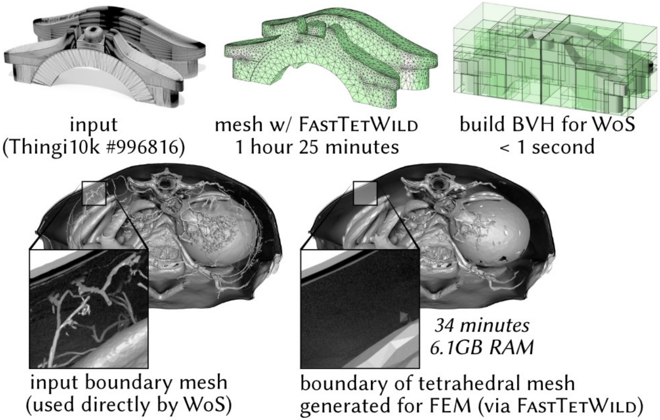
# RMC voor lineaire ODEs

Isidoor Pinillo Esquivel

# Grid-Free Monte Carlo

## Sawhney e.a. 2022



# Monte Carlo

$$\int_0^1 f(s)ds = E[f(U)] \tag{1}$$

$$\approx \frac{1}{n}\sum_{j=1}^{n} f(U_j) \tag{2}$$

$$\text{met } U_j \sim \text{Uniform}(0,1) \tag{3}$$

# Monte Carlo Fout Analyse

$$\text{fout} = O_p\left(\sqrt{\frac{\text{Var}(f(U))}{n}}\right) \text{ (CLT)} \quad (4)$$

# Monte Carlo Fout Analyse

$$\text{fout} = O_p\left(\sqrt{\frac{\text{Var}(f(U))}{n}}\right) \text{(CLT)} \quad (4)$$

$$\sqrt{\text{Var}(f(U))} \quad (5)$$

$$= \sqrt{E[(f(U) - E[f(U)])^2]} \quad (6)$$

$$= ||f - E[f(U)]||_2 \quad (7)$$

$$\sim ||f||_2 \quad (8)$$

# Waarom Monte Carlo?

- Paralleliseerbaar
- Hoge dimensie
- Complexe geometrie

# Waarom ODEs?

- Grid-free + tijdafhankelijkheid?
- ODEs simpeler als PDEs

# Stochastic Gradient Descent

SGD $=$ GD $+$ unbiased gradients

# Stochastic Gradient Descent

SGD = GD + unbiased gradients

$$f(x) = \frac{1}{n}\sum_{j=1}^{n} f_j(x) \tag{9}$$

$$ \tag{10}$$

$$ \tag{11}$$

# Stochastic Gradient Descent

SGD = GD + unbiased gradients

$$f(x) = \frac{1}{n}\sum_{j=1}^{n} f_j(x) \tag{9}$$

$$\nabla f(x) = \frac{1}{n}\sum_{j=1}^{n} \nabla f_j(x) \tag{10}$$

$$\tag{11}$$

# Stochastic Gradient Descent

SGD = GD + unbiased gradients

$$f(x) = \frac{1}{n}\sum_{j=1}^{n} f_j(x) \tag{9}$$

$$\nabla f(x) = \frac{1}{n}\sum_{j=1}^{n} \nabla f_j(x) \tag{10}$$

$$= E[\nabla f_J(x)] \tag{11}$$

# Russische Roulette Voorbeeld

$$Z = U + \frac{f(U)}{1000} \tag{12}$$
$$U \sim \text{Uniform}(0,1), \quad f \text{ duur} \tag{13}$$
$$\tag{14}$$
$$\tag{15}$$

