Assignment Levy processes

Abstract

We elaborate some results from chapter 25 of Sato's *Lévy processes and infinitely divisible distributions* Sato 2013. Additions are denoted in blue, while deletions are signified in red.

Proof of Existence of Moments from Smoothness of Characteristic Function

Definition 0.0.1 (Δ_h^k)
We define $\forall h > 0 : \Delta_h$ (h-central difference) of $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$\Delta_h f(t) = f(t+h) - f(t-h), \tag{1}$$

and $\forall k \in \mathbb{N} : \Delta_h^k$ is defined as applying Δ_h applying k times and $k = 0$ as the identity. Note that Δ_h^k is a linear operator. We will always take differences in respect to the t variable.

Lemma 0.0.2 ($\Delta_h^k e^{at}$)

$$\forall k \in \mathbb{N} : \Delta_h^k e^{at} = e^{at}\left(e^{ah} - e^{-ah}\right)^k. \tag{2}$$

Proof. This follows simply by induction. $k = 0$ is trivial. Base case:

$$\begin{aligned} \Delta_h e^{at} &= \left(e^{a(t+h)} - e^{a(t-h)}\right) && (3)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right). && (4) \end{aligned}$$

Induction hypothesis:

$$\Delta_h^{k-1} e^{at} = e^{at}\left(e^{ah} - e^{-ah}\right)^{k-1}. \tag{5}$$

Induction step:

$$\begin{aligned} \Delta_h^k e^{at} &= \Delta_h \Delta_h^{k-1}(e^{at}) && (6)\\ &= \Delta_h\left(e^{at}\left(e^{ah} - e^{-ah}\right)^{k-1}\right) && (7)\\ &= \Delta_h\left(e^{at}\right)\left(e^{ah} - e^{-ah}\right)^{k-1} && (8)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right)\left(e^{ah} - e^{-ah}\right)^{k-1} && (9)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right)^k. && (10) \end{aligned}$$

$\square$

Theorem 0.0.3 (central finite difference formula)
$\forall k \in \mathbb{N}\ \forall$ k-continuously differentiable f's at 0:

$$f^{(k)}(0) = \lim_{h\to 0} \frac{\Delta_h^k f(t)|_{t=0}}{(2h)^k}. \tag{11}$$

Proof. Follows by the mean value theorem.

$$\begin{aligned}
&\lim_{h\to 0} \frac{\Delta_h^k f(t)|_{t=0}}{(2h)^k} && (12)\\
&= \lim_{h\to 0} \frac{\Delta_h \Delta_h^{k-1} f(t)|_{t=0}}{(2h)^k} && (13)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-1} f(t)(|_{t=h} - |_{t=-h})}{(2h)^k} && (14)\\
&= \lim_{h\to 0} \frac{2h(\Delta_h^{k-1} f(t))'|_{t=z_h^1}}{(2h)^k}, z_h^1 \in [-h, h] && (15)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-1} f'(t)|_{t=z_h^1}}{(2h)^{k-1}}, z_h^1 \in [-h, h] && (16)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-2} f^{(2)}(t)|_{t=z_h^2}}{(2h)^{k-2}}, z_h^2 \in [z_h^1 - h, z_h^1 + h] \subset [-2h, 2h] && (17)\\
&\ldots && (18)\\
&= \lim_{h\to 0} f^{(k)}(t)|_{t=z_h^k}, z_h^k \in [-kh, kh] && (19)\\
&= \lim_{h\to 0} f^{(k)}(z_h^k) = f^{(k)}(0). && (20)
\end{aligned}$$

Last line follows because f^k is continuous at 0. □

Lemma 0.0.4
$\forall k \in \mathbb{N} : \phi_X(t)$ is $2k$-time continuous differentiable $\Rightarrow E[X^{2k}] < \infty$

Proof. Use the central finite difference formula and previous lemma:

$$\begin{aligned}
|\phi_X^{(2k)}(0)| &= \left| \lim_{h\to 0} \frac{\Delta_h^{2k}|_{t=0} \phi_X(t)}{(2h)^{2k}} \right| && (21)\\
&= \left| \lim_{h\to 0} \frac{\Delta_h^{2k}|_{t=0} E\left[e^{itX}\right]}{(2h)^{2k}} \right| && (22)\\
&= \left| \lim_{h\to 0} \frac{E\left[\Delta_h^{2k}|_{t=0} e^{itX}\right]}{(2h)^{2k}} \right| && (23)\\
&= \left| \lim_{h\to 0} \frac{E\left[e^{itX}|_{t=0}(e^{ihX} - e^{-ihX})^{2k}\right]}{(2h)^{2k}} \right| && (24)\\
&= \left| \lim_{h\to 0} \frac{E\left[(e^{ihX} - e^{-ihX})^{2k}\right]}{(2h)^{2k}} \right| && (25)\\
&= \left| \lim_{h\to 0} \frac{1}{i^{2k}} E\left[\frac{(e^{ihX} - e^{-ihX})^{2k}}{(2h)^{2k}}\right] \right| && (26)\\
&= \lim_{h\to 0} E\left[\frac{\sin(Xh)^{2k}}{h^{2k}}\right] && (27)
\end{aligned}$$

$$= \lim_{h \to 0} E\left[X^{2k} \frac{\sin(Xh)^{2k}}{X^{2k}h^{2k}}\right] \tag{28}$$

$$= \lim_{h \to 0} E\left[X^{2k} \operatorname{sinc}(Xh)^{2k}\right] \tag{29}$$

$$\geq E\left[X^{2k}\right] \tag{30}$$

Where last line follows by Fatou's lemma. □

25. Moments

We define the g-moment of a random variable and discuss finiteness of the g-moment of X_t for a Lévy process $\{X_t\}$.

