See Exercises 29.4–29.6 for further information.

Proof of theorem. Use the translation invariance of the Lebesgue measure. Then

$$\int P[\,x + X_t \in B\,]\mathrm{d}x = \int \mathrm{d}x \int 1_B(x+y)\mu^t(\mathrm{d}y) = \int \mu^t(\mathrm{d}y)\,\mathrm{Leb}(B-y)$$
$$= \int \mu^t(\mathrm{d}y)\,\mathrm{Leb}(B) = \mathrm{Leb}(B).$$

This is (24.16) for the Lebesgue measure. □

25. Moments

We define the g-moment of a random variable and discuss finiteness of the g-moment of X_t for a Lévy process $\{X_t\}$.

DEFINITION 25.1. Let $g(x)$ be a nonnegative measurable function on $\mathbb{R}^d$. We call $\int g(x)\mu(\mathrm{d}x)$ the *g-moment* of a measure μ on $\mathbb{R}^d$. We call $E[g(X)]$ the *g-moment* of a random variable X on $\mathbb{R}^d$.

DEFINITION 25.2. A function $g(x)$ on $\mathbb{R}^d$ is called *submultiplicative* if it is nonnegative and there is a constant $a > 0$ such that

(25.1) $$g(x+y) \le a g(x) g(y) \quad \text{for } x, y \in \mathbb{R}^d.$$

A function bounded on every compact set is called *locally bounded.*

THEOREM 25.3 (g-Moment). *Let g be a submultiplicative, locally bounded, measurable function on $\mathbb{R}^d$. Then, finiteness of the g-moment is not a time dependent distributional property in the class of Lévy processes. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure ν. Then, X_t has finite g-moment for every $t > 0$ if and only if $[\nu]_{\{|x|>1\}}$ has finite g-moment.*

The following facts indicate the wide applicability of the theorem.

PROPOSITION 25.4. (i) *The product of two submultiplicative functions is submultiplicative.*

(ii) *If $g(x)$ is submultiplicative on $\mathbb{R}^d$, then so is $g(cx+\gamma)^\alpha$ with $c \in \mathbb{R}$, $\gamma \in \mathbb{R}^d$, and $\alpha > 0$.*

(iii) *Let $0 < \beta \le 1$. Then the following functions are submultiplicative:*

$$|x| \vee 1,\ |x_j| \vee 1,\ x_j \vee 1,\ \exp(|x|^\beta),\ \exp(|x_j|^\beta),$$
$$\exp((x_j \vee 0)^\beta),\ \log(|x| \vee \mathrm{e}),\ \log(|x_j| \vee \mathrm{e}),\ \log(x_j \vee \mathrm{e}),$$
$$\log\log(|x| \vee \mathrm{e}^{\mathrm{e}}),\ \log\log(|x_j| \vee \mathrm{e}^{\mathrm{e}}),\ \log\log(x_j \vee \mathrm{e}^{\mathrm{e}}).$$

Here x_j is the jth component of x.

Proof. (i) Immediate from the definition.

(ii) Let $g_1(x) = g(cx)$, $g_2(x) = g(x+\gamma)$, and $g_3(x) = g(x)^\alpha$. Then it follows from (25.1) that $g_1(x+y) \leq ag_1(x)g_1(y)$, $g_2(x+y) \leq a^2 g(-\gamma)g_2(x)g_2(y)$, and $g_3(x+y) \leq a^\alpha g_3(x)g_3(y)$.

(iii) Let $h(u)$ be a positive increasing function on $\mathbb{R}$ such that, for some $b \geq 0$, $h(u)$ is flat on $(-\infty, b]$ and $\log h(u)$ is concave on $[b, \infty)$. Then $h(u)$ is submultiplicative on $\mathbb{R}$. In fact, for $u, v \geq b$, the function $f(u) = \log h(u)$ satisfies

$$f(u+b) - f(u) \leq f(2b) - f(b),$$
$$f(u+v) - f(v) \leq f(u+b) - f(b),$$

and hence

$$f(u+v) \leq f(u+b) - f(b) + f(v) \leq f(2b) - 2f(b) + f(u) + f(v),$$

which shows

$$h(u+v) \leq \text{const}\, h(u)h(v). \tag{25.2}$$

It follows that (25.2) holds for all $u, v \in \mathbb{R}$. The functions $u \vee 1$, $\exp((u \vee 0)^\beta)$, $\log(u \vee \mathrm{e})$, and $\log\log(u \vee \mathrm{e}^\mathrm{e})$ fulfill the conditions on $h(u)$. By (25.2) and by the increasingness of h, the functions $h(|x|)$, $h(|x_j|)$, and $h(x_j)$ are submultiplicative on $\mathbb{R}^d$. □

We prove Theorem 25.3 after three lemmas.

