Lecture 7: Hypothesis Test of Linear Regression with a Single Regressor

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Outline

The question after estimation

$$\widehat{TestScore} = 698.93 - 2.28 \times STR \tag{1}$$

- Now the question faced by the superintendent of the California elementary school districts is whether the estimated coefficient on STR is valid.
- In the terminology of statistics, his question is whether β_1 is statistically significantly different from zero.

Step 1: set up the two-sided hypothesis

$$H_0 : \beta_1 = \beta_{1,0}, H_1 : \beta_1 \neq \beta_{1,0}$$

Step 2: Compute the t-statistic

- The general form of the t-statistic is

$$t = \frac{\text{estimator} - \text{hypothesized value}}{\text{standard error of the estimator}} \tag{2}$$

- The t-statistics for testing β_1 is

$$t = \frac{\hat{\beta}_1 - \beta_{1,0}}{SE(\hat{\beta}_1)} \tag{3}$$

The standard error of $\hat{\beta}_1$ is calculated as

$$SE(\hat{\beta}_1) = \sqrt{\hat{\sigma}^2_{\hat{\beta}_1}} \tag{4}$$

where

$$\hat{\sigma}^2_{\hat{\beta}_1} = \frac{1}{n} \frac{\frac{1}{n-2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \hat{u}_i^2}{\left[\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2\right]^2} \tag{5}$$

How to understand the equation for $\hat{\sigma}^2_{\hat{\beta}_1}$

- $\hat{\sigma}^2_{\hat{\beta}_1}$ is the estimator of the variance of $\hat{\beta}_1$, i.e., $\mathrm{Var}(\hat{\beta}_1)$.
- The variance of $\hat{\beta}_1$ is

$$\sigma^2_{\hat{\beta}_1} = \frac{1}{n} \frac{\mathrm{Var}\left((X_i - \mu_X)u_i\right)}{\left(\mathrm{Var}(X_i)\right)^2}$$

- The denominator in $\hat{\sigma}^2_{\hat{\beta}_1}$ is a consistent estimator of $\mathrm{Var}(X_i)^2$.
- The numerator in $\hat{\sigma}^2_{\hat{\beta}_1}$ is a consistent estimator of $\mathrm{Var}((X_i - \mu_X)u_i)$.
- The standard error computed as $\hat{\sigma}^2_{\hat{\beta}_1}$ is the heteroskedasticity-robust standard error.

Step 3: compute the p-value

- The p-value is the probability of observing a value of $\hat{\beta}_1$ at least as different from $\beta_{1,0}$ as the estimate actually computed ($\hat{\beta}_1^{act}$), assuming that the null hypothesis is correct.

$$
\begin{aligned}
p\text{-value} &= \mathrm{Pr}_{H_0}\left(|\hat{\beta}_1 - \beta_{1,0}| > |\hat{\beta}_1^{act} - \beta_{1,0}|\right) \\
&= \mathrm{Pr}_{H_0}\left(\left|\frac{\hat{\beta}_1 - \beta_{1,0}}{SE(\hat{\beta}_1)}\right| > \left|\frac{\hat{\beta}_1^{act} - \beta_{1,0}}{SE(\hat{\beta}_1)}\right|\right) \\
&= \mathrm{Pr}_{H_0}\left(|t| > |t^{act}|\right)
\end{aligned}
$$

Step 3: compute the p-value (cont'd)

- With a large sample, the t statistic is approximately distributed as a standard normal random variable. Therefore, we can compute

$$p\text{-value} = \Pr\left(|t| > |t^{act}|\right) = 2\Phi(-|t^{act}|)$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution.

- The null hypothesis is rejected at the 5% significance level if the p-value < 0.05 or, equivalently, $|t^{act}| > 1.96$.

