

# The use/abuse of copulas in actuarial science and finance

Isidoor Pinillo Esquivel

- Literature overview
- Modelling dependence
- Copulas
- Tools
- Conclusion

- “Correlation and Dependence in Risk Management”, Embrechts, McNeil, and Straumann [EMS02]
- “Understanding Relationships Using Copulas”, Frees and Valdez [FV98]
- “THE DEVIL IS IN THE TAILS: ACTUARIAL MATHEMATICS AND THE SUBPRIME MORTGAGE CRISIS”, Donnelly and Embrechts [DE]
- *From Dependence to Causation*, Lopez-Paz [Lop16]

# Favorite course on yt with a section on dependence

$$\sigma^2 = \mathbb{E} \left( g(X, X') g(Y, Y') \right) \quad \begin{matrix} (X, Y) \sim P \\ (X', Y') \sim P \end{matrix}$$

$$g(X, X') = d(X, X') - 2 \int d(X, u) dP(u) + \iint d(u, v) dP(u) dP(v)$$

**Lecture 14: Graphical Models**

**R** Ryan T  
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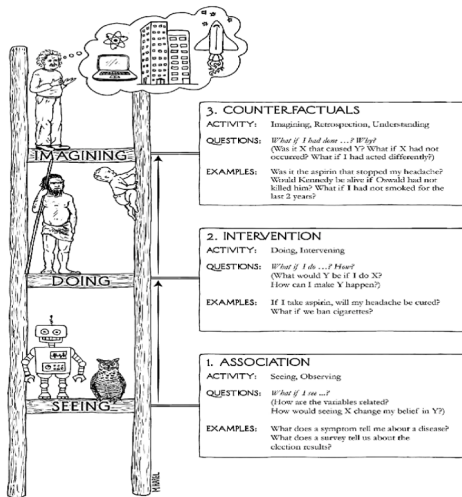
1,689 views Mar 20, 2015 [Spring 2015: Statistical Machine Learning \(10-702/36-702\)](#)  
Lecture Date: 03/16/15

## The Ladder of Causality

“Actual” Causality

“Causality-in-mean”

Statistics



# Modelling dependence 1

- Modelling with multivariate RVs  $\rightarrow$  density estimation
- Usually only data and domain knowledge
- Non-parametric vs parametric
- Data and computational limitations
- Approximate assumptions
- Issues at tails
- Marginals + copulas

# Multivariate CLT

**Theorem 3.9 (Central Limit Theorem in  $\mathbb{R}^k$ )** Let  $X_n, n \geq 1$  be i.i.d. random vectors with  $\mathbb{E}|X_1|^2 < \infty$  and  $\mathbb{E}X_1 = 0$ . Set  $S_n = \sum_{j=1}^n X_j, n \geq 1$ . Then we have:

$$S_n/\sqrt{n} \xrightarrow{d} Y \sim N(0, \Sigma),$$

where  $\Sigma = \text{Cov}(X_1)$ .

**Proof** In view of Corollary 3.1 it is sufficient to show for any  $t \in \mathbb{R}^k$ ,

$$\langle S_n/\sqrt{n}, t \rangle \xrightarrow{d} \langle Y, t \rangle. \quad (3.5)$$

If  $t = 0$ , this is trivial. If  $t \neq 0$ , we can write  $\langle S_n/\sqrt{n}, t \rangle = \sum_{j=1}^n \langle X_j, t \rangle / \sqrt{n}$ .

Obviously, the random variables  $\langle X_j, t \rangle, j \geq 1$  are i.i.d and we have  $\mathbb{E}\langle X_1, t \rangle = \langle \mathbb{E}X_1, t \rangle = 0$ . Moreover, we have  $\text{Var}(\langle X_1, t \rangle) = \langle t, \Sigma t \rangle =: \sigma_t^2$ . If  $\sigma_t^2 > 0$ , it follows from the formula for the characteristic function of  $Y$  that  $\langle Y, t \rangle$  is normal( $0, \sigma_t^2$ )-distributed. By the 1-dimensional CLT we then have

$$\sum_{j=1}^n \langle X_j, t \rangle / \sqrt{n} \xrightarrow{d} \langle Y, t \rangle,$$

and we see that (3.5) holds in this case.

If  $\sigma_t^2 = 0$ , we have with probability one,  $\langle S_n/\sqrt{n}, t \rangle = \langle Y, t \rangle = 0$ , and (3.5) is trivial.  $\square$

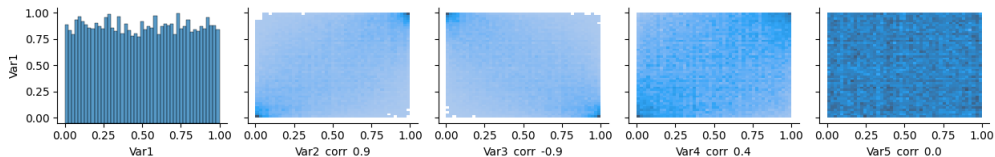
# Gaussian copula

Sklar's theorem  $\rightarrow$  transform the marginals to uniforms

Same amount parameters as the Gaussian distribution

No parametric assumption on marginals compared to assuming Gaussian

emperical 2d gaussian copula density different corr





# Spherical/elliptical distributions 1

Spherical distributions have a spherical/elliptical symmetry or

$$X =_d RU. \quad (1)$$

with  $R > 0$ ,  $U \perp\!\!\!\perp R$  uniform on unit sphere.

