, we have focused on the return series that are stationary.
- In some studies, interest rates, foreign exchange rates, or the price series of an asset are of interest, tend to be nonstationary. For a price series, the nonstationarity is mainly due to the fact that there is no fixed level for the price.
- In the time series literature, such a nonstationary series is called *unit-root nonstationary time series*.
  - The best-known example of unit-root nonstationary time series is the random walk model.

Random Walk I

- A time series p_t is a random walk if it satisfies

$$p_t = p_{t-1} + a_t$$

  where p_0 is a real number denoting the starting value of the process and a_t is a white noise series.
- If a_t has a symmetric distribution around 0, then conditional on p_{t-1}, p_t has a 50-50 chance to go up or down, implying that p_t would go up or down at random.
  - If we treat the random walk model as a special AR(1) model, then the coefficient of p_{t-1} is unity, which does not satisfy the weak stationarity condition of an AR(1) model.
    - A random walk series is, therefore, not weakly stationary, and we call it a **unit-root nonstationary** time series.

Random Walk II

- The random walk model has been widely considered as a statistical model for the movement of logged stock prices. Under such a model, the stock price is not predictable or mean reverting.
  - The 1-step ahead forecast of model at the forecast origin h is

$$\hat{p}_h(1) = E(p_{h+1}|p_h, p_{h-1}, \dots) = p_h$$

    which is the log price of the stock at the forecast origin. Such a forecast has no practical value.
  - The 2-step ahead forecast is

$$\begin{aligned}\hat{p}_h(2) &= E(p_{h+2}|p_h, p_{h-1}, \dots) = p_h \\ &= E(p_{h+1} + a_{h+2}|p_h, p_{h-1}, \dots) \\ &= E(p_{h+1}|p_h, p_{h-1}, \dots) \\ &= p_h\end{aligned}$$

    which again is the log price at the forecast origin.

Random Walk III

- In fact, for any forecast horizon $k > 0$, we have

$$\hat{p}_h(k) = p_h.$$

For all forecast horizons, point forecasts of a random walk model are simply the value of the series at the forecast origin. Therefore, the process is not mean reverting.

- The MA representation of the random walk model has several important practical implications.

$$p_t = a_t + a_{t-1} + \dots$$

- The k-step ahead forecast error is

$$e_h(k) = a_{h+l} + \dots + a_{h+1}.$$

Random Walk IV

- $Var(e_h(k)) = k\sigma_a^2$, goes to infinity as $k \to \infty$.
- The usefulness of point forecast diminishes as k increases, and the model is unpredictable.
- The model has a strong memory as it remembers all of the past shocks.

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Random Walk with Drift I

- The log return series of a market index tends to have a small and positive mean. This implies that the model for the log price is

$$p_t = \mu + p_{t-1} + a_t$$

where $\mu = E(p_t - p_{t-1})$. It represents the time trend of the log price and is often referred to as the **drift** of the model.

Random Walk with Drift II

- Assume that the initial log price is p_0

$$
\begin{aligned}
p_1 &= \mu + p_0 + a_1 \\
p_2 &= \mu + p_1 + a_2 = 2\mu + p_0 + a_2 + a_1 \\
\vdots & \quad \vdots \\
p_t &= t\mu + p_0 + a_t + a_{t-1} + \ldots + a_1
\end{aligned}
$$

- It is a time trend $t\mu$ and a pure random walk $\sum_{i=1}^{t} a_t$.
- A positive slope μ implies that the log price eventually goes to infinity.

Example: 3M Stock Price I

To illustrate the effect of the drift parameter on the price series, we consider the monthly log stock returns of the 3M Company from February 1946 to December 2008. As shown by the sample EACF in Table 2.5, the series has no significant serial correlation. The series thus follows the simple model

$$x_t = 0.0103 + a_t, \quad \hat{\sigma}_a = 0.0637, \tag{2.38}$$

where 0.0103 is the sample mean of x_t and has a standard error 0.0023. The mean of the monthly log returns of 3M stock is, therefore, significantly different from 0 at the 1% level. As a matter of fact, the one-sample test of zero mean shows a t-ratio of 4.44 with p-value close to 0. We use the log return series to construct two log price series, namely,

$$p_t = \sum_{i=1}^{t} x_i \quad \text{and} \quad p_t^* = \sum_{i=1}^{t} a_i,$$

Example: 3M Stock Price II

where a_i is the mean-corrected log return in Equation (2.38) (i.e., $a_t = x_t - 0.0103$). The p_t is the log price of 3M stock, assuming that the initial log price is 0 (i.e., the log price of January 1946 was 0). The p_t^* is the corresponding log price if the mean of log returns was 0. Figure 2.16 shows the time plots of p_t and p_t^*, as well as a straight line $y_t = 0.0103 \times t + 1946$, where t is the time sequence of the returns and 1946 is the starting year of the stock. From the plots, the importance of the constant 0.0103 in Equation (2.38) is evident. In addition, as expected, it represents the slope of the upward trend of p_t.