# Russische Roulette Voorbeeld

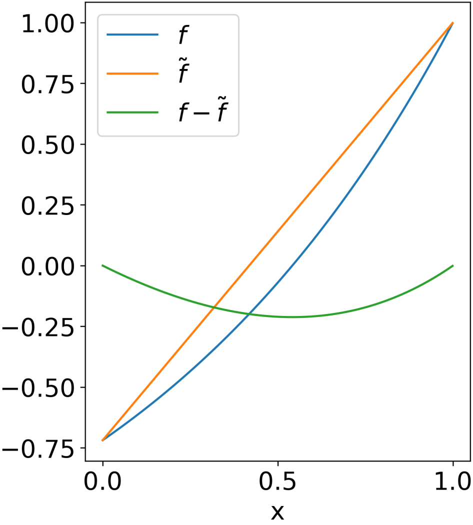
$$Z = U + \frac{f(U)}{1000} \tag{12}$$

$$U \sim \text{Uniform}(0, 1), \quad f \text{ duur} \tag{13}$$

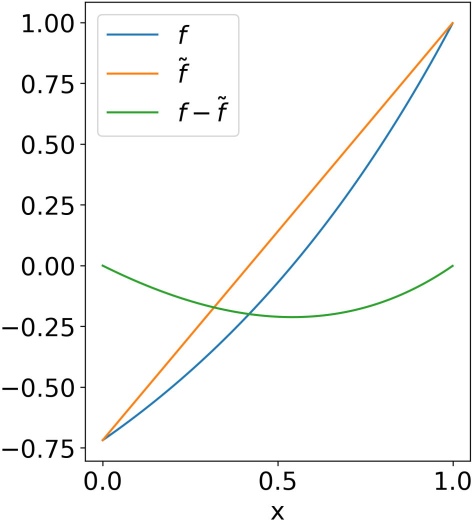
$$\tilde{Z} = U + B\left(\frac{1}{100}\right)\frac{f(U)}{10} \tag{14}$$

$$B(p) \sim \text{Bernoulli}(p) \tag{15}$$

# Control Variates

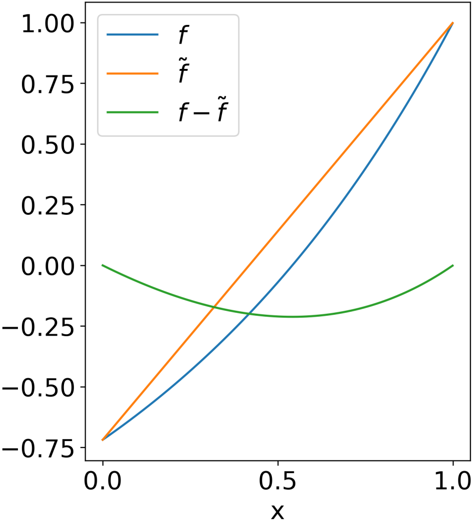


# Control Variates




$$f \approx \tilde{f} \to$$

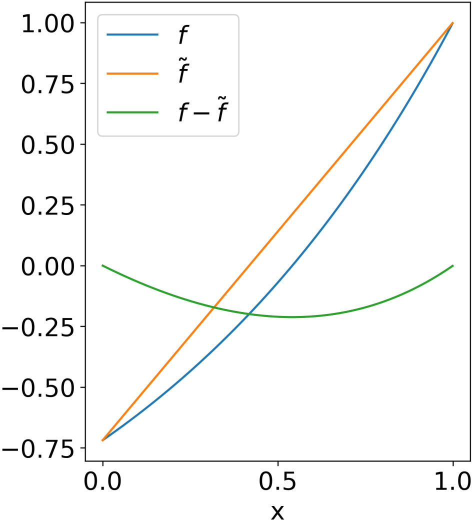
# Control Variates




$$f \approx \tilde{f} \rightarrow$$

$$||f - \tilde{f}||_2 \leq ||f||_2$$

# Control Variates




$$f \approx \tilde{f} \rightarrow$$

$$||f - \tilde{f}||_2 \leq ||f||_2$$

weet $\int \tilde{f}(s)ds$

bv $\tilde{f}$ lineair

# Trapezium Monte Carlo

Trap MC = Russische Roulette + Control Variates

# Trapezium Monte Carlo

Trap MC = Russische Roulette + Control Variates

$$\int_{x}^{x+\Delta x} f(s)ds \tag{16}$$

(17)

(18)

(19)

# Trapezium Monte Carlo

Trap MC = Russische Roulette + Control Variates

$$\int_x^{x+\Delta x} f(s)ds \tag{16}$$

$$= \int_x^{x+\Delta x} \tilde{f}(s)ds + \int_x^{x+\Delta x} f(s) - \tilde{f}(s)ds \tag{17}$$

$$\tag{18}$$

$$\tag{19}$$

# Trapezium Monte Carlo

Trap MC = Russische Roulette + Control Variates

$$\int_x^{x+\Delta x} f(s)ds \tag{16}$$

$$= \int_x^{x+\Delta x} \tilde{f}(s)ds + \int_x^{x+\Delta x} f(s) - \tilde{f}(s)ds \tag{17}$$

$$= \Delta x \frac{f(x) + f(x+\Delta x)}{2} + E\left[ IB\left(\frac{1}{l}\right)(f(S_x) - \tilde{f}(S_x))\right] \tag{18}$$

$$\tag{19}$$

# Trapezium Monte Carlo

Trap MC = Russische Roulette + Control Variates

$$\int_x^{x+\Delta x} f(s)ds \tag{16}$$

$$= \int_x^{x+\Delta x} \tilde{f}(s)ds + \int_x^{x+\Delta x} f(s) - \tilde{f}(s)ds \tag{17}$$

$$= \Delta x \frac{f(x) + f(x+\Delta x)}{2} + E\left[ IB\left(\frac{1}{l}\right)(f(S_x) - \tilde{f}(S_x))\right] \tag{18}$$

$$\text{waar } \mathrm{l} = \text{ RR rate}, S_x \sim \text{Uniform}(x, x+\Delta x) \tag{19}$$

