Chapter 3

Row Echelon Form

3.1 Introduction to Row Echelon Form

The Row Echelon Form (REF) represents another fundamental canonical form in matrix theory. While the Normal Form employs both row and column operations to achieve its simplified structure, the Row Echelon Form uses exclusively row operations, making it particularly valuable in contexts where preserving column relationships is necessary.

Definition and Basic Properties

Definition 3.1. *A matrix is in* ***Row Echelon Form*** *if it satisfies the following conditions:*

1. *All rows consisting entirely of zeros are at the bottom of the matrix.*
2. *The leading entry (the first non-zero element) of each non-zero row appears to the right of the leading entry of the row above it.*
3. *The leading entry in each non-zero row is 1.*
4. *All entries in a column below a leading 1 are zeros.*

Formally, if we denote the position of the leading entry in row i as j_i, then $j_1 < j_2 < \cdots < j_r$ where r is the number of non-zero rows.

Important Properties of Row Echelon Form

- The number of non-zero rows in a Row Echelon Form equals the rank of the matrix.
- A matrix can be transformed into Row Echelon Form using only elementary row operations.
- The Row Echelon Form of a matrix is not unique (unlike Reduced Row Echelon Form).
- The positions of the leading 1s correspond to the pivots in the Gaussian elimination process.
- The Row Echelon Form reveals the dimensions of the row space and nullity of the matrix.

$$A_{REF} = \begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 3.1: A general matrix in Row Echelon Form. The positions marked with asterisks (*) can be any values. The red dots indicate the leading 1s, which form a staircase pattern.

Comparison with Normal Form

While both Row Echelon Form and Normal Form serve to simplify matrices and reveal their fundamental properties, they differ significantly in approach and resulting structure.

Row Echelon Form vs. Normal Form

Row Echelon Form	Normal Form
Uses only elementary row operations	Uses both row and column operations
Preserves linear dependence relationships among columns	Changes both row and column relationships
Not unique for a given matrix	Unique for a given matrix (rank determines the form)
Has the form $\begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix}$ where F represents free variables	Has the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ with zeros in all non-pivot positions
Directly applicable to solving systems of linear equations	More theoretical; useful for matrix classification
Preserves column space	Simplifies the matrix structure completely

The key distinction is that Row Echelon Form maintains the original structure of the column relationships, which is crucial when working with systems of linear equations. The entries to the right of each leading 1 in REF (the matrix F above) contain significant information about the solution space of the corresponding system.

In contrast, Normal Form uses both row and column operations to create a more symmetric, simplified structure where all non-diagonal elements are zero. This makes Normal Form particularly useful for matrix classification and determining matrix equivalence, while Row Echelon Form is better suited for computational applications like solving linear systems.

3.2 Visual Approach and Additional Solved Examples

The process of transforming a matrix into Row Echelon Form (REF) follows similar steps to the Normal Form reduction but uses only row operations. This section provides a visual guide to the reduction process followed by worked examples with different matrix types.

Visualization of the Reduction Process

STEP 1:

$$\begin{array}{ccccc} \boxed{a_{11}} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array}$$

Find your first pivot element at position (1,1).
If $a_{11} = 0$, locate a non-zero element elsewhere in the column and use row swaps to move it to position (1,1).

STEP 2:

$$\begin{matrix} 1 & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{matrix}$$

Transform the pivot element to 1 by dividing the entire first row by a_{11} or by swapping rows.
This gives us our first proper pivot at position (1,1).

STEP 3:

$$\begin{matrix} 1 & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a'_{m2} & a'_{m3} & \cdots & a'_{mn} \end{matrix}$$

Zero out all elements below the pivot by subtracting multiples of the first row from each row below:
$R_i \leftarrow R_i - a_{i1} \cdot R_1$
for each row $i = 2, 3, \ldots, m$

STEP 4:

$$\begin{matrix} 1 & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & \boxed{a'_{22}} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a'_{m2} & a'_{m3} & \cdots & a'_{mn} \end{matrix}$$

Move to the next pivot position (2,2).
Repeat steps 1-3 on the remaining submatrix (excluding the first row).
If $a'_{22} = 0$, look for a non-zero element below in the same column or move to the next column.

