Multigrid for solving complex-valued Helmholtz problems

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1 Failure of the Multigrid method for Helmholtz problems: analysis

1.1 Discretization

(a)

$$\begin{aligned} 10 &\leq \lambda \ \#\text{gridpoints} \Leftrightarrow \\ 10 &\leq \frac{2\pi}{\sqrt{|\sigma|}} \frac{1}{h^d} \Leftrightarrow \\ \sqrt{|\sigma|} h^d &\leq \frac{2\pi}{10} \approx 0.625. \end{aligned}$$

(b)

$$\# \text{ roosterpunten} = \frac{10\sqrt{600}}{2\pi}.$$

1.2 1D model problem

(a)
To proof : $H^{2h} \neq I_h^{2h} H^h I_{2h}^h$.

$$\begin{aligned} H^{2h} &= H_n = A_n + \sigma id_n \\ H^h &= H_{2n} = A_{2n} + \sigma id_{2n} \end{aligned}$$

Assume that $A_n = I_h^{2h} A_{2n} I_{2h}^h = R_{2n} A_{2n} I_n$. By linearity it is sufficient to proof:

$$\begin{aligned} \sigma id_n &\neq \sigma R_{2n} id_{2n} I_n \Leftrightarrow \\ id_n &\neq R_{2n} I_n \Leftarrow \\ (id_n)_{00} = 1 &\neq \frac{3}{4} = (R_{2n} I_n)_{00} \end{aligned}$$

First equivalence follows from $\sigma \neq 0$. The assumption and the last inequality depends on the definition of restriction and interpolation.
(b)

See code/main.ipynb for code and plots. We implemented $f(t) = \delta(t - 0.5)$ by concentrating all the mass into the middle element of f_n.
(c)
There exists a closed formula for eigenvalues and eigenvectors of tridiagonal toeplitz matrix. It is just tedious to use. Alternatively the eigenvalues and eigenvectors can be derived from the Poisson problem ($\sigma = 0$) because

$$\begin{aligned} Av &= \lambda v \Rightarrow \\ (A + \sigma id)v &= Av + \sigma v \\ &= (\lambda + \sigma)v \end{aligned}$$

i.e. eigenvectors stay the same and eigenvalues get shifted by σ.

(d)
See code/main.ipynb for the plot. $\sigma = 0$ is the boundary where H goes from indefinite to definite.

1.3 LFA analysis of the ω-Jacobi smoother

(a)
For grid points without a neighboring boundary point there H acts like following stencil:

$$H_n = n^2[-1 \quad 2 + \frac{\sigma}{n^2} \quad -1].$$

So R_ω works element wise the following way on the error:

$$e_j^{m+1} = (1 - \omega)e_j^m + \frac{\omega n^2}{2n^2 + \sigma}(e_{j-1}^m + e_{j+1}^m). \tag{1}$$

Very similar to the analysis for the Poisson equation. Note that we haven't used that σ is real. Doing the LFA subsitution $e_j^{(m)} = \mathcal{A}(m)e^{ij\theta}$:

$$\begin{aligned} A(m+1) &= A(m)\left(1 - \omega + \frac{\omega n^2}{2n^2 + \sigma}(e^{-i\theta} + e^{i\theta})\right) \\ &= A(m)\left(1 - \omega + 2\cos(\theta)\frac{\omega n^2}{2n^2 + \sigma}\right) \end{aligned}$$

The factor in behind of $A(m)$ is the amplification factor $G(\theta)$.
(b)
$\sigma = 0$ reduces back to the LFA we did for the Poisson equation. $\theta \approx 0 \Rightarrow \cos(\theta) \approx 1 + O\left(\frac{1}{n^2}\right) \Rightarrow G(\theta) \approx 1 - O\left(\frac{1}{n^2}\right)$ so smooth modes are preserved for big n. (c)
See code/main.ipynb for the plot.

(d)
Maximum of $G(\theta)$ is achieved at $\theta = 0$ because $G(\theta)$ is just an increasing function of $\cos(\theta)$. This means that $\rho = 1 - \omega + 2\frac{\omega n^2}{2n^2+\sigma} \approx 1.05$ which suggests that weighted jacobi wouldn't converge.

1.4 Spectral analysis of the two-grid correction scheme

(a)
Note that

$$R_{2n} w_k^{2n} = \begin{cases} a(k) w_k^n & \text{if } k \leq n \\ a(n-k) w_k^n & \text{if } k > n \end{cases} \tag{2}$$

and

$$I_n R_{2n} v_{2n} \approx v_{2n}. \tag{3}$$

if v_{2n} is smooth sometimes by definition. We checked this numerically by plotting the reconstruction error. For linear interpolation and the corresponding restriction this follows from bounds on reconstruction loss for Lagrange interpolation. To check the normalizing constant try $v_{2n} = 1$.

$$\begin{aligned} TG w_k^{2n} &= (id - I_n H_n^{-1} R_{2n} H_{2n}) w_k^{2n} \\ &= w_k^{2n} - I_n H_n^{-1} R_{2n} \lambda_k(H_{2n}) w_k^{2n} \\ &= w_k^{2n} - I_n H_n^{-1} a(k) w_k^n \lambda_k(H_{2n}) \\ &= w_k^{2n} - I_n a(k) w_k^n \lambda_k(H_n^{-1}) \lambda_k(H_{2n}) \\ &= w_k^{2n} - I_n R_{2n} w_k^{2n} \lambda_k(H_n^{-1}) \lambda_k(H_{2n}) \\ &\approx w_k^{2n} - w_k^{2n} \lambda_k(H_n^{-1}) \lambda_k(H_{2n}) \\ &\approx w_k^{2n} (1 - \lambda_k(H_n^{-1}) \lambda_k(H_{2n})) \end{aligned}$$

(b)
We already analytically derived the eigenvalues for H_n. For the plot see code.
TG iterations may amplify smooth modes when $\rho_k > 1$.
(c)
See code for the plots. $\rho_k > 1$ when the sign changes. In the previous case the index closest to the sign change is $k = 6$. No the smoother leaves smooth modes almost unchanged.

2 Solving the complex-valued Helmholtz problem using Multigrid

2.1 1D model problem

2.2 LFA analysis of the ω-Jacobi smoother

2.3 Spectral analysis of the two-grid correction scheme

2.4 2D model problem

2.5 Aggressive coarsening

3 Multigrid as a preconditioner for Krylov subspace methods

3.1 MG-GMRES for the indefinite Helmholtz problem