The use/abuse of copulas in actuarial science and finance

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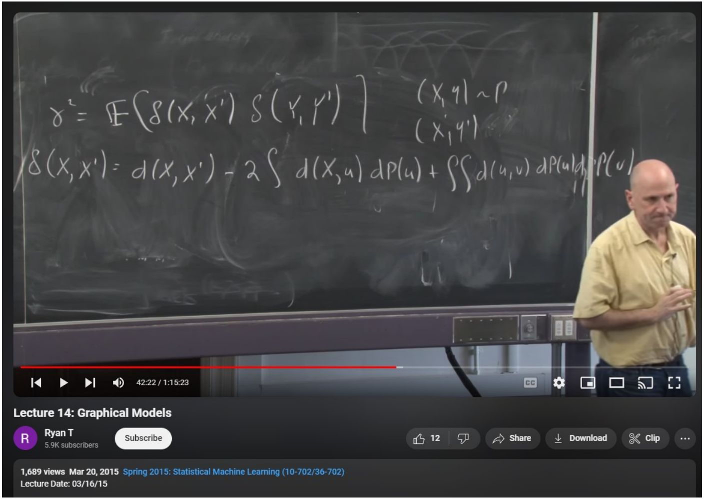
Overview

- Literature overview
- Modelling dependence
- Copulas
- Tools
- Conclusion

Papers

- "Correlation and Dependence in Risk Management", Embrechts, McNeil, and Straumann [EMS02]
- "Understanding Relationships Using Copulas", Frees and Valdez [FV98]
- "THE DEVIL IS IN THE TAILS: ACTUARIAL MATHEMATICS AND THE SUBPRIME MORTGAGE CRISIS", Donnelly and Embrechts [DE]
- *From Dependence to Causation*, Lopez-Paz [Lop16]

Favorite course on yt with a section on dependence



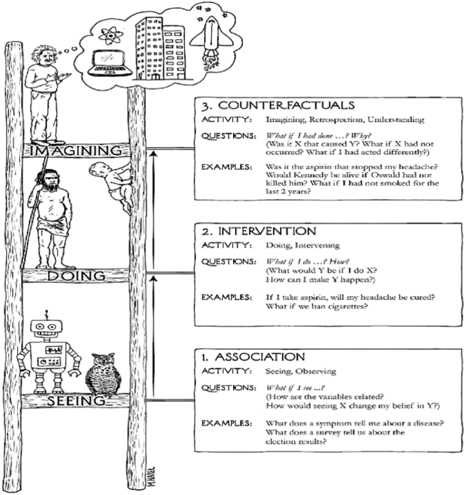
Dependence

The Ladder of Causality

"Actual" Causality

"Causality-in-mean"

Statistics



Modelling dependence 1

- Modelling with multivariate RVs $\rightarrow$ density estimation
- Usually only data and domain knowledge
- Non-parametric vs parametric
- Data and computational limitations
- Approximate assumptions
- Issues at tails
- Marginals $+$ copulas

Multivariate CLT

Theorem 3.9 (Central Limit Theorem in $\mathbb{R}^k$) *Let $X_n, n \geq 1$ be i.i.d. random vectors with $\mathbb{E}|X_1|^2 < \infty$ and $\mathbb{E}X_1 = 0$. Set $S_n = \sum_{j=1}^n X_j, n \geq 1$. Then we have:*

$$S_n/\sqrt{n} \xrightarrow{d} Y \sim N(0, \Sigma),$$

where $\Sigma = \mathrm{Cov}(X_1)$.

