Assignment Levy processes

Abstract

We elaborate some results from chapter 25 of Sato's *Lévy processes and infinitely divisible distributions* Sato 2013. Chapter 25 of Sato 2013 discusses properties of g-moments of Lévy processes and it relation with the Lévy measure.

1 Proof of Existence of Moments from Smoothness of Characteristic Function

In this section we prove the partial reverse of the existence of moments implies smoothness of the characteristic function.

Definition 1.0.1 (Δ_h^k)
We define $\forall h > 0 : \Delta_h$ (h-central difference) of $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$\Delta_h f(t) = f(t+h) - f(t-h), \tag{1}$$

and $\forall k \in \mathbb{N} : \Delta_h^k$ is defined as applying Δ_h applying k times and $k = 0$ as the identity. Note that Δ_h^k is a linear operator. We will always take differences in respect to the t variable.

Lemma 1.0.2 ($\Delta_h^k e^{at}$)

$$\forall k \in \mathbb{N} : \Delta_h^k e^{at} = e^{at} \left(e^{ah} - e^{-ah} \right)^k. \tag{2}$$

Proof. This follows simply by induction. $k = 0$ is trivial. Base case:

$$\begin{aligned} \Delta_h e^{at} &= \left(e^{a(t+h)} - e^{a(t-h)} \right) && (3) \\ &= e^{at} \left(e^{ah} - e^{-ah} \right). && (4) \end{aligned}$$

Induction hypothesis:

$$\Delta_h^{k-1} e^{at} = e^{at} \left(e^{ah} - e^{-ah} \right)^{k-1}. \tag{5}$$

Induction step:

$$\begin{aligned} \Delta_h^k e^{at} &= \Delta_h \Delta_h^{k-1} (e^{at}) && (6) \\ &= \Delta_h \left(e^{at} \left(e^{ah} - e^{-ah} \right)^{k-1} \right) && (7) \\ &= \Delta_h \left(e^{at} \right) \left(e^{ah} - e^{-ah} \right)^{k-1} && (8) \\ &= e^{at} \left(e^{ah} - e^{-ah} \right) \left(e^{ah} - e^{-ah} \right)^{k-1} && (9) \\ &= e^{at} \left(e^{ah} - e^{-ah} \right)^k. && (10) \end{aligned}$$

□

Theorem 1.0.3 (central finite difference formula)
$\forall k \in \mathbb{N}\ \forall$ k-continuously differentiable f's at 0:

$$f^{(k)}(0) = \lim_{h\to 0} \frac{\Delta_h^k f(t)|_{t=0}}{(2h)^k}. \tag{11}$$

Proof. Follows by the mean value theorem.

$$\begin{aligned}
&\lim_{h\to 0} \frac{\Delta_h^k f(t)|_{t=0}}{(2h)^k} && (12)\\
&= \lim_{h\to 0} \frac{\Delta_h \Delta_h^{k-1} f(t)|_{t=0}}{(2h)^k} && (13)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-1} f(t)(|_{t=h} - |_{t=-h})}{(2h)^k} && (14)\\
&= \lim_{h\to 0} \frac{2h(\Delta_h^{k-1} f(t))'|_{t=z_h^1}}{(2h)^k}, z_h^1 \in [-h, h] && (15)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-1} f'(t)|_{t=z_h^1}}{(2h)^{k-1}}, z_h^1 \in [-h, h] && (16)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-2} f^{(2)}(t)|_{t=z_h^2}}{(2h)^{k-2}}, z_h^2 \in [z_h^1 - h, z_h^1 + h] \subset [-2h, 2h] && (17)\\
&\ \ldots && (18)\\
&= \lim_{h\to 0} f^{(k)}(t)|_{t=z_h^k}, z_h^k \in [-kh, kh] && (19)\\
&= \lim_{h\to 0} f^{(k)}(z_h^k) = f^{(k)}(0). && (20)
\end{aligned}$$

Last line follows because f^k is continuous at 0. This proof may need some modification because we need to assume that there exist a neighborhood around 0 where f is k-times continuously differentiable instead only k-times continuously differentiable at 0. □

Lemma 1.0.4
$\forall k \in \mathbb{N} : \phi_X(t)$ is $2k$-time continuous differentiable $\Rightarrow E[X^{2k}] < \infty$

Proof. Use the central finite difference formula and previous lemma:

$$\begin{aligned}
|\phi_X^{(2k)}(0)| &= \left| \lim_{h\to 0} \frac{\Delta_h^{2k}|_{t=0} \phi_X(t)}{(2h)^{2k}} \right| && (21)\\
&= \left| \lim_{h\to 0} \frac{\Delta_h^{2k}|_{t=0} E\left[e^{itX}\right]}{(2h)^{2k}} \right| && (22)\\
&= \left| \lim_{h\to 0} \frac{E\left[\Delta_h^{2k}|_{t=0} e^{itX}\right]}{(2h)^{2k}} \right| && (23)\\
&= \left| \lim_{h\to 0} \frac{E\left[e^{itX}|_{t=0}(e^{ihX} - e^{-ihX})^{2k}\right]}{(2h)^{2k}} \right| && (24)\\
&= \left| \lim_{h\to 0} \frac{E\left[(e^{ihX} - e^{-ihX})^{2k}\right]}{(2h)^{2k}} \right| && (25)\\
&= \left| \lim_{h\to 0} \frac{1}{i^{2k}} E\left[\frac{(e^{ihX} - e^{-ihX})^{2k}}{(2h)^{2k}} \right] \right| && (26)
\end{aligned}$$

