

## Lecture 2: The ARCH Model

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# Outline

- 1 The Volatility of Asset Returns
- 2 The Structure of a Volatility Model

# Volatility is measured with the conditional variance

- Volatility here refers to the **conditional variance** of a time series.
- For a return series  $\{r_t\}$ , we are now interested in

$$\sigma_t^2 = \text{Var}(r_t \mid F_{t-1})$$

where  $F_{t-1}$  is the information set at time  $t - 1$ .

# Characteristics of volatility (1)

- 1 There exist **volatility clusters**. That is, volatility may be high for certain time periods and low for other periods.

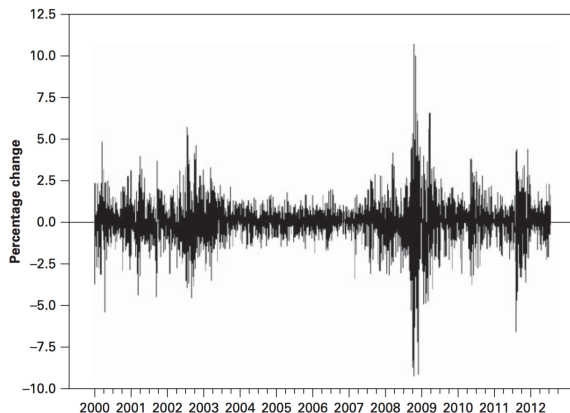


Figure: Percentage Change in the NYSE U.S.100 stock price index

## Characteristics of volatility (2)

- 1 Volatility evolves over time in a continuous manner. That is, volatility jumps are rare.

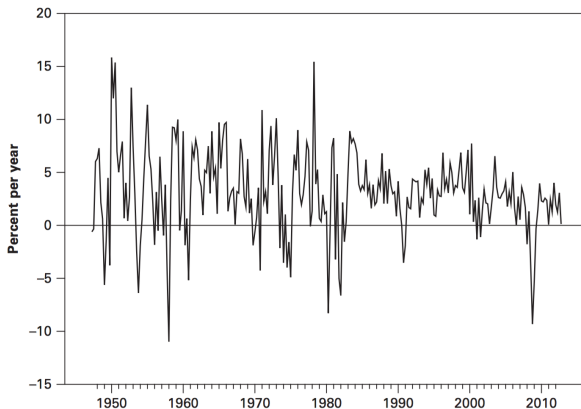


Figure: Annualized Growth Rate of Real GDP

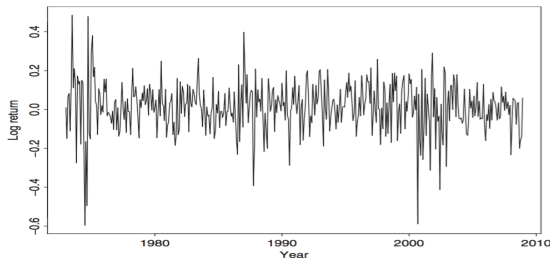
## Characteristics of volatility (3)

- 1 Volatility does not diverge to infinity. That is, volatility varies within some fixed range. Statistically speaking, this means that volatility is often stationary.
- 1 Volatility seems to react differently to a big price increase or a big price drop, referred to as the leverage effect.

# The basic idea of building a volatility model

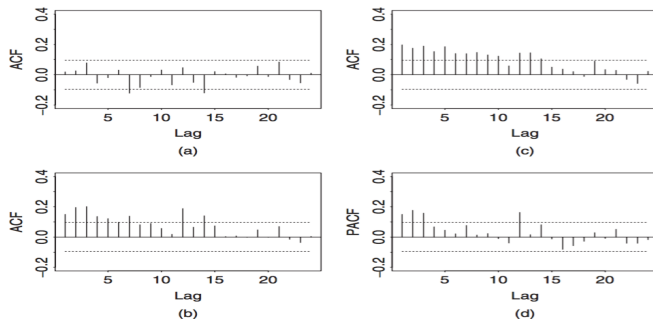
Consider the log return series  $\{r_t\}$ . The basic idea of a volatility model is

- $\{r_t\}$  may appear to be either serially uncorrelated or serially correlated with a minor order.
- However,  $\{r_t\}$  is a dependent series and the dependence arises from its conditional variance.



**Figure:** Time plot of monthly log returns of Intel stock from January 1973 to December 2008

# The sample ACF of $\{r_t\}$ and $\{r_t^2\}$



**Figure:** Sample ACF and PACF of various functions of monthly log stock returns of Intel Corporation from January 1973 to December 2008: (a) ACF of the log returns, (b) ACF of the squared log returns, (c) ACF of the absolute log returns, and (d) PACF of the squared log returns.



# Decompose $r_t$ into the mean and variance equations

To capture the dependence in a time series through its second moment but not the mean, we model the mean process and the variance process separately.

