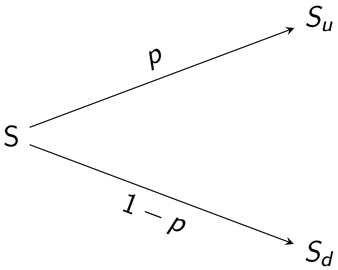
4. Discrete-time models

A two-period, two-state world

Consider the following model with

- Two periods: time $t \in \{0, 1\}$
- One share S
- One riskless asset
- At time $t = 0$, the share is worth S euros
- There are two possible states of the world at time $t = 1$
- At time $t = 1$, the share is either worth S_u or S_d euros
- The riskless asset has a rate of return of $1 + r$, $r > 0$: 1 euro today will yield $1 + r$ euros at time $t = 1$.

A two-period, two-state world



We will consider the problem of pricing a call option C on S.

The buyer of such an option pays a price C at time $t = 0$ which gives him the right, but not the obligation, to buy the underlying asset S at time $t = 1$ at a price K (the strike or exercise price) which is determined at time $t = 0$, at the settlement of the contract.

Hedging portfolio and absence of opportunities of arbitrage

A portfolio H is characterized by a pair (α, Δ) where α is a euro amount invested in the riskless asset and Δ is the number of shares which the investor holds.

If (α, Δ) defines the portfolio H at time $t = 0$, H is worth $\alpha + \Delta S$ euros. At time $t = 1$, this portfolio is worth

$$\alpha(1 + r) + \Delta S_u \text{ if the share has gone up,}$$

$$\alpha(1 + r) + \Delta S_d \text{ if the share has gone down.}$$

Portfolio is called a *hedging portfolio* if it replicates the option, i.e., if

$$\alpha(1+r) + \Delta S_u = C_u = \max(S_u - K, 0)$$
$$\alpha(1+r) + \Delta S_d = C_d = \max(S_d - K, 0).$$

Solving this linear system, one finds a pair (α^*, Δ^*) such that

$$\alpha^* = \frac{1}{1+r}\left(C_u - \frac{C_u - C_d}{S_u - S_d} S_u\right)$$
$$\Delta^* = \frac{C_u - C_d}{S_u - S_d}.$$

Let H^* be the portfolio corresponding to the "replicating" tuple (α^*, Δ^*). Then we have

$$
\begin{aligned}
H^* &= \alpha^* + \Delta^* S \\
&= \frac{1}{1+r}\left(C_u - \frac{C_u - C_d}{S_u - S_d} S_u\right) + \frac{C_u - C_d}{S_u - S_d} S \\
&= \frac{1}{1+r}\left[C_u - \frac{C_u S_u}{S_u - S_d} + \frac{C_u S(1+r)}{S_u - S_d} + \frac{C_d S_u}{S_u - S_d} - \frac{C_d S(1+r)}{S_u - S_d}\right] \\
&= \frac{1}{1+r}\left[C_u\left(\frac{S(1+r) - S_d}{S_u - S_d}\right) + C_d\left(\frac{S_u - S(1+r)}{S_u - S_d}\right)\right].
\end{aligned}
$$

Define

$$q = \frac{S(1+r) - S_d}{S_u - S_d}.$$

Then we have

$$1 - q = \frac{S_u - S(1+r)}{S_u - S_d}$$

from which we derive that

$$H^* = \alpha^* + \Delta^* S = \frac{1}{1+r}\left[qC_u + (1-q)C_d\right].$$

We are going to prove that the option price C is equal to the value of the hedging portfolio H^* at time $t = 0$, that is

$$C = H^*.$$

The proof relies on the absence of arbitrage hypothesis (A.O.A) (see Definition 3.1).

Theorem 4.1

If there are no opportunities of arbitrage, then we have

$$C = H^*.$$

Proof. Assume that the option price $C > H^*$, then there exists an opportunity of arbitrage. Indeed, we can construct an arbitrage portfolio as follows:

<u>At time $t = 0$</u>

1. Sell the option at price C.
2. Buy a portfolio (α^*, Δ^*) at price H^* and invest $C - H^*$ at rate r.

Because $C = \alpha^* \ +(C - H^*) + \Delta^* S$, the total initial investment is zero.

