Assignment Levy processes

Abstract

We elaborate some results from chapter 25 of Sato's *Lévy processes and infinitely divisible distributions* Sato 2013. Chapter 25 of Sato 2013 discusses properties of g-moments of Lévy processes and their relation with the Lévy measure.

1 Proof of Existence of Moments from Smoothness of Characteristic Function

In this section we prove the partial reverse of the existence of moments implies smoothness of the characteristic function.

Definition 1.0.1 (Δ_h^k)
We define $\forall h > 0 : \Delta_h$ (h-central difference) of $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$\Delta_h f(t) = f(t+h) - f(t-h), \tag{1}$$

and $\forall k \in \mathbb{N} : \Delta_h^k$ is defined as applying Δ_h applying k times and $k = 0$ as the identity. Note that Δ_h^k is a linear operator. We will always take differences in respect to the t variable.

Lemma 1.0.2 ($\Delta_h^k e^{at}$)

$$\forall k \in \mathbb{N} : \Delta_h^k e^{at} = e^{at} \left(e^{ah} - e^{-ah}\right)^k. \tag{2}$$

Proof. This follows simply by induction. $k = 0$ is trivial. Base case:

$$\begin{aligned} \Delta_h e^{at} &= \left(e^{a(t+h)} - e^{a(t-h)}\right) & (3)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right). & (4) \end{aligned}$$

Induction hypothesis:

$$\Delta_h^{k-1} e^{at} = e^{at}\left(e^{ah} - e^{-ah}\right)^{k-1}. \tag{5}$$

Induction step:

$$\begin{aligned} \Delta_h^k e^{at} &= \Delta_h \Delta_h^{k-1}(e^{at}) & (6)\\ &= \Delta_h \left(e^{at}\left(e^{ah} - e^{-ah}\right)^{k-1}\right) & (7)\\ &= \Delta_h \left(e^{at}\right)\left(e^{ah} - e^{-ah}\right)^{k-1} & (8)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right)\left(e^{ah} - e^{-ah}\right)^{k-1} & (9)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right)^{k}. & (10) \end{aligned}$$

□

Theorem 1.0.3 (central finite difference formula)
$\forall k \in \mathbb{N}\ \forall\ k$-continuously differentiable f's at 0:

$$f^{(k)}(0) = \lim_{h\to 0} \frac{\Delta_h^k f(t)|_{t=0}}{(2h)^k}. \tag{11}$$

Proof. Follows by the mean value theorem.

$$\begin{aligned}
&\lim_{h\to 0} \frac{\Delta_h^k f(t)|_{t=0}}{(2h)^k} && (12)\\
&= \lim_{h\to 0} \frac{\Delta_h \Delta_h^{k-1} f(t)|_{t=0}}{(2h)^k} && (13)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-1} f(t)(|_{t=h} - |_{t=-h})}{(2h)^k} && (14)\\
&= \lim_{h\to 0} \frac{2h(\Delta_h^{k-1} f(t))'|_{t=z_h^1}}{(2h)^k}, z_h^1 \in [-h, h] && (15)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-1} f'(t)|_{t=z_h^1}}{(2h)^{k-1}}, z_h^1 \in [-h, h] && (16)\\
&= \lim_{h\to 0} \frac{\Delta_h^{k-2} f^{(2)}(t)|_{t=z_h^2}}{(2h)^{k-2}}, z_h^2 \in [z_h^1 - h, z_h^1 + h] \subset [-2h, 2h] && (17)\\
&\dots && (18)\\
&= \lim_{h\to 0} f^{(k)}(t)|_{t=z_h^k}, z_h^k \in [-kh, kh] && (19)\\
&= \lim_{h\to 0} f^{(k)}(z_h^k) = f^{(k)}(0). && (20)
\end{aligned}$$

Last line follows because f^k is continuous at 0. This proof may need some modification because we need to assume that there exist a neighborhood around 0 where f is k-times continuously differentiable instead only k-times continuously differentiable at 0. □

Lemma 1.0.4
$\forall k \in \mathbb{N} : \phi_X(t)$ is $2k$-time continuous differentiable $\Rightarrow E[X^{2k}] < \infty$