Application to test scores

$$\widehat{TestScore} = \underset{(10.4)}{698.9} - \underset{(0.52)}{2.28} \times STR,\ R^2 = 0.051,\ SER = 1.86$$

- The heteroskedasticity-robust standard errors are reported in the parentheses.
- The null hypothesis against the alternative one as

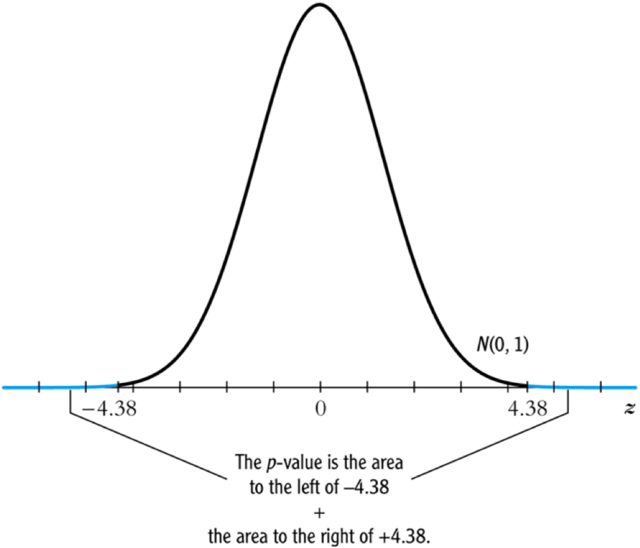
$$H_0 : \beta_1 = 0, H_1 : \beta_1 \neq 0$$

- The t-statistics is

$$t = \frac{\hat{\beta}_1}{SE(\hat{\beta}_1)} = \frac{-2.28}{0.52} = -4.38 < -1.96$$

- The p-value associated with $t^{act} = -4.38$ is approximately 0.00001, which is far less than 0.05. So we reject the null hypothesis.

Rejecting the null hypothesis



The one-sided hypotheses

- In some cases, it is appropriate to use a one-sided hypothesis test. For example, the superintendent of the California school districts want to know whether an increase in class sizes has a negative effect on test scores, that is, $\beta_1 < 0$.
- For such a test, we can set up the null hypothesis and the one-sided alternative hypothesis as

$$H_0 : \beta_1 = \beta_{1,0} \text{ vs. } H_1 : \beta_1 < \beta_{1,0}$$

The one-sided left-tail test

- The t-statistic is the same as in a two-sided test

$$t = \frac{\hat{\beta}_1 - \beta_{1,0}}{SE(\hat{\beta}_1)}$$

- Since we test $\beta_1 < \beta_{1,0}$, if this is true, the t-statistics should be statistically significantly less than zero.
- The p-value is computed as $\Pr(t < t^{act}) = \Phi(t^{act})$.
- The null hypothesis is rejected at the 5% significance level when p-value < 0.05 or $t^{act} < -1.645$.
- In the application of test scores, the t-statistics is -4.38, which is less than -1.645 and -2.33. Thus, the null hypothesis is rejected at the 1% level.

Two equivalent definitions of confidence intervals

- Recall that a 95% confidence interval for β_1 has two equivalent definitions:
  1. It is the set of values of β_1 that cannot be rejected using a two-sided hypothesis test with a 5% significance level.
  2. It is an interval that has a 95% probability of containing the true value of β_1.

Construct the 95% confidence interval for β_1

- We can obtain the 95% confidence interval for β_1 using the t statistic and the acceptance region at the 5% significant level.
- The acceptance region is $-1.96 \leq \frac{\hat{\beta}_1 - \beta_1}{SE(\hat{\beta}_1)} \leq 1.96$
- The 95% confidence interval for β_1 is

$$\left[\hat{\beta}_1 - 1.96 SE(\hat{\beta}_1),\ \hat{\beta}_1 + 1.96 SE(\hat{\beta}_1)\right]$$

The application to test scores

- In the application to test scores, given that $\hat{\beta}_1 = -2.28$ and $SE(\hat{\beta}_1) = 0.52$, the 95% confidence interval for β_1 is

$$-2.28 \pm 1.96 \times 0.52, \text{ or } -3.30 \leq \beta_1 \leq -1.26$$

- The confidence interval only spans over the negative region, implying that the null hypothesis of $\beta_1 = 0$ can be rejected at the 5% significance level.

