# RMC voor lineaire ODEs

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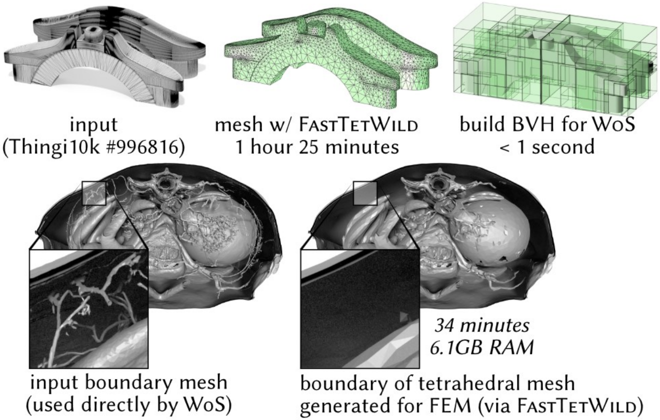
# Grid-Free Monte Carlo

## Sawhney e.a. 2022



# Monte Carlo

$$\int_0^1 f(s)ds = E[f(U)] \tag{1}$$

$$\approx \frac{1}{n}\sum_{j=1}^{n} f(U_j) \tag{2}$$

$$\text{met } U_j \sim \text{Uniform}(0,1) \tag{3}$$

# Monte Carlo Fout Analyse

$$\text{fout} = O_p\left(\sqrt{\frac{\text{Var}(f(U))}{n}}\right) \text{(CLT)} \quad (4)$$

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$$\sqrt{\mathrm{Var}(f(U))} \quad (5)$$

$$= \sqrt{E[(f(U) - E[f(U)])^2]} \quad (6)$$

$$= ||f - E[f(U)]||_2 \quad (7)$$

$$\sim ||f||_2 \quad (8)$$

# Waarom Monte Carlo?

- paralleliseerbaar
- dimensie onafhankelijke convergentie
- complexe geometrie

# Waarom ODEs?

- grid-free + tijdafhankelijkheid?
- ODEs simpeler als PDEs

# Stochastic Gradient Descent

SGD = GD + unbiased gradients

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$$f(x) = \frac{1}{n}\sum_{j=1}^{n} f_j(x) \tag{9}$$

$$\tag{10}$$

$$\tag{11}$$

# Stochastic Gradient Descent

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$$\nabla f(x) = \frac{1}{n}\sum_{j=1}^{n} \nabla f_j(x) \tag{10}$$

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$$= E[\nabla f_J(x)] \tag{11}$$

# Russische Roulette Voorbeeld

$$Z = U + \frac{f(U)}{1000} \tag{12}$$
$$U \sim \text{Uniform}(0,1), \quad f \text{ duur} \tag{13}$$
$$\tag{14}$$
$$\tag{15}$$

# Russische Roulette Voorbeeld

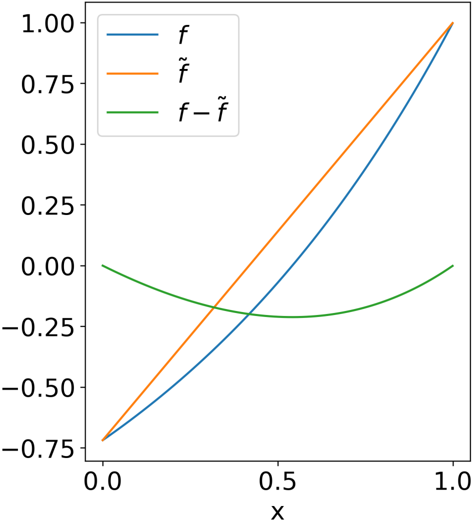
$$Z = U + \frac{f(U)}{1000} \tag{12}$$

$$U \sim \text{Uniform}(0,1), \quad f \text{ duur} \tag{13}$$

$$\tilde{Z} = U + B\left(\frac{1}{100}\right)\frac{f(U)}{10} \tag{14}$$

$$B(p) \sim \text{Bernoulli}(p) \tag{15}$$

# Control Variates



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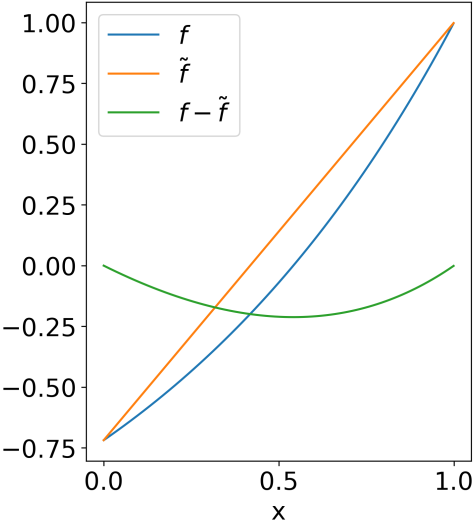