Proof. Use the central finite difference formula and previous lemma:

$$\begin{aligned}
|\phi_X^{(2k)}(0)| &= \left| \lim_{h\to 0} \frac{\Delta_h^{2k}|_{t=0} \phi_X(t)}{(2h)^{2k}} \right| && (21)\\
&= \left| \lim_{h\to 0} \frac{\Delta_h^{2k}|_{t=0} E\left[e^{itX}\right]}{(2h)^{2k}} \right| && (22)\\
&= \left| \lim_{h\to 0} \frac{E\left[\Delta_h^{2k}|_{t=0} e^{itX}\right]}{(2h)^{2k}} \right| && (23)\\
&= \left| \lim_{h\to 0} \frac{E\left[e^{itX}|_{t=0}(e^{ihX} - e^{-ihX})^{2k}\right]}{(2h)^{2k}} \right| && (24)\\
&= \left| \lim_{h\to 0} \frac{E\left[(e^{ihX} - e^{-ihX})^{2k}\right]}{(2h)^{2k}} \right| && (25)\\
&= \left| \lim_{h\to 0} \frac{1}{i^{2k}} E\left[\frac{(e^{ihX} - e^{-ihX})^{2k}}{(2h)^{2k}}\right] \right| && (26)
\end{aligned}$$

$$= \lim_{h\to 0} E\left[\frac{\sin(Xh)^{2k}}{h^{2k}}\right] \tag{27}$$

$$= \lim_{h\to 0} E\left[X^{2k}\frac{\sin(Xh)^{2k}}{X^{2k}h^{2k}}\right] \tag{28}$$

$$= \lim_{h\to 0} E\left[X^{2k}\,\mathrm{sinc}(Xh)^{2k}\right] \tag{29}$$

$$\geq E\left[X^{2k}\right] \tag{30}$$

Where last line follows by Fatou's lemma. □

2 Submultiplicative Functions

In this section we introduce submultiplicative functions and prove some properties of them.

Definition 2.0.1 (submultiplicative function)
A function $g(x)$ on $\mathbb{R}^d$ is submultiplicative if and only if it is nonnegative and $\exists a \in \mathbb{R} > 0:$

$$g(x+y) \leq ag(x)g(y) \text{ for } x, y \in \mathbb{R}^d. \tag{31}$$

We will use following lemma implicitly in the following proofs.

Lemma 2.0.2 (induction on submultiplicativity)
Let $g(x)$ be a submultiplicative with constant a then:

$$\forall (x_i)_i \subset \mathbb{R}^d : g\left(\sum_{i=1}^{n} x_i\right) \leq a^{n-1}\prod_{i=1}^{n} g(x_i). \tag{32}$$

Proof. This follows by induction on n.
Base case: $n = 2$ is true by submultiplicativity.
Induction hypothesis: assume $2 \leq n \leq k$.
Induction step: $n = k+1$:

$$\forall (x_i)_i \subset \mathbb{R}^d : \tag{33}$$

$$g\left(\sum_{i=1}^{k+1} x_i\right) = g\left(\sum_{i=1}^{k} x_i + x_{k+1}\right) \tag{34}$$

$$\leq ag\left(\sum_{i=1}^{k} x_i\right) g(x_{k+1}) \text{ by submultiplicativity} \tag{35}$$

$$\leq aa^{k-1}\prod_{i=1}^{k} g\left(x_i\right) g(x_{k+1}) \text{ by induction hypothesis} \tag{36}$$

$$\leq a^k \prod_{i=1}^{k+1} g\left(x_i\right). \tag{37}$$

□

Lemma 2.0.3
The product of two submultiplicative functions is submultiplicative.

Proof. Let $g, h : \mathbb{R}^d \to \mathbb{R}$ be submultiplicative functions then:

$$\forall x, y \in \mathbb{R}^d : g(x+y)h(x+y) \leq a_1 g(x)g(y)a_2 h(x)h(y) \leq a_1 a_2 g(x)h(x)g(y)h(y).$$

So $gh : \mathbb{R}^d \to \mathbb{R}$ is submultiplicative. □

Lemma 2.0.4
If $g(x)$ is submultiplicative on $\mathbb{R}^d$, then so is $g(cx+\gamma)^\alpha$ with $c \in \mathbb{R}$, $\gamma \in \mathbb{R}^d$, and $\alpha > 0$.

