Chapter 6

Vector Integral Calculus

Vector integral calculus extends differential calculus by introducing integral operations on vector fields. This chapter covers line integrals, conservative fields, and fundamental theorems connecting different types of integrals.

6.1 Line Integrals

6.1.1 Line Integrals of Vector Fields

Definition 6.1 (Line Integral of a Vector Field). *For a vector field $\vec{F}(\vec{r})$ and a smooth curve C parameterized by $\vec{r}(t)$ for $t \in [a, b]$:*

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt \tag{6.1}$$

Line Integral Evaluation

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (2y+3)\hat{\imath} + (xz)\hat{\jmath} + (yz-x)\hat{k}$ and C is the straight line from $(0, 0, 0)$ to $(2, 1, 1)$.
Parameterize the curve: $\vec{r}(t) = (2t, t, t)$ for $t \in [0, 1]$
$\frac{d\vec{r}}{dt} = (2, 1, 1)dt$
Substituting: $\vec{F}(\vec{r}(t)) = (2t + 3, 2t^2, t^2 - 2t)$
Therefore:

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 [(2t + 3)(2) + (2t^2)(1) + (t^2 - 2t)(1)]dt \tag{6.2}$$

$$= \int_0^1 [4t + 6 + 2t^2 + t^2 - 2t]dt \tag{6.3}$$

$$= \int_0^1 [3t^2 + 2t + 6]dt \tag{6.4}$$

$$= [t^3 + t^2 + 6t]_0^1 = 8 \tag{6.5}$$

6.1.2 Properties of Line Integrals

1. **Linearity**: $\int_C (a\vec{F} + b\vec{G}) \cdot d\vec{r} = a \int_C \vec{F} \cdot d\vec{r} + b \int_C \vec{G} \cdot d\vec{r}$
2. **Additivity**: $\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$

3. **Orientation**: $\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$

6.2 Conservative Vector Fields

6.2.1 Path Independence

Definition 6.2 (Conservative Vector Field). *A vector field $\vec{F}$ is conservative if:*

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \tag{6.6}$$

for any closed curve C.

6.2.2 Test for Conservative Fields

For a vector field $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, check if $\nabla \times \vec{F} = \vec{0}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{6.7}$$

Checking Conservative Field

Determine if $\vec{F} = (4xy + z^3)\hat{i} + (2x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is conservative.
Calculate $\nabla \times \vec{F}$:

$$\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y} = -1 - (-1) = 0 \tag{6.8}$$

$$\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} = 3z^2 - 3z^2 = 0 \tag{6.9}$$

$$\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = 4x - 4x = 0 \tag{6.10}$$

Since all components are zero, the field is conservative.

6.2.3 Finding Scalar Potential

For a conservative field, find ϕ such that $\vec{F} = \nabla\phi$ using:

$$\phi = \int F_1 dx + \int F_2 dy + \int F_3 dz \tag{6.11}$$

- Integrate F_1 with respect to x treating y, z constant
- From F_2, integrate terms free from x with respect to y
- From F_3, integrate terms free from x and y with respect to z

Finding Scalar Potential

For $\vec{F} = (2xy + z^3)\hat{i} + (x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$:
From F_1: $\int(2xy + z^3)dx = x^2y + xz^3$
From F_2: Terms without x: $-z$, so $\int(-z)dy = -zy$

From F_3: No terms without x and y
Therefore: $\phi = x^2y + xz^3 - zy + C$

6.3 Work and Conservative Forces

For a conservative force field, work is path-independent:

$$W = \int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A) \tag{6.12}$$

Work Calculation

Find work done by $\vec{F} = (2xz^3 + 6y)\hat{i} + (6x - 2yz)\hat{j} + (3x^2z^2 - y^2)\hat{k}$ from $(0, 0, 0)$ to $(1, 1, 1)$.
First verify conservative: $\nabla \times \vec{F} = \vec{0}$
Find potential: $\phi = x^2z^3 + 6xy - y^2z + C$
Calculate work:

$$W = \phi(1, 1, 1) - \phi(0, 0, 0) \tag{6.13}$$
$$= [1 + 6 - 1] - [0] \tag{6.14}$$
$$= 6 \tag{6.15}$$

6.4 Green's Theorem

Theorem 6.3 (Green's Theorem)**.** *For a vector field $\vec{F} = P\hat{i} + Q\hat{j}$ and a closed curve C enclosing region R:*

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{6.16}$$

Applying Green's Theorem

Evaluate $\oint_C 3ydx + 2xdy$ where C is the boundary of $0 \leq x \leq 2\pi$, $0 \leq y \leq \sin x$.
Using Green's theorem:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - 3 = -1 \tag{6.17}$$
$$\oint_C 3ydx + 2xdy = \iint_R (-1)dA \tag{6.18}$$
$$= -\int_0^{2\pi} \int_0^{\sin x} dydx \tag{6.19}$$
$$= -\int_0^{2\pi} \sin x dx \tag{6.20}$$
$$= -[-\cos x]_0^{2\pi} = 0 \tag{6.21}$$

