Definition 0.0.1 (Δ_h^k)
We define $\forall h > 0 : \Delta_h$ (h-central difference) of $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$\Delta_h f(t) = f(t+h) - f(t-h), \tag{1}$$

and $\forall k \in \mathbb{N} : \Delta_h^k$ is defined as applying Δ_h applying k times and $k = 0$ as the identity. Note that Δ_h^k is a linear operator. We will always take differences in respect to the t variable.

Lemma 0.0.2 ($\Delta_h^k e^{at}$)

$$\forall k \in \mathbb{N} : \Delta_h^k e^{at} = e^{at}\left(e^{ah} - e^{-ah}\right)^k. \tag{2}$$

Proof. This follows simply by induction. $k = 0$ is trivial. Base case:

$$\begin{aligned}\Delta_h e^{at} &= \left(e^{a(t+h)} - e^{a(t-h)}\right) &(3)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right). &(4)\end{aligned}$$

Induction hypothesis:

$$\Delta_h^{k-1} e^{at} = e^{at}\left(e^{ah} - e^{-ah}\right)^{k-1}. \tag{5}$$

Induction step:

$$\begin{aligned}\Delta_h^k e^{at} &= \Delta_h \Delta_h^{k-1}(e^{at}) &(6)\\ &= \Delta_h\left(e^{at}\left(e^{ah} - e^{-ah}\right)^{k-1}\right) &(7)\\ &= \Delta_h\left(e^{at}\right)\left(e^{ah} - e^{-ah}\right)^{k-1} &(8)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right)\left(e^{ah} - e^{-ah}\right)^{k-1} &(9)\\ &= e^{at}\left(e^{ah} - e^{-ah}\right)^k. &(10)\end{aligned}$$

$\square$

Lemma 0.0.3 (Δc)

$$\forall c \in \mathbb{R} : \Delta_h c = c - c = 0. \tag{11}$$

Lemma 0.0.4 ($\Delta_h t^n$)
$\forall n > 0 \in \mathbb{N} : \Delta_h t^n$ is a polynomial of degree $n-1$ with leading coefficient $2hn$.

Proof.

$$\Delta_h t^n = (t+h)^n - (t-h)^n. \tag{12}$$

So $\Delta_h t^n$ is a polynomial, verify the degree and leading coefficient by looking at its derivatives (or use the binomial theorem):

$$\begin{aligned}\frac{d^n}{dt^n}\left((t+h)^n - (t-h)^n\right) &= n! - n! = 0. &(13)\\ \frac{d^{n-1}}{dt^{n-1}}\left((t+h)^n - (t-h)^n\right) &= n!(t+h) - n!(t-h). &(14)\\ &= 2hn! \neq 0. &(15)\end{aligned}$$

$\square$

Lemma 0.0.5 ($\Delta_h^k t^n$)
$\forall n, k \in \mathbb{N}, n < k : \Delta_h^k t^n = 0.$

Proof. Follows from the previous 2 lemmas and linearity of Δ_h^k. □

Lemma 0.0.6 ($\Delta_h^k t^k$)
$\forall k \in \mathbb{N} : \Delta_h^k t^k = (2h)^k k!.$

Proof. Proof by induction. Base case $k = 0$ is trivial. Induction hypothesis:

$$\forall l < k : \Delta_h^l t^l = (2h)^l l!. \tag{16}$$

Induction step, use previous lemmas:

$$\begin{aligned}
\Delta_h^k t^k &= \Delta_h^{k-1} \Delta_h t^k && (17)\\
&= \Delta_h^{k-1} \left((t+h)^k - (t-h)^k\right) && (18)\\
&= \Delta_h^{k-1} \left(2hkt^{k-1} + ...\right) && (19)\\
&= \Delta_h^{k-1} \left(2hkt^{k-1}\right) && (20)\\
&= 2h\Delta_h^{k-1} \left(t^{k-1}\right) && (21)\\
&= 2hk(2h)^{k-1}(k-1)! && (22)\\
&= (2h)^k k! && (23)
\end{aligned}$$

□

Theorem 0.0.7 (central finite difference formula)
$\forall k \in \mathbb{N}\ \forall$ k-continuously differentiable f's at 0:

$$f^{(k)}(0) = \lim_{h \to 0} \frac{\Delta_h^k|_{t=0} f(t)}{(2h)^k}. \tag{24}$$

Proof. Follows from Taylor's theorem and previous lemma.

$$\begin{aligned}
&\lim_{h \to 0} \frac{\Delta_h^k|_{t=0} f(t)}{(2h)^k} && (25)\\
&= \lim_{h \to 0} \frac{\Delta_h^k|_{t=0} \left(\sum_{n=0}^{k-1} \frac{f^{(n)}(0)}{n!} t^n + \left(\frac{f^{(k)}(0)}{k!} + h_k(t)\right) t^k\right)}{(2h)^k} && (26)\\
&= \lim_{h \to 0} \frac{\Delta_h^k|_{t=0} \left(\left(\frac{f^{(k)}(0)}{k!} + h_k(t)\right) t^k\right)}{(2h)^k} && (27)\\
&= \lim_{h \to 0} \frac{\left(\frac{f^{(k)}(0)}{k!} + h_k(t)\right) \Delta_h^k|_{t=0} \left(t^k\right)}{(2h)^k} && (28)\\
&= \lim_{h \to 0} \frac{\left(\frac{f^{(k)}(0)}{k!} + h_k(t)\right) (2h)^k k!}{(2h)^k} && (29)\\
&= \lim_{h \to 0} f^{(k)}(0) + h_k(t) k! && (30)\\
&= f^{(k)}(0) && (31)
\end{aligned}$$

□

Lemma 0.0.8
$\forall k \in \mathbb{N} : \phi_X(t)$ is $2k$-time continuous differentiable $\Rightarrow E[X^{2k}] < \infty$

Proof. Use the central finite difference formula and previous lemma's:

$$
\begin{aligned}
|\phi_X^{(2k)}(0)| &= \left| \lim_{h\to 0} \frac{\Delta_h^{2k}|_{t=0}\phi_X(t)}{(2h)^{2k}} \right| && (32)\\
&= \left| \lim_{h\to 0} \frac{\Delta_h^{2k}|_{t=0}E\left[e^{itX}\right]}{(2h)^{2k}} \right| && (33)\\
&= \left| \lim_{h\to 0} \frac{E\left[\Delta_h^{2k}|_{t=0}e^{itX}\right]}{(2h)^{2k}} \right| && (34)\\
&= \left| \lim_{h\to 0} \frac{E\left[e^{itX}|_{t=0}(e^{ihX} - e^{-ihX})^{2k}\right]}{(2h)^{2k}} \right| && (35)\\
&= \left| \lim_{h\to 0} \frac{E\left[(e^{ihX} - e^{-ihX})^{2k}\right]}{(2h)^{2k}} \right| && (36)\\
&= \left| \lim_{h\to 0} \frac{1}{i^{2k}} E\left[\frac{(e^{ihX} - e^{-ihX})^{2k}}{(2h)^{2k}}\right] \right| && (37)\\
&= \lim_{h\to 0} E\left[\frac{\sin(Xh)^{2k}}{h^{2k}}\right] && (38)\\
&= \lim_{h\to 0} E\left[X^{2k}\frac{\sin(Xh)^{2k}}{X^{2k}h^{2k}}\right] && (39)\\
&= \lim_{h\to 0} E\left[X^{2k}\,\mathrm{sinc}(Xh)^{2k}\right] && (40)\\
&\geq E\left[X^{2k}\right] && (41)
\end{aligned}
$$

Where last line follows by Fatou's lemma. □