Multigrid for solving complex-valued Helmholtz problems

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1 Failure of the Multigrid method for Helmholtz problems: analysis

1.1 Discretization

(a)

$$\begin{aligned} 10 &\leq \lambda \ \#\text{gridpoints} \Leftrightarrow \\ 10 &\leq \frac{2\pi}{\sqrt{|\sigma|}} \frac{1}{h^d} \Leftrightarrow \\ \sqrt{|\sigma|} h^d &\leq \frac{2\pi}{10} \approx 0.625. \end{aligned}$$

(b)

$$\# \text{ roosterpunten} = \frac{10\sqrt{600}}{2\pi}.$$

1.2 1D model problem

(a)
To proof : $H^{2h} \neq I_h^{2h} H^h I_{2h}^h$.

$$\begin{aligned} H^{2h} &= H_n = A_n + \sigma id_n \\ H^h &= H_{2n} = A_{2n} + \sigma id_{2n} \end{aligned}$$

Assume that $A_n = I_h^{2h} A_{2n} I_{2h}^h = R_{2n} A_{2n} I_n$. By linearity it is sufficient to proof:

$$\begin{aligned} \sigma id_n &\neq \sigma R_{2n} id_{2n} I_n \Leftrightarrow \\ id_n &\neq R_{2n} I_n \Leftarrow \\ (id_n)_{00} = 1 &\neq \frac{3}{4} = (R_{2n} I_n)_{00} \end{aligned}$$

First equivalence follows from $\sigma \neq 0$. The assumption and the last inequality depends on the definition of restriction and interpolation.
(b)

See code/main.ipynb for code and plots. We implemented $f(t) = \delta(t - 0.5)$ by concentrating all the mass into the middle element of f_n.
(c)
There exists a closed formula for eigenvalues and eigenvectors of tridiagonal toeplitz matrix. It is just tedious to use. Alternatively the eigenvalues and eigenvectors can be derived from the Poisson problem ($\sigma = 0$) because

$$\begin{aligned} Av &= \lambda v \Rightarrow \\ (A + \sigma id)v &= Av + \sigma v \\ &= (\lambda + \sigma)v \end{aligned}$$

i.e. eigenvectors stay the same and eigenvalues get shifted by σ.

(d)
See code/main.ipynb for the plot. $\sigma = 0$ is the boundary where H goes from indefinite to definite.

1.3 LFA analysis of the ω-Jacobi smoother

(a)
For grid points without a neighboring boundary point there H acts like following stencil:

$$H_n = n^2[-1 \quad 2 + \frac{\sigma}{n^2} \quad -1].$$

So R_ω works element wise the following way on the error:

$$e_j^{m+1} = (1 - \omega)e_j^m + \frac{\omega n^2}{2n^2 + \sigma}(e_{j-1}^m + e_{j+1}^m). \tag{1}$$

Very similar to the analysis for the Poisson equation. Note that we haven't used that σ is real. Doing the LFA subsitution $e_j^{(m)} = \mathcal{A}(m)e^{ij\theta}$:

$$\begin{aligned} A(m+1) &= A(m)\left(1 - \omega + \frac{\omega n^2}{2n^2 + \sigma}(e^{-i\theta} + e^{i\theta})\right) \\ &= A(m)\left(1 - \omega + 2\cos(\theta)\frac{\omega n^2}{2n^2 + \sigma}\right) \end{aligned}$$

The factor in behind of $A(m)$ is the amplification factor $G(\theta)$.
(b)
$\sigma = 0$ reduces back to the LFA we did for the Poisson equation. $\theta \approx 0 \Rightarrow \cos(\theta) \approx 1 + O\left(\frac{1}{n^2}\right) \Rightarrow G(\theta) \approx 1 - O\left(\frac{1}{n^2}\right)$ so smooth modes are preserved for big n. (c)
See code/main.ipynb for the plot.

(d)
Maximum of $G(\theta)$ is achieved at $\theta = 0$ because $G(\theta)$ is just an increasing function of $\cos(\theta)$. This means that $\rho = 1 - \omega + 2\frac{\omega n^2}{2n^2+\sigma} \approx 1.05$ which suggests that weighted jacobi wouldn't converge.