$$= \lim_{h\to 0} E\left[\frac{\sin(Xh)^{2k}}{h^{2k}}\right] \tag{27}$$

$$= \lim_{h\to 0} E\left[X^{2k}\frac{\sin(Xh)^{2k}}{X^{2k}h^{2k}}\right] \tag{28}$$

$$= \lim_{h\to 0} E\left[X^{2k}\,\mathrm{sinc}(Xh)^{2k}\right] \tag{29}$$

$$\geq E\left[X^{2k}\right] \tag{30}$$

Where last line follows by Fatou's lemma. □

2 Submultiplicative Functions

In this section we introduce submultiplicative functions and prove some properties of them.

Definition 2.0.1 (submultiplicative function)
A function $g(x)$ on $\mathbb{R}^d$ is submultiplicative if and only if it is nonnegative and $\exists a \in \mathbb{R} > 0$:

$$g(x+y) \leq a g(x) g(y) \text{ for } x, y \in \mathbb{R}^d. \tag{31}$$

Lemma 2.0.2 (induction on submultiplicativity)
Let $g(x)$ be a submultiplicative with constant a then:

$$\forall (x_i)_i \subset \mathbb{R}^d : g\left(\sum_{i=1}^n x_i\right) \leq a^{n-1}\prod_{i=1}^n g(x_i). \tag{32}$$

Proof. This follows by induction on n.
Base case: $n = 2$ is true by submultiplicativity.
Induction hypothesis: assume $2 \leq n \leq k$.
Induction step: $n = k + 1$:

$$\forall (x_i)_i \subset \mathbb{R}^d : \tag{33}$$

$$g\left(\sum_{i=1}^{k+1} x_i\right) = g\left(\sum_{i=1}^{k} x_i + x_{k+1}\right) \tag{34}$$

$$\leq a g\left(\sum_{i=1}^{k} x_i\right) g(x_{k+1}) \text{ by submultiplicativity} \tag{35}$$

$$\leq a a^{k-1}\prod_{i=1}^{k} g\left(x_i\right) g(x_{k+1}) \text{ by induction hypothesis} \tag{36}$$

$$\leq a^k \prod_{i=1}^{k+1} g\left(x_i\right). \tag{37}$$

□

Lemma 2.0.3
The product of two submultiplicative functions is submultiplicative.

Proof. Let $g, h : \mathbb{R}^d \to \mathbb{R}$ be submultiplicative functions then:

$$\forall x, y \in \mathbb{R}^d : g(x+y)h(x+y) \leq a_1 g(x)g(y)a_2 h(x)h(y) \leq a_1 a_2 g(x)h(x)g(y)h(y).$$

So $gh : \mathbb{R}^d \to \mathbb{R}$ is submultiplicative. □

Lemma 2.0.4
If $g(x)$ is submultiplicative on $\mathbb{R}^d$, then so is $g(cx+\gamma)^\alpha$ with $c \in \mathbb{R}$, $\gamma \in \mathbb{R}^d$, and $\alpha > 0$.

Proof. Let $g_1(x) = g(cx), g_2(x) = g(x+\gamma)$, and $g_3(x) = g(x)^\alpha$.

$$
\begin{aligned}
&\forall x, y \in \mathbb{R}^d : && (38)\\
&g_1(x+y) = g(c(x+y)) = g(cx+cy) \le ag(cx)g(cy) = ag_1(x)g_1(y) && (39)\\
&g_2(x+y) = g(x+y+\gamma) = g(x+\gamma+y+\gamma-\gamma) \le a^2 g(-\gamma)g(x+\gamma)g(y+\gamma) && (40)\\
&\qquad \le a^2 g(-\gamma)g_2(x)g_2(y) && (41)\\
&g_3(x+y) = g(x+y)^\alpha \le (ag(x)g(y))^\alpha = a^\alpha g(x)^\alpha g(y)^\alpha = a^\alpha g_3(x)g_3(y). && (42)
\end{aligned}
$$

□

Example 2.0.5 (submultiplicative functions)
The class of submultiplicative functions is broad. Let $0 < \beta \le 1$. Then the following functions are submultiplicative:

$$|x| \vee 1, |x_j| \vee 1, x_j \vee 1, \exp\left(|x|^\beta\right), \exp\left(|x_j|^\beta\right) \tag{43}$$

$$\exp\left((x_j \vee 0)^\beta\right), \log(|x| \vee e), \log\left(|x_j| \vee e\right), \log\left(x_j \vee \mathrm{e}\right) \tag{44}$$

$$\log\log\left(|x| \vee e^e\right), \log\log\left(|x_j| \vee e^e\right), \log\log\left(x_j \vee e^e\right) \tag{45}$$

Proof. See Sato 2013 page 159 PROPOSITION 25.4 (iii). □

3 Finiteness of the g-moments of a Lévy process

In this section we characterize the finiteness of the g-moment of a Lévy process in terms of the Lévy measure.

