Chapter 2

Introduction to Functions of Several Variables, Limit, Continuity and Partial Derivatives

2.1 Functions of Several Variables

2.1.1 Basic Definitions

In many real-world phenomena, quantities depend on multiple independent variables. For instance, the temperature at a point in a room depends on three spatial coordinates, or a company's profit might depend on the prices of multiple products.

Definition 2.1 (Function of Several Variables)**.** *A function f of n variables assigns to each ordered n-tuple $(x_1, x_2, \ldots, x_n)$ in its domain $D \subset \mathbb{R}^n$ exactly one real number, denoted by $f(x_1, x_2, \ldots, x_n)$.*

Examples of Functions of Several Variables

$$f(x,y) = x^2 + y^2 \quad \text{(Paraboloid)} \tag{2.1}$$
$$g(x,y) = \sin(xy) \quad \text{(Oscillatory Surface)} \tag{2.2}$$
$$h(x,y,z) = xyz \quad \text{(Three-variable Product)} \tag{2.3}$$
$$p(x,y,z) = \frac{1}{x^2 + y^2 + z^2} \quad \text{(Inverse Square Law)} \tag{2.4}$$

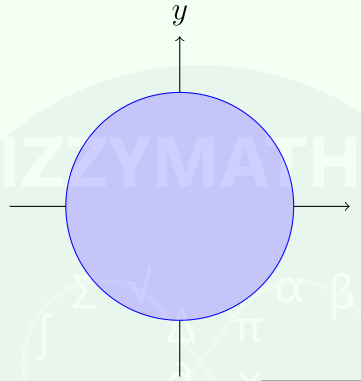
2.1.2 Domain and Range

The domain of a function is the set of all input values for which the function is defined. For functions of several variables, this is a subset of $\mathbb{R}^n$.

Finding Domains

For $f(x,y) = \sqrt{1 - x^2 - y^2}$:

- The expression under the square root must be non-negative: $1 - x^2 - y^2 \geq 0$
- Rearranging: $x^2 + y^2 \leq 1$
- This is the unit disk in $\mathbb{R}^2$: $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$

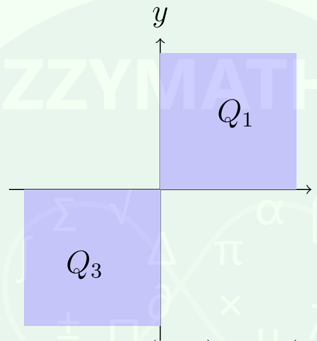


Domain of $f(x,y) = \sqrt{1 - x^2 - y^2}$

Finding Domains

For $g(x,y) = \ln(xy)$:

- The argument of the logarithm must be positive: $xy > 0$
- This means either both x and y are positive, or both are negative
- $D = \{(x,y) \mid xy > 0\} = \{(x,y) \mid (x > 0 \text{ and } y > 0) \text{ or } (x < 0 \text{ and } y < 0)\}$



Domain of $g(x,y) = \ln(xy)$

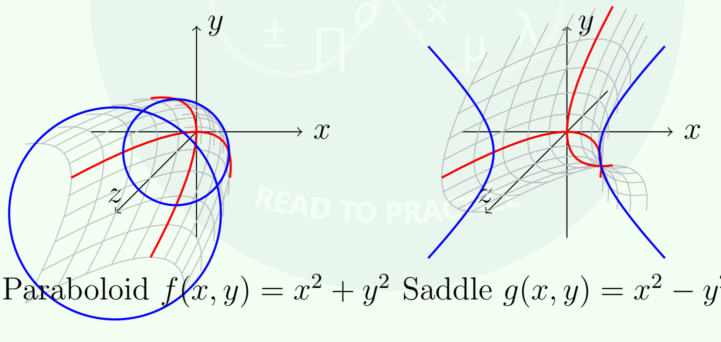
2.1.3 Visualization of Functions of Several Variables

Functions of Two Variables

For $f : \mathbb{R}^2 \to \mathbb{R}$, the graph is a surface in three-dimensional space:

$$\{(x, y, f(x,y)) \mid (x,y) \in D\}$$

Visualizing Common Functions



Paraboloid $f(x,y) = x^2 + y^2$ Saddle $g(x,y) = x^2 - y^2$

Level Curves and Level Surfaces

Definition 2.2 (Level Curves and Surfaces)**.**

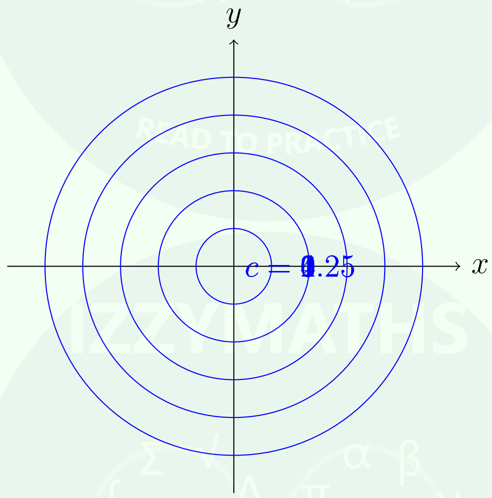
- *For a function $f(x, y)$ of two variables, a level curve is the set of all points (x, y) such that $f(x, y) = c$ for some constant c.*
- *For a function $f(x, y, z)$ of three variables, a level surface is the set of all points (x, y, z) such that $f(x, y, z) = c$ for some constant c.*

Level Curves

For $f(x, y) = x^2 + y^2$, the level curves are:

$$x^2 + y^2 = c$$

These are circles centered at the origin with radius $\sqrt{c}$.