Confidence intervals for predicted effects of changing X

- β_1 is the marginal effect of X on Y. That is, when X changes by ΔX, Y changes by $\beta_1 \Delta X$.
- So the 95% confidence interval for the change in Y when X changes by ΔX is

$$\begin{aligned} &\left[\hat{\beta}_1 - 1.96SE(\hat{\beta}_1),\ \hat{\beta}_1 + 1.96SE(\hat{\beta}_1)\right] \times \Delta X \\ = &\left[\hat{\beta}_1 \Delta X - 1.96SE(\hat{\beta}_1)\Delta X,\ \hat{\beta}_1 \Delta X + 1.96SE(\hat{\beta}_1)\Delta X\right] \end{aligned}$$

A binary variable

- A binary variable takes on values of one if some condition is true and zero otherwise, which is also called a dummy variable, a categorical variable, or an indicator variable.

$$D_i = \begin{cases} 1, & \text{if the } i^{th} \text{ subject is female} \\ 0, & \text{if the } i^{th} \text{ subject is male} \end{cases}$$

The linear regression model with a binary regressor

$$Y_i = \beta_0 + \beta_1 D_i + u_i,\ i = 1, \dots, n \tag{6}$$

- β_1 is estimated by the OLS estimation method in the same way as a continuous regressor.

Interpretation of the regression coefficients

- Given that the assumption $E(u_i|D_i) = 0$ holds, we have two population regression functions:
  - When $D_i = 1$, $E(Y_i|D_i = 1) = \beta_0 + \beta_1$
  - When $D_i = 0$, $E(Y_i|D_i = 0) = \beta_0$
- $\beta_1 = E(Y_i|D_i = 1) - E(Y_i|D_i = 0)$, i.e., the difference in the population means between two groups.

Hypothesis tests and confidence intervals

- The null v.s. alternative hypothesis

$$H_0: \beta_1 = 0 \text{ vs. } H_1: \beta_1 \neq 0$$

- The t-statistic

$$t = \frac{\hat{\beta}_1}{SE(\hat{\beta}_1)}$$

- The 95% confidence interval

$$\hat{\beta}_1 \pm 1.96 SE(\hat{\beta}_1)$$

Application to test scores

- We use a binary variable D to represent small and large classes.

$$D_i = \begin{cases} 1, & \text{if } STR_i < 20 \text{ (small classes)} \\ 0, & \text{if } STR_i \geq 20 \text{ (large classes)} \end{cases}$$

- Using the OLS estimation, the estimated regression function is

$$\widehat{TestScore} = \underset{(1.3)}{650.0} - \underset{(1.8)}{7.4}\ D,\ R^2 = 0.037,\ SER = 18.7$$

Application to test scores (cont'd)

- The t-statistic for β_1 is $t = 7.4/1.8 = 4.04 > 1.96$ so that β_1 is significantly different from zero.
  - The test score in small classes are on average 7.4 higher than that in large classes.
- The confidence interval for the difference is $7.4 \pm 1.96 \times 1.8 = (3.9, 10.9)$.

Homoskedasticity

- The error term u_i is homoskedastic if the conditional variance of u_i given X_i is constant for all $i = 1, \ldots, n$.
- Mathematically, it says $\mathrm{Var}(u_i|X_i) = \sigma^2$, for $i = 1, \ldots, n$, i.e., the variance of u_i for all i is a constant and does not depend on X_i.

Heteroskedasticity

- The error term u_i is heteroskedastic if the conditional variance of u_i given X_i changes on X_i for $i = 1, \ldots, n$.
- $\mathrm{Var}(u_i|X_i) = \sigma_i^2$, for $i = 1, \ldots, n$.
- A multiplicative form of heteroskedasticity is $\mathrm{Var}(u_i|X_i) = \sigma^2 f(X_i)$ where $f(X_i)$ is a function of X_i, for example, $f(X_i) = X_i$ as a simplest case.