6.5 Stokes' Theorem

Theorem 6.4 (Stokes' Theorem). *For a vector field $\vec{F}$ and a surface S with boundary curve C:*

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} \tag{6.22}$$

Using Stokes' Theorem

For $\vec{F} = (y^2)\hat{i} + (x^2)\hat{j} - (x+z)\hat{k}$ and triangular surface with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$:
Calculate $\nabla \times \vec{F} = 0\hat{i} + 1\hat{j} + (2x-2y)\hat{k}$
For surface in xy-plane: $d\vec{S} = \hat{k}\,dx\,dy$
Therefore:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (2x-2y)\,dx\,dy \tag{6.23}$$

$$= \int_0^1 \int_0^{1-u} (2u-2v)\,dv\,du \tag{6.24}$$

$$= 0 \tag{6.25}$$

6.6 Divergence Theorem

Theorem 6.5 (Divergence Theorem). *For a vector field $\vec{F}$ and a volume V with surface S:*

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V (\nabla \cdot \vec{F})\,dV \tag{6.26}$$

Applying Divergence Theorem

For $\vec{F} = (x\hat{i} - y\hat{j} + (z^2-1)\hat{k})$ and cylindrical volume $x^2+y^2=4$, $0 \le z \le 1$:
Calculate $\nabla \cdot \vec{F} = 1 - 1 + 2z = 2z$
Using cylindrical coordinates:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V 2z\,dV \tag{6.27}$$

$$= \int_0^1 \int_0^{2\pi} \int_0^2 2z \cdot r\,dr\,d\theta\,dz \tag{6.28}$$

$$= 4\pi \tag{6.29}$$

6.7 Key Applications

6.7.1 Circulation and Flux

- Circulation: $\oint_C \vec{F} \cdot d\vec{r}$ measures the tendency to rotate around C
- Flux: $\iint_S \vec{F} \cdot d\vec{S}$ measures flow through surface S

6.7.2 Physical Interpretations

- Line integrals: Work done by forces along paths
- Surface integrals: Flux of electric or magnetic fields
- Volume integrals: Mass, charge, or energy content

6.7.3 Fundamental Theorem Connections

The theorems establish relationships between different integral types:

$$\begin{tikzcd}
\oint_C \vec{F} \cdot d\vec{r} \arrow[d, "\text{Stokes}"] & \iint_S \vec{F} \cdot d\vec{S} \arrow[d, "\text{Gauss}"] \\
\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} & \iiint_V (\nabla \cdot \vec{F})\, dV
\end{tikzcd}$$

$$\begin{tikzcd}
\int_A^B \vec{F} \cdot d\vec{r} \arrow[d, "\text{FTC}"] \\
\phi(B) - \phi(A)
\end{tikzcd}$$

These theorems form the foundation for applications in physics, engineering, and mathematics.

6.8 Solved Examples on Vector Integral

Example: Evaluating Line Integral

Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = (2y+3)\hat{\imath} + (xz)\hat{\jmath} + (yz-x)\hat{k}$ and C: the straight line joining (0,0,0) and (2,1,1).

Solution

Step 1: Parameterize the curve C.
For a straight line from point A(0,0,0) to point B(2,1,1), we can parameterize as:

$$\vec{r}(t) = (1-t)\vec{A} + t\vec{B}$$

where $0 \leq t \leq 1$.

$$\begin{aligned}\vec{r}(t) &= (1-t)(0,0,0) + t(2,1,1) \\ &= (2t, t, t)\end{aligned}$$

So: $x = 2t$, $y = t$, $z = t$
Step 2: Find $d\vec{r}$.

$$d\vec{r} = \frac{d\vec{r}}{dt}dt = (2,1,1)dt$$

Step 3: Express $\vec{F}$ in terms of the parameter t.

$\vec{F} = (2y+3)\hat{i} + (xz)\hat{j} + (yz-x)\hat{k}$
Substituting $x = 2t$, $y = t$, $z = t$:

$$\begin{aligned}\vec{F}(t) &= (2t+3)\hat{i} + (2t\cdot t)\hat{j} + (t\cdot t - 2t)\hat{k}\\ &= (2t+3)\hat{i} + (2t^2)\hat{j} + (t^2-2t)\hat{k}\end{aligned}$$

Step 4: Calculate $\vec{F}\cdot d\vec{r}$.

$$\begin{aligned}\vec{F}\cdot d\vec{r} &= [(2t+3)\hat{i} + (2t^2)\hat{j} + (t^2-2t)\hat{k}]\cdot[(2,1,1)dt]\\ &= [(2t+3)\cdot 2 + (2t^2)\cdot 1 + (t^2-2t)\cdot 1]dt\\ &= [4t+6+2t^2+t^2-2t]dt\\ &= [3t^2+2t+6]dt\end{aligned}$$

Step 5: Evaluate the line integral.

