Assignment Levy processes

Abstract

We elaborate some results from chapter 25 of Sato's *Lévy processes and infinitely divisible distributions* Sato 2013. Additions are denoted in blue, while deletions are signified in red.

Proof of Existence of Moments from Smoothness of Characteristic Function

Definition 0.0.1 (Δ_h^k)
We define $\forall h > 0 : \Delta_h$ (h-central difference) of $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$\Delta_h f(t) = f(t+h) - f(t-h), \tag{1}$$

and $\forall k \in \mathbb{N} : \Delta_h^k$ is defined as applying Δ_h applying k times and $k = 0$ as the identity. Note that Δ_h^k is a linear operator. We will always take differences in respect to the t variable.

Lemma 0.0.2 ($\Delta_h^k e^{at}$)

$$\forall k \in \mathbb{N} : \Delta_h^k e^{at} = e^{at}\left(e^{ah} - e^{-ah}\right)^k. \tag{2}$$

Proof. This follows simply by induction. $k = 0$ is trivial. Base case:

$$\begin{aligned} \Delta_h e^{at} &= \left(e^{a(t+h)} - e^{a(t-h)}\right) &(3)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right). &(4)\end{aligned}$$

Induction hypothesis:

$$\Delta_h^{k-1} e^{at} = e^{at}\left(e^{ah} - e^{-ah}\right)^{k-1}. \tag{5}$$

Induction step:

$$\begin{aligned} \Delta_h^k e^{at} &= \Delta_h \Delta_h^{k-1}(e^{at}) &(6)\\ &= \Delta_h\left(e^{at}\left(e^{ah} - e^{-ah}\right)^{k-1}\right) &(7)\\ &= \Delta_h\left(e^{at}\right)\left(e^{ah} - e^{-ah}\right)^{k-1} &(8)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right)\left(e^{ah} - e^{-ah}\right)^{k-1} &(9)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right)^k. &(10)\end{aligned}$$

□

Theorem 0.0.3 (central finite difference formula)
$\forall k \in \mathbb{N}\ \forall$ k-continuously differentiable f's at 0:

$$f^{(k)}(0) = \lim_{h\to 0} \frac{\Delta_h^k f(t)|_{t=0}}{(2h)^k}. \tag{11}$$

Proof. Follows by the mean value theorem.

$$\begin{aligned}
&\lim_{h\to 0} \frac{\Delta_h^k f(t)|_{t=0}}{(2h)^k} && (12)\\
&= \lim_{h\to 0} \frac{\Delta_h \Delta_h^{k-1} f(t)|_{t=0}}{(2h)^k} && (13)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-1} f(t)(|_{t=h} - |_{t=-h})}{(2h)^k} && (14)\\
&= \lim_{h\to 0} \frac{2h(\Delta_h^{k-1} f(t))'|_{t=z_h^1}}{(2h)^k}, z_h^1 \in [-h, h] && (15)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-1} f'(t)|_{t=z_h^1}}{(2h)^{k-1}}, z_h^1 \in [-h, h] && (16)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-2} f^{(2)}(t)|_{t=z_h^2}}{(2h)^{k-2}}, z_h^2 \in [z_h^1 - h, z_h^1 + h] \subset [-2h, 2h] && (17)\\
&\ldots && (18)\\
&= \lim_{h\to 0} f^{(k)}(t)|_{t=z_h^k}, z_h^k \in [-kh, kh] && (19)\\
&= \lim_{h\to 0} f^{(k)}(z_h^k) = f^{(k)}(0). && (20)
\end{aligned}$$

Last line follows because f^k is continuous at 0. $\square$

Lemma 0.0.4
$\forall k \in \mathbb{N} : \phi_X(t)$ is $2k$-time continuous differentiable $\Rightarrow E[X^{2k}] < \infty$

Proof. Use the central finite difference formula and previous lemma:

$$\begin{aligned}
|\phi_X^{(2k)}(0)| &= \left| \lim_{h\to 0} \frac{\Delta_h^{2k}|_{t=0} \phi_X(t)}{(2h)^{2k}} \right| && (21)\\
&= \left| \lim_{h\to 0} \frac{\Delta_h^{2k}|_{t=0} E\left[e^{itX}\right]}{(2h)^{2k}} \right| && (22)\\
&= \left| \lim_{h\to 0} \frac{E\left[\Delta_h^{2k}|_{t=0} e^{itX}\right]}{(2h)^{2k}} \right| && (23)\\
&= \left| \lim_{h\to 0} \frac{E\left[e^{itX}|_{t=0}(e^{ihX} - e^{-ihX})^{2k}\right]}{(2h)^{2k}} \right| && (24)\\
&= \left| \lim_{h\to 0} \frac{E\left[(e^{ihX} - e^{-ihX})^{2k}\right]}{(2h)^{2k}} \right| && (25)\\
&= \left| \lim_{h\to 0} \frac{1}{i^{2k}} E\left[\frac{(e^{ihX} - e^{-ihX})^{2k}}{(2h)^{2k}} \right] \right| && (26)\\
&= \lim_{h\to 0} E\left[\frac{\sin(Xh)^{2k}}{h^{2k}} \right] && (27)
\end{aligned}$$