LEMMA 25.5. *If $g(x)$ is submultiplicative and locally bounded, then*

$$g(x) \leq b\mathrm{e}^{c|x|} \tag{25.3}$$

with some $b > 0$ and $c > 0$.

Proof. Choose b in such a way that $\sup_{|x|\leq 1} g(x) \leq b$ and $ab > 1$. If $n - 1 < |x| \leq n$, then

$$g(x) \leq a^{n-1} g(\tfrac{1}{n}x)^n \leq a^{n-1}b^n \leq b(ab)^{|x|},$$

which shows (25.3). □

LEMMA 25.6. *Let μ be an infinitely divisible distribution on $\mathbb{R}$ with Lévy measure ν supported on a bounded set. Then $\widehat{\mu}(z)$ can be extended to an entire function on $\mathbb{C}$.*

Proof. There is a finite $a > 0$ such that $S_\nu \subset [-a, a]$. The Lévy representation of $\widehat{\mu}(z)$ is written as

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}Az^2 + \int_{[-a,a]} (\mathrm{e}^{\mathrm{i}zx} - 1 - \mathrm{i}zx)\nu(\mathrm{d}x) + \mathrm{i}\gamma' z\right]$$

with some $\gamma' \in \mathbb{R}$. The right-hand side is meaningful even if z is complex. Denote this function by $\Phi(z)$. Then $\Phi(z)$ is an entire function, since we can exchange the order of integration and differentiation. □

LEMMA 25.7. *If μ is a probability measure on $\mathbb{R}$ and $\widehat{\mu}(z)$ is extendible to an entire function on $\mathbb{C}$, then μ has finite exponential moments, that is, it has finite $\mathrm{e}^{c|x|}$-moment for every $c > 0$.*

Proof. It follows from Proposition 2.5(x) that $\alpha_n = \int x^n \mu(\mathrm{d}x)$ and $\beta_n = \int |x|^n \mu(\mathrm{d}x)$ are finite for any $n \geq 1$. Since $\dfrac{\mathrm{d}^n \widehat{\mu}}{\mathrm{d}z^n}(0) = \mathrm{i}^n \alpha_n$, we have

$$\widehat{\mu}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{i}^n \alpha_n z^n,$$

the radius of convergence of the right-hand side being infinite. Notice that $\beta_{2k} = \alpha_{2k}$ and $\beta_{2k+1} \leq \frac{1}{2}(\alpha_{2k+2} + \alpha_{2k})$, since $|x|^{2k+1} \leq \frac{1}{2}(x^{2k+2} + x^{2k})$. It follows that

$$\int \mathrm{e}^{c|x|} \mu(\mathrm{d}x) = \sum_{n=0}^{\infty} \frac{1}{n!} \beta_n c^n < \infty,$$

completing the proof. $\square$

Proof of Theorem 25.3. Let $\nu_0 = [\nu]_{\{|x| \leq 1\}}$ and $\nu_1 = [\nu]_{\{|x|>1\}}$. Construct independent Lévy processes $\{X_t^0\}$ and $\{X_t^1\}$ on $\mathbb{R}^d$ such that $\{X_t\} \stackrel{\mathrm{d}}{=} \{X_t^0 + X_t^1\}$ and $\{X_t^1\}$ is compound Poisson with Lévy measure ν_1. Let μ_0 and μ_1 be the distributions of X_1^0 and X_1^1, respectively.

Suppose that X_t has finite g-moment for some $t > 0$. It follows from

$$E[g(X_t)] = \iint g(x+y) {\mu_0}^t(\mathrm{d}x) {\mu_1}^t(\mathrm{d}y)$$

that $\int g(x+y) {\mu_1}^t(\mathrm{d}y) < \infty$ for some x. This means

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(x+y) {\nu_1}^n(\mathrm{d}y) < \infty.$$

Since $g(y) \leq a g(-x) g(x+y) \leq ab\mathrm{e}^{c|x|} g(x+y)$ by Lemma 25.5, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int g(y) {\nu_1}^n(\mathrm{d}y) < \infty. \tag{25.4}$$

Hence $\int g(y) \nu_1(\mathrm{d}y) < \infty$.

