RMC voor lineaire ODEs

Isidoor Pinillo Esquivel

Introductie

$$y_t = A(t)y(t). \tag{1}$$

Euler:

(4)

(6)

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Euler:

$$\frac{\tilde{y}(t_k + \Delta t) - \tilde{y}(t_k)}{\Delta t} = A(t_k)\tilde{y}(t_k) \tag{2}$$

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Y(T_{k+1}) &= (I + \Delta t A(T_{k+1}))Y(T_k) \qquad (5)\\
T_{k+1} &= T_k + \mathrm{Exp}(\Delta t) \qquad (6)
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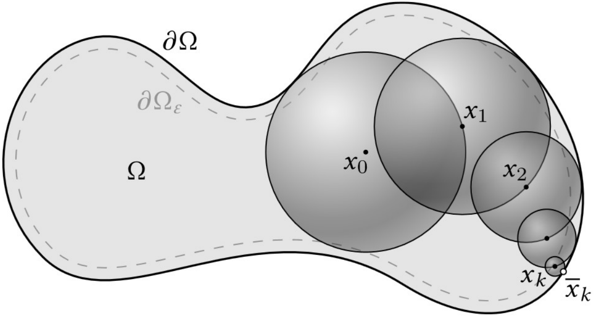
\end{aligned}$$

$$y(T_k) = E[Y(T_k)]. \tag{7}$$

Overzicht

Motivatie

Veralgemenen van WoS algoritme van (Sawhney e.a. 2022) naar tijd



Random sample efficientie van optelling

Probleem: benader $\bar{x} = \frac{1}{k}\sum_{i=1}^{k} x_i$ met $n << k$ van x_i's $\in [0, 1]$ (symmetrisch in x_i)

[0] Heinrich en Novak 2001.

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Vertrouwensinterval kans $= 1 \sim \frac{k-n}{k} = 1 - \frac{n}{k} \approx 1$ (worst case $\frac{1}{k}$ uit elkaar)

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Samplen of termen willekeurig weg laten (Russische roulette)

$$\bar{x} \cong \frac{1}{n}\sum_{i=1}^{n} x_{I_i} \cong \frac{1}{k}\sum_{i=1}^{k} B_i x_i. \tag{8}$$

$\left(E[B_i] = 1, P[B_i = 0] = 1 - \frac{n}{k}\right)$

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Variantie $=$ RMSE $\sim O\left(\frac{1}{\sqrt{n}}\right)$ en ook vertrouwensintervallen kans < 1 (CLT of Chebychev's ongelijkheid)

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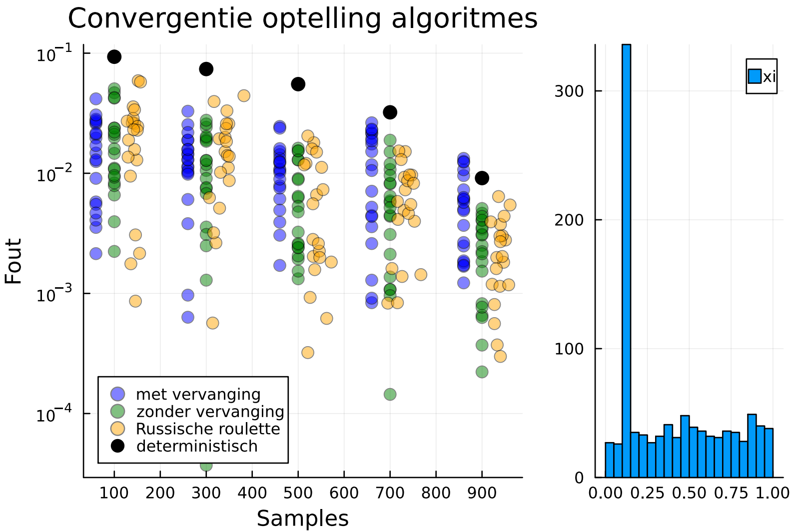
Optelling algoritmes (plot)