Homoskedasticity and heteroskedasticity compared

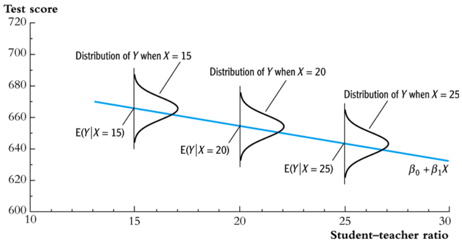
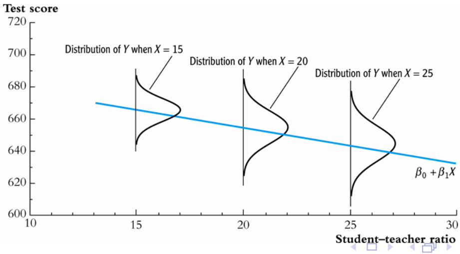


Figure: Homoskedasticity



Mathematical implications of homoskedasticity

Unbiasedness, consistency, and the asymptotic distribution

- As long as the least squares assumptions holds, whether the error term, u_i, is homoskedastic or heteroskedastic does not affect unbiasedness, consistency, and the asymptotic normal distribution of the OLS estimators.
  - The unbiasedness requires that $E(u_i|X_i) = 0$
  - The consistency requires that $E(X_i u_i) = 0$, which is true if $E(u_i|X_i) = 0$.
  - The asymptotic normal distribution requires additionally that $\mathrm{Var}((X_i - \mu_X)u_i) < \infty$, which still holds as long as Assumption 3 holds.

Mathematical implications of homoskedasticity (cont'd)

Efficiency

- The existence of heteroskedasticity affects the enfficiency of the OLS estimator
  - Suppose $\hat{\beta}_1$ and $\tilde{\beta}_1$ are both unbiased estimators of β_1. Then, $\hat{\beta}_1$ is said to be more efficient than $\tilde{\beta}_1$ if

$$\mathrm{Var}(\hat{\beta}_1) < \mathrm{Var}(\tilde{\beta}_1)$$

  - When the errors are homoskedastic, the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are the most efficient among all estimators that are linear in $Y_1, \ldots, Y_n$ and are unbiased, conditional on $X_1, \ldots, X_n$.

The homoskedasticity-only variance formula

- Recall that we can write $\hat{\beta}_1$ as

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_i (X_i - \bar{X}) u_i}{\sum_i (X_i - \bar{X})^2}$$

- If u_i for $i = 1, \ldots, n$ is homoskedastic and σ^2 is known, then

$$\sigma^2_{\hat{\beta}_1} = \mathrm{Var}(\hat{\beta}_1 | X_i) = \frac{\sum_i (X_i - \bar{X})^2 \mathrm{Var}(u_i | X_i)}{\left[\sum_i (X_i - \bar{X})^2\right]^2} = \frac{\sigma^2}{\sum_i (X_i - \bar{X})^2} \tag{7}$$

The homoskedasticity-only variance when σ^2 is unknown

- When σ^2 is unknown, then we use $s_u^2 = 1/(n-2)\sum_i \hat{u}_i^2$ as an estimator of σ^2.
- The homoskedasticity-only estimator of the variance of $\hat{\beta}_1$ is

$$\tilde{\sigma}^2_{\hat{\beta}_1} = \frac{s_u^2}{\sum_i (X_i - \bar{X})^2} \tag{8}$$

- The homoskedasticity-only standard error is $SE(\hat{\beta}_1) = \sqrt{\tilde{\sigma}^2_{\hat{\beta}_1}}$.