Proof In view of Corollary 3.1 it is sufficient to show for any $t \in \mathbb{R}^k$,

$$\langle S_n/\sqrt{n}, t\rangle \xrightarrow{d} \langle Y, t\rangle. \tag{3.5}$$

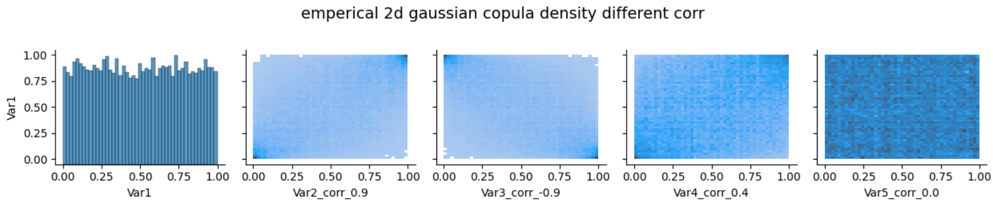
If $t = 0$, this is trivial. If $t \neq 0$, we can write $\langle S_n/\sqrt{n}, t\rangle = \sum_{j=1}^n \langle X_j, t\rangle/\sqrt{n}$. Obviously, the random variables $\langle X_j, t\rangle, j \geq 1$ are i.i.d and we have $\mathbb{E}\langle X_1, t\rangle = \langle \mathbb{E}X_1, t\rangle = 0$. Moreover, we have $\mathrm{Var}(\langle X_1, t\rangle) = \langle t, \Sigma t\rangle =: \sigma_t^2$. If $\sigma_t^2 > 0$, it follows from the formula for the characteristic function of Y that $\langle Y, t\rangle$ is normal$(0, \sigma_t^2)$-distributed. By the 1-dimensional CLT we then have

$$\sum_{j=1}^n \langle X_j, t\rangle/\sqrt{n} \xrightarrow{d} \langle Y, t\rangle,$$

and we see that (3.5) holds in this case.
If $\sigma_t^2 = 0$, we have with probability one, $\langle S_n/\sqrt{n}, t\rangle = \langle Y, t\rangle = 0$, and (3.5) is trivial. $\square$

Gaussian copula

Sklar's theorem $\rightarrow$ transform the marginals to uniforms
Same amount parameters as the Gaussian distribution
No parametric assumption on marginals compared to assuming Gaussian



Spherical/elliptical distributions 1

Spherical distributions have a spherical/elliptical symmetry or

$$X =_d RU. \tag{1}$$

with $R > 0, U \perp\!\!\!\perp R$ uniform on unit sphere.

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Parametrized by mean, covariance matrix and R.
Semi-parametric multivariate distributions

Spherical/elliptical distributions 2

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- An affine transformation of an elliptical distribution is also elliptical.
- The sum in the set of elliptical distributions with the same generator is closed.
- Marginal and conditional distributions of the components of elliptical distributions are elliptical.

Spherical/elliptical distributions 2

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In [EMS02] they show that linear portfolios where individual risk together are jointly elliptical distributed, **risk measures lose structure**, simplifying risk management tasks. Specifically they show that VaR is equivalent to variance risk analysis.

Tail dependence 1

Tail dependence ≈ dependence in the tails . The upper tail dependence (λ) between X and Y is defined as follows:

$$\lim_{\alpha \to 1-} P[Y > F_Y^{-1}(\alpha) \mid X > F_X^{-1}(\alpha)] = \lambda. \tag{2}$$

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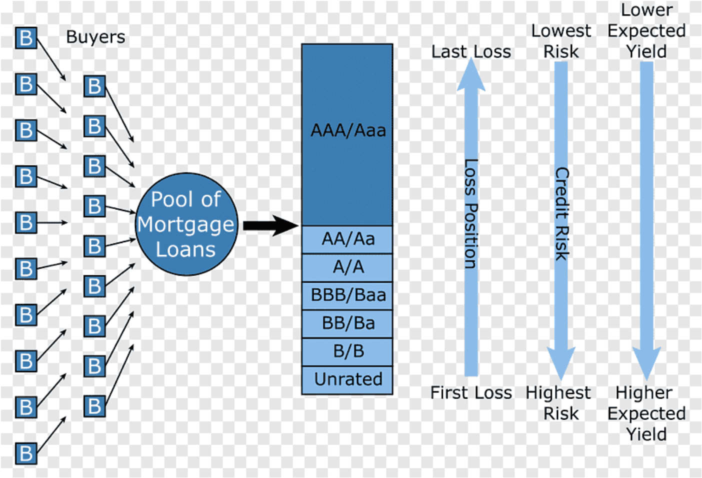
[EMS02] show different ways to calculate tail dependence. They show that normal distributions have no tail dependence.