DEFINITION 25.1. Let $g(x)$ be a nonnegative measurable function on $\mathbb{R}^d$. We call $\int g(x)\mu(\mathrm{d}x)$ the g-moment of a measure μ on $\mathbb{R}^d$. We call $E[g(X)]$ the g-moment of a random variable X on $\mathbb{R}^d$.

DEFINITION 25.2. A function $g(x)$ on $\mathbb{R}^d$ is called submultiplicative if it is nonnegative and there is a constant $a > 0$ such that

$$g(x+y) \leq a g(x) g(y) \text{ for } x, y \in \mathbb{R}^d \tag{25.1}$$

A function bounded on every compact set is called locally bounded.

LEMMA 25.i1 Let $g(x)$ be a submultiplicative with constant a then:

$$\forall (x_i)_i \subset \mathbb{R}^d : g\left(\sum_{i=1}^{n} x_i\right) \leq a^{n-1} \prod_{i=1}^{n} g(x_i).$$

Proof. This follows by induction on n.
Base case: $n = 2$ is true by submultiplicativity.
Induction hypothesis: assume $2 \leq n \leq k$.
Induction step: $n = k + 1$:

$$\begin{aligned}
&\forall (x_i)_i \subset \mathbb{R}^d : \\
g\left(\sum_{i=1}^{k+1} x_i\right) &= g\left(\sum_{i=1}^{k} x_i + x_{k+1}\right) \\
&\leq a g\left(\sum_{i=1}^{k} x_i\right) g(x_{k+1}) \text{ by submultiplicativity} \\
&\leq a a^{k-1} \prod_{i=1}^{k} g\left(x_i\right) g(x_{k+1}) \text{ by induction hypothesis} \\
&\leq a^k \prod_{i=1}^{k+1} g\left(x_i\right).
\end{aligned}$$

□

THEOREM 25.3 (g-Moment). Let g be a submultiplicative, locally bounded, measurable function on $\mathbb{R}^d$. Then, finiteness of the g-moment is not a time dependent distributional property in the class of Lévy processes. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with

Lévy measure ν. Then, X_t has finite g-moment for every $t > 0$ if and only if $[\nu]_{\{|x|>1\}}$ has finite g-moment.

The following facts indicate the wide applicability of the theorem.

PROPOSITION 25.4.
(i) The product of two submultiplicative functions is submultiplicative.
(ii) If $g(x)$ is submultiplicative on $\mathbb{R}^d$, then so is $g(cx+\gamma)^\alpha$ with $c \in \mathbb{R}$, $\gamma \in \mathbb{R}^d$, and $\alpha > 0$.
(iii) Let $0 < \beta \le 1$. Then the following functions are submultiplicative:

$$|x| \vee 1, |x_j| \vee 1, x_j \vee 1, \exp\left(|x|^\beta\right), \exp\left(|x_j|^\beta\right)$$
$$\exp\left((x_j \vee 0)^\beta\right), \log(|x| \vee e), \log\left(|x_j| \vee e\right), \log\left(x_j \vee \mathrm{e}\right)$$
$$\log\log\left(|x| \vee e^e\right), \log\log\left(|x_j| \vee e^e\right), \log\log\left(x_j \vee e^e\right)$$

Here x_j is the j th component of x.

Proof. (i) Immediate from the definition.
Let $g, h : \mathbb{R}^d \to \mathbb{R}$ be submultiplicative functions then:

$$\forall x, y \in \mathbb{R}^d : g(x+y)h(x+y) \le a_1 g(x)g(y)a_2 h(x)h(y) \le a_1 a_2 g(x)h(x)g(y)h(y).$$

So $gh : \mathbb{R}^d \to \mathbb{R}$ is submultiplicative.

(ii) Let $g_1(x) = g(cx), g_2(x) = g(x+\gamma)$, and $g_3(x) = g(x)^\alpha$. Then it follows from (25.1) that $g_1(x+y) \le a g_1(x) g_1(y)$, $g_2(x+y) \le a^2 g(-\gamma) g_2(x) g_2(y)$, and $g_3(x+y) \le a^\alpha g_3(x) g_3(y)$.

$$\begin{aligned}
&\forall x, y \in \mathbb{R}^d : \\
&g_1(x+y) = g(c(x+y)) = g(cx+cy) \le a g(cx) g(cy) = a g_1(x) g_1(y) \\
&g_2(x+y) = g(x+y+\gamma) = g(x+\gamma+y+\gamma-\gamma) \le a^2 g(-\gamma) g(x+\gamma) g(y+\gamma) \\
&\qquad\qquad \le a^2 g(-\gamma) g_2(x) g_2(y) \\
&g_3(x+y) = g(x+y)^\alpha \le (a g(x) g(y))^\alpha = a^\alpha g(x)^\alpha g(y)^\alpha = a^\alpha g_3(x) g_3(y).
\end{aligned}$$

