The use/abuse of copulas in actuarial science and finance

Abstract

This is an assignment for the actuarial models course. The assignment is to summarize [EMS02], [FV98], [DE] and discuss the following:

- The purpose is to understand the impact of the assumption regarding the dependence structure between risk factors.
- This is done by means of the concept of copulas.
- In particular, we study the impact of misused copulas and correlation in the valuation of collateralized debt obligations (CDO's).

1 Overview

[EMS02] introduces linear dependency, copulas, comonotonicity and rank correlation which already are introduced in the course. Also spherical/elliptical distributions and tail dependence are introduced. Spherical/elliptical distributions generalize the normal distribution and has good properties for common dependency measures. Tail dependence quantifies dependency in the tails. They also discuss some common dependency fallacies and simulation of copulas.

[FV98] also introduces copulas, present examples, simulation and fitting of copulas. Fitting copulas is done by choosing a parametric model guided by non-parametric density estimation and parametric estimation is done with MLE and AIC. Main examples are joint mortality modeling and modeling insurance company indemnity claims. And finally also introducing stochastic orders and distortion functions which we also covered previously.

[DE] also introduces copulas but primarily focuses on the 2007-2008 financial crisis. It mainly discusses modeling Collateralized Debt Obligations (CDOs) and trenching, the impact of incorrectly modeling dependence, highlighting the consequences of incorrectly modelling default clustering. This is illustrated through an example of modeling the default probability of bonds under the assumption of identical pairwise correlation.

2 Introduction to new concepts

2.1 Spherical/elliptical distributions

Spherical/elliptical distributions have a spherical/elliptical symmetry. I.e. we can characterize a spherical distributions X as follows:

$$X =_d RU. \tag{1}$$

with R a positive random variable and U independent of R a random vector uniformly distributed on the unit sphere. Elliptical distributions can be obtained as affine transformations of spherical distributions. Elliptical distributions are fully characterized by their mean, covariance matrix and the characteristic function of normalized R (normalize such that the mean = 1) also called generator when the covariance exists ($E[R^2] < \infty$). So it is a semi-parametric family of distributions with limited 1 dimensional non-parametric random variable.

Here are some obvious properties of elliptical distributions:

- An affine transformation of an elliptical distribution is also elliptical.
- The sum in the set of elliptical distributions with the same generator is closed.
- Marginal and conditional distributions of the components of elliptical distributions are elliptical. The intuition for this is the same as for normal distributions, the intersection between ellipsoids and planes are also ellipsoids.

In [EMS02] they show that linear portfolios where individual risk together are jointly elliptical distributed, risk measures lose structure, simplifying risk management tasks. Specifically they show that VaR is equivalent to variance risk analysis. This emphasizes that the assumption of elliptical distributions is a strong assumption and normal distributions are even stronger.

2.2 Tail dependence

Making assumptions is dangerous. Data trades-off with assumptions. Determining high dimensional structure requires a lot of data. Dependence is a high dimensional structure and at tails we have little data both by definition. So naturally making assumptions about the dependence structure at the tails is a risky business. Tail dependence is a way to quantify the dependence in the tails . The upper tail dependence (λ) between X and Y is defined as follows:

$$\lim_{\alpha \to 1-} P[Y > F_2^{-1}(\alpha) \mid X > F_1^{-1}(\alpha)] = \lambda. \tag{2}$$

When X and Y are continuous distribution we can express this in terms of their copula:

$$\lim_{\alpha\to 1-} P[Y > F_2^{-1}(\alpha) \mid X > F_1^{-1}(\alpha)] \tag{3}$$

$$= \lim_{\alpha\to 1-} P[U_2 > \alpha \mid U_1 > \alpha] \tag{4}$$

$$= \lim_{\alpha\to 1-} \frac{P[U_1 > \alpha, U_2 > \alpha]}{P[U_1 > \alpha]} \tag{5}$$

$$= \lim_{\alpha\to 1-} \frac{1 - P[(U_1 > \alpha, U_2 > \alpha)^c]}{1-\alpha} \tag{6}$$

$$= \lim_{\alpha\to 1-} \frac{1 - P[(U_1 \leq \alpha) \cup (U_2 \leq \alpha)]}{1-\alpha} \tag{7}$$

$$= \lim_{\alpha\to 1-} \frac{1 - P[U_1 \leq \alpha] - P[U_2 \leq \alpha] + P[U_1 \leq \alpha, U_2 \leq \alpha]}{1-\alpha} \tag{8}$$

$$= \lim_{\alpha\to 1-} \frac{1 - 2\alpha + C(\alpha,\alpha)}{1-\alpha}. \tag{9}$$

[EMS02] show different ways to calculate tail dependence. They show that normal distributions have no tail dependence.

2.3 CDO's

CDOs, or Collateralized Debt Obligations, are financial tools that pool various debt assets, such as mortgages, and repackage them into discrete tranches for investors. I.e. a CDO at a trench t running from $s_t \to s_{t+1}$ is $SI_{s_t \leq S \leq s_{t+1}}$ with $S = \sum_i X_i$ and X_i the individual debt assets. Apart from the moral hazard, modeling CDO's in particularly in senior trenches is hard due to having to model dependence at the tails. [DE] underscores the role of inadequate modelling as a contributing factor to the financial crisis of $2007 - 2008$.

3 Link with lecture's topics

There is a lot of overlap with the course. Copulas, stochastic orders, distortion functions, risk measures.

4 Possible applications

In [DE] there is a summary of the main contributions from mathematics to economics and finance:

- understanding and clarifying models used in economics;
- making heuristic methods mathematically precise;
- highlighting model conditions and restrictions on applicability;
- working out numerous explicit examples;
- leading the way for stress-testing and robustness properties, and

- offering a relevant and challenging field of research on its own.

The main application of coplas is a theoretical tool for studying dependence. Theorems about copulas help understanding dependence and choosing better assumptions. The main example is that Gaussian copulas not always have desirable properties.

Copulas can also be used to visualize low dimensional dependence structures of continuous data containing more information then any individual dependence measure. We visualize following causal analysis from `https://www.pywhy.org/dowhy/main/example_notebooks/gcm_rca_microservice_architecture.html`. as an example.

`./copula example/pairplot.png`

Figure 1: This a histogram pairplot of the data. This should capture the pairwise dependence structures in the data but the non-uniform marginals makes it hard to spot independence only obvious dependence structure is comonotonicity.

References

[DE] Catherine Donnelly and Paul Embrechts. "THE DEVIL IS IN THE TAILS: ACTUARIAL MATHEMATICS AND THE SUBPRIME MORTGAGE CRISIS". en. In: ().

[EMS02] Paul Embrechts, Alexander J. McNeil, and Daniel Straumann. "Correlation and Dependence in Risk Management: Properties and Pitfalls". en. In: *Risk Management*. Ed. by M. A. H. Dempster. 1st ed. Cambridge University Press, Jan. 2002, pp. 176–223. ISBN: 978-0-521-78180-0 978-0-511-61533-7 978-0-521-16963-9. DOI: 10.1017/CBO9780511615337.008. URL: https://www.cambridge.org/core/product/identifier/CBO9780511615337A013/type/book_part (visited on 04/21/2024).

[FV98] Edward W. Frees and Emiliano A. Valdez. "Understanding Relationships Using Copulas". en. In: *North American Actuarial Journal* 2.1 (Jan. 1998), pp. 1–25. ISSN: 1092-0277, 2325-0453. DOI: 10.1080/10920277.1998.10595667. URL: http://www.tandfonline.com/doi/abs/10.1080/10920277.1998.10595667 (visited on 04/05/2024).