Proof. Let $g_1(x) = g(cx), g_2(x) = g(x+\gamma)$, and $g_3(x) = g(x)^\alpha$.

$$
\begin{aligned}
&\forall x, y \in \mathbb{R}^d : && (38)\\
&g_1(x+y) = g(c(x+y)) = g(cx+cy) \le a g(cx) g(cy) = a g_1(x) g_1(y) && (39)\\
&g_2(x+y) = g(x+y+\gamma) = g(x+\gamma+y+\gamma-\gamma) \le a^2 g(-\gamma) g(x+\gamma) g(y+\gamma) && (40)\\
&\qquad \le a^2 g(-\gamma) g_2(x) g_2(y) && (41)\\
&g_3(x+y) = g(x+y)^\alpha \le (a g(x) g(y))^\alpha = a^\alpha g(x)^\alpha g(y)^\alpha = a^\alpha g_3(x) g_3(y). && (42)
\end{aligned}
$$

□

Example 2.0.5 (submultiplicative functions)
The class of submultiplicative functions is broad. Let $0 < \beta \le 1$. Then the following functions are submultiplicative:

$$|x| \vee 1, |x_j| \vee 1, x_j \vee 1, \exp\left(|x|^\beta\right), \exp\left(|x_j|^\beta\right) \tag{43}$$

$$\exp\left((x_j \vee 0)^\beta\right), \log(|x| \vee e), \log\left(|x_j| \vee e\right), \log\left(x_j \vee \mathrm{e}\right) \tag{44}$$

$$\log\log\left(|x| \vee e^e\right), \log\log\left(|x_j| \vee e^e\right), \log\log\left(x_j \vee e^e\right) \tag{45}$$

Proof. See Sato 2013 page 159 PROPOSITION 25.4 (iii). □

3 Finiteness of the g-moments of a Lévy process

In this section we characterize the finiteness of the g-moment of a Lévy process in terms of the Lévy measure.

Definition 3.0.1 (g-moment)
Let $g(x)$ be a nonnegative measurable function on $\mathbb{R}^d$. We call $\int g(x)\mu(\mathrm{d}x)$ the g-moment of a measure μ on $\mathbb{R}^d$. We call $E[g(X)]$ the g-moment of a random variable X on $\mathbb{R}^d$.

Definition 3.0.2 (locally bounded function)
A function is locally bounded if and only if it is bounded on every compact set.

The main result of the assignment is following theorem.

Theorem 3.0.3 (charaterization of finiteness of the g-moment of a Lévy process)
Let g be a submultiplicative, locally bounded, measurable function on $\mathbb{R}^d$ and let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure ν. Then, X_t has finite g-moment for every $t > 0$ if and only if $[\nu]_{\{|x|>1\}}$ has finite g-moment.

Corollary 3.0.4 (time independent property)
Let g be a submultiplicative, locally bounded, measurable function on $\mathbb{R}^d$. Then, finiteness of the g-moment is not a time dependent distributional property in the class of Lévy processes.

Proof. Follows directly from Theorem 3.0.3. □

We prove Theorem 3.0.3 after three lemmas.

Lemma 3.0.5 (exponential bound for submultiplicative functions)
If $g(x)$ is submultiplicative and locally bounded, then

$$g(x) \leq b e^{c|x|} \tag{46}$$

with some $b > 0$ and $c > 0$.

Proof. $\sup_{|x|\leq 1} g(x) < \infty$ because g is locally bounded, let $b = \max(\sup_{|x|\leq 1} g(x), \frac{1}{a} + 1)$ such that $\sup_{|x|\leq 1} g(x) \leq b$ and $ab > 1$. $\forall x$ set $n_x = \lceil |x| \rceil$ implying $n_x - 1 < |x| \leq n_x, |\frac{x}{n_x}| \leq 1$ and $g(\frac{x}{n_x}) \leq b$, then

$$g(x) = g\left(\sum_{i=1}^{n_x} \frac{x}{n_x}\right) \leq a^{n_x - 1} g\left(\frac{x}{n_x}\right)^{n_x} \leq a^{n_x-1} b^{n_x} \leq b(ab)^{|x|} \leq b e^{\ln(ab)|x|}. \tag{47}$$

□

Lemma 3.0.6
Let μ be an infinitely divisible distribution on $\mathbb{R}$ with Lévy measure ν supported on a bounded set $S \subset [-a, a]$ for some $a > 0 \in \mathbb{R}$. Then $\widehat{\mu}(t)$ can be extended to an entire function on $\mathbb{C}$.