Cash flows at time $t = 1$, depending on the state of the world

- S_u
  1. $K - S_u$ (buyer exercises the option)
  2. $S_u - K + (C - H^*)(1 + r)$ (portfolio value + riskless investment)
- S_d
  1. 0 (buyer does not exercise the option)
  2. $(C - H^*)(1 + r) > 0$ (portfolio value + riskless investment)

In both states of the world, the total cash flow equals

$$(C - H^*)(1 + r) > 0.$$

Starting from a zero investment, we end up with a strictly positive portfolio value in all states of the world, which is an arbitrage opportunity. Since the same type of reasoning holds for $C < H^*$, we must have that $C = H^*$ under the A.O.A. hypothesis. $\square$

Remark 4.2

A.O.A. $\Longrightarrow S_d \leqslant (1+r)S \leqslant S_u$

Proof. If $(1+r)S < S_d$, at time $t = 0$, one can borrow S at rate r and buy a share at price S.

At time $t = 1$, one sells the share at price $S_1 \geqslant S_d$ and repays its loan for $(1+r)S$.

The agent earned $S_1 - (1+r)S > 0$ while investing nothing, which contradicts the A.O.A. A similar reasoning holds for $(1+r)S > S_u$, from which the remark follows. $\square$

Similarly, for a put option, we have

$$P = \tfrac{1}{1+r}\left(qP_u + (1-q)P_d\right)$$

where

$$\begin{aligned} P_u &= \max(0, K - S_u), \\ P_d &= \max(0, K - S_d), \\ q &= \frac{1}{S_u - S_d}\left((1+r)S - S_d\right). \end{aligned}$$

Risk-neutral probability

Note that the A.O.A. implies that

$$q = \frac{1}{S_u - S_d} \left[(1+r)S - S_d\right] \in [0, 1]. \tag{1}$$

Furthermore, this can also be written as

$$(1+r)S = qS_u + (1-q)\, S_d. \tag{2}$$

The left-hand side of (2) is the gain realized when investing S euros in the riskless asset.The right-hand side is the expected gain when buying a share at S euros, if the upper state of the world has a probability q and a lower state of the world has a probability $(1-q)$.

This means that we are in a **risk-neutral** model: the pricing probability q is such that the underlying asset S has an expected return equal to the riskless rate r.

Theorem 4.3

The fair price of an option is equal to the expectation of the discounted payoff with respect to the risk-neutral probability.

Proof. For a call option, the discounted payoffs in each possible state of the world are

$$\tfrac{1}{1+r} C_u \quad \text{and} \quad \tfrac{1}{1+r} C_d.$$

We have already shown that $q = P\left[S_1 = S_u\right]$ and $1 - q = P\left[S_1 = S_d\right]$ induces a *risk-neutral* probability distribution Q. So the expectation of the discounted payoff with respect to this risk-neutral probability can be written as

$$\mathrm{E}^Q\left[\tfrac{(S_1-K)_+}{1+r}\right] = \tfrac{1}{1+r}\left(qC_u + (1-q)C_d\right)$$

which is precisely the arbitrage-free price of a call option. $\square$

State prices

We can define the state price π as follows:

$$\pi = \frac{q}{1+r}.$$

We then have

$$C = \pi C_u + (1-\pi)\, C_d,$$

where π is called "state price" because it can be interpreted as the price to pay at time $t = 0$ to get 1 euro at time $t = 1$ in state u (just consider a derivative that yields 1 euro in state u and zero otherwise). Similarly, $1 - \pi$ is the state price associated with state d.