$$= \lim_{h \to 0} E\left[X^{2k} \frac{\sin(Xh)^{2k}}{X^{2k}h^{2k}}\right] \tag{28}$$

$$= \lim_{h \to 0} E\left[X^{2k} \operatorname{sinc}(Xh)^{2k}\right] \tag{29}$$

$$\geq E\left[X^{2k}\right] \tag{30}$$

Where last line follows by Fatou's lemma. □

25. Moments

We define the g-moment of a random variable and discuss finiteness of the g-moment of X_t for a Lévy process $\{X_t\}$.

DEFINITION 25.1. Let $g(x)$ be a nonnegative measurable function on $\mathbb{R}^d$. We call $\int g(x)\mu(\mathrm{d}x)$ the g-moment of a measure μ on $\mathbb{R}^d$. We call $E[g(X)]$ the g-moment of a random variable X on $\mathbb{R}^d$.

DEFINITION 25.2. A function $g(x)$ on $\mathbb{R}^d$ is called submultiplicative if it is nonnegative and there is a constant $a > 0$ such that

$$g(x+y) \leq a g(x) g(y) \text{ for } x, y \in \mathbb{R}^d \tag{25.1}$$

A function bounded on every compact set is called locally bounded.

LEMMA 25.i1 Let $g(x)$ be a submultiplicative with constant a then:

$$\forall (x_i)_i \subset \mathbb{R}^d : g\left(\sum_{i=1}^{n} x_i\right) \leq a^{n-1} \prod_{i=1}^{n} g(x_i).$$

Proof. This follows by induction on n.
Base case: $n = 2$ is true by submultiplicativity.
Induction hypothesis: assume $2 \leq n \leq k$.
Induction step: $n = k + 1$:

$$\begin{aligned}
&\forall (x_i)_i \subset \mathbb{R}^d : \\
g\left(\sum_{i=1}^{k+1} x_i\right) &= g\left(\sum_{i=1}^{k} x_i + x_{k+1}\right) \\
&\leq a g\left(\sum_{i=1}^{k} x_i\right) g(x_{k+1}) \text{ by submultiplicativity} \\
&\leq a a^{k-1} \prod_{i=1}^{k} g\left(x_i\right) g(x_{k+1}) \text{ by induction hypothesis} \\
&\leq a^{k} \prod_{i=1}^{k+1} g\left(x_i\right).
\end{aligned}$$

□

THEOREM 25.3 (g-Moment). Let g be a submultiplicative, locally bounded, measurable function on $\mathbb{R}^d$. Then, finiteness of the g-moment is not a time dependent distributional property in the class of Lévy processes. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with

Lévy measure ν. Then, X_t has finite g-moment for every $t > 0$ if and only if $[\nu]_{\{|x|>1\}}$ has finite g-moment.

The following facts indicate the wide applicability of the theorem.

PROPOSITION 25.4.
(i) The product of two submultiplicative functions is submultiplicative.
(ii) If $g(x)$ is submultiplicative on $\mathbb{R}^d$, then so is $g(cx+\gamma)^\alpha$ with $c \in \mathbb{R}$, $\gamma \in \mathbb{R}^d$, and $\alpha > 0$.
(iii) Let $0 < \beta \leq 1$. Then the following functions are submultiplicative:

$$|x| \vee 1, |x_j| \vee 1, x_j \vee 1, \exp\left(|x|^\beta\right), \exp\left(|x_j|^\beta\right)$$
$$\exp\left((x_j \vee 0)^\beta\right), \log(|x| \vee e), \log\left(|x_j| \vee e\right), \log\left(x_j \vee \mathrm{e}\right)$$
$$\log\log\left(|x| \vee e^e\right), \log\log\left(|x_j| \vee e^e\right), \log\log\left(x_j \vee e^e\right)$$

Here x_j is the j th component of x.

Proof. (i) Immediate from the definition.
Let $g, h : \mathbb{R}^d \to \mathbb{R}$ be submultiplicative functions then:

$$\forall x, y \in \mathbb{R}^d : g(x+y)h(x+y) \leq a_1 g(x)g(y)a_2 h(x)h(y) \leq a_1 a_2 g(x)h(x)g(y)h(y).$$

So $gh : \mathbb{R}^d \to \mathbb{R}$ is submultiplicative.

