General Approach to Blind Source Separation

Xi-Ren Cao, *Fellow, IEEE,* and Ruey-wen Liu, *Life Fellow, IEEE*

***Abstract*—This paper identifies and studies two major issues in the blind source separation problem: separability and separation principles. We show that separability is an intrinsic property of the measured signals and can be described by the concept of m-row decomposability introduced in this paper; we also show that separation principles can be developed by using the structure characterization theory of random variables. In particular, we show that these principles can be derived concisely and intuitively by applying the Darmois–Skitovich theorem, which is well known in statistical inference theory and psychology. Some new insights are gained for designing blind source separation filters.**

I. Introduction

BLIND SOURCE separation is a fundamental problem in signal processing. Consider a number of source signals coming from different sources and a number of receivers. Each receiver receives a linear combination of these source signals; combinations of these are called *measured signals*. Neither the structure of the linear combinations nor the source signals are known to the receivers. In this environment, we want to identify the linear combinations (blind identification problem) and decouple the linear combinations (blind source decoupling).

This area has been very active, relevant works include [1]–[3], [7]–[10], and [12]–[21]. Surprisingly, this seemingly impossible problem has elegant solutions. It also has a broad range of applications in many areas, such as array signal processing [14], seismic signal processing, identification of MA processing [15], and blind equalization [20].

To distinguish the problem of source separation from that of signals separation (correlation in time domain is normally exploited in the latter but not in the former), the blind source separation problem is also called *independent component analysis* [3]. The goal of this problem is to design a filter, with the measured signals as its inputs, such that its output signals are as independent as possible. There are many works dealing with this subject (see, e.g., [1]–[3], [5], [12], [13], [15]–[17], [19]). Our purpose is to provide a comprehensive study on the feasibility of decoupling and the basic principles for identification. We show that there are two major issues associated with this approach. First, it is possible to separate the source signals only if the measured signals satisfy some conditions. This issue is called the *separability*, which is an intrinsic property of the unknown combination in the measured signals. The second problem is, given a set of separable measured signals, what are the principles that we can apply to verify whether the source signals have been separated? This is a nontrivial question because both the source signals and the form of the combinations are unknown. We call these principles the *separation principles*.

While some results derived in the paper regarding separation principles already exist in the literature, we intend to provide a clear picture as well as a uniform framework for the fundamental theory, which covers the existing results as special cases, and to develop some new insights and results by using this approach. In particular, our results include the case where only one output signal is independent of others and the case where the number of measured signals is less than the number of the source signals. These results provide some new insights that are useful for designing simpler blind source separation filters. In [17], it has been shown that under some circumstances it is possible, by using higher order statistics, to estimate the distinct angles of arrivals for more sources than sensors. Our approach applies to the model presented in this paper (see Section II) and hence the problem is different from the one in [17].

We first define the concept of m-row decomposability of a matrix. We prove that in principle the n source signals can be separated into m groups if and only if the combination matrix is m-row decomposable. When $m = n$, this means that the matrix has a full rank and the source signals can be completely separated. When $m < n$, necessary and sufficient conditions for m-row decomposability are derived. By applying these conditions, some special results are obtained. We show that when the number of measured signals (hence the number of sensors) l is less than that of the source signals n, it is possible to separate the source signals into l groups only if there are $n - l$ column vectors in the combination matrix such that each of them is proportional to one of the remaining l column vectors. We show, by an example, that it is possible to use a set of sensors whose number is less than that of the source signals to retrieve any source signal, one at a time, by carefully designing the location of the sensors.

We then show that the basic separation principles can be derived by using the structure characterization theory of random variables (see Section IV for a detailed description). Briefly speaking, the characterization problem can be stated as follows. Given a linear structure of a random vector whose

Manuscript received July 27, 1994; revised August 17, 1995. This work was supported by Hong Kong UPGC under grant HKUST 599/94E and by the Office of Naval Research under Contract N00014-91-J-1461. The associate editor coordinating the review of this paper and approving it for publication was Prof. Stanley J. Reeves.

X.-R. Cao is with the Hong Kong University of Science and Technology, Kowloon, Hong Kong.

R. Liu is with the University of Notre Dame, Notre Dame, IN 46556 USA.

Publisher Item Identifier S 1053-587X(96)02394-8.

components are independent, can we characterize the linear structure or the probability distributions of the random vector? This problem has been studied extensively with respect to its applications to statistic inference and psychology (see e.g., [11]); however, it requires some specific considerations to apply the theory to blind source separation. In particular, we show that the fundamental separation principles can be derived and explained in a clear and concise manner by using the Darmois–Skitovich theorem, which is itself very simple.

Section II states the problem, Section III deals with separability, and Section IV studies the separation principles. The paper concludes with a summary in Section V.

II. The Problem

Let $\boldsymbol{s} = (s_1, s_2, \cdots, s_n)^T$ with $s_i, i = 1, 2, \cdots, n$ being n mutually independent random processes whose exact distribution functions are unknown. These $s_i, i = 1, 2, \cdots, n$ represent n unknown *source signals*. Suppose that we have l sensors; the output of each sensor, denoted as $x_j, j = 1, 2, \cdots, l$, measures a linear combination of the n signals. In a vector form, this is expressed as

$$\boldsymbol{x} = B\boldsymbol{s} \tag{1}$$

where $\boldsymbol{x} = (x_1, \cdots, x_l)^T$ and B is an $l \times n$ matrix. $x_i, i = 1, 2 \cdots, l$, are called the *measured signals* and $\boldsymbol{x}$ is the information that is available, B and $\boldsymbol{s}$ are unknown, and B is not necessary to be of full rank. We wish to find the condition under which it is possible to retrieve $\boldsymbol{s}$ from $\boldsymbol{x}$ and shall develop general principles for source separation.

In blind source separation, a linear filter is designed that uses $\boldsymbol{x}$ as the input. The output of the filter is an m-dimensional random vector $\boldsymbol{y} = (y_1, \cdots, y_m)^T$ that satisfies

$$\boldsymbol{y} = C\boldsymbol{x} \tag{2}$$

where C is an $m \times l$ matrix, whose components can be chosen at our discretion. Combining both (1) and (2) together, we get

$$\boldsymbol{y} = A\boldsymbol{s} \tag{3}$$

where $A = C \times B$ is an $m \times n$ matrix. Note that we have only partial control of A, because B is either unknown and predetermined, or we have only a very little control of it by choosing the locations of the sensors (see (2)).

If the source signals are corrupted by noises, then the measured signals are

$$\boldsymbol{x} = B\boldsymbol{s} + \boldsymbol{v}$$

where $\boldsymbol{v} = (v_1, \cdots, v_n)^T$ represents noises, which are assumed to be mutually independent to each other (spatially white) and to the source signals. This equation can be rewritten as

$$\boldsymbol{x} = B'\boldsymbol{s}'$$

where $B' = (B, I)$ and $\boldsymbol{s}, = (\boldsymbol{s}^T, \boldsymbol{v}^T)^T$. This is in the same form as (1). Thus, when $l < n$, (1) covers signals with additive noises as special cases (see Section IV-C).

