RMC voor lineaire ODEs

Isidoor Pinillo Esquivel

Introductie

$$y_t = A(t)y(t). \tag{1}$$

Euler:

$$\begin{aligned}
\frac{\tilde{y}(t_k + \Delta t) - \tilde{y}(t_k)}{\Delta t} &= A(t_k)\tilde{y}(t_k) \quad (2)\\
\tilde{y}(t_{k+1}) &= (I + \Delta t A(t_k))\tilde{y}(t_k) \quad (3)\\
Y(T_{k+1}) &= (I + \Delta t A(T_{k+1}))Y(T_k) \quad (4)
\end{aligned}$$

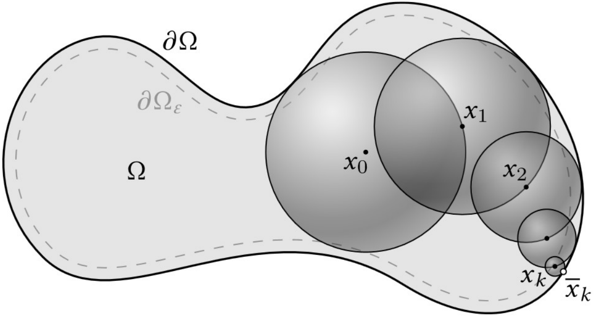
$T_{k+1} = T_k + \mathrm{Exp}(\Delta t)$, exponentiële verdeelde stappen.

$$y(T_k) = E[Y(T_k)]. \tag{5}$$

Overzicht

Motivatie

Veralgemenen van WoS algoritme van (Sawhney e.a. 2022) naar tijd



Monte Carlo

Random sample efficientie van optelling

Probleem: benader $\bar{x} = \frac{1}{k}\sum_{i=1}^{k} x_i$ met $n << k$ van x_i's $\in [0,1]$ (symmetrisch in x_i)

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Vertrouwensinterval kans $= 1 \sim \frac{k-n}{k} = 1 - \frac{n}{k} \approx 1$
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Samplen of termen willekeurig weg laten (Russische roulette)

$$\bar{x} \cong \frac{1}{n}\sum_{i=1}^{n} x_{I_i} \cong \frac{1}{k}\sum_{i=1}^{k} B_i x_i. \tag{6}$$

$\left(E[B_i] = 1, P[B_i = 0] = 1 - \frac{n}{k}\right)$

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Variantie $=$ RMSE $\sim O\left(\frac{1}{\sqrt{n}}\right)$ en ook vertrouwensintervallen kans < 1 (CLT of Chebychev's ongelijkheid)

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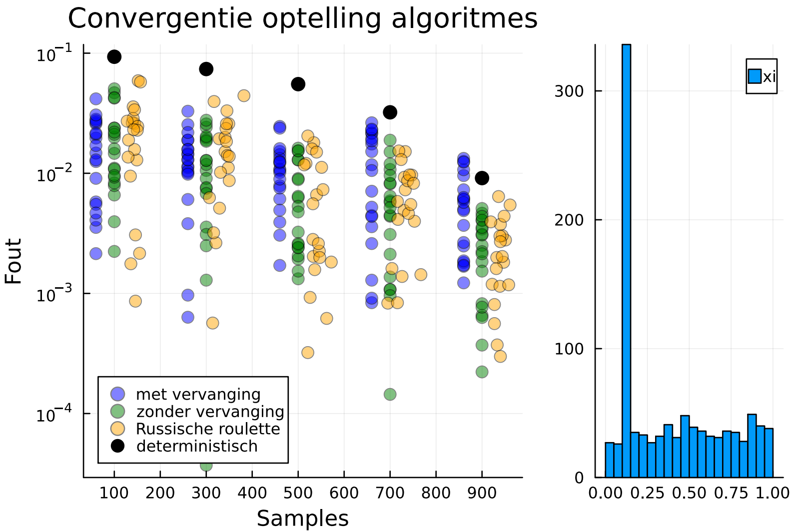
Optelling algoritmes (plot)



