

Multigrid for solving complex-valued Helmholtz problems

Isidoor Pinillo Esquivel

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1 Failure of the Multigrid method for Helmholtz problems: analysis

1.1 Discretization

(a)

$$\begin{aligned} 10 \leq \lambda \text{ \#gridpoints} &\Leftrightarrow \\ 10 \leq \frac{2\pi}{\sqrt{|\sigma|}} \frac{1}{h^d} &\Leftrightarrow \\ \sqrt{|\sigma|} h^d &\leq \frac{2\pi}{10} \approx 0.625. \end{aligned}$$

(b)

$$\text{\# roosterpunten} = \frac{10\sqrt{600}}{2\pi}.$$

1.2 1D model problem

(a)

To proof : $H^{2h} \neq I_h^{2h} H^h I_{2h}^h$.

$$\begin{aligned} H^{2h} &= H_n = A_n + \sigma id_n \\ H^h &= H_{2n} = A_{2n} + \sigma id_{2n} \end{aligned}$$

Assume that $A_n = I_h^{2h} A_{2n} I_{2h}^h = R_{2n} A_{2n} I_n$. By linearity it is sufficient to proof:

$$\begin{aligned} \sigma id_n &\neq \sigma R_{2n} id_{2n} I_n \Leftrightarrow \\ id_n &\neq R_{2n} I_n \Leftarrow \\ (id_n)_{00} &= 1 \neq \frac{3}{4} = (R_{2n} I_n)_{00} \end{aligned}$$

First equivalence follows from $\sigma \neq 0$. The assumption and the last inequality depends on the definition of restriction and interpolation.

(b)

See code/main.ipynb for code and plots. We implemented $f(t) = \delta(t - 0.5)$ by concentrating all the mass into the middle element of f_n .

(c)

There exists a closed formula for eigenvalues and eigenvectors of tridiagonal toeplitz matrix. It is just tedious to use. Alternatively the eigenvalues and eigenvectors can be derived from the Poisson problem ($\sigma = 0$) because

$$\begin{aligned} Av &= \lambda v \Rightarrow \\ (A + \sigma id)v &= Av + \sigma v \\ &= (\lambda + \sigma)v \end{aligned}$$

i.e. eigenvectors stay the same and eigenvalues get shifted by σ .

(d)

See code/main.ipynb for the plot. $\sigma = 0$ is the boundary where H goes from indefinite to definite.

1.3 LFA analysis of the ω -Jacobi smoother

(a)

For grid points without a neighboring boundary point there H acts like following stencil:

$$H_n = n^2 \begin{bmatrix} -1 & 2 + \frac{\sigma}{n^2} & -1 \end{bmatrix}.$$

So R_ω works element wise the following way on the error:

$$e_j^{m+1} = (1 - \omega)e_j^m + \frac{\omega n^2}{2n^2 + \sigma}(e_{j-1}^m + e_{j+1}^m). \quad (1)$$

Very similar to the analysis for the Poisson equation. Note that we haven't used that σ is real. Doing the LFA substitution $e_j^{(m)} = \mathcal{A}(m)e^{ij\theta}$:

$$\begin{aligned} A(m+1) &= A(m) \left(1 - \omega + \frac{\omega n^2}{2n^2 + \sigma}(e^{-i\theta} + e^{i\theta}) \right) \\ &= A(m) \left(1 - \omega + 2 \cos(\theta) \frac{\omega n^2}{2n^2 + \sigma} \right) \end{aligned}$$

The factor in behind of $A(m)$ is the amplification factor $G(\theta)$.

(b)

$\sigma = 0$ reduces back to the LFA we did for the Poisson equation. $\theta \approx 0 \Rightarrow \cos(\theta) \approx 1 + O\left(\frac{1}{n^2}\right) \Rightarrow G(\theta) \approx 1 - O\left(\frac{1}{n^2}\right)$ so smooth modes are preserved for big n .

(c) See code/main.ipynb for the plot.

(d)

Maximum of $G(\theta)$ is achieved at $\theta = 0$ because $G(\theta)$ is just an increasing function of $\cos(\theta)$. This means that $\rho = 1 - \omega + 2 \frac{\omega n^2}{2n^2 + \sigma} \approx 1.05$ which suggests that weighted jacobi wouldn't converge.