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^1 (3t^2+2t+6)dt\\ &= \left[t^3+t^2+6t\right]_0^1\\ &= (1+1+6)-(0+0+0)\\ &= 8\end{aligned}$$

Therefore, $\int_C \vec{F}\cdot d\vec{r} = 8$.

Example: Evaluating Line Integral with Parametric Curve

Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = (3x^2)\hat{i} + (2xz-y)\hat{j} + (z)\hat{k}$ and C: $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from $t = 0$ to $t = 1$.

Solution

Step 1: The curve C is given parametrically as:

$$x = 2t^2, \quad y = t, \quad z = 4t^2 - t$$

with $0 \le t \le 1$.
Step 2: Find $d\vec{r}$.

$$\begin{aligned}\frac{dx}{dt} &= 4t\\ \frac{dy}{dt} &= 1\\ \frac{dz}{dt} &= 8t-1\end{aligned}$$

Therefore:

$$d\vec{r} = \frac{d\vec{r}}{dt}dt = (4t, 1, 8t-1)dt$$

Step 3: Express $\vec{F}$ in terms of the parameter t.
$\vec{F} = (3x^2)\hat{i} + (2xz - y)\hat{j} + (z)\hat{k}$
Substituting $x = 2t^2$, $y = t$, $z = 4t^2 - t$:

$$\begin{aligned} F_1 &= 3x^2 = 3(2t^2)^2 = 3 \cdot 4t^4 = 12t^4 \\ F_2 &= 2xz - y = 2(2t^2)(4t^2 - t) - t = 4t^2(4t^2 - t) - t \\ &= 16t^4 - 4t^3 - t \\ F_3 &= z = 4t^2 - t \end{aligned}$$

Step 4: Calculate $\vec{F} \cdot d\vec{r}$.

$$\begin{aligned} \vec{F} \cdot d\vec{r} &= [(12t^4)\hat{i} + (16t^4 - 4t^3 - t)\hat{j} + (4t^2 - t)\hat{k}] \cdot [(4t, 1, 8t - 1)dt] \\ &= [(12t^4) \cdot 4t + (16t^4 - 4t^3 - t) \cdot 1 + (4t^2 - t) \cdot (8t - 1)]dt \\ &= [48t^5 + 16t^4 - 4t^3 - t + (4t^2 - t)(8t - 1)]dt \end{aligned}$$

Expanding $(4t^2 - t)(8t - 1)$:

$$\begin{aligned} (4t^2 - t)(8t - 1) &= 32t^3 - 4t^2 - 8t^2 + t \\ &= 32t^3 - 12t^2 + t \end{aligned}$$

Continue:

$$\begin{aligned} \vec{F} \cdot d\vec{r} &= [48t^5 + 16t^4 - 4t^3 - t + 32t^3 - 12t^2 + t]dt \\ &= [48t^5 + 16t^4 + 28t^3 - 12t^2]dt \end{aligned}$$

Step 5: Evaluate the line integral.

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 (48t^5 + 16t^4 + 28t^3 - 12t^2)dt \\ &= \left[\frac{48t^6}{6} + \frac{16t^5}{5} + \frac{28t^4}{4} - \frac{12t^3}{3}\right]_0^1 \\ &= \left[8t^6 + \frac{16t^5}{5} + 7t^4 - 4t^3\right]_0^1 \\ &= 8 + \frac{16}{5} + 7 - 4 \\ &= 11 + \frac{16}{5} \\ &= \frac{55 + 16}{5} \\ &= \frac{71}{5} \end{aligned}$$

Therefore, $\int_C \vec{F} \cdot d\vec{r} = \frac{71}{5}$.

Example: Evaluating Line Integral along Parabolic Path

If $f = (2x + y^2)\hat{i} + (3y - 4x)\hat{j}$ then evaluate $\int_C \vec{F} \cdot d\vec{r}$ around the parabolic arc $y = x^2$ joining (0,0) and (1,1).