The heteroskedasticity-robust standard error

- The heteroskedasticity-robust standard error is

$$SE(\hat{\beta}_1) = \sqrt{\hat{\sigma}^2_{\hat{\beta}_1}}$$

where

$$\hat{\sigma}^2_{\hat{\beta}_1} = \frac{1}{n} \frac{\frac{1}{n-2}\sum_{i=1}^{n}(X_i - \bar{X})^2 \hat{u}_i^2}{\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right]^2}$$

which is also referred to as Eicker-Huber-White standard errors.

What does this mean in practice?

- Heteroskedasticity is common in cross-sectional data. It is always safer to report the heteroskedasticity-robust standard errors and use these to compute the robust t-statistic.
- In most software, the default setting is to report the homoskedasticity-only standard errors. Therefore, you need to manually add the option for the robust estimation.
    - In R, you can use the following codes

```
library(lmtest)
model1 <- lm(testscr ~ str, data = classdata)
coeftest(model1, vcov = vcovHC(model1, type="HC1"))
```

The Gauss-Markov conditions

For $\mathbf{X} = [X_1, \ldots, X_n]$

1. $E(u_i|\mathbf{X}) = 0$ (The exogeneity assumption)
2. $\mathrm{Var}(u_i|\mathbf{X}) = \sigma_u^2$, $0 < \sigma_u^2 < \infty$ (The homoskedasticity assumption)
3. $E(u_i u_j|\mathbf{X}) = 0$, $i \neq j$ (The uncorrelation assumption)

The least squares assumptions

- We have already known the least squares assumptions: for $i = 1, \ldots, n$,
  1. $E(u_i|X_i) = 0$
  2. (X_i, Y_i) are i.i.d., and
  3. Large outliers are unlikely.

From the three Least Squares Assumptions and the homoskedasticity assumption to the Gauss-Markov conditions

- All the Gauss-Markov conditions, except for the homoskedasticity assumption, can be derived from the least squares assumptions.
  - The least squares assumptions (1) and (2) imply $E(u_i|\mathbf{X}) = E(u_i|X_i) = 0$.
  - The least squares assumptions (1) and (2) imply $\mathrm{Var}(u_i|\mathbf{X}) = \mathrm{Var}(u_i|X_i)$.
  - With the homoskedasticity assumption, $\mathrm{Var}(u_i|X_i) = \sigma_u^2$, the least squares assumption (3) then implies $0 < \sigma_u^2 < \infty$.
  - The least squares assumptions (1) and (2) imply that $E(u_i u_j|\mathbf{X}) = E(u_i u_j|X_i, X_j) = E(u_i|X_i)E(u_j|X_j) = 0$.

The Gauss-Markov Theorem

- The Gauss-Markov Theorem for $\hat{\beta}_1$:
  If the Gauss-Markov conditions hold, then the OLS estimator $\hat{\beta}_1$ is the Best (most efficient) Linear conditionally Unbiased Estimator (BLUE).
- The theorem can also be applied to $\hat{\beta}_0$.

The linear estimators of β_1

- Any linear estimator $\tilde{\beta}_1$, it can be written as

$$\tilde{\beta}_1 = \sum_{i=1}^{n} a_i Y_i \tag{9}$$

where the weights a_i for $i = 1, \dots, n$ depend on $X_1, \dots, X_n$ but not on $Y_1, \dots, Y_n$.

The linear conditionally unbiased estimators

- $\tilde{\beta}_1$ is conditionally unbiased means that

$$E(\tilde{\beta}_1|\mathbf{X}) = \beta_1 \tag{10}$$

- By the Gauss-Markov conditions, we can have

$$\begin{aligned} E(\tilde{\beta}_1|\mathbf{X}) &= \sum_i a_i E(\beta_0 + \beta_1 X_i + u_i|\mathbf{X}) \\ &= \beta_0 \sum_i a_i + \beta_1 \sum_i a_i X_i \end{aligned}$$

