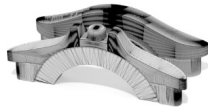


# RMC voor lineaire ODEs

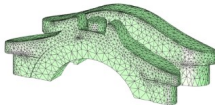
Isidoor Pinillo Esquivel

# Grid-Free Monte Carlo

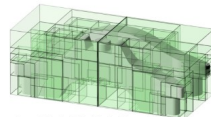
Sawhney e.a. 2022



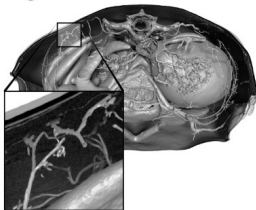
input  
(Thing10k #996816)



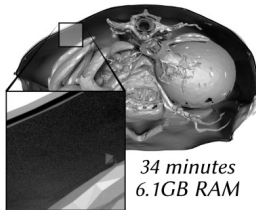
mesh w/ FASTTETWILD  
1 hour 25 minutes



build BVH for WoS  
< 1 second



input boundary mesh  
(used directly by WoS)



34 minutes  
6.1GB RAM  
boundary of tetrahedral mesh  
generated for FEM (via FASTTETWILD)

# Monte Carlo

$$\int_0^1 f(s) ds = E[f(U)] \quad (1)$$

$$\approx \frac{1}{n} \sum_{j=1}^n f(U_j) \quad (2)$$

$$\text{met } U_j \sim \text{Uniform}(0, 1) \quad (3)$$

# Monte Carlo Fout Analyse

$$f_{\text{out}} = O_p \left( \sqrt{\frac{\text{Var}(f(U))}{n}} \right) \text{ (CLT) } (4)$$

# Monte Carlo Fout Analyse

$$\text{fout} = O_p \left( \sqrt{\frac{\text{Var}(f(U))}{n}} \right) \text{ (CLT)} \quad (4)$$

$$\sqrt{\text{Var}(f(U))} \quad (5)$$

$$= \sqrt{E[(f(U) - E[f(U)])^2]} \quad (6)$$

$$= \|f - E[f(U)]\|_2 \quad (7)$$

$$\sim \|f\|_2 \quad (8)$$

# Waarom Monte Carlo?

- paralleliseerbaar
- dimensie onafhankelijke convergentie
- complexe geometrie

# Waarom ODEs?

- grid-free + tijdafhankelijkheid?
- ODEs simpeler als PDEs

# Stochastic Gradient Descent

$\text{SGD} = \text{GD} + \text{unbiased gradients}$



# Stochastic Gradient Descent

SGD = GD + unbiased gradients

$$f(x) = \frac{1}{n} \sum_{j=1}^n f_j(x) \quad (9)$$

(10)

(11)

# Stochastic Gradient Descent

SGD = GD + unbiased gradients

$$f(x) = \frac{1}{n} \sum_{j=1}^n f_j(x) \quad (9)$$

$$\nabla f(x) = \frac{1}{n} \sum_{j=1}^n \nabla f_j(x) \quad (10)$$

$$(11)$$

# Stochastic Gradient Descent

SGD = GD + unbiased gradients

$$f(x) = \frac{1}{n} \sum_{j=1}^n f_j(x) \quad (9)$$

$$\nabla f(x) = \frac{1}{n} \sum_{j=1}^n \nabla f_j(x) \quad (10)$$

$$= E[\nabla f_J(x)] \quad (11)$$

# Russische Roulette Voorbeeld

$$Z = U + \frac{f(U)}{1000} \quad (12)$$

$$U \sim \text{Uniform}(0, 1), \quad f \text{ duur} \quad (13)$$

$$(14)$$

$$(15)$$

# Russische Roulette Voorbeeld

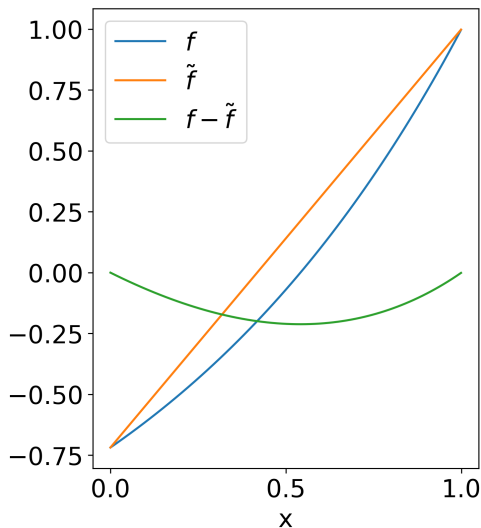
$$Z = U + \frac{f(U)}{1000} \quad (12)$$

