Lecture 7: Hypothesis Test of Linear Regression with a Single Regressor

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Outline

The question after estimation

$$\widehat{TestScore} = 698.93 - 2.28 \times STR \tag{1}$$

- Now the question faced by the superintendent of the California elementary school districts is whether the estimated coefficient on *STR* is valid.
- In the terminology of statistics, his question is whether β_1 is statistically significantly different from zero.

Step 1: set up the two-sided hypothesis

$$H_0 : \beta_1 = \beta_{1,0}, H_1 : \beta_1 \neq \beta_{1,0}$$

Step 2: Compute the t-statistic

- The general form of the t-statistic is

$$t = \frac{\text{estimator} - \text{hypothesized value}}{\text{standard error of the estimator}} \tag{2}$$

- The t-statistics for testing β_1 is

$$t = \frac{\hat{\beta}_1 - \beta_{1,0}}{SE(\hat{\beta}_1)} \tag{3}$$

The standard error of $\hat{\beta}_1$ is calculated as

$$SE(\hat{\beta}_1) = \sqrt{\hat{\sigma}^2_{\hat{\beta}_1}} \tag{4}$$

where

$$\hat{\sigma}^2_{\hat{\beta}_1} = \frac{1}{n} \frac{\frac{1}{n-2}\sum_{i=1}^{n}(X_i - \bar{X})^2 \hat{u}_i^2}{\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right]^2} \tag{5}$$

How to understand the equation for $\hat{\sigma}^2_{\hat{\beta}_1}$

- $\hat{\sigma}^2_{\hat{\beta}_1}$ is the estimator of the variance of $\hat{\beta}_1$, i.e., $\mathrm{Var}(\hat{\beta}_1)$.
- The variance of $\hat{\beta}_1$ is

$$\sigma^2_{\hat{\beta}_1} = \frac{1}{n}\frac{\mathrm{Var}\left((X_i - \mu_X)u_i\right)}{\left(\mathrm{Var}(X_i)\right)^2}$$

- The denominator in $\hat{\sigma}^2_{\hat{\beta}_1}$ is a consistent estimator of $\mathrm{Var}(X_i)^2$.
- The numerator in $\hat{\sigma}^2_{\hat{\beta}_1}$ is a consistent estimator of $\mathrm{Var}((X_i - \mu_X)u_i)$.
- The standard error computed as $\hat{\sigma}^2_{\hat{\beta}_1}$ is the heteroskedasticity-robust standard error.

Step 3: compute the p-value

- The p-value is the probability of observing a value of $\hat{\beta}_1$ at least as different from $\beta_{1,0}$ as the estimate actually computed ($\hat{\beta}_1^{act}$), assuming that the null hypothesis is correct.

$$
\begin{aligned}
p\text{-value} &= \Pr_{H_0}\left(|\hat{\beta}_1 - \beta_{1,0}| > |\hat{\beta}_1^{act} - \beta_{1,0}|\right) \\
&= \Pr_{H_0}\left(\left|\frac{\hat{\beta}_1 - \beta_{1,0}}{SE(\hat{\beta}_1)}\right| > \left|\frac{\hat{\beta}_1^{act} - \beta_{1,0}}{SE(\hat{\beta}_1)}\right|\right) \\
&= \Pr_{H_0}\left(|t| > |t^{act}|\right)
\end{aligned}
$$

Step 3: compute the p-value (cont'd)

- With a large sample, the t statistic is approximately distributed as a standard normal random variable. Therefore, we can compute

$$p\text{-value} = \Pr\left(|t| > |t^{act}|\right) = 2\Phi(-|t^{act}|)$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution.

- The null hypothesis is rejected at the 5% significance level if the p-value < 0.05 or, equivalently, $|t^{act}| > 1.96$.

Application to test scores

$$\widehat{TestScore} = \underset{(10.4)}{698.9} - \underset{(0.52)}{2.28} \times STR,\ R^2 = 0.051,\ SER = 1.86$$

- The heteroskedasticity-robust standard errors are reported in the parentheses.
- The null hypothesis against the alternative one as

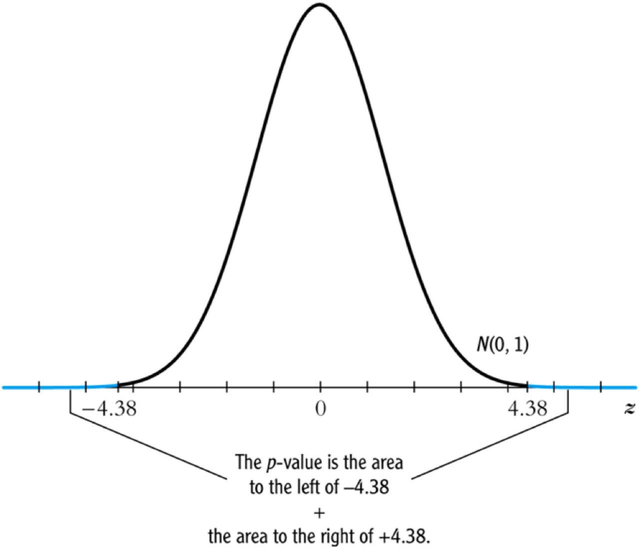
$$H_0 : \beta_1 = 0, H_1 : \beta_1 \neq 0$$

- The t-statistics is

$$t = \frac{\hat{\beta}_1}{SE(\hat{\beta}_1)} = \frac{-2.28}{0.52} = -4.38 < -1.96$$

- The p-value associated with $t^{act} = -4.38$ is approximately 0.00001, which is far less than 0.05. So we reject the null hypothesis.

Rejecting the null hypothesis



The one-sided hypotheses

- In some cases, it is appropriate to use a one-sided hypothesis test. For example, the superintendent of the California school districts want to know whether an increase in class sizes has a negative effect on test scores, that is, $\beta_1 < 0$.
- For such a test, we can set up the null hypothesis and the one-sided alternative hypothesis as

$$H_0 : \beta_1 = \beta_{1,0} \text{ vs. } H_1 : \beta_1 < \beta_{1,0}$$

The one-sided left-tail test

- The t-statistic is the same as in a two-sided test

$$t = \frac{\hat{\beta}_1 - \beta_{1,0}}{SE(\hat{\beta}_1)}$$

- Since we test $\beta_1 < \beta_{1,0}$, if this is true, the t-statistics should be statistically significantly less than zero.
- The p-value is computed as $\Pr(t < t^{act}) = \Phi(t^{act})$.
- The null hypothesis is rejected at the 5% significance level when p-value < 0.05 or $t^{act} < -1.645$.
- In the application of test scores, the t-statistics is -4.38, which is less than -1.645 and -2.33. Thus, the null hypothesis is rejected at the 1% level.