(iii) Let $h(u)$ be a positive increasing function on $\mathbb{R}$ such that, for some $b \ge 0, h(u)$ is constant on $(-\infty, b]$ and $\log h(u)$ is concave on $[b, \infty)$. Then $h(u)$ is submultiplicative on $\mathbb{R}$. In fact, for $u, v \ge b \ge 0$, by concaveness the function $f(u) = \log h(u)$ satisfies

$$\frac{f(u+b) - f(u)}{b} \le \frac{f(2b) - f(b)}{b} \Leftrightarrow f(u+b) - f(u) \le f(2b) - f(b)$$
$$\frac{f(u+v) - f(v)}{u} \le \frac{f(u+b) - f(b)}{u} \Leftrightarrow f(u+v) - f(v) \le f(u+b) - f(b)$$

the first line is for $b \neq 0, b = 0$ is trivial and hence

$$f(u+v) \le f(u+b) - f(b) + f(v) \le f(2b) - 2f(b) + f(u) + f(v) \Leftrightarrow$$
$$h(u+v) \le \frac{h(2b)}{h(b)^2} h(u) h(v)$$

which shows

$$h(u+v) \le \text{ const } h(u)h(v) \tag{25.2}$$

It follows (we miss the cases for $u \geq b \geq v, b \geq u, v$) that (25.2) holds for all $u, v \in \mathbb{R}$. The functions $u \vee 1, \exp\left((u \vee 0)^{\beta}\right)$, $\log(u \vee \mathrm{e})$, and $\log\log\left(u \vee \mathrm{e}^{\mathrm{e}}\right)$ fulfil the conditions on $h(u)$. By (25.2) and by the increasingness of h, the functions $h(|x|), h\left(|x_j|\right)$, and $h\left(x_j\right)$ are submultiplicative on $\mathbb{R}^d$.

□

We prove Theorem 25.3 after three lemmas.

LEMMA 25.5. If $g(x)$ is submultiplicative and locally bounded, then

$$g(x) \leq b e^{c|x|} \tag{25.3}$$

with some $b > 0$ and $c > 0$.

Proof. Choose b in such a way that $\sup_{|x| \leq 1} g(x) \leq b$ and $ab > 1$. If $n - 1 < |x| \leq n$, then $\sup_{|x|\leq 1} g(x) < \infty$ because g is locally bounded, let $b = \max(\sup_{|x|\leq 1} g(x), \frac{1}{a} + 1)$ such that $\sup_{|x|\leq 1} g(x) < b$ and $ab > 1$. $\forall x$ set $n_x = \lceil |x| \rceil$ implying $n_x - 1 < |x| \leq n_x$, $\frac{x}{n_x} \in \{u | |u| \leq 1\}$, then

$$g(x) = g\left(\sum_{i=1}^{n_x} \frac{x}{n_x}\right) \leq a^{n_x - 1} g\left(\frac{x}{n_x}\right)^{n_x} \leq a^{n_x-1} b^{n_x} \leq b(ab)^{|x|} \leq b e^{\ln(ab)|x|}$$

which shows (25.3). □

LEMMA 25.6. Let μ be an infinitely divisible distribution on $\mathbb{R}$ with Lévy measure ν supported on a bounded set $S \subset [-a, a]$. Then $\widehat{\mu}(t)$ can be extended to an entire function on $\mathbb{C}$.

Proof. There is a finite $a > 0$ such that $S_\nu \subset [-a, a]$. The Lévy representation of $\widehat{\mu}(t)$ is written as

$$\begin{aligned}
\widehat{\mu}(t) &= \exp\left[-\frac{1}{2}At^2 + \int_{[-a,a]} \left(\mathrm{e}^{itx} - 1 - itx\right)\nu(dx) + i\gamma' t\right] \\
&= \exp\left[-\frac{1}{2}At^2 + i\gamma' t\right] \exp\left[\int_{-a}^{a} \left(\mathrm{e}^{itx} - 1 - itx\right)\nu(dx)\right]
\end{aligned}$$

with some $\gamma' \in \mathbb{R}$. The right-hand side is meaningful even if t is complex. Denote this function by $\Phi(t)$. Then $\Phi(t)$ is an entire function, since we can exchange the order of integration and differentiation. As products, sums, compositions and integrals of entire functions are entire (integrals follow from Morera's theorem), exp and polynomials are entire it is sufficient to show that $\int_{-a}^{a} \left(\mathrm{e}^{itx} - 1 - itx\right)\nu(dx)$ is finite for $t \in \mathbb{C}$:

$$\begin{aligned}
&\left|\int_{-a}^{a} \left(\mathrm{e}^{itx} - 1 - itx\right)\nu(dx)\right| \\
&\leq \int_{-a}^{a} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) \\
&\leq \int_{-1}^{1} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) + \int_{[-a,a]\setminus[-1,1]} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) \\
&\leq \int_{-1}^{1} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) + \sup_{x \in [-a,a]\setminus[-1,1]} \left(|\mathrm{e}^{itx} - 1 - itx|\right) \int_{[-a,a]\setminus[-1,1]} \nu(dx) \\
&\leq \int_{-1}^{1} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) + I_2
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{-1}^{1} \left| \sum_{n=2}^{\infty} \frac{(ixt)^n}{n!} \right| \nu(dx) + I_2 \\
&\leq \int_{-1}^{1} \sum_{n=2}^{\infty} \frac{|x|^n |t|^n}{n!} \nu(dx) + I_2 \\
&\leq \int_{-1}^{1} |x|^2 \sum_{n=2}^{\infty} \frac{|x|^{n-2} |t|^n}{n!} \nu(dx) + I_2 \\
&\leq \int_{-1}^{1} |x|^2 \sum_{n=2}^{\infty} \frac{|t|^n}{n!} \nu(dx) + I_2 \\
&\leq \sum_{n=2}^{\infty} \frac{|t|^n}{n!} \int_{-1}^{1} |x|^2 \nu(dx) + I_2 \\
&\leq (e^{|t|} - |t| - 1) \int_{-1}^{1} |x|^2 \nu(dx) + I_2 \\
&< \infty.
\end{aligned}
$$