Solution

Step 1: Parameterize the curve C.
The curve is $y = x^2$ from (0,0) to (1,1). We can parameterize as:

$$x = t, \quad y = t^2$$

where $0 \leq t \leq 1$.
Step 2: Find $d\vec{r}$.

$$\frac{dx}{dt} = 1$$
$$\frac{dy}{dt} = 2t$$

Therefore:

$$d\vec{r} = \frac{d\vec{r}}{dt}dt = (1, 2t)dt$$

Step 3: Express $\vec{F}$ in terms of the parameter t.
$\vec{F} = (2x + y^2)\hat{i} + (3y - 4x)\hat{j}$
Substituting $x = t$, $y = t^2$:

$$F_1 = 2x + y^2 = 2t + (t^2)^2 = 2t + t^4$$
$$F_2 = 3y - 4x = 3t^2 - 4t$$

Step 4: Calculate $\vec{F} \cdot d\vec{r}$.

$$\begin{aligned}\vec{F} \cdot d\vec{r} &= [(2t + t^4)\hat{i} + (3t^2 - 4t)\hat{j}] \cdot [(1, 2t)dt] \\ &= [(2t + t^4) \cdot 1 + (3t^2 - 4t) \cdot 2t]dt \\ &= [2t + t^4 + 2t(3t^2 - 4t)]dt \\ &= [2t + t^4 + 6t^3 - 8t^2]dt \\ &= [t^4 + 6t^3 - 8t^2 + 2t]dt\end{aligned}$$

Step 5: Evaluate the line integral.

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^1 (t^4+6t^3-8t^2+2t)dt\\ &= \left[\frac{t^5}{5}+\frac{6t^4}{4}-\frac{8t^3}{3}+\frac{2t^2}{2}\right]_0^1\\ &= \left[\frac{t^5}{5}+\frac{3t^4}{2}-\frac{8t^3}{3}+t^2\right]_0^1\\ &= \frac{1}{5}+\frac{3}{2}-\frac{8}{3}+1\\ &= \frac{6}{30}+\frac{45}{30}-\frac{80}{30}+\frac{30}{30}\\ &= \frac{6+45-80+30}{30}\\ &= \frac{1}{30}\end{aligned}$$

Therefore, $\int_C \vec{F}\cdot d\vec{r} = \frac{1}{30}$.

Example: Finding Work Done and Checking if Force is Conservative

Find work done by the force $\vec{F} = (2xz^3+6y)\hat{i}+(6x-2yz)\hat{j}+(3x^2z^2-y^2)\hat{k}$ in moving a particle from (0,0,0) to (1,1,1). Is the force field is conservative.

Solution

Step 1: Check if the force field is conservative.
A force field is conservative if $\nabla\times\vec{F}=\vec{0}$.
Given: $\vec{F} = (2xz^3+6y)\hat{i}+(6x-2yz)\hat{j}+(3x^2z^2-y^2)\hat{k}$

$$\nabla\times\vec{F} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xz^3+6y & 6x-2yz & 3x^2z^2-y^2\end{vmatrix}$$

For i-component:

$$\begin{aligned}\frac{\partial}{\partial y}(3x^2z^2-y^2)-\frac{\partial}{\partial z}(6x-2yz) &= -2y-(-2y)\\ &= -2y+2y\\ &= 0\end{aligned}$$

For j-component:

$$\begin{aligned}\frac{\partial}{\partial z}(2xz^3+6y)-\frac{\partial}{\partial x}(3x^2z^2-y^2) &= 6xz^2-6xz^2\\ &= 0\end{aligned}$$

For k-component:

$$\begin{aligned}\frac{\partial}{\partial x}(6x-2yz)-\frac{\partial}{\partial y}(2xz^3+6y) &= 6-6\\ &= 0\end{aligned}$$

Since $\nabla \times \vec{F} = \vec{0}$, the force field is conservative.
Step 2: Find the scalar potential ϕ such that $\vec{F} = -\nabla\phi$.
Using our formula:

$$\phi = -\left[\int F_1 dx + \int F_2 dy + \int F_3 dz\right]$$

From $F_1 = 2xz^3 + 6y$:

$$\int F_1 dx = \int (2xz^3 + 6y)dx = x^2z^3 + 6xy$$

From $F_2 = 6x - 2yz$: - Terms free from x: $-2yz$

$$\int F_2 dy = \int (-2yz)dy = -y^2z$$

From $F_3 = 3x^2z^2 - y^2$: - Terms free from x and y: none
Therefore:

$$\phi = -(x^2z^3 + 6xy - y^2z) + C = -x^2z^3 - 6xy + y^2z + C$$

Step 3: Calculate the work done.
For a conservative force, work done is:

$$\begin{aligned} W &= \phi(1,1,1) - \phi(0,0,0) \\ &= [-1^2 \cdot 1^3 - 6 \cdot 1 \cdot 1 + 1^2 \cdot 1] - [-0^2 \cdot 0^3 - 6 \cdot 0 \cdot 0 + 0^2 \cdot 0] \\ &= [-1 - 6 + 1] - [0] \\ &= -6 \end{aligned}$$

Therefore, the work done is $W = -6$ and the force field is conservative.

