10. Backward Stochastic Differential Equations

Introduction

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- Backward Stochastic differential equations (BSDEs) are mathematically advanced, but these equations nonetheless epitomize very elegantly the fundamental mathematical backbone of derivative pricing theory
- Deep learning algorithms for solving BSDEs have made these a tool of choice for solving high dimensional option pricing problems efficiently
- Contrary to ordinary differential equations (ODEs), stochastic differential equations with a **terminal** condition (BSDEs) are fundamentally different stochastic differential equation with an **initial** condition ((forward) SDEs). They are the subject of a separate mathematical theory.

1. Backward stochastic differential equations (BSDEs)

A backward stochastic differential equation is as follows:

$$\begin{cases} -dY_t = f\left(t, Y_t, Z_t\right) dt - Z_t dW_t, \quad t \in [0, T] \\ Y_T = \xi \end{cases}$$

where $(Y, Z)_{t\in[0,T]}$ takes values in $\mathbb{R}^m \times \mathbb{R}^{m\times d}$, $f(\omega, t, y, z)$ is the generator (or driver) and the $\mathcal{F}_T$-measurable random variable $\xi = Y_T$ is the terminal condition of the BSDE.

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In integral form, we have

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s\right) ds - \int_t^T Z_s dW_s.$$

Existence and uniqueness

Assume that $f \equiv 0$, a solution (Y, Z) of the preceding BSDE satisfies:

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Taking conditional expectations with respect to the filtration at time t on both sides, we have that

$$Y_t = \mathbb{E}[Y_T|\mathcal{F}_t] = \mathbb{E}[\xi|\mathcal{F}_t]$$

By the properties of conditional expectations, Y is a martingale.

By the martingale representation theorem, there exists a unique process Z such that

$$\begin{aligned} Y_t &= \mathbb{E}[\xi|\mathcal{F}_t] \\ &= Y_0 + \int_0^t Z_s dW_s \\ &= \mathbb{E}[\xi|\mathcal{F}_0] + \int_0^t Z_s dW_s \\ &= \mathbb{E}[\xi] + \int_0^t Z_s dW_s. \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
\xi &= Y_T \\
&= Y_0 + \int_0^T Z_s dW_s \\
&= Y_0 + \int_0^t Z_s dW_s + \int_t^T Z_s dW_s \\
&= Y_t + \int_t^T Z_s dW_s,
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For this simple case, we see that an adapted solution to the BSDE can only be given by a pair, that is, the Z component is needed to ensure that the process Y is adapted. In a sense, the Z component "steers" the system and is thus called the control process. One cannot simply revert time as in deterministic ODEs, as the filtration can only go in one direction.

BSDEs and conditional expectations

The following result formalizes the connection between BSDEs and conditional expectations. Consider the following *linear* BSDE:

$$\begin{cases} -dY_t = [\phi_t + Y_t\beta_t + Z_t\gamma_t]dt - Z_t dW_t, \quad t \in [0, T] \\ Y_T = \xi. \end{cases}$$

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The above LBSDE has a unique solution (Y, Z), and Y is given by the closed formula

$$Y_t = \mathbb{E}\left[\xi\Gamma_T^t + \int_t^T \Gamma_s^t \varphi_s ds \mid \mathcal{F}_t\right]$$

where the process Γ_s^t is the solution of the forward linear SDE

$$\begin{cases} d\Gamma_s^t = \Gamma_s^t[\beta_s ds + \gamma_s dW_s] \quad s \in [t, T] \\ \Gamma_t^t = 1. \end{cases}$$

Option pricing

Take the typical setup for continuous-time asset pricing and assume the following dynamics for a riskless asset (money market instrument or bond)

$$dS_t^0 = S_t^0 r_t dt$$

and a risky asset

$$dS_t = S_t[\mu_t dt + \sigma_t dW_t],$$

where W is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{P}$ is said to be the "objective" probability measure.