$$f \approx \tilde{f} \rightarrow$$

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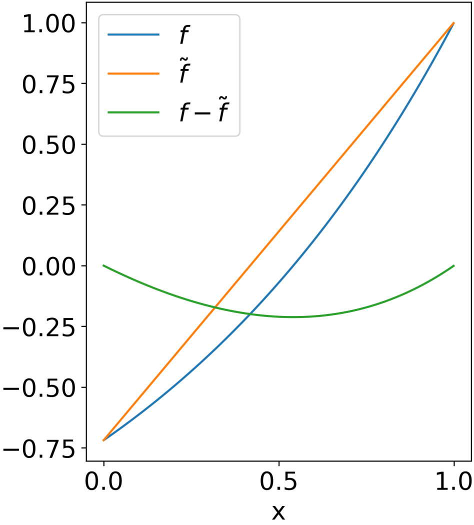


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$$||f - \tilde{f}||_2 \leq ||f||_2$$

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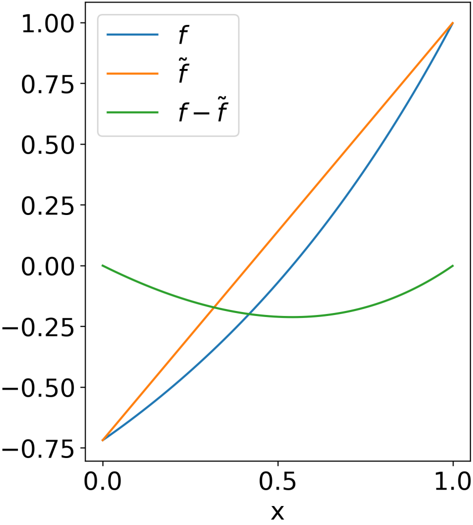


$$f \approx \tilde{f} \rightarrow$$

$$||f - \tilde{f}||_2 \leq ||f||_2$$

weet $\int \tilde{f}(s)ds$

bv $\tilde{f}$ lineair

# Trapezium Monte Carlo

Trap MC = Russische Roulette + Control Variates

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$$\int_{x}^{x+\Delta x} f(s)ds \tag{16}$$

(17)

(18)

(19)

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Trap MC = Russische Roulette + Control Variates

$$\int_x^{x+\Delta x} f(s)ds \tag{16}$$

$$= \int_x^{x+\Delta x} \tilde{f}(s)ds + \int_x^{x+\Delta x} f(s) - \tilde{f}(s)ds \tag{17}$$

(18)

(19)

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$$= \Delta x \frac{f(x) + f(x+\Delta x)}{2} + E\left[ IB\left(\frac{1}{l}\right)(f(S_x) - \tilde{f}(S_x))\right] \tag{18}$$

$$\tag{19}$$

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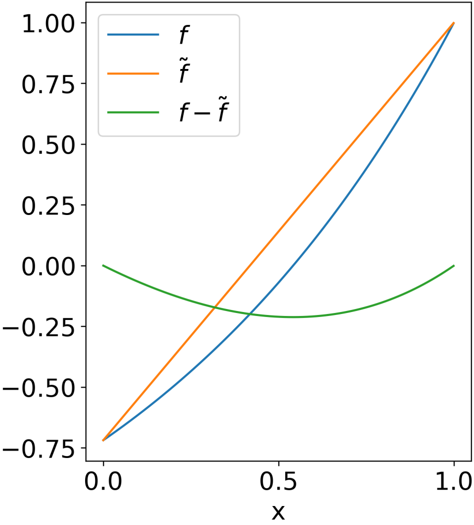
$$\text{waar } l = \text{ RR rate}, S_x \sim \text{Uniform}(x, x+\Delta x) \tag{19}$$