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Parametrized by mean, covariance matrix and  $R$ .

Semi-parametric multivariate distributions

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- Marginal and conditional distributions of the components of elliptical distributions are elliptical.

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In [EMS02] they show that linear portfolios where individual risk together are jointly elliptical distributed, **risk measures lose structure**, simplifying risk management tasks. Specifically they show that VaR is equivalent to variance risk analysis.

# Tail dependence 1

Tail dependence  $\approx$  dependence in the tails . The upper tail dependence ( $\lambda$ ) between  $X$  and  $Y$  is defined as follows:

$$\lim_{\alpha \rightarrow 1-} P[Y > F_Y^{-1}(\alpha) \mid X > F_X^{-1}(\alpha)] = \lambda. \quad (2)$$

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[EMS02] show different ways to calculate tail dependence. They show that normal distributions have no tail dependence.



## Tail dependence 2

When  $X$  and  $Y$  are continuous distribution we can express this in terms of their copula:

$$\lim_{\alpha \rightarrow 1-} P[Y > F_Y^{-1}(\alpha) \mid X > F_X^{-1}(\alpha)] \quad (3)$$

$$= \lim_{\alpha \rightarrow 1-} P[U_2 > \alpha \mid U_1 > \alpha] \quad (4)$$

$$= \lim_{\alpha \rightarrow 1-} \frac{P[U_1 > \alpha, U_2 > \alpha]}{P[U_1 > \alpha]} \quad (5)$$

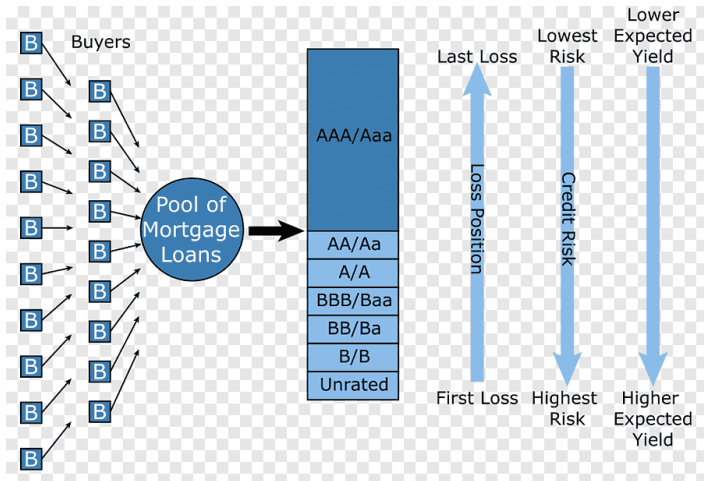
$$= \lim_{\alpha \rightarrow 1-} \frac{1 - P[(U_1 > \alpha, U_2 > \alpha)^c]}{1 - \alpha} \quad (6)$$

$$= \lim_{\alpha \rightarrow 1-} \frac{1 - P[(U_1 \leq \alpha) \cup (U_2 \leq \alpha)]}{1 - \alpha} \quad (7)$$

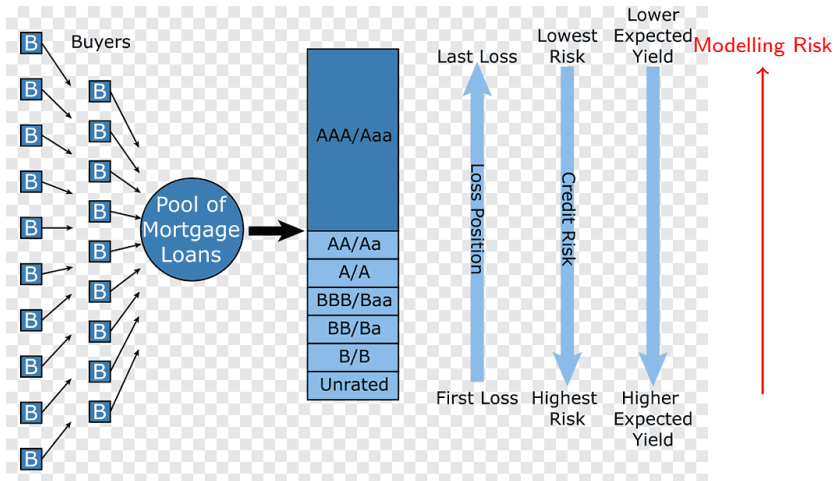
$$= \lim_{\alpha \rightarrow 1-} \frac{1 - P[U_1 \leq \alpha] - P[U_2 \leq \alpha] + P[U_1 \leq \alpha, U_2 \leq \alpha]}{1 - \alpha} \quad (8)$$

$$= \lim_{\alpha \rightarrow 1-} \frac{1 - 2\alpha + C(\alpha, \alpha)}{1 - \alpha}. \quad (9)$$