Option pricing

Assume there exists a bounded-valued process θ, called *risk premium* (or *market price of risk*) such that

$$\frac{\mu_t - r_t}{\sigma_t} = \theta_t.$$

Under these assumptions, the market is complete.

Option pricing

Let us define a wealth process $V = \pi_t^0 + \pi_t$ representing the sum of the amounts invested in the riskless and risky asset, respectively. V is said to be self-financing if it satisfies the following linear SDE

$$\begin{aligned} dV_t &= r_t V_t dt + \pi_t(\mu_t - r_t)dt + \pi_t \sigma_t dW_t. \\ &= r_t V_t dt + \pi_t \sigma_t [\theta_t dt + dW_t]. \end{aligned}$$

Option pricing

We say that X is a replicating portfolio for the $\mathcal{F}_T$-measurable contingent claim ξ if X is a self-financing strategy such that

$$\left\{ \begin{array}{l} dX_t = [r_t X_t + \pi_t \sigma_t \theta_t] dt + \pi_t \sigma_t dW_t, \quad t \in [0, T] \\ X_T = \xi. \end{array} \right.$$

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Letting $Z = \pi_t \sigma_t$, we get

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This is a linear BSDE, the solution of which is thus

$$X_t = \mathbb{E}\left[\xi \Gamma_T^t \mid \mathcal{F}_t\right]$$

where Γ_s^t satisfies

$$\begin{cases} d\Gamma_s^t = \Gamma_s^t [r_s ds + \theta_s dW_s] \quad s \in [t, T] \\ \Gamma_t^t = 1. \end{cases}$$

Option pricing

Using Itô's formula, we have that

$$\Gamma_t = \exp - \left[\int_0^t r_s ds + \int_0^t \theta_t dW_s + \frac{1}{2} \int_0^t \theta_s^2 ds \right].$$

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Let $\mathbb{Q}$ be a probability measure with Radon-Nikodym derivative with respect to $\mathbb{P}$ given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp - \left[\int_0^t \theta_t dW_s + \frac{1}{2} \int_0^t \theta_s^2 ds \right],$$

then we get

$$X_t = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_s ds} \xi | \mathcal{F}_t].$$

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then we get

$$X_t = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_s ds} \xi | \mathcal{F}_t].$$

This is the well-known result that the fair price of a contingent claim ξ is the expectation of the discounted value of its payoff under the risk-neutral probability measure $\mathbb{Q}$.

Option pricing

$\mathbb{Q}$ is called a risk-neutral probability measure because under $\mathbb{Q}$, Girsanov theorem says that

$$dW_t^{\mathbb{Q}} = dW_t + \theta_t dt$$

is a Brownian motion and X has risk-neutral dynamics

$$\begin{aligned} dX_t &= [r_t X_t + Z_t \theta_t] dt + Z_t dW_t \\ &= [r_t X_t + Z_t \theta_t] dt + Z_t [dW_t^{\mathbb{Q}} - \theta_t dt] \\ &= r_t X_t dt + Z_t dW_t^{\mathbb{Q}}. \end{aligned}$$

The expected rate of return of X is indeed the riskless rate r.

Non-linear expectations and risk measures

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- Extending this idea to non-linear BSDE leads to the concept of non-linear expectation
- Dynamic risk measures also relate closely to BSDEs

Forward backward stochastic differential equations

Consider the following BSDE

$$\begin{cases} -dY_t = f\left(t, X_t, Y_t, Z_t\right) dt - Z_t dW_t, \quad t \in [0, T] \\ Y_T = \Phi(X_t), \end{cases}$$

where the pair of processes (Y, Z) is the solution of the BSDE and X is the solution of a SDE

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t \in [0, T] \\ X_0 = x. \end{cases}$$

Forward backward stochastic differential equations

Consider the following BSDE

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$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t \in [0, T] \\ X_0 = x. \end{cases}$$

Note how the generator has changed from $f(\omega, t, y, z)$ to $f(t, x, y, z)$. This pair of equations is called a Markovian BSDE, or (uncoupled) FBSDE.

