# Answers: Lévy-Khintchine formula

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## (1)

WTS: $D$ is the unit ball and

$$\nu(\{0\}) = 0 \textit{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty, \tag{1}$$

set

$$f(t,x) = (\exp(i\langle t, x\rangle) - 1 - i\langle t, x\rangle I_D(x)). \tag{2}$$

then following function is continuous:

$$t \mapsto \int_{\mathbb{R}^d} f(t,x)\nu(dx). \tag{3}$$

*Proof.* Take $t_k \to t \implies \sup_{k>0}(t_k) < \infty, \sup_{k>0}(|t_k|^2) < \infty$ (if the supremum is $\infty$ there exist a subsequence that converges to $\infty$ which is a contradiction) and

$$|f(t_k, x)| \leq \frac{1}{2}|t_k|^2|x|^2 I_D(x) + 2I_{D^c}(x) \tag{4}$$

$$\leq \frac{1}{2}\sup_k \left(|t_k|^2\right)|x|^2 I_D(x) + 2I_{D^c}(x). \tag{5}$$

so $f(t_k, x)$ is dominated (5) which is $\nu$-integrable because of (1). Now by the dominated convergence theorem we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} f(t_k, x)\nu(dx) \tag{6}$$

$$= \int_{\mathbb{R}^d} \lim_{k\to\infty} f(t_k, x)\nu(dx) \tag{7}$$

$$= \int_{\mathbb{R}^d} f(t,x)\nu(dx). \tag{8}$$

$\square$

## (2)

WTS: compound Poisson process has a Lévy-Khintchine representation.

*Proof.* To see this rewrite the characteristic function of a compound Poisson process $X$:

$$\phi_X(t) = \exp\left(\lambda \int_{\mathbb{R}^d} (\exp(i\langle t, x\rangle))Q(dx) - 1\right) \tag{9}$$

$$= \exp\left(\lambda \int_{\mathbb{R}^d} (\exp(i\langle t, x\rangle) - 1)Q(dx)\right) \tag{10}$$

$$= \exp\left(\int_{\mathbb{R}^d} (\exp(i\langle t, x\rangle) - 1)\lambda Q(dx)\right) \tag{11}$$

$$= \exp\left(\int_{\mathbb{R}^d} (\exp(i\langle t, x\rangle) - 1 - i\langle t, x\rangle I_D(x) + i\langle t, x\rangle I_D(x))\lambda Q(dx)\right) \tag{12}$$

$$= \exp\left(\int_{\mathbb{R}^d} (\exp(i\langle t, x\rangle) - 1 - i\langle t, x\rangle I_D(x))\lambda Q(dx) + i\langle t, \int_D x\lambda Q(dx)\rangle\right) \tag{13}$$

□

## (3)

Question: What can you say about the infinitely divisible distribution if $\nu$ the Lévy measure is finite?
Answer: The infinitely divisible distribution is the sum of a independent normal distribution and a compound Poisson distribution.

$$\exp\left[-\frac{1}{2}\langle t, At\rangle + i\langle \gamma, t\rangle + \int_{\mathbb{R}^d} \left(\exp(i\langle t, x\rangle) - 1 - i\langle t, x\rangle I_D(x)\right) \nu(dx)\right] \tag{14}$$

$$= \exp\left[-\frac{1}{2}\langle t, At\rangle + i\langle \gamma, t\rangle + \int_{\mathbb{R}^d} \left(\exp(i\langle t, x\rangle) - 1\right) \nu(dx) - \int_{\mathbb{R}^d} \left(i\langle t, x\rangle I_D(x)\right) \nu(dx)\right] \tag{15}$$

$$= \exp\left[-\frac{1}{2}\langle t, At\rangle + i\langle \gamma - \int_D x\nu(dx), t\rangle\right] \exp\left[\int_{\mathbb{R}^d} \left(\exp(i\langle t, x\rangle) - 1\right) \nu(dx)\right] \tag{16}$$

$$= \exp\left[-\frac{1}{2}\langle t, At\rangle + i\langle \gamma - \int_D x\nu(dx), t\rangle\right] \exp\left[\int_{\mathbb{R}^d} \left(\exp(i\langle t, x\rangle) - 1\right) \nu(\mathbb{R}^d)\frac{\nu(dx)}{\nu(\mathbb{R}^d)}\right]. \tag{17}$$

The terms of the product are easily recognized as the characteristic functions of a normal distribution and a compound Poisson distribution.

## (4)

WTS:

$$s_k > 0 \to \infty \implies -\frac{1}{2}\langle t, At\rangle + i\langle \gamma, t\rangle s_k^{-1} + \int_{\mathbb{R}^d} \frac{\exp(is_k\langle t, x\rangle) - 1 - is_k\langle t, x\rangle I_D(x)}{s_k^2}\nu(dx) \to \frac{-\langle t, At\rangle}{2}. \tag{18}$$

*Proof.* Assume $s_k \to \infty$ set

$$f(t, x) = (\exp(i\langle t, x\rangle) - 1 - i\langle t, x\rangle I_D(x)), \tag{19}$$

then all we have to show is that (the second term goes to zero)

$$\int_{\mathbb{R}^d} f(s_k t, x)\nu(dx) \to 0. \tag{20}$$

Use the same dominator (5):

$$\left|\frac{f(s_k t, x)}{s_k^2}\right| \leq \frac{\frac{1}{2}|s_k t|^2 |x|^2 I_D(x) + 2I_{D^c}(x)}{s_k^2} \tag{21}$$

$$\leq \frac{1}{2}|t|^2|x|^2 I_D(x) + \frac{2I_{D^c(x)}}{\inf_k s_k^2}. \tag{22}$$

