

**Definition 0.0.1** ( $\Delta_h^k$ )

We define  $\forall h > 0 : \Delta_h$  ( $h$ -central difference) of  $f : \mathbb{R} \rightarrow \mathbb{R}$  as follows:

$$\Delta_h f(t) = f(t+h) - f(t-h), \quad (1)$$

and  $\forall k \in \mathbb{N} : \Delta_h^k$  is defined as applying  $\Delta_h$  applying  $k$  times and  $k = 0$  as the identity. Note that  $\Delta_h^k$  is a linear operator. We will always take differences in respect to the  $t$  variable.

**Lemma 0.0.2** ( $\Delta_h^k e^{at}$ )

$$\forall k \in \mathbb{N} : \Delta_h^k e^{at} = e^{at} (e^{ah} - e^{-ah})^k. \quad (2)$$

*Proof.* This follows simply by induction.  $k = 0$  is trivial. Base case:

$$\Delta_h e^{at} = (e^{a(t+h)} - e^{a(t-h)}) \quad (3)$$

$$= e^{at} (e^{ah} - e^{-ah}). \quad (4)$$

Induction hypothesis:

$$\Delta_h^{k-1} e^{at} = e^{at} (e^{ah} - e^{-ah})^{k-1}. \quad (5)$$

Induction step:

$$\Delta_h^k e^{at} = \Delta_h \Delta_h^{k-1} (e^{at}) \quad (6)$$

$$= \Delta_h \left( e^{at} (e^{ah} - e^{-ah})^{k-1} \right) \quad (7)$$

$$= \Delta_h (e^{at}) (e^{ah} - e^{-ah})^{k-1} \quad (8)$$

$$= e^{at} (e^{ah} - e^{-ah}) (e^{ah} - e^{-ah})^{k-1} \quad (9)$$

$$= e^{at} (e^{ah} - e^{-ah})^k. \quad (10)$$

□

**Theorem 0.0.3** (central finite difference formula)

$\forall k \in \mathbb{N} \forall k$ -continuously differentiable  $f$ 's at 0:  $\exists \delta > 0 : \forall |s| < \delta :$

$$f^{(k)}(s) = \lim_{h \rightarrow 0} \frac{\Delta_h^k f(t)|_{t=s}}{(2h)^k}. \quad (11)$$

*Proof.* Follows by induction and the mean value theorem. Base case  $k = 0$  is trivial.

Induction hypothesis:

$$\exists \delta > 0 : \forall |s| < \delta : \forall l < k : f^{(l)}(s) = \lim_{h \rightarrow 0} \frac{\Delta_h^l f(t)|_{t=s}}{(2h)^l}. \quad (12)$$

Induction step:

$$\lim_{h \rightarrow 0} \frac{\Delta_h^k f(t)|_{t=s}}{(2h)^k} \quad (13)$$

$$= \lim_{h \rightarrow 0} \frac{\Delta_h \Delta_h^{k-1} f(t)|_{t=s}}{(2h)^k} \quad (14)$$

$$= \lim_{h \rightarrow 0} \frac{\Delta_h^{k-1} f(t)(|_{t=s+h} - |_{t=s-h})}{(2h)^k} \quad (15)$$

$$= \lim_{h \rightarrow 0} \frac{2h(\Delta_h^{k-1} f(t))'|_{t=z_h}, z_h \in [s-h, s+h]}{(2h)^k} \quad (16)$$

$$= \lim_{h \rightarrow 0} \frac{\Delta_h^{k-1} f'(t)|_{t=z_h}}{(2h)^{k-1}} = f'^{(k-1)}(s) = f^{(k)}(s). \quad (17)$$

The induction hypothesis may be applied because  $\Delta_h^{k-1} f'(t)$  is  $k-1$ -continuously differentiable. □

**Lemma 0.0.4**

$\forall k \in \mathbb{N} : \phi_X(t)$  is  $2k$ -time continuous differentiable  $\Rightarrow E[X^{2k}] < \infty$

*Proof.* Use the central finite difference formula and previous lemma's:

$$|\phi_X^{(2k)}(0)| = \left| \lim_{h \rightarrow 0} \frac{\Delta_h^{2k} |_{t=0} \phi_X(t)}{(2h)^{2k}} \right| \quad (18)$$

$$= \left| \lim_{h \rightarrow 0} \frac{\Delta_h^{2k} |_{t=0} E[e^{itX}]}{(2h)^{2k}} \right| \quad (19)$$

$$= \left| \lim_{h \rightarrow 0} \frac{E[\Delta_h^{2k} |_{t=0} e^{itX}]}{(2h)^{2k}} \right| \quad (20)$$

$$= \left| \lim_{h \rightarrow 0} \frac{E[e^{itX} |_{t=0} (e^{ihX} - e^{-ihX})^{2k}]}{(2h)^{2k}} \right| \quad (21)$$

$$= \left| \lim_{h \rightarrow 0} \frac{E[(e^{ihX} - e^{-ihX})^{2k}]}{(2h)^{2k}} \right| \quad (22)$$

$$= \left| \lim_{h \rightarrow 0} \frac{1}{i^{2k}} E \left[ \frac{(e^{ihX} - e^{-ihX})^{2k}}{(2h)^{2k}} \right] \right| \quad (23)$$

$$= \lim_{h \rightarrow 0} E \left[ \frac{\sin(Xh)^{2k}}{h^{2k}} \right] \quad (24)$$

$$= \lim_{h \rightarrow 0} E \left[ X^{2k} \frac{\sin(Xh)^{2k}}{X^{2k} h^{2k}} \right] \quad (25)$$

$$= \lim_{h \rightarrow 0} E[X^{2k} \text{sinc}(Xh)^{2k}] \quad (26)$$

$$\geq E[X^{2k}] \quad (27)$$

Where last line follows by Fatou's lemma. □