Lecture 2: Review of Probability

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Outline

Experiments and outcomes

- An experiment is the processes that generate random results
- The outcomes of an experiment are its mutually exclusive potential results.
- Example: tossing a coin. The outcome is either getting a head(H) or a tail(T) but not both.

Sample space and events

- A sample space consists of all the outcomes from an experiment, denoted with the set S.
  - $S = \{H, T\}$ in the tossing-coin experiment.
- An event is a subset of the sample space.
- Getting a head is an event, which is $\{H\} \subset \{H, T\}$.

An intuitive definition of probability

- The probability of an event is the proportion of the time that the event will occur in the long run.
- For example, we toss a coin for n times and get m heads. When n is very large, we can say that the probability of getting a head in a toss is m/n.

An axiomatic definition of probability

- A probability of an event A in the sample space S, denoted as $\Pr(A)$, is a function that assign A a real number in $[0, 1]$, satisfying the following three conditions:
  1. $0 \leq \Pr(A) \leq 1$.
  2. $\Pr(S) = 1$.
  3. For any disjoint sets, A and B, that is A and B have no element in common, $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

The definition of random variables

- A random variable is a numerical summary associated with the outcomes of an experiment.
- You can also think of a random variable as a function mapping from an event ω in the sample space Ω to the real line.

An illustration of random variables

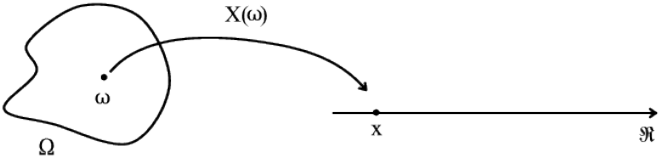


Figure: An illustration of random variable

Discrete and continuous random variables

Random variables can take different types of values

- A discrete random variables takes on a discrete set of values, like $0, 1, 2, \ldots, n$
- A continuous random variable takes on a continuum of possble values, like any value in the interval (a, b).

The probability distribution for a discrete random variable

- The probability distribution of a discrete random variable is the list of all possible values of the variable and the probability that each value will occur. These probabilities sum to 1.
- The probability mass function. Let X be a discrete random variable. The probability distribution of X (or the probability mass function), $p(x)$, is

$$p(x) = \mathrm{Pr}(X = x)$$

- The axioms of probability require that
  1. $0 \leq p(x) \leq 1$
  2. 2) $\sum_{i=1}^{n} p(x_i) = 1$.

An example of the probability distribution of a discrete random variable

Table: An illustration of the probability distribution of a discrete random variable

X	1	2	3	Sum
$\mathrm{P}(x)$	0.25	0.75	0.25	1

Definition of the c.d.f.

- The cumulative probability distribution (or the cumulative distribution function, c.d.f.):
  Let $F(x)$ be the c.d.f of X. Then $F(x) = \Pr(X \leq x)$.

Table: An illustration of the c.d.f. of a discrete random variable

X	1	2	3	Sum
$\mathrm{P}(x)$	0.25	0.50	0.25	1
C.d.f.	0.25	0.75	1	–

An illustration of the c.d.f. of a discrete random variable

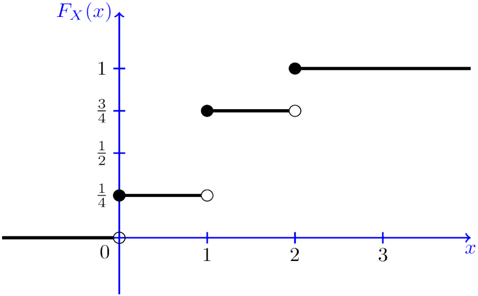


Figure: The c.d.f. of a discrete random variable

Bernouli distribution

The Bernoulli distribution

$$G = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

Definition of the c.d.f. and the p.d.f.

- The cumulative distribution function of a continous random variable is defined as it is for a discrete random variable.

$$F(x) = \mathrm{Pr}(X \leq x)$$

- The probability density function (p.d.f.) of X is the function that satisfies

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t \text{ for all } x$$

Properties of the c.d.f.