Put-call parity revisited

It is clear that

$$(S_1 - K)_+ - (K - S_1)_+ = S_1 - K.$$

Taking the expectation with respect to risk-neutral measure and discounting it follows that

$$C - P = \mathsf{E}^Q\left[\tfrac{S_1}{1+r}\right] - \tfrac{K}{1+r}$$

which, by definition of the risk-neutral probability (see (2)), yields the put-call parity

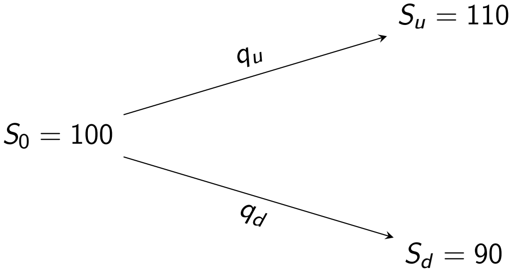
$$C - P = S - \frac{K}{1+r}.$$

Incomplete markets

In the above model, the market is complete because the risk-neutral probability measure is unique and hence the fair price is also unique. This is because we have two assets (a stock and a bond) and two states of the world, corresponding to S_u and S_d. The hedging argument then leads to a system of two equations with two unknowns.

Now, consider in a trinomial model, where, say, the possible states of the world correspond to S_u, S_m and S_d. In that set-up, there are infinitely many risk-neutral probability measures, and so many fair prices, because we only have two assets (a stock and a bond) but this time three states of the world - that is, a system of three equations with two unknowns. Perfect replication is impossible and only a fair interval of option prices can be determined - not a single price.

To illustrate this, let us consider a very simple set-up, where $r = 0$ and



We consider a call option on S whose payoff at time $t = 1$ is $C_1 = \max(S_1 - K, 0)$, with $K = 100$. In this set-up, we are looking for the price of the call at time $t = 0$, that is C_0.

First, we calculate the risk-neutral probability q. By definition, we know that

$$\mathsf{E}^{Q}[S_1] = S_0,$$

because the interest rate r is zero. Therefore, remembering that $q_u + q_d = 1$, we have

$$\begin{aligned} 100 &= 110q_u + 90q_d \\ &= 110q_u + 90(1 - q_u) \end{aligned}$$

from which we deduce that

$$q_u = q_d = \frac{1}{2}.$$

The price of the call at time $t = 0$ is given by the risk-neutral expectation of its (discounted) payoff at maturity, that is,

$$\begin{aligned} C_0 &= \mathsf{E}^{Q}[C_1] \\ &= \frac{1}{2}\max(110 - 100, 0) + \frac{1}{2}\max(90 - 100, 0) \\ &= 5. \end{aligned}$$

Now, suppose we have a trinomial model with a third possible state of the world S_m:

$$
\begin{tikzcd}
 & S_u = 110 \\
S_0 = 100 \arrow[ur, "q_u"] \arrow[r, "q_m"] \arrow[dr, "q_d"'] & S_m = 100 \\
 & S_d = 90
\end{tikzcd}
$$

By definition of the risk-neutral probability, we have $q_u + q_d + q_m = 1$ and

$$\begin{aligned}
100 &= q_u 110 + q_d 90 + q_m 100 \\
100 &= q_u 110 + q_d 90 + 100(1 - q_u - q_d) \\
0 &= 10q_u - 10q_d \\
q_u &= q_d = q, q_m = 1 - 2q.
\end{aligned}$$

From $q_m \geq 0$, we deduce that $q \in [0, \frac{1}{2}]$ and

$$\begin{aligned}
C_0 &= E^Q[C_1] \\
&= qC_u + qC_d + (1 - 2q)C_m \\
&= 10q
\end{aligned}$$

from which it follows that $C_0 \in [0, 5]$.

The three-world universe is an example of an incomplete market, that is, a market where portfolios cannot be arranged to give precisely the desired pay-off, and it is characteristic of incomplete markets that the price of an option can only be shown to **lie in an interval** rather than being forced to take a precise value.

The market price of such an option would then be determined within the range of possible prices by the risk-preferences of traders in the market rather than mathematics.

A two-period binomial model

The binomial option pricing model was introduced by Cox, Ross and Rubinstein in 1979 in their article "**Option Pricing : A simplified approach**".

The binomial model provides a simple and intuitive numerical method to value options.

In particular, the primary practical use of the binomial model is to price American-style options (on assets that pay dividends), but it can be extended to more complex options as well.

The binomial model is a particular case of a tree model where each state of the world is followed, in the next period, by two states of the world (the so-called upper and lower states of the world), each having its own probability of occurrence.