We implicitly used $|x| < 1 \implies |x|^{n-2} < 1$,$|\mathrm{e}^{itx} - 1 - itx|$ continuous in x so achieves maximum on compact sets (Weierstrass theorem) and

$$
\int_{\mathbb{R}} (|x|^2 \wedge 1) \nu(dx) < \infty \implies \int_{-1}^{1} |x|^2 \nu(dx) < \infty, \int_{[-a,a] \setminus [-1,1]} \nu(dx) < \infty.
$$

□

LEMMA 25.7. If μ is a probability measure on $\mathbb{R}$ and $\widehat{\mu}(z)$ is extendible to an entire function on $\mathbb{C}$, then μ has finite exponential moments, that is, it has finite $e^{c|x|}$-moment for every $c > 0$.

Proof. It follows from Proposition 2.5(x) (see 0.0.4) that $\alpha_n = \int x^n \mu(\mathrm{d}x)$ and $\beta_n = \int |x|^n \mu(\mathrm{d}x)$ are finite for any $n \geq 1$. Since $\frac{\mathrm{d}^n \widehat{\mu}}{\mathrm{d}z^n}(0) = \mathrm{i}^n \alpha_n$, we have

$$
\widehat{\mu}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{i}^n \alpha_n z^n
$$

the radius of convergence of the right-hand side being infinite therefore $\forall z : \left| \frac{z^n \alpha_n}{n!} \right| = \frac{|z|^n}{n!} \alpha_n \to 0$. Notice that $\beta_{2k} = \alpha_{2k}$ and $\beta_{2k+1} \leq \frac{1}{2} (\alpha_{2k+2} + \alpha_{2k})$, since $|x|^{2k+1} \leq \frac{1}{2} (x^{2k+2} + x^{2k})$. It follows that $\forall c : \frac{2^n c^n}{n!} \beta_n \to 0 \implies \exists n_c : \forall n > n_c : \frac{2^n c^n}{n!} \beta_n < c$ and

$$
\begin{aligned}
\int \mathrm{e}^{c|x|} \mu(\mathrm{d}x) &= \int \sum_{n=0}^{\infty} \frac{c^n |x|^n}{n!} \mu(\mathrm{d}x) \\
&= \sum_{n=0}^{\infty} \frac{c^n}{n!} \int |x|^n \mu(\mathrm{d}x) \\
&\leq \sum_{n=0}^{\infty} \frac{1}{n!} \beta_n c^n \\
&\leq \sum_{n=0}^{n_c} \frac{1}{n!} \beta_n c^n + \sum_{n=n_c}^{\infty} \frac{1}{n!} \beta_n c^n
\end{aligned}
$$

$$\leq \sum_{n=0}^{n_c} \frac{1}{n!}\beta_n c^n + \sum_{n=n_c}^{\infty} \frac{c}{2^n}$$
$$< \infty$$

completing the proof.

□

Proof of Theorem 25.3. Let $\nu_0 = [\nu]_{\{|x|\leq 1\}}$ and $\nu_1 = [\nu]_{\{|x|>1\}}$. Construct independent Lévy processes $\{X_t^0\}$ and $\{X_t^1\}$ on $\mathbb{R}^d$ such that $\{X_t\} =_d \{X_t^0 + X_t^1\}$ and $\{X_t^1\}$ is compound Poisson with Lévy measure ν_1. Let μ_0^t and μ_1^t be the distributions of X_t^0 and X_t^1, respectively.

Suppose that X_t has finite g-moment for some $t > 0$. It follows from

$$\infty > E\left[g\left(X_t\right)\right] = E\left[g\left(X_t^0 + X_t^1\right)\right] = \iint g(x+y)\mu_0^t(dx)\mu_1^t(dy)$$

that $\int g(x+y)\mu_1^t(dy) < \infty$ for some x. This means Because μ_1^t is a compound Poisson we have (see (24.1) condition on the Poisson distribution of the compound Poisson process)

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(x+y)\nu_1^n(dy) < \infty.$$

Since $g(y) \leq ag(-x)g(x+y) \leq abe^{c|x|}g(x+y)$ by Lemma 25.5, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(y)\nu_1^n(dy) < \infty. \tag{25.4}$$

Hence $\int g(y)\nu_1(dy) < \infty$.