Definition 3.0.1 (g-moment)
Let $g(x)$ be a nonnegative measurable function on $\mathbb{R}^d$. We call $\int g(x)\mu(\mathrm{d}x)$ the g-moment of a measure μ on $\mathbb{R}^d$. We call $E[g(X)]$ the g-moment of a random variable X on $\mathbb{R}^d$.

Definition 3.0.2 (locally bounded function)
A function is locally bounded if and only if it is bounded on every compact set.

The main result of the assignment is following theorem.

Theorem 3.0.3 (charaterization of finiteness of the g-moment of a Lévy process)
Let g be a submultiplicative, locally bounded, measurable function on $\mathbb{R}^d$ and let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure ν. Then, X_t has finite g-moment for every $t > 0$ if and only if $[\nu]_{\{|x|>1\}}$ has finite g-moment.

Corollary 3.0.4 (time independent property)
Let g be a submultiplicative, locally bounded, measurable function on $\mathbb{R}^d$. Then, finiteness of the g-moment is not a time dependent distributional property in the class of Lévy processes.

Proof. Follows directly from Theorem 3.0.3. □

We prove Theorem 3.0.3 after three lemmas.

Lemma 3.0.5 (exponential bound for submultiplicative functions)
If $g(x)$ is submultiplicative and locally bounded, then

$$g(x) \leq be^{c|x|} \tag{46}$$

with some $b > 0$ and $c > 0$.

Proof. $\sup_{|x|\leq 1} g(x) < \infty$ because g is locally bounded, let $b = \max(\sup_{|x|\leq 1} g(x), \frac{1}{a} + 1)$ such that $\sup_{|x|\leq 1} g(x) \leq b$ and $ab > 1$. $\forall x$ set $n_x = \lceil |x| \rceil$ implying $n_x - 1 < |x| \leq n_x, |\frac{x}{n_x}| \leq 1$ and $g(\frac{x}{n_x}) \leq b$, then

$$g(x) = g\left(\sum_{i=1}^{n_x} \frac{x}{n_x}\right) \leq a^{n_x - 1} g\left(\frac{x}{n_x}\right)^{n_x} \leq a^{n_x - 1} b^{n_x} \leq b(ab)^{|x|} \leq be^{\ln(ab)|x|}. \tag{47}$$

□

Lemma 3.0.6
Let μ be an infinitely divisible distribution on $\mathbb{R}$ with Lévy measure ν supported on a bounded set $S \subset [-a, a]$ for some $a > 0 \in \mathbb{R}$. Then $\widehat{\mu}(t)$ can be extended to an entire function on $\mathbb{C}$.

Proof. The Lévy representation of $\widehat{\mu}(t)$ is

$$\widehat{\mu}(t) = \exp\left[-\frac{1}{2}At^2 + \int_{[-a,a]} \left(\mathrm{e}^{itx} - 1 - itx\right) \nu(dx) + i\gamma' t\right] \tag{48}$$

$$= \exp\left[-\frac{1}{2}At^2 + i\gamma' t\right] \exp\left[\int_{-a}^{a} \left(\mathrm{e}^{itx} - 1 - itx\right) \nu(dx)\right] \tag{49}$$

with some $\gamma' \in \mathbb{R}$.
As products, sums, compositions and integrals of entire functions are entire (integrals follow from Morera's theorem), exp and polynomials are entire it is sufficient to show that $\int_{-a}^{a} \left(\mathrm{e}^{itx} - 1 - itx\right) \nu(dx)$ is finite for $t \in \mathbb{C}$:

$$\left|\int_{-a}^{a} \left(\mathrm{e}^{itx} - 1 - itx\right) \nu(dx)\right| \tag{50}$$

$$\leq \int_{-a}^{a} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) \tag{51}$$

$$\leq \int_{-1}^{1} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) + \int_{[-a,a]\setminus[-1,1]} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) \tag{52}$$

$$\leq \int_{-1}^{1} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) + \sup_{x\in[-a,a]\setminus[-1,1]} \left(|\mathrm{e}^{itx} - 1 - itx|\right) \int_{[-a,a]\setminus[-1,1]} \nu(dx) \tag{53}$$

$$\leq \int_{-1}^{1} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) + I_2 \tag{54}$$

$$\leq \int_{-1}^{1} \left|\sum_{n=2}^{\infty} \frac{(ixt)^n}{n!}\right| \nu(dx) + I_2 \tag{55}$$

$$\leq \int_{-1}^{1} \sum_{n=2}^{\infty} \frac{|x|^n |t|^n}{n!} \nu(dx) + I_2 \tag{56}$$

$$\leq \int_{-1}^{1} |x|^2 \sum_{n=2}^{\infty} \frac{|x|^{n-2}|t|^n}{n!} \nu(dx) + I_2 \tag{57}$$

$$\leq \int_{-1}^{1} |x|^2 \sum_{n=2}^{\infty} \frac{|t|^n}{n!} \nu(dx) + I_2 \tag{58}$$

$$\leq \sum_{n=2}^{\infty} \frac{|t|^n}{n!} \int_{-1}^{1} |x|^2 \nu(dx) + I_2 \tag{59}$$

$$\leq (e^{|t|} - |t| - 1) \int_{-1}^{1} |x|^2 \nu(dx) + I_2 \tag{60}$$

$$< \infty. \tag{61}$$

We implicitly used $|x| < 1 \implies |x|^{n-2} < 1$,$|\mathrm{e}^{itx} - 1 - itx|$ continuous in x so achieves maximum on compact sets (Weierstrass theorem) and

$$\int_{\mathbb{R}} (|x|^2 \wedge 1)\nu(dx) < \infty \implies \int_{-1}^{1} |x|^2 \nu(dx) < \infty, \int_{[-a,a]\setminus[-1,1]} \nu(dx) < \infty. \tag{62}$$