PDE Equivalence

Under appropriate assumptions, there exists a unique solution (X, Y, Z) to the above FBSDE. Define u as

$$u(t, \mathbf{x}) := Y_t^{t,\mathbf{x}} = \mathbb{E}\left[\Phi\left(X_T^{t,\mathbf{x}}\right) + \int_t^T f\left(s, X_s^{t,\mathbf{x}}, Y_s^{t,\mathbf{x}}, Z_s^{t,\mathbf{x}}\right) ds\right]$$

where $(X^{(t,x)}, Y^{(t,x)}, Z^{(t,x)})$ denotes the adapted solution to the FBSDE, restricted to $[t, T]$ with $X_t^{(t,x)} = x$.

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where $(X^{(t,x)}, Y^{(t,x)}, Z^{(t,x)})$ denotes the adapted solution to the FBSDE, restricted to $[t, T]$ with $X_t^{(t,x)} = x$. Then u is a solution of the parabolic PDE

$$\begin{cases} ((\partial/\partial t) + \mathcal{L}_t)\, u(t, \mathbf{x}) + f\left(t, \mathbf{x}, u(t, \mathbf{x}), (\nabla u(t, \mathbf{x}))^\top \sigma(t, \mathbf{x})\right) = 0 \\ u(T, \mathbf{x}) = \Phi(\mathbf{x}), \end{cases}$$

PDE Equivalence

where $\mathcal{L}$ denotes the second-order differential operator

$$\mathcal{L}_t u(t,x) = \frac{1}{2}\,\mathsf{Tr}\left(\sigma\sigma^{\mathrm{T}}(t,x)\nabla^2 u(t,x)\right) + \nabla u(t,x)\cdot\mu(t,x).$$

From this, we can deduce that

$$Y_t = u(t,X_t), \quad Z_t = \nabla u(t,\mathbf{x}))^\top \sigma(t,\mathbf{x}).$$

This equivalence is the well-known non-linear Feynman-Kac formula.

Numerical approximations of BSDEs

Let the time step $\Delta_i^N = t_i - t_{i-1}$ and $\Delta W_i^N = W_{t_i} - W_{t_{i-1}}$ for $i = 1, \ldots, N$. The discrete time approximation of the forward process X using the Euler scheme is given by

$$\begin{cases} X_{t_0}^N = X_{t_0} \\ X_{t_i}^N = X_{t_{i-1}}^N + b\left(t_{i-1}, X_{t_{i-1}}^N\right)\Delta_i^N + \sigma\left(X_{t_{i-1}}^N\right)\Delta W_i^N. \end{cases}$$

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Consider now a BSDE on a one step interval $[t_i, t_{i+1}]$

$$\begin{aligned} Y_{t_i} &= Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f\left(s, X_s, Y_s, Z_s\right) ds - \int_{t_i}^{t_{i+1}} Z_S dW_s \\ &\approx Y_{t_{i+1}}^N + \Delta_i^N f\left(t_i, X_{t_i}^N, Y_{t_i}^N, Z_{t_i}^N\right) - \Delta W_i^N Z_{t_i}, \end{aligned}$$

where the righ-hand side of the last equation denotes the Euler discrete-time approximation $Y_{t_i}^N$.

Numerical approximations of BSDEs

Taking conditional expectations of both sides gives

$$Y_{t_i}^N = \mathbb{E}\left[Y_{t_{i+1}}^N + \Delta_i^N f\left(t_i, X_{t_i}^N, Y_{t_i}^N, Z_{t_i}^N\right) \Big| \mathcal{F}_{t_i}\right],$$

because $Y_{t_i}^N$ is $\mathcal{F}_{t_i}$-measurable. The explicit version of the approximation is given by

$$Y_{t_i}^N = \mathbb{E}\left[Y_{t_{i+1}}^N + \Delta_i^N f\left(t_i, X_{t_i}^N, Y_{t_{i+1}}^N, Z_{t_i}^N\right) \Big| \mathcal{F}_{t_i}\right].$$