Equation (3) shows that $\boldsymbol{y}$ has a linear structure. Ideally if, for example, the first row of A is $(1, 0, \cdots, 0)$, then we have $y_1 = s_1$, i.e., we can successfully retrieve the first source signal; or in a better situation, if A is an identity matrix, then $y_i = s_i$ for all $i = 1, 2, \cdots, n$. From this discussion, the problem becomes whether it is possible and if so, how, to obtain a C such that each row of A contains only one nonzero entry (a more general discussion is given in Section III).

Therefore, there are two major issues in this problem. The first one is the *separability*, i.e., the existence of a C such that A can separate the source signals. This depends purely on the structure of B. In general, (1) represents a set of l linear equations in n unknowns. If the rank of B equals n (we must have $l \geq n$ in this case), then there always exists a C such that $CB = A = I$, where I is an identity matrix. That is, the set of sources is separable. This is the assumption used in much of the literature including [3], [5], [15], and [12]. However, the problem is more complicated when $l < n$. In this paper, we shall provide a uniform framework for both cases $l < n$ and $l \geq n$. A formal study of this is in Section III.

The second issue is that even if such a C exists, because B and $\boldsymbol{s}$ are unknown, there is no way to measure the matrix $A = C \times B$; i.e., one does not know A explicitly. The issue therefore becomes how to determine the structure of A by applying some principles; we refer to these principles as the *separation principles*.

Some research has been done in developing separation principles. Comon [3] studied the case $m = n = l$; it was shown that if the n components of $\boldsymbol{y}$ are pairwise independent and $\boldsymbol{s}$ has at most one component with Gaussian distribution, then the $n \times n$ matrix A is a generalized permutation matrix, i.e., each row and column contains only one nonzero entry. In this paper, we shall extend these results to more general cases and show that the basic principles can be derived concisely by using only one simple theorem in structure characterization theory, namely the Darmois–Skitovich theorem.

In the approach where separation principles are applied, B may not need to be explicitly estimated. However, it is important to note there are many algorithms where B is directly estimated from the observed signals $\boldsymbol{x}$ by using higher-order statistics, and the source signals $\boldsymbol{s}$ are then obtained symbolically as $A\boldsymbol{s} = B^{-1}B\boldsymbol{s}$ where B^{-1} denotes a general inverse of B(see [5], [6], and [15]–[18]).

For reference, we list in Table I the notations used in this paper.

III. Separability

Source separability is an intrinsic property of the linear combinations in the measured signals, i.e., it depends on the structure of B.

We first state some terminology. The set of n integers $W = \{1, 2, \cdots, n\}$ can be partitioned into $m, m \leq n$ disjoint subsets $W_i, i = 1, 2, \cdots, m$ (i.e., $W_i \cap W_j = \emptyset, i \neq j$, and $\cup_{i=1}^{m} W_i = W$). An $m \times n$ matrix A is called a *partition*

TABLE I
NOTATIONS

Notations	Explanations
n	Number of source signals
l	Number of sensors (or measured signals)
m	Number of outputs of a filter
B	$l \times n$ matrix
C	$m \times l$ matrix
A	$m \times m$ matrix, $A = C \times B$
$\boldsymbol{s}$	m-dimensional vector of source signals
$\boldsymbol{x}$	l-dimensional vector of measured signals, $\boldsymbol{x} = B\boldsymbol{s}$
$\boldsymbol{y}$	m-dimensional vector of output signals,$\boldsymbol{y} = C\boldsymbol{x} = A\boldsymbol{s}$
$\boldsymbol{b}_i$	ith column of B
W	Set of integers $\{1, 2, \cdots, n\}$
W_i	ith subset of the partition of W

matrix if its (i,j)th entry $a_{i,j} = 1$ if $j \in W_i$, and $a_{i,j} = 0$ if $j \notin W_i$. For a given m and n, the number of different partition matrices is

$$\sum_{n_1+\cdots+n_m=n} \frac{n!}{n_1!n_2!\cdots n_m!}$$

which equals the number of choices that n_i integers among $1, 2, \cdots, n$ are picked up for set $W_i, i = 1, 2, \cdots, m$. A is of full row rank, if and only if no subset $W_i, i = 1, 2, \cdots, m$, is an empty set. A full rank partition matrix is simply a matrix of zero's and one's in which each column has only one nonzero entry, and each row has at least one nonzero entry. To make the discussion meaningful, we assume that A is of full row rank.

A set of random variables $s_1, s_2, \cdots, s_n$ is said to be mutually independent if the probability distribution functions satisfy $P(s_1, s_2, \cdots, s_n) = \Pi_{i=1}^n P(s_i)$; A set of random variables $y_1, y_2, \cdots, y_m$ is said to be pairwise independent if $P(y_i, y_j) = P(y_i)P(y_j)$ for any pair of $i, j = 1, 2, \cdots, m, i \neq j$. Mutual independence implies pairwise independence; but the converse is not true.

A *diagonal matrix* is a matrix whose nondiagonal entries are all zeros. An $m \times n$ *generalized partition matrix* is a product of an $m \times n$ partition matrix and an $n \times n$ nonsingular diagonal matrix. In a generalized partition matrix the zero entries are the same as those in a partition matrix, but the nonzero entries may have any nonzero values.

If A is a partition matrix, then (3) becomes

$$y_i = \sum_{j \in W_i} s_j, \qquad i = 1, 2, \cdots, m$$
$$W_i \cap W_j = \emptyset, \qquad i \neq j.$$

That is, using A we can separate (decouple) the source signals into m groups; each output of the filter is the sum of the signals in one group. If for some $i, W_i = \{k\}$, then $y_i = s_k$; i.e., by using A, we can separate out one signal s_k. If A is a generalized partition matrix, then y_i is a linear combination of the signals in subset $W_i, i = 1, 2, \cdots, m$. Since we do not care about the amplitude of a signal, we shall discuss the general case with generalized partition matrices.

If $m = n$, then every subset $W_i, i = 1, 2, \cdots, n$, contains only one element, the partition matrix becomes a *permutation matrix*, and the generalized partition matrix becomes a *generalized permutation matrix*. In a generalized permutation matrix, each column and each row has only one nonzero entry; in a permutation matrix, this nonzero entry equals one. A generalized permutation matrix is a product of permutation matrix and a nonsingular diagonal matrix. If A is a generalized permutation matrix, then we can separate out all the source signals. The problem for $m = n = l$ has been studied, for example, in [3], [5], and [15].

Definition 1: An $l \times n$ matrix B is called *m-row decomposable*, or simply *decomposable*, if there exists an $m \times l$ matrix C such that $A = C \times B$ is an $m \times n$ generalized partition matrix.

For maximum decomposability, we assume that the $m \times n$ matrix A is of full row rank. Obviously, if $n \geq m$, we need $\rho(B) \geq m$ and $\rho(C) \geq m$, where $\rho(X)$ denotes the rank of a matrix X. In addition, if such a C exists, then the rows of the generalized partition matrix A are in the space spanned by the row vectors of B.

If B is m-row decomposable, then in principle it is possible to separate the n source signals into m groups. If $m = n$, we have a complete separation. Obviously, if a matrix is m-row decomposable, it must be $(m-1)$-row decomposable. (For example, we can make an $(m-1) \times n$ generalized partition matrix from an $m \times n$ matrix by adding its two rows together.) $m > n$ is not a realistic case, since one cannot get more than n source signals from the measured signals; the corresponding partition matrix contains $m - n$ zero rows.