□

Lemma 3.0.7 (entire characteristic functions imply finite exponential moments)
If μ is a probability measure on $\mathbb{R}$ and $\widehat{\mu}(z)$ is extendible to an entire function on $\mathbb{C}$, then μ has finite exponential moments, that is, it has finite $e^{c|x|}$-moment for every $c > 0$.

Proof. As entire functions are infinitely differentiable it follows from Lemma 1.0.4 that $\alpha_{2n} = \int x^{2n} \mu(\mathrm{d}x) = \int |x|^{2n} \mu(\mathrm{d}x)$ are finite for any $n \geq 1$ and by Hölder's inequality the uneven (absolute) moments $(\beta_{2n+1}), \alpha_{2n+1}$ are also finite. Since $\frac{\mathrm{d}^n \widehat{\mu}}{\mathrm{d}z^n}(0) = \mathrm{i}^n \alpha_n$, we have

$$\widehat{\mu}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{i}^n \alpha_n z^n$$

as $\widehat{\mu}(z)$ is entire the radius of convergence of the right-hand side is infinite therefore $\forall z \in \mathbb{C} : \left|\frac{z^n \alpha_n}{n!}\right| = \frac{|z|^n}{n!} \alpha_n \to 0$ and again by Hölder's inequality also $\forall c \in \mathbb{R} : \frac{2^n c^n}{n!} \beta_n \to 0 \implies \forall \delta > 0 \in \mathbb{R}, \exists n_c \in \mathbb{N}, \forall n > n_\delta \in \mathbb{N} : \frac{2^n c^n}{n!} \beta_n < \delta$. Now the exponential moments are easily bounded:

$$\int \mathrm{e}^{c|x|} \mu(\mathrm{d}x) = \int \sum_{n=0}^{\infty} \frac{c^n |x|^n}{n!} \mu(\mathrm{d}x) \tag{63}$$

$$= \sum_{n=0}^{\infty} \frac{c^n}{n!} \int |x|^n \mu(\mathrm{d}x) \tag{64}$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{n!} \beta_n c^n \tag{65}$$

$$\leq \sum_{n=0}^{n_\delta} \frac{1}{n!} \beta_n c^n + \sum_{n=n_\delta}^{\infty} \frac{1}{n!} \beta_n c^n \tag{66}$$

$$\leq \sum_{n=0}^{n_\delta} \frac{1}{n!} \beta_n c^n + \sum_{n=n_\delta}^{\infty} \frac{\delta}{2^n} \tag{67}$$

$$< \infty. \tag{68}$$

Exchanging the sum and the integral is justified by Tonelli's theorem as the summand is positive. □

Corollary 3.0.8

A Lévy process with a Lévy measure supported on a bounded set has finite exponential moments.

Proof. Follows from Lemma's 1.0.4, 3.0.6. □

Proof of Theorem 25.3. Let $\nu_0 = [\nu]_{\{|x|\le 1\}}$ and $\nu_1 = [\nu]_{\{|x|>1\}}$. Construct independent Lévy processes $\{X_t^0\}$ and $\{X_t^1\}$ on $\mathbb{R}^d$ such that $\{X_t\} =_d \{X_t^0 + X_t^1\}$ and $\{X_t^1\}$ is compound Poisson with Lévy measure ν_1. Let μ_0^t and μ_1^t be the distributions of X_t^0 and X_t^1, respectively.
Suppose that X_t has finite g-moment for some $t > 0$. It follows from

$$\infty > E\left[g\left(X_t\right)\right] = E\left[g\left(X_t^0 + X_t^1\right)\right] = \iint g(x+y)\mu_0^t(dx)\mu_1^t(dy)$$

that $\int g(x+y)\mu_1^t(dy) < \infty$ for some x. This means Because μ_1^t is a compound Poisson we have (see (24.1) condition on the Poisson distribution of the compound Poisson process)

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(x+y)\nu_1^n(dy) < \infty.$$

Since $g(y) \le ag(-x)g(x+y) \le abe^{c|x|}g(x+y)$ by Lemma 25.5, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(y)\nu_1^n(dy) < \infty. \tag{25.4}$$

Hence $\int g(y)\nu_1(dy) < \infty$.
Conversely, suppose that $\int g(y)\nu_1(dy) = G < \infty$. Let us prove that $E\left[g\left(X_t\right)\right] < \infty$ for every t. By the submultiplicativity,

$$\begin{aligned} E\left[g\left(X_t\right)\right] &= E\left[g\left(X_t^0 + X_t^1\right)\right] \\ &\le aE\left[g\left(X_t^0\right)g\left(X_t^1\right)\right] \\ &= aE\left[g\left(X_t^0\right)\right]E\left[g\left(X_t^1\right)\right] \\ &\le abE\left[e^{c\left|X_t^0\right|}\right]E\left[g\left(X_t^1\right)\right] \end{aligned}$$