$$U \sim \text{Uniform}(0, 1), \quad f \text{ duur} \quad (13)$$

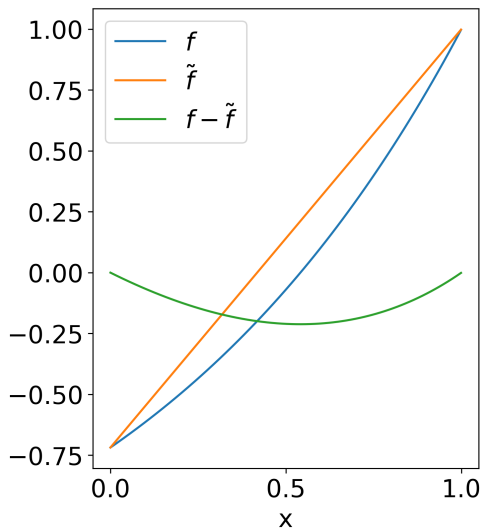
$$\tilde{Z} = U + B \left( \frac{1}{100} \right) \frac{f(U)}{10} \quad (14)$$

$$B(p) \sim \text{Bernoulli}(p) \quad (15)$$

# Control Variates

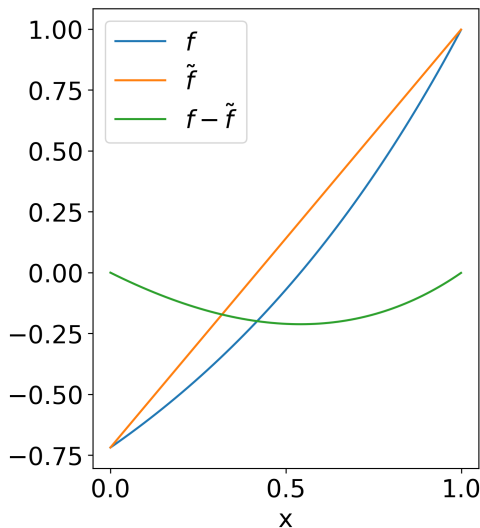


# Control Variates



$$f \approx \tilde{f} \rightarrow$$

# Control Variates

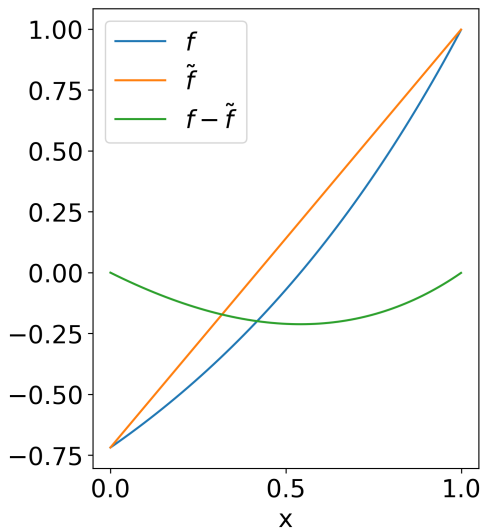


$$f \approx \tilde{f} \rightarrow$$

$$\|f - \tilde{f}\|_2 \leq \|f\|_2$$



# Control Variates



$$f \approx \tilde{f} \rightarrow$$

$$\|f - \tilde{f}\|_2 \leq \|f\|_2$$

weet  $\int \tilde{f}(s) ds$   
bv  $\tilde{f}$  lineair

# Trapezium Monte Carlo

Trap MC = Russische Roulette + Control Variates

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$$\int_x^{x+\Delta x} f(s) ds \quad (16)$$

(17)

(18)

(19)

# Trapezium Monte Carlo

Trap MC = Russische Roulette + Control Variates

$$\int_x^{x+\Delta x} f(s) ds \quad (16)$$

$$= \int_x^{x+\Delta x} \tilde{f}(s) ds + \int_x^{x+\Delta x} f(s) - \tilde{f}(s) ds \quad (17)$$