Conversely, suppose that $\int g(y) \nu_1(\mathrm{d}y) < \infty$. Let us prove that $E[g(X_t)] < \infty$ for every t. By the submultiplicativity,

$$\begin{aligned}\int g(y) {\nu_1}^n(\mathrm{d}y) &= \int \cdots \int g(y_1 + \cdots + y_n) \nu_1(\mathrm{d}y_1) \ldots \nu_1(dy_n)\\ &\leq a^{n-1} \left(\int g(y) \nu_1(\mathrm{d}y) \right)^n.\end{aligned}$$

Hence we have (25.4) for every t. That is, X_t^1 has finite g-moment. Since

$$E[g(X_t)] \le abE[\mathrm{e}^{c|X_t^0|}]\, E[g(X_t^1)]$$

by (25.1) and (25.3), it remains only to show that $E[\mathrm{e}^{c|X_t^0|}] < \infty$. Let $X_j^0(t)$, $1 \le j \le d$, be the components of X_t^0. Then

$$E[\mathrm{e}^{c|X_t^0|}] \le E\left[\exp\left(c\sum_{j=1}^{d}|X_j^0(t)|\right)\right] \le E\left[\prod_{j=1}^{d}(\mathrm{e}^{cX_j^0(t)} + \mathrm{e}^{-cX_j^0(t)})\right],$$

which is written as a sum of a finite number of terms of the form $E[\exp X_t^\sharp]$ with $X_t^\sharp$ being a linear combination of $X_j^0(t)$, $1 \le j \le d$. Since $\{X_t^\sharp\}$ is a Lévy process on $\mathbb{R}$ with Lévy measure supported on a bounded set (use Proposition 11.10), $E[\exp X_t^\sharp]$ is finite by virtue of Lemmas 25.6 and 25.7. This proves all statements in the theorem. □

COROLLARY 25.8. *Let $\alpha > 0$, $0 < \beta \le 1$, and $\gamma \ge 0$. None of the properties $\int |x|^\alpha \mu(\mathrm{d}x) < \infty$, $\int (0 \vee \log|x|)^\alpha \mu(\mathrm{d}x) < \infty$, and $\int |x|^\gamma \mathrm{e}^{\alpha|x|^\beta}\mu(\mathrm{d}x) < \infty$ is time dependent in the class of Lévy processes. For a Lévy process on $\mathbb{R}^d$ with Lévy measure ν, each of the properties is expressed by the corresponding property of $[\nu]_{\{|x|>1\}}$.*

This follows from Theorem 25.3 and Proposition 25.4.

REMARK 25.9. There is a nonnegative measurable function $g(x)$ satisfying (25.3) such that finiteness of the g-moment is a time dependent distributional property in the class of Lévy processes. For example, let $g(x) = (1 \wedge |x|^{-\alpha})\mathrm{e}^{|x|}$ with $\alpha > 0$. Consider a Γ-process $\{X_t\}$ with $EX_1 = 1$. Then it is easy to see that $E[g(X_t)] < \infty$ if and only if $t < \alpha$. This process has $\nu = x^{-1}\mathrm{e}^{-x}1_{(0,\infty)}(x)\mathrm{d}x$, so that ν_1 has finite g-moment (Example 8.10).

EXAMPLE 25.10. Let $\{X_t\}$ be a non-trivial semi-stable process on $\mathbb{R}^d$ with index $\alpha \in (0, 2)$. Then, for every $t > 0$, $E[|X_t|^\eta]$ is finite or infinite according as $0 < \eta < \alpha$ or $\eta \ge \alpha$, respectively. To see this, notice that the argument in the proofs of Theorem 13.15 and Proposition 14.5 gives

$$\int_{S_n(b)} |x|^\eta \nu(\mathrm{d}x) = b^{n(\eta-\alpha)} \int_{S_0(b)} |x|^\eta \nu(\mathrm{d}x),$$