Let $B = [[b_{j,i}]_{j=1}^l]_{i=1}^n$ and $\boldsymbol{b}_i = (b_{1,i}, b_{2,i}, \cdots, b_{l,i})^T, i = 1, 2, \cdots, n$, be the ith column vector of B. Suppose that $W_i, i = 1, 2, \cdots, m$ is a partition of $\{1, 2, \cdots, n\}$; W_i contains n_i elements, $\Sigma_{i=1}^m n_i = n$. This partition corresponds to m disjoint groups of the vectors $\boldsymbol{b}_k, k = 1, 2, \cdots, n$, denoted as $G_i = \{\boldsymbol{b}_j \colon j \in W_i\}, i = 1, 2, \cdots, m$. Let the spaces spanned by G_i be $S_i \equiv S_{W_i}, i = 1, 2, \cdots, m$. Let the direct sum of all S_j except S_k (i.e., $j = 1, \cdots, k-1, k+1, \cdots, m$) be S_{-k}, and the space spanned by all $\boldsymbol{b}_i, i = 1, 2, \cdots, n$ be $S \equiv S_W$.

Let d_i be the dimension of S_i, d_{-k} be the dimension of S_{-k}, and d the dimension of S. Then $d \leq l, d_i \leq n_i, n \geq d_k + d_{-k} \geq d$ and $n \geq \Sigma_{i=1}^m d_i \geq d$.

Theorem 1: An $l \times n$ matrix B is m-row decomposable if and only if its n column vectors $\boldsymbol{b}_i, i = 1, 2, \cdots, n$, can be partitioned into m disjoint groups $G_i, i = 1, 2, \cdots, m$, such that for all $k, k = 1, 2, \cdots, m$, if $j \in W_k$ then $\boldsymbol{b}_j \notin S_{-k}$.

Proof: We first prove that the condition is sufficient. For any k, there must exist d'_k vectors, $n_k \geq d_k \geq d'_k = d - d_{-k}$, denoted as $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{d'_k}$, that are orthogonal to each other and are also orthogonal to S_{-k}. If $\boldsymbol{b}_j \notin S_{-k}$ for all $j \in W_k$, there must exist a vector $\boldsymbol{c}_k$ that is orthogonal to S_{-k} and is nonorthogonal to all $\boldsymbol{b}_j, j \in W_k$. The existence of $\boldsymbol{c}_k$ can be proved as follows. Let

$$\boldsymbol{c}_k = \sum_{i=1}^{d'_k} \alpha_i \boldsymbol{e}_i$$

which is orthogonal to S_{-k}. Any set of $\alpha_i, i = 1, 2, \cdots, d'_k$ such that

$$\boldsymbol{c}_k^T \boldsymbol{b}_j = \sum_{i=1}^{d'_k} (\boldsymbol{e}_i^T \boldsymbol{b}_j)\alpha_i \neq 0, \qquad j \in W_k \tag{4}$$

can be chosen as the desired $\boldsymbol{c}_k$. The coefficient matrix of the above equation is $D = [[\boldsymbol{e}_i^T \boldsymbol{b}_j]_{j\in W_k}]_{i=1}^{d'_k}$. Given a j, because $\boldsymbol{b}_j \notin S_{-k}$, the entries $\boldsymbol{e}_i^T \boldsymbol{b}_j, i = 1, 2, \cdots, d'_k$, cannot be all zeros; i.e., every row vector of D is nonzero. Since the right hand side of (4) can be any nonzero numbers, the existence of $\alpha_i, i = 1, 2, \cdots, d'_k$, is obvious.

Let C be an $m \times l$ matrix whose kth row vector is $\boldsymbol{c}_k^T$. It is then easy to verify that $C \times B$ is a generalized partition matrix; the jth entry in its kth row is nonzero if and only if $j \in W_k$.

Next, we prove the necessity of the condition. Suppose that there is a matrix C such that $C \times B$ is a generalized partition matrix corresponding to partition $W_i, i = 1, 2, \cdots, m$. Each W_i corresponds to a subspace S_i in the l-dimensional space; S_i is composed of $\boldsymbol{b}_j, j \in W_i$. Denote the kth row vector of C as $\boldsymbol{c}_k$. Then $\boldsymbol{c}_k$ is orthogonal to S_{-k} and for any $j \in W_k, \boldsymbol{c}_k^T \boldsymbol{b}_j \neq 0$. Therefore, we must have $\boldsymbol{b}_j \notin S_{-k}$ for all $j \in W_k$. □

In words, matrix B is m-row decomposable if and only if its columns can be grouped into m disjoint groups such that the column vectors in each group are independent of the vectors in all the other groups. Note that the spaces spanned by different groups may overlap. The theorem requires only that the column vectors do not lie in the joint part.

Example 1: This example illustrates the theorem. Consider

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

where $\boldsymbol{b}_1 = (1, 0, 0)^T, \boldsymbol{b}_2 = (0, 0, 1)^T, \boldsymbol{b}_3 = (1, 1, 0)^T$ and $\boldsymbol{b}_4 = (0, 1, 1)^T$. Let $G_1 = (b_1, b_2)$ and $G_2 = (b_3, b_4)$. Then G_1 and G_2 are the partition satisfying the condition of Theorem 1. We can choose, for example, $\boldsymbol{c}_1^T = (0, 1, 0)^T$ and $\boldsymbol{c}_2^T = (1, -1, 1)^T$ and

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

It is easy to verify that

$$A = C \times B = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

is a partition matrix.

The partition is not unique. For example, if we choose $G_1 = (b_1, b_3), G_2 = (b_2, b_4), \boldsymbol{c}_1 = (0, 0, 1)^T$ and $\boldsymbol{c}_2 = (1, 0, 0)^T$, then we have

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$A = C \times B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Next, consider

$$B = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

where $\boldsymbol{b}_1 = (0, 0, 1),^T \boldsymbol{b}_2 = (1, 1, 0),^T \boldsymbol{b}_3 = (1, 1, 1),^T$, and $\boldsymbol{b}_4 = (-1, -1, 1)^T$. No matter how we partition the four column vectors into two groups, there is always one vector in one group that belongs to the space spanned by the other group. Thus, B is not two-row decomposable. □

Corollary 1: If B is of full column rank, then it is m-row decomposable for any $m \leq n$.

Proof: This is a direct consequence of Theorem 1. □

The condition in Corollary 1 implies $l \geq n$. The matrix C can be determined as follows. Let C' be an $n \times l$ matrix such that $C' \times B = I$. The ith row vector of C' is orthogonal to all the jth, $j \neq i$, column vectors of B. Let $W_k, k = 1, 2, \cdots, m$ be a partition of $\{1, 2, \cdots, n\}$. Define $c_{k,i} = \Sigma_{j\in W_k} c'_{j,i}, i = 1, 2, \cdots, l$. Let $C = [[c_{k,j}]_{k=1}^m]_{j=1}^l$ be an $m \times l$ matrix. Then it is easy to verify that $C \times B$ is a partition matrix corresponding to the partition $W_i, i = 1, 2, \cdots, n$.

Therefore, if $l \geq n$ (i.e., the number of sensors is greater or equal to the number of source signals), it is possible to design a filter such that its m outputs $y_k, k = 1, 2, \cdots, m, m \leq n$ separate the source signals into m groups. When $m = n$, the source signals are completely separated out. We may also choose $m = 2$. That is, we can design a filter that has only two outputs; one contains $n' < n$ source signals and the other, $n - n'$ source signals. If $n' = 1$, the filter can separate one signal out from the remaining $n - 1$ signals.