(ii) Let $g_1(x) = g(cx), g_2(x) = g(x+\gamma)$, and $g_3(x) = g(x)^\alpha$. Then it follows from (25.1) that $g_1(x+y) \leq a g_1(x) g_1(y)$, $g_2(x+y) \leq a^2 g(-\gamma) g_2(x) g_2(y)$, and $g_3(x+y) \leq a^\alpha g_3(x) g_3(y)$.

$$\begin{aligned}
&\forall x, y \in \mathbb{R}^d : \\
&g_1(x+y) = g(c(x+y)) = g(cx+cy) \leq a g(cx) g(cy) = a g_1(x) g_1(y) \\
&g_2(x+y) = g(x+y+\gamma) = g(x+\gamma+y+\gamma-\gamma) \leq a^2 g(-\gamma) g(x+\gamma) g(y+\gamma) \\
&\qquad\qquad \leq a^2 g(-\gamma) g_2(x) g_2(y) \\
&g_3(x+y) = g(x+y)^\alpha \leq (a g(x) g(y))^\alpha = a^\alpha g(x)^\alpha g(y)^\alpha = a^\alpha g_3(x) g_3(y).
\end{aligned}$$

(iii) Let $h(u)$ be a positive increasing function on $\mathbb{R}$ such that, for some $b \geq 0, h(u)$ is constant on $(-\infty, b]$ and $\log h(u)$ is concave on $[b, \infty)$. Then $h(u)$ is submultiplicative on $\mathbb{R}$. In fact, for $u, v \geq b \geq 0$, by concaveness the function $f(u) = \log h(u)$ satisfies

$$\frac{f(u+b)-f(u)}{b} \leq \frac{f(2b)-f(b)}{b} \Leftrightarrow f(u+b)-f(u) \leq f(2b)-f(b)$$
$$\frac{f(u+v)-f(v)}{u} \leq \frac{f(u+b)-f(b)}{u} \Leftrightarrow f(u+v)-f(v) \leq f(u+b)-f(b)$$

the first line is for $b \neq 0, b = 0$ is trivial and hence

$$\begin{aligned}
&f(u+v) \leq f(u+b)-f(b)+f(v) \leq f(2b)-2f(b)+f(u)+f(v) \Leftrightarrow \\
&h(u+v) \leq \frac{h(2b)}{h(b)^2} h(u)h(v)
\end{aligned}$$

which shows

$$h(u+v) \leq \text{ const } h(u)h(v) \tag{25.2}$$

It follows (we miss the cases for $u \geq b \geq v, b \geq u, v$) that (25.2) holds for all $u, v \in \mathbb{R}$. The functions $u \vee 1, \exp\left((u \vee 0)^{\beta}\right), \log(u \vee \mathrm{e})$, and $\log\log\left(u \vee \mathrm{e}^{\mathrm{e}}\right)$ fulfil the conditions on $h(u)$. By (25.2) and by the increasingness of h, the functions $h(|x|), h\left(|x_j|\right)$, and $h\left(x_j\right)$ are submultiplicative on $\mathbb{R}^d$.

$\square$

We prove Theorem 25.3 after three lemmas.

LEMMA 25.5. If $g(x)$ is submultiplicative and locally bounded, then

$$g(x) \leq b e^{c|x|} \tag{25.3}$$

with some $b > 0$ and $c > 0$.

Proof. Choose b in such a way that $\sup_{|x|\leq 1} g(x) \leq b$ and $ab > 1$. If $n-1 < |x| \leq n$, then $\exists b : \sup_{|x|\leq 1} g(x) \leq b$ and $ab > 1$ because g is locally bounded. $\forall x$ set $n_x = \lceil |x| \rceil$ implying $n_x - 1 < |x| \leq n_x$, $\frac{x}{n_x} \in \{u \| u| \leq 1\}$, then

$$g(x) = g\left(\sum_{i=1}^{n_x} \frac{x}{n_x}\right) \leq a^{n_x - 1} g\left(\frac{x}{n_x}\right)^{n_x} \leq a^{n_x-1} b^{n_x} \leq b(ab)^{|x|} \leq b e^{\ln(ab)|x|}$$

which shows (25.3). $\square$

LEMMA 25.6. Let μ be an infinitely divisible distribution on $\mathbb{R}^d$ with Lévy measure ν supported on a bounded set S. Then $\widehat{\mu}(t)$ can be extended to an entire function on $\mathbb{C}^d$.