Conversely, suppose that $\int g(y)\nu_1(dy) = G < \infty$. Let us prove that $E\left[g\left(X_t\right)\right] < \infty$ for every t. By the submultiplicativity,

$$\begin{aligned} E\left[g\left(X_t\right)\right] &= E\left[g\left(X_t^0 + X_t^1\right)\right] \\ &\leq aE\left[g\left(X_t^0\right)g\left(X_t^1\right)\right] \\ &= aE\left[g\left(X_t^0\right)\right]E\left[g\left(X_t^1\right)\right] \\ &\leq abE\left[\mathrm{e}^{c\left|X_t^0\right|}\right]E\left[g\left(X_t^1\right)\right] \end{aligned}$$

So it is sufficient to show that $E\left[g\left(X_t^1\right)\right] < \infty$ and $E\left[\mathrm{e}^{c\left|X_t^0\right|}\right] < \infty$

$$\begin{aligned} E[g(X_t^1)] &= \int g(x)\mu_1^t(dx) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(x)\nu_1^n(dx) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int g\left(\sum_j x_j\right) \prod_j \nu_1\left(dx_j\right) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} a^{n-1} \int \prod_j g\left(x_j\right) \prod_j \nu_1\left(dx_j\right) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} a^{n-1} \left(\int g(x)\nu_1(dx)\right)^n \end{aligned}$$

$$= a^{-1}\sum_{n=0}^{\infty}\frac{t^n}{n!}a^nG^n$$
$$= a^{-1}e^{aGt} < \infty.$$

$$\int g(y)\nu_1^n(dy) = \int\cdots\int g\left(y_1+\cdots+y_n\right)\nu_1\left(dy_1\right)\ldots\nu_1\left(dy_n\right)$$
$$\leq a^{n-1}\left(\int g(y)\nu_1(dy)\right)^n$$

Hence we have (25.4) for every t. That is, X_t^1 has finite g-moment. Since by (25.1) and (25.3), it remains only to show that $E\left[e^{c|X_t^0|}\right] < \infty$.

Let $X_j^0(t)$, $1 \leq j \leq d$, be the components of X_t^0. Then

$$E\left[e^{c|X_t^0|}\right] \leq E\left[\exp\left(c\sum_{j=1}^{d}|X_j^0(t)|\right)\right] \leq E\left[\prod_{j=1}^{d}\left(e^{cX_j^0(t)}+e^{-cX_j^0(t)}\right)\right]$$
$$\leq E\left[\sum_{\alpha_j\in\{-1,1\}} e^{c\sum_{j=1}^{d}\alpha_jX_j^0(t)}\right] \leq \sum_{\alpha_j\in\{-1,1\}} E\left[e^{c\sum_{j=1}^{d}\alpha_jX_j^0(t)}\right]$$

which is written as a sum of a finite number of terms of the form $E\left[\exp X_t^\sharp\right]$ with $X_t^\sharp$ being a linear combination of $X_j^0(t), 1 \leq j \leq d$. Since $\left\{X_t^\sharp\right\}$ is a Lévy process on $\mathbb{R}$ with Lévy measure supported on a bounded set (use Proposition 11.10), $E\left[\exp X_t^\sharp\right]$ is finite by virtue of Lemmas 25.6 and 25.7. This proves all statements in the theorem.

Since X_t^0 is a Lévy process with a Lévy measure supported on a bounded set, by proposition 11.10 is $\sum_{j=1}^{d}\alpha_jX_j^0(t)$ a Lévy process on $\mathbb{R}$ with a Lévy measure supported on a bounded set therefore it has finite (exponential) moments by lemma 25.7. I.e. $E\left[e^{c|X_t^0|}\right] \leq E\left[e^{c\sum_{j=1}^{d}\alpha_jX_j^0(t)}\right] < \infty$. □

Corollary 25.8. Let $\alpha > 0, 0 < \beta \leq 1$, and $\gamma \geq 0$. None of the properties $\int|x|^\alpha\mu(\mathrm{d}x) < \infty, \int(0\vee\log|x|)^\alpha\mu(\mathrm{d}x) < \infty$, and $\int|x|^\gamma e^{\alpha|x|^\beta}\mu(\mathrm{d}x) < \infty$ is time dependent in the class of Lévy processes. For a Lévy process on $\mathbb{R}^d$ with Lévy measure ν, each of the properties is expressed by the corresponding property of $[\nu]_{\{|x|>1\}}$.

Proof. This follows from Theorem 25.3 and Proposition 25.4. □

EXAMPLE 25.12. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ generated by (A,ν,γ). In components, $X_t = \left(X_j(t)\right), \gamma = (\gamma_j)$, and $A = (A_{jk})$. Then X_t has finite mean for $t > 0$ if and only if $\int_{\{|x|>1\}}|x|\nu(\mathrm{d}x) < \infty$. When this condition is met, we can find $m_j(t) = E\left[X_j(t)\right]$ expressed as

$$m_j(t) = t\left(\int_{|x|>1}x_j\nu(\mathrm{d}x)+\gamma_j\right) = t\gamma_{1,j}, \quad j = 1,\ldots,d \tag{25.7}$$

differentiating $\widehat{\mu}(z)$ (Proposition 2.5(ix) , $m_j(t) = \frac{1}{i}\partial z_j|_{z=0}\widehat{\mu}_t(z)$).