and hence $\int_{|x|>1} |x|^\eta \nu(\mathrm{d}x) < \infty$ if and only if $\eta < \alpha$; apply Corollary 25.8. In particular, for a stable process on $\mathbb{R}$ with parameters (α, β, τ, c) (Definition 14.16), $E[X_t] = \tau t$ if $1 < \alpha < 2$ (use Proposition 2.5(ix)). The following explicit results are known. If $0 < \alpha < 1$ and $\{X_t\}$ is a stable subordinator with $E[\mathrm{e}^{-uX_t}] = \mathrm{e}^{-tc'u^\alpha}$ (Example 24.12), then, for $-\infty < \eta < \alpha$,

$$E[X_t^\eta] = (tc')^{\eta/\alpha}\frac{\Gamma(1-\frac{\eta}{\alpha})}{\Gamma(1-\eta)}, \tag{25.5}$$

which is shown by Wolfe [**508**] and Shanbhag and Sreehari [**419**] (Exercise 29.17). If $0 < \alpha < 2$ and $\{X_t\}$ is symmetric and α-stable on $\mathbb{R}$ with $E[\mathrm{e}^{\mathrm{i}zX_t}] = \mathrm{e}^{-tc|z|^\alpha}$ (Theorem 14.14), then, for $-1 < \eta < \alpha$,

$$E[|X_t|^\eta] = (tc)^{\eta/\alpha} \frac{2^\eta \Gamma(\frac{1+\eta}{2})\Gamma(1-\frac{\eta}{\alpha})}{\sqrt{\pi}\,\Gamma(1-\frac{\eta}{2})}, \tag{25.6}$$

as is shown in [**419**].

EXAMPLE 25.11. If $\{X_t\}$ is a Lévy process on $\mathbb{R}$ with Lévy measure supported on $(-\infty, 0]$, then $E[\mathrm{e}^{cX_t}] < \infty$ for every $c > 0$ and $t > 0$. Use Theorem 25.3 for $g(x) = \mathrm{e}^{cx}$. For instance, a stable process on $\mathbb{R}$ with $1 \le \alpha < 2$ and $\beta = -1$ satisfies this assumption although it has support $\mathbb{R}$ for every $t > 0$ (Theorem 24.10(i)).

EXAMPLE 25.12. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ generated by (A, ν, γ). In components, $X_t = (X_j(t))$, $\gamma = (\gamma_j)$, and $A = (A_{jk})$. Then X_t has finite mean for $t > 0$ if and only if $\int_{|x|>1} |x|\nu(\mathrm{d}x) < \infty$. When this condition is met, we can find $m_j(t) = E[X_j(t)]$ expressed as

$$m_j(t) = t\left(\int_{|x|>1} x_j \nu(\mathrm{d}x) + \gamma_j\right) = t\gamma_{1,j}, \qquad j = 1, \dots, d, \tag{25.7}$$

differentiating $\widehat{\mu}(z)$ (Proposition 2.5(ix)). Here $\gamma_{1,j}$ is the jth component of the center γ_1 in (8.8). Similarly, $E[|X_t|^2] < \infty$ for all $t > 0$ if and only if $\int_{|x|>1} |x|^2\nu(\mathrm{d}x) < \infty$. In this case,

$$v_{jk}(t) = E[(X_j(t) - m_j(t))(X_k(t) - m_k(t))], \qquad j, k = 1, \dots, d,$$

the (j, k) elements of the covariance matrix of $X(t)$, are expressed as

$$v_{jk}(t) = t\left(A_{jk} + \int_{\mathbb{R}^d} x_j x_k \nu(\mathrm{d}x)\right). \tag{25.8}$$

Theorem 25.3 shows that, for a Lévy process $\{X_t\}$ with Lévy measure ν, the tails of P_{X_t} and ν have a kind of similarity. Are they actually equivalent in some class of Lévy processes? This question was answered by Embrecht, Goldie, and Veraverbeke [**109**] for subordinators. We state their result without proof in two remarks below.

