

25. Moments

We define the g -moment of a random variable and discuss finiteness of the g -moment of X_t for a Lévy process $\{X_t\}$.

DEFINITION 25.1. Let $g(x)$ be a nonnegative measurable function on \mathbb{R}^d . We call $\int g(x)\mu(dx)$ the g -moment of a measure μ on \mathbb{R}^d . We call $E[g(X)]$ the g -moment of a random variable X on \mathbb{R}^d .

DEFINITION 25.2. A function $g(x)$ on \mathbb{R}^d is called submultiplicative if it is nonnegative and there is a constant $a > 0$ such that

$$g(x+y) \leq ag(x)g(y) \text{ for } x, y \in \mathbb{R}^d \quad (25.1)$$

A function bounded on every compact set is called locally bounded.

THEOREM 25.3 (g -Moment). Let g be a submultiplicative, locally bounded, measurable function on \mathbb{R}^d . Then, finiteness of the g -moment is not a time dependent distributional property in the class of Lévy processes. Let $\{X_t\}$ be a Lévy process on \mathbb{R}^d with Lévy measure ν . Then, X_t has finite g -moment for every $t > 0$ if and only if $[\nu]_{\{|x|>1\}}$ has finite g -moment.

The following facts indicate the wide applicability of the theorem.

PROPOSITION 25.4.

- (i) The product of two submultiplicative functions is submultiplicative.
- (ii) If $g(x)$ is submultiplicative on \mathbb{R}^d , then so is $g(cx + \gamma)^\alpha$ with $c \in \mathbb{R}$, $\gamma \in \mathbb{R}^d$, and $\alpha > 0$.
- (iii) Let $0 < \beta \leq 1$. Then the following functions are submultiplicative:

$$\begin{aligned} & |x| \vee 1, |x_j| \vee 1, x_j \vee 1, \exp(|x|^\beta), \exp(|x_j|^\beta) \\ & \exp((x_j \vee 0)^\beta), \log(|x| \vee e), \log(|x_j| \vee e), \log(x_j \vee e) \\ & \log \log(|x| \vee e^e), \log \log(|x_j| \vee e^e), \log \log(x_j \vee e^e) \end{aligned}$$

Here x_j is the j th component of x .

Proof. (i) Immediate from the definition.

(ii) Let $g_1(x) = g(cx)$, $g_2(x) = g(x + \gamma)$, and $g_3(x) = g(x)^\alpha$. Then it follows from (25.1) that $g_1(x+y) \leq ag_1(x)g_1(y)$, $g_2(x+y) \leq a^2g(-\gamma)g_2(x)g_2(y)$, and $g_3(x+y) \leq a^\alpha g_3(x)g_3(y)$.

(iii) Let $h(u)$ be a positive increasing function on \mathbb{R} such that, for some $b \geq 0$, $h(u)$ is flat on $(-\infty, b]$ and $\log h(u)$ is concave on $[b, \infty)$. Then $h(u)$ is submultiplicative on \mathbb{R} . In fact, for $u, v \geq b$, the function $f(u) = \log h(u)$ satisfies

$$\begin{aligned} f(u+b) - f(u) &\leq f(2b) - f(b) \\ f(u+v) - f(v) &\leq f(u+b) - f(b) \end{aligned}$$

and hence

$$f(u+v) \leq f(u+b) - f(b) + f(v) \leq f(2b) - 2f(b) + f(u) + f(v)$$

which shows

$$h(u+v) \leq \text{const } h(u)h(v) \quad (25.2)$$

It follows that (25.2) holds for all $u, v \in \mathbb{R}$. The functions $u \vee 1$, $\exp((u \vee 0)^\beta)$, $\log(u \vee e)$, and $\log \log(u \vee e^e)$ fulfil the conditions on $h(u)$. By (25.2) and by the increasingness of h , the functions $h(|x|)$, $h(|x_j|)$, and $h(x_j)$ are submultiplicative on \mathbb{R}^d . \square

We prove Theorem 25.3 after three lemmas.