(18)

(19)

# Trapezium Monte Carlo

Trap MC = Russische Roulette + Control Variates

$$\int_x^{x+\Delta x} f(s) ds \quad (16)$$

$$= \int_x^{x+\Delta x} \tilde{f}(s) ds + \int_x^{x+\Delta x} f(s) - \tilde{f}(s) ds \quad (17)$$

$$= \Delta x \frac{f(x) + f(x + \Delta x)}{2} + E \left[ IB \left( \frac{1}{I} \right) (f(S_x) - \tilde{f}(S_x)) \right] \quad (18)$$

$$(19)$$

# Trapezium Monte Carlo

Trap MC = Russische Roulette + Control Variates

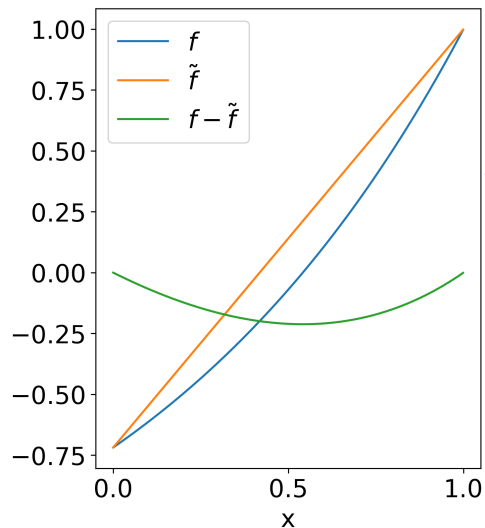
$$\int_x^{x+\Delta x} f(s) ds \quad (16)$$

$$= \int_x^{x+\Delta x} \tilde{f}(s) ds + \int_x^{x+\Delta x} f(s) - \tilde{f}(s) ds \quad (17)$$

$$= \Delta x \frac{f(x) + f(x + \Delta x)}{2} + E \left[ IB \left( \frac{1}{I} \right) (f(S_x) - \tilde{f}(S_x)) \right] \quad (18)$$

waar  $I$  = RR rate,  $S_x \sim \text{Uniform}(x, x + \Delta x)$  (19)

# Trapezium Monte Carlo



$$\begin{aligned} & \int_x^{x+\Delta x} f(s) ds \\ &= \Delta x \frac{f(x) + f(x + \Delta x)}{2} \\ &+ E \left[ IB \left( \frac{1}{I} \right) (f(S_x) - \tilde{f}(S_x)) \right] \end{aligned}$$

# Trapezium Monte Carlo

$$\int_a^b f(s) ds \quad (20)$$

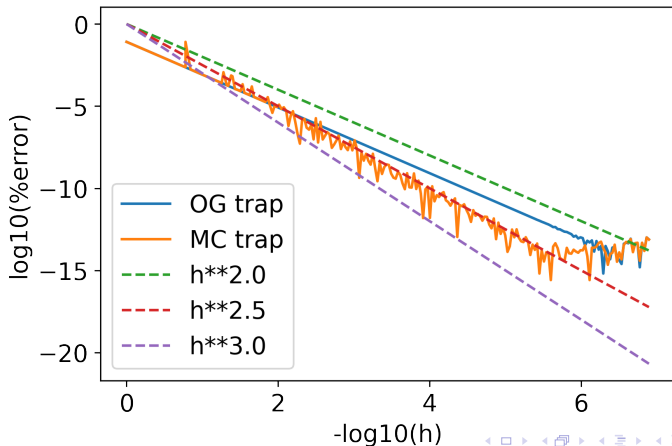
$$\approx \Delta x \sum_x \frac{f(x) + f(x + \Delta x)}{2} \quad (21)$$

$$+ IB \left( \frac{1}{I} \right) \left( f(S_x) - f(x) - \frac{S_x - x}{\Delta x} (f(x + \Delta x) - f(x)) \right) \quad (22)$$