DEFINITION 25.13. A probability measure μ on $[0, \infty)$ is called *subexponential* if $\mu(x, \infty) > 0$ for every x and

$$\lim_{x\to\infty} \frac{\mu^n(x, \infty)}{\mu(x, \infty)} = n \quad \text{for } n = 2, 3, \dots. \tag{25.9}$$

The class of probability measures satisfying (25.9) above was introduced by Chistyakov [**65**]. The condition can be weakened. Specifically, if

$$\limsup_{x\to\infty} \frac{\mu^2(x, \infty)}{\mu(x, \infty)} \le 2,$$

then μ is subexponential. The meaning of (25.9) is as follows. Let $\{Z_j\}$ be independent nonnegative random variables each with distribution μ and let $S_n = \sum_{j=1}^n Z_j$ and $M_n = \max_{1\le j\le n} Z_j$. Then μ satisfies (25.9) if and only if

$$P[\,S_n > x\,] \sim P[\,M_n > x\,], \quad x \to \infty, \quad \text{for } n = 2, 3, \ldots.$$

In fact, $P[\,S_n > x\,] = \mu^n(x,\infty)$ and

$$\begin{aligned} P[\,M_n > x\,] &= \textstyle\sum_{j=1}^n P[\,X_1 \le x, \ldots, X_{j-1} \le x, X_j > x\,] \\ &= \textstyle\sum_{j=1}^n \mu(0,x]^{j-1}\mu(x,\infty) \sim n\mu(x,\infty) \quad \text{as } x \to \infty. \end{aligned}$$

REMARK 25.14. A basic result on subexponentiality is as follows. If $\{X_t\}$ be a subordinator with Lévy measure ν, then the following conditions are equivalent [109]:

(1) $\nu(1,\infty) > 0$ and $\frac{1}{\nu(1,\infty)}[\nu]_{(1,\infty)}$ is subexponential;
(2) P_{X_t} is subexponential for every $t > 0$;
(3) P_{X_t} is subexponential for some $t > 0$;
(4) $P[\,X_t > x\,] \sim t\nu(x,\infty)$, $x \to \infty$, for every $t > 0$;
(5) $P[\,X_t > x\,] \sim t\nu(x,\infty)$, $x \to \infty$, for some $t > 0$.

Some of the consequences of subexponentiality are as follows. Let μ be a subexponential probability measure on $[0,\infty)$. Then,

(1) for any $y \in \mathbb{R}$, $\mu(x-y,\infty)/\mu(x,\infty) \to 1$ as $x \to \infty$;
(2) for every $\varepsilon > 0$, $\int_{[0,\infty)} e^{\varepsilon x}\mu(dx) = \infty$;
(3) if μ' is a probability measure on $[0,\infty)$ satisfying $\lim_{x\to\infty} \frac{\mu'(x,\infty)}{\mu(x,\infty)} = c$ for some $c \in (0,\infty)$, then μ' is subexponential.

A function $L(x)$ is called *slowly varying* at ∞ if $L(x) \ne 0$ and $L(cx) \sim L(x)$, $x \to \infty$, for any $c > 0$. A function $f(x)$ is called *regularly varying of index* η at ∞ if $f(x) = x^\eta L(x)$ with $L(x)$ slowly varying at ∞.

REMARK 25.15. A sufficient condition for subexponentiality is as follows. If μ is a probability measure on $[0,\infty)$ such that $\mu(x,\infty)$ is regularly varying of index $-\alpha$ at ∞ with some $\alpha \ge 0$, then μ is subexponential. In the case of an infinitely divisible distribution with Lévy measure ν, we can also apply this to $\frac{1}{\nu(1,\infty)}[\nu]_{(1,\infty)}$. For example, the Pareto distribution (Remark 8.12) and one-sided stable distributions (by the form of the Lévy measures in Remark 14.4) are subexponential. As examples not covered by this sufficient condition, the Weibull distribution with parameter $0 < \alpha < 1$ and the log-normal distribution in Remark 8.12 are subexponential.

For related results and references on subexponentiality, see the recent book [110] of Embrecht, Klüppelberg, and Mikosch.

REMARK 25.16. Grübel [156] extends a part of the assertions in Remark 25.14 as follows. Let $h(x)$ be a nonnegative continuous function on $[0,\infty)$ decreasing to 0 as $x \to \infty$ such that

$$-\int_0^x h(x-y)dh(y) = O(h(x)), \quad x \to \infty. \tag{25.10}$$

Let μ be an infinitely divisible distribution on $\mathbb{R}$ and let ν be its Lévy measure. Then the following hold as $x \to \infty$: $\mu(x,\infty) = O(h(x))$ if and only if $\nu(x,\infty) = O(h(x))$; $\mu(x,\infty) = o(h(x))$ if and only if $\nu(x,\infty) = o(h(x))$. Examples of functions $h(x)$ satisfying the conditions above are $h(x) = (1+x)^{-\alpha}(1+\log(1+x))^{-\beta}$ with $\alpha > 0$, $\beta \geq 0$ or with $\alpha = 0$, $\beta > 0$, and $h(x) = \mathrm{e}^{-cx^{\alpha}}$ with $c > 0$, $0 < \alpha < 1$. A sufficient condition for (25.10) is that $\sup_x \frac{h(x)}{h(2x)} < \infty$.