Proof. There is a finite $a > 0$ such that $S_\nu \subset [-a, a]$. The Lévy representation of $\widehat{\mu}(t)$ is written as

$$\widehat{\mu}(t) = \exp\left[-\frac{1}{2}\langle t, At\rangle + \int_S \left(\mathrm{e}^{\mathrm{i}\langle t,x\rangle} - 1 - \mathrm{i}\langle t, x\rangle\right) \nu(\mathrm{d}x) + \mathrm{i}\langle \gamma', t\rangle\right]$$

with some $\gamma' \in \mathbb{R}^d$. The right-hand side is meaningful even if t is complex. Denote this function by $\Phi(t)$. Then $\Phi(t)$ is an entire function in each of its variables, since we can exchange the order of integration and differentiation and innerproducts are analytical. Thereby $\Phi(t)$ is also entire by Hartogs's theorem. $\square$

LEMMA 25.7. If μ is a probability measure on $\mathbb{R}$ and $\widehat{\mu}(z)$ is extendible to an entire function on $\mathbb{C}$, then μ has finite exponential moments, that is, it has finite $e^{c|x|}$-moment for every $c > 0$.

Proof. It follows from Proposition 2.5(x) (see 0.0.4) that $\alpha_n = \int x^n \mu(\mathrm{d}x)$ and $\beta_n = \int |x|^n \mu(\mathrm{d}x)$ are finite for any $n \geq 1$. Since $\frac{\mathrm{d}^n \widehat{\mu}}{\mathrm{d}z^n}(0) = \mathrm{i}^n \alpha_n$, we have

$$\widehat{\mu}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{i}^n \alpha_n z^n$$

the radius of convergence of the right-hand side being infinite. Notice that $\beta_{2k} = \alpha_{2k}$ and $\beta_{2k+1} \leq \frac{1}{2}\left(\alpha_{2k+2} + \alpha_{2k}\right)$, since $|x|^{2k+1} \leq \frac{1}{2}\left(x^{2k+2} + x^{2k}\right)$. It follows that

$$\int \mathrm{e}^{|x|} \mu(\mathrm{d}x) = \sum_{n=0}^{\infty} \frac{1}{n!} \beta_n c^n < \infty$$

completing the proof.

$\square$

Proof of Theorem 25.3. Let $\nu_0 = [\nu]_{\{|x|\leq 1\}}$ and $\nu_1 = [\nu]_{\{|x|>1\}}$. Construct independent Lévy processes $\{X_t^0\}$ and $\{X_t^1\}$ on $\mathbb{R}^d$ such that $\{X_t\} =_d \{X_t^0 + X_t^1\}$ and $\{X_t^1\}$ is compound Poisson with Lévy measure ν_1. Let μ_0^t and μ_1^t be the distributions of X_t^0 and X_t^1, respectively.

Suppose that X_t has finite g-moment for some $t>0$. It follows from

$$E\left[g\left(X_t\right)\right] = E\left[g\left(X_t^0 + X_t^1\right)\right] = \iint g(x+y)\mu_0^t(dx)\mu_1^t(dy)$$

that $\int g(x+y)\mu_1^t(dy) < \infty$ for some x. This means (see (24.1) in sato)

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(x+y)\nu_1^n(dy) < \infty$$

Since $g(y) \leq a g(-x) g(x+y) \leq a b e^{c|x|} g(x+y)$ by Lemma 25.5, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(y)\nu_1^n(dy) < \infty \tag{25.4}$$

Hence $\int g(y)\nu_1(dy) < \infty$.

Conversely, suppose that $\int g(y)\nu_1(dy) < \infty$. Let us prove that $E\left[g\left(X_t\right)\right] < \infty$ for every t. By the submultiplicativity,

$$\begin{aligned}\int g(y)\nu_1^n(dy) &= \int\cdots\int g\left(y_1+\cdots+y_n\right)\nu_1\left(dy_1\right)\ldots\nu_1\left(dy_n\right)\\ &\leq a^{n-1}\left(\int g(y)\nu_1(dy)\right)^n\end{aligned}$$

Hence we have (25.4) for every t. That is, X_t^1 has finite g-moment. Since

$$E\left[g\left(X_t\right)\right] \leq E\left[g\left(X_t^0 + X_t^1\right)\right] \leq abE\left[\mathrm{e}^{c\left|X_t^0\right|}\right] E\left[g\left(X_t^1\right)\right]$$

by (25.1) and (25.3), it remains only to show that $E\left[e^{c\left|X_t^0\right|}\right] < \infty$. Let $X_j^0(t)$, $1 \leq j \leq d$, be the components of X_t^0. Then