LEMMA 25.5. If $g(x)$ is submultiplicative and locally bounded, then

$$g(x) \leq be^{c|x|} \quad (25.3)$$

with some $b > 0$ and $c > 0$.

Proof. Choose b in such a way that $\sup_{|x| \leq 1} g(x) \leq b$ and $ab > 1$. If $n-1 < |x| \leq n$, then

$$g(x) \leq a^{n-1} g\left(\frac{1}{n}x\right)^n \leq a^{n-1} b^n \leq b(ab)^{|x|}$$

which shows (25.3). \square

LEMMA 25.6. Let μ be an infinitely divisible distribution on \mathbb{R} with Lévy measure ν supported on a bounded set. Then $\hat{\mu}(z)$ can be extended to an entire function on \mathbb{C} .

Proof. There is a finite $a > 0$ such that $S_\nu \subset [-a, a]$. The Lévy representation of $\hat{\mu}(z)$ is written as

$$\hat{\mu}(z) = \exp \left[-\frac{1}{2}Az^2 + \int_{[-a,a]} (e^{izx} - 1 - izx) \nu(dx) + i\gamma'z \right]$$

with some $\gamma' \in \mathbb{R}$. The right-hand side is meaningful even if z is complex. Denote this function by $\Phi(z)$. Then $\Phi(z)$ is an entire function, since we can exchange the order of integration and differentiation. \square

LEMMA 25.7. If μ is a probability measure on \mathbb{R} and $\hat{\mu}(z)$ is extendible to an entire function on \mathbb{C} , then μ has finite exponential moments, that is, it has finite $e^{c|x|}$ -moment for every $c > 0$.

Proof. It follows from Proposition 2.5(x) that $\alpha_n = \int x^n \mu(dx)$ and $\beta_n = \int |x|^n \mu(dx)$ are finite for any $n \geq 1$. Since $\frac{d^n \hat{\mu}}{dz^n}(0) = i^n \alpha_n$, we have

$$\hat{\mu}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} i^n \alpha_n z^n$$

the radius of convergence of the right-hand side being infinite. Notice that $\beta_{2k} = \alpha_{2k}$ and $\beta_{2k+1} \leq \frac{1}{2}(\alpha_{2k+2} + \alpha_{2k})$, since $|x|^{2k+1} \leq \frac{1}{2}(x^{2k+2} + x^{2k})$. It follows that

$$\int e^{|x|} \mu(dx) = \sum_{n=0}^{\infty} \frac{1}{n!} \beta_n c^n < \infty$$

completing the proof. \square

Proof of Theorem 25.3. Let $\nu_0 = [\nu]_{\{|x| \leq 1\}}$ and $\nu_1 = [\nu]_{\{|x| > 1\}}$. Construct independent Lévy processes $\{X_t^0\}$ and $\{X_t^1\}$ on \mathbb{R}^d such that $\{X_t\} =_d \{X_t^0 + X_t^1\}$ and $\{X_t^1\}$ is compound Poisson with Lévy measure ν_1 . Let μ_0 and μ_1 be the distributions of X_1^0 and X_1^1 , respectively.

Suppose that X_t has finite g -moment for some $t > 0$. It follows from

$$E[g(X_t)] = \iint g(x+y)\mu_0^t(dx)\mu_1^t(dy)$$

that $\int g(x+y)\mu_1^t(dy) < \infty$ for some x . This means

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(x+y)\nu_1^n(dy) < \infty$$

Since $g(y) \leq ag(-x)g(x+y) \leq abe^{c|x|}g(x+y)$ by Lemma 25.5, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(y)\nu_1^n(dy) < \infty \quad (25.4)$$

Hence $\int g(y)\nu_1(dy) < \infty$.