Example: Using Green's Theorem to Evaluate Line Integral

Using Green's theorem evaluate $\int_C [cosy\hat{i} + x(1 - siny)\hat{j}] \cdot d\vec{r}$ where C is closed curve $x^2 + y^2 = 1$, $z = 0$

Solution

Step 1: Green's theorem states:
For a closed curve C enclosing region R:

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Step 2: Identify P and Q from the given vector field.
Given: $\vec{F} = cosy\hat{i} + x(1 - siny)\hat{j}$
Therefore:

$$\begin{aligned} P &= cosy \\ Q &= x(1 - siny) \end{aligned}$$

Step 3: Calculate $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[x(1 - siny)] = 1 - siny$$
$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[cosy] = -siny$$

Therefore:

$$\begin{aligned}\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= (1 - siny) - (-siny)\\ &= 1 - siny + siny\\ &= 1\end{aligned}$$

Step 4: Set up the double integral.
The curve C: $x^2 + y^2 = 1$ is a circle of radius 1 centered at origin.
Therefore:

$$\int_C [cosydx + x(1 - siny)dy] = \iint_R 1\, dx\, dy$$

Step 5: Evaluate the double integral.
The region R is a disk of radius 1:

$$\iint_R 1\, dx\, dy = \text{Area of circle} = \pi r^2 = \pi(1)^2 = \pi$$

Therefore, using Green's theorem:

$$\int_C [cosy\hat{i} + x(1 - siny)\hat{j}] \cdot d\vec{r} = \pi$$

Example: Applying Green's Theorem to Region Boundary

Apply Green's theorem to evaluate $\int_C 3ydx + 2xdy$ where C is boundary of $0 \leq x \leq 2\pi; 0 \leq y \leq sinx$

Solution

Step 1: Identify P and Q from the given line integral.
Given: $\int_C 3ydx + 2xdy$
Therefore:

$$P = 3y$$
$$Q = 2x$$

Step 2: Apply Green's theorem.
Green's theorem states:

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Calculate the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[2x] = 2$$
$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[3y] = 3$$

Therefore:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - 3 = -1$$

Step 3: Set up the double integral.
The region R is bounded by:

$$0 \le x \le 2\pi$$
$$0 \le y \le \sin x$$

Therefore:

$$\int_C 3ydx + 2xdy = \iint_R (-1)dxdy = -\iint_R dxdy$$

Step 4: Evaluate the double integral.

$$\begin{aligned}
-\iint_R dxdy &= -\int_0^{2\pi}\int_0^{\sin x} dydx \\
&= -\int_0^{2\pi} [y]_0^{\sin x} dx \\
&= -\int_0^{2\pi} \sin x dx \\
&= -[-\cos x]_0^{2\pi} \\
&= -[(-\cos 2\pi) - (-\cos 0)] \\
&= -[(-1) - (-1)] \\
&= -[0] \\
&= 0
\end{aligned}$$

Therefore, $\int_C 3ydx + 2xdy = 0$

Example: Using Green's Theorem for Circle

Using Green's theorem evaluate $\int_C [cosx\, siny - 4y]dx + [sinx\, cosy]dy$ where C is circle $x^2 + y^2 = 1$

Solution

Step 1: Identify P and Q from the given line integral.
Given: $\int_C [cosx\, siny - 4y]dx + [sinx\, cosy]dy$

Therefore:

$$P = cosx\ siny - 4y$$
$$Q = sinx\ cosy$$

Step 2: Apply Green's theorem.
Green's theorem states:

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Calculate the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[sinx\ cosy] = cosx\ cosy$$
$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[cosx\ siny - 4y] = cosx\ cosy - 4$$

Therefore:

$$\begin{aligned}\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= cosx\ cosy - (cosx\ cosy - 4)\\ &= cosx\ cosy - cosx\ cosy + 4\\ &= 4\end{aligned}$$

Step 3: Set up the double integral.
The region R is a disk of radius 1 centered at origin, bounded by $x^2 + y^2 = 1$.
Therefore:

$$\int_C [cosx\ siny - 4y]dx + [sinx\ cosy]dy = \iint_R 4\,dx\,dy$$

Step 4: Evaluate the double integral.

$$\begin{aligned}\iint_R 4\,dx\,dy &= 4\iint_R dx\,dy\\ &= 4 \times \text{Area of unit circle}\\ &= 4 \times \pi\\ &= 4\pi\end{aligned}$$

Therefore, $\int_C [cosx\ siny - 4y]dx + [sinx\ cosy]dy = 4\pi$

Example: Surface Integral over Paraboloid

Evaluate, $\iint_s (\nabla \times \vec{F}) \cdot d\vec{S}$ for the surface of paraboloid $z = 4 - x^2 - y^2$ $(z \geq 0)$ where $\vec{F} = y^2\hat{i} + z\hat{j} + xy\hat{k}$

Solution

Step 1: Use Stokes' theorem.