Numerical approximations of BSDEs

The discrete-time approximation for the Z component of the BSDE solution is given by multiplying both sides by ΔW_i^N and taking conditional expectations, denoting $\mathbb{E}_{t_i}[\cdot] = \mathbb{E}\left[\cdot \mid \mathcal{F}_{t_i}\right]$

$$\begin{aligned}
\mathbb{E}_{t_i}\left[\Delta W_i^N Y_{t_i}^N\right] &= \mathbb{E}_{t_i}\left[\Delta W_i^N Y_{t_{i+1}}\right] \\
&+ \mathbb{E}_{t_i}\left[\Delta W_i^N \int_{t_i}^{t_{i+1}} f\left(s, X_s, Y_s, Z_s\right) ds\right] \\
&- \mathbb{E}_{t_i}\left[\Delta W_i^N \int_{t_i}^{t_{i+1}} Z_S dW_s\right] \\
&= \mathbb{E}_{t_i}\left[\Delta W_i^N Y_{t_{i+1}}\right] - \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} dW_s \int_{t_i}^{t_{i+1}} Z_S dW_s\right] \\
&= \mathbb{E}_{t_i}\left[\Delta W_i^N Y_{t_{i+1}}\right] - \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z_S ds\right].
\end{aligned}$$

Numerical approximations of BSDEs

Because the left-hand side is equal to zero (a Brownian motion normally distributed increments, i.e. with expectation equal to zero), it follows that

$$Z_{t_i}^N = \frac{1}{\Delta_i^N}\mathbb{E}\left[\Delta W_i^N Y_{t_{i+1}} | \mathcal{F}_{t_i}\right].$$

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Putting everything together, we get

$$\begin{cases} Y_{t_N}^N = \Phi\left(X_{t_N}^N\right), \\ Z_{t_i}^N = \frac{1}{\Delta_i^N} \mathbb{E}\left[\Delta W_i^N Y_{t_{i+1}}^N \mid \mathcal{F}_{t_i}\right], \\ Y_{t_i}^N = \mathbb{E}\left[Y_{t_{i+1}}^N + \Delta_i^N f\left(t_i, X_{t_i}^N, Y_{t_{i+1}}^N, Z_{t_i}^N\right) \mid \mathcal{F}_{t_i}\right]. \end{cases}$$

Least-squares regression based method

This approach could also be called Longstaff-Schwarz, referring to the seminal paper using least-squares to estimate the conditional expected payoff, and compute the continuation value, of an American option at eacht time step of the Monte Carlo simulation.

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For each time point t_k, the idea is to choose a $\mathbb{R}^K$-valued deterministic base e, given e.g. by a sequence of appropriate polynomials of size K. The conditional expectation is approximated by choosing the coefficients α which minimize the mean, over all simulated paths M, of the distance between the discretized quantity to approximate evaluated at time t_{k+1} and $\alpha \cdot e$.

Least-squares regression based method

Algorithm 2

Set $y_n^{n,M,K}(\cdot) = \Phi$

for $k = (n-1)$ to 1 **do**

$$\alpha_{i,k}^{M,K} = \underset{\alpha}{\operatorname{argmin}} \frac{1}{M} \sum_{m=1}^{M} \left| y_{k+1}^{n,M,K}(X_{k+1}^{\pi,m}) \frac{\Delta W_k^{i,m}}{\Delta_k} - \alpha \cdot e_{i,k}^{K}(X_k^{\pi,m}) \right|^2, \quad i \in \{1, \cdots, d\}.$$

Put $z_k^{n,M,K} = (z_{1,k}^{n,M,K}, \cdots, z_{d,k}^{n,M,K})$ where $z_{i,k}^{n,M,K}(\cdot) = \alpha_{i,k}^{M,K} \cdot e_{i,k}^{K}(\cdot)$.