Continuing with the same notation as formerly, we denote by S_{ud} the price of the stock at time $t = 2$ after an increase of the stock price between $t = 0$ and $t = 1$ and a decrease between $t = 1$ and $t = 2$.

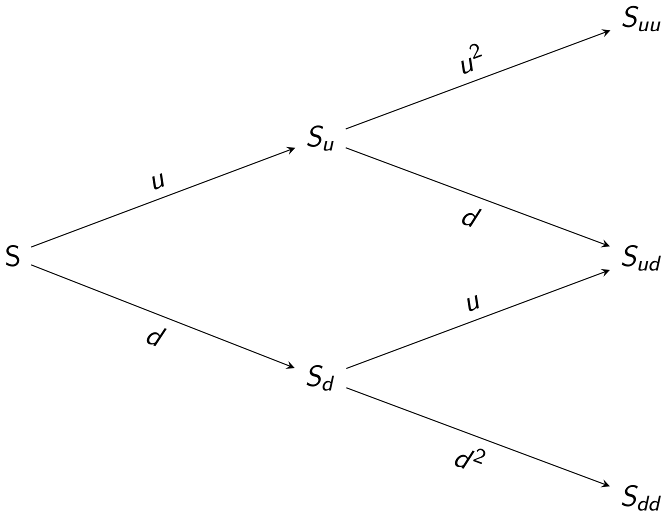
The tree is recombining in the sense that $S_{ud} = S_{du}$ or $S_{udd} = S_{dud}$, so that only the number of rises determines the stock price.

We first consider a two-period binomial model. We make the following assumptions:

- We only have two assets :
  - One riskless asset whose return r does neither depend on the state of the world nor on the time period;
  - One risky asset, say a share, whose price follows the preceding dynamics.
- We assume that

$$S_d \neq S_u; \quad S_{d^2} \neq S_{u^2} \neq S_{du} \ldots$$

- We also assume that S_d is the product of S by a return d: $S_d = dS$. Similarly, we have $S_u = uS$.
- We assume that the A.O.A. holds and, in particular, that $d < 1 + r < u$.


S_{uu}
u^2
S_u
u
d
S
S_{ud}
u
d
S_d
d^2
S_{dd}

Consider a call option C with strike K which matures at time $t = 2$.

If the underlying stock price at time $t = 2$ is $u^2 S$, the payoff of the call will be $C_{uu} = \max\left(0, u^2 S - K\right)$. We have similar formulas for C_{ud} and C_{dd}.

We calculate the price of the option at each point in time, going backwards from the maturity. For example, at time $t = 1$, we have

$$C_u = \frac{1}{1+r}\left\{qC_{uu} + (1-q)\, C_{ud}\right\},$$

with

$$q = \frac{1}{u-d}((1+r) - d).$$

Similarly,

$$C_d = \frac{1}{1+r}\left\{qC_{du} + (1-q)\,C_{d^2}\right\}.$$

The same reasoning applies between $t = 0$ and $t = 1$:

$$C = \tfrac{1}{1+r}\left\{qC_u + (1-q)\,C_d\right\}.$$

Substituting C_u and C_d, as well as C_{u^2}, C_{d^2} and C_{ud}, we get

$$C = \frac{1}{(1+r)^2}\{q^2(u^2S-K)_+ + 2q(1-q)(udS-K)_+ \\ + (1-q)^2(d^2S-K)_+\}.$$

Example

In the file "Two Period Binomial Tree.xls", we illustrate the above concepts by pricing a European call option with the following parameters

$$S = 100, K = 100, u = 1.75, d = 0.75, r = 0.25.$$

using different techniques: risk-neutral probabilities, state prices, hedging portfolio, etc.

American options

As we have seen in Chapter 3 (Arbitrage), we always have

$$C_0 > S_0 - K$$

for a European call option, meaning that it will have a positive time value if $r > 0$. Since American options cannot be worth less than equivalent European options it follows that American call options on non-dividend paying stock will not be exercised early because selling the option will always dominate exercising it. Intuitively an early exercise will involving paying the strike price earlier and this is undesirable.

However, American put options can be exercised early because this will involve receiving the strike price earlier. The two period binomial model can be used to illustrate this possibility.