$$\widehat{\mu}_t(z) = \exp\left[-t\left(\frac{1}{2}\langle z,Az\rangle + i\langle\gamma,z\rangle + \int_{\mathbb{R}^d}\left(\exp(i\langle z,x\rangle)-1-i\langle z,x\rangle I_D(x)\right)\nu(dx)\right)\right],$$

$$
\begin{aligned}
\partial z_j|_{z=0}\left(-\frac{1}{2}\langle z, Az\rangle\right) &= -\frac{1}{2}\langle \partial z_j(z), Az|_{z=0}\rangle - \frac{1}{2}\langle z|_{z=0}, A\partial z_j(z)\rangle = 0,\\
\partial z_j|_{z=0}\left(i\langle \gamma, z\rangle\right) &= i\langle \gamma, \partial z_j|_{z=0}(z)\rangle = i\langle \gamma, e_j\rangle = i\gamma_j,\\
\partial z_j|_{z=0}\left(\int_{\mathbb{R}^d}(\exp(i\langle z,x\rangle) - 1 - i\langle z,x\rangle I_D(x))\nu(dx)\right)&\\
&= \int_{\mathbb{R}^d}(ix_j - ix_j I_D(x))\nu(dx)\\
&= i\int_{D^c} x_j\nu(dx)
\end{aligned}
$$

The differentiation of the integral is calculated by using the definition of the partial derivative, L'Hôpital's rule, the dominated convergence theorem and $\int(|x|^2 \wedge 1)\nu(dx) < \infty$.

Here $\gamma_{1,j}$ is the j th component of the center γ_1 in (8.8). Similarly, $E\left[|X_t|^2\right] < \infty$ for all $t > 0$ if and only if $\int_{|x|>1}|x|^2\nu(\mathrm{d}x) < \infty$. In this case,

$$
\begin{aligned}
v_{jk}(t) &= E\left[(X_j(t) - m_j(t))\,(X_k(t) - m_k(t))\right], \quad j,k = 1,\ldots,d\\
&= E[X_j(t)X_k(t)] - m_j(t)m_k(t)\\
&= -\partial z_j\partial z_k \widehat{\mu}_t(z)|_{z=0} - m_j(t)m_k(t)\\
&= -\partial z_j\left(\partial z_k(\ln\widehat{\mu}_t(z))\widehat{\mu}_t(z)\right)|_{z=0} - m_j(t)m_k(t)\\
&= -\partial z_j\partial z_k|_{z=0}(\ln\widehat{\mu}_t(z))\widehat{\mu}_t(z) - \partial z_j(\ln\widehat{\mu}_t(z))\partial z_k(\ln\widehat{\mu}_t(z))\widehat{\mu}_t(z)|_{z=0} - m_j(t)m_k(t)\\
&= -\partial z_j\partial z_k|_{z=0}(\ln\widehat{\mu}_t(z))\cancelto{1}{\widehat{\mu}_t(z)} - \cancelto{im_j(t)}{\partial z_j(\ln\widehat{\mu}_t(z))}\cancelto{im_k(t)}{\partial z_k(\ln\widehat{\mu}_t(z))}\cancelto{1}{\widehat{\mu}_t(z)}|_{z=0} - m_j(t)m_k(t)\\
&= -\partial z_j\partial z_k|_{z=0}(\ln\widehat{\mu}_t(z))
\end{aligned}
$$

We didn't check that integration and differentiation can be exchanged.

the (j,k) elements of the covariance matrix of $X(t)$ calculated with the cumulant generating function, are expressed as

$$
v_{jk}(t) = t\left(A_{jk} + \int_{\mathbb{R}^d} x_j x_k \nu(\mathrm{d}x)\right) \tag{25.8}
$$

$$
\begin{aligned}
-\partial z_j\partial z_k\left(-\frac{t}{2}\langle z, Az\rangle\right) &= \frac{t}{2}\partial z_j\left(\langle e_k, Az\rangle + \langle z, Ae_k\rangle\right)\\
&= \frac{t}{2}\left(\langle e_k, Ae_j\rangle + \langle e_j, Ae_k\rangle\right)\\
&= tA_{jk}\\
-\partial z_j\partial z_k\left(i\langle \gamma, z\rangle\right) &= -i\partial z_j\langle \gamma, e_k\rangle\\
&= 0\\
-\partial z_j\partial z_k|_{z=0}\left(t\int_{\mathbb{R}^d}\left(\exp(i\langle z,x\rangle) - 1 - i\langle z,x\rangle I_D(x)\right)\nu(dx)\right)&\\
&= -t\int_{\mathbb{R}^d}\partial z_j\partial z_k|_{z=0}\exp(i\langle z,x\rangle)\nu(dx)\\
&= t\int_{\mathbb{R}^d} x_j x_k\nu(dx)
\end{aligned}
$$

Theorem 25.3 shows that, for a Lévy process $\{X_t\}$ with Lévy measure ν, the tails of P_{X_t} and ν have a kind of similarity. Are they actually equivalent in some class of Lévy processes? This question was answered by Embrecht, Goldie, and Veraverbeke [109] for subordinators. We state their result without proof in two remarks below.

THEOREM 25.17 (Exponential moment). Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ generated by (A, ν, γ). Let

$$C = \left\{ c \in \mathbb{R}^d : \int_{|x|>1} \mathrm{e}^{\langle c,x\rangle} \nu(\mathrm{d}x) < \infty \right\}$$

(i) The set C is convex and contains the origin.

(ii) $c \in C$ if and only if $E\mathrm{e}^{\langle c,X_t\rangle} < \infty$ for some $t > 0$ or, equivalently, for every $t > 0$.