Stokes' theorem states:

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

where C is the boundary curve of surface S.
Step 2: Find the boundary curve C.
The boundary of the paraboloid is where $z = 0$:

$$\begin{aligned} 4 - x^2 - y^2 &= 0 \\ x^2 + y^2 &= 4 \end{aligned}$$

This is a circle of radius 2 centered at origin in the xy-plane (z = 0).
Step 3: Parameterize the boundary curve C.
Use the standard parameterization for a circle:

$$\begin{aligned} x &= 2\cos t \\ y &= 2\sin t \\ z &= 0 \end{aligned}$$

where $0 \leq t \leq 2\pi$.
Step 4: Calculate $d\vec{r}$.

$$\begin{aligned} \frac{dx}{dt} &= -2\sin t \\ \frac{dy}{dt} &= 2\cos t \\ \frac{dz}{dt} &= 0 \end{aligned}$$

Therefore:

$$d\vec{r} = (-2\sin t, 2\cos t, 0)dt$$

Step 5: Express $\vec{F}$ in terms of parameter t.
$\vec{F} = y^2\hat{i} + z\hat{j} + xy\hat{k}$
Substituting $x = 2\cos t$, $y = 2\sin t$, $z = 0$:

$$\begin{aligned} \vec{F}(t) &= (2\sin t)^2\hat{i} + 0\hat{j} + (2\cos t)(2\sin t)\hat{k} \\ &= 4\sin^2 t\hat{i} + 0\hat{j} + 4\sin t\cos t\hat{k} \end{aligned}$$

Step 6: Calculate $\vec{F} \cdot d\vec{r}$.

$$\begin{aligned} \vec{F} \cdot d\vec{r} &= [4\sin^2 t\hat{i} + 0\hat{j} + 4\sin t\cos t\hat{k}] \cdot [(-2\sin t, 2\cos t, 0)dt] \\ &= [4\sin^2 t \cdot (-2\sin t) + 0 \cdot 2\cos t + 4\sin t\cos t \cdot 0]dt \\ &= [-8\sin^3 t]dt \end{aligned}$$

Step 7: Evaluate the line integral.

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-8\sin^3 t)dt$$
$$= -8\int_0^{2\pi} \sin^3 t dt$$

Using the identity $\sin^3 t = \frac{3\sin t - \sin 3t}{4}$:

$$= -8\int_0^{2\pi} \frac{3\sin t - \sin 3t}{4} dt$$
$$= -2\int_0^{2\pi} (3\sin t - \sin 3t)dt$$
$$= -2\left[(-3\cos t + \frac{\cos 3t}{3})\right]_0^{2\pi}$$
$$= -2\left[(3\cos 2\pi - \frac{\cos 6\pi}{3}) - (3\cos 0 - \frac{\cos 0}{3})\right]$$
$$= -2\left[(3 - \frac{1}{3}) - (3 - \frac{1}{3})\right]$$
$$= -2[0]$$
$$= 0$$

Therefore, $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0$

Example: Applying Stokes' Theorem to Ellipsoid Surface

Evaluate, $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$, where $\vec{F} = (x^3 - y^3)\hat{i} - (xyz)\hat{j} + (y^3)\hat{k}$ and S is the surface $x^2 + 4y^2 + z^2 - 2x = 4$ above the plane x = 0.

Solution

Step 1: Apply Stokes' theorem.
Stokes' theorem states:

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

where C is the boundary curve of surface S.
Step 2: Find the boundary curve C.
The boundary of the surface is where it intersects the plane x = 0:

$$x^2 + 4y^2 + z^2 - 2x = 4$$
$$\text{Substituting } x = 0:$$
$$0 + 4y^2 + z^2 - 0 = 4$$
$$4y^2 + z^2 = 4$$
$$y^2 + \frac{z^2}{4} = 1$$

This is an ellipse in the yz-plane at x = 0.
Step 3: Parameterize the boundary curve C.

Parametric equations for the ellipse:

$$\begin{aligned} x &= 0 \\ y &= \cos t \\ z &= 2\sin t \end{aligned}$$

where $0 \leq t \leq 2\pi$.
Step 4: Calculate $d\vec{r}$.

$$\begin{aligned} \frac{dx}{dt} &= 0 \\ \frac{dy}{dt} &= -\sin t \\ \frac{dz}{dt} &= 2\cos t \end{aligned}$$

Therefore:

$$d\vec{r} = (0, -\sin t, 2\cos t)dt$$

Step 5: Express $\vec{F}$ in terms of parameter t.
$\vec{F} = (x^3 - y^3)\hat{\imath} - (xyz)\hat{\jmath} + (y^3)\hat{k}$
Substituting $x = 0$, $y = \cos t$, $z = 2\sin t$:

$$\begin{aligned} \vec{F}(t) &= (0 - \cos^3 t)\hat{\imath} - (0)\hat{\jmath} + (\cos^3 t)\hat{k} \\ &= -\cos^3 t\hat{\imath} + 0\hat{\jmath} + \cos^3 t\hat{k} \end{aligned}$$