$$\begin{aligned}E\left[e^{c\left|X_t^0\right|}\right] &\leq E\left[\exp\left(c\sum_{j=1}^d\left|X_j^0(t)\right|\right)\right] \leq E\left[\prod_{j=1}^d\left(e^{cX_j^0(t)} + e^{-cX_j^0(t)}\right)\right]\\ &\leq E\left[\sum_{\alpha_j\in\{-1,1\}} e^{c\sum_{j=1}^d \alpha_j X_j^0(t)}\right] \leq \sum_{\alpha_j\in\{-1,1\}} E\left[e^{c\sum_{j=1}^d \alpha_j X_j^0(t)}\right]\end{aligned}$$

which is written as a sum of a finite number of terms of the form $E\left[\exp X_t^\sharp\right]$ with $X_t^\sharp$ being a linear combination of $X_j^0(t), 1 \leq j \leq d$. Since $\left\{X_t^\sharp\right\}$ is a Lévy process on $\mathbb{R}$ with Lévy measure supported on a bounded set (use Proposition 11.10), $E\left[\exp X_t^\sharp\right]$ is finite by virtue of Lemmas 25.6 and 25.7. This proves all statements in the theorem.

Since X_t^0 is a Lévy process with a Lévy measure supported on a bounded set, $X_j^0(t)$ is a Lévy process on $\mathbb{R}$ with a Lévy measure supported on a bounded set therefore it has finite (exponential) moments by (25.7) and so do any linear combinations $\sum_{j=1}^d \alpha_j X_j^0(t)$. I.e. $E\left[e^{c\left|X_t^0\right|}\right] \leq E\left[e^{c\sum_{j=1}^d \alpha_j X_j^0(t)}\right] < \infty$. ◻

Corollary 25.8. Let $\alpha > 0, 0 < \beta \leq 1$, and $\gamma \geq 0$. None of the properties $\int |x|^{\alpha} \mu(\mathrm{d}x) < \infty$, $\int (0 \vee \log |x|)^{\alpha} \mu(\mathrm{d}x) < \infty$, and $\int |x|^{\gamma} e^{\alpha |x|^{\beta}} \mu(\mathrm{d}x) < \infty$ is time dependent in the class of Lévy processes. For a Lévy process on $\mathbb{R}^d$ with Lévy measure ν, each of the properties is expressed by the corresponding property of $[\nu]_{\{|x|>1\}}$.

Proof. This follows from Theorem 25.3 and Proposition 25.4. □

EXAMPLE 25.12. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ generated by (A, ν, γ). In components, $X_t = (X_j(t)), \gamma = (\gamma_j)$, and $A = (A_{jk})$. Then X_t has finite mean for $t > 0$ if and only if $\int_{\{|x|>1\}} |x| \nu(\mathrm{d}x) < \infty$. When this condition is met, we can find $m_j(t) = E[X_j(t)]$ expressed as

$$m_j(t) = t \left(\int_{|x|>1} x_j \nu(\mathrm{d}x) + \gamma_j \right) = t\gamma_{1,j}, \quad j = 1, \ldots, d \tag{25.7}$$

differentiating $\widehat{\mu}(z)$ (Proposition 2.5(ix) , $m_j(t) = \frac{1}{i} \partial z_j|_{z=0} \widehat{\mu}_t(z)$).

$$\begin{aligned}
\widehat{\mu}_t(z) &= \exp\left[-t \left(\frac{1}{2} \langle z, Az \rangle + i \langle \gamma, z \rangle + \int_{\mathbb{R}^d} \left(\exp(i\langle z, x \rangle) - 1 - i \langle z, x \rangle I_D(x) \right) \nu(dx) \right) \right], \\
\partial z_j|_{z=0} \left(-\frac{1}{2} \langle z, Az \rangle \right) &= -\frac{1}{2} \langle \partial z_j(z), Az|_{z=0} \rangle - \frac{1}{2} \langle z|_{z=0}, A\partial z_j(z) \rangle = 0, \\
\partial z_j|_{z=0} \left(i \langle \gamma, z \rangle \right) &= i \langle \gamma, \partial z_j|_{z=0}(z) \rangle = i \langle \gamma, e_j \rangle = i\gamma_j, \\
\partial z_j|_{z=0} \left(\int_{\mathbb{R}^d} (\exp(i\langle z, x \rangle) - 1 - i\langle z, x \rangle I_D(x)) \nu(dx) \right) & \\
&= \int_{\mathbb{R}^d} (ix_j - ix_j I_D(x)) \nu(dx) \\
&= i \int_{D^c} x_j \nu(dx)
\end{aligned}$$