Conversely, suppose that $\int g(y)\nu_1(dy) < \infty$. Let us prove that $E[g(X_t)] < \infty$ for every t . By the submultiplicativity,

$$\begin{aligned} \int g(y)\nu_1^n(dy) &= \int \cdots \int g(y_1 + \cdots + y_n) \nu_1(dy_1) \cdots \nu_1(dy_n) \\ &\leq a^{n-1} \left(\int g(y)\nu_1(dy) \right)^n \end{aligned}$$

Hence we have (25.4) for every t . That is, X_t^1 has finite g -moment. Since

$$E[g(X_t)] \leq abE[e^{c|X_t^0|}] E[g(X_t^1)]$$

by (25.1) and (25.3), it remains only to show that $E[e^{c|X_t^0|}] < \infty$. Let $X_j^0(t)$, $1 \leq j \leq d$, be the components of X_t^0 . Then

$$E[e^{c|X_t^0|}] \leq E \left[\exp \left(c \sum_{j=1}^d |X_j^0(t)| \right) \right] \leq E \left[\prod_{j=1}^d (e^{cX_j^0(t)} + e^{-cX_j^0(t)}) \right]$$

which is written as a sum of a finite number of terms of the form $E[\exp X_t^\sharp]$ with X_t^\sharp being a linear combination of $X_j^0(t)$, $1 \leq j \leq d$. Since $\{X_t^\sharp\}$ is a Lévy process on \mathbb{R} with Lévy measure supported on a bounded set (use Proposition 11.10), $E[\exp X_t^\sharp]$ is finite by virtue of Lemmas 25.6 and 25.7. This proves all statements in the theorem. \square

Corollary 25.8. Let $\alpha > 0$, $0 < \beta \leq 1$, and $\gamma \geq 0$. None of the properties $\int |x|^\alpha \mu(dx) < \infty$, $\int (0 \vee \log |x|)^\alpha \mu(dx) < \infty$, and $\int |x|^\gamma e^{\alpha|x|^\beta} \mu(dx) < \infty$ is time dependent in the class of Lévy processes. For a Lévy process on \mathbb{R}^d with Lévy measure ν , each of the properties is expressed by the corresponding property of $[\nu]_{\{|x| > 1\}}$.

Proof. This follows from Theorem 25.3 and Proposition 25.4. \square

REMARK 25.9. There is a nonnegative measurable function $g(x)$ satisfying (25.3) such that finiteness of the g -moment is a time dependent distributional property in the class of Lévy processes. For example, let $g(x) = (1 \wedge |x|^{-\alpha}) e^{|x|}$ with $\alpha > 0$. Consider a Γ -process $\{X_t\}$ with $E[X_1] = 1$. Then it is easy to see that $E[g(X_t)] < \infty$ if and only if $t < \alpha$. This process has $\nu = x^{-1} e^{-x} 1_{(0,\infty)}(x) dx$, so that ν_1 has finite g -moment (Example 8.10).

EXAMPLE 25.10. Let $\{X_t\}$ be a non-trivial semi-stable process on \mathbb{R}^d with index $\alpha \in (0, 2)$. Then, for every $t > 0$, $E[|X_t|^\eta]$ is finite or infinite according as $0 < \eta < \alpha$ or $\eta \geq \alpha$, respectively. To see this, notice that the argument in the proofs of Theorem 13.15 and Proposition 14.5 gives

$$\int_{S_n(b)} |x|^\eta \nu(dx) = b^{n(\eta-\alpha)} \int_{S_0(b)} |x|^\eta \nu(dx)$$

and hence $\int_{|x|>1} |x|^\eta \nu(dx) < \infty$ if and only if $\eta < \alpha$; apply Corollary 25.8. In particular, for a stable process on \mathbb{R} with parameters (α, β, τ, c) (Definition 14.16), $E[X_t] = \tau t$ if $1 < \alpha < 2$ (use Proposition 2.5(ix)). The following explicit results are known. If $0 < \alpha < 1$ and $\{X_t\}$ is a stable subordinator with $E[e^{-uX_t}] = e^{-t\epsilon'u^\alpha}$ (Example 24.12), then, for $-\infty < \eta < \alpha$,

$$E[X_t^\eta] = (tc')^{\eta/\alpha} \frac{\Gamma(1 - \frac{\eta}{\alpha})}{\Gamma(1 - \eta)} \quad (25.5)$$

