Multigrid for solving complex-valued Helmholtz problems

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1 Failure of the Multigrid method for Helmholtz problems: analysis

1.1 Discretization

(a)

$$
\begin{aligned}
10 &\leq \lambda \ \#\text{gridpoints} \Leftrightarrow \\
10 &\leq \frac{2\pi}{\sqrt{|\sigma|}} \frac{1}{h^d} \Leftrightarrow \\
\sqrt{|\sigma|} h^d &\leq \frac{2\pi}{10} \approx 0.625.
\end{aligned}
$$

(b)

$$
\#\ \text{roosterpunten} = \frac{10\sqrt{600}}{2\pi}.
$$

1.2 1D model problem

(a)
To proof : $H^{2h} \neq I_h^{2h} H^h I_{2h}^h$.

$$
\begin{aligned}
H^{2h} &= H_n = A_n + \sigma id_n \\
H^h &= H_{2n} = A_{2n} + \sigma id_{2n}
\end{aligned}
$$

Assume that $A_n = I_h^{2h} A_{2n} I_{2h}^h = R_{2n} A_{2n} I_n$. By linearity it is sufficient to proof:

$$
\begin{aligned}
\sigma id_n &\neq \sigma R_{2n} id_{2n} I_n \Leftrightarrow \\
id_n &\neq R_{2n} I_n \Leftarrow \\
(id_n)_{00} = 1 &\neq \frac{3}{4} = (R_{2n} I_n)_{00}
\end{aligned}
$$

First equivalence follows from $\sigma \neq 0$. The assumption and the last inequality depends on the definition of restriction and interpolation.
(b)

See code/main.ipynb for code and plots. We implemented $f(t) = \delta(t - 0.5)$ by concentrating all the mass into the middle element of f_n.
(c)
There exists a closed formula for eigenvalues and eigenvectors of tridiagonal toeplitz matrix. It is just tedious to use. Alternatively the eigenvalues and eigenvectors can be derived from the Poisson problem ($\sigma = 0$) because

$$\begin{aligned} Av &= \lambda v \Rightarrow \\ (A + \sigma id)v &= Av + \sigma v \\ &= (\lambda + \sigma)v \end{aligned}$$

i.e. eigenvectors stay the same and eigenvalues get shifted by σ.

(d)
See code/main.ipynb for the plot. $\sigma = 0$ is the boundary where H goes from indefinite to definite.

1.3 LFA analysis of the ω-Jacobi smoother

(a)
For grid points without a neighboring boundary point there H acts like following stencil:

$$H_n = n^2[-1 \quad 2 + \frac{\sigma}{n^2} \quad -1].$$

So R_ω works element wise the following way on the error:

$$e_j^{m+1} = (1 - \omega)e_j^m + \frac{\omega n^2}{2n^2 + \sigma}(e_{j-1}^m + e_{j+1}^m). \tag{1}$$

Very similar to the analysis for the Poisson equation. Note that we haven't used that σ is real. Doing the LFA subsitution $e_j^{(m)} = \mathcal{A}(m)e^{ij\theta}$:

$$\begin{aligned} A(m+1) &= A(m)\left(1 - \omega + \frac{\omega n^2}{2n^2 + \sigma}(e^{-i\theta} + e^{i\theta})\right) \\ &= A(m)\left(1 - \omega + 2\cos(\theta)\frac{\omega n^2}{2n^2 + \sigma}\right) \end{aligned}$$

The factor in behind of $A(m)$ is the amplification factor $G(\theta)$.
(b)
$\sigma = 0$ reduces back to the LFA we did for the Poisson equation. $\theta \approx 0 \Rightarrow \cos(\theta) \approx 1 + O\left(\frac{1}{n^2}\right) \Rightarrow G(\theta) \approx 1 - O\left(\frac{1}{n^2}\right)$ so smooth modes are preserved for big n. (c)
See code/main.ipynb for the plot.

(d)
Maximum of $G(\theta)$ is achieved at $\theta = 0$ because $G(\theta)$ is just an increasing function of $\cos(\theta)$. This means that $\rho = 1 - \omega + 2\frac{\omega n^2}{2n^2+\sigma} \approx 1.05$ which suggests that weighted jacobi wouldn't converge.