Level curves for $f(x, y) = x^2 + y^2$

Level Surfaces

For $f(x, y, z) = x^2 + y^2 + z^2$, the level surfaces are:

$$x^2 + y^2 + z^2 = c$$

These are spheres centered at the origin with radius $\sqrt{c}$.

Level curves and surfaces provide valuable insights into the behavior of functions of several variables. The spacing between level curves indicates the steepness of the function—closely spaced curves suggest a steep gradient.

2.2 Limits and Continuity for Functions of Several Variables

2.2.1 Limits of Functions of Several Variables

Definition 2.3 (Limit of a Function of Two Variables)**.** *We say $\lim_{(x,y)\to(a,b)} f(x, y) = L$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that:*

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x, y) - L| < \varepsilon$$

Evaluating Limits

To determine if a limit exists:

1. Check if approaching the point along different paths gives the same result
2. If all paths yield the same value, the limit likely exists
3. If different paths give different values, the limit does not exist

Example 1: $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

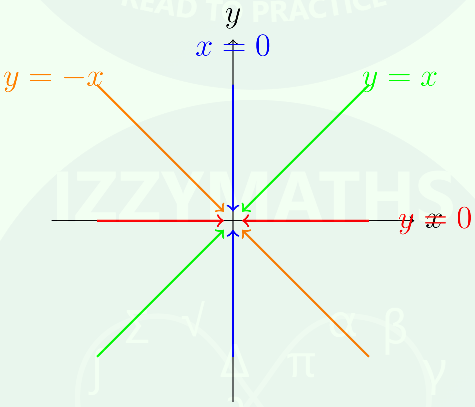
- Along x-axis ($y = 0$): $\lim = 0$
- Along y-axis ($x = 0$): $\lim = 0$
- Along $y = x$: $\lim = \lim_{x\to 0} \frac{x^3}{2x^2} = \lim_{x\to 0} \frac{x}{2} = 0$

Since all paths lead to 0, the limit exists and equals 0.

Example 2: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

- Along axes: $\lim = 0$
- Along $y = x$: $\lim = \frac{1}{2}$
- Along $y = -x$: $\lim = -\frac{1}{2}$

Since different paths give different values, this limit does not exist.



Different paths yield different limits

2.2.2 Continuity of Functions of Several Variables

Definition 2.4 (Continuity at a Point). *A function $f(x, y)$ is continuous at a point (a, b) if:*

1. *$f(a, b)$ is defined*
2. *$\lim_{(x,y)\to(a,b)} f(x, y)$ exists*
3. *$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$*

Basic Properties of Continuous Functions

If $f(x, y)$ and $g(x, y)$ are continuous at (a, b), then:

1. $f(x, y) + g(x, y)$ is continuous at (a, b)
2. $cf(x, y)$ is continuous at (a, b) for any constant c
3. $f(x, y) \cdot g(x, y)$ is continuous at (a, b)
4. $\frac{f(x,y)}{g(x,y)}$ is continuous at (a, b) if $g(a, b) \neq 0$

2.3 Partial Derivatives

2.3.1 Definition and Geometric Interpretation

Definition 2.5 (Partial Derivatives). *For a function $f(x, y)$, the partial derivatives are defined as:*

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} \tag{2.5}$$

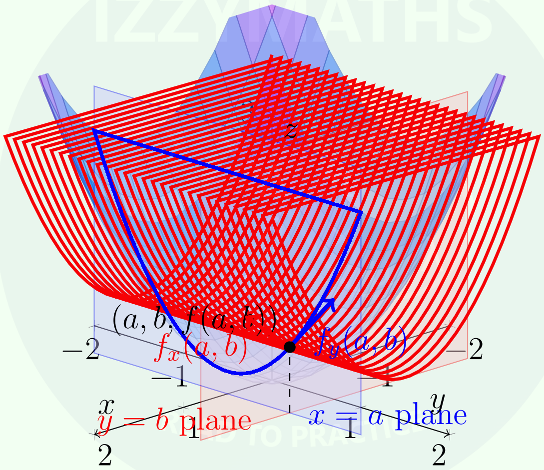
$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h} \tag{2.6}$$

Alternative notations: $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$

Geometric Interpretation of Partial Derivatives

For a function $z = f(x, y)$ representing a surface in 3D space:

- $f_x(a, b)$ is the slope of the tangent line to the curve formed by intersecting the surface with the vertical plane $y = b$.
- $f_y(a, b)$ is the slope of the tangent line to the curve formed by intersecting the surface with the vertical plane $x = a$.



2.3.2 Higher-Order Partial Derivatives

Definition 2.6 (Second-Order Partial Derivatives). *For a function $f(x, y)$, the second-order partial derivatives are:*

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \tag{2.7}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \tag{2.8}$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \tag{2.9}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \tag{2.10}$$

Clairaut's Theorem

If $f(x, y)$ has continuous second-order partial derivatives at (a, b), then the mixed partial derivatives are equal: $f_{xy}(a, b) = f_{yx}(a, b)$.

Computing Higher-Order Partial Derivatives

For $f(x, y) = x^3y^2 + x^2y^3$:
First-order partial derivatives:

$$f_x = 3x^2y^2 + 2xy^3 \tag{2.11}$$
$$f_y = 2x^3y + 3x^2y^2 \tag{2.12}$$

Second-order mixed partial derivatives:

$$f_{xy} = \frac{\partial}{\partial y}(3x^2y^2 + 2xy^3) = 6x^2y + 6xy^2 \tag{2.13}$$
$$f_{yx} = \frac{\partial}{\partial x}(2x^3y + 3x^2y^2) = 6x^2y + 6xy^2 \tag{2.14}$$

Note that $f_{xy} = f_{yx}$, confirming Clairaut's theorem.