- For the equation above being satisfied with any β_0 and β_1, we must have

$$\sum_i a_i = 0 \text{ and } \sum_i a_i X_i = 1$$

The OLS esimator $\hat{\beta}_1$ is a linear conditionally unbiased estimator

- $\hat{\beta}_1 = \frac{\sum_i (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2} = \frac{\sum_i (X_i - \bar{X}) Y_i}{\sum_i (X_i - \bar{X})^2} = \sum_i \hat{a}_i Y_i$
  where the weights are

$$\hat{a}_i = \frac{X_i - \bar{X}}{\sum_i (X_i - \bar{X})^2}, \text{ for } i = 1, \ldots, n$$

- Since $\hat{\beta}_1$ is a linear conditionally unbiased estimator, we must have

$$\sum_i \hat{a}_i = 0 \text{ and } \sum_i \hat{a}_i X_i = 1$$

which can be simply verified.

A scratch of the proof of the Gauss-Markov theorem

- A key in the proof of the Gauss-Markov theorem is that we can rewrite the expression of any linear conditionally unbiased estimator $\tilde{\beta}_1$ as

$$\tilde{\beta}_1 = \sum_i a_i Y_i = \sum_i (\hat{a}_i + d_i) Y_i = \hat{\beta}_1 + \sum_i d_i Y_i$$

- The goal of the proof is to show that

$$\mathrm{Var}(\hat{\beta}_1 | \mathbf{X}) \leq \mathrm{Var}(\tilde{\beta}_1 | \mathbf{X})$$

  The equality holds only when $\tilde{\beta}_1 = \hat{\beta}_1$.
- The proof of the Gauss-Markov theorem is in Appendix 5.2.

The limitations of the Gauss-Markov theorem

The Gauss-Markov conditions may hold in practice.

- $E(u \mid X) \neq 0$
  - Cases: omitted variable, endogeneity
  - Consequence: the OLS is biased
  - Solution: adding more X's, IV method
- $\mathrm{Var}(u_i|X)$ not constant
  - Cases: heteroskedasticity
  - Consequence: the OLS is inefficient
  - Solution: WLS, GLS, HCCME
- $E(u_i u_j|X) \neq 0$
  - Cases: autocorrelation
  - Consequence: the OLS is inefficient
  - Solution: GLS, HAC

The classical assumptions of the least squares estimation

- The classical assumptions of the least squares estimation:
  For $i = 1, 2, \ldots, n$
  - Assumption 1: $E(u_i|X_i) = 0$ (exogeneity of X)
  - Assumption 2: (X_i, Y_i) are i.i.d. (IID of X, Y)
  - Assumption 3: $0 < E(X_i^4) < \infty$ and $0 < E(Y_i^4) < \infty$ (No large outliers)
  - Extended Assumption 4: $\mathrm{Var}(u_i|X_i) = \sigma_u^2,$ and $0 < \sigma_u^2 < \infty$ (homoskedasticity)
  - Extended Assumption 5: $u_i|X_i \sim N(0, \sigma_u^2)$ (normality)

The t-statistic is for β_1

- The null v.s. alternative hypotheses:

$$H_0 : \beta_1 = \beta_{1,0} \text{ vs } H_1 : \beta_1 \neq \beta_{1,0}$$

- The t-statistic:

$$t = \frac{\hat{\beta}_1 - \beta_{1,0}}{\hat{\sigma}_{\hat{\beta}_1}}$$

where $\hat{\sigma}^2_{\hat{\beta}_1} = \frac{s_u^2}{\sum_i (X_i - \bar{X})^2}$ and $s_u^2 = \frac{1}{n-2} \sum_i \hat{u}_i^2 = SER^2$.

The Student-t distribution of t

- When the classical least squares assumptions hold, the t-statistic has the exact distribution of the Student's t distribution with $(n-2)$ degrees of freedom.

$$t = \frac{\hat{\beta}_1 - \beta_{1,0}}{\hat{\sigma}_{\hat{\beta}_1}} \sim t(n-2)$$