# Trapezium Monte Carlo

$$\int_0^1 e^s ds, \quad \Delta x = h, \quad l = 100 \quad (23)$$



# Trapezium Monte Carlo

$$\|f\|_{\infty} = O(h^k) \quad (24)$$

$$\|f\|_2 = O(h^{k+\frac{d}{2}}) \quad (25)$$

$$(26)$$

$$(27)$$

# Trapezium Monte Carlo

$$\|f\|_{\infty} = O(h^k) \quad (24)$$

$$\|f\|_2 = O(h^{k+\frac{d}{2}}) \quad (25)$$

$$\|f\|_2 \leq \sqrt{\|f\|_1 \|f\|_{\infty}} \quad (26)$$

$$(27)$$

# Trapezium Monte Carlo

$$\|f\|_{\infty} = O(h^k) \quad (24)$$

$$\|f\|_2 = O(h^{k+\frac{d}{2}}) \quad (25)$$

$$\|f\|_2 \leq \sqrt{\|f\|_1 \|f\|_{\infty}} \quad (26)$$

$\downarrow$

$$\|f\|_1 = O(h^{k+d})? \quad (27)$$

# Trapezium Monte Carlo

$$\|f\|_{\infty} = O(h^k) \quad (24)$$

$$\|f\|_2 = O(h^{k+\frac{d}{2}}) \quad (25)$$

$$\|f\|_2 \leq \sqrt{\|f\|_1 \|f\|_{\infty}} \quad (26)$$

$\downarrow$

$$\|f\|_1 = O(h^{k+d})? \quad (27)$$

Heinrich en Novak 2001

# RMC Voorbeeld

$$y' = y \quad (28)$$

$$(29)$$

$$(30)$$

$$(31)$$

$$(32)$$

# RMC Voorbeeld

$$y' = y \quad (28)$$

$$y(t) = y(0) + \int_0^t y(s) ds \quad (29)$$

$$(30)$$

$$(31)$$

$$(32)$$

# RMC Voorbeeld

$$y' = y \quad (28)$$

$$y(t) = y(0) + \int_0^t y(s) ds \quad (29)$$

$$\text{wil } Y : E[Y(t)] = y(t) \quad (30)$$

$$(31)$$

$$(32)$$



# RMC Voorbeeld

$$y' = y \quad (28)$$

$$y(t) = y(0) + \int_0^t y(s) ds \quad (29)$$

$$\text{wil } Y : E[Y(t)] = y(t) \quad (30)$$

$$Y(t) = y(0) + tY(S) \quad (31)$$

$$S \sim \text{Uniform}(0, t) \quad (32)$$

# RMC Voorbeeld

$$Y(t) = y(0) + tY(S) \quad (33)$$