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Trend-Stationary Time Series

A closely related model that exhibits linear trend is the trend-stationary time series model,

$$p_t = \beta_0 + \beta_1 t + x_t,$$

where x_t is a stationary time series, for example, a stationary AR(p) series. Here, p_t grows linearly in time with rate β_1 and hence can exhibit behavior similar to that of a random walk model with drift. However, there is a major difference between the two models. To see this, suppose that p_0 is fixed. The random walk model with drift assumes that the mean $E(p_t) = p_0 + \mu t$ and variance $\text{Var}(p_t) = t\sigma_a^2$, both of them are time dependent. On the other hand, the trend-stationary model assumes the mean $E(p_t) = \beta_0 + \beta_1 t$, which depends on time, and variance $\text{Var}(p_t) = \text{Var}(x_t)$, which is finite and time invariant. The trend-stationary series can be transformed into a stationary one by removing the time trend via a simple linear regression analysis. For

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General Unit-Root Nonstationary Models I

- Consider an ARMA model. If one extends the model by allowing the AR polynomial to have 1 as a characteristic root, then the model becomes the well-known ***autoregressive integrated moving average*** (ARIMA) model.
- An ARIMA model is said to be unit-root nonstationary because its AR polynomial has a unit root.
    - Similar to a random walk model, an ARIMA model has strong memory because the ψ_i coefficients in its MA representation do not decay over time to 0, implying that the past shock a_{t-i} of the model has a permanent effect on the series.

General Unit-Root Nonstationary Models II

$$x_t = \mu + \psi(B)a_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots$$

$$\psi(B) = \frac{\theta(B)}{\phi(B)}.$$

For instance, if $\phi(B) = 1 - \phi_1 B$ and $\theta(B) = 1 - \theta_1 B$, then

$$\psi(B) = \frac{1 - \theta_1 B}{1 - \phi_1 B} = 1 + (\phi_1 - \theta_1)B + \phi_1(\phi_1 - \theta_1)B^2 + \phi_1^2(\phi_1 - \theta_1)B^3 + \cdots$$

$$\pi(B) = \frac{1 - \phi_1 B}{1 - \theta_1 B} = 1 - (\phi_1 - \theta_1)B - \theta_1(\phi_1 - \theta_1)B^2 - \theta_1^2(\phi_1 - \theta_1)B^3 - \cdots.$$

- A conventional approach for handling unit-root nonstationarity is to use differencing.

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Differencing I

- A time series y_t is said to be an ARIMA(p, 1, q) process if the change series

$$c_t = y_t - y_{t-1} = (1 - B)y_t$$

follows a stationary and invertible ARMA(p, q) model.

  - In finance, price series are commonly believed to be nonstationary, but the log return series,

$$x_t = \ln(P_t) - \ln(P_{t-1})$$

is stationary.

  - In this case, the log price series is unit-root nonstationary and hence can be treated as an ARIMA process.

Differencing II

- The idea of transforming a nonstationary series into a stationary one by considering its change series is called differencing in the time series literature.
  - $c_t = y_t - y_{t-1}$ is referred to as the first differenced series of y_t.
- A time series y_t may contain multiple unit roots and needs to be differenced multiple times to become stationary.
  - For example, if both y_t and its first differenced series $c_t = y_t - y_{t-1}$ are unit-root nonstationary, but

$$s_t = c_t - c_{t-1} = y_t - 2y_{t-1} + y_{t-2}$$

    is weakly stationary,
    - y_t has double unit roots,
    - s_t is the second differenced series of y_t.
    - If s_t follows an ARMA(p, q) model, then y_t is an ARIMA(p, 2, q) process.

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Dickey-Fuller Test I

- To test whether the log price p_t of an asset follows a random walk or a random walk with drift, we employ the models

$$
\begin{aligned}
p_t &= \phi_1 p_{t-1} + e_t, && (1)\\
p_t &= \phi_0 + \phi_1 p_{t-1} + e_t, && (2)
\end{aligned}
$$

  where e_t denotes the error term.
- Consider the null hypothesis $H_0: \phi_1 = 1$ versus the alternative hypothesis $H_a: \phi_1 < 1$.
  - This is the well-known unit-root testing problem (Dickey and Fuller, 1979).

Dickey-Fuller Test II

- A convenient test statistic is the t ratio of the LS estimate of ϕ_1 under the null hypothesis

$$\hat{\phi}_1 = \frac{\sum_{t=1}^{T} p_{t-1} p_t}{\sum_{t=1}^{T} p_{t-1}^2}, \ \hat{\sigma}_e^2 = \frac{\sum_{t=1}^{T} (p_t - \hat{\phi}_1 p_{t-1})^2}{T-1},$$

  where $p_0 = 0$ and T is the sample size.