Here $\gamma_{1,j}$ is the j th component of the center γ_1 in (8.8). Similarly, $E\left[|X_t|^2\right] < \infty$ for all $t > 0$ if and only if $\int_{|x|>1} |x|^2 \nu(\mathrm{d}x) < \infty$. In this case,

$$\begin{aligned}
v_{jk}(t) &= E\left[(X_j(t) - m_j(t)) (X_k(t) - m_k(t)) \right], \quad j, k = 1, \ldots, d \\
&= E[X_j(t) X_k(t)] - m_j(t) m_k(t) \\
&= -\partial z_j \partial z_k \widehat{\mu}_t(z)|_{z=0} - m_j(t) m_k(t) \\
&= -\partial z_j \left(\partial z_k (\ln \widehat{\mu}_t(z)) \widehat{\mu}_t(z) \right)|_{z=0} - m_j(t) m_k(t) \\
&= -\partial z_j \partial z_k|_{z=0} (\ln \widehat{\mu}_t(z)) \widehat{\mu}_t(z) - \partial z_j (\ln \widehat{\mu}_t(z)) \partial z_k (\ln \widehat{\mu}_t(z)) \widehat{\mu}_t(z)|_{z=0} - m_j(t) m_k(t) \\
&= -\partial z_j \partial z_k|_{z=0} (\ln \widehat{\mu}_t(z)) \cancelto{1}{\widehat{\mu}_t(z)} - \cancelto{im_j(t)}{\partial z_j (\ln \widehat{\mu}_t(z))} \cancelto{im_k(t)}{\partial z_k (\ln \widehat{\mu}_t(z))} \cancelto{1}{\widehat{\mu}_t(z)}|_{z=0} - m_j(t) m_k(t) \\
&= -\partial z_j \partial z_k|_{z=0} (\ln \widehat{\mu}_t(z))
\end{aligned}$$

the (j, k) elements of the covariance matrix of $X(t)$ calculated with the cumulant generating function, are expressed as

$$v_{jk}(t) = t \left(A_{jk} + \int_{\mathbb{R}^d} x_j x_k \nu(\mathrm{d}x) \right) \tag{25.8}$$

$$-\partial z_j \partial z_k \left(-\frac{t}{2} \langle z, Az \rangle \right) = \frac{t}{2} \partial z_j \left(\langle e_k, Az \rangle + \langle z, Ae_k \rangle \right)$$

$$
\begin{aligned}
&= \frac{t}{2}\left(\langle e_k, Ae_j\rangle + \langle e_j, Ae_k\rangle\right)\\
&= tA_{jk}\\
-\partial z_j \partial z_k \left(i\langle \gamma, z\rangle\right) &= -i\partial z_j \langle \gamma, e_k\rangle\\
&= 0\\
-\partial z_j \partial z_k|_{z=0} \left(t\int_{\mathbb{R}^d} \left(\exp(i\langle z,x\rangle) - 1 - i\langle z,x\rangle I_D(x)\right)\nu(dx)\right)&\\
&= -t\int_{\mathbb{R}^d} \partial z_j \partial z_k|_{z=0} \exp(i\langle z,x\rangle)\nu(dx)\\
&= t\int_{\mathbb{R}^d} x_j x_k \nu(dx)
\end{aligned}
$$

Theorem 25.3 shows that, for a Lévy process $\{X_t\}$ with Lévy measure ν, the tails of P_{X_t} and ν have a kind of similarity. Are they actually equivalent in some class of Lévy processes? This question was answered by Embrecht, Goldie, and Veraverbeke [109] for subordinators. We state their result without proof in two remarks below.

THEOREM 25.17 (Exponential moment). Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ generated by (A,ν,γ). Let

$$
C = \left\{ c \in \mathbb{R}^d : \int_{|x|>1} \mathrm{e}^{\langle c,x\rangle}\nu(\mathrm{d}x) < \infty \right\}
$$

(i) The set C is convex and contains the origin.

(ii) $c \in C$ if and only if $E\mathrm{e}^{\langle c,X_t\rangle} < \infty$ for some $t>0$ or, equivalently, for every $t>0$.

(iii) If $w \in \mathbb{C}^d$ is such that $\operatorname{Re} w \in C$, then

$$
\Psi(w) = \frac{1}{2}\langle w, Aw\rangle + \int_{\mathbb{R}^d} \left(\mathrm{e}^{\langle w,x\rangle} - 1 - \langle w,x\rangle 1_D(x)\right)\nu(\mathrm{d}x) + \langle \gamma, w\rangle \tag{25.11}
$$

is definable, $E\left|\mathrm{e}^{(w,X_t)}\right| < \infty$, and

$$
E\left[\mathrm{e}^{\langle \boldsymbol{w},X_t\rangle}\right] = \mathrm{e}^{t\Psi(w)} \tag{25.12}
$$

Proof. (i) Obviously C contains the origin. If c_1 and c_2 are in C, then, for any $0<r<1$ and $s=1-r$,

$$
\int_{|x|>1} e^{\langle rc_1+sc_2,x\rangle}\nu(\mathrm{d}x) \le \left(\int_{|x|>1} e^{\langle c_1,x\rangle}\nu(\mathrm{d}x)\right)^{\tau} \left(\int_{|x|>1} e^{\langle c_2,x\rangle}\nu(\mathrm{d}x)\right)^{s} < \infty
$$

by Hölder's inequality. Hence C is convex.