which is shown by Wolfe [508] and Shanbhag and Sreehari [419] (Exercise 29.17). If $0 < \alpha < 2$ and $\{X_t\}$ is symmetric and α -stable on \mathbb{R} with $E[e^{iX_t}] = e^{-tc|z|^\alpha}$ (Theorem 14.14), then, for $-1 < \eta < \alpha$,

$$E[|X_t|^\eta] = (tc)^{\eta/\alpha} \frac{2^\eta \Gamma(\frac{1+\eta}{2}) \Gamma(1 - \frac{\eta}{\alpha})}{\sqrt{\pi} \Gamma(1 - \frac{\eta}{2})} \quad (25.6)$$

as is shown in [419].

EXAMPLE 25.11. If $\{X_t\}$ is a Lévy process on \mathbb{R} with Lévy measure supported on $(-\infty, 0]$, then $E[e^{cX_t}] < \infty$ for every $c > 0$ and $t > 0$. Use Theorem 25.3 for $g(x) = e^{cx}$. For instance, a stable process on \mathbb{R} with $1 \leq \alpha < 2$ and $\beta = -1$ satisfies this assumption although it has support \mathbb{R} for every $t > 0$ (Theorem 24.10(i)).

EXAMPLE 25.12. Let $\{X_t\}$ be a Lévy process on \mathbb{R}^d generated by $(A, \nu, 1)$. In components, $X_t = (X_j(t))$, $\gamma = (\gamma_j)$, and $A = (A_{jk})$. Then X_t has finite mean for $t > 0$ if and only if $\int_{\{|x|>1\}} |x| \nu(dx) < \infty$. When this condition is met, we can find $m_j(t) = E[X_j(t)]$ expressed as

$$m_j(t) = t \left(\int_{|x|>1} x_j \nu(dx) + \gamma_j \right) = t\gamma_{1,j}, \quad j = 1, \dots, d \quad (25.7)$$

differentiating $\hat{\mu}(z)$ (Proposition 2.5(ix)). Here $\gamma_{1,j}$ is the j th component of the center γ_1 in (8.8). Similarly, $E[|X_t|^2] < \infty$ for all $t > 0$ if and only if $\int_{|x|>1} |x|^2 \nu(dx) < \infty$. In this case,

$$v_{jk}(t) = E[(X_j(t) - m_j(t))(X_k(t) - m_k(t))], \quad j, k = 1, \dots, d$$

the (j, k) elements of the covariance matrix of $X(t)$, are expressed as

$$v_{jk}(t) = t \left(A_{jk} + \int_{\mathbb{R}^d} x_j x_k \nu(dx) \right) \quad (25:8)$$

Theorem 25.3 shows that, for a Lévy process $\{X_t\}$ with Lévy measure ν , the tails of P_{X_t} and ν have a kind of similarity. Are they actually equivalent in some class of Lévy processes? This question was answered by Embrecht, Goldie, and Veraverbeke [109] for subordinators. We state their result without proof in two remarks below.

Definition 25.13. A probability measure μ on $[0, \infty)$ is called subexponential if $\mu(x, \infty) > 0$ for every x and

$$\lim_{x \rightarrow \infty} \frac{\mu^n(x, \infty)}{\mu(x, \infty)} = n \quad \text{for } n = 2, 3, \dots \quad (25.9)$$

The class of probability measures satisfying (25.9) above was introduced by Chistyakov [65]. The condition can be weakened. Specifically, if

$$\limsup_{x \rightarrow \infty} \frac{\mu^2(x, \infty)}{\mu(x, \infty)} \leq 2$$

then μ is subexponential. The meaning of (25.9) is as follows. Let $\{Z_j\}$ be independent nonnegative random variables each with distribution μ and let $S_n = \sum_{j=1}^n Z_j$ and $M_n = \max_{1 \leq j \leq n} Z_j$. Then μ satisfies (25.9) if and only if

$$P[S_n > x] \sim P[M_n > x], \quad x \rightarrow \infty, \quad \text{for } n = 2, 3, \dots$$