# Trapezium Monte Carlo




$$\int_x^{x+\Delta x} f(s)ds$$

$$= \Delta x \frac{f(x) + f(x+\Delta x)}{2}$$

$$+ E\left[IB\left(\frac{1}{l}\right)(f(S_x) - \tilde{f}(S_x))\right]$$

# Trapezium Monte Carlo

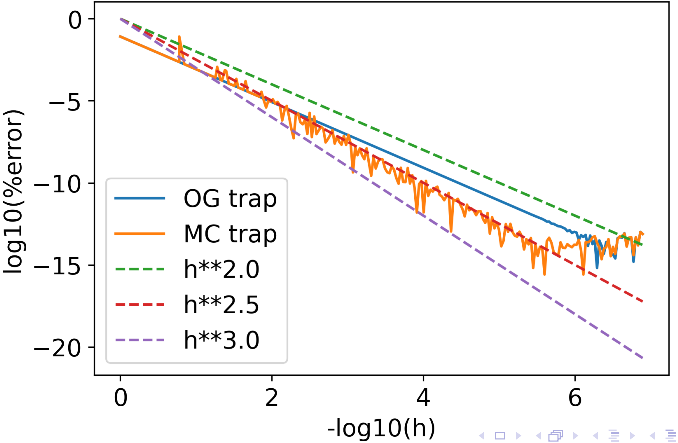
$$\int_a^b f(s)ds \tag{20}$$

$$\approx \Delta x \sum_x \frac{f(x) + f(x + \Delta x)}{2} \tag{21}$$

$$+ IB\left(\frac{1}{l}\right)\left(f(S_x) - f(x) - \frac{S_x - x}{\Delta x}(f(x + \Delta x) - f(x))\right) \tag{22}$$

# Trapezium Monte Carlo

$$\int_0^1 e^s ds, \quad \Delta x = h, \quad l = 100 \tag{23}$$



# Trapezium Monte Carlo

$$\|f\|_\infty = O_p(h^k) \tag{24}$$

$$\|f\|_2 = O_p(h^{k+\frac{d}{2}}) \tag{25}$$

(26)

(27)

(28)

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$$\tag{28}$$

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$$\|f\|_1 = O_p\left(h^{k+d}\right)? \tag{28}$$

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Heinrich en Novak 2001

# RMC Voorbeeld

$$y' = y \tag{29}$$

(30)

(31)

(32)

(33)

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$$y' = y \tag{29}$$

$$y(t) = y(0) + \int_0^t y(s)ds \tag{30}$$

$$\tag{31}$$

$$\tag{32}$$

$$\tag{33}$$

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$$\text{wil } Y : E[Y(t)] = y(t) \tag{31}$$

$$\tag{32}$$

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$$y(t) = y(0) + \int_0^t y(s)ds \tag{30}$$

$$\text{wil } Y : E[Y(t)] = y(t) \tag{31}$$

$$Y(t) = y(0) + tY(S) \tag{32}$$

$$S \sim \text{Uniform}(0, t) \tag{33}$$

# RMC Voorbeeld

$$Y(t) = y(0) + tY(S) \tag{34}$$

(35)

(36)

(37)

(38)

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$$\infty \text{ recursie} \tag{35}$$

$$\tag{36}$$

$$\tag{37}$$

$$\tag{38}$$

# RMC Voorbeeld

$$Y(t) = y(0) + tY(S) \tag{34}$$

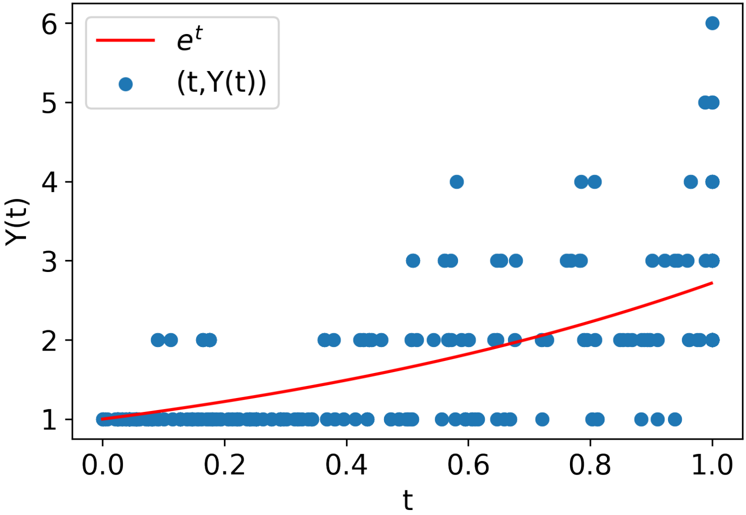
$$\infty \text{ recursie} \tag{35}$$

$$Y(t) = 1 + B(t)Y(S) \tag{36}$$

$$t < 1 \tag{37}$$

$$B(t) \sim \text{Bernoulli}(t) \tag{38}$$

# RMC Voorbeeld



# RRMC Voorbeeld

$$y' = y, \quad y(0) = 1, \quad (t_n) \tag{39}$$

$$y(t) = y(t_n) + \int_{t_n}^{t} y(s)ds \tag{40}$$

$$Y_n(t) = y(t_n) + (t - t_n)Y_n((t - t_n)U + t_n) \tag{41}$$

$$Y_n(t) = y_n + (t - t_n)Y_n((t - t_n)U + t_n) \tag{42}$$

$$y_n = \begin{cases} Y_{n-1}(t_n) & \text{if } n \neq 0 \\ y(t_0) & \text{if } n = 0 \end{cases} \tag{43}$$

# Unbiased Non-Linearity

- exponentiele voorbeeld + screenshot paper
- VRE
- Feynman-Kac formule
- Magnus series