When $g(x) = \mathrm{e}^{\langle c,x\rangle}$, the g-moment of a Lévy process is explicitly expressible. We define, for $w = (w_j)_{1\leq j\leq d}$ and $v = (v_j)_{1\leq j\leq d}$ in $\mathbb{C}^d$, the inner product $\langle w,v\rangle = \sum_{j=1}^{d} w_j v_j$ (not the Hermitian inner product $\sum_{j=1}^{d} w_j \overline{v_j}$). We write $\operatorname{Re} w = (\operatorname{Re} w_j)_{1\leq j\leq d} \in \mathbb{R}^d$. Let $D = \{x \in \mathbb{R}^d \colon |x| \leq 1\}$.

THEOREM 25.17 (Exponential moment). *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ generated by (A,ν,γ). Let*

$$C = \left\{c \in \mathbb{R}^d \colon \int_{|x|>1} \mathrm{e}^{\langle c,x\rangle}\nu(\mathrm{d}x) < \infty\right\}.$$

(i) *The set C is convex and contains the origin.*

(ii) *$c \in C$ if and only if $E\mathrm{e}^{\langle c,X_t\rangle} < \infty$ for some $t > 0$ or, equivalently, for every $t > 0$.*

(iii) *If $w \in \mathbb{C}^d$ is such that $\operatorname{Re} w \in C$, then*

$$\Psi(w) = \frac{1}{2}\langle w, Aw\rangle + \int_{\mathbb{R}^d} (\mathrm{e}^{\langle w,x\rangle} - 1 - \langle w,x\rangle 1_D(x))\nu(\mathrm{d}x) + \langle \gamma, w\rangle \tag{25.11}$$

is definable, $E|\mathrm{e}^{\langle w,X_t\rangle}| < \infty$, and

$$E[\mathrm{e}^{\langle w,X_t\rangle}] = \mathrm{e}^{t\Psi(w)}. \tag{25.12}$$

Proof. (i) Obviously C contains the origin. If c_1 and c_2 are in C, then, for any $0 < r < 1$ and $s = 1-r$,

$$\int_{|x|>1} \mathrm{e}^{\langle rc_1+sc_2,x\rangle}\nu(\mathrm{d}x) \leq \left(\int_{|x|>1} \mathrm{e}^{\langle c_1,x\rangle}\nu(\mathrm{d}x)\right)^r \left(\int_{|x|>1} \mathrm{e}^{\langle c_2,x\rangle}\nu(\mathrm{d}x)\right)^s < \infty$$

by Hölder's inequality. Hence C is convex.

(ii) The function $g(x) = \mathrm{e}^{\langle c,x\rangle}$ is clearly submultiplicative. Hence Theorem 25.3 gives the assertion.

(iii) Any linear transformation U of $\mathbb{R}^d$ to $\mathbb{R}^d$ can be uniquely extended to a linear transformation of $\mathbb{C}^d$ to $\mathbb{C}^d$. Regarding U as a $d\times d$ matrix, it is easy to see that $\langle w, Uv\rangle = \langle U'w, v\rangle$ for $w, v \in \mathbb{C}^d$, where U' is the transpose of U. Now let $\operatorname{Re} w \in C$. Then $\int_{|x|>1} |\mathrm{e}^{\langle w,x\rangle}|\nu(\mathrm{d}x) = \int_{|x|>1} \mathrm{e}^{\langle \operatorname{Re} w,x\rangle}\nu(\mathrm{d}x) < \infty$, which shows that $\Psi(w)$ of (25.11) is definable and finite. Also, $E|\mathrm{e}^{\langle w,X_t\rangle}| = E\mathrm{e}^{\langle \operatorname{Re} w,X_t\rangle} < \infty$ by (ii). Let us show (25.12) in three steps.