So it is sufficient to show that $E\left[g\left(X_t^1\right)\right] < \infty$ and $E\left[e^{c\left|X_t^0\right|}\right] < \infty$

$$\begin{aligned} E[g(X_t^1)] &= \int g(x)\mu_1^t(dx) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(x)\nu_1^n(dx) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int g\left(\sum_j x_j\right) \prod_j \nu_1\left(dx_j\right) \\ &\le \sum_{n=0}^{\infty} \frac{t^n}{n!} a^{n-1} \int \prod_j g\left(x_j\right) \prod_j \nu_1\left(dx_j\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} a^{n-1} \left(\int g(x) \nu_1(dx) \right)^n \\
&= a^{-1} \sum_{n=0}^{\infty} \frac{t^n}{n!} a^n G^n \\
&= a^{-1} e^{aGt} < \infty.
\end{aligned}
$$

$$
\begin{aligned}
\int g(y) \nu_1^n(dy) &= \int \cdots \int g\left(y_1 + \cdots + y_n\right) \nu_1\left(dy_1\right) \ldots \nu_1\left(dy_n\right) \\
&\leq a^{n-1} \left(\int g(y) \nu_1(dy) \right)^n
\end{aligned}
$$

Hence we have (25.4) for every t. That is, X_t^1 has finite g-moment. Since by (25.1) and (25.3), it remains only to show that $E\left[e^{c\left|X_t^0\right|}\right] < \infty$.

Let $X_j^0(t)$, $1 \leq j \leq d$, be the components of X_t^0. Then

$$
\begin{aligned}
E\left[e^{c\left|X_t^0\right|}\right] &\leq E\left[\exp\left(c \sum_{j=1}^{d} \left|X_j^0(t)\right|\right)\right] \leq E\left[\prod_{j=1}^{d} \left(e^{cX_j^0(t)} + e^{-cX_j^0(t)}\right)\right] \\
&\leq E\left[\sum_{\alpha_j \in \{-1,1\}} e^{c \sum_{j=1}^{d} \alpha_j X_j^0(t)}\right] \leq \sum_{\alpha_j \in \{-1,1\}} E\left[e^{c \sum_{j=1}^{d} \alpha_j X_j^0(t)}\right]
\end{aligned}
$$

which is written as a sum of a finite number of terms of the form $E\left[\exp X_t^\sharp\right]$ with $X_t^\sharp$ being a linear combination of $X_j^0(t), 1 \leq j \leq d$. Since $\left\{X_t^\sharp\right\}$ is a Lévy process on $\mathbb{R}$ with Lévy measure supported on a bounded set (use Proposition 11.10), $E\left[\exp X_t^\sharp\right]$ is finite by virtue of Lemmas 25.6 and 25.7. This proves all statements in the theorem.

Since X_t^0 is a Lévy process with a Lévy measure supported on a bounded set, by proposition 11.10 is $\sum_{j=1}^{d} \alpha_j X_j^0(t)$ a Lévy process on $\mathbb{R}$ with a Lévy measure supported on a bounded set therefore it has finite (exponential) moments by lemma 25.7. I.e. $E\left[e^{c\left|X_t^0\right|}\right] \leq E\left[e^{c \sum_{j=1}^{d} \alpha_j X_j^0(t)}\right] < \infty$. □

Corollary 25.8. Let $\alpha > 0, 0 < \beta \leq 1$, and $\gamma \geq 0$. None of the properties $\int |x|^\alpha \mu(\mathrm{d}x) < \infty$, $\int (0 \vee \log |x|)^\alpha \mu(\mathrm{d}x) < \infty$, and $\int |x|^\gamma e^{\alpha |x|^\beta} \mu(\mathrm{d}x) < \infty$ is time dependent in the class of Lévy processes. For a Lévy process on $\mathbb{R}^d$ with Lévy measure ν, each of the properties is expressed by the corresponding property of $[\nu]_{\{|x|>1\}}$.

Proof. This follows from Theorem 25.3 and Proposition 25.4. □

EXAMPLE 25.12. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ generated by (A, ν, γ). In components, $X_t = \left(X_j(t)\right), \gamma = \left(\gamma_j\right)$, and $A = \left(A_{jk}\right)$. Then X_t has finite mean for $t > 0$ if and only if $\int_{\{|x|>1\}} |x| \nu(\mathrm{d}x) < \infty$. When this condition is met, we can find $m_j(t) = E\left[X_j(t)\right]$ expressed as

$$
m_j(t) = t\left(\int_{|x|>1} x_j \nu(\mathrm{d}x) + \gamma_j\right) = t\gamma_{1,j}, \quad j = 1, \ldots, d \tag{25.7}
$$

differentiating $\widehat{\mu}(z)$ (Proposition 2.5(ix) , $m_j(t) = \frac{1}{i} \partial z_j|_{z=0} \widehat{\mu_t}(z)$).