In fact, $P[S_n > x] = \mu^n(x, \infty)$ and

$$\begin{aligned} P[M_n > x] &= \sum_{j=1}^n P[X_1 \leq x, \dots, X_{j-1} \leq x, X_j > x] \\ &= \sum_{j=1}^n \mu(0, x)^{j-1} \mu(x, \infty) \sim n\mu(x, \infty) \quad \text{as } x \rightarrow \infty \end{aligned}$$

REMARK 25.14. A basic result on subexponentiality is as follows. If $\{X_t\}$ be a subordinator with Lévy measure ν , then the following conditions are equivalent [109]:

- (1) $\nu(1, \infty) > 0$ and $\frac{1}{\nu(1, \infty)}[\nu]_{(1, \infty)}$ is subexponential;
- (2) P_{X_t} is subexponential for every $t > 0$;
- (3) P_{X_t} is subexponential for some $t > 0$;
- (4) $P[X_t > x] \sim t\nu(x, \infty), x \rightarrow \infty$, for every $t > 0$;
- (5) $P[X_t > x] \sim t\nu(x, \infty), x \rightarrow \infty$, for some $t > 0$.

Some of the consequences of subexponentiality are as follows. Let μ be a subexponential probability measure on $[0, \infty)$. Then,

- (1) for any $y \in \mathbb{R}$, $\mu(x - y, \infty)/\mu(x, \infty) \rightarrow 1$ as $x \rightarrow \infty$;

(2) for every $\varepsilon > 0$, $\int_{[0,\infty)} e^{\varepsilon x} \mu(dx) = \infty$;

(3) if μ' is a probability measure on $[0, \infty)$ satisfying $\lim_{x \rightarrow \infty} \frac{\mu'(x, \infty)}{\mu(x, \infty)} = c$ for some $c \in (0, \infty)$, then μ' is subexponential.

A function $L(x)$ is called slowly varying at ∞ if $L(x) \neq 0$ and $L(cx) \sim L(x)$, $x \rightarrow \infty$, for any $c > 0$. A function $f(x)$ is called regularly varying of index η at ∞ if $f(x) = x^\eta L(x)$ with $L(x)$ slowly varying at ∞ .

REMARK 25.15. A sufficient condition for subexponentiality is as follows. If μ is a probability measure on $[0, \infty)$ such that $\mu(x, \infty)$ is regularly varying of index $-\alpha$ at ∞ with some $\alpha \geq 0$, then μ is subexponential. In the case of an infinitely divisible distribution with Lévy measure ν , we can also apply this to $\frac{1}{\nu(1, \infty)}[\nu]_{(1, \infty)}$. For example, the Pareto distribution (Remark 8.12) and oned-sided stable distributions (by the form of the Lévy measures in Remark 14.4) are subexponential. As examples not covered by this sufficient condition, the Weibull distribution with parameter $0 < \alpha < 1$ and the log-normal distribution in Remark 8.12 are subexponential.

For related results and references on subexponentiality, see the recent book [110] of Embrecht, Klüppelberg, and Mikosch.

REMARK 25.16. Gribel [156] extends a part of the assertions in Remark 25.14 as follows. Let $h(x)$ be a nonnegative continuous function on $[0, \infty)$ decreasing to 0 as $x \rightarrow \infty$ such that

$$-\int_0^x h(x-y)dh(y) = O(h(x)), \quad x \rightarrow \infty \quad (25.10)$$

Let μ be an infinitely divisible distribution on \mathbb{R} and let ν be its Lévy measure. Then the following hold as $x \rightarrow \infty$: $\mu(x, \infty) = O(h(x))$ if and only if $\nu(x, \infty) = O(h(x))$; $\mu(x, \infty) = o(h(x))$ if and only if $\nu(x, \infty) = o(h(x))$. Examples of functions $h(x)$ satisfying the conditions above are $h(x) = (1+x)^{-\alpha}(1+\log(1+x))^{-\beta}$ with $\alpha > 0, \beta \geq 0$ or with $\alpha = 0, \beta > 0$, and $h(x) = e^{-cx^\alpha}$ with $c > 0, 0 < \alpha < 1$. A sufficient condition for (25.10) is that $\sup_x \frac{h(x)}{h(2x)} < \infty$.