- For both discrete and continuous random variable, $F(X)$ must satisfy the following properties:
  1. $F(+\infty) = 1$ and $F(-\infty) = 0$ ($F(x)$ is bounded between 0 and 1)
  2. $x > y \Rightarrow F(x) \geq F(y)$ ($F(x)$ is nondecreasing)
- By the definition of the c.d.f., we can conveniently calculate probabilities, such as,
  - $\mathrm{P}(x > a) = 1 - \mathrm{P}(x \leq a) = 1 - F(a)$
  - $\mathrm{P}(a < x \leq b) = F(b) - F(a)$.

The c.d.f. and p.d.f. of a normal distribution

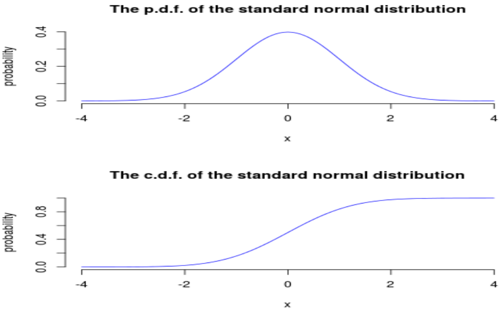


Figure: The p.d.f. and c.d.f. of a continuous random variable (the normal distribution)

The expected value

- The expected value of a random variable, X, denoted as $\mathrm{E}(X)$, is the long-run average of the random variable over many repeated trials or occurrences, which is also called the expectation or the mean.
- The expected value measures the centrality of a random variable.

Mathematical definition

- For a discrete random variable

$$\mathrm{E}(X) = \sum_{i=1}^{n} x_i \mathrm{Pr}(X = x_i)$$

- e.g. The expectation of a Bernoulli random variable, G,

$$\mathrm{E}(G) = 1 \cdot p + 0 \cdot (1 - p) = p$$

- For a continuous random variable

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x) \mathrm{d}x$$

Definition of variance and standard deviation

- The variance of a random variable X measures its average deviation from its own expected value.
- Let $\mathrm{E}(X) = \mu_X$. Then the variance of X,

$$\begin{aligned}
\mathrm{Var}(X) &= \sigma_X^2 = \mathrm{E}(X - \mu_X)^2 \\
&= \begin{cases} \sum_{i=1}^{n} (x - \mu_X)^2 \mathrm{Pr}(X = x_i) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x) \mathrm{d}x & \text{if } X \text{ is continuous} \end{cases}
\end{aligned}$$

- The standard deviation of X: $\sigma_X = \sqrt{\mathrm{Var}(X)}$

Computing variance

- A convenient formula for calculating the variance is

$$\mathrm{Var}(X) = \mathrm{E}(X - \mu_X)^2 = \mathrm{E}(X^2) - \mu_X^2$$

- The variance of a Bernoulli random variable, G

$$\mathrm{Var}(G) = (1-p)^2 p + (0-p)^2(1-p) = p(1-p)$$

- The expectation and variance of a linear function of X. Let $Y = a + bX$, then
  - $\mathrm{E}(Y) = a + \mathrm{E}(X)$
  - $\mathrm{Var}(Y) = \mathrm{Var}(a + bX) = b^2\mathrm{Var}(X)$.

Definition of the moments of a distribution

kth moment: The kth moment of the distribution of X is $\mathrm{E}(X^k)$. So, the expectation is the "first" moment of X.

kth central moment: The kth central moment of the distribution of X with its mean μ_X is $\mathrm{E}(X-\mu_X)^k$. So, the variance is the second central moment of X.

A caveat

It is important to remember that not all the moments of a distribution exist.

Skewness

- The skewness of a distribution provides a mathematical way to describe how much a distribution deviates from symmetry.

$$\text{Skewness} = \mathrm{E}(X - \mu_X)^3 / \sigma_X^3$$

- A symmetric distribution has a skewness of zero.
- The skewness can be either positive or negative.
- That $\mathrm{E}(X - \mu_X)^3$ is divided by σ_X^3 is to make the skewness measure unit free.

