Multigrid for solving complex-valued Helmholtz problems

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1 Failure of the Multigrid method for Helmholtz problems: analysis

1.1 Discretization

(a)

$$\begin{aligned} 10 &\leq \lambda \text{ \#gridpoints} \Leftrightarrow \\ 10 &\leq \frac{2\pi}{\sqrt{|\sigma|}} \frac{1}{h^d} \Leftrightarrow \\ \sqrt{|\sigma|} h^d &\leq \frac{2\pi}{10} \approx 0.625. \end{aligned}$$

(b)

$$\# \text{ roosterpunten} = \frac{10\sqrt{600}}{2\pi}.$$

1.2 1D model problem

(a)
To proof : $H^{2h} \neq I_h^{2h} H^h I_{2h}^h$.

$$\begin{aligned} H^{2h} &= H_n = A_n + \sigma id_n \\ H^h &= H_{2n} = A_{2n} + \sigma id_{2n} \end{aligned}$$

Assume that $A_n = I_h^{2h} A_{2n} I_{2h}^h = R_{2n} A_{2n} I_n$. By linearity it is sufficient to proof:

$$\begin{aligned} \sigma id_n &\neq \sigma R_{2n} id_{2n} I_n \Leftrightarrow \\ id_n &\neq R_{2n} I_n \Leftarrow \\ (id_n)_{00} = 1 &\neq \frac{3}{4} = (R_{2n} I_n)_{00} \end{aligned}$$

First equivalence follows from $\sigma \neq 0$. The assumption and the last inequality depends on the definition of restriction and interpolation.
(b)

See code/main.ipynb for code and plots.
(c)
There exists a closed formula for eigenvalues and eigenvectors of tridiagonal toeplitz matrix. It is just tedious to use. Alternatively the eigenvalues and eigenvectors can be derived from the Poisson problem ($\sigma = 0$) because

$$\begin{aligned} Av &= \lambda v \Rightarrow \\ (A + \sigma id)v &= Av + \sigma v \\ &= (\lambda + \sigma)v \end{aligned}$$

i.e. eigenvectors stay the same and eigenvalues get shifted by σ.

(d)
See code/main.ipynb for the plot. $\sigma = 0$ is the boundary where H goes from indefinite to definite.

1.3 LFA analysis of the ω-Jacobi smoother

1.4 Spectral analysis of the two-grid correction scheme

2 Solving the complex-valued Helmholtz problem using Multigrid

2.1 1D model problem

2.2 LFA analysis of the ω-Jacobi smoother

2.3 Spectral analysis of the two-grid correction scheme

2.4 2D model problem

2.5 Aggressive coarsening

3 Multigrid as a preconditioner for Krylov subspace methods

3.1 MG-GMRES for the indefinite Helmholtz problem