Definition 0.0.1 (Δ_h^k)
We define $\forall h > 0 : \Delta_h$ (h-central difference) of $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$\Delta_h f(t) = f(t+h) - f(t-h), \tag{1}$$

and $\forall k \in \mathbb{N} : \Delta_h^k$ is defined as applying Δ_h applying k times and $k = 0$ as the identity. Note that Δ_h^k is a linear operator. We will always take differences in respect to the t variable.

Lemma 0.0.2 ($\Delta_h^k e^{at}$)

$$\forall k \in \mathbb{N} : \Delta_h^k e^{at} = e^{at} \left(e^{ah} - e^{-ah}\right)^k. \tag{2}$$

Proof. This follows simply by induction. $k = 0$ is trivial. Base case:

$$\begin{aligned} \Delta_h e^{at} &= \left(e^{a(t+h)} - e^{a(t-h)}\right) &(3)\\ &= e^{at} \left(e^{ah} - e^{-ah}\right). &(4) \end{aligned}$$

Induction hypothesis:

$$\Delta_h^{k-1} e^{at} = e^{at} \left(e^{ah} - e^{-ah}\right)^{k-1}. \tag{5}$$

Induction step:

$$\begin{aligned} \Delta_h^k e^{at} &= \Delta_h \Delta_h^{k-1} (e^{at}) &(6)\\ &= \Delta_h \left(e^{at} \left(e^{ah} - e^{-ah}\right)^{k-1}\right) &(7)\\ &= \Delta_h \left(e^{at}\right) \left(e^{ah} - e^{-ah}\right)^{k-1} &(8)\\ &= e^{at} \left(e^{ah} - e^{-ah}\right) \left(e^{ah} - e^{-ah}\right)^{k-1} &(9)\\ &= e^{at} \left(e^{ah} - e^{-ah}\right)^k. &(10) \end{aligned}$$

□

Theorem 0.0.3 (central finite difference formula)
$\forall k \in \mathbb{N}\ \forall$ k-continuously differentiable f's at 0:

$$f^{(k)}(0) = \lim_{h \to 0} \frac{\Delta_h^k f(t)|_{t=0}}{(2h)^k}. \tag{11}$$

Proof. Follows by induction and the mean value theorem. Base case $k = 0$ is trivial. Induction hypothesis:

$$\forall l < k : f^{(l)}(0) = \lim_{h \to 0} \frac{\Delta_h^l f(t)|_{t=0}}{(2h)^l}. \tag{12}$$

Induction step:

$$
\begin{aligned}
&\lim_{h\to 0}\frac{\Delta_h^k f(t)|_{t=0}}{(2h)^k} && (13)\\
&=\lim_{h\to 0}\frac{\Delta_h\Delta_h^{k-1} f(t)|_{t=0}}{(2h)^k} && (14)\\
&=\lim_{h\to 0}\frac{\Delta_h^{k-1} f(t)(|_{t=h}-|_{t=-h})}{(2h)^k} && (15)\\
&=\lim_{h\to 0}\frac{2h(\Delta_h^{k-1} f(t))'|_{t=z_h}}{(2h)^k}, z_h\in[-h,h] && (16)\\
&=\lim_{h\to 0}\frac{\Delta_h^{k-1} f'(t)|_{t=z_h}}{(2h)^{k-1}} = f'^{(k-1)}(0)=f^{(k)}(0). && (17)
\end{aligned}
$$

The induction hypothesis may be applied because $\Delta_h^{k-1}f'(t)$ is $k-1$-continuously differentiable.

□

Lemma 0.0.4

$\forall k\in\mathbb{N}:\phi_X(t)$ is $2k$-time continuous differentiable $\Rightarrow E[X^{2k}]<\infty$

Proof. Use the central finite difference formula and previous lemma's:

$$
\begin{aligned}
|\phi_X^{(2k)}(0)| &= \left|\lim_{h\to 0}\frac{\Delta_h^{2k}|_{t=0}\phi_X(t)}{(2h)^{2k}}\right| && (18)\\
&= \left|\lim_{h\to 0}\frac{\Delta_h^{2k}|_{t=0}E\left[e^{itX}\right]}{(2h)^{2k}}\right| && (19)\\
&= \left|\lim_{h\to 0}\frac{E\left[\Delta_h^{2k}|_{t=0}e^{itX}\right]}{(2h)^{2k}}\right| && (20)\\
&= \left|\lim_{h\to 0}\frac{E\left[e^{itX}|_{t=0}(e^{ihX}-e^{-ihX})^{2k}\right]}{(2h)^{2k}}\right| && (21)\\
&= \left|\lim_{h\to 0}\frac{E\left[(e^{ihX}-e^{-ihX})^{2k}\right]}{(2h)^{2k}}\right| && (22)\\
&= \left|\lim_{h\to 0}\frac{1}{i^{2k}}E\left[\frac{(e^{ihX}-e^{-ihX})^{2k}}{(2h)^{2k}}\right]\right| && (23)\\
&= \lim_{h\to 0}E\left[\frac{\sin(Xh)^{2k}}{h^{2k}}\right] && (24)\\
&= \lim_{h\to 0}E\left[X^{2k}\frac{\sin(Xh)^{2k}}{X^{2k}h^{2k}}\right] && (25)\\
&= \lim_{h\to 0}E\left[X^{2k}\,\mathrm{sinc}(Xh)^{2k}\right] && (26)\\
&\geq E\left[X^{2k}\right] && (27)
\end{aligned}
$$

Where last line follows by Fatou's lemma. □