2.4 Solved Examples

Example 1: Implicit Function

Problem: If $e^u = \tan x + \tan y$, prove that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2$

Detailed Solution

We'll solve this step-by-step using implicit differentiation.
Step 1: Start with the given equation and compute $\frac{\partial u}{\partial x}$.
We have $e^u = \tan x + \tan y$. Taking the partial derivative with respect to x on both sides:

$$\frac{\partial}{\partial x}(e^u) = \frac{\partial}{\partial x}(\tan x + \tan y) \tag{2.15}$$
$$\tag{2.16}$$

Using the chain rule for the left side:

$$e^u \cdot \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\tan x) + \frac{\partial}{\partial x}(\tan y) \tag{2.17}$$

Since y is treated as a constant when taking $\frac{\partial}{\partial x}$:

$$e^u \cdot \frac{\partial u}{\partial x} = \sec^2 x + 0 \tag{2.18}$$
$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{e^u} \tag{2.19}$$

Step 2: Similarly, compute $\frac{\partial u}{\partial y}$.

$$\frac{\partial}{\partial y}(e^u) = \frac{\partial}{\partial y}(\tan x + \tan y) \tag{2.20}$$

$$e^u \cdot \frac{\partial u}{\partial y} = 0 + \sec^2 y \tag{2.21}$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{e^u} \tag{2.22}$$

Step 3: Now we need to evaluate $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y}$.
Substituting our expressions:

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} = \sin 2x \cdot \frac{\sec^2 x}{e^u} + \sin 2y \cdot \frac{\sec^2 y}{e^u} \tag{2.23}$$

$$= \frac{1}{e^u}\left(\sin 2x \cdot \sec^2 x + \sin 2y \cdot \sec^2 y\right) \tag{2.24}$$

Step 4: Use trigonometric identities to simplify.
Recall the double angle formula: $\sin 2x = 2\sin x\cos x$
Also, $\sec^2 x = \frac{1}{\cos^2 x}$
Therefore:

$$\sin 2x \cdot \sec^2 x = 2\sin x\cos x \cdot \frac{1}{\cos^2 x} \tag{2.25}$$

$$= 2\sin x \cdot \frac{1}{\cos x} \tag{2.26}$$

$$= 2\tan x \tag{2.27}$$

Similarly, $\sin 2y \cdot \sec^2 y = 2\tan y$
Step 5: Substitute back into our expression.

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} = \frac{1}{e^u}(2\tan x + 2\tan y) \tag{2.28}$$

$$= \frac{2}{e^u}(\tan x + \tan y) \tag{2.29}$$

Step 6: Recall our original equation $e^u = \tan x + \tan y$.
Therefore:

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} = \frac{2}{e^u} \cdot e^u \tag{2.30}$$

$$= 2 \tag{2.31}$$

This proves that $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} = 2$.

Example 2: Second Order Partial Derivatives

Problem: If $U = (x^2 - y^2)f(xy)$, then show that $U_{xx} + U_{yy} = (x^4 - y^4)f''(xy)$

Detailed Solution

We'll find the second partial derivatives of U and combine them as required.
Step 1: First, let's set $z = xy$ to simplify our work. Then $U = (x^2 - y^2)f(z)$ where $z = xy$.
Step 2: Find the first partial derivative with respect to x.
Using the product rule:

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}[(x^2 - y^2)f(z)] \tag{2.32}$$

$$= \frac{\partial}{\partial x}(x^2 - y^2) \cdot f(z) + (x^2 - y^2) \cdot \frac{\partial}{\partial x}[f(z)] \tag{2.33}$$

For the first term: $\frac{\partial}{\partial x}(x^2 - y^2) = 2x$
For the second term, we need to use the chain rule since $z = xy$:

$$\frac{\partial}{\partial x}[f(z)] = f'(z) \cdot \frac{\partial z}{\partial x} \tag{2.34}$$

$$= f'(z) \cdot y \tag{2.35}$$

Substituting back:

$$\frac{\partial U}{\partial x} = 2x \cdot f(z) + (x^2 - y^2) \cdot f'(z) \cdot y \tag{2.36}$$

$$= 2xf(z) + y(x^2 - y^2)f'(z) \tag{2.37}$$

Step 3: Find the second partial derivative with respect to x (i.e., U_{xx}).

$$U_{xx} = \frac{\partial}{\partial x}\left[2xf(z) + y(x^2 - y^2)f'(z)\right] \tag{2.38}$$

We'll handle each term separately:

$$\frac{\partial}{\partial x}[2xf(z)] = 2f(z) + 2x\frac{\partial}{\partial x}[f(z)] \tag{2.39}$$

$$= 2f(z) + 2xf'(z) \cdot y \tag{2.40}$$

$$= 2f(z) + 2xyf'(z) \tag{2.41}$$

For the second term:

$$\frac{\partial}{\partial x}[y(x^2 - y^2)f'(z)] = y\frac{\partial}{\partial x}[(x^2 - y^2)f'(z)] \tag{2.42}$$

$$= y\left[\frac{\partial}{\partial x}(x^2 - y^2) \cdot f'(z) + (x^2 - y^2) \cdot \frac{\partial}{\partial x}[f'(z)]\right] \tag{2.43}$$

$$= y\left[2x \cdot f'(z) + (x^2 - y^2) \cdot \frac{\partial}{\partial x}[f'(z)]\right] \tag{2.44}$$

Using the chain rule again:

$$\frac{\partial}{\partial x}[f'(z)] = f''(z) \cdot \frac{\partial z}{\partial x} \tag{2.45}$$

$$= f''(z) \cdot y \tag{2.46}$$

Therefore:

$$\frac{\partial}{\partial x}[y(x^2-y^2)f'(z)] = y[2x\cdot f'(z) + (x^2-y^2)\cdot f''(z)\cdot y] \tag{2.47}$$
$$= 2xy\cdot f'(z) + y^2(x^2-y^2)f''(z) \tag{2.48}$$

