

Multigrid for solving complex-valued Helmholtz problems

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1 Failure of the Multigrid method for Helmholtz problems: analysis

1.1 Discretization

(a)

$$\begin{aligned} 10 &\leq \lambda \# \text{gridpoints} \Leftrightarrow \\ 10 &\leq \frac{2\pi}{\sqrt{|\sigma|}} \frac{1}{h^d} \Leftrightarrow \\ \sqrt{|\sigma|} h^d &\leq \frac{2\pi}{10} \approx 0.625. \end{aligned}$$

(b)

$$\# \text{ roosterpunten} = \frac{10\sqrt{600}}{2\pi}.$$

1.2 1D model problem

(a)

To proof : $H^{2h} \neq I_h^{2h} H^h I_{2h}^h$.

$$\begin{aligned} H^{2h} &= H_n = A_n + \sigma id_n \\ H^h &= H_{2n} = A_{2n} + \sigma id_{2n} \end{aligned}$$

Assume that $A_n = I_h^{2h} A_{2n} I_{2h}^h = R_{2n} A_{2n} I_n$. By linearity it is sufficient to proof:

$$\begin{aligned} \sigma id_n &\neq \sigma R_{2n} id_{2n} I_n \Leftrightarrow \\ id_n &\neq R_{2n} I_n \Leftarrow \\ (id_n)_{00} &= 1 \neq \frac{3}{4} = (R_{2n} I_n)_{00} \end{aligned}$$

First equivalence follows from $\sigma \neq 0$. The assumption and the last inequality depends on the definition of restriction and interpolation.

(b)

See code/main.ipynb for code and plots.

(c)

There exists a closed formula for eigenvalues and eigenvectors of tridiagonal toeplitz matrix. It is just tedious to use. Alternatively the eigenvalues and eigenvectors can be derived from the Poisson problem ($\sigma = 0$) because

$$\begin{aligned} Av &= \lambda v \Rightarrow \\ (A + \sigma id)v &= Av + \sigma v \\ &= (\lambda + \sigma)v \end{aligned}$$

i.e. eigenvectors stay the same and eigenvalues get shifted by σ .

(d)

See code/main.ipynb for the plot. $\sigma = 0$ is the boundary where H goes from indefinite to definite.

1.3 LFA analysis of the ω -Jacobi smoother

(a)

For grid points without a neighboring boundary point there H acts like following stencil:

$$H_n = n^2 \begin{bmatrix} -1 & 2 + \frac{\sigma}{n^2} & -1 \end{bmatrix}.$$

So R_ω works element wise the following way on the error:

$$e_j^{m+1} = (1 - \omega)e_j^m + \frac{\omega n^2}{2n^2 + \sigma}(e_{j-1}^m + e_{j+1}^m). \quad (1)$$

Very similar to the analysis for the Poisson equation. Note that we haven't used that σ is real. Doing the LFA substitution $e_j^{(m)} = \mathcal{A}(m)e^{ij\theta}$:

$$\begin{aligned} A(m+1) &= A(m) \left(1 - \omega + \frac{\omega n^2}{2n^2 + \sigma}(e^{-i\theta} + e^{i\theta}) \right) \\ &= A(m) \left(1 - \omega + 2 \cos(\theta) \frac{\omega n^2}{2n^2 + \sigma} \right) \end{aligned}$$

The factor in behind of $A(m)$ is the amplification factor $G(\theta)$.

(b)

$\sigma = 0$ reduces back to the LFA we did for the Poisson equation. $\theta \approx 0 \Rightarrow \cos(\theta) \approx 1 + O\left(\frac{1}{n^2}\right) \Rightarrow G(\theta) \approx 1 - O\left(\frac{1}{n^2}\right)$ so smooth modes are preserved for big n . (c) See code/main.ipynb for the plot.

(d)

Maximum of $G(\theta)$ is achieved at $\theta = 0$ because $G(\theta)$ is just an increasing function of $\cos(\theta)$. This means that $\rho = 1 - \omega + 2 \frac{\omega n^2}{2n^2 + \sigma} \approx 1.05$ which suggests that weighted jacobi wouldn't converge.

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