Consider a put option in our example with a strike price $K = 100$. The value of this put option at the final nodes is 0, 0 and 43.75. Thus the value of the put option following an up movement in the first period is 0 as the option can never get back in the money.

However, following a down movement in the first period the value of the option using risk neutral valuation is 17.5. But early exercise of the option would give 25. Thus early exercise is the better alternative and the option must have a value of 25 if it is of the American type.

At the initial node the option is thus worth 10 if it is an American option and 7 if it is a European option that cannot be exercised early at the end of the first period.

See file "Two Period Binomial Tree.xls".

n-period binomial model

In the one-period binomial model there are two possible end values and in the two-period binomial model there are three possible end values. Extrapolating we have that in the n-period binomial model there are $n+1$ possible end values.

The number of ways that each value is reached is determined by Pascal's triangle:

Period 0						1					
Period 1					1		1				
Period 2				1		2		1			
Period 3			1		3		3		1		
Period 4		1		4		6		4		1	
Period 5	1		5		10		10		5		1

The numbers in Pascal's triangle are called the binomial coefficients and are defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $(n+1)$ is the row and $(k+1)$ is the column of the corresponding number in the triangle. So for example, if $n = 4$ and $k = 2$ then

$$\binom{n}{k} = \frac{4!}{2!(4-2)!} = 6,$$

which is indeed the number at row 5 and column 3 in the triangle. In this context, binomial coefficients show the number of ways returns can be generated by k up returns and $n-k$ down returns.

In our binomial model there are two outcomes for the stock price: "up" or "down". We can treat this as a random variable and associate with each event a probability.

Let p be the probability of a "up"-state and $1 - p =$ be the probability of a "down"-state. The probability of having k up states after n periods is given by

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k},$$

which is a binomial variable with parameter p. Thus for example, the probability of having 2 "up" states in 4 four periods is $6p^2(1-p)^2$.

We shall need to know the probability of having more than a certain number of "up"-states. Let $B_p(x)$ denote the probability that the binomial random variable with parameter p is greater than x after n periods. That is the probability that we have x or more than x "up"-states in n periods. This is given by

$$B(n, p, x) = \sum_{k=x}^{n} \binom{n}{k} p^k (1-p)^{n-k}.$$

So for example the probability that we have two or more up states in four periods is given by

$$\binom{4}{2} p^2(1-p)^2 + \binom{4}{3} p^3(1-p)^1 + \binom{4}{4} p^4(1-p)^0 =$$
$$6p^2(1-p)^2 + 4p^3(1-p)^1 + p^4.$$

Therefore, in a n-period binomial model, the expected value of a call option after n periods is given by the equation

$$C = \frac{1}{(1+r)^n} \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} \left(u^k d^{n-k} S - K \right)_+ \tag{3}$$

The right-hand side of (3) is the expectation of the discounted option payoff with respect to a binomial distribution with parameters n and q:

$$C = \tfrac{1}{(1+r)^n} \mathrm{E}^q \left[(S_n - K)_+ \right].$$

Indeed, $\ln\left(S_n/(Sd^n)\right)/\ln\left(u/d\right)$ is a random variable with binomial distribution with parameters n and q :

$$P\left[S_n = u^j d^{n-j} S\right] = P\left[\ln\left(S_n/(Sd^n)\right)/\ln\left(u/d\right) = j\right] = \binom{n}{j} q^j \left(1-q\right)^{n-j}.$$

We are going to rewrite (3) a little differently. Let

$$\eta = \inf \left\{ j \in \mathbb{N} \mid u^j d^{n-j} S - K > 0 \right\}.$$

And let $[\alpha]$ be the integer part of α, that is

$$[\alpha] \leq \alpha < [\alpha] + 1.$$

Because

$$
\begin{aligned}
& u^j d^{n-j} S - K > 0 \\
\iff & \frac{u^j}{d^j} > \frac{K}{Sd^n} \\
\iff & j > \frac{\ln\left(K/(Sd^n)\right)}{\ln(u/d)}
\end{aligned}
$$

Then

$$
\eta = \left[\frac{\ln(K/(Sd^n))}{\ln(u/d)}\right] + 1.
$$

η is the minimum number of up movements necessary during the n periods for the option to end up in-the-money.