Combining the results:

$$U_{xx} = 2f(z) + 2xyf'(z) + 2xy\cdot f'(z) + y^2(x^2-y^2)f''(z) \tag{2.49}$$
$$= 2f(z) + 4xyf'(z) + y^2(x^2-y^2)f''(z) \tag{2.50}$$

Step 4: Similarly, find the first partial derivative with respect to y.
Using the product rule:

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}[(x^2-y^2)f(z)] \tag{2.51}$$
$$= \frac{\partial}{\partial y}(x^2-y^2)\cdot f(z) + (x^2-y^2)\cdot\frac{\partial}{\partial y}[f(z)] \tag{2.52}$$

For the first term: $\frac{\partial}{\partial y}(x^2-y^2) = -2y$
For the second term, using the chain rule:

$$\frac{\partial}{\partial y}[f(z)] = f'(z)\cdot\frac{\partial z}{\partial y} \tag{2.53}$$
$$= f'(z)\cdot x \tag{2.54}$$

Therefore:

$$\frac{\partial U}{\partial y} = -2y\cdot f(z) + (x^2-y^2)\cdot f'(z)\cdot x \tag{2.55}$$
$$= -2yf(z) + x(x^2-y^2)f'(z) \tag{2.56}$$

Step 5: Find the second partial derivative with respect to y (i.e., U_{yy}).

$$U_{yy} = \frac{\partial}{\partial y}\left[-2yf(z) + x(x^2-y^2)f'(z)\right] \tag{2.57}$$

Again, handling each term:

$$\frac{\partial}{\partial y}[-2yf(z)] = -2f(z) - 2y\frac{\partial}{\partial y}[f(z)] \tag{2.58}$$
$$= -2f(z) - 2yf'(z)\cdot x \tag{2.59}$$
$$= -2f(z) - 2xyf'(z) \tag{2.60}$$

For the second term:

$$\frac{\partial}{\partial y}[x(x^2-y^2)f'(z)] = x\frac{\partial}{\partial y}[(x^2-y^2)f'(z)] \tag{2.61}$$
$$= x\left[\frac{\partial}{\partial y}(x^2-y^2)\cdot f'(z) + (x^2-y^2)\cdot\frac{\partial}{\partial y}[f'(z)]\right] \tag{2.62}$$
$$= x\left[-2y\cdot f'(z) + (x^2-y^2)\cdot\frac{\partial}{\partial y}[f'(z)]\right] \tag{2.63}$$

Using the chain rule:

$$\frac{\partial}{\partial y}[f'(z)] = f''(z) \cdot \frac{\partial z}{\partial y} \tag{2.64}$$
$$= f''(z) \cdot x \tag{2.65}$$

Therefore:

$$\frac{\partial}{\partial y}[x(x^2 - y^2)f'(z)] = x[-2y \cdot f'(z) + (x^2 - y^2) \cdot f''(z) \cdot x] \tag{2.66}$$
$$= -2xy \cdot f'(z) + x^2(x^2 - y^2)f''(z) \tag{2.67}$$

Combining the results:

$$U_{yy} = -2f(z) - 2xyf'(z) - 2xy \cdot f'(z) + x^2(x^2 - y^2)f''(z) \tag{2.68}$$
$$= -2f(z) - 4xyf'(z) + x^2(x^2 - y^2)f''(z) \tag{2.69}$$

Step 6: Add the second partial derivatives to find $U_{xx} + U_{yy}$.

$$U_{xx} + U_{yy} = \left[2f(z) + 4xyf'(z) + y^2(x^2 - y^2)f''(z)\right] + \left[-2f(z) - 4xyf'(z) + x^2(x^2 - y^2)f''(z)\right] \tag{2.70}$$
$$= 2f(z) - 2f(z) + 4xyf'(z) - 4xyf'(z) + y^2(x^2 - y^2)f''(z) + x^2(x^2 - y^2)f''(z) \tag{2.71}$$

The first four terms cancel out:

$$U_{xx} + U_{yy} = y^2(x^2 - y^2)f''(z) + x^2(x^2 - y^2)f''(z) \tag{2.72}$$
$$= (x^2 - y^2)f''(z)(y^2 + x^2) \tag{2.73}$$
$$= (x^2 - y^2)f''(xy)(x^2 + y^2) \tag{2.74}$$

Step 7: Simplify further by expanding.

$$U_{xx} + U_{yy} = (x^2 - y^2)f''(xy)(x^2 + y^2) \tag{2.75}$$
$$= (x^4 + x^2y^2 - x^2y^2 - y^4)f''(xy) \tag{2.76}$$
$$= (x^4 - y^4)f''(xy) \tag{2.77}$$

Thus, we have proven that $U_{xx} + U_{yy} = (x^4 - y^4)f''(xy)$.

Example 3: Differential Operator

Problem: If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

Detailed Solution

For simplicity, let's denote the expression inside the logarithm as $f(x, y, z)$:

$$f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$$

Step 1: Let's verify a key property of $f(x, y, z)$.