Monte Carlo integratie

Integratie $\approx$ sommatie, integreerbare $f : \mathbb{R} \to [0,1]$:

$$\int_0^1 f(s)ds = E[f(U)] \tag{7}$$

$$\cong \frac{1}{n}\sum_{j=1}^{n} f(U_j) \tag{8}$$

$$\text{met } U_j \sim \text{Uniform}(0,1) \tag{9}$$

Overzicht integratie

Beschouw $f : [0,1]^d \to [0,1]$ k-keer Lipschitz-differentieerbaar:

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	$k = -1$	$0 \leq k \leq 2$	$3 \leq k \leq 5$	$k > 5$
$d = 1$	MC	CVMC	?	FFT
$2 \leq d \leq 5$	MC	CVMC	CVMC	?
$d > 5$	MC	MC	MC	MC

Tabel: Overzicht integratie algoritmes.

Main Poisson algoritme

$$y_t = A(t)y, \quad A(t) \text{ meetbaar en } \sup_t ||A(t)|| < \infty \Leftrightarrow \tag{11}$$

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$$

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$$
y(t) = \int_0^{e^{-\sigma t}} y(0) d\tau + \int_{e^{-\sigma t}}^1 \left(I + \frac{A(s)}{\sigma}\right) y(s) d\tau. \tag{15}
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sample τ uniform, equivalent aan een Poisson proces te samplen.

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Doe dit recursief verder:

$$Y(t) = \begin{cases} y(0) & \text{als } e^{-\sigma t} \leq \tau \\ \left(I + \frac{A(S)}{\sigma} \right) Y(S) & \text{anders} \end{cases}, \tag{17}$$

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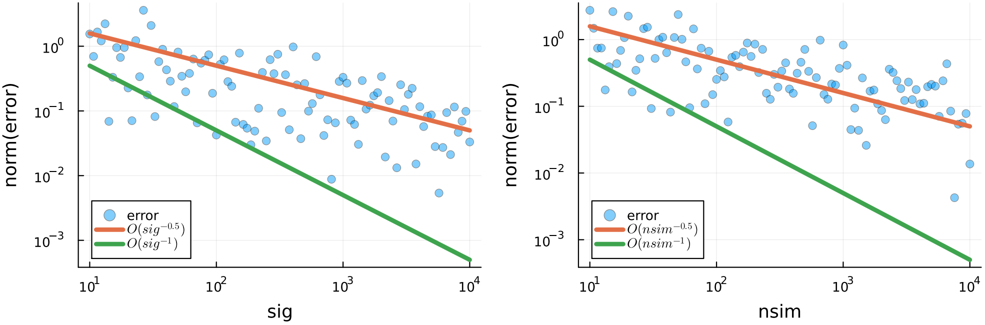
$$y(t) = E\left[\prod_{k=1}^{N_t} \left(I + \frac{A(T_k)}{\sigma} \right) \right] y(0). \tag{18}$$

Main Poisson algoritme (convergentie)



Figuur: Realizaties van de norm van de fout, met *nsim* $= 1$ en σ variabel of *nsim* variabel en $\sigma = 1$. Hoeveelheid samples van A per simulatie = Poisson(σ).

Main Poisson algoritme (opmerkingen)

$$y(t) \cong Y(t) = \prod_{k=1}^{N_t} \left(I + \frac{A(T_k)}{\sigma} \right) y(0). \tag{19}$$

- TB: $E[Y(t)], Var[Y(t)] < \infty$
- Parallelle complexiteit
- Volgorde matrix vermenigvuldingen $v^T y(t)$
- Inspiratie uit Acebrón en Ribeiro 2016

Limitaties

- Stabiliteit $\rightarrow$ Kettunen e.a. 2021 (efficiënte unbiased $e^{\int A(s)ds}y(0)$)
- Biased voor non-lineaire ODEs

Toekomstig Werk

- Random ODEs
- Specifieke ODEs