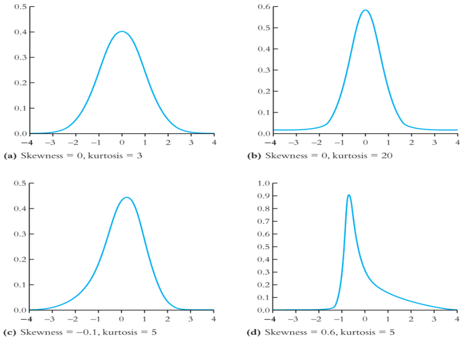
Kurtosis

- The kurtosis of the distribution of a random variable X measures how much of the variance of X arises from extreme values, which makes the distribution have "heavy" tails.

$$\text{Kurtosis} = \mathrm{E}(X - \mu_X)^4 / \sigma_X^4$$

- The kurtosis must be positive.
- The kurtosis of the normal distribution is 3. So a distribution that has its kurtosis exceeding 3 is called heavy-tailed.
- The kurtosis is also unit free.

An illustration of skewness and kurtosis



- All four distributions have a mean of zero and a variance of one, while (a) and (b) are symmetric and (b)-(d) are heavy-tailed.

The joint and marginal distributions

The joint probability function of two discrete random variables

- The joint distribution of two random variables X and Y is

$$p(x, y) = \Pr(X = x, Y = y)$$

- $p(x, y)$ must satisfy
  1. $p(x, y) \geq 0$
  2. $\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$ for all possible combinations of values of X and Y.

The joint probability function of two continuous random variables

- For two continuous random variables, X and Y, the counterpart of $p(x, y)$ is the joint probability density function, $f(x, y)$, such that
  1. $f(x, y) \geq 0$
  2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$

The marginal probability distribution

- The marginal probability distribution of a random variable X is simply the probability distribution of its own.
- For a discrete random variable, we can compute the marginal distribution of X as

$$\Pr(X = x) = \sum_{i=1}^{n} \Pr(X, Y = y_i) = \sum_{i=1}^{n} p(x, y_i)$$

- For a continuous random variable, the marginal distribution is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

An example of joint and marginal distributions

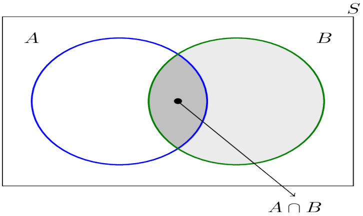
Table: Joint and marginal distributions of raining and commuting time

	Rain ($X = 0$)	No rain ($X = 1$)	Total
Long commute ($Y = 0$)	0.15	0.07	0.22
Short commute ($Y = 1$)	0.15	0.63	0.78
Total	0.30	0.70	1

Conditional probability

- For any two events A and B, the conditional probability of A given B is defined as

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

The conditional probability distribution

- The conditional distribution of a random variable Y given another random variable X is $\mathrm{Pr}(Y|X=x)$.
- The formula to compute it is

$$\mathrm{Pr}(Y|X=x) = \frac{\mathrm{Pr}(X=x, Y)}{\mathrm{Pr}(X=x)}$$

- For continuous random variables X and Y, we define the conditional density function as

$$f(y|x) = \frac{f(x,y)}{f_X(x)}$$

The conditional expectation

- The conditional expectation of Y given X is the expected value of the conditional distribution of Y given X.
- For discrete random variables, the conditional mean of Y given $X = x$ is

$$\mathrm{E}(Y \mid X = x) = \sum_{i=1}^{n} y_i \mathrm{Pr}(Y \mid X = x)$$

- For continuous random variables, it is computed as

$$\int_{-\infty}^{\infty} y f(y \mid x)\, dy$$

- The expected mean of commuting time given it is raining is $0 \times 0.1 + 1 \times 0.9 = 0.9$.

The law of iterated expectation

- The law of iterated expectation:

$$\mathrm{E}(Y) = E\left[\mathrm{E}(Y|X)\right]$$

- It says that the mean of Y is the weighted average of the conditional expectation of Y given X, weighted by the probability distribution of X. That is,

$$\mathrm{E}(Y) = \sum_{i=1}^{n} \mathrm{E}(Y \mid X = x_i)\mathrm{Pr}(X = x_i)$$

- If $\mathrm{E}(X|Y) = 0$, then $\mathrm{E}(X) = E\left[\mathrm{E}(X|Y)\right] = 0$.