$$\alpha_{0,k}^{M,K} = \underset{\alpha}{\operatorname{argmin}} \frac{1}{M} \sum_{m=1}^{M} \left| y_{k+1}^{n,M,K}(X_{k+1}^{\pi,m}) + \Delta_k f(t_k, X_k^{\pi,m}, y_{k+1}^{n,M,K}(X_{k+1}^{\pi,m}), z_k^{n,M,K}(X_k^{\pi,m})) - \alpha \cdot e_{0,k}^{K}(X_k^{\pi,m}) \right|^2.$$

Put $y_k^{n,M,K}(\cdot) = \alpha_k^{M,K} \cdot e_{0,k}^{K}(\cdot)$.

end

Return $Y_0^{\pi,M,K} = \frac{1}{M} \sum_{m=1}^{M} (y_1^{n,M,K}(X_1^{\pi,m}) + \Delta_1 f(t_0, \mathbf{x}, y_1^{n,M,K}(X_1^{\pi,m}), z_1^{n,M,K}(X_1^{\pi,m})))$.

Figure 1: BSDE least-squares algorithm

These algorithms are well studied, and for large K, M, they converge to the solution of the considered BSDE.

Deep BSDEs

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Given that (deep) neural networks are universal approximators, one could think of them as another technique to approximate the conditional expectations in the least-squares regression approach.

Therefore, next to the Euler discretization of X above, we first define the forward discretization of Y, because we need to be able to compute Y_T based on an initial value y

$$\begin{aligned}\bar{Y}^n_{t_{k+1}} = \bar{Y}^n_{t_k} - f\left(t_k, \bar{X}^n_{t_k}, \bar{Y}^n_{t_k}, z_k\left(\bar{X}^n_{t_k}\right)\right)(t_{k+1} - t_k) \\ + z_k\left(\bar{X}^n_{t_k}\right)\left(W_{t_{k+1}} - W_{t_k}\right), \quad \mathrm{k} \in \{0, 1, \cdots, n-1\},\end{aligned}$$

with initial state $Y^n_{t_0} = y$.

Deep BSDEs

The minimization problem we consider is then

$$
\begin{aligned}
&\inf_{y,Z} \mathbb{E}\left[\left|\Phi\left(X_T\right) - Y_T^{y,Z}\right|^2\right] \\
&\approx \inf_{y,\{z_k\}_k} \mathbb{E}\left[\left|\Phi\left(\bar{X}_T^n\right) - \bar{Y}_T^n\right|^2\right] \\
&\approx \inf_{y,\{z_k\}_k} \frac{1}{M}\sum_{l=1}^{M}\left|\Phi\left(\bar{X}_{T,l}^n\right) - \bar{Y}_{T,l}^n\right|^2,
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for sufficiently lare M.

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\end{aligned}
$$

for sufficiently lare M. Deep learning comes into play when, at each time step $k \in \{1, \cdots, n\}$, one approximate the target continuous function z_k by a neural network parametrized by θ_k (for a suitable dimension depending on the number of layers).

Deep BSDEs

Finally, we seek the parameter Θ (which contains the parameters θ_k for all time steps k via the following minimization

$$\inf_{\Theta} \frac{1}{M} \sum_{l=1}^{M} \left| \Phi \left(\bar{X}_{T,l}^{n} \right) - \bar{Y}_{T,l}^{n}(\Theta) \right|^2$$

using, for example, stochastic gradient descent recursively for a suitable learning rate.

Deep BSDEs

- Remember that $Y_t = u(t, X_t), Z_t = \nabla u(t, \mathbf{x}))^\top \sigma(t, \mathbf{x})$, so we actually minimize over the gradient of the solution as the *policy function*

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- This gradient has a particular significance in finance, as it gives the **hedge** of the derivative
- The policy function Z maybe have a high dimension and still yield reasonable computation times

Deep BSDEs

- Remember that $Y_t = u(t, X_t), Z_t = \nabla u(t, \mathbf{x}))^\top \sigma(t, \mathbf{x})$, so we actually minimize over the gradient of the solution as the *policy function*
- This gradient has a particular significance in finance, as it gives the **hedge** of the derivative
- The policy function Z maybe have a high dimension and still yield reasonable computation times
- Other architectures are able to give the price and the hedge at all points in time;