

Answers: Lévy-Khintchine formula

April 20, 2024

(1)

WTS: D is the unit ball and

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty, \quad (1)$$

set

$$f(t, x) = (\exp(i\langle t, x \rangle) - 1 - i\langle t, x \rangle I_D(x)). \quad (2)$$

then following function is continuous:

$$t \mapsto \int_{\mathbb{R}^d} f(t, x) \nu(dx). \quad (3)$$

Proof. Take $t_k \rightarrow t \implies \sup_{k>0} (t_k) < \infty, \sup_{k>0} (|t_k|^2) < \infty$ (if the supremum is ∞ there exist a subsequence that converges to ∞ which is a contradiction) and

$$|f(t_k, x)| \leq \frac{1}{2} |t_k|^2 |x|^2 I_D(x) + 2I_{D^c}(x) \quad (4)$$

$$\leq \frac{1}{2} \sup_k (|t_k|^2) |x|^2 I_D(x) + 2I_{D^c}(x). \quad (5)$$

so $f(t_k, x)$ is dominated (5) which is ν -integrable because of (1). Now by the dominated convergence theorem we have

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} f(t_k, x) \nu(dx) \quad (6)$$

$$= \int_{\mathbb{R}^d} \lim_{k \rightarrow \infty} f(t_k, x) \nu(dx) \quad (7)$$

$$= \int_{\mathbb{R}^d} f(t, x) \nu(dx). \quad (8)$$

□

(2)

WTS: compound Poisson process has a Lévy-Khintchine representation.

Proof. To see this rewrite the characteristic function of a compound Poisson process X :

$$\phi_X(t) = \exp \left(\lambda \int_{\mathbb{R}^d} (\exp(i\langle t, x \rangle)) Q(dx) - 1 \right) \quad (9)$$

$$= \exp \left(\lambda \int_{\mathbb{R}^d} (\exp(i\langle t, x \rangle) - 1) Q(dx) \right) \quad (10)$$

$$= \exp \left(\int_{\mathbb{R}^d} (\exp(i\langle t, x \rangle) - 1) \lambda Q(dx) \right) \quad (11)$$

$$= \exp \left(\int_{\mathbb{R}^d} (\exp(i\langle t, x \rangle) - 1 - i\langle t, x \rangle I_D(x) + i\langle t, x \rangle I_D(x)) \lambda Q(dx) \right) \quad (12)$$

$$= \exp \left(\int_{\mathbb{R}^d} (\exp(i\langle t, x \rangle) - 1 - i\langle t, x \rangle I_D(x)) \lambda Q(dx) + i\langle t, \int_D x \lambda Q(dx) \rangle \right) \quad (13)$$

□

(3)

Question: What can you say about the infinitely divisible distribution if ν the Lévy measure is finite?

Answer: The infinitely divisible distribution is the sum of a independent normal distribution and a compound Poisson distribution.

$$\exp \left[-\frac{1}{2} \langle t, At \rangle + i\langle \gamma, t \rangle + \int_{\mathbb{R}^d} (\exp(i\langle t, x \rangle) - 1 - i\langle t, x \rangle I_D(x)) \nu(dx) \right] \quad (14)$$

$$= \exp \left[-\frac{1}{2} \langle t, At \rangle + i\langle \gamma, t \rangle + \int_{\mathbb{R}^d} (\exp(i\langle t, x \rangle) - 1) \nu(dx) - \int_{\mathbb{R}^d} (i\langle t, x \rangle I_D(x)) \nu(dx) \right] \quad (15)$$

$$= \exp \left[-\frac{1}{2} \langle t, At \rangle + i\langle \gamma - \int_D x \nu(dx), t \rangle \right] \exp \left[\int_{\mathbb{R}^d} (\exp(i\langle t, x \rangle) - 1) \nu(dx) \right] \quad (16)$$

$$= \exp \left[-\frac{1}{2} \langle t, At \rangle + i\langle \gamma - \int_D x \nu(dx), t \rangle \right] \exp \left[\int_{\mathbb{R}^d} (\exp(i\langle t, x \rangle) - 1) \nu(\mathbb{R}^d) \frac{\nu(dx)}{\nu(\mathbb{R}^d)} \right]. \quad (17)$$

The terms of the product are easily recognized as the characteristic functions of a normal distribution and a compound Poisson distribution.