Conditional variance

- With the conditional mean of Y given X, we can compute the conditional variance as

$$\mathrm{Var}(Y \mid X = x) = \sum_{i=1}^{n} [y_i - \mathrm{E}(Y \mid X = x)]^2 \,\mathrm{Pr}(Y = y_i \mid X = x)$$

- From the law of iterated expectation, we can get the following

$$\mathrm{Var}(Y) = \mathrm{E}(\mathrm{Var}(Y \mid X)) + \mathrm{Var}(\mathrm{E}(Y \mid X))$$

Independent random variables

- Two random variables X and Y are independently distributed, or independent, if knowing the value of one of the variable provides no information about the other.
- Mathematically, it means that

$$\Pr(Y = y \mid X = x) = \Pr(Y = y)$$

- If X and Y are independent

$$\Pr(Y = y, X = x) = \Pr(X = x)\Pr(Y = y)$$

Independence between two continuous random variable

- For two continuous random variables, X and Y, they are independent if

$$f(x|y) = f_X(x) \text{ or } f(y|x) = f_Y(y)$$

- It follows that if X and Y are independent

$$f(x, y) = f(x|y)f_Y(y) = f_X(x)f_Y(y)$$

Covariance

- The covariance of two discrete random variables X and Y is

$$\begin{aligned}\mathrm{Cov}(X, Y) = \sigma_{XY} &= \mathrm{E}(X - \mu_X)(Y - \mu_Y) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - \mu_X)(y_j - \mu_Y)\mathrm{Pr}(X = x_i, Y = y_j)\end{aligned}$$

- For continous random variables, the covariance of X and Y is

$$\mathrm{Cov}(X, Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - \mu_X)(y - \mu_y)f(x, y)dxdy$$

- The covariance can also be computed as

$$\mathrm{Cov}(X, Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$$

Correlation coefficient

- The correlation coefficient of X and Y is

$$\mathrm{corr}(X,Y) = \rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{[\mathrm{Var}(X)\mathrm{Var}(Y)]^{1/2}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

- $-1 \leq \mathrm{corr}(X,Y) \leq 1$.
- $\mathrm{corr}(X,Y) = 0$ (or $\mathrm{Cov}(X,Y) = 0$) means that X and Y are uncorrelated.
- Since $\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$, when X and Y are uncorrelated, then $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$.

Independence and uncorrelation

- If X and Y are independent, then

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i-\mu_X)(y_j-\mu_Y)\mathrm{Pr}(X=x_i)\mathrm{Pr}(Y=y_j)\\ &= \sum_{i=1}^{n}(x_i-\mu_X)\mathrm{Pr}(X=x_i)\sum_{j=1}^{m}(y_j-\mu_y)\mathrm{Pr}(Y=y_j)\\ &= 0\times 0 = 0\end{aligned}$$

- That is, if X and Y are independent, they must be uncorrelated.
- However, the converse is not true. If X and Y are uncorrelated, there is a possibility that they are actually dependent.

Conditional mean and correlation

- If X and Y are independent, then we must have $\mathrm{E}(Y \mid X) = \mathrm{E}(Y) = \mu_Y$
- Then, we can prove that $\mathrm{Cov}(X, Y) = 0$ and $\mathrm{corr}(X, Y) = 0$.

$$\begin{aligned} \mathrm{E}(XY) &= \mathrm{E}(\mathrm{E}(XY \mid X)) = \mathrm{E}(X\mathrm{E}(Y \mid X)) \\ &= \mathrm{E}(X)\mathrm{E}(Y \mid X) = \mathrm{E}(X)\mathrm{E}(Y) \end{aligned}$$

It follows that $\mathrm{Cov}(X, Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = 0$ and $\mathrm{corr}(X, Y) = 0$.

Some useful operations

The following properties of $\mathrm{E}(\cdot)$, $\mathrm{Var}(\cdot)$ and $\mathrm{Cov}(\cdot)$ are useful in calculation,

$$
\begin{aligned}
\mathrm{E}(a + bX + cY) &= a + b\mu_X + c\mu_Y \\
\mathrm{Var}(aX + bY) &= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY} \\
\mathrm{Cov}(a + bX + cV, Y) &= b\sigma_{XY} + c\sigma_{VY}
\end{aligned}
$$