Proof. The Lévy representation of $\widehat{\mu}(t)$ is

$$\widehat{\mu}(t) = \exp\left[-\frac{1}{2}At^2 + \int_{[-a,a]} \left(\mathrm{e}^{itx} - 1 - itx\right) \nu(dx) + i\gamma' t\right] \tag{48}$$

$$= \exp\left[-\frac{1}{2}At^2 + i\gamma' t\right] \exp\left[\int_{-a}^{a} \left(\mathrm{e}^{itx} - 1 - itx\right) \nu(dx)\right] \tag{49}$$

with some $\gamma' \in \mathbb{R}$.
As products, sums, compositions and integrals of entire functions are entire (integrals follow from Morera's theorem), exp and polynomials are entire it is sufficient to show that $\int_{-a}^{a} \left(\mathrm{e}^{itx} - 1 - itx\right) \nu(dx)$ is finite for $t \in \mathbb{C}$:

$$\left|\int_{-a}^{a} \left(\mathrm{e}^{itx} - 1 - itx\right) \nu(dx)\right| \tag{50}$$

$$\leq \int_{-a}^{a} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) \tag{51}$$

$$\leq \int_{-1}^{1} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) + \int_{[-a,a]\setminus[-1,1]} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) \tag{52}$$

$$\leq \int_{-1}^{1} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) + \sup_{x \in [-a,a]\setminus[-1,1]} \left(|\mathrm{e}^{itx} - 1 - itx|\right) \int_{[-a,a]\setminus[-1,1]} \nu(dx) \tag{53}$$

$$\leq \int_{-1}^{1} |\mathrm{e}^{itx} - 1 - itx| \nu(dx) + I_2 \tag{54}$$

$$\leq \int_{-1}^{1} \left|\sum_{n=2}^{\infty} \frac{(ixt)^n}{n!}\right| \nu(dx) + I_2 \tag{55}$$

$$\leq \int_{-1}^{1} \sum_{n=2}^{\infty} \frac{|x|^n |t|^n}{n!} \nu(dx) + I_2 \tag{56}$$

$$\leq \int_{-1}^{1} |x|^2 \sum_{n=2}^{\infty} \frac{|x|^{n-2}|t|^n}{n!} \nu(dx) + I_2 \tag{57}$$

$$\leq \int_{-1}^{1} |x|^2 \sum_{n=2}^{\infty} \frac{|t|^n}{n!} \nu(dx) + I_2 \tag{58}$$

$$\leq \sum_{n=2}^{\infty} \frac{|t|^n}{n!} \int_{-1}^{1} |x|^2 \nu(dx) + I_2 \tag{59}$$

$$\leq (e^{|t|} - |t| - 1) \int_{-1}^{1} |x|^2 \nu(dx) + I_2 \tag{60}$$

$$< \infty. \tag{61}$$

We implicitly used $|x| < 1 \implies |x|^{n-2} < 1$,$|\mathrm{e}^{itx} - 1 - itx|$ continuous in x so achieves maximum on compact sets (Weierstrass theorem) and

$$\int_{\mathbb{R}} (|x|^2 \wedge 1) \nu(dx) < \infty \implies \int_{-1}^{1} |x|^2 \nu(dx) < \infty, \int_{[-a,a] \setminus [-1,1]} \nu(dx) < \infty. \tag{62}$$

□

Lemma 3.0.7 (entire characteristic functions imply finite exponential moments)
If μ is a probability measure on $\mathbb{R}$ and $\widehat{\mu}(z)$ is extendible to an entire function on $\mathbb{C}$, then μ has finite exponential moments, that is, it has finite $e^{c|x|}$-moment for every $c > 0$.