$$
\begin{aligned}
\widehat{\mu}_t(z) &= \exp\left[-t\left(\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(\exp(i\langle z, x\rangle) - 1 - i\langle z, x\rangle I_D(x)\right)\nu(dx)\right)\right], \\
\partial z_j|_{z=0}\left(-\frac{1}{2}\langle z, Az\rangle\right) &= -\frac{1}{2}\langle \partial z_j(z), Az|_{z=0}\rangle - \frac{1}{2}\langle z|_{z=0}, A\partial z_j(z)\rangle = 0, \\
\partial z_j|_{z=0}\left(i\langle \gamma, z\rangle\right) &= i\langle \gamma, \partial z_j|_{z=0}(z)\rangle = i\langle \gamma, e_j\rangle = i\gamma_j, \\
\partial z_j|_{z=0}&\left(\int_{\mathbb{R}^d}(\exp(i\langle z, x\rangle) - 1 - i\langle z, x\rangle I_D(x))\nu(dx)\right) \\
&= \int_{\mathbb{R}^d}(ix_j - ix_j I_D(x))\nu(dx) \\
&= i\int_{D^c} x_j \nu(dx)
\end{aligned}
$$

The differentiation of the integral is calculated by using the definition of the partial derivative, L'Hôpital's rule, the dominated convergence theorem and $\int(|x|^2 \wedge 1)\nu(dx) < \infty$.
Here $\gamma_{1,j}$ is the j th component of the center γ_1 in (8.8). Similarly, $E\left[|X_t|^2\right] < \infty$ for all $t > 0$ if and only if $\int_{|x|>1}|x|^2\nu(\mathrm{d}x) < \infty$. In this case,

$$
\begin{aligned}
v_{jk}(t) &= E\left[\left(X_j(t) - m_j(t)\right)\left(X_k(t) - m_k(t)\right)\right], \quad j, k = 1, \ldots, d \\
&= E[X_j(t)X_k(t)] - m_j(t)m_k(t) \\
&= -\partial z_j \partial z_k \widehat{\mu}_t(z)|_{z=0} - m_j(t)m_k(t) \\
&= -\partial z_j\left(\partial z_k(\ln\widehat{\mu}_t(z))\widehat{\mu}_t(z)\right)|_{z=0} - m_j(t)m_k(t) \\
&= -\partial z_j \partial z_k|_{z=0}(\ln\widehat{\mu}_t(z))\widehat{\mu}_t(z) - \partial z_j(\ln\widehat{\mu}_t(z))\partial z_k(\ln\widehat{\mu}_t(z))\widehat{\mu}_t(z)|_{z=0} - m_j(t)m_k(t) \\
&= -\partial z_j \partial z_k|_{z=0}(\ln\widehat{\mu}_t(z))\cancelto{1}{\widehat{\mu}_t(z)} - \cancelto{im_j(t)}{\partial z_j(\ln\widehat{\mu}_t(z))}\cancelto{im_k(t)}{\partial z_k(\ln\widehat{\mu}_t(z))}\;\cancelto{1}{\widehat{\mu}_t(z)}|_{z=0} - m_j(t)m_k(t) \\
&= -\partial z_j \partial z_k|_{z=0}(\ln\widehat{\mu}_t(z))
\end{aligned}
$$

We didn't check that integration and differentiation can be exchanged.
the (j, k) elements of the covariance matrix of $X(t)$ calculated with the cumulant generating function, are expressed as

$$
v_{jk}(t) = t\left(A_{jk} + \int_{\mathbb{R}^d} x_j x_k \nu(\mathrm{d}x)\right) \tag{25.8}
$$

$$
\begin{aligned}
-\partial z_j \partial z_k\left(-\frac{t}{2}\langle z, Az\rangle\right) &= \frac{t}{2}\partial z_j\left(\langle e_k, Az\rangle + \langle z, Ae_k\rangle\right) \\
&= \frac{t}{2}\left(\langle e_k, Ae_j\rangle + \langle e_j, Ae_k\rangle\right) \\
&= tA_{jk} \\
-\partial z_j \partial z_k\left(i\langle \gamma, z\rangle\right) &= -i\partial z_j\langle \gamma, e_k\rangle \\
&= 0 \\
-\partial z_j \partial z_k|_{z=0}&\left(t\int_{\mathbb{R}^d}\left(\exp(i\langle z, x\rangle) - 1 - i\langle z, x\rangle I_D(x)\right)\nu(dx)\right) \\
&= -t\int_{\mathbb{R}^d}\partial z_j \partial z_k|_{z=0}\exp(i\langle z, x\rangle)\nu(dx) \\
&= t\int_{\mathbb{R}^d} x_j x_k \nu(dx)
\end{aligned}
$$

Theorem 25.3 shows that, for a Lévy process $\{X_t\}$ with Lévy measure ν, the tails of P_{X_t} and ν have a kind of similarity. Are they actually equivalent in some class of Lévy processes? This question was answered by Embrecht, Goldie, and Veraverbeke [109] for subordinators. We state their result without proof in two remarks below.

THEOREM 25.17 (Exponential moment). Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ generated by (A, ν, γ). Let

$$C = \left\{ c \in \mathbb{R}^d : \int_{|x|>1} \mathrm{e}^{\langle c,x\rangle} \nu(\mathrm{d}x) < \infty \right\}$$

(i) The set C is convex and contains the origin.