1.4 Spectral analysis of the two-grid correction scheme

(a)
It is easily seen that

$$R_{2n} = cS_{2n}(3id - A_{2n}). \tag{2}$$

with $c \in \mathbb{R}_0$, $S_{2n} : \mathbb{R}^{2n-1} \to \mathbb{R}^{n-1} : (v_j)_{j \le 2n-2} \to (v_{2j+1})_{j \le n-2}$ subsampling uneven components. Using that $S_{2n} w_k^{2n} = w_k^n$ if $k < n$ it is easily seen that:

$$\begin{aligned}
R_{2n} w_k^{2n} &= cS_{2n}(3id - A_{2n}) w_k^{2n} \\
&= c(3 - \lambda_k(A_{2n})) S_{2n} w_k^{2n} \\
&= a(n,k) w_k^n.
\end{aligned}$$

In our case I_n is linear interpolation, reconstuction error for lagrange interpolation can be bounded using Taylors theorem.

$$I_n S_{2n} v_{2n} \approx c v_{2n}.. \tag{3}$$

For w_k^{2n} smooth and combining with previous argument we have:

$$I_n R_{2n} w_{2n} \approx c_1 w_{2n}. \tag{4}$$

To check the normalizing constant try $v_{2n} = 1 \Rightarrow c_1 = 1$.
We also checked these facts numerically by plotting see code. Now doing spectral analysis of TG is straight forward:

$$\begin{aligned}
TGw_k^{2n} &= (id - I_n H_n^{-1} R_{2n} H_{2n}) w_k^{2n} \\
&= w_k^{2n} - I_n H_n^{-1} R_{2n} \lambda_k(H_{2n}) w_k^{2n} \\
&= w_k^{2n} - I_n H_n^{-1} a(n,k) w_k^n \lambda_k(H_{2n}) \\
&= w_k^{2n} - I_n a(n,k) w_k^n \lambda_k(H_n^{-1}) \lambda_k(H_{2n}) \\
&= w_k^{2n} - I_n R_{2n} w_k^{2n} \lambda_k(H_n^{-1}) \lambda_k(H_{2n}) \\
&\approx w_k^{2n} - w_k^{2n} \lambda_k(H_n^{-1}) \lambda_k(H_{2n}) \\
&\approx w_k^{2n} (1 - \lambda_k(H_n^{-1}) \lambda_k(H_{2n}))
\end{aligned}$$

(b)
We already analytically derived the eigenvalues for H_n. For the plot see code.
TG iterations may amplify smooth modes when $\rho_k > 1$.
(c)
See code for the plots. $\rho_k > 1$ when the sign changes. In the previous case the index closest to the sign change is $k = 6$. No, the smoother leaves smooth modes almost unchanged.

2 Solving the complex-valued Helmholtz problem using Multigrid

2.1 1D model problem

(a)
see code

(b)
For a point source it may not be obvious but the solutions for complex shifted problem is very similar.
(c)
see code

2.2 LFA analysis of the ω-Jacobi smoother

(a)
Already answered in previous question. ρ is still $|G(\theta)|$ almost the same reasoning.

$$|G(\theta)| = |a + b\cos(\theta) + c\cos(\theta)i| \tag{5}$$

with $a, b \in \mathbb{R}^+$ and $c \in \mathbb{R}$ still gets optimized when $\cos(\theta)$ gets optimized. Using that argument requires that $2n^2 + R(\sigma) \geq 0$ which follows from the criterium on n we placed at the start.
(b)
Depending on β the smoother may be stable.
(c)
We think $|G(\pi)|$ or $|G(\frac{\pi}{2})|$. We have numerical evidence not a proof yet. (d)
Eyeballing the plot we made $\omega \approx 0.65$ is good.

2.3 Spectral analysis of the two-grid correction scheme

(a)
Already did that. See code. Well with the formula not numerical eigenvalues ...
(b)
The instability in ρ_k dampens.

2.4 2D model problem

(a)
Solve happens in the code. (b)
Less iterations are needed for the same amount of error. This is because σ the source of the problem get relatively smaller to n^2 in our convergence factors, asymptotically behavior should go back the Laplace problem.

2.5 Aggressive coarsening

(a)
Not sure here is our guess:(TG = two grid,FG= four grid)

$$FG = id - I_{2n} I_n H_n^{-1} R_{2n} R_{4n} H_{4n}. \tag{6}$$

Not sure what is meant by the eigenmode analysis but what we previously did does generalizes. The eigenmodes that are complementary with w_k^{4n} on the n grid are: $w_{4n-k}^{4n}, w_{2n-k}^{4n}, w_{2n+k}^{4n}$.
(b)

Our solver diverged. Basically we are skipping a smoothing step which makes the interpolation worse. Also relevant is that convergence for this problem is better on larger grids. In the FG eigenmode analysis, interpolation error is worse and eigenvalues are less similar which suggest that only the smoothest modes of the errors get dampend. Accuracy drops but the trade off is that aggressive coarsening is less expensive about $\frac{1}{2^d}$ cheaper on everything but the operations on the base grid. (Half of the geometric series is gone.)

3 Multigrid as a preconditioner for Krylov subspace methods

3.1 MG-GMRES for the indefinite Helmholtz problem