Proof. As entire functions are infinitely differentiable it follows from Lemma 1.0.4 that $\alpha_{2n} = \int x^{2n} \mu(\mathrm{d}x) = \int |x|^{2n} \mu(\mathrm{d}x)$ are finite for any $n \geq 1$ and by Hölder's inequality the uneven (absolute) moments $(\beta_{2n+1}), \alpha_{2n+1}$ are also finite. Since $\frac{\mathrm{d}^n \widehat{\mu}}{\mathrm{d}z^n}(0) = \mathrm{i}^n \alpha_n$, we have

$$\widehat{\mu}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{i}^n \alpha_n z^n$$

as $\widehat{\mu}(z)$ is entire the radius of convergence of the right-hand side is infinite therefore $\forall z \in \mathbb{C} : \left|\frac{z^n \alpha_n}{n!}\right| = \frac{|z|^n}{n!} \alpha_n \to 0$ and again by Hölder's inequality also $\forall c \in \mathbb{R} : \frac{2^n c^n}{n!} \beta_n \to 0 \implies \forall \delta > 0 \in \mathbb{R}, \exists n_c \in \mathbb{N}, \forall n > n_\delta \in \mathbb{N} : \frac{2^n c^n}{n!} \beta_n < \delta$. Now the exponential moments are easily bounded:

$$\int \mathrm{e}^{c|x|} \mu(\mathrm{d}x) = \int \sum_{n=0}^{\infty} \frac{c^n |x|^n}{n!} \mu(\mathrm{d}x) \tag{63}$$

$$= \sum_{n=0}^{\infty} \frac{c^n}{n!} \int |x|^n \mu(\mathrm{d}x) \tag{64}$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{n!} \beta_n c^n \tag{65}$$

$$\leq \sum_{n=0}^{n_\delta} \frac{1}{n!} \beta_n c^n + \sum_{n=n_\delta}^{\infty} \frac{1}{n!} \beta_n c^n \tag{66}$$

$$\leq \sum_{n=0}^{n_\delta} \frac{1}{n!} \beta_n c^n + \sum_{n=n_\delta}^{\infty} \frac{\delta}{2^n} \tag{67}$$

$$< \infty. \tag{68}$$

Exchanging the sum and the integral is justified by Tonelli's theorem as the summand is positive. □

Corollary 3.0.8
A Lévy process with a Lévy measure supported on a bounded set has finite exponential moments.

Proof. Follows from Lemma's 3.0.7, 3.0.6. □

Theorem 3.0.9 (compound Poisson process expansion)
Let Y_t be compound Poisson process with Lévy measure ν then

$$P_{Y_t} = e^{-t\int \nu(dx)} \sum_{n=0}^{\infty} \frac{t^n}{n!} \nu^n. \tag{69}$$

Proof.

$$\begin{aligned} P[Y_t \in B] &= P\left[\sum_{j=0}^{N_t} \varepsilon_j \in B\right] &(70)\\ &= \sum_{n=0}^{\infty} P\left[\sum_{j=0}^{n} \varepsilon_j \in B\right] P[N_t = n] &(71)\\ &= \sum_{n=0}^{\infty} P\left[\sum_{j=0}^{n} \varepsilon_j \in B\right] e^{-t\int \nu(dx)} \frac{t^n}{n!} &(72)\end{aligned}$$

□

Proof of Theorem 3.0.3. Let $\nu_0 = [\nu]_{\{|x|\le 1\}}$ and $\nu_1 = [\nu]_{\{|x|>1\}}$. Let $\left\{X_t^i\right\}$ be independent Lévy processes $\mathbb{R}^d$ such that $\{X_t\} =_d \left\{X_t^0 + X_t^1\right\}$ and $\left\{X_t^1\right\}$ is compound Poisson with Lévy measure ν_1. Let μ_0^t and μ_1^t be the distributions of X_t^0 and X_t^1, respectively.
Suppose that X_t has a finite g-moment for some $t > 0$. Then

$$\infty > E\left[g\left(X_t\right)\right] = E\left[g\left(X_t^0 + X_t^1\right)\right] = \iint g(x+y)\mu_1^t(dy)\mu_0^t(dx). \tag{73}$$

If $\forall x \in \mathbb{R}^d : \int g(x+y)\mu_1^t(dy) = \infty \implies E[g(X_t)] = \infty$ which is a contradiction. So $\exists x \in \mathbb{R}^d : \int g(x+y)\mu_1^t(dy) < \infty$. Because μ_1^t is a compound Poisson we have by theorem 3.0.9:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(x+y)\nu_1^n(dy) < \infty. \tag{74}$$