Note that $f(x, y, z)$ can be factored as:

$$f(x, y, z) = x^3 + y^3 + z^3 - 3xyz \tag{2.78}$$
$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz) \tag{2.79}$$

This factorization is a well-known algebraic identity, which we can verify by expansion.
Step 2: Let's define $D = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ for brevity.
We need to compute D^2u.
Step 3: First, we'll find Du.
Using the chain rule:

$$Du = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \tag{2.80}$$
$$= \frac{1}{f} \cdot \frac{\partial f}{\partial x} + \frac{1}{f} \cdot \frac{\partial f}{\partial y} + \frac{1}{f} \cdot \frac{\partial f}{\partial z} \tag{2.81}$$
$$= \frac{1}{f} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \right) \tag{2.82}$$

Let's compute the partial derivatives of f:

$$\frac{\partial f}{\partial x} = 3x^2 - 3yz \tag{2.83}$$
$$\frac{\partial f}{\partial y} = 3y^2 - 3xz \tag{2.84}$$
$$\frac{\partial f}{\partial z} = 3z^2 - 3xy \tag{2.85}$$

Therefore:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 3x^2 - 3yz + 3y^2 - 3xz + 3z^2 - 3xy \tag{2.86}$$
$$= 3(x^2 + y^2 + z^2 - xy - yz - xz) \tag{2.87}$$

From Step 1, we know that $f = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)$
Therefore:

$$Du = \frac{1}{f} \cdot 3(x^2 + y^2 + z^2 - xy - yz - xz) \tag{2.88}$$
$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - xz)}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)} \tag{2.89}$$
$$= \frac{3}{x + y + z} \tag{2.90}$$

Step 4: Now, we'll find $D^2u = D(Du) = D\left(\frac{3}{x+y+z}\right)$.

Using the quotient rule:

$$\begin{aligned}
D\left(\frac{3}{x+y+z}\right) &= 3 \cdot D\left(\frac{1}{x+y+z}\right) && (2.91)\\
&= 3 \cdot \frac{D(1)\cdot(x+y+z) - 1 \cdot D(x+y+z)}{(x+y+z)^2} && (2.92)\\
&= 3 \cdot \frac{0-(1+1+1)}{(x+y+z)^2} && (2.93)\\
&= 3 \cdot \frac{-3}{(x+y+z)^2} && (2.94)\\
&= \frac{-9}{(x+y+z)^2} && (2.95)
\end{aligned}$$

Therefore, we have proven:

$$D^2u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Example 4

Problem: If $z = \tan(y+ax) + (y-ax)^{3/2}$, find the value of $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2}$.

Detailed Solution

For clarity, let's break the function into two parts:

$$z = \underbrace{\tan(y+ax)}_{u(x,y)} + \underbrace{(y-ax)^{3/2}}_{v(x,y)}$$

Step 1: Let's find the second-order partial derivatives of the first term $u(x,y) = \tan(y+ax)$.
Let's define $w = y + ax$, so $u = \tan(w)$.
First, we'll find $\frac{\partial u}{\partial x}$:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial \tan(w)}{\partial w} \cdot \frac{\partial w}{\partial x} && (2.96)\\
&= \sec^2(w) \cdot a && (2.97)\\
&= a\sec^2(y+ax) && (2.98)
\end{aligned}$$

Now for $\frac{\partial^2 u}{\partial x^2}$:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left[a\sec^2(y+ax)\right] && (2.99)\\
&= a\frac{\partial}{\partial x}\left[\sec^2(y+ax)\right] && (2.100)\\
&= a \cdot \frac{\partial \sec^2(w)}{\partial w} \cdot \frac{\partial w}{\partial x} && (2.101)\\
&= a \cdot 2\sec^2(w)\tan(w) \cdot a && (2.102)\\
&= 2a^2\sec^2(y+ax)\tan(y+ax) && (2.103)
\end{aligned}$$

Next, we'll find $\frac{\partial u}{\partial y}$:

$$\frac{\partial u}{\partial y} = \frac{\partial \tan(w)}{\partial w} \cdot \frac{\partial w}{\partial y} \tag{2.104}$$
$$= \sec^2(w) \cdot 1 \tag{2.105}$$
$$= \sec^2(y + ax) \tag{2.106}$$

And for $\frac{\partial^2 u}{\partial y^2}$:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\sec^2(y + ax)\right] \tag{2.107}$$
$$= \frac{\partial \sec^2(w)}{\partial w} \cdot \frac{\partial w}{\partial y} \tag{2.108}$$
$$= 2\sec^2(w)\tan(w) \cdot 1 \tag{2.109}$$
$$= 2\sec^2(y + ax)\tan(y + ax) \tag{2.110}$$

Step 2: Now let's find the second-order partial derivatives of the second term $v(x, y) = (y - ax)^{3/2}$.
Let's define $t = y - ax$, so $v = t^{3/2}$.
First, we'll find $\frac{\partial v}{\partial x}$:

$$\frac{\partial v}{\partial x} = \frac{\partial t^{3/2}}{\partial t} \cdot \frac{\partial t}{\partial x} \tag{2.111}$$
$$= \frac{3}{2}t^{1/2} \cdot (-a) \tag{2.112}$$
$$= -\frac{3a}{2}(y - ax)^{1/2} \tag{2.113}$$

Now for $\frac{\partial^2 v}{\partial x^2}$:

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left[-\frac{3a}{2}(y - ax)^{1/2}\right] \tag{2.114}$$
$$= -\frac{3a}{2} \cdot \frac{\partial}{\partial x}\left[(y - ax)^{1/2}\right] \tag{2.115}$$
$$= -\frac{3a}{2} \cdot \frac{1}{2}(y - ax)^{-1/2} \cdot (-a) \tag{2.116}$$
$$= \frac{3a^2}{4}(y - ax)^{-1/2} \tag{2.117}$$

Next, we'll find $\frac{\partial v}{\partial y}$:

$$\frac{\partial v}{\partial y} = \frac{\partial t^{3/2}}{\partial t} \cdot \frac{\partial t}{\partial y} \tag{2.118}$$
$$= \frac{3}{2}t^{1/2} \cdot 1 \tag{2.119}$$
$$= \frac{3}{2}(y - ax)^{1/2} \tag{2.120}$$