Step 1. Let e_1 be the unit vector with first component 1. Assume that $e_1 \in C$. Let us prove (25.12) for all $w = (w_j)_{1\leq j\leq d}$ with $\operatorname{Re} w_1 \in [0,1]$ and

$\operatorname{Re} w_j = 0$, $2 \le j \le d$. Fix $t > 0$ and $w_2, \ldots, w_d \in \mathbb{C}$ with $\operatorname{Re} w_j = 0$, $2 \le j \le d$, and regard w_1 as variable in $F = \{w_1 \in \mathbb{C} \colon \operatorname{Re} w_1 \in [0,1]\}$. Consider $f(w_1) = E\mathrm{e}^{\langle w, X_t\rangle}$. Then $f(w_1)$ is continuous on F, since

$$|\mathrm{e}^{\langle w, X(t)\rangle}| = \mathrm{e}^{(\operatorname{Re} w_1)X_1(t)} \le (\operatorname{Re} w_1)\mathrm{e}^{X_1(t)} + (1 - \operatorname{Re} w_1) \le \mathrm{e}^{X_1(t)} + 1$$

by the convexity of $\mathrm{e}^{uX_1(t)}$ in u, where $X_1(t)$ is the first component of $X(t)$. Moreover, $f(w_1)$ is analytic in the interior of F, since it is the limit of the analytic functions $E[\mathrm{e}^{\langle w, X_t\rangle};\ |X_t| \le n]$ as $n \to \infty$. Similarly, $h(w_1) = \mathrm{e}^{t\Psi(w)}$ is continuous on F and analytic in the interior of F. If $\operatorname{Re} w_1 = 0$, then $f(w_1) = h(w_1)$, which is the Lévy–Khintchine representation of P_{X_t}. Therefore, as in the proof of Theorem 24.11, the principle of reflection and the uniqueness theorem yield (25.12) when $\operatorname{Re} w_1 \in [0,1]$.

Step 2. Let U be a linear transformation from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Let $Y_t = UX_t$. Then $\{Y_t\}$ is a Lévy process with generating triplet (A_U, ν_U, γ_U) by Proposition 11.10. Write

$$C_U = \left\{c \in \mathbb{R}^d \colon \int_{|x|>1} \mathrm{e}^{\langle c, x\rangle} \nu_U(\mathrm{d}x) < \infty\right\}.$$

Since $\nu_U = \nu U^{-1}$, we have $C_U = (U')^{-1}C$. Given $w \in \mathbb{C}^d$ satisfying $\operatorname{Re} w \in C$, let $v = (U')^{-1}w$. Then $\operatorname{Re} v \in C_U$. Define

$$\Psi_U(v) = \frac{1}{2}\langle v, A_U v\rangle + \int_{\mathbb{R}^d} (\mathrm{e}^{\langle v, x\rangle} - 1 - \langle v, x\rangle 1_D(x))\nu_U(\mathrm{d}x) + \langle \gamma_U, v\rangle.$$

We claim that if

$$E[\mathrm{e}^{\langle v, Y_t\rangle}] = \mathrm{e}^{t\Psi_U(v)}, \tag{25.13}$$

then w satisfies (25.12). In fact, $\langle v, Y_t\rangle = \langle (U^{-1})'w, Y_t\rangle = \langle w, X_t\rangle$ and

$$\begin{aligned}\Psi_U(v) &= \frac{1}{2}\langle U'v, AU'v\rangle + \int (\mathrm{e}^{\langle v, Ux\rangle} - 1 - \langle v, Ux\rangle 1_D(x))\nu(\mathrm{d}x) + \langle U\gamma, v\rangle \\ &= \Psi(w)\end{aligned}$$

by (11.8)–(11.10). That is, (25.13) is identical with (25.12).

Step 3. Given $w \in \mathbb{C}^d$ satisfying $\operatorname{Re} w \in C$, we shall show (25.12). If $\operatorname{Re} w = 0$, there is nothing to prove. Assume $\operatorname{Re} w \ne 0$. Choose a linear transformation U from $\mathbb{R}^d$ onto $\mathbb{R}^d$ such that $\operatorname{Re} w = U'e_1$. Consider the Lévy process $Y_t = UX_t$. Since $C_U = (U')^{-1}C$, we have $e_1 \in C_U$. We know, by Step 1, that, if $v \in \mathbb{C}^d$ satisfies $\operatorname{Re} v = e_1$, then (25.13) holds. Hence, by the result of Step 2, w satisfies (25.12). □

We close this section with a discussion of the g-moments of $\sup_{s\in[0,t]} |X_s|$.