When $g(x) = e^{\langle c, x \rangle}$, the g -moment of a Lévy process is explicitly expressible. We define, for $w = (w_j)_{1 \leq j \leq d}$ and $v = (v_j)_{1 \leq j \leq d}$ in \mathbb{C}^d , the inner product $\langle w, v \rangle = \sum_{j=1}^d w_j v_j$ (not the Hermitian inner product $\sum_{j=1}^d w_j \bar{v}_j$). We write $\operatorname{Re} w = (\operatorname{Re} w_j)_{1 \leq j \leq d} \in \mathbb{R}^d$. Let $D = \{x \in \mathbb{R}^d : |x| \leq 1\}$.

THEOREM 25.17 (Exponential moment). Let $\{X_t\}$ be a Lévy process on \mathbb{R}^d generated by (A, ν, γ) . Let

$$C = \left\{ c \in \mathbb{R}^d : \int_{|x| > 1} e^{\langle c, x \rangle} \nu(dx) < \infty \right\}$$

- (i) The set C is convex and contains the origin.
- (ii) $c \in C$ if and only if $E e^{\langle c, X_t \rangle} < \infty$ for some $t > 0$ or, equivalently, for every $t > 0$.
- (iii) If $w \in \mathbb{C}^d$ is such that $\operatorname{Re} w \in C$, then

$$\Psi(w) = \frac{1}{2} \langle w, Aw \rangle + \int_{\mathbb{R}^d} \left(e^{\langle w, x \rangle} - 1 - \langle w, x \rangle 1_D(x) \right) \nu(dx) + \langle \gamma, w \rangle \quad (25.11)$$

is definable, $E |e^{\langle w, X_t \rangle}| < \infty$, and

$$E \left[e^{\langle w, X_t \rangle} \right] = e^{t\Psi(w)} \quad (25.12)$$

Proof. (i) Obviously C contains the origin. If c_1 and c_2 are in C , then, for any $0 < r < 1$ and $s = 1 - r$,

$$\int_{|x|>1} e^{\langle rc_1 + sc_2, x \rangle} \nu(dx) \leq \left(\int_{|x|>1} e^{\langle c_1, x \rangle} \nu(dx) \right)^\tau \left(\int_{|x|>1} e^{\langle c_2, x \rangle} \nu(dx) \right)^s < \infty$$

by Hölder's inequality. Hence C is convex.

(ii) The function $g(x) = e^{\langle c, x \rangle}$ is clearly submultiplicative. Hence Theorem 25.3 gives the assertion.

(iii). Any linear transformation U of \mathbb{R}^d to \mathbb{R}^d can be uniquely extended to a linear transformation of \mathbb{C}^d to \mathbb{C}^d . Regarding U as a $d \times d$ matrix, it is easy to see that $\langle w, Uv \rangle = \langle U'w, v \rangle$ for $w, v \in \mathbb{C}^d$, where U' is the transpose of U . Now let $\operatorname{Re} w \in C$. Then $\int_{|x|>1} |e^{\langle w, x \rangle}| \nu(dx) = \int_{|x|>1} e^{\langle \operatorname{Re} w, x \rangle} \nu(dx) < \infty$, which shows that $\Psi(w)$ of (25.11) is definable and finite. Also, $E |e^{\langle w, X_t \rangle}| = E e^{\langle \operatorname{Re} w, X_t \rangle} < \infty$ by (ii). Let us show (25.12) in three steps.