Tail dependence 2

When X and Y are continuous distribution we can express this in terms of their copula:

$$\begin{aligned}
&\lim_{\alpha\to 1-} P[Y > F_Y^{-1}(\alpha) \mid X > F_X^{-1}(\alpha)] && (3)\\
&= \lim_{\alpha\to 1-} P[U_2 > \alpha \mid U_1 > \alpha] && (4)\\
&= \lim_{\alpha\to 1-} \frac{P[U_1 > \alpha, U_2 > \alpha]}{P[U_1 > \alpha]} && (5)\\
&= \lim_{\alpha\to 1-} \frac{1 - P[(U_1 > \alpha, U_2 > \alpha)^c]}{1-\alpha} && (6)\\
&= \lim_{\alpha\to 1-} \frac{1 - P[(U_1 \leq \alpha) \cup (U_2 \leq \alpha)]}{1-\alpha} && (7)\\
&= \lim_{\alpha\to 1-} \frac{1 - P[U_1 \leq \alpha] - P[U_2 \leq \alpha] + P[U_1 \leq \alpha, U_2 \leq \alpha]}{1-\alpha} && (8)\\
&= \lim_{\alpha\to 1-} \frac{1 - 2\alpha + C(\alpha,\alpha)}{1-\alpha}. && (9)
\end{aligned}$$

Modelling CDOs



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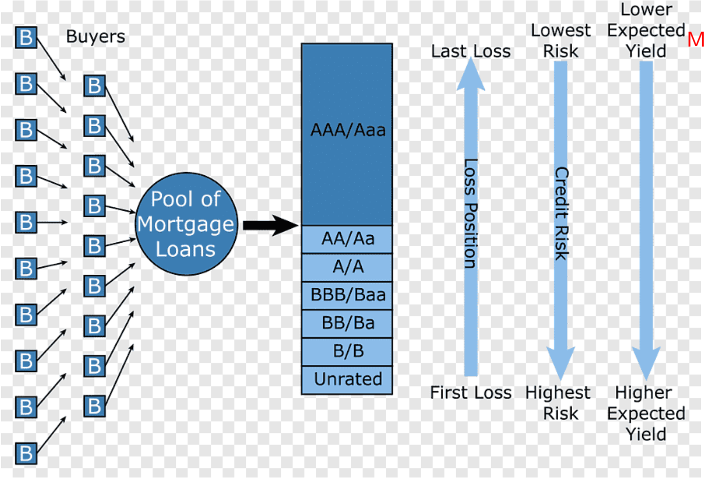
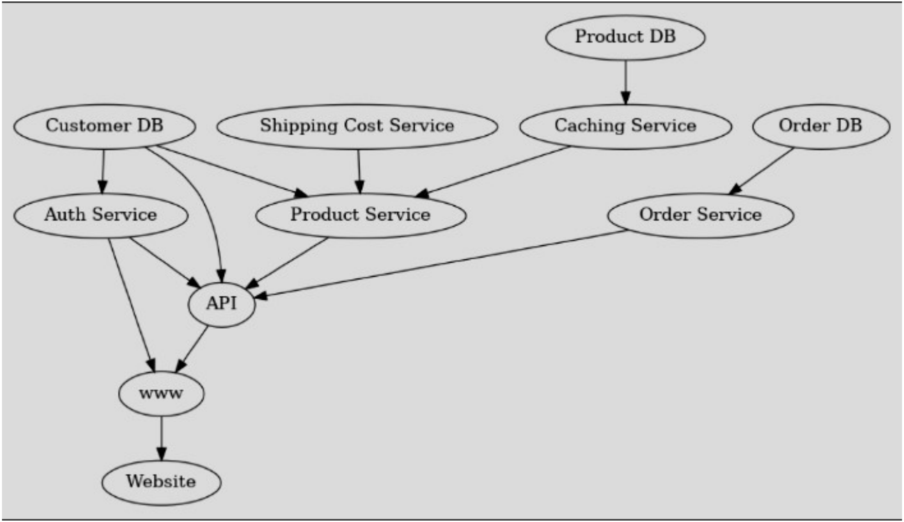
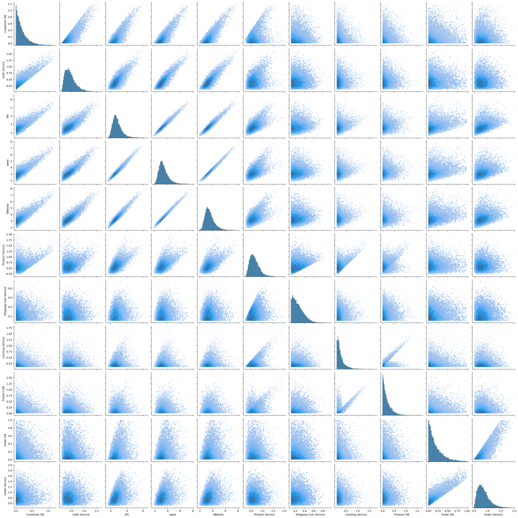