This yields

$$C = \tfrac{1}{(1+r)^n} \sum_{k=\eta}^{n} \binom{n}{k} q^k (1-q)^{n-k} \left(u^k d^{n-k} S - K \right)$$

which we can rewrite as

$$C = S \sum_{k=\eta}^{n} \binom{n}{k} \left(\frac{qu}{1+r} \right)^k \left(\frac{d-dq}{1+r} \right)^{n-k} - \frac{K}{(1+r)^n} B(n, \eta, q) .$$

Because

$$
\begin{aligned}
&\frac{1+r-d}{u-d} = q, \\
&\Longleftrightarrow 1+r-d = qu - qd \\
&\Longleftrightarrow 1+r-qu = d - qd \\
&\Longleftrightarrow 1-\frac{qu}{1+r} = \frac{d-qd}{1+r},
\end{aligned}
$$

we have the following theorem:

Theorem 4.4 *The price of a European call option in the binomial model is given by*

$$
C = SB(n,\eta,\frac{qu}{1+r}) - \frac{K}{(1+r)^n} B\left(n,\eta,q\right).
$$

We illustrate the above formula with the previous example:

$$C = SB\left(n, \eta, \frac{qu}{1+r}\right) - \frac{K}{(1+r)^n} B(n, \eta, q)$$

$$\eta = \left[\frac{\ln(K/(Sd^n))}{\ln(u/d)}\right] + 1 = \left[\frac{\ln\left(100/\left(100(0.75)^2\right)\right)}{\ln(1.75/0.75)}\right] + 1 = 1$$

$$\frac{qu}{1+r} = 0.5\frac{1.75}{1.25} = 0.7$$

$$B(2,1,0.7) = \binom{2}{1} 0.7x0.3 + \binom{2}{2} 0.7^2 = 0.91$$

$$B(2,1,0.5) = \binom{2}{1} 0.5x0.5 + \binom{2}{2} 0.5^2 = 0.75$$

$$C = 100B(2,1,0.7) - \frac{100}{(1.25)^2} B(2,1,0.5) = 43.$$

Delta

At this stage, it is appropriate to introduce delta, an important parameter (sometimes referred to as a "Greek letter" or simply a "Greek") in the pricing and hedging of options. The delta (Δ) of a stock option is the ratio of the change in the price of the stock option to the change in the price of the underlying stock:

$$\Delta = \frac{C_u - C_d}{S_u - S_d}.$$

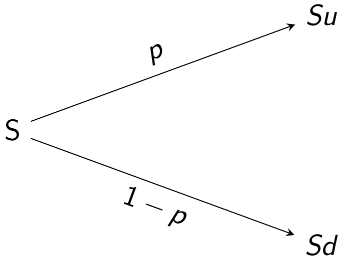
It is the number of units of the stock we should hold for each option shorted in order to create a riskless portfolio. It is the same as the Δ introduced earlier in this chapter.

The construction of a riskless portfolio is sometimes referred to as delta hedging. The delta of a call option is positive, whereas the delta of a put option is negative.

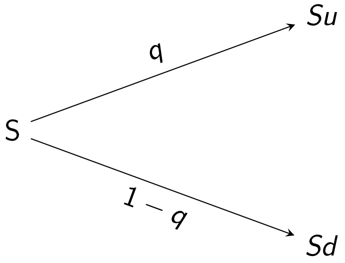
The two-step examples show that delta changes over time. Thus, in order to maintain a riskless hedge using an option and the underlying stock, we need to adjust our holdings in the stock periodically.

Matching volatility with u and d

In practice, when constructing a binomial tree to represent the movements in a stock price, we choose the parameters u and d to match the volatility of the stock price. A question that arises is whether we should match volatility in the real world or the risk-neutral world. As we will now show, this does not matter. For small Δt and particular values of u and d, the volatility being assumed is the same in both the real world and the risk-neutral world. The figure below show the stock price movements over one step of a binomial tree in the real world.



The figure below shows the stock price movements over one step of a binomial tree in the risk-neutral world.



We assume that the time step is of length Δt.