(ii) $c \in C$ if and only if $E\mathrm{e}^{\langle c,X_t\rangle} < \infty$ for some $t > 0$ or, equivalently, for every $t > 0$.

(iii) If $w \in \mathbb{C}^d$ is such that $\operatorname{Re} w \in C$, then

$$\Psi(w) = \frac{1}{2}\langle w, Aw\rangle + \int_{\mathbb{R}^d} \left(\mathrm{e}^{\langle w,x\rangle} - 1 - \langle w, x\rangle 1_D(x) \right) \nu(\mathrm{d}x) + \langle \gamma, w\rangle \tag{25.11}$$

is definable, $E\left|\mathrm{e}^{(w,X_t)}\right| < \infty$, and

$$E\left[\mathrm{e}^{\langle \boldsymbol{w},X_t\rangle}\right] = \mathrm{e}^{t\Psi(w)} \tag{25.12}$$

Proof. (i) Obviously C contains the origin. If c_1 and c_2 are in C, then, for any $0 < r < 1$ and $s = 1 - r$,

$$\int_{|x|>1} e^{\langle rc_1+sc_2,x\rangle} \nu(\mathrm{d}x) \le \left(\int_{|x|>1} e^{\langle c_1,x\rangle} \nu(\mathrm{d}x) \right)^{\tau} \left(\int_{|x|>1} e^{\langle c_2,x\rangle} \nu(\mathrm{d}x) \right)^{s} < \infty$$

by Hölder's inequality. Hence C is convex.

(ii) The function $g(x) = \mathrm{e}^{(c,x)}$ is clearly submultiplicative. Hence Theorem 25.3 gives the assertion.

(iii). Any linear transformation U of $\mathbb{R}^d$ to $\mathbb{R}^d$ can be uniquely extended to a linear transformation of $\mathbb{C}^d$ to $\mathbb{C}^d$. Regarding U as a $d \times d$ matrix, it is easy to see that $\langle w, Uv\rangle = \langle U'w, v\rangle$ for $w, v \in \mathbb{C}^d$, where U' is the transpose of U. Now let $\operatorname{Re} w \in C$. Then $\int_{|x|>1} \left|\mathrm{e}^{\langle \boldsymbol{w},x\rangle}\right| \nu(\mathrm{d}x) = \int_{|x|>1} \mathrm{e}^{\langle \operatorname{Re} w,x\rangle} \nu(\mathrm{d}x) < \infty$, which shows that $\Psi(w)$ of (25.11) is definable and finite. Also, $E\left|\mathrm{e}^{\langle w,X_t\rangle}\right| = Ee^{\langle Rev,X_t\rangle} < \infty$ by (ii). Let us show (25.12) in three steps.

Step 1. Let e_1 be the unit vector with first component 1. Assume that $e_1 \in C$. Let us prove (25.12) for all $w = (w_j)_{1 \le j \le d}$ with $\operatorname{Re} w_1 \in [0, 1]$ and $\operatorname{Re} w_j = 0, 2 \le j \le d$. Fix $t > 0$ and $w_2, \ldots, w_d \in \mathbb{C}$ with $\operatorname{Re} w_j = 0,\ 2 \le j \le d$, and regard w_1 as variable in $F = \{w_1 \in \mathbb{C} : \operatorname{Re} w_1 \in [0, 1]\}$. Consider $f(w_1) = E\mathrm{e}^{\langle \boldsymbol{w},X_t\rangle}$. Then $f(w_1)$ is continuous on F, since

$$\left|e^{\langle \boldsymbol{w},X(t)\rangle}\right| = e^{(\operatorname{Re} w_1)X_1(t)} \le (\operatorname{Re} w_1)\, e^{X_1(t)} + (1 - \operatorname{Re} w_1) \le e^{X_1(t)} + 1$$

by the convexity of $\mathrm{e}^{uX_1(t)}$ in u, where $X_1(t)$ is the first component of $X(t)$. Moreover, $f(w_1)$ is analytic in the interior of F, since it is the limit of the analytic functions $E\left[e^{\langle \boldsymbol{v},X_t\rangle}; |X_t| \le n\right]$ as $n \to \infty$. Similarly, $h(w_1) = \mathrm{e}^{\mathrm{tw}(w)}$ is continuous on F and analytic

in the interior of F. If $\operatorname{Re} w_1 = 0$, then $f(w_1) = h(w_1)$, which is the Lévy-Khintchine representation of P_{X_i}. Therefore, as in the proof of Theorem 24.11, the principle of reflection and the uniqueness theorem yield (25.12) when $\operatorname{Re} w_1 \in [0,1]$.
Step 2. Let U be a linear transformation from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Let $Y_t = UX_t$. Then $\{Y_t\}$ is a Lévy process with generating triplet (A_U, ν_U, γ_U) by Proposition 11.10. Write

$$C_U = \left\{ c \in \mathbb{R}^d : \int_{|x|>1} e^{\langle c,x\rangle} \nu_U(\,\mathrm{d}x) < \infty \right\}$$

Since $\nu_U = \nu U^{-1}$, we have $C_U = (U')^{-1} C$. Given $w \in \mathbb{C}^d$ satisfying Re $w \in C$, let $v = (U')^{-1} w$. Then $\operatorname{Re} v \in C_U$. Define

$$\Psi_U(v) = \frac{1}{2} \langle v, A_U v\rangle + \int_{\mathbb{R}^d} \left(\mathrm{e}^{\langle v,x\rangle} - 1 - \langle v, x\rangle 1_D(x) \right) \nu_U(\,\mathrm{d}x) + \langle \gamma_U, v\rangle$$