When $l < n$, the n column vectors of B are not independent, and B cannot be n-row decomposable. Thus, a complete separation is not possible, i.e., there exists no linear filter that can provide simultaneously the n source signals separately. However, it is entirely possible to have filters whose $m(<n)$ outputs are a partition of m complementary subsets of the set of the source signals. Some special cases are studied in the following corollaries.

Corollary 2: If $l < n$, then B is l-row decomposable if and only if it has l independent column vectors and each of the other column vectors is proportional to one of these l vectors.

Proof: The conditions in this corollary imply that $\boldsymbol{b}_i, i = 1, 2, \cdots, n$ can be partitioned into l disjoint groups, each spanning a 1-D subspace, and these subspaces are distinct. From Theorem 1, the conditions are sufficient. Next, we prove necessity. Suppose B is l-row decomposable. Then, according to Theorem 1, the column vectors of B can be partitioned into l disjoint groups $G_i, i = 1, 2, \cdots, l$. We claim that each subspace spanned by $G_i, S_i, i = 1, 2, \cdots, l$ must be 1-D. Suppose the opposite is true. Without loss of generality, we assume that S_1 is 2-D. Then, according to Theorem 1, the vectors in G_2 are not in S_1; thus, the space spanned by S_1 and S_2 must be at least 3-D. Continuing this process, we know that the space spanned by $S_1, S_2, \cdots, S_{l-1}$, which is S_{-l}, must be l-D, i.e., $S_{-l} = S$. Therefore, we must have $\boldsymbol{b}_j \in S_{-l}$ for all $j \in S_l$. This contradicts Theorem 1. □

Take a 2×3 matrix B as an example. If we choose C to be the inverse of $[b_{i,j}]_{i,j=1}^2$, then

$$A = C \times B = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \end{pmatrix}.$$

If $(b_{1,3}, b_{2,3})$ is not proportional to $(b_{1,1}, b_{2,1})$ or $(b_{1,2}, b_{2,2})$, then $\alpha \neq 0$ and $\beta \neq 0$. A is not a generalized partition matrix. In this case, it is not possible to design a filter to separate out the three source signals into two complementary groups from the two measured signals.

Theorem 2: B is two-row decomposable with the first row being a unit vector whose kth entry is unity and other entries are zeros if and only if the kth column vector does not belong to the space spanned by all the other column vectors. □

This theorem is straightforward. Take $k = 1$ as an example. Let $W_1 = \{1\}, W_2 = \{2, \cdots, n\}$. Choosing a vector $\boldsymbol{c}_1$ that is orthogonal to S_2 and a vector $\boldsymbol{c}_2$ such that $\boldsymbol{c}_2^T \boldsymbol{b}_1 = 0$, and letting

$$C = \begin{pmatrix} \boldsymbol{c}_1^T \\ \boldsymbol{c}_2^T \end{pmatrix}$$

we have

$$A = C \times B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \beta_2 & \beta_3 & \cdots & \beta_n \end{pmatrix}.$$

The practical application of this theorem is significant: The filter successfully separates out one source signal s_k from the remaining signals.

A direct consequence of Theorem 2 is that the number of sources that can be separated equals the number of column vectors of B, each of which is independent of the remaining $n - 1$ vectors. It is obviously important to estimate this number and to determine which separated component is a source signal and which one is still a mixture. This general problem remains open. Theorem 4 and Corollary 5 in Section IV provide principles for determining which column vector has been separated out for some special cases when $m = n$.

In summary, separability depends on the structure of the unknown combination of $\boldsymbol{x} = B\boldsymbol{s}$. When $l < n$, it is generally not possible to design filters to simultaneously separate out all the n signals from l measured signals. However, if the structure of matrix B satisfies some properties described in Theorems 1 and 2 and Corollary 2, the source signals may be separated into several complementary groups. Thus, in some special cases it is possible to separate out some of the source signals by using fewer sensors. For example, it may be possible to use two sensors to separate one signal out of a set of n source signals. The following example illustrates this idea.

Example 2: Suppose that we have three sources geographically located as shown by the black dots in Fig. 1, and we only have two sensors. For illustrative purposes, we assume that the problem is a narrow-band problem such that the differences in signal propagation delays can be ignored. We first place these two sensors, shown as the circles in the figure, on the line that has the equal distance to two of the sources. Let $r_{i,j}$ be the distance of sensor x_i to source $s_j, i = 1, 2$ and $j = 1, 2, 3$. We have $r_{1,2} = r_{1,3}$ and $r_{2,2} = r_{2,3}$. Suppose that the amplitude of the signals received by a sensor is a function of the distance between the sensor and the source. Then the measured signals are

$$\begin{aligned} x_1 &= f(r_{1,1})s_1 + f(r_{1,2})s_2 + f(r_{1,3})s_3 \\ x_2 &= f(r_{2,1})s_1 + f(r_{2,2})s_2 + f(r_{2,3})s_3. \end{aligned}$$

That is,

$$B = \begin{pmatrix} f(r_{1,1}) & f(r_{1,2}) & f(r_{1,3}) \\ f(r_{2,1}) & f(r_{2,2}) & f(r_{2,3}) \end{pmatrix}.$$

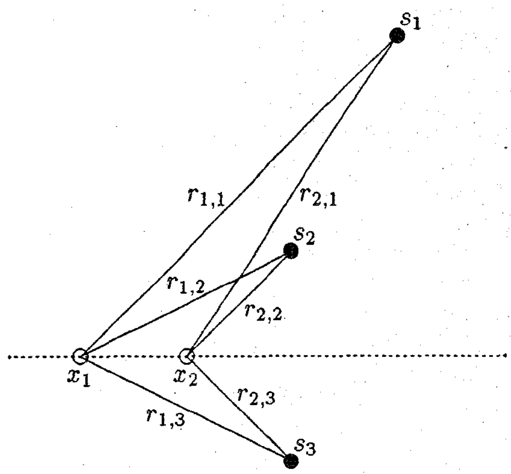


Fig. 1. Example for two sensors and three sources.

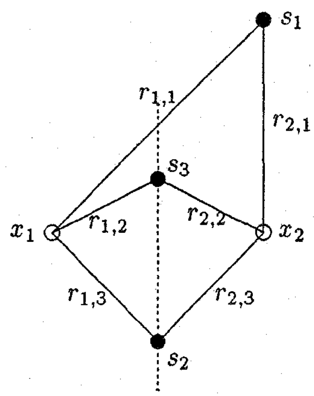


Fig. 2. Another example for two sensors and three sources.

The second and third columns are the same. Both Corollary 2 and Theorem 2 can be applied to this case. In fact, if C is the inverse of the square matrix composed of the first two columns of B, then A is a partition matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \alpha \end{pmatrix}.$$

To actually achieve this structure for A, we can use a filter with two inputs x_1 and x_2 such that its two outputs y_1 and y_2 are independent (see the next section). Thus, $y_1 = s_1$. That is, we have separated out the first signal. By placing the two sensors on the line that has an equal distance to s_1 and s_3, we may use these two sensors to produce signal s_2. s_3 can be obtained in a similar way. Note, however, the approach is not as efficient as that which uses three sensors. In fact, in this approach to separate out two signals we have to place two sensors at four different sensor locations. In general, with $l < n$ sensors, at most $l - 1$ sources can be separated out.