And for $\frac{\partial^2 v}{\partial y^2}$:

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{3}{2}(y-ax)^{1/2}\right] \tag{2.121}$$
$$= \frac{3}{2}\cdot\frac{\partial}{\partial y}\left[(y-ax)^{1/2}\right] \tag{2.122}$$
$$= \frac{3}{2}\cdot\frac{1}{2}(y-ax)^{-1/2}\cdot 1 \tag{2.123}$$
$$= \frac{3}{4}(y-ax)^{-1/2} \tag{2.124}$$

Step 3: Calculate $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2}$.
For the first part $u(x,y) = \tan(y+ax)$:

$$\frac{\partial^2 u}{\partial x^2} - a^2\frac{\partial^2 u}{\partial y^2} = 2a^2\sec^2(y+ax)\tan(y+ax) - a^2\cdot 2\sec^2(y+ax)\tan(y+ax) \tag{2.125}$$
$$= 0 \tag{2.126}$$

For the second part $v(x,y) = (y-ax)^{3/2}$:

$$\frac{\partial^2 v}{\partial x^2} - a^2\frac{\partial^2 v}{\partial y^2} = \frac{3a^2}{4}(y-ax)^{-1/2} - a^2\cdot\frac{3}{4}(y-ax)^{-1/2} \tag{2.127}$$
$$= 0 \tag{2.128}$$

Therefore, $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0 + 0 = 0$.

Example: Equality of Mixed Partial Derivatives

Problem: If $u = \log(x^2+y^2)$, verify that $u_{xy} = u_{yx}$.

Detailed Solution

This problem asks us to verify Clairaut's theorem, which states that if a function has continuous second-order partial derivatives, then the mixed partial derivatives are equal, i.e., $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.
Let's compute these derivatives for $u = \log(x^2+y^2)$.
Step 1: First, find $\frac{\partial u}{\partial x}$.
Using the chain rule:

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial x}(x^2+y^2) \tag{2.129}$$
$$= \frac{1}{x^2+y^2}\cdot 2x \tag{2.130}$$
$$= \frac{2x}{x^2+y^2} \tag{2.131}$$

Step 2: Now, find $\frac{\partial^2 u}{\partial y \partial x}$ (i.e., u_{yx}).

We differentiate $\frac{\partial u}{\partial x}$ with respect to y:

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right) \tag{2.132}$$

$$\tag{2.133}$$

Using the quotient rule:

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(x^2+y^2)\cdot\frac{\partial}{\partial y}(2x) - 2x\cdot\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} \tag{2.134}$$

$$= \frac{(x^2+y^2)\cdot 0 - 2x\cdot 2y}{(x^2+y^2)^2} \tag{2.135}$$

$$= \frac{-4xy}{(x^2+y^2)^2} \tag{2.136}$$

Step 3: Next, find $\frac{\partial u}{\partial y}$.
Using the chain rule again:

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial y}(x^2+y^2) \tag{2.137}$$

$$= \frac{1}{x^2+y^2}\cdot 2y \tag{2.138}$$

$$= \frac{2y}{x^2+y^2} \tag{2.139}$$

Step 4: Finally, find $\frac{\partial^2 u}{\partial x \partial y}$ (i.e., u_{xy}).
We differentiate $\frac{\partial u}{\partial y}$ with respect to x:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}\right) \tag{2.140}$$

$$\tag{2.141}$$

Using the quotient rule:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{(x^2+y^2)\cdot\frac{\partial}{\partial x}(2y) - 2y\cdot\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} \tag{2.142}$$

$$= \frac{(x^2+y^2)\cdot 0 - 2y\cdot 2x}{(x^2+y^2)^2} \tag{2.143}$$

$$= \frac{-4xy}{(x^2+y^2)^2} \tag{2.144}$$

Step 5: Compare the results.
We found:

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-4xy}{(x^2+y^2)^2} \tag{2.145}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-4xy}{(x^2+y^2)^2} \tag{2.146}$$

Since $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$, we have verified that $u_{xy} = u_{yx}$ for the function $u = \log(x^2+y^2)$.
This illustrates Clairaut's Theorem, which states that for a function with continuous mixed partial derivatives, the order of differentiation doesn't matter.

Example 5

Problem: If $u = \tan^{-1}\left(\frac{x}{y}\right)$, then show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Detailed Solution

We need to verify that the mixed partial derivatives of $u = \tan^{-1}\left(\frac{x}{y}\right)$ are equal.

Step 1: First, find $\frac{\partial u}{\partial x}$.
Using the chain rule with $\frac{d}{dt}\tan^{-1}(t) = \frac{1}{1+t^2}$:

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right) \tag{2.147}$$

$$= \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} \tag{2.148}$$

$$= \frac{1}{\frac{y^2+x^2}{y^2}} \cdot \frac{1}{y} \tag{2.149}$$

$$= \frac{y^2}{y^2+x^2} \cdot \frac{1}{y} \tag{2.150}$$

$$= \frac{y}{y^2+x^2} \tag{2.151}$$

Step 2: Now, find $\frac{\partial^2 u}{\partial y \partial x}$.
We differentiate $\frac{\partial u}{\partial x}$ with respect to y:

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{y}{y^2+x^2}\right) \tag{2.152}$$

Using the quotient rule:

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(y^2+x^2)\cdot\frac{\partial}{\partial y}(y) - y\cdot\frac{\partial}{\partial y}(y^2+x^2)}{(y^2+x^2)^2} \tag{2.153}$$

$$= \frac{(y^2+x^2)\cdot 1 - y\cdot 2y}{(y^2+x^2)^2} \tag{2.154}$$

$$= \frac{y^2+x^2-2y^2}{(y^2+x^2)^2} \tag{2.155}$$

$$= \frac{x^2-y^2}{(y^2+x^2)^2} \tag{2.156}$$

Step 3: Next, find $\frac{\partial u}{\partial y}$.
Using the chain rule:

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) \tag{2.157}$$

$$= \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) \tag{2.158}$$

$$= \frac{y^2}{y^2+x^2} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) \tag{2.159}$$

Using the quotient rule for $\frac{\partial}{\partial y}\left(\frac{x}{y}\right)$:

$$\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{y \cdot \frac{\partial}{\partial y}(x) - x \cdot \frac{\partial}{\partial y}(y)}{y^2} \tag{2.160}$$

$$= \frac{y \cdot 0 - x \cdot 1}{y^2} \tag{2.161}$$

$$= -\frac{x}{y^2} \tag{2.162}$$

Therefore:

$$\frac{\partial u}{\partial y} = \frac{y^2}{y^2 + x^2} \cdot \left(-\frac{x}{y^2}\right) \tag{2.163}$$

$$= -\frac{x}{y^2 + x^2} \tag{2.164}$$

Step 4: Finally, find $\frac{\partial^2 u}{\partial x \partial y}$.
We differentiate $\frac{\partial u}{\partial y}$ with respect to x:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-\frac{x}{y^2 + x^2}\right) \tag{2.165}$$

Using the quotient rule:

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{(y^2 + x^2) \cdot \frac{\partial}{\partial x}(x) - x \cdot \frac{\partial}{\partial x}(y^2 + x^2)}{(y^2 + x^2)^2} \tag{2.166}$$

$$= -\frac{(y^2 + x^2) \cdot 1 - x \cdot 2x}{(y^2 + x^2)^2} \tag{2.167}$$

$$= -\frac{y^2 + x^2 - 2x^2}{(y^2 + x^2)^2} \tag{2.168}$$

$$= -\frac{y^2 - x^2}{(y^2 + x^2)^2} \tag{2.169}$$

$$= \frac{x^2 - y^2}{(y^2 + x^2)^2} \tag{2.170}$$

Step 5: Compare the results.
We found:

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{x^2 - y^2}{(y^2 + x^2)^2} \tag{2.171}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{(y^2 + x^2)^2} \tag{2.172}$$

Since $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$, we have verified that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the function $u = \tan^{-1}\left(\frac{x}{y}\right)$.
This confirms Clairaut's Theorem, which states that if the mixed partial derivatives are continuous, then they are equal regardless of the order of differentiation.

Example 6

Problem: If $u = 4e^{-6x}\sin(pt-6x)$ satisfies $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, then find the value of p.

Detailed Solution

We need to find the value of p such that the function $u = 4e^{-6x}\sin(pt-6x)$ satisfies the differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.
Step 1: Find $\frac{\partial u}{\partial t}$.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[4e^{-6x}\sin(pt-6x)\right] \tag{2.173}$$
$$= 4e^{-6x}\cdot\frac{\partial}{\partial t}[\sin(pt-6x)] \tag{2.174}$$
$$= 4e^{-6x}\cdot\cos(pt-6x)\cdot\frac{\partial}{\partial t}(pt-6x) \tag{2.175}$$
$$= 4e^{-6x}\cdot\cos(pt-6x)\cdot p \tag{2.176}$$
$$= 4pe^{-6x}\cos(pt-6x) \tag{2.177}$$

Step 2: Find $\frac{\partial u}{\partial x}$ (we'll need this for the second derivative).

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[4e^{-6x}\sin(pt-6x)\right] \tag{2.178}$$
$$\tag{2.179}$$

Using the product rule:

$$\frac{\partial u}{\partial x} = 4\cdot\frac{\partial}{\partial x}[e^{-6x}]\cdot\sin(pt-6x) + 4e^{-6x}\cdot\frac{\partial}{\partial x}[\sin(pt-6x)] \tag{2.180}$$
$$= 4\cdot(-6e^{-6x})\cdot\sin(pt-6x) + 4e^{-6x}\cdot\cos(pt-6x)\cdot\frac{\partial}{\partial x}(pt-6x) \tag{2.181}$$
$$= -24e^{-6x}\sin(pt-6x) + 4e^{-6x}\cdot\cos(pt-6x)\cdot(-6) \tag{2.182}$$
$$= -24e^{-6x}\sin(pt-6x) - 24e^{-6x}\cos(pt-6x) \tag{2.183}$$

Step 3: Find $\frac{\partial^2 u}{\partial x^2}$.
We differentiate $\frac{\partial u}{\partial x}$ with respect to x:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[-24e^{-6x}\sin(pt-6x) - 24e^{-6x}\cos(pt-6x)\right] \tag{2.184}$$
$$\tag{2.185}$$

Let's handle each term separately:
For the first term:

$$\frac{\partial}{\partial x}\left[-24e^{-6x}\sin(pt-6x)\right] = -24\cdot\frac{\partial}{\partial x}[e^{-6x}]\cdot\sin(pt-6x) - 24e^{-6x}\cdot\frac{\partial}{\partial x}[\sin(pt-6x)] \tag{2.186}$$
$$= -24\cdot(-6e^{-6x})\cdot\sin(pt-6x) - 24e^{-6x}\cdot\cos(pt-6x)\cdot(-6) \tag{2.187}$$
$$= 144e^{-6x}\sin(pt-6x) + 144e^{-6x}\cos(pt-6x) \tag{2.188}$$