We claim that if

$$E\left[e^{\langle v_i Y_t\rangle}\right] = \mathrm{e}^{t\Psi_U(v)} \tag{25.13}$$

then w satisfies (25.12). In fact, $\langle v, Y_t\rangle = \left\langle \left\langle U^{-1}\right\rangle' w, Y_t \right\rangle = \langle w, X_t\rangle$ and

$$\begin{aligned}\Psi_U(v) &= \frac{1}{2} \left\langle U'v, AU'v\right\rangle + \int \left(\mathrm{e}^{\langle v,Ux\rangle} - 1 - \langle v, Ux\rangle 1_D(x) \right) v(\,\mathrm{d}x) + \langle U\gamma, v\rangle \\ &= \Psi(w)\end{aligned}$$

by (11.8)-(11.10). That is, (25.13) is identical with (25.12).

Step 3. Given $w \in \mathbb{C}^d$ satisfying $\operatorname{Re} w \in C$, we shall show (25.12). If Re $w = 0$, there is nothing to prove. Assume Re $w \neq 0$. Choose a linear transformation U from $\mathbb{R}^d$ onto $\mathbb{R}^d$ such that $\operatorname{Re} w = U'e_1$. Consider the Lévy process $Y_t = UX_t$. Since $C_U = (U')^{-1} C$, we have $e_1 \in C_U$. We know, by Step 1, that, if $v \in \mathbb{C}^d$ satisfies $\operatorname{Re} v = e_1$, then (25.13) holds. Hence, by the result of Step 2, w satisfies (25.12).

$\square$

We close this section with a discussion of the g-moments of $\sup_{s\in[0,t]} |X_s|$

THEOREM 25.18. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$. Define

$$X_t^* = \sup_{s\in[0,t]} |X_s| \tag{25.14}$$

Let $g(r)$ be a nonnegative continuous submultiplicative function on $[0,\infty)$, increasing to ∞ as $r \to \infty$. Then the following four statements are equivalent.
(1) $E\left[g\left(X_t^*\right)\right] < \infty$ for some $t > 0$.

(2) $E\left[g\left(X_t^*\right)\right] < \infty$ for every $t > 0$.

(3) $E\left[g\left(|X_t|\right)\right] < \infty$ for some $t > 0$.

(4) $E\left[g\left(|X_t|\right)\right] < \infty$ for every $t > 0$.

Proof. Since $g(|x|)$ is submultiplicative on $\mathbb{R}^d$, (3) and (4) are equivalent by Theorem 25.3. As $|X_t| \leq X_t^* \implies g(|X_t|) \leq g(X_t^*)$, all we have to show is that, for any fixed $t > 0$, $E[g(|X_t|)] < \infty$ implies $E[g(X_t^*)] < \infty$. We claim that, for any $a > 0$ and $b > 0$,

$$P[X_t^* > a + b] \leq P[|X_t| > a] / P[X_t^* \leq b/2] \tag{25.15}$$

Fix t and let $t_{n,j} = jt/2^n$ for $j = 1, \ldots, 2^n$ and $X_{(n)}^* = \max_{1 \leq j \leq 2^n} \left|X_{t_{n,j}}\right|$. Choosing $Z_j(s) = Z_j = X_{t_{n,j}} - X_{t_{n,j-1}}$ in Lemma 20.2 and using Remark 20.3, we have

$$P\left[X_{(n)}^* > a + b\right] \leq P[|X_t| > a] / P\left[X_{(n)}^* \leq b/2\right]$$

in (20.2). Hence, letting $n \to \infty$, we get (25.15). Choose $b > 0$ such that $P[X_t^* \leq b/2] > 0$. Let $\tilde{g}(r)$ be a continuous increasing function on $[0, \infty)$ such that $\bar{g}(0) = 0$ and $\tilde{g}(r) = g(r)$ for $r \geq 1$. Apply Lemma 17.6 to $k(r) = 1 - P||X_t| \leq r]$ and $l(r) = \tilde{g}(r)$. Then

$$\int_{0+}^{\infty} P[|X_t| > r] \, \mathrm{d}\tilde{g}(r) = \int_{(0,\infty)} \tilde{g}(r) P[|X_t| \in \mathrm{d}r] = E[\tilde{g}(|X_t|)]$$

14 follows from (25.15) that

$$\int_{0+}^{\infty} P[X_t^* > r + b] \, \mathrm{d}\tilde{g}(r) \leq E[\tilde{g}(|X_t|)] / P[X_t^* \leq b/2]$$

The integral in the left-hand side equals

$$\int_{(0,\infty)} \tilde{g}(r) P[X_t^* - b \in \mathrm{d}r] = E[\tilde{g}(X_t^* - b); X_t^* > b]$$

similarly. Hence, if $E[g(|X_t|)] < \infty$, then $E[g(X_t^* - b); X_t^* > b] < \infty$ and, by the submultiplicativity of $g, E[g(X_t^*)] < \infty$.

$\square$

References

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