1.4 Spectral analysis of the two-grid correction scheme

(a)
It is easily seen that

$$R_{2n} = cS_{2n}(3id - A_{2n}). \tag{2}$$

with $c \in \mathbb{R}_0$, $S_{2n} : \mathbb{R}^{2n-1} \to \mathbb{R}^{n-1} : (v_j)_{j\leq 2n-2} \to (v_{2j+1})_{j\leq n-2}$ subsampling uneven components. Using that $S_{2n}w_k^{2n} = w_k^n$ if $k < n$ it is easily seen that:

$$\begin{aligned}
R_{2n}w_k^{2n} &= cS_{2n}(3id - A_{2n})w_k^{2n} \\
&= c(3 - \lambda_k(A_{2n}))S_{2n}w_k^{2n} \\
&= a(n,k)w_k^n.
\end{aligned}$$

In our case I_n is linear interpolation, reconstuction error for lagrange interpolation can be bounded using Taylors theorem.

$$I_nS_{2n}v_{2n} \approx cv_{2n}.. \tag{3}$$

For w_k^{2n} smooth and combining with previous argument we have:

$$I_nR_{2n}w_{2n} \approx c_1w_{2n}. \tag{4}$$

To check the normalizing constant try $v_{2n} = 1 \Rightarrow c_1 = 1$.
We also checked these facts numerically by plotting see code. Now doing spectral analysis of TG is straight forward:

$$\begin{aligned}
TGw_k^{2n} &= (id - I_nH_n^{-1}R_{2n}H_{2n})w_k^{2n} \\
&= w_k^{2n} - I_nH_n^{-1}R_{2n}\lambda_k(H_{2n})w_k^{2n} \\
&= w_k^{2n} - I_nH_n^{-1}a(n,k)w_k^n\lambda_k(H_{2n}) \\
&= w_k^{2n} - I_na(n,k)w_k^n\lambda_k(H_n^{-1})\lambda_k(H_{2n}) \\
&= w_k^{2n} - I_nR_{2n}w_k^{2n}\lambda_k(H_n^{-1})\lambda_k(H_{2n}) \\
&\approx w_k^{2n} - w_k^{2n}\lambda_k(H_n^{-1})\lambda_k(H_{2n}) \\
&\approx w_k^{2n}(1 - \lambda_k(H_n^{-1})\lambda_k(H_{2n}))
\end{aligned}$$

(b)
We already analytically derived the eigenvalues for H_n. For the plot see code.
TG iterations may amplify smooth modes when $\rho_k > 1$.
(c)
See code for the plots. $\rho_k > 1$ when the sign changes. In the previous case the index closest to the sign change is $k = 6$. No, the smoother leaves smooth modes almost unchanged.

2 Solving the complex-valued Helmholtz problem using Multigrid

2.1 1D model problem

(a)
see code

(b)
For a point source it may not be obvious but the solutions for complex shifted problem is very similar.
(c)
see code

2.2 LFA analysis of the ω-Jacobi smoother

(a)
Already answered in previous question. ρ is still $|G(\theta)|$ almost the same reasoning.

$$|G(\theta)| = |a + b\cos(\theta) + c\cos(\theta)i| \tag{5}$$

with $a, b \in \mathbb{R}^+$ and $c \in \mathbb{R}$ still gets optimized when $\cos(\theta)$ gets optimized. Using that argument requires that $2n^2 + R(\sigma) \geq 0$ which follows from the criterium on n we placed at the start.
(b)
Depending on β the smoother may be stable.
(c)
We think $|G(\pi)|$ or $|G(\frac{\pi}{2})|$. We have numerical evidence not a proof yet. (d)
Eyeballing the plot we made $\omega \approx 0.65$ is good.

2.3 Spectral analysis of the two-grid correction scheme

(a)
Already did that. See code. Well with the formula not numerical eigenvalues ...
(b)
The instability in ρ_k dampens.

2.4 2D model problem

2.5 Aggressive coarsening

3 Multigrid as a preconditioner for Krylov subspace methods

3.1 MG-GMRES for the indefinite Helmholtz problem