Step 6: Calculate $\vec{F} \cdot d\vec{r}$.

$$\begin{aligned} \vec{F} \cdot d\vec{r} &= [-\cos^3 t\hat{\imath} + 0\hat{\jmath} + \cos^3 t\hat{k}] \cdot [(0, -\sin t, 2\cos t)dt] \\ &= [(-\cos^3 t) \cdot 0 + 0 \cdot (-\sin t) + \cos^3 t \cdot 2\cos t]dt \\ &= [2\cos^4 t]dt \end{aligned}$$

Step 7: Evaluate the line integral.

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} 2\cos^4 t dt \\ &= 2\int_0^{2\pi} \cos^4 t dt \end{aligned}$$

Using the formula $\cos^4 t = \frac{3+4\cos(2t)+\cos(4t)}{8}$:

$$\begin{aligned}
&= 2\int_0^{2\pi} \frac{3+4\cos(2t)+\cos(4t)}{8} dt \\
&= \frac{1}{4}\int_0^{2\pi} (3+4\cos(2t)+\cos(4t))dt \\
&= \frac{1}{4}\left[3t + 2\sin(2t) + \frac{\sin(4t)}{4}\right]_0^{2\pi} \\
&= \frac{1}{4}[(6\pi+0+0)-(0+0+0)] \\
&= \frac{6\pi}{4} \\
&= \frac{3\pi}{2}
\end{aligned}$$

Therefore, $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \frac{3\pi}{2}$

Example: Using Stokes' Theorem for Triangular Path

Using Stoke's theorem evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (y^2)\hat{i} + (x^2)\hat{j} - (x+z)\hat{k}$ and C is the triangle with vertices (0,0,0), (1,0,0), (0,1,0).

Solution

Step 1: Apply Stokes' theorem.
Stokes' theorem states:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

where S is the surface bounded by curve C.
Step 2: Calculate $\nabla \times \vec{F}$.
Given: $\vec{F} = (y^2)\hat{i} + (x^2)\hat{j} - (x+z)\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

For i-component:

$$\frac{\partial}{\partial y}[-(x+z)] - \frac{\partial}{\partial z}[x^2] = 0 - 0 = 0$$

For j-component:

$$\frac{\partial}{\partial z}[y^2] - \frac{\partial}{\partial x}[-(x+z)] = 0 - (-1) = 1$$

For k-component:

$$\frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial y}[y^2] = 2x - 2y$$

Therefore:

$$\nabla \times \vec{F} = 0\hat{i} + 1\hat{j} + (2x - 2y)\hat{k}$$

Step 3: Parameterize the surface S.
The triangle has vertices (0,0,0), (1,0,0), (0,1,0), so it lies in the plane x + y = 1, z = 0.
We can parameterize as:

$$x = u, \quad y = v, \quad z = 0$$

where $0 \leq u \leq 1$, $0 \leq v \leq 1 - u$.
Step 4: Find $d\vec{S}$.
For the surface in the xy-plane:

$$d\vec{S} = \pm\hat{k}\, du\, dv$$

Since the normal points in the positive z-direction:

$$d\vec{S} = \hat{k}\, du\, dv$$

Step 5: Calculate $(\nabla \times \vec{F}) \cdot d\vec{S}$.

$$\begin{aligned}(\nabla \times \vec{F}) \cdot d\vec{S} &= [0\hat{i} + 1\hat{j} + (2x - 2y)\hat{k}] \cdot [\hat{k}\, du\, dv] \\ &= (2x - 2y)\, du\, dv \\ &= (2u - 2v)\, du\, dv\end{aligned}$$

Step 6: Evaluate the surface integral.

$$\begin{aligned}\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} &= \int_0^1 \int_0^{1-u} (2u - 2v)\, dv\, du \\ &= \int_0^1 \left[2uv - v^2\right]_0^{1-u} du \\ &= \int_0^1 \left[2u(1-u) - (1-u)^2\right] du \\ &= \int_0^1 \left[2u - 2u^2 - (1 - 2u + u^2)\right] du \\ &= \int_0^1 \left[2u - 2u^2 - 1 + 2u - u^2\right] du \\ &= \int_0^1 (4u - 3u^2 - 1)\, du \\ &= \left[2u^2 - u^3 - u\right]_0^1 \\ &= (2 - 1 - 1) - (0) \\ &= 0\end{aligned}$$