Fig. 2 shows another configuration, where source 1 (or 2) has an equal distance to the two sensors, i.e., $r_{1,2} = r_{2,2}$ and $r_{1,3} = r_{2,3}$. In this case, the second and third columns are proportional to each other. We can also use this configuration to retrieve any source signal, one at a time.

In practice, the problem may be much more complicated than the case discussed above. For example, the time delay may not be negligible; in this case, wideband techniques must be used. In addition, the exact locations of the sources may not be known and would have to be estimated. Therefore, the application of Theorem 2 depends on real problems and requires further research. □

Further research is needed for determining the value of l and the configurations when $n > 3$.

IV. Separation Principles

In the last section, we found the conditions for the structure of B under which the source signals are separable by linear filters. As was mentioned in the Introduction, because both the matrix B and the signal $\boldsymbol{s}$ are unknown, there is no way to measure the matrix $A = C \times B$; i.e., one does not know A explicitly. Therefore, some principles have to be developed to determine the structure of A. This is the topic of this section.

We show that the basic principles for all the cases can be derived from a simple theorem, which is stated at the beginning of the next subsection.

A. Preliminary Results

Darmois–Skitovich Theorem [11]: Let $\boldsymbol{s} = (s_1, \cdots, s_n)^T$ be an n-dimensional ($n \geq 2$) random vector whose components are mutually independent, and

$$\begin{aligned} y_1 &= a_{1,1}s_1 + a_{1,2}s_2 + \cdots + a_{1,n}s_n \\ y_2 &= a_{2,1}s_1 + a_{2,2}s_2 + \cdots + a_{2,n}s_n. \end{aligned}$$

Suppose y_1 and y_2 are independent. If $a_{1,i} \neq 0$ and $a_{2,i} \neq 0$, then s_i has a Gaussian distribution. □

To prove the main results, we shall also need a simple lemma derived from orthogonality.

Let $\sigma_i^2, i = 1, 2, \cdots, n$, be the variances of $s_i, i = 1, 2, \cdots, n$, respectively. The covariances of y_i and $y_j, i, j = 1, 2, \cdots, m$, are

$$\mathrm{Cov}(y_i, y_j) = \sum_{k=1}^{n} a_{i,k}a_{j,k}\sigma_k^2.$$

Without loss of generality, we assume that all $\sigma_k^2, k = 1, 2, \cdots, n$, and the variances of $y_i, i = 1, 2, \cdots, m$, equal one. Suppose that y_i and y_j are independent; then they are uncorrelated. Thus

$$\sum_{k=1}^{n} a_{i,k}^2 = 1, \qquad i = 1, 2, \cdots, m$$

and

$$\sum_{k=1}^{n} a_{i,k}a_{j,k} = 0, \qquad i \neq j. \tag{5}$$

In this case, the ith and jth rows of A are orthogonal to each other.

If $m = n$, then A is an orthogonal matrix. That is, $A^{-1} = A^T$.

Lemma 1: If, in a nonsingular $n \times n$ matrix A, the first row is orthogonal to the remaining $n - 1$ rows, then $a_{1,k} = 0$ for all $k = 2, \cdots, n$ if and only if $a_{k,1} = 0$ for all $k = 2, \cdots, n$.□

The proof is straightforward and will be omitted. One direct consequence of the lemma is the following corollary.

Corollary 3: If A is an orthogonal matrix, then an entry is the only nonzero entry in its row (or column), if and only if it is the only nonzero entry in its column (or row). □

We shall first discuss the basic case, i.e., $m = n$, in the next subsection.

B. The Case $m = n$

When $m = n$, if $y_i, i = 1, 2, \cdots, m$ are pairwise independent, then matrix A is an orthogonal matrix. Let $A_i, i = 1, \cdots, k$ be square matrices and define $\mathrm{diag}\{A_1, A_2, \cdots, A_k\}$ be the square matrix with matrices $A_i, i = 1, 2, \cdots, k$, being at the diagonal position and all other entries being zero.

From Corollary 3, if an entry is the only nonzero entry in its column in A, then it is the only nonzero entry in its row and vice versa. Thus, by renumbering the indices of s_i and y_j, we can decompose A into the form

$$A = \begin{pmatrix} A' & O \\ O & I \end{pmatrix}$$

where O represents a zero matrices whose all entries are zeros and the identity matrix I may be of null dimensions. A' may be further decomposed into

$$A' = \begin{pmatrix} A_1 & O \\ D & A_2 \end{pmatrix}.$$

A' is orthogonal, since A is. In addition, A_1 is nonsingular; thus, D is a zero matrix (it also may be of null dimensions). Continuing the decomposition, we finally get $A = \mathrm{diag}\{A_1, A_2, \cdots, A_k, I\}$ where A_i's are orthogonal matrices. We can assume that the dimension of each A_k is greater than one (if A_i has only one entry, it can be included in I). Let n_i be the dimension of $A_i, i = 1, 2, \cdots, k + 1$, with $A_{k+1} = I$. Let $\boldsymbol{y} = (\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{k+1}), \boldsymbol{s} = (\boldsymbol{s}_1, \cdots, \boldsymbol{s}_{k+1})$ where $\boldsymbol{y}_j$ and $\boldsymbol{s}_j$ consist of components that correspond to $A_i, i = 1, 2, \cdots, k + 1$, i.e.,

$$\boldsymbol{y}_i = A_i\boldsymbol{s}_i, \qquad i = 1, 2, \cdots, k + 1.$$

Every column in $A_i, i < k + 1$ has more than one nonzero entry. Let $y_{i,j}$ and $s_{i,j}$ be the jth components of $\boldsymbol{y}_i$ and $\boldsymbol{s}_i$, respectively, $j = 1, 2, \cdots, n_i, i = 1, 2, \cdots, k + 1$. Then $\boldsymbol{y}_i = (y_{i,1}, \cdots, y_{i,n_i})^T$ and $\boldsymbol{s}_i = (s_{i,1}, \cdots, s_{i,n_i})^T$.

The above discussion leads to the following theorem, which characterizes the linear structure represented in A.

Theorem 3: Let $\boldsymbol{y} = A\boldsymbol{s}$, where A is a nonsingular $n \times n$ matrix, $\boldsymbol{s} = (s_1, \cdots, s_n)^T$ is an n-dimensional vector whose components are mutually independent, and $\boldsymbol{y} = (y_1, \cdots, y_n)^T$ is an n-dimensional vector whose components are pairwise independent. Suppose that every component of $\boldsymbol{s}$ and $\boldsymbol{y}$ has unit variance. Then

1) By reordering the components of $\boldsymbol{s}$ and $\boldsymbol{y}, A$ may be reduced to a block diagonal form, i.e., $A = \mathrm{diag}\{A_1, \cdots, A_k, I\}$. The dimension of each A_i is greater than one, and $A_i, i = 1, 2, \cdots, k$ are orthogonal matrices.
2) The components of $\boldsymbol{s}$ corresponding to $I, s_{k+1,j}, j = 1, 2, \cdots, n_{k+1}$ may have any distribution (Gaussian or non-Gaussian).
3) Every component of $\boldsymbol{s}$ corresponding to $A_i, s_{i,j}, j = 1, 2, \cdots, n_i, i = 1, \cdots, k$ has a Gaussian distribution.