# Trapezium Monte Carlo

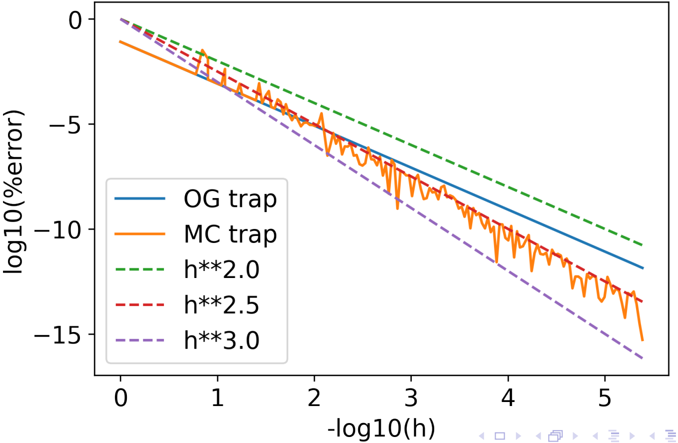
$$\int_a^b f(s)ds \tag{20}$$

$$\approx \Delta x \sum_x \frac{f(x) + f(x + \Delta x)}{2} \tag{21}$$

$$+ IB\left(\frac{1}{l}\right)\left(f(S_x) - f(x) - \frac{S_x - x}{\Delta x}(f(x + \Delta x) - f(x))\right) \tag{22}$$

# Trapezium Monte Carlo

$$\int_0^1 e^s ds, \quad \Delta x = h, \quad l = 100 \tag{23}$$



# RMC Voorbeeld

$$y' = y \tag{24}$$

(25)

(26)

(27)

(28)

# RMC Voorbeeld

$$y' = y \tag{24}$$

$$y(t) = y(0) + \int_0^t y(s)ds \tag{25}$$

$$\tag{26}$$

$$\tag{27}$$

$$\tag{28}$$

# RMC Voorbeeld

$$y' = y \tag{24}$$

$$y(t) = y(0) + \int_0^t y(s)ds \tag{25}$$

$$\text{wil } Y : E[Y(t)] = y(t) \tag{26}$$

$$\tag{27}$$

$$\tag{28}$$

# RMC Voorbeeld

$$y' = y \tag{24}$$

$$y(t) = y(0) + \int_0^t y(s)ds \tag{25}$$

$$\text{wil } Y : E[Y(t)] = y(t) \tag{26}$$

$$Y(t) = y(0) + tY(S) \tag{27}$$

$$S \sim \text{Uniform}(0, t) \tag{28}$$

# RMC Voorbeeld

$$Y(t) = y(0) + tY(S) \tag{29}$$

(30)

(31)

(32)

(33)

# RMC Voorbeeld

$$Y(t) = y(0) + tY(S) \tag{29}$$

$$\infty \text{ recursie} \tag{30}$$

$$\tag{31}$$

$$\tag{32}$$

$$\tag{33}$$

# RMC Voorbeeld

$$Y(t) = y(0) + tY(S) \tag{29}$$

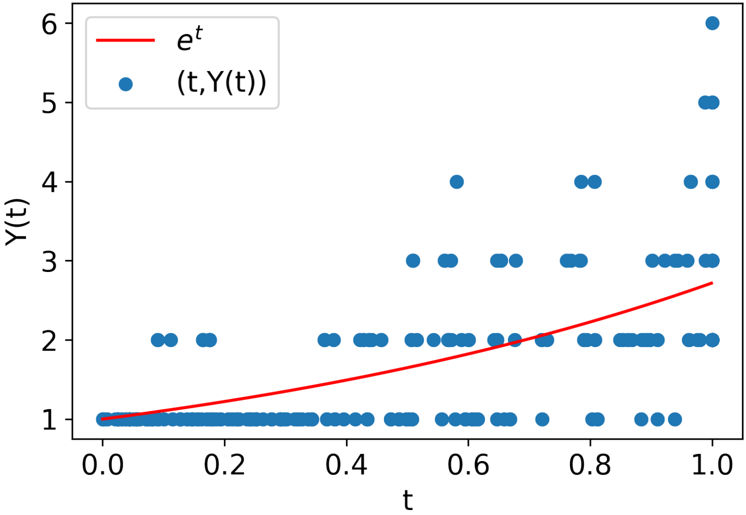
$$\infty \text{ recursie} \tag{30}$$

$$Y(t) = 1 + B(t)Y(S) \tag{31}$$

$$t < 1 \tag{32}$$

$$B(t) \sim \text{Bernoulli}(t) \tag{33}$$

# RMC Plot



# Recursie in Recursie

- Next Flight (Rendering)
- Stochastic Variance Reduced Gradient Descent

$$\nabla f(x) = E[\nabla f_J(x) - \nabla f_J(\tilde{x})] + \nabla f(\tilde{x}) \tag{34}$$