(iii) If $w \in \mathbb{C}^d$ is such that $\operatorname{Re} w \in C$, then

$$\Psi(w) = \frac{1}{2}\langle w, Aw\rangle + \int_{\mathbb{R}^d} \left(\mathrm{e}^{\langle w,x\rangle} - 1 - \langle w, x\rangle 1_D(x) \right) \nu(\mathrm{d}x) + \langle \gamma, w\rangle \tag{25.11}$$

is definable, $E\left|\mathrm{e}^{(w,X_t)}\right| < \infty$, and

$$E\left[\mathrm{e}^{\langle \boldsymbol{w},X_t\rangle}\right] = \mathrm{e}^{t\Psi(w)} \tag{25.12}$$

Proof. (i) Obviously C contains the origin. If c_1 and c_2 are in C, then, for any $0 < r < 1$ and $s = 1 - r$,

$$\int_{|x|>1} e^{\langle rc_1+sc_2,x\rangle} \nu(\mathrm{d}x) \leq \left(\int_{|x|>1} e^{\langle c_1,x\rangle} \nu(\mathrm{d}x) \right)^{\tau} \left(\int_{|x|>1} e^{\langle c_2,x\rangle} \nu(\mathrm{d}x) \right)^{s} < \infty$$

by Hölder's inequality. Hence C is convex.

(ii) The function $g(x) = \mathrm{e}^{(c,x)}$ is clearly submultiplicative. Hence Theorem 25.3 gives the assertion.

(iii). Any linear transformation U of $\mathbb{R}^d$ to $\mathbb{R}^d$ can be uniquely extended to a linear transformation of $\mathbb{C}^d$ to $\mathbb{C}^d$. Regarding U as a $d \times d$ matrix, it is easy to see that $\langle w, Uv\rangle = \langle U'w, v\rangle$ for $w, v \in \mathbb{C}^d$, where U' is the transpose of U. Now let $\operatorname{Re} w \in C$. Then $\int_{|x|>1} \left|\mathrm{e}^{\langle \boldsymbol{w},x\rangle}\right| \nu(\mathrm{d}x) = \int_{|x|>1} \mathrm{e}^{\langle \operatorname{Re} w,x\rangle} \nu(\mathrm{d}x) < \infty$, which shows that $\Psi(w)$ of (25.11) is definable and finite. Also, $E\left|\mathrm{e}^{\langle w,X_t\rangle}\right| = Ee^{(Rev,X_t)} < \infty$ by (ii). Let us show (25.12) in three steps.

Step 1. Let e_1 be the unit vector with first component 1. Assume that $e_1 \in C$. Let us prove (25.12) for all $w = (w_j)_{1\leq j\leq d}$ with $\operatorname{Re} w_1 \in [0, 1]$ and $\operatorname{Re} w_j = 0, 2 \leq j \leq d$. Fix $t > 0$ and $w_2, \ldots, w_d \in \mathbb{C}$ with $\operatorname{Re} w_j = 0$, $2 \leq j \leq d$, and regard w_1 as variable in $F = \{w_1 \in \mathbb{C} : \operatorname{Re} w_1 \in [0, 1]\}$. Consider $f(w_1) = E\mathrm{e}^{\langle \boldsymbol{w},X_t\rangle}$. Then $f(w_1)$ is continuous on F, since

$$\left| e^{\langle \boldsymbol{w},X(t)\rangle} \right| = e^{(\operatorname{Re} w_1) X_1(t)} \leq (\operatorname{Re} w_1)\, e^{X_1(t)} + (1 - \operatorname{Re} w_1) \leq e^{X_1(t)} + 1$$

by the convexity of $\mathrm{e}^{uX_1(t)}$ in u, where $X_1(t)$ is the first component of $X(t)$. Moreover, $f(w_1)$ is analytic in the interior of F, since it is the limit of the analytic functions $E\left[e^{\langle \boldsymbol{v},X_t\rangle}; |X_t| \leq n\right]$ as $n \to \infty$. Similarly, $h(w_1) = \mathrm{e}^{\mathrm{tw}(w)}$ is continuous on F and analytic in the interior of F. If $\operatorname{Re} w_1 = 0$, then $f(w_1) = h(w_1)$, which is the Lévy-Khintchine representation of P_{X_i}. Therefore, as in the proof of Theorem 24.11, the principle of reflection and the uniqueness theorem yield (25.12) when $\operatorname{Re} w_1 \in [0, 1]$.

Step 2. Let U be a linear transformation from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Let $Y_t = UX_t$. Then $\{Y_t\}$ is a Lévy process with generating triplet (A_U, ν_U, γ_U) by Proposition 11.10. Write

$$C_U = \left\{ c \in \mathbb{R}^d : \int_{|x|>1} e^{\langle c,x \rangle} \nu_U(\mathrm{~d}x) < \infty \right\}$$

Since $\nu_U = \nu U^{-1}$, we have $C_U = (U')^{-1} C$. Given $w \in \mathbb{C}^d$ satisfying Re $w \in C$, let $v = (U')^{-1} w$. Then $\operatorname{Re} v \in C_U$. Define

$$\Psi_U(v) = \frac{1}{2} \langle v, A_U v \rangle + \int_{\mathbb{R}^d} \left(\mathrm{e}^{\langle v,x \rangle} - 1 - \langle v, x \rangle 1_D(x) \right) \nu_U(\mathrm{~d}x) + \langle \gamma_U, v \rangle$$