(ii) The function $g(x) = \mathrm{e}^{(c,x)}$ is clearly submultiplicative. Hence Theorem 25.3 gives the assertion.

(iii). Any linear transformation U of $\mathbb{R}^d$ to $\mathbb{R}^d$ can be uniquely extended to a linear transformation of $\mathbb{C}^d$ to $\mathbb{C}^d$. Regarding U as a $d\times d$ matrix, it is easy to see that $\langle w, Uv\rangle = \langle U'w, v\rangle$ for $w,v \in \mathbb{C}^d$, where U' is the transpose of U. Now let $\operatorname{Re} w \in C$.

Then $\int_{|x|>1}\left|\mathrm{e}^{\langle \boldsymbol{w},x\rangle}\right|\nu(\mathrm{d}x)=\int_{|x|>1}\mathrm{e}^{\langle \operatorname{Re} w,x\rangle}\nu(\mathrm{d}x)<\infty$, which shows that $\Psi(w)$ of (25.11) is definable and finite. Also, $E\left|\mathrm{e}^{\langle w,X_t\rangle}\right|=Ee^{\langle Rev,X_t\rangle}<\infty$ by (ii). Let us show (25.12) in three steps.

Step 1. Let e_1 be the unit vector with first component 1. Assume that $e_1\in C$. Let us prove (25.12) for all $w=(w_j)_{1\le j\le d}$ with $\operatorname{Re} w_1\in[0,1]$ and
$\operatorname{Re} w_j=0, 2\le j\le d$. Fix $t>0$ and $w_2,\ldots,w_d\in\mathbb{C}$ with $\operatorname{Re} w_j=0,\ 2\le j\le d$, and regard w_1 as variable in $F=\{w_1\in\mathbb{C}:\operatorname{Re} w_1\in[0,1]\}$. Consider $f(w_1)=E\mathrm{e}^{\langle \boldsymbol{w},X_t\rangle}$. Then $f(w_1)$ is continuous on F, since

$$\left|e^{\langle \boldsymbol{w},X(t)\rangle}\right|=e^{(\operatorname{Re} w_1)X_1(t)}\le(\operatorname{Re} w_1)\,e^{X_1(t)}+(1-\operatorname{Re} w_1)\le e^{X_1(t)}+1$$

by the convexity of $\mathrm{e}^{uX_1(t)}$ in u, where $X_1(t)$ is the first component of $X(t)$. Moreover, $f(w_1)$ is analytic in the interior of F, since it is the limit of the analytic functions $E\left[e^{\langle \boldsymbol{v},X_t\rangle};|X_t|\le n\right]$ as $n\to\infty$. Similarly, $h(w_1)=\mathrm{e}^{\mathrm{tw}(w)}$ is continuous on F and analytic in the interior of F. If $\operatorname{Re} w_1=0$, then $f(w_1)=h(w_1)$, which is the Lévy-Khintchine representation of P_{X_i}. Therefore, as in the proof of Theorem 24.11, the principle of reflection and the uniqueness theorem yield (25.12) when $\operatorname{Re} w_1\in[0,1]$.

Step 2. Let U be a linear transformation from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Let $Y_t=UX_t$. Then $\{Y_t\}$ is a Lévy process with generating triplet (A_U,ν_U,γ_U) by Proposition 11.10. Write

$$C_U=\left\{c\in\mathbb{R}^d:\int_{|x|>1}e^{\langle c,x\rangle}\nu_U(\,\mathrm{d}x)<\infty\right\}$$

Since $\nu_U=\nu U^{-1}$, we have $C_U=(U')^{-1}C$. Given $w\in\mathbb{C}^d$ satisfying Re $w\in C$, let $v=(U')^{-1}w$. Then $\operatorname{Re} v\in C_U$. Define

$$\Psi_U(v)=\frac{1}{2}\langle v,A_Uv\rangle+\int_{\mathbb{R}^d}\left(\mathrm{e}^{\langle v,x\rangle}-1-\langle v,x\rangle 1_D(x)\right)\nu_U(\,\mathrm{d}x)+\langle\gamma_U,v\rangle$$

We claim that if

$$E\left[e^{\langle v_i Y_t\rangle}\right]=\mathrm{e}^{t\Psi_U(v)}\tag{25.13}$$

then w satisfies (25.12). In fact, $\langle v,Y_t\rangle=\left\langle\left\langle U^{-1}\right\rangle' w,Y_t\right\rangle=\langle w,X_t\rangle$ and

$$\begin{aligned}\Psi_U(v)&=\frac{1}{2}\left\langle U'v,AU'v\right\rangle+\int\left(\mathrm{e}^{\langle v,Ux\rangle}-1-\langle v,Ux\rangle 1_D(x)\right)v(\,\mathrm{d}x)+\langle U\gamma,v\rangle\\&=\Psi(w)\end{aligned}$$

by (11.8)-(11.10). That is, (25.13) is identical with (25.12).