# RRMC Voorbeeld

$$t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t_7 \quad t_8 \quad t_9 \quad t_{10}$$

# RRMC Voorbeeld

$t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t_7 \quad t_8 \quad t_9 \quad t_{10}$

$$y_n = \begin{cases} Y_{n-1}(t_n, y_{n-1}) & \text{als } n \neq 0 \\ y(t_0) & \text{als } n = 0 \end{cases} \qquad (35)\ (36)$$

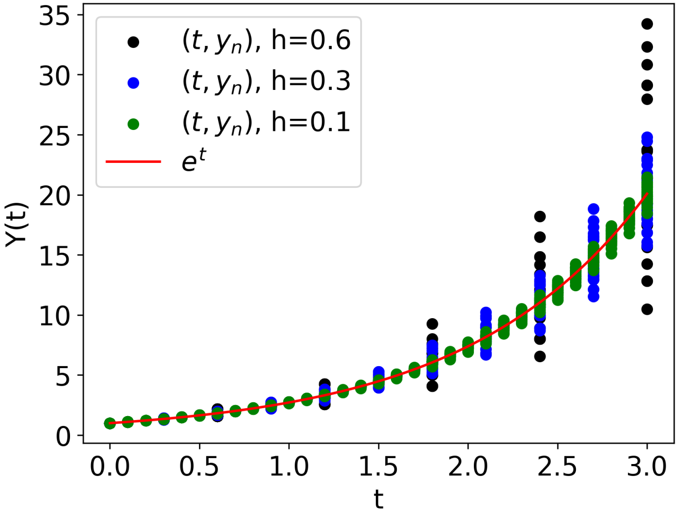
# RRMC Voorbeeld

$$t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t_7 \quad t_8 \quad t_9 \quad t_{10}$$

$$y_n = \begin{cases} Y_{n-1}(t_n, y_{n-1}) & \text{als } n \neq 0 \\ y(t_0) & \text{als } n = 0 \end{cases} \tag{35}$$

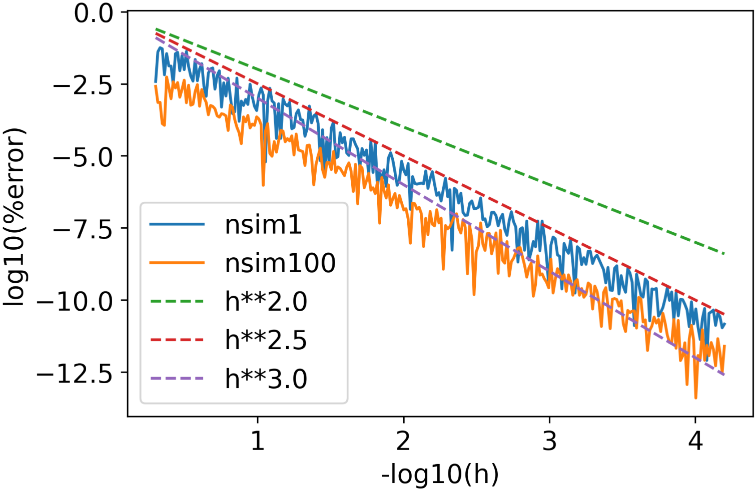
$$Y_n(t, y_n) = y_n + \Delta t Y_n(S_n, y_n) \tag{36}$$

# RRMC Plot



35
30
25
20
15
10
5
0
Y(t)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
t
$(t, y_n)$, h=0.6
$(t, y_n)$, h=0.3
$(t, y_n)$, h=0.1
$e^t$

# CV RRMC Plot



0.0
−2.5
−5.0
−7.5
−10.0
−12.5
log10(%error)
1
2
3
4
-log10(h)
nsim1
nsim100
h**2.0
h**2.5
h**3.0

# Limitaties

- Stabiliteit $\rightarrow$ Kettunen e.a. 2021 (efficiënte unbiased $e^{\int A(s)ds}y(0)$ )
- Biased voor non-lineaire ODEs

# Toekomstig Werk

- Random ODEs
- Specifieke ODEs