1.4 Spectral analysis of the two-grid correction scheme

(a)

It is easily seen that

$$R_{2n} = cS_{2n}(3id - A_{2n}). \quad (2)$$

with $c \in \mathbb{R}_0$, $S_{2n} : \mathbb{R}^{2n-1} \rightarrow \mathbb{R}^{n-1} : (v_j)_{j \leq 2n-2} \rightarrow (v_{2j+1})_{j \leq n-2}$ subsampling uneven components. Using that $S_{2n}w_k^{2n} = w_k^n$ if $k < n$ it is easily seen that:

$$\begin{aligned} R_{2n}w_k^{2n} &= cS_{2n}(3id - A_{2n})w_k^{2n} \\ &= c(3 - \lambda_k(A_{2n}))S_{2n}w_k^{2n} \\ &= a(n, k)w_k^n. \end{aligned}$$

In our case I_n is linear interpolation, reconstruction error for lagrange interpolation can be bounded using Taylors theorem.

$$I_n S_{2n} v_{2n} \approx c v_{2n}. \quad (3)$$

For w_k^{2n} smooth and combining with previous argument we have:

$$I_n R_{2n} w_{2n} \approx c_1 w_{2n}. \quad (4)$$

To check the normalizing constant try $v_{2n} = 1 \Rightarrow c_1 = 1$.

We also checked these facts numerically by plotting see code. Now doing spectral analysis of TG is straight forward:

$$\begin{aligned} TGw_k^{2n} &= (id - I_n H_n^{-1} R_{2n} H_{2n}) w_k^{2n} \\ &= w_k^{2n} - I_n H_n^{-1} R_{2n} \lambda_k(H_{2n}) w_k^{2n} \\ &= w_k^{2n} - I_n H_n^{-1} a(n, k) w_k^n \lambda_k(H_{2n}) \\ &= w_k^{2n} - I_n a(n, k) w_k^n \lambda_k(H_n^{-1}) \lambda_k(H_{2n}) \\ &= w_k^{2n} - I_n R_{2n} w_k^{2n} \lambda_k(H_n^{-1}) \lambda_k(H_{2n}) \\ &\approx w_k^{2n} - w_k^{2n} \lambda_k(H_n^{-1}) \lambda_k(H_{2n}) \\ &\approx w_k^{2n} (1 - \lambda_k(H_n^{-1}) \lambda_k(H_{2n})) \end{aligned}$$

(b)

We already analytically derived the eigenvalues for H_n . For the plot see code.

TG iterations may amplify smooth modes when $\rho_k > 1$.

(c)

See code for the plots. $\rho_k > 1$ when the sign changes. In the previous case the index closest to the sign change is $k = 6$. No, the smoother leaves smooth modes almost unchanged.

2 Solving the complex-valued Helmholtz problem using Multigrid

2.1 1D model problem

(a)

see code

(b)

For a point source it may not be obvious but the solutions for complex shifted problem is very similar.

(c)

see code

2.2 LFA analysis of the ω -Jacobi smoother

(a)

Already answered in previous question. ρ is still $|G(\theta)|$ almost the same reasoning.

$$|G(\theta)| = |a + b \cos(\theta) + c \cos(\theta)i| \quad (5)$$

with $a, b \in \mathbb{R}^+$ and $c \in \mathbb{R}$ still gets optimized when $\cos(\theta)$ gets optimized. Using that argument requires that $2n^2 + R(\sigma) \geq 0$ which follows from the criterium on n we placed at the start.

(b)

Depending on β the smoother may be stable.

(c)

We think $|G(\pi)|$ or $|G(\frac{\pi}{2})|$. We have numerical evidence not a proof yet. (d)

Eyeballing the plot we made $\omega \approx 0.65$ is good.

2.3 Spectral analysis of the two-grid correction scheme

(a)

Already did that. See code. Well with the formula not numerical eigenvalues ...

(b)

The instability in ρ_k dampens.

2.4 2D model problem

(a)

Solve happens in the code. (b)

Less iterations are needed for the same amount of error. This is because σ the source of the problem get relatively smaller to n^2 in our convergence factors, asymptotically behavior should go back the Laplace problem.

2.5 Aggressive coarsening

(a)

Not sure here is our guess:

$$FG = id - I_{2n}I_nH_n^{-1}R_{2n}R_{4n}H_{4n}. \quad (6)$$

Not sure what is meant by the eigenmode analysis but what we previously did does generalizes. The eigenmodes that are complementary with w_k^{4n} on the n grid are: $w_{4n-k}^{4n}, w_{2n-k}^{4n}, w_{2n+k}^{4n}$.

3 Multigrid as a preconditioner for Krylov subspace methods

3.1 MG-GMRES for the indefinite Helmholtz problem