If $\inf_k s_k^2 = 0$ then because $s_k^2 > 0$ there exist a subsequence that converges to 0 which is a contradiction with $s_k > 0 \to \infty$. Again we can apply the dominated convergence theorem:

$$\lim_{k\to\infty}\left|\int_{\mathbb{R}^d} \frac{\exp(is_k\langle t,x\rangle) - 1 - is_k\langle t,x\rangle I_D(x)}{s_k^2}\nu(dx)\right| \tag{23}$$

$$\leq \lim_{k\to\infty}\int_{\mathbb{R}^d}\left|\frac{\exp(is_k\langle t,x\rangle) - 1 - is_k\langle t,x\rangle I_D(x)}{s_k^2}\right|\nu(dx) \tag{24}$$

$$\leq \int_{\mathbb{R}^d}\lim_{k\to\infty}\left|\frac{\exp(is_k\langle t,x\rangle) - 1 - is_k\langle t,x\rangle I_D(x)}{s_k^2}\right|\nu(dx) \tag{25}$$

$$\leq \int_{\mathbb{R}^d}\lim_{k\to\infty}\frac{|\exp(is_k\langle t,x\rangle) - 1 - is_k\langle t,x\rangle I_D(x)|}{s_k^2}\nu(dx) \tag{26}$$

$$\leq \int_{\mathbb{R}^d}\lim_{k\to\infty}\frac{|\exp(is_k\langle t,x\rangle) - 1| + |is_k\langle t,x\rangle I_D(x)|}{s_k^2}\nu(dx) \tag{27}$$

$$\leq \int_{\mathbb{R}^d}\lim_{k\to\infty}\frac{2 + |is_k\langle t,x\rangle I_D(x)|}{s_k^2}\nu(dx) \tag{28}$$

$$\to 0 \tag{29}$$

□

## (5)

WTS: Set:

$$f(t,x) = (\exp(i\langle t,x\rangle) - 1 - i\langle t,x\rangle I_D(x)), \tag{30}$$

then

$$\int_{|x|\geq\frac{1}{n}} f(s_k t, x)\nu(dx) \to_n \int_{\mathbb{R}^d} f(s_k t, x)\nu(dx). \tag{31}$$

*Proof.* Use the same dominator (5) and apply dominated convergence theorem. □

## (6)

WTS: $\int_{[-h,h]^d}\psi_n(t)dt \in \mathbb{R}$ without calculating it.

*Proof.* Didn't manage to prove it without calculating it. But here is an idea: we know because of the Lévy-Khintchine formula (which we can't get without the calculation) it is easily seen that $\psi(t)$ is conjugate to $\psi(-t)$, $|\psi| = 1$ and we are integrating/ summing over a even domain ($t \in D \implies -t \in D$) so the complex parts cancel out. □

## (7)

Question: Why can we apply Fubini's theorem in line 1 page 23.
Answer: Because the conditions are satisfied this is already well explained in the notes.

## (8)

Question: Why is line 15 on page 13 easy to see?
Answer: It is a product of multiple sinc functions which are individually bounded between $-1$ and 1 and reach their maximum at 0 which is 1. So the product is continuous also bounded between $-1$ and 1 and only reaches 1 in 0.

## (9)

WTS: If $\psi$ is continuous at 0 and $\psi(0) = 0$ then $\forall \epsilon, \exists h_\epsilon$ :

$$-(2h_\epsilon)^{-d} \int_{[-h_\epsilon, h_\epsilon]^d} \psi(t)dt < \frac{\epsilon}{4}. \tag{32}$$

*Proof.* Because $\psi$ is continuous at 0

$$\forall \epsilon, \exists \delta_\epsilon, \forall x \in B(0, \delta_\epsilon) : |\psi(x)| < \frac{\epsilon}{4}. \tag{33}$$

Choose $h_\epsilon = \frac{\delta_\epsilon}{\sqrt{d}}$ so that $[-h_\epsilon, h_\epsilon]^d \subset B(0, \delta_\epsilon)$ then

$$\begin{aligned}
& -(2h_\epsilon)^{-d} \int_{[-h_\epsilon, h_\epsilon]^d} \psi(t)dt && (34)\\
& \leq \left| (2h_\epsilon)^{-d} \int_{[-h_\epsilon, h_\epsilon]^d} \psi(t)dt \right| && (35)\\
& \leq (2h_\epsilon)^{-d} \int_{[-h_\epsilon, h_\epsilon]^d} |\psi(t)|\, dt && (36)\\
& < (2h_\epsilon)^{-d} \int_{[-h_\epsilon, h_\epsilon]^d} \frac{\epsilon}{4} dt && (37)\\
& = \frac{\epsilon}{4}. && (38)
\end{aligned}$$

□

## (10)

Question: Formula (2.12). Check why we had to insert the extra term $\langle t, x\rangle^2/2$?
Answer: This $\langle t, x\rangle^2/2$ term is to make $f_t$ continuous and bounded to then apply weak convergence.