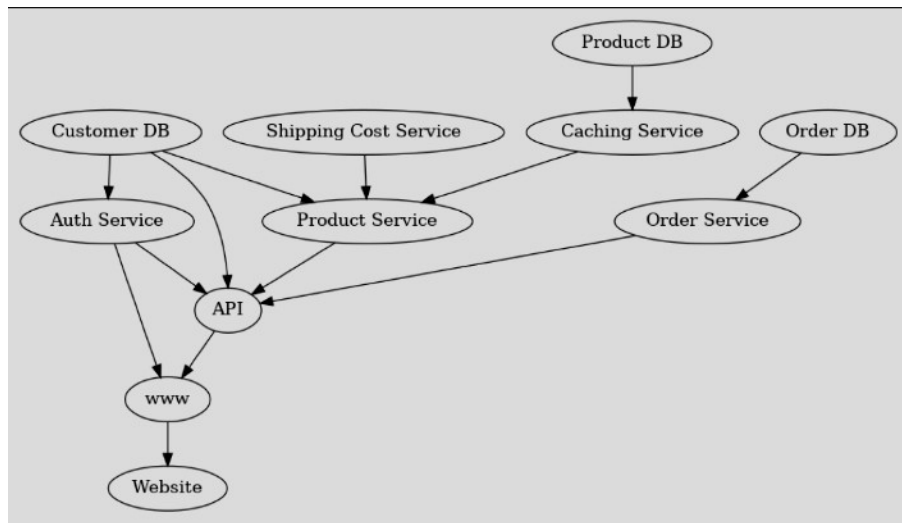
# Modelling CDOs



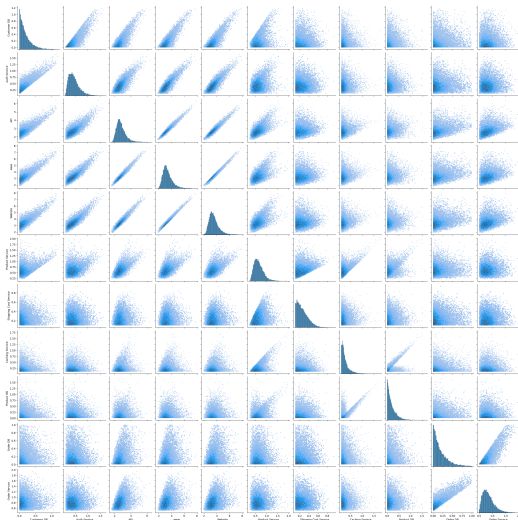
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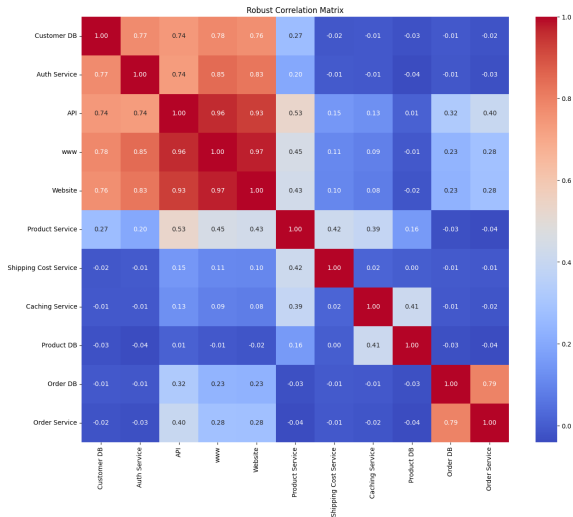
# Visualizing with copulas 1



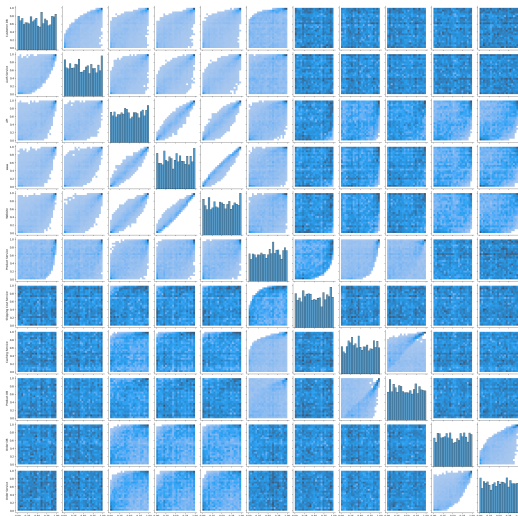
# Visualizing with copulas 2



# Visualizing with copulas 3



## Visualizing with copulas 4



- Reliant on parameteric assumptions in high dimensions
- Copulas don't always retain smoothness/other properties
- Difficulties with multimode distributions
- Density estimation (breaking Vapnik's principle)
- High dimensional dependence is unintuitive and complicated



- Copulas are a powerful tool for modelling dependence
- Theoretical tools for understanding dependence