The expected stock price at the end of the first time step in the real world is $Se^{\mu \Delta t}$, where μ is the expected return. On the tree the expected stock price at this time is

$$pSu + (1-p)Sd.$$

In order to match the expected return on the stock with the tree's parameters, we must therefore have

$$pSu + (1-p)Sd = Se^{\mu \Delta t}$$

or

$$p = \frac{e^{\mu \Delta t} - d}{u - d}. \tag{4}$$

As we will explain later, the volatility σ of a stock price is defined so that $\sigma\sqrt{\Delta t}$ is the standard deviation of the return on the stock price in a short period of time of length Δt. Equivalently, the variance of the return is $\sigma^2 \Delta t$.

Recalling that the variance of a random variable X is equal to $\mathrm{E}[X^2] - \mathrm{E}[X]^2$, the variance of the stock price return on the tree is

$$pu^2 + (1-p)d^2 - [pu + (1-p)d]^2.$$

(Strictly speaking, the returns are $u - 1$ and $d - 1$ but subtracting 1 from a random variable has no impact on the variance)

In order to match the stock price volatility with the tree's parameters, we must therefore have

$$pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 = \sigma^2 \Delta t. \tag{5}$$

Substituting (4) in (5) gives

$$e^{\mu \Delta t}(u + d) - ud - e^{2\mu \Delta t} = \sigma^2 \Delta t.$$

Using the series expansion $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ and ignoring the terms in Δt^2 and higher powers of Δt, one solution to this equation is

$$u = e^{\sigma \sqrt{\Delta t}}$$

$$d = e^{-\sigma \sqrt{\Delta t}}.$$

These are the values of u and d proposed by Cox, Ross, and Rubinstein (1979) for matching volatility.

Now, considering the risk-neutral probabilities, the expected stock price at the end of the time step is $Se^{r\Delta t}$. The variance of the stock price return in the risk-neutral world is

$$qu^2 + (1-q)d^2 - [qu + (1-q)d]^2 = e^{r\Delta t}(u+d) - ud - e^{2r\Delta t}.$$

Substituting $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$, we find this equals $\sigma^2\Delta t$ when terms in Δt^2 and higher powers of Δt are ignored.

This analysis shows that when we move from the real world to the risk-neutral world the expected return on the stock changes, but its volatility remains the same (at least in the limit as Δt tends to zero). This is an illustration of an important general result known as *Girsanov's theorem*.

When we move from a world with one set of risk preferences to a world with another set of risk preferences, the expected growth rates in variables change, but their volatilities remain the same. Moving from one set of risk preferences to another is sometimes referred to as changing the measure.

The real-world measure is sometimes referred to as the P-measure, while the risk-neutral world measure is referred to as the Q-measure. In that sense, p is the probability under the P-measure and q is the probability under the Q-measure.

Binomial tree formulas

When the length of the time step on a binomial tree is Δt, we should match volatility by setting

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\Delta t}} \\
d &= e^{-\sigma\sqrt{\Delta t}} \\
q &= \frac{e^{r\Delta t} - d}{u - d}.
\end{aligned}
$$

Convergence to the Black-Scholes formula

When binomial trees are used in practice, the life of the option is typically divided into 30 or more time steps. In each time step there is a binomial stock price movement. With 30 time steps there are 31 terminal stock prices and 2^{30}, or about 1 billion, possible stock price paths are implicitly considered.

As the number of time steps is increased (so that Δt becomes smaller), the binomial tree model makes the same assumptions about stock price behavior as the Black– Scholes model, which will be presented in a later chapter. When the binomial tree is used to price a European option, the price converges to the Black–Scholes price, as expected, as the number of time steps is increased. This is stated in the next theorem.

Theorem 4.5 *The price of a European call option with strike K and maturity T in a binomial tree model with Cox, Ross and Rubinstein parametrization converges to the Black-Scholes price as the number of time steps n approaches infinity, that is,*

$$e^{-rT} \sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} \max(Su^j d^{n-j} - K, 0)$$

converges to the Black-Scholes price

$$\begin{aligned} c &= SN(d_1) - Ke^{-rT} N(d_2), \\ d_1 &= \frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

as $n \to \infty$, with N being the cumulative normal distribution function.