Monte Carlo integratie

Integratie $\approx$ sommatie, integreerbare $f : \mathbb{R} \to [0,1]$:

$$\int_0^1 f(s)ds = E[f(U)] \tag{9}$$

$$\cong \frac{1}{n}\sum_{j=1}^{n} f(U_j) \tag{10}$$

$$\text{met } U_j \sim \text{Uniform}(0,1) \tag{11}$$

Main Poisson algoritme

$$y_t = A(t)y, \quad A(t) \text{ meetbaar en } \sup_t ||A(t)|| < \infty \Leftrightarrow \tag{12}$$

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$$y(t) = e^{-\sigma t} y(0) + \int_0^t e^{(s-t)\sigma} \left((\sigma I + A(s))\, y(s)\right) ds, \tag{15}$$

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$$y(t) = \int_0^{e^{-\sigma t}} y(0) d\tau + \int_{e^{-\sigma t}}^1 \left(I + \frac{A(s)}{\sigma} \right) y(s) d\tau. \tag{16}$$

Main Poisson algoritme (recursie)

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doe dit recursief verder:

$$Y(t) = \begin{cases} y(0) & \text{als } e^{-\sigma t} \leq \tau \\ \left(I + \frac{A(S)}{\sigma} \right) Y(S) & \text{anders} \end{cases}, \tag{18}$$

met $S = t + \frac{\ln(\tau)}{\sigma}$.

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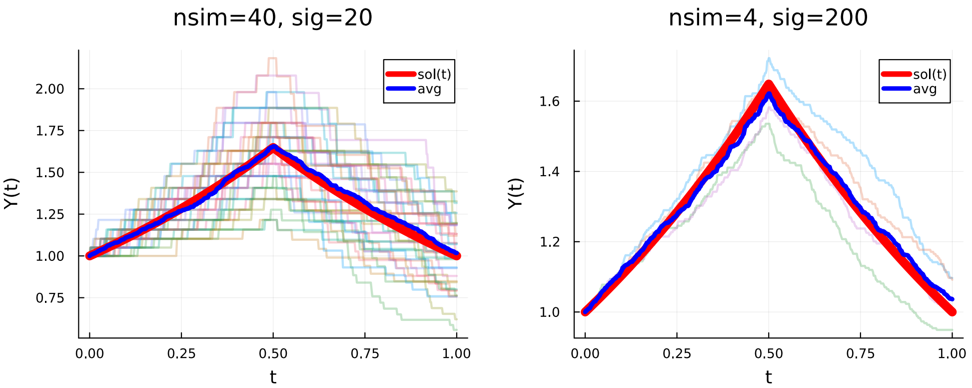
met $S = t + \frac{\ln(\tau)}{\sigma}$.

$$y(t) = E\left[\prod_{k=1}^{N_t} \left(I + \frac{A(T_k)}{\sigma} \right) \right] y(0). \tag{19}$$

Main Poisson algoritme (convergentie)



Figuur: Realisaties $Y(t)$ met $A(t) = \begin{cases} 1 & \text{voor } t < 0.5, \\ -1 & \text{voor } t \geq 0.5. \end{cases}$ voor verschillende *nsim* en *sig*.

Main Poisson algoritme (opmerkingen)

$$y(t) \cong Y(t) = \prod_{k=1}^{N_t} \left(I + \frac{A(T_k)}{\sigma} \right) y(0). \tag{20}$$

- TB: $E[||Y(t)||], Var[||Y(t)||] < \infty$, wet totale verwachting/variantie
- Parallelle complexiteit
- Inspiratie uit Acebrón en Ribeiro 2016

1. Introductie
2. Motivatie
3. Monte Carlo
4. Main Poisson algoritme
5. Geavanceerde methoden
6. **Conclusie**

Limitaties

- Stabiliteit $\rightarrow$ Kettunen e.a. 2021 (efficiënte unbiased $e^{\int A(s)ds}y(0)$)
- Biased voor non-lineaire ODEs

Toekomstig Werk

- Random ODEs
- Specifieke ODEs