Step 3. Given $w\in\mathbb{C}^d$ satisfying $\operatorname{Re} w\in C$, we shall show (25.12). If Re $w=0$, there is nothing to prove. Assume Re $w\neq 0$. Choose a linear transformation U from $\mathbb{R}^d$ onto $\mathbb{R}^d$ such that $\operatorname{Re} w=U'e_1$. Consider the Lévy process $Y_t=UX_t$. Since $C_U=(U')^{-1}C$, we have $e_1\in C_U$. We know, by Step 1, that, if $v\in\mathbb{C}^d$ satisfies $\operatorname{Re} v=e_1$, then (25.13) holds. Hence, by the result of Step 2, w satisfies (25.12).

□

We close this section with a discussion of the g-moments of $\sup_{s\in[0,t]}|X_s|$

THEOREM 25.18. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$. Define

$$X_t^* = \sup_{s\in[0,t]} |X_s| \tag{25.14}$$

Let $g(r)$ be a nonnegative continuous submultiplicative function on $[0, \infty)$, increasing to ∞ as $r \to \infty$. Then the following four statements are equivalent.

(1) $E[g(X_t^*)] < \infty$ for some $t > 0$.

(2) $E[g(X_t^*)] < \infty$ for every $t > 0$.

(3) $E[g(|X_t|)] < \infty$ for some $t > 0$.

(4) $E[g(|X_t|)] < \infty$ for every $t > 0$.

Proof. Since $g(|x|)$ is submultiplicative on $\mathbb{R}^d$, (3) and (4) are equivalent by Theorem 25.3. As $|X_t| \leq X_t^* \implies g(|X_t|) \leq g(X_t^*)$, all we have to show is that, for any fixed $t > 0$, $E[g(|X_t|)] < \infty$ implies $E[g(X_t^*)] < \infty$. We claim that, for any $a > 0$ and $b > 0$,

$$P[X_t^* > a + b] \leq P[|X_t| > a] / P[X_t^* \leq b/2] \tag{25.15}$$

Fix t and let $t_{n,j} = jt/2^n$ for $j = 1, \ldots, 2^n$ and $X_{(n)}^* = \max_{1\leq j\leq 2^n} \left|X_{t_{n,j}}\right|$. Choosing $Z_j(s) = Z_j = X_{t_{n,j}} - X_{t_{n,j-1}}$ in Lemma 20.2 and using Remark 20.3, we have

$$P\left[X_{(n)}^* > a + b\right] \leq P[|X_t| > a] / P\left[X_{(n)}^* \leq b/2\right]$$

in (20.2). Hence, letting $n \to \infty$, we get (25.15). Choose $b > 0$ such that $P[X_t^* \leq b/2] > 0$. Let $\tilde{g}(r)$ be a continuous increasing function on $[0, \infty)$ such that $\bar{g}(0) = 0$ and $\tilde{g}(r) = g(r)$ for $r \geq 1$. Apply Lemma 17.6 to $k(r) = 1 - P||X_t| \leq r]$ and $l(r) = \tilde{g}(r)$. Then

$$\int_{0+}^{\infty} P[|X_t| > r] \, \mathrm{d}\tilde{g}(r) = \int_{(0,\infty)} \tilde{g}(r) P[|X_t| \in \mathrm{d}r] = E[\tilde{g}(|X_t|)]$$

14 follows from (25.15) that

$$\int_{0+}^{\infty} P[X_t^* > r + b] \, \mathrm{d}\tilde{g}(r) \leq E[\tilde{g}(|X_t|)] / P[X_t^* \leq b/2]$$

The integral in the left-hand side equals

$$\int_{(0,\infty)} \tilde{g}(r) P[X_t^* - b \in \mathrm{d}r] = E[\tilde{g}(X_t^* - b); X_t^* > b]$$

similarly. Hence, if $E[g(|X_t|)] < \infty$, then $E[g(X_t^* - b); X_t^* > b] < \infty$ and, by the submultiplicativity of g, $E[g(X_t^*)] < \infty$.

□

References

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