(4)

WTS:

$$s_k > 0 \rightarrow \infty \implies -\frac{1}{2} \langle t, At \rangle + i\langle \gamma, t \rangle s_k^{-1} + \int_{\mathbb{R}^d} \frac{\exp(is_k \langle t, x \rangle) - 1 - is_k \langle t, x \rangle I_D(x)}{s_k^2} \nu(dx) \rightarrow \frac{-\langle t, At \rangle}{2}. \quad (18)$$

Proof. Assume $s_k \rightarrow \infty$ set

$$f(t, x) = (\exp(i\langle t, x \rangle) - 1 - i\langle t, x \rangle I_D(x)), \quad (19)$$

then all we have to show is that (the second term goes to zero)

$$\int_{\mathbb{R}^d} f(s_k t, x) \nu(dx) \rightarrow 0. \quad (20)$$

Use the same dominator (5):

$$\left| \frac{f(s_k t, x)}{s_k^2} \right| \leq \frac{\frac{1}{2}|s_k t|^2|x|^2 I_D(x) + 2I_{D^c}(x)}{s_k^2} \quad (21)$$

$$\leq \frac{1}{2}|t|^2|x|^2 I_D(x) + \frac{2I_{D^c}(x)}{\inf_k s_k^2}. \quad (22)$$

If $\inf_k s_k^2 = 0$ then because $s_k^2 > 0$ there exist a subsequence that converges to 0 which is a contradiction with $s_k > 0 \rightarrow \infty$. Again we can apply the dominated convergence theorem:

$$\lim_{k \rightarrow \infty} \left| \int_{\mathbb{R}^d} \frac{\exp(is_k \langle t, x \rangle) - 1 - is_k \langle t, x \rangle I_D(x)}{s_k^2} \nu(dx) \right| \quad (23)$$

$$\leq \lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} \left| \frac{\exp(is_k \langle t, x \rangle) - 1 - is_k \langle t, x \rangle I_D(x)}{s_k^2} \right| \nu(dx) \quad (24)$$

$$\leq \int_{\mathbb{R}^d} \lim_{k \rightarrow \infty} \left| \frac{\exp(is_k \langle t, x \rangle) - 1 - is_k \langle t, x \rangle I_D(x)}{s_k^2} \right| \nu(dx) \quad (25)$$

$$\leq \int_{\mathbb{R}^d} \lim_{k \rightarrow \infty} \frac{|\exp(is_k \langle t, x \rangle) - 1 - is_k \langle t, x \rangle I_D(x)|}{s_k^2} \nu(dx) \quad (26)$$

$$\leq \int_{\mathbb{R}^d} \lim_{k \rightarrow \infty} \frac{|\exp(is_k \langle t, x \rangle) - 1| + |is_k \langle t, x \rangle I_D(x)|}{s_k^2} \nu(dx) \quad (27)$$

$$\leq \int_{\mathbb{R}^d} \lim_{k \rightarrow \infty} \frac{2 + |is_k \langle t, x \rangle I_D(x)|}{s_k^2} \nu(dx) \quad (28)$$

$$\rightarrow 0 \quad (29)$$

□

(5)

WTS: Set:

$$f(t, x) = (\exp(i \langle t, x \rangle) - 1 - i \langle t, x \rangle I_D(x)), \quad (30)$$

then

$$\int_{|x| \geq \frac{1}{n}} f(s_k t, x) \nu(dx) \rightarrow_n \int_{\mathbb{R}^d} f(s_k t, x) \nu(dx). \quad (31)$$

Proof. Use the same dominator (5) and apply dominated convergence theorem. □

(6)

WTS: $\int_{[-h, h]^d} \psi_n(t) dt \in \mathbb{R}$ without calculating it.

Proof. Didn't manage to prove it without calculating it. But here is an idea: we know because of the Lévy-Khintchine formula (which we can't get without the calculation) it is easily seen that $\psi(t)$ is conjugate to $\psi(-t)$, $|\psi| = 1$ and we are integrating/ summing over a even domain ($t \in D \implies -t \in D$) so the complex parts cancel out. □

(7)

Question: Why can we apply Fubini's theorem in line 1 page 23.

Answer: Because the conditions are satisfied this is already well explained in the notes.

(8)

Question: Why is line 15 on page 13 easy to see?

Answer: It is a product of multiple sinc functions which are individually bounded between -1 and 1 and reach their maximum at 0 which is 1 . So the product is continuous also bounded between -1 and 1 and only reaches 1 in 0 .

(9)