Since $g(y) \le ag(-x)g(x+y) \le abe^{c|x|}g(x+y)$ by Lemma 3.0.5, we get

$$\begin{aligned} \infty &> \sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(x+y)\nu_1^n(dy) &(75)\\ &\ge a^{-1}b^{-1}e^{-c|x|} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(y)\nu_1^n(dy) &(76)\\ &\ge a^{-1}b^{-1}e^{-c|x|} \frac{t^n}{n!} \int g(y)\nu_1^n(dy) \quad \forall n \in \mathbb{N} &(77)\end{aligned}$$

In particular $n=1$ hence $\int g(y)\nu_1(dy)<\infty$.
Conversely, suppose that $\int g(y)\nu_1(dy)=G<\infty$ then we have to prove that $E\left[g\left(X_t\right)\right]$ $<\infty$ for every t.

$$\begin{aligned}
E\left[g\left(X_t\right)\right] &= E\left[g\left(X_t^0+X_t^1\right)\right] && (78)\\
&\leq aE\left[g\left(X_t^0\right)g\left(X_t^1\right)\right] \quad \text{by submultiplicativity} && (79)\\
&= aE\left[g\left(X_t^0\right)\right]E\left[g\left(X_t^1\right)\right] \quad \text{by independence} && (80)\\
&\leq abE\left[\mathrm{e}^{\mathrm{c}\left|X_t^0\right|}\right]E\left[g\left(X_t^1\right)\right] \quad \text{by Lemma 3.0.5} && (81)
\end{aligned}$$

So it is sufficient to show that $E\left[g\left(X_t^1\right)\right]<\infty$ and $E\left[\mathrm{e}^{\mathrm{c}\left|X_t^0\right|}\right]<\infty$

$$\begin{aligned}
E[g(X_t^1)] &= \int g(x)\mu_1^t(dx) && (82)\\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\int g(x)\nu_1^n(dx) \quad \text{by theorem 3.0.9} && (83)\\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\int g\left(\sum_j x_j\right)\prod_j \nu_1\left(dx_j\right) && (84)\\
&\leq \sum_{n=0}^{\infty}\frac{t^n}{n!}a^{n-1}\int \prod_j g\left(x_j\right)\prod_j \nu_1\left(dx_j\right) \quad \text{by submultiplicativity} && (85)\\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}a^{n-1}\left(\int g(x)\nu_1(dx)\right)^n && (86)\\
&= a^{-1}\sum_{n=0}^{\infty}\frac{t^n}{n!}a^nG^n && (87)\\
&= a^{-1}e^{aGt}<\infty. && (88)
\end{aligned}$$

Let $X_j^0(t)$, $1\leq j\leq d$, be the components of X_t^0. Then

$$\begin{aligned}
E\left[e^{c\left|X_t^0\right|}\right] &\leq E\left[\exp\left(c\sum_{j=1}^{d}\left|X_j^0(t)\right|\right)\right]\leq E\left[\prod_{j=1}^{d}\left(e^{cX_j^0(t)}+e^{-cX_j^0(t)}\right)\right] && (89)\\
&\leq E\left[\sum_{\alpha_j\in\{-1,1\}} e^{c\sum_{j=1}^{d}\alpha_jX_j^0(t)}\right]\leq \sum_{\alpha_j\in\{-1,1\}}E\left[e^{c\sum_{j=1}^{d}\alpha_jX_j^0(t)}\right] && (90)
\end{aligned}$$

Since X_t^0 is a Lévy process with a Lévy measure supported on a bounded set therefore $\sum_{j=1}^{d}\alpha_jX_j^0(t)$ is also a Lévy process on $\mathbb{R}$ with a Lévy measure supported on a bounded set by Sato 2013 page 65 PROPOSITION 11.10 therefore it has finite (exponential) moments by corollary 3.0.8. I.e. $E\left[e^{c\left|X_t^0\right|}\right]\leq\sum_{\alpha_j\in\{-1,1\}}E\left[e^{c\sum_{j=1}^{d}\alpha_jX_j^0(t)}\right]<\infty$. □

Corollary 3.0.10
Let $\alpha>0, 0<\beta\leq 1$, and $\gamma\geq 0$. None of the properties $\int|x|^\alpha\mu(\mathrm{d}x)<\infty, \int(0\vee\log|x|)^\alpha\mu(\mathrm{d}x)<\infty$, and $\int|x|^\gamma e^{\alpha|x|^\beta}\mu(\mathrm{d}x)<\infty$ is time dependent in the class of Lévy processes. For a Lévy process on $\mathbb{R}^d$ with Lévy measure ν, each of the properties is expressed by the corresponding property of $[\nu]_{\{|x|>1\}}$.