Step 1. Let e_1 be the unit vector with first component 1. Assume that $e_1 \in C$. Let us prove (25.12) for all $w = (w_j)_{1 \leq j \leq d}$ with $\operatorname{Re} w_1 \in [0, 1]$ and $\operatorname{Re} w_j = 0, 2 \leq j \leq d$. Fix $t > 0$ and $w_2, \dots, w_d \in \mathbb{C}$ with $\operatorname{Re} w_j = 0, 2 \leq j \leq d$, and regard w_1 as variable in $F = \{w_1 \in \mathbb{C} : \operatorname{Re} w_1 \in [0, 1]\}$. Consider $f(w_1) = E e^{\langle w, X_t \rangle}$. Then $f(w_1)$ is continuous on F , since

$$\left| e^{\langle w, X(t) \rangle} \right| = e^{(\operatorname{Re} w_1) X_1(t)} \leq (\operatorname{Re} w_1) e^{X_1(t)} + (1 - \operatorname{Re} w_1) \leq e^{X_1(t)} + 1$$

by the convexity of $e^{uX_1(t)}$ in u , where $X_1(t)$ is the first component of $X(t)$. Moreover, $f(w_1)$ is analytic in the interior of F , since it is the limit of the analytic functions $E [e^{\langle v, X_t \rangle}; |X_t| \leq n]$ as $n \rightarrow \infty$. Similarly, $h(w_1) = e^{t\psi(w)}$ is continuous on F and analytic in the interior of F . If $\operatorname{Re} w_1 = 0$, then $f(w_1) = h(w_1)$, which is the Lévy-Khintchine representation of P_{X_i} . Therefore, as in the proof of Theorem 24.11, the principle of reflection and the uniqueness theorem yield (25.12) when $\operatorname{Re} w_1 \in [0, 1]$.

Step 2. Let U be a linear transformation from \mathbb{R}^d onto \mathbb{R}^d . Let $Y_t = UX_t$. Then $\{Y_t\}$ is a Lévy process with generating triplet (A_U, ν_U, γ_U) by Proposition 11.10. Write

$$C_U = \left\{ c \in \mathbb{R}^d : \int_{|x|>1} e^{\langle c, x \rangle} \nu_U(dx) < \infty \right\}$$

Since $\nu_U = \nu U^{-1}$, we have $C_U = (U')^{-1} C$. Given $w \in \mathbb{C}^d$ satisfying $\operatorname{Re} w \in C$, let $v = (U')^{-1} w$. Then $\operatorname{Re} v \in C_U$. Define

$$\Psi_U(v) = \frac{1}{2} \langle v, A_U v \rangle + \int_{\mathbb{R}^d} \left(e^{\langle v, x \rangle} - 1 - \langle v, x \rangle 1_D(x) \right) \nu_U(dx) + \langle \gamma_U, v \rangle$$

We claim that if

$$E \left[e^{\langle v, Y_t \rangle} \right] = e^{t \Psi_U(v)} \quad (25.13)$$

then w satisfies (25.12). In fact, $\langle v, Y_t \rangle = \left\langle \langle U^{-1} \rangle' w, Y_t \right\rangle = \langle w, X_t \rangle$ and

$$\begin{aligned} \Psi_U(v) &= \frac{1}{2} \langle U'v, AU'v \rangle + \int \left(e^{\langle v, Ux \rangle} - 1 - \langle v, Ux \rangle 1_D(x) \right) v(\mathrm{d}x) + \langle U\gamma, v \rangle \\ &= \Psi(w) \end{aligned}$$

by (11.8)-(11.10). That is, (25.13) is identical with (25.12).

Step 3. Given $w \in \mathbb{C}^d$ satisfying $\operatorname{Re} w \in C$, we shall show (25.12). If $\operatorname{Re} w = 0$, there is nothing to prove. Assume $\operatorname{Re} w \neq 0$. Choose a linear transformation U from \mathbb{R}^d onto \mathbb{R}^d such that $\operatorname{Re} w = U'e_1$. Consider the Lévy process $Y_t = UX_t$. Since $C_U = (U')^{-1}C$, we have $e_1 \in C_U$. We know, by Step 1, that, if $v \in \mathbb{C}^d$ satisfies $\operatorname{Re} v = e_1$, then (25.13) holds. Hence, by the result of Step 2, w satisfies (25.12). \square

We close this section with a discussion of the g -moments of $\sup_{s \in [0, t]} |X_s|$

THEOREM 25.18. Let $\{X_t\}$ be a Lévy process on \mathbb{R}^d . Define

$$X_t^* = \sup_{s \in [0, t]} |X_s| \quad (25.14)$$

Let $g(r)$ be a nonnegative continuous submultiplicative function on $[0, \infty)$, increasing to ∞ as $r \rightarrow \infty$. Then the following four statements are equivalent.