For a return series  $\{r_t\}$ , we can model it as

$$r_t = \mu_t + a_t \quad (1)$$

where  $\mu_t$  represents the conditional mean and  $a_t$  is modeled to capture the conditional variance.

# The mean equation

$$\mu_t = E(r_t \mid F_{t-1}) = \sum_{i=1}^p \phi_i y_{t-i} - \sum_{i=1}^q \theta_i a_{t-i} \quad (2)$$

$$y_t = r_t - \phi_0 - \sum_{i=1}^k \beta_i x_{it}$$

$F_{t-1}$  is the information set at time  $t - 1$ .

If you combine these two equations, and let  $\mu_t = r_t - a_t$ , you will find that it is just an ARMA( $p, q$ ) model with additional regressors  $x_{it}$ .

# The variance equation

Denote the conditional variance of  $r_t$  with  $\sigma_t^2$ .

$$\begin{aligned}\sigma_t^2 &= \text{Var}(r_t \mid F_{t-1}) = E((r_t - E(r_t \mid F_{t-1})))^2 \mid F_{t-1}) \\ &= E((r_t - \mu_t)^2 \mid F_{t-1}) \\ &= \text{Var}(a_t \mid F_{t-1})\end{aligned}$$

# The variance equation (cont'd)

- If we assume that  $E(a_t | F_{t-1}) = 0$ , we can see that  $\sigma_t^2 = E(a_t^2 | F_{t-1})$ .

This result motivates us to use the series of  $\{a_t^2\}$  to model the conditional variance  $\sigma_t^2$ .

- The simplest model is a linear model, like the following

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2$$

- Let  $a_t^2 = \sigma_t^2 + \nu_t$  where  $\nu_t$  is a white noise series. The above equation turns into an AR( $m$ ) model for  $\{a_t^2\}$  as follows

$$a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2 + \nu_t$$

This equation represents the essential idea of an ARCH model with just a little modification.

# The procedure of building a volatility model

Building a volatility model for an asset return series consists of four steps:

- 1 Specify a mean equation by testing for serial dependence in the data and, if necessary, building an econometric model (e.g., an ARMA model) for the return series to remove any linear dependence.
- 2 Use the squared residuals of the mean equation to test for ARCH effects.
- 3 Specify a volatility model if ARCH effects are statistically significant, and perform a joint estimation of the mean and volatility equations.
- 4 Check the fitted model carefully and refine it if necessary.

# Testing for the presence of ARCH effect

## The Ljung-Box test for the series of $a_t^2$

Upon obtaining the residuals from the estimation of an adequate mean equation, we can use the squared residuals  $\{\hat{a}_t^2\}$  to test the existence of autocorrelation.

- The Ljung-Box test is used to test the null hypothesis
$$H_0 : \rho_1 = \cdots = \rho_m = 0.$$
- The  $Q(m)$  statistic is calculated and compared with the critical value from  $\chi^2(m)$  distribution at the desired significance level.
- The rejection of the null hypothesis implies that there is autoregressive conditional heteroskedastic (ARCH) effect.

# The LM test

## An auxiliary regression

We estimate a  $AR(m)$  model regarding  $\{\hat{a}_t^2\}$ , that is,

$$\hat{a}_t^2 = \alpha_0 + \alpha_1 \hat{a}_{t-1}^2 + \cdots + \alpha_m \hat{a}_{t-m}^2 + e_t$$

## The LM test

With this model, we test the joint hypothesis

$$H_0 : \alpha_1 = \cdots = \alpha_m = 0$$

- The LM statistic is  $NR^2$  where  $N$  is the sample size of this regression and  $R^2$  is the coefficient of the determination of this regression.
- Given the null hypothesis is true, this statistic follows a  $\chi^2(m)$  distribution.

# The LM test (cont'd)

Alternatively, we can use F statistic to test the joint hypothesis.

- Let  $SSR_0 = \sum_{t=m+1}^T (\hat{a}_t^2 - \bar{\omega})^2$ , where  $\bar{\omega} = (1/T) \sum_{t=1}^T \hat{a}_t^2$ .
- Let  $SSR_1 = \sum_{t=m+1}^T \hat{e}_t^2$  where  $\hat{e}_t$  is the residuals from the regression.
- The F statistic is

$$F = \frac{(SSR_0 - SSR_1)/m}{SSR_1/(T - 2m - 1)} \sim F(m, T - 2m - 1)$$

- Rejecting the null hypothesis motivates us to model the possible ARCH effect.



# An example

Go back to Figure 4. Since the return series is already stationary, we directly test the squared return series to check the ARCH effect.

- In the LM test of the ARCH effect,  $F = 53.62$  and the p value is close to zero.
- The Ljung–Box statistics of the  $a_t^2$  series also shows strong ARCH effects with  $Q(12) = 89.85$ , the p value of which is close to zero.
- Therefore, we can confirm that the return series of Intel stock has an ARCH effect, and next we need to model such an effect.