Proof: Part 1 follows directly from the above discussion. From 1, we can denote any component of $\boldsymbol{s}$ as $s_j = s_{i,l}, j = 1, \cdots, n, i = 1, \cdots, k+1$ and $l = 1, \cdots, n_i$. If $i = k+1$, we have $y_j = s_j$. Obviously, s_j may have any distribution. This proves 2. To prove 3, we assume $i \leq k$. Since A is nonsingular, there must be at least one $a_{u,j} \neq 0$. From the structure of A, this $a_{u,j}$ is an entry in A_i. Next, From Corollary 3, there must be at least one $a_{v,j}, v \neq u$ such that $a_{v,j} \neq 0$. Thus, we have

$$y_u = \cdots + a_{u,j}s_j + \cdots$$

and

$$y_v = \cdots + a_{v,j}s_j + \cdots$$

with $a_{u,j}a_{v,j} \neq 0$. Applying the Darmois–Skitovich theorem to the pair of y_u and y_v, we conclude that s_j must have a Gaussian distribution. □

In the theorem, $y_i, i = 1, 2, \cdots, n$ are required to be only pairwise independent, which is weaker than mutually independent. One direct consequence of the theorem is that the number of Gaussian signals in $y_i, i = 1, 2, \cdots, n$, is the same as that of the Gaussian signals in $s_i, i = 1, 2, \cdots, n$.

Corollary 4: Under the conditions of Theorem 3, if at most one component of $\boldsymbol{s}$ is Gaussian distributed, then A is a permutation matrix.

Proof: If k, the number of the submatrices A_i (whose rank is more than one) in the decomposition of Theorem 3, is greater than zero, then there must be at least two components whose distributions are Gaussian. Thus, $k = 0$. Therefore, by reordering the components, A becomes I. That is, A is a permutation matrix. □

The result in Corollary 4 has been derived before (see, e.g., [3]); similar results are obtained in [5], and [17]. The implication of this corollary in blind source separation is straightforward: given l measured signals, $x_i, i = 1, 2, \cdots, l$ that are linear functions of n source signals s_i, which are non-Gaussian except possibly one, if we can design a filter such that the n output signals are pairwise independent, then the set of output signals must be the same as that of the source signals. This can be viewed as a separation principle: Although we do not know the exact form of B, C, and A, but if the output signals are independent, then we have separated out the source signals. Note that in Corollary 4, we assume that there is only one Gaussian component and the signals do not contain noises. If source signals contain Gaussian noises, then the noises can be viewed as special individual signals; the situation becomes the case $l < n$ and the noises cannot be separated out (see the discussion in Section IV-C). Finally, the decomposition in Theorem 3 is related to the equivalence class of identifiable MA(q) channels (see, e.g., [5] and [15] for details).

Theorem 3 can be viewed as an extension of the existing results to more general cases. The assumption of a single Gaussian source is not required. The theorem says that if the outputs of a filter are pairwise independent, then the outputs are either individual non-Gaussian sources, individual Gaussian noises, or mixtures of Gaussian noises.

To verify that the outputs of a filter are pairwise independent is not trivial. In [10], an adaptive filter based on neural network was proposed, and it was proved in [13] that under certain conditions on this filter, matrix A, which is time varying, converges to a generalized permutation matrix. Other references include [1] and [3].

Note that Theorem 3 has nothing to do with matrix B; therefore, separation principles cannot replace separability issues. That is, only if B is row decomposable, then it is possible to design a filter such that the outputs are pairwise independent. Otherwise, the filter will never converge to a permutation matrix A. In particular, if $l < n$, no filter can provide n independent outputs.

In the next theorem, we study the case where only one output signal is independent of the remaining output signals.

Theorem 4: Let $\boldsymbol{y} = A\boldsymbol{s}$ where $\boldsymbol{y} = (y_1, \cdots, y_n)^T$ and $\boldsymbol{s} = (s_1, \cdots, s_n)^T$ are two n-dimensional random vectors. The components of $\boldsymbol{s}$ are mutually independent and y_1 is independent of all $y_j, j = 2, \cdots, n$ pairwise. Suppose that A is nonsingular, s_1 is non-Gaussian, and every component of $\boldsymbol{s}$ and $\boldsymbol{y}$ has a unit variance. Let A_{-1} be the $(n-1) \times (n-1)$ matrix obtained by crossing out the first column and the first row in A, i.e., $A_{-1} = [a_{i,j}]_{i,j=2}^n$. Then, either

1) $y_1 = s_1$ and A_{-1} is nonsingular; or
2) s_1 is a linear function of $y_j, j = 2, \cdots, n, y_1$ is a linear function of $s_j, j = 2, \cdots, n$, (i.e., $a_{1,1} = 0,$) and A_{-1} is singular.

Proof: 1. Suppose $a_{1,1} \neq 0$. Because s_1 is non-Gaussian, applying the Darmois–Skitovich theorem to the two equations corresponding to y_1 and $y_j, j = 2, \cdots, n$ leads to the conclusion that $a_{j,1} = 0, j = 2, \cdots, n$. By Lemma 1, we have $a_{1,j} = 0$ for $j = 2, \cdots, n$. Thus, $y_1 = a_{1,1}s_1$. Since y_1 and s_1 have the same variance, $y_1 = s_1$. From this, $A = \mathrm{diag}\{1, A_{-1}\}$. A_{-1} is nonsingular since A is.

2. Suppose $a_{1,1} = 0$. Thus, $y_1 = a_{1,2}s_2 + \cdots + a_{1,n}s_n$ and y_1 is independent of s_1. By inverting the matrix A, we get $s_1 = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n$. Since y_1 is independent of s_1 and $y_j, j = 2, \cdots, n$, we have $\alpha_1 = 0$ and $s_1 = \alpha_2 y_2 + \cdots + \alpha_n y_n$.

Next, we prove that A_{-1} is singular. Suppose the opposite is true, i.e., A_{-1} is nonsingular. Let $\boldsymbol{a} = (a_{1,2}, \cdots, a_{1,n}),^T$ $\boldsymbol{b} = (a_{2,1}, \cdots, a_{n,1}),^T$ $\boldsymbol{y}_{-1} = (y_2, \cdots, y_n)^T$, and $\boldsymbol{s}_{-1} = (s_2, \cdots, s_n)^T$. With $a_{1,1} = 0, \boldsymbol{y} = A\boldsymbol{s}$ takes the form

$$\begin{pmatrix} y_1 \\ \boldsymbol{y}_{-1} \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{a}^T \\ \boldsymbol{b} & A_{-1} \end{pmatrix} \begin{pmatrix} s_1 \\ \boldsymbol{s}_{-1} \end{pmatrix}. \tag{6}$$

Since we assume $|A_{-1}| \neq 0$, there exists a nonzero vector $\boldsymbol{c} = (c_2, \cdots, c_n)^T$ such that $-\boldsymbol{a}^T = \boldsymbol{c}^T A_{-1}$. Let $w = \boldsymbol{c}^T\boldsymbol{b}$.