THEOREM 25.18. *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$. Define*

$$X_t^* = \sup_{s\in[0,t]} |X_s|. \tag{25.14}$$

Let $g(r)$ be a nonnegative continuous submultiplicative function on $[0,\infty)$, increasing to ∞ as $r \to \infty$. Then the following four statements are equivalent.

(1) $E[g(X_t^*)] < \infty$ *for some* $t > 0$.
(2) $E[g(X_t^*)] < \infty$ *for every* $t > 0$.
(3) $E[g(|X_t|)] < \infty$ *for some* $t > 0$.
(4) $E[g(|X_t|)] < \infty$ *for every* $t > 0$.

Proof. Since $g(|x|)$ is submultiplicative on $\mathbb{R}^d$, (3) and (4) are equivalent by Theorem 25.3. As $|X_t| \le X_t^*$, all we have to show is that, for any fixed $t > 0$, $E[g(|X_t|)] < \infty$ implies $E[g(X_t^*)] < \infty$. We claim that, for any $a > 0$ and $b > 0$,

$$P[X_t^* > a+b] \le P[|X_t| > a]/P[X_t^* \le b/2]. \tag{25.15}$$

Fix t and let $t_{n,j} = jt/2^n$ for $j = 1, \ldots, 2^n$ and $X_{(n)}^* = \max_{1\le j\le 2^n} |X_{t_{n,j}}|$. Choosing $Z_j(s) = Z_j = X_{t_{n,j}} - X_{t_{n,j-1}}$ in Lemma 20.2 and using Remark 20.3, we have

$$P[X_{(n)}^* > a+b] \le P[|X_t| > a]/P[X_{(n)}^* \le b/2]$$

in (20.2). Hence, letting $n \to \infty$, we get (25.15). Choose $b > 0$ such that $P[X_t^* \le b/2] > 0$. Let $\widetilde{g}(r)$ be a continuous increasing function on $[0,\infty)$ such that $\widetilde{g}(0) = 0$ and $\widetilde{g}(r) = g(r)$ for $r \ge 1$. Apply Lemma 17.6 to $k(r) = 1 - P[|X_t| \le r]$ and $l(r) = \widetilde{g}(r)$. Then

$$\int_{0+}^{\infty} P[|X_t| > r] d\widetilde{g}(r) = \int_{(0,\infty)} \widetilde{g}(r)\, P[|X_t| \in dr] = E[\widetilde{g}(|X_t|)].$$

It follows from (25.15) that

$$\int_{0+}^{\infty} P[X_t^* > r+b] d\widetilde{g}(r) \le E[\widetilde{g}(|X_t|)]/P[X_t^* \le b/2].$$

The integral in the left-hand side equals

$$\int_{(0,\infty)} \widetilde{g}(r) P[X_t^* - b \in dr] = E[\widetilde{g}(X_t^* - b);\ X_t^* > b],$$

similarly. Hence, if $E[g(|X_t|)] < \infty$, then $E[g(X_t^* - b);\ X_t^* > b] < \infty$ and, by the submultiplicativity of g, $E[g(X_t^*)] < \infty$. □

REMARK 25.19. Let $d = 1$. Doob [**93**], p. 337, shows an explicit bound:

$$E[(X_t^*)^\alpha] \le 8E[|X_t|^\alpha] \quad \text{for } \alpha \ge 1, \tag{25.16}$$

provided that $E|X_t| < \infty$ and $EX_t = 0$. This is true not only for Lévy processes but also for additive processes on $\mathbb{R}$. Define the supremum process $M_t = \sup_{0\le s\le t} X_s$. If $\frac{1}{\nu(1,\infty)}[\nu]_{(1,\infty)}$ is subexponential, then $P[M_t > x]/\nu(x,\infty) \to t$ as $x \to \infty$. This is due to Berman [**18**] and Rosinski and Samorodnitsky [**382**]. Extension of this result in the case where the right tail of ν is lighter is studied in Braverman and Samorodnitsky [**57**] and Braverman [**56**].