Modelling Risk

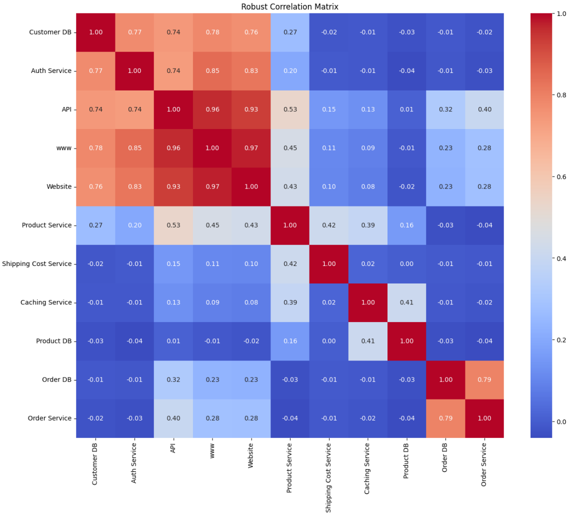
Visualizing with copulas 1



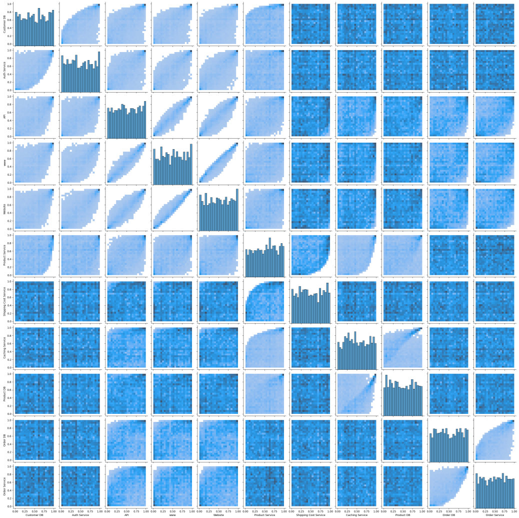
Visualizing with copulas 2



Visualizing with copulas 3



Visualizing with copulas 4



Copula cons

- Reliant on parameteric assumptions in high dimensions
- Copulas don't always retain smoothness/other properties
- Difficulties with multimode distributions
- Density estimation (breaking Vapnik's principle)
- High dimensional dependence is unintuitive and complicated

Conclusion