Then we have

$$(1, \boldsymbol{c}^T)A = (w, 0, \cdots, 0).$$

Because A is nonsingular, $w \neq 0$. Finally, multiplying both sides of $\boldsymbol{y} = A\boldsymbol{s}$ on the right by $(1, \boldsymbol{c}^T)$, we get

$$s_1 = \frac{1}{w}(y_1 + \boldsymbol{c}^T \boldsymbol{y}_{-1}).$$

This contradicts the fact that s_1 does not depend on y_1. □

As an example, we consider the case $m = n = 3$. We assume that s_1 is non-Gaussian and y_1 is independent of y_2 and y_3 pairwise. Without considering possible permutations, there are only two cases, as follows:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \tag{7}$$

or

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & \phi & \psi \\ \theta & \alpha & \beta \\ \vartheta & \gamma & \delta \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}. \tag{8}$$

If, furthermore, s_2 is non-Gaussian, then in case (7) we cannot further improve our knowledge of matrix A because we do not know if y_2 and y_3 are independent; but in case (8), we must have either $\phi = \beta = \delta = 0$ and $\psi = 1$ or $\psi = \alpha = \gamma = 0$ and $\phi = 1$. Thus, A has essentially the same form as in (7).

Corollary 4 is a special case of Theorem 4. For example, we consider

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}.$$

In this case, $s_1 = y_2' = y_2$, a special case of a combination of y_2 and y_3, and A_{-1} is singular. If we reorder the components by setting, say, $s_1^* = s_2, s_2^* = s_3$ and $s_3^* = s_1$, then

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1^* \\ s_2^* \\ s_3^* \end{pmatrix}.$$

We have $y_1 = s_1^*$ and A_{-1} is nonsingular.

Corollary 5: In Theorem 4, if all $s_i, i = 1, 2, \cdots, n$, except possibly one, are non-Gaussian, then $A = \mathrm{diag}\{1, A_{-1}\}$ or a matrix obtained by permuting its rows or columns.

Proof: If s_i is non-Gaussian, by applying the Darmois–Skitovich theorem to y_1 and any $y_j, j = 2, \cdots, n$, we conclude that either $a_{1,i} = 0$ or the ith column vector is $(1, 0, \cdots, 0)^T$. Because y_1 is independent of all $y_j, j = 2, \cdots, n$, pairwise, the column vector corresponding to the only possible Gaussian component must have the same property. Since A is nonsingular, there is one and only one column that has the form $(1, 0, \cdots, 0)^T$. This completes the proof. □

The engineering explanation of Theorem 4 is that if y_1 is independent of $y_j, j = 2, \cdots, n$, pairwise, then s_1 either equals y_1 or can be retrieved by using $y_2, \cdots, y_n$. If, in addition, all the source signals are non-Gaussian except one, then by Corollary 5 there must be one y_j that equals one of the signals. In blind source separation, this means that if we can design a filter such that one output is independent of the other $n - 1$ outputs pairwise, then the output must contain one of the source signals.

This has two possible applications. First, if we want to obtain only one source signal, then we only need to design a filter such that y_1 is independent of others (the problem of which particular signal is separated out is to be solved). Second, in case we want to obtain all the source signals, we may implement blind source separation step by step, in each step, one source signal is separated. More specifically, at the first step we apply the filter to the n sources and separate out one signal, say $y_1 = s_i$. According to Corollary 5, the rest outputs $y_2, \cdots, y_n$ are linear combinations of $s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n$. In the next step, we apply the filter to $y_2, \cdots, y_n$ and another source signal can be separated out. The procedure continues until all the signals are separated out. The filter used in each step is simpler than the one that separates all the signals at once.

Theorem 1 and Corollary 1 indicate that when $m = n = l$ the source signals are separable if and only if B is of full rank; Theorems 2, and 3 and Corollary 4 describe the principles for designing filters.

C. The Case $m < n$

We assume that A is of full row rank. As discussed in Section III, when $m < n, \boldsymbol{y}$ does not contain enough information to separate all source signals. But we still can determine the structure of the matrix A if the components of $\boldsymbol{y}$ are pairwise independent.

Theorem 5: Let $\boldsymbol{y} = A\boldsymbol{s}$ where $\boldsymbol{y} = (y_1, \cdots, y_m)^T$ is an m-dimensional random vector and $\boldsymbol{s} = (s_1, \cdots, s_n)^T$ is an n-dimensional random vector $m \leq n$. The components of $\boldsymbol{s}$ are mutually independent and those of $\boldsymbol{y}$ are pairwise independent. If at most one component of $\boldsymbol{s}$ is Gaussian distributed, then A is a generalized partition matrix in which each row contains at least one nonzero entry.

Proof: By the Darmois-Skitovich theorem, each column in A corresponding to a non-Gaussian source signal contains only one nonzero entry. Because of the orthogonal property (5), the column corresponding to the unique Gaussian source signal also contains only one nonzero entry. Since A is of rank m, all the row vectors have to be nonzero. This concludes the proof. □

Theorem 5 claims that given l linear functions of the source signals, if we can design a filter such that its outputs are pairwise independent, then the filter separates the n source signals into m nonempty disjoint groups. Again, this is possible only if the combination matrix B is m-row decomposable. Otherwise, there exists no linear filter whose m outputs are pairwise independent. In other words, the filter designed in [13] will not converge.

In general, if in Theorem 5 the number of Gaussian-distributed components (noises) in $\boldsymbol{s}$ is $d > 1$, the A must take

TABLE II
MAIN RESULTS

	Separability	Separation Principles	
$m = n = l$ (no. of sensors equals no. of sources)	B is of full rank	More than one Gaussian noise	(A) Theorems 3 (B) Theorem 4
		A single Gaussian noise	(A) Corollary 4 (B) Corollary 5
$m \leq l < n$ (no. of sensors is less than no. of sources)	m-row decomposability Def. 1, Thms 1, 2, Cor. 2	More than one Gaussian noise	The example at the end of Section 4.3
		A single Gaussian noise	Theorem 5

Case A: All outputs are pairwisely independent.
Case B: One output is pairwisely independent to all the other outputs.

the following form:

$$A = (A', D)$$

where A' is an $m \times (n-d)$ generalized partition matrix and D is an $m \times d$ orthogonal matrix. A' may contain zero rows; and D contains at most d nonzero rows. These results can be easily obtained from the Darmois-Skitovich theorem. A possible example is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & c_{1,1} & c_{1,2} & c_{1,3} \\ 0 & 0 & 0 & 0 & 1 & 1 & c_{2,1} & c_{2,2} & c_{2,3} \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$$

in which $m = 4, n = 9$, and $d = 3$; $c_{i,j}, i, j = 1, 2, 3$ form an orthogonal matrix.

V. CONCLUSION

We have investigated two major issues in blind source separation: separability and separation principles. Separability refers to the intrinsic property that a set of mixed-source signals can be decoupled by linear filters. Separation principles provide guidelines for designing filters. When B has a full rank, the separability condition has been given before (e.g., [1]–[3], [15], and [17]). This paper extends the results to the case where B is not of full rank.

Necessary and sufficient conditions have been developed for source signals to be separable. It was shown that with $l < n$ sensors, at most $l-1$ signals can be separated out. An example shows that if we carefully design the locations of the sensors (i.e., control the form of the combination matrix B), we can separate out a desired source signal with less than n sensors.

The basic separation principles were developed by following a uniform approach based on the structure characterization theory of random variables. The core theorem is the Darmois–Skitovich theorem, which is well known in statistical inference and psychology.