$$(34)$$

$$(35)$$

$$(36)$$

$$(37)$$

# RMC Voorbeeld

$$Y(t) = y(0) + tY(S) \quad (33)$$

$$\infty \text{ recursie} \quad (34)$$

$$(35)$$

$$(36)$$

$$(37)$$

# RMC Voorbeeld

$$Y(t) = y(0) + tY(S) \quad (33)$$

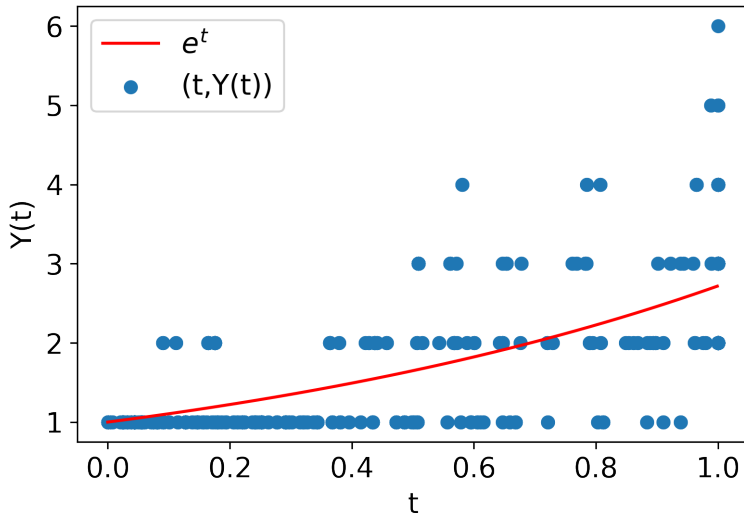
$$\infty \text{ recursie} \quad (34)$$

$$Y(t) = 1 + B(t)Y(S) \quad (35)$$

$$t < 1 \quad (36)$$

$$B(t) \sim \text{Bernoulli}(t) \quad (37)$$

# RMC Plot



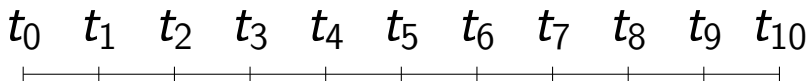
# Recurisie in Recursie

- Stochastic Variance Reduced Gradient Descent

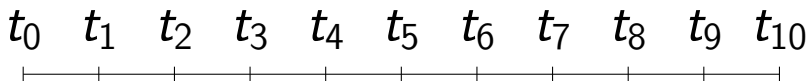
$$\begin{aligned}\nabla f(x) \\ &= E[\nabla f_J(x) - \nabla f_J(\tilde{x})] + \nabla f(\tilde{x})\end{aligned}\tag{38}$$

- Next Flight (Rendering)

# RRMC Voorbeeld



# RRMC Voorbeeld



$$y_n = \begin{cases} Y_{n-1}(t_n, y_{n-1}) & \text{als } n \neq 0 \\ y(t_0) & \text{als } n = 0 \end{cases}$$

(39)

(40)



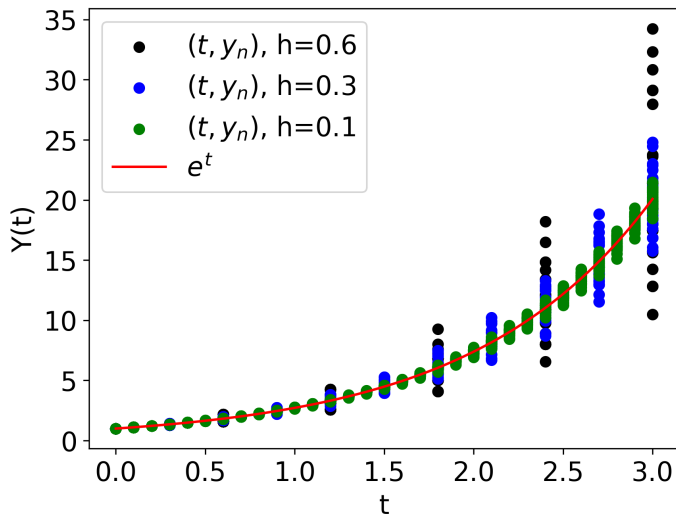
# RRMC Voorbeeld

$$\begin{array}{cccccccccccc} t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 & t_9 & t_{10} \\ | & | & | & | & | & | & | & | & | & | & | \end{array}$$

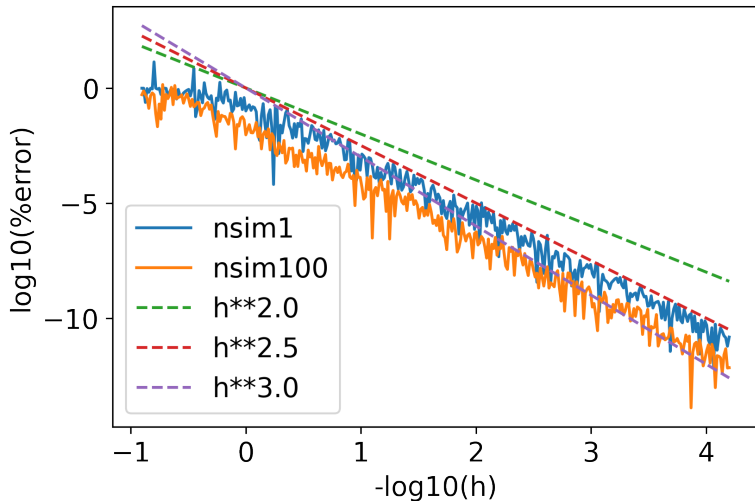
$$y_n = \begin{cases} Y_{n-1}(t_n, y_{n-1}) & \text{als } n \neq 0 \\ y(t_0) & \text{als } n = 0 \end{cases} \quad (39)$$

$$Y_n(t, y_n) = y_n + \Delta t Y_n(S_n, y_n) \quad (40)$$

# RRMC Plot



# CV RRMC Plot



# Limitaties

- Stabiliteit  $\rightarrow$  exp integrator
- Efficiëntie  $\rightarrow$  ???