- (1) $E[g(X_t^*)] < \infty$ for some $t > 0$.
- (2) $E[g(X_t^*)] < \infty$ for every $t > 0$.
- (3) $E[g(|X_t|)] < \infty$ for some $t > 0$.
- (4) $E[g(|X_t|)] < \infty$ for every $t > 0$.

Proof. Since $g(|x|)$ is submultiplicative on \mathbb{R}^d , (3) and (4) are equivalent by Theorem 25.3. As $|X_t| \leq X_t^*$, all we have to show is that, for any fixed $t > 0$, $E[g(|X_t|)] < \infty$ implies $E[g(X_t^*)] < \infty$. We claim that, for any $a > 0$ and $b > 0$, \square

$$P[X_t^* > a + b] \leq P[|X_t| > a] / P[X_t^* \leq b/2] \quad (25.15)$$

Fix t and let $t_{n,j} = jt/2^n$ for $j = 1, \dots, 2^n$ and $X_{(n)}^* = \max_{1 \leq j \leq 2^n} |X_{t_{n,j}}|$. Choosing $Z_j(s) = Z_j = X_{t_{n,j}} - X_{t_{n,j-1}}$ in Lemma 20.2 and using Remark 20.3, we have

$$P[X_{(n)}^* > a + b] \leq P[|X_t| > a] / P[X_{(n)}^* \leq b/2]$$

in (20.2). Hence, letting $n \rightarrow \infty$, we get (25.15). Choose $b > 0$ such that $P[X_t^* \leq b/2] > 0$. Let $\tilde{g}(r)$ be a continuous increasing function on $[0, \infty)$ such that $\tilde{g}(0) = 0$ and $\tilde{g}(r) = g(r)$ for $r \geq 1$. Apply Lemma 17.6 to $k(r) = 1 - P[|X_t| \leq r]$ and $l(r) = \tilde{g}(r)$. Then

$$\int_{0+}^{\infty} P[|X_t| > r] d\tilde{g}(r) = \int_{(0,\infty)} \tilde{g}(r) P[|X_t| \in dr] = E[\tilde{g}(|X_t|)]$$

14 follows from (25.15) that

$$\int_{0+}^{\infty} P[X_t^* > r + b] d\tilde{g}(r) \leq E[\tilde{g}(|X_t|)] / P[X_t^* \leq b/2]$$

The integral in the left-hand side equals

$$\int_{(0,\infty)} \tilde{g}(r) P[X_t^* - b \in dr] = E[\tilde{g}(X_t^* - b); X_t^* > b]$$

similarly. Hence, if $E[g(|X_t|)] < \infty$, then $E[g(X_t^* - b); X_t^* > b] < \infty$ and, by the submultiplicativity of g , $E[g(X_t^*)] < \infty$.

Remark 25.19. Let $d = 1$. Doob [93], p. 337, shows an explicit bound:

$$E[(X_t^*)^\alpha] \leq 8E[|X_t|^\alpha] \text{ for } \alpha \geq 1 \quad (25.16)$$

provided that $E|X_t| < \infty$ and $EX_t = 0$. This is true not only for Lévy processes but also for additive processes on \mathbb{R} . Define the supremum process $M_t = \sup_{0 \leq s \leq t} X_s$. If $\frac{1}{\nu(1,\infty)}[\nu]_{(1,\infty)}$ is subexponential, then $P[M_t > x] / \nu(x, \infty) \rightarrow t$ as $x \rightarrow \infty$. This is due to Berman [18] and Rosinski and Samorodnitsky [382]. Extension of this result in the case where the right tail of ν is lighter is studied in Braverman and Samorodnitsky [57] and Braverman [56].