The development of the separation principles is concise and intuitive. Besides covering some existing results, the approach also leads to new insights that may allow us to design simpler filters with less independence requirements (Corollary 5) or with less outputs (the case $m < n$).

The results derived in this paper can be considered types of "existence theorems," and the practical implementations for blind source separation have to be developed individually. Higher order statistics can be used to determine pairwise independence of outputs. This has been discussed in many previous works, e.g., [1]–[3], [15], and [17]. In an ongoing work, an approach based on characteristic functions is being proposed. With this approach, the order of statistics that is needed to determine matrix C can be found under different conditions. The higher order statistics can be obtained by taking snapshots of the measured signals. In the approach, neither the source signals nor the noises are assumed to be temporally white.

Finally, Table II summarizes the main results of this paper.

REFERENCES

[1] J. Cardoso, "Source separation using high order moments," in *Proc. IEEE ICASSP*, vol. 4, 1989, pp. 2109–2112.

[2] P. Comon, "Separation of sources using higher-order cumulants," *SPIE Conf. Adv. Algorithms Architec. Signal Processing*, vol. IV, San Diego, CA, 1989, pp. 170–181.

[3] ——, "Independent component analysis, a new concept?" *Signal Processing*, vol. 36, pp. 287–314, 1994.

[4] G. Darmois, "Analyse generale des liaisons stochastiques," *Rev. Inst. Int. Stat.*, vol. 21, 1953, pp. 2–8.

[5] G. B. Giannakis, Y. Inouye, and J. M. Mendel, "Cumulant based identification of multichannel moving-average methods," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 783–787, 1989.

[6] G. B. Giannakis and M. K. Tsatsanis, "A unifying maximum likelihood view of cumulant and polyspectral measures for non-Gaussian signal classification and estimation," *IEEE Trans. Inform. Theory*, vol. 38, pp. 386–406, 1992.

[7] R. D. Gitlin and S. B. Weinstein, "Fractionally-spaced equalization: An improved digital transversal equalizer," *Bell Syst. Tech. J.*, vol. 60, pp. 275–296, 1981.

[8] J. Herault, C. Jutten, and B. Ans, "Detection de grandeurs primitives dans un message composite par une architecture de calcul neuromimetique en apprentissage non supervise," in *Proc. Xeme colloque GRETSI*, Nice, France, 1985, pp. 1017–1022.

[9] J. J. Hopfield, "On factory computation and object perception," in *Proc. Nat. Acad. Sci.*, vol. 88, 1991, pp. 6462.

[10] C. Jutten and J. Herault, "Blind separation of sources, part i: an adaptive algorithm based on neuromimetic architecture," *Signal Processing*, vol. 28, pp. 1–10, 1991.

[11] A. M. Kagan, J. V. Linnik, and C. R. Rao, *Characterization Problems in Mathematical Statistics*. New York: Wiley, 1973.

[12] B. Laheld and J. F. Cardoso, "Adaptive source separation without prewhitening," in *Proc. EUSIPCO*, Edinburgh, Scotland, 1994, pp. 183–186.

[13] X. T. Ling, Y. F. Huang, and R. Liu, "A neural network for blind signal separation," presented at the IEEE Int. Symp. Circ. Syst., 1994.

[14] V. C. Soon, L. Tong, Y. F. Huang, and R. Liu, "A robust method for wideband signal separation", in *Proc. 1993 IEEE Int. Symp. Circ. Syst.*, Chicago, IL, 1993, pp. 703–706.

[15] A. Swami, G. Giannakis, and S. Shamsunder, "Multichannel ARMA Processes," *IEEE Trans. Signal Processing*, vol. 42, pp. 898–913, 1994.

[16] A. Swami and J. M. Mendel, "ARMA parameter estimation using only output cumulants," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, pp. 1257–1265, 1992.

[17] S. Shamsunder and G. Giannakis, "Modeling of non-gaussian array data using cumulants: doa estimation of more sources with less sensors," *Signal Processing*, vol. 30, 279–297, 1993.

[18] L. Tong, Y. Inouye, and R. Liu, "Waveform-preserving blind estimation of multiple independent sources" *IEEE Trans. Signal Processing*, vol. 41, 2461–2470, 1993.

[19] L. Tong, R. Liu, V. C. Soon, and J. F. Huang, "Indeterminacy and identifiability of blind identification," *IEEE Trans. Circ. Syst.*, pp. 409–509, 1991.

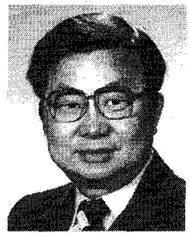
[20] L. Tong, G. Xu, and T. Kailath, "Fast blind equalization via antenna arrays," in *Proc. ICASSP*, 1993, pp. 272–275.

[21] G. Xu and T. Kailath, "Direction-of-arrival estimation via exploitation of cyclostationarity—a combination of temporal and spatial processing," *IEEE Trans. Signal Processing*, vol. 40, pp. 1775–1786, 1992.

Xi-Ren Cao (SM'89–F'96) received the M.S. and Ph.D. degrees from Harvard University, Cambridge, MA, in 1981 and 1984, respectively. He was a research fellow at Harvard from January 1984 to April 1986.

He worked as a principal/consultant engineer and supervisor/manager in the System Performance Group of Digital Equipment Corporation, Marlboro, MA, from May 1986 to October 1993. Since October 1993, he is with the Hong Kong University of Science and Technology, Kowloon, Hong Kong. He is the author of the book *Realization Probabilities—the Dynamics of Queuing Systems,*(Springer Verlag, 1994), and is the coauthor, with Y. C. Ho, of the book *Perturbation Analysis of Discrete-Event Dynamic Systems* (Kluwer, 1991).

Dr. Cao received the Outstanding Transactions Paper Award in 1987 from the IEEE Control System Society and is the 1990 corecipient of the Outstanding Publication Award of the College of Simulation, Institution of Management Science. He served as associate editors of IEEE TRANSACTIONS ON AUTOMATIC CONTROL and IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY, and is on the editorial board of the *Journal of Discrete-Event Dynamic Systems.* He serves as an area editor of the *International Journal of Operations and Quantitative Management.* He is the chairman of the technical committee of networks of the IEEE Control Systems Society.



Ruey-wen Liu (LF'95) received the B.S., M.S., and Ph.D. degrees from the University of Illinois, Urbana, all in electrical engineering.

Since 1960, he has been with the University of Notre Dame, and is currently the Frank M. Freimann Professor of Electrical and Computer Engineering. He was a Visiting Professor at National Taiwan University (Spring 1969), the University of California at Berkeley (1965–1966, 1977–1978, and 1984–1985), and Hong Kong University of Science and Technology (1991–92). His research interests have focused on nonlinear dynamics, fault diagnosis of analog circuits, and feedback control design; his most recent interests are in the areas of temporal neural networks and blind signal processing.

Dr. Liu was awarded the IEEE Fellow in 1981. From 1989 to 1991, he has served as Editor of IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS. He was President of the IEEE Circuits and Systems Society in 1995. He served as a member of the overseas advisory committee for *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences of Japan* (from 1991 to 1997). He received an honorary professorship from the Chinese Academy of Science, Beijing, China (1988), and the Fu-dan University in Shanghai, China.