Therefore, $\int_C \vec{F} \cdot d\vec{r} = 0$

Example: Using Divergence Theorem for Parallelepiped

Use divergence theorem and evaluate, $\iint_S \vec{f} \cdot d\vec{s}$ where $\vec{f} = \sin x\hat{i} + (2 - \cos y)\hat{j}$ and S is the total surface area of parallelepiped bounded by x=0, x=3, y=0, y=2, z=0, z=3

Solution

Step 1: Apply the divergence theorem.
The divergence theorem states:

$$\iint_S \vec{f} \cdot d\vec{s} = \iiint_V (\nabla \cdot \vec{f})\, dV$$

where V is the volume enclosed by surface S.
Step 2: Calculate the divergence of $\vec{f}$.
Given: $\vec{f} = \sin x\hat{i} + (2 - \cos y)\hat{j} + 0\hat{k}$

$$\begin{aligned}
\nabla \cdot \vec{f} &= \frac{\partial}{\partial x}(\sin x) + \frac{\partial}{\partial y}(2 - \cos y) + \frac{\partial}{\partial z}(0) \\
&= \cos x + \sin y + 0 \\
&= \cos x + \sin y
\end{aligned}$$

Step 3: Set up the volume integral.
The volume V is a rectangular parallelepiped with:

$$0 \leq x \leq 3, \quad 0 \leq y \leq 2, \quad 0 \leq z \leq 3$$

Step 4: Evaluate the volume integral.

$$\begin{aligned}
\iiint_V (\nabla \cdot \vec{f})\, dV &= \int_0^3 \int_0^2 \int_0^3 (\cos x + \sin y)\, dz\, dy\, dx \\
&= \int_0^3 \int_0^2 (\cos x + \sin y) \cdot z \Big|_0^3 dy\, dx \\
&= \int_0^3 \int_0^2 3(\cos x + \sin y)\, dy\, dx \\
&= 3 \int_0^3 \int_0^2 (\cos x + \sin y)\, dy\, dx \\
&= 3 \int_0^3 [y \cos x - \cos y]_0^2\, dx \\
&= 3 \int_0^3 [2\cos x - \cos 2 + \cos 0]\, dx \\
&= 3 \int_0^3 [2\cos x - \cos 2 + 1]\, dx \\
&= 3\,[2\sin x - x\cos 2 + x]_0^3 \\
&= 3\,[(2\sin 3 - 3\cos 2 + 3) - (0 - 0 + 0)] \\
&= 3(2\sin 3 - 3\cos 2 + 3) \\
&= 6\sin 3 - 9\cos 2 + 9
\end{aligned}$$

Therefore, $\iint_S \vec{f} \cdot d\vec{s} = 6\sin 3 - 9\cos 2 + 9$

Example: Using Divergence Theorem for Cylindrical Surface

Use divergence theorem and evaluate, $\iint_S (x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}) \cdot d\vec{s}$, where S is the surface z = 0, z = 1, and $x^2 + y^2 = 4$.

Solution

Step 1: Apply the divergence theorem.
The divergence theorem states:

$$\iint_S \vec{F} \cdot d\vec{s} = \iiint_V (\nabla \cdot \vec{F})\, dV$$

where V is the volume enclosed by surface S.
Step 2: Calculate the divergence of $\vec{F}$.
Given: $\vec{F} = x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}$

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2 - 1)\\ &= 1 - 1 + 2z\\ &= 2z\end{aligned}$$

Step 3: Identify the volume V.
The volume V is a cylinder bounded by: - Circular top surface: z = 1, $x^2 + y^2 = 4$ - Circular bottom surface: z = 0, $x^2 + y^2 = 4$ - Lateral surface: $x^2 + y^2 = 4$, $0 \leq z \leq 1$
Step 4: Convert to cylindrical coordinates.
For cylindrical coordinates:

$$\begin{aligned}x &= r\cos\theta\\ y &= r\sin\theta\\ dV &= r\,dr\,d\theta\,dz\end{aligned}$$

The region becomes:

$$0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 1$$

Step 5: Evaluate the volume integral.

$$
\begin{aligned}
\iiint_V (\nabla \cdot \vec{F})\, dV &= \iiint_V 2z\, dV \\
&= \int_0^1 \int_0^{2\pi} \int_0^2 2z \cdot r\, dr\, d\theta\, dz \\
&= \int_0^1 \int_0^{2\pi} 2z \left[\frac{r^2}{2}\right]_0^2 d\theta\, dz \\
&= \int_0^1 \int_0^{2\pi} 2z \cdot 2\, d\theta\, dz \\
&= \int_0^1 4z \cdot 2\pi\, dz \\
&= 8\pi \int_0^1 z\, dz \\
&= 8\pi \left[\frac{z^2}{2}\right]_0^1 \\
&= 8\pi \cdot \frac{1}{2} \\
&= 4\pi
\end{aligned}
$$

Therefore, $\iint_S (x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}) \cdot d\vec{s} = 4\pi$