We claim that if

$$E\left[e^{\langle v_i Y_t \rangle} \right] = \mathrm{e}^{t \Psi_U(v)} \tag{25.13}$$

then w satisfies (25.12). In fact, $\langle v, Y_t \rangle = \left\langle \left\langle U^{-1} \right\rangle' w, Y_t \right\rangle = \langle w, X_t \rangle$ and

$$\begin{aligned} \Psi_U(v) &= \frac{1}{2} \langle U'v, AU'v \rangle + \int \left(\mathrm{e}^{\langle v,Ux \rangle} - 1 - \langle v, Ux \rangle 1_D(x) \right) v(\mathrm{~d}x) + \langle U\gamma, v \rangle \\ &= \Psi(w) \end{aligned}$$

by (11.8)-(11.10). That is, (25.13) is identical with (25.12).

Step 3. Given $w \in \mathbb{C}^d$ satisfying $\operatorname{Re} w \in C$, we shall show (25.12). If Re $w = 0$, there is nothing to prove. Assume Re $w \neq 0$. Choose a linear transformation U from $\mathbb{R}^d$ onto $\mathbb{R}^d$ such that $\operatorname{Re} w = U' e_1$. Consider the Lévy process $Y_t = UX_t$. Since $C_U = (U')^{-1} C$, we have $e_1 \in C_U$. We know, by Step 1, that, if $v \in \mathbb{C}^d$ satisfies $\operatorname{Re} v = e_1$, then (25.13) holds. Hence, by the result of Step 2, w satisfies (25.12).

$\square$

We close this section with a discussion of the g-moments of $\sup_{s \in [0,t]} |X_s|$

THEOREM 25.18. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$. Define

$$X_t^* = \sup_{s \in [0,t]} |X_s| \tag{25.14}$$

Let $g(r)$ be a nonnegative continuous submultiplicative function on $[0, \infty)$, increasing to ∞ as $r \to \infty$. Then the following four statements are equivalent.

(1) $E[g(X_t^*)] < \infty$ for some $t > 0$.

(2) $E[g(X_t^*)] < \infty$ for every $t > 0$.

(3) $E[g(|X_t|)] < \infty$ for some $t > 0$.

(4) $E[g(|X_t|)] < \infty$ for every $t > 0$.

Proof. Since $g(|x|)$ is submultiplicative on $\mathbb{R}^d$, (3) and (4) are equivalent by Theorem 25.3. As $|X_t| \leq X_t^* \implies g(|X_t|) \leq g(X_t^*)$, all we have to show is that, for any fixed $t > 0$, $E[g(|X_t|)] < \infty$ implies $E[g(X_t^*)] < \infty$. We claim that, for any $a > 0$ and $b > 0$,

$$P[X_t^* > a + b] \leq P[|X_t| > a] / P[X_t^* \leq b/2] \tag{25.15}$$

Fix t and let $t_{n,j} = jt/2^n$ for $j = 1, \ldots, 2^n$ and $X_{(n)}^* = \max_{1 \leq j \leq 2^n} \left|X_{t_{n,j}}\right|$. Choosing $Z_j(s) = Z_j = X_{t_{n,j}} - X_{t_{n,j-1}}$ in Lemma 20.2 and using Remark 20.3, we have

$$P\left[X_{(n)}^* > a + b\right] \leq P[|X_t| > a] / P\left[X_{(n)}^* \leq b/2\right]$$

in (20.2). Hence, letting $n \to \infty$, we get (25.15). Choose $b > 0$ such that $P[X_t^* \leq b/2] > 0$. Let $\tilde{g}(r)$ be a continuous increasing function on $[0, \infty)$ such that $\bar{g}(0) = 0$ and $\tilde{g}(r) = g(r)$ for $r \geq 1$. Apply Lemma 17.6 to $k(r) = 1 - P||X_t| \leq r]$ and $l(r) = \tilde{g}(r)$. Then

$$\int_{0+}^{\infty} P[|X_t| > r] \,\mathrm{d}\tilde{g}(r) = \int_{(0,\infty)} \tilde{g}(r) P[|X_t| \in \mathrm{d}r] = E[\tilde{g}(|X_t|)]$$

14 follows from (25.15) that

$$\int_{0+}^{\infty} P[X_t^* > r + b] \,\mathrm{d}\tilde{g}(r) \leq E[\tilde{g}(|X_t|)] / P[X_t^* \leq b/2]$$

The integral in the left-hand side equals

$$\int_{(0,\infty)} \tilde{g}(r) P[X_t^* - b \in \mathrm{d}r] = E[\tilde{g}(X_t^* - b); X_t^* > b]$$

similarly. Hence, if $E[g(|X_t|)] < \infty$, then $E[g(X_t^* - b); X_t^* > b] < \infty$ and, by the submultiplicativity of g, $E[g(X_t^*)] < \infty$.

□

References

Sato, Ken-iti (2013). *Lévy processes and infinitely divisible distributions*. en. Revisted edition, corrected paperback edition. Cambridge studies in advanced mathematics 68. Cambridge: Cambridge Univ. Press. ISBN: 978-0-521-55302-5 978-1-107-65649-9.