Proof. This follows from Theorem 3.0.3 and Example 2.0.5. $\square$

Lemma 3.0.11 (formulas for the first 2 moments of a Lévy process)
Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ generated by (A, ν, γ). In components, $X_t = (X_j(t))\,, \gamma = (\gamma_j)$, and $A = (A_{jk})$. Then $\int_{\{|x_j|>1\}} |x_j| \nu(\mathrm{d}x) < \infty \implies m_j(t) = E\left[X_j(t)\right] < \infty$ and can be expressed as

$$m_j(t) = t\left(\int_{|x|>1} x_j \nu(\mathrm{d}x) + \gamma_j\right). \tag{91}$$

Similarly $\int_{\{|x_j|,|x_k|>1\}} |x_j||x_k|\nu(\mathrm{d}x), \int_{\{|x_j|>1\}} |x_j|^2\nu(\mathrm{d}x), \int_{\{|x_k|>1\}} |x_k|^2\nu(\mathrm{d}x) < \infty \implies v_{jk} = E\left[\left(X_j(t) - m_j(t)\right)\left(X_k(t) - m_k(t)\right)\right], v_{jj}, v_{kk} < \infty$ and can be expressed as

$$v_{jk}(t) = t\left(A_{jk} + \int_{\mathbb{R}^d} x_j x_k \nu(\mathrm{d}x)\right). \tag{92}$$

Proof. The finiteness conditions follow from using the fact that projections of Lévy processes are again Lévy processes Sato 2013 page 65 PROPOSITION 11.10 and Theorem 3.0.3. The formulas are derived from the derivatives of the characteristic function or cumulant function. I.e.

$$\begin{aligned}
m_j(t) &= -i\partial z_j|_{z=0}\widehat{\mu}_t(z) && (93)\\
&= -i\partial z_j|_{z=0} e^{\ln(\widehat{\mu}_t(z))} && (94)\\
&= -i\partial z_j|_{z=0} \ln\left(\widehat{\mu}_t(z)\right) \cancelto{1}{\widehat{\mu}_t(0)} && (95)\\
&= -i\partial z_j|_{z=0} \ln\left(\widehat{\mu}_t(z)\right), && (96)\\
v_{jk}(t) &= E\left[\left(X_j(t) - m_j(t)\right)\left(X_k(t) - m_k(t)\right)\right] && (97)\\
&= E[X_j(t)X_k(t)] - m_j(t)m_k(t) && (98)\\
&= -\partial z_j \partial z_k \widehat{\mu}_t(z)|_{z=0} - m_j(t)m_k(t) && (99)\\
&= -\partial z_j\left(\partial z_k(\ln\widehat{\mu}_t(z))\widehat{\mu}_t(z)\right)|_{z=0} - m_j(t)m_k(t) && (100)\\
&= -\partial z_j\partial z_k|_{z=0}(\ln\widehat{\mu}_t(z))\widehat{\mu}_t(z) - \partial z_j(\ln\widehat{\mu}_t(z))\partial z_k(\ln\widehat{\mu}_t(z))\widehat{\mu}_t(z)|_{z=0} - m_j(t)m_k(t) && (101)\\
&= -\partial z_j\partial z_k|_{z=0}(\ln\widehat{\mu}_t(z))\cancelto{1}{\widehat{\mu}_t(z)} - \cancelto{im_j(t)}{\partial z_j(\ln\widehat{\mu}_t(z))}\cancelto{im_k(t)}{\partial z_k(\ln\widehat{\mu}_t(z))}\cancelto{1}{\widehat{\mu}_t(z)}|_{z=0} - m_j(t)m_k(t) && (102)\\
&= -\partial z_j\partial z_k|_{z=0}(\ln\widehat{\mu}_t(z)) && (103)
\end{aligned}$$

The cumulant function of a Lévy process is given by the Lévy-Khintchine formula:

$$\frac{\ln\widehat{\mu}_t(z)}{t} = -\frac{1}{2}\langle z, Az\rangle + i\langle\gamma, z\rangle + \int_{\mathbb{R}^d}\left(\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle I_D(x)\right)\nu(\mathrm{d}x). \tag{104}$$

The first 2 terms are easily differentiated:

Operator	$-\frac{1}{2}\langle z, Az\rangle$	$i\langle\gamma, z\rangle$
$\partial z_j\|_{z=0}$	0	γ_j
$\partial z_j\partial z_k\|_{z=0}$	$-A_{jk}$	0

Let $I(z) = \int_{\mathbb{R}^d}\left(\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle I_D(x)\right)\nu(\mathrm{d}x)$ then

$$\begin{aligned}
\partial z_j|_{z=0} I(z) &= \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}^d} e^{ihx_j} - ihx_j I_D(x) - 1\nu(dx) && (105)\\
&= \lim_{h\to 0}\frac{1}{h}\int_D e^{ihx_j} - ihx_j - 1\nu(dx) + \frac{1}{h}\int_{D^c} e^{ihx_j} - 1\nu(dx) && (106)
\end{aligned}$$

$$= \cancelto{0}{\lim_{h\to 0}\int_D \sum_{n=2}^{\infty}\frac{(ihx_j)^n}{n!h}\nu(dx)} + \lim_{h\to 0}\frac{1}{h}\int_{D^c} ihx_j + e^{ihx_j} - ihx_j - 1\nu(dx) \tag{107}$$

$$= \lim_{h\to 0}\int_{D^c} ix_j + \frac{e^{ihx_j} - ihx_j - 1}{h}\nu(dx) \tag{108}$$

$$= \int_{D^c} ix_j\nu(dx) + \cancelto{0}{\lim_{h\to 0}\int_{D^c}\frac{e^{ihx_j} - ihx_j - 1}{h}\nu(dx)} \tag{109}$$

$$= \int_{D^c} ix_j\nu(dx) \tag{110}$$

In line (107) the canceling follows from $|x_j| < |x|$ and doing similar calculations as line (55) to line (60) in proof of Lemma 3.0.6 then taking the limit. We still have to elaborate the cancellation on line (109):

$$\left|\lim_{h\to 0}\int_{D^c}\frac{e^{ihx_j} - ihx_j - 1}{h}\nu(dx)\right| \tag{111}$$

$$\leq \lim_{h\to 0}\int_{D^c}\left|\frac{e^{ihx_j} - ihx_j - 1}{h}\right|\nu(dx) \tag{112}$$

$$= \lim_{h\to 0}\int_{D^c}\left|\frac{ix_j\int_0^h e^{isx_j} - 1ds}{h}\right|\nu(dx) \tag{113}$$

$$\leq \lim_{h\to 0}\int_{D^c}\frac{h|x_j|\sup_{s\in[0,h]}\left|e^{isx_j} - 1\right|}{h}\nu(dx) \tag{114}$$

$$\leq \lim_{h\to 0}\int_{D^c}|x_j|\sup_{s\in[0,h]}\left|e^{isx_j} - 1\right|\nu(dx) \tag{115}$$

$$\leq \lim_{h\to 0}\int_{D^c\cap\{|x_j|\leq 1\}}|x_j|\sup_{s\in[0,h]}\left|e^{isx_j} - 1\right|\nu(dx) + \lim_{h\to 0}\int_{D^c\cap\{|x_j|>1\}}|x_j|\sup_{s\in[0,h]}\left|e^{isx_j} - 1\right|\nu(dx) \tag{116}$$

$$\leq \lim_{h\to 0}\int_{D^c\cap\{|x_j|\leq 1\}}\sup_{s\in[0,h]}\left|e^{isx_j} - 1\right|\nu(dx) + \lim_{h\to 0}\int_{D^c\cap\{|x_j|>1\}}|x_j|\sup_{s\in[0,h]}\left|e^{isx_j} - 1\right|\nu(dx) \tag{117}$$

$$\leq 0 \tag{118}$$

On line (117) the integrals are dominated by $\int_{D^c} 2\nu(dx), \int_{|x_j|>1} 2|x_j|\nu(dx) < \infty$ and the dominated convergence theorem can be applied. After exchanging both limits are 0.
$\partial z_j\partial z_k|_{z=0}I(z) = \int_{\mathbb{R}^d}\partial z_j\partial z_k|_{z=0}\exp(i\langle z,x\rangle)\nu(dx) = -\int_{\mathbb{R}^d}x_jx_k\nu(dx)$ where exchanging the integral and the derivative isn't elaborated. □

References

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