- The **Dickey-Fuller** test

$$DF = \frac{\hat{\phi}_1 - 1}{std(\hat{\phi}_1)}$$

Dickey-Fuller Test III

- If e_t is a white noise series with finite moments of order slightly greater than 2, then the DF-statistic converges to a function of the standard Brownian motion as $\mathrm{T} \to \infty$;
- If ϕ_0 is 0 but second model is employed, the resulting t-ratio for testing $\phi_1 = 1$ will converge to another non-standard asymptotic distribution.
  - In either case, simulation is used to obtain critical values of the test statistics;
- If $\phi_0 \neq 0$ and second model is used, then the t ratio for testing $\phi_1 = 1$ is asymptotically normal. However, large sample sizes are needed for the asymptotic normal distribution to hold.

Augmented Dickey-Fuller Test I

- For many economic time series, ARIMA(p, d, q) models might be more appropriate than the simple model used above.
- In the econometric literature, AR(p) models are often used.
  - To verify the existence of a unit root in an AR(p) process, one may perform the test $H_0: \beta = 1$ versus $H_a: \beta < 1$ using the regression

$$x_t = c_t + \beta x_{t-1} + \sum_{i=1}^{p-1} \phi_i \Delta x_{t-i} + e_t, \tag{3}$$

where c_t is a deterministic function of the time index t and $\Delta x_j = x_j - x_{j-1}$ is the differenced series of x_t.

  - In practice, c_t can be 0 or a constant or $c_t = \omega_0 + \omega_1 t$.

Augmented Dickey-Fuller Test II

- Let $\hat{\beta}$ denoting the LS estimate of β, the t-ratio of $\hat{\beta}-1$,

$$ADF = \frac{\hat{\beta}-1}{std(\hat{\beta})}.$$

  - This is the well-known **augmented Dickey–Fuller** unit-root test.

- The regression equation can also be rewritten as

$$\Delta x_t = c_t + \beta_c x_{t-1} + \sum_{i=1}^{p-1} \phi_i \Delta x_{t-i} + e_t, \tag{4}$$

where $\beta_c = \beta - 1$. The equivalent hypothesis $H_0: \ \beta_c = 0$ versus $H_a: \ \beta_c < 0$.

Augmented Dickey-Fuller Test III

- The intuition behind the test is that if the series is integrated then the lagged level of the series (x_{t-1}) will provide no relevant information in predicting the change in x_t besides the one obtained in the lagged changes (Δx_{t-k}). In this case the $\beta = 0$ and null hypothesis is not rejected.
- By including lags of the order p the ADF formulation allows for higher-order autoregressive processes.
  - The lag length p has to be determined when applying the test.
    - One possible approach is to test down from high orders and examine the t-values on coefficients.
    - An alternative approach is to examine information criteria.
- There are alternative unit root tests such as the Phillips–Perron test (PP).

Example I

Example 2.4. Consider the log series of US quarterly GDP from 1947.I to 2008.IV. The series exhibits an upward trend, showing the growth of US economy and has high sample serial correlations; see the lower-left panel of Figure 2.17. The first differenced series, representing the growth rate of US GDP and also shown in Figure 2.17, seems to vary around a fixed mean level, even though the variability appears to be smaller in recent years. To confirm the observed phenomenon, we apply the augmented Dicky–Fuller unit-root test to the log series. On the basis of the sample PACF of the differenced series shown in Figure 2.17, we choose $p = 10$. Other values of p are also used, but they do not alter the conclusion of the test. With $p = 10$, ADF-test statistic is -1.611 with p-value 0.457, indicating that the unit-root hypothesis cannot be rejected. $\square$

Example II

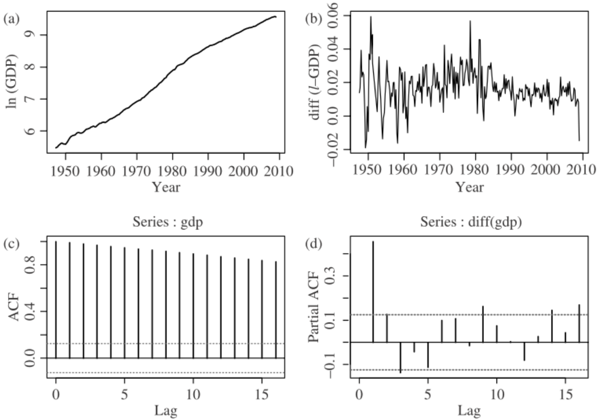

Figure 2.17. Log series of US quarterly GDP from 1947.I to 2008.IV: (a) Time plot the the logged GDP series, (b) sample ACF of the log GDP data, (c) time plot of the first differenced series, and (d) sample PACF of the differenced series.