STEP 5:

$$\begin{matrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & 1 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{matrix}$$

After completing the process, the matrix is in Row Echelon Form.
The asterisks (*) represent arbitrary values that remain in the upper right portion of the matrix.
The rank r equals the number of pivots (leading 1s).

Color Legend

Current pivot | Pivot = 1 | Zeroed element

Key Differences from Normal Form Reduction

- Unlike Normal Form reduction, we do not perform column operations to zero out elements to the right of pivots.
- The resulting matrix has leading 1s with zeros below them, but may have non-zero elements above and to the right of the pivots.
- The reduction focuses on creating a "staircase" pattern of pivots without fully simplifying the matrix.
- If we encounter a zero element at a potential pivot position, we must either find a non-zero element in the same column below or move to the next column.

Additional Solved Examples

2×2 Full Rank Matrix Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 9 \end{bmatrix}$.

- **Step 1:** Prepare (1,1). $a_{11} = 2 \neq 0$. OK.
- **Step 2:** Make Pivot '1': $R_1 \to \frac{1}{2}R_1$

$$\implies \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$$

- **Step 3:** Zeros Below: $R_2 \to R_2 - 3R_1$

$$\implies \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

- **Step 4:** Move to (2,2). $a_{22} = 3 \neq 0$. OK.
- **Step 5:** Make Pivot '1': $R_2 \to \frac{1}{3}R_2$

$$\implies \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Result: The Row Echelon Form is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Rank $r = 2$.

Note that unlike the Normal Form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the Row Echelon Form retains the non-zero element above the second pivot.

3×4 Matrix with Rank 2 Let $B = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 5 & 4 \\ 1 & 1 & 0 & 7 \end{bmatrix}$.

- **Step 1:** Prepare (1,1). $a_{11} = 1 \neq 0$. OK.
- **Step 2:** Make Pivot '1': Already 1.
- **Step 3:** Zeros Below: $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$.

$$\implies \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

- **Step 4:** Move to (2,2). $a_{22} = 0$ and entire column below is 0. Move to position (2,3).
- **Step 5:** Make Pivot '1': $a_{23} = 1$. Already 1.

- **Step 6:** Zeros Below: $R_3 \to R_3 + 2R_2$.

$$\implies \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Result: The Row Echelon Form is $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Rank $r = 2$.

Notice that non-zero elements remain to the right of pivots and in the first row, reflecting the system's solution space structure.

3×3 Matrix Requiring Row Swaps Let $C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 5 & 4 \end{bmatrix}$.

- **Step 1:** Prepare (1,1). $a_{11} = 0$. Look below: $a_{21} = 1 \neq 0$. Swap rows $R_1 \leftrightarrow R_2$.

$$\implies \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 5 & 4 \end{bmatrix}$$

- **Step 2:** Make Pivot '1': Already 1.
- **Step 3:** Zeros Below: $R_3 \to R_3 - 2R_1$.

$$\implies \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

- **Step 4:** Move to (2,2). $a_{22} = 1 \neq 0$. OK.
- **Step 5:** Make Pivot '1': Already 1.
- **Step 6:** Zeros Below: $R_3 \to R_3 - R_2$.

$$\implies \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Result: The Row Echelon Form is $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Rank $r = 2$.

This example demonstrates the need for row swaps when the potential pivot position contains a zero.

4×3 Matrix with Dependent Rows Let $D = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 0 & 0 & 1 \\ 3 & 6 & 1 \end{bmatrix}$.

- **Step 1:** Prepare (1,1). $a_{11} = 1 \neq 0$. OK.
- **Step 2:** Make Pivot '1': Already 1.

- **Step 3:** Zeros Below: $R_2 \to R_2 - 2R_1$; $R_4 \to R_4 - 3R_1$. (Row 3 already has 0).

$$\implies \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

- **Step 4:** Move to (2,2). $a_{22} = 0$ and entire column below is 0. Move to position (2,3).
- **Step 5:** Make Pivot '1': $a_{23} = 1$. Already 1.
- **Step 6:** Zeros Below: $R_3 \to R_3 - R_2$; $R_4 \to R_4 + 2R_2$.

$$\implies \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result: The Row Echelon Form is $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Rank $r = 2$.

This matrix has more rows than columns, and two rows are linearly dependent on the others.