For the second term:

$$\frac{\partial}{\partial x}\left[-24e^{-6x}\cos(pt-6x)\right] = -24\cdot\frac{\partial}{\partial x}[e^{-6x}]\cdot\cos(pt-6x) - 24e^{-6x}\cdot\frac{\partial}{\partial x}[\cos(pt-6x)] \quad (2.189)$$

$$= -24\cdot(-6e^{-6x})\cdot\cos(pt-6x) - 24e^{-6x}\cdot(-\sin(pt-6x))\cdot(-6) \quad (2.190)$$

$$= 144e^{-6x}\cos(pt-6x) - 144e^{-6x}\sin(pt-6x) \quad (2.191)$$

Combining the two terms:

$$\frac{\partial^2 u}{\partial x^2} = 144e^{-6x}\sin(pt-6x) + 144e^{-6x}\cos(pt-6x) + 144e^{-6x}\cos(pt-6x) - 144e^{-6x}\sin(pt-6x) \quad (2.192)$$

$$= 144e^{-6x}\sin(pt-6x) - 144e^{-6x}\sin(pt-6x) + 144e^{-6x}\cos(pt-6x) + 144e^{-6x}\cos(pt-6x) \quad (2.193)$$

$$= 0 + 288e^{-6x}\cos(pt-6x) \quad (2.194)$$

$$= 288e^{-6x}\cos(pt-6x) \quad (2.195)$$

Step 4: Apply the differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.
We have:

$$\frac{\partial u}{\partial t} = 4pe^{-6x}\cos(pt-6x) \quad (2.196)$$

$$\frac{\partial^2 u}{\partial x^2} = 288e^{-6x}\cos(pt-6x) \quad (2.197)$$

Since these must be equal:

$$4pe^{-6x}\cos(pt-6x) = 288e^{-6x}\cos(pt-6x) \quad (2.198)$$

(2.199)

Dividing both sides by $e^{-6x}\cos(pt-6x)$ (assuming it's non-zero):

$$4p = 288 \quad (2.200)$$

$$p = \frac{288}{4} = 72 \quad (2.201)$$

Therefore, the value of p is 72.

Example 7

Problem: If $U = f(x^2+y^2)$ then, show that $y^2U_{xx} - 2xyU_{xy} + x^2U_{yy} = xU_x + yU_y$.

Detailed Solution

Step 1: Let's define $t = x^2 + y^2$ to simplify our work. Then $U = f(t)$.
Step 2: Find the first-order partial derivatives of U using the chain rule.

For $\frac{\partial U}{\partial x}$:

$$\frac{\partial U}{\partial x} = \frac{dU}{dt} \cdot \frac{\partial t}{\partial x} \tag{2.202}$$
$$= f'(t) \cdot 2x \tag{2.203}$$
$$= 2xf'(t) \tag{2.204}$$

Similarly, for $\frac{\partial U}{\partial y}$:

$$\frac{\partial U}{\partial y} = \frac{dU}{dt} \cdot \frac{\partial t}{\partial y} \tag{2.205}$$
$$= f'(t) \cdot 2y \tag{2.206}$$
$$= 2yf'(t) \tag{2.207}$$

Step 3: Find the second-order partial derivatives.
For $\frac{\partial^2 U}{\partial x^2}$ (i.e., U_{xx}):

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}(2xf'(t)) \tag{2.208}$$
$$= 2f'(t) + 2x \cdot \frac{\partial}{\partial x}(f'(t)) \tag{2.209}$$
$$= 2f'(t) + 2x \cdot f''(t) \cdot \frac{\partial t}{\partial x} \tag{2.210}$$
$$= 2f'(t) + 2x \cdot f''(t) \cdot 2x \tag{2.211}$$
$$= 2f'(t) + 4x^2 f''(t) \tag{2.212}$$

For the mixed derivative $\frac{\partial^2 U}{\partial x \partial y}$ (i.e., U_{xy}):

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial y}(2xf'(t)) \tag{2.213}$$
$$= 2x \cdot \frac{\partial}{\partial y}(f'(t)) \tag{2.214}$$
$$= 2x \cdot f''(t) \cdot \frac{\partial t}{\partial y} \tag{2.215}$$
$$= 2x \cdot f''(t) \cdot 2y \tag{2.216}$$
$$= 4xyf''(t) \tag{2.217}$$

For $\frac{\partial^2 U}{\partial y^2}$ (i.e., U_{yy}):

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}(2yf'(t)) \tag{2.218}$$
$$= 2f'(t) + 2y \cdot \frac{\partial}{\partial y}(f'(t)) \tag{2.219}$$
$$= 2f'(t) + 2y \cdot f''(t) \cdot \frac{\partial t}{\partial y} \tag{2.220}$$
$$= 2f'(t) + 2y \cdot f''(t) \cdot 2y \tag{2.221}$$
$$= 2f'(t) + 4y^2 f''(t) \tag{2.222}$$

Step 4: Evaluate the left side of the equation $y^2 U_{xx} - 2xyU_{xy} + x^2 U_{yy}$.

Substituting the expressions we found:

$$
\begin{aligned}
y^2U_{xx} - 2xyU_{xy} + x^2U_{yy} &= y^2(2f'(t) + 4x^2f''(t)) - 2xy(4xyf''(t)) + x^2(2f'(t) + 4y^2f''(t)) && (2.223)\\
&= 2y^2f'(t) + 4y^2x^2f''(t) - 8x^2y^2f''(t) + 2x^2f'(t) + 4x^2y^2f''(t) && (2.224)\\
&= 2y^2f'(t) + 2x^2f'(t) + 4y^2x^2f''(t) - 8x^2y^2f''(t) + 4x^2y^2f''(t) && (2.225)\\
&= 2y^2f'(t) + 2x^2f'(t) + 0 && (2.226)\\
&= 2(x^2 + y^2)f'(t) && (2.227)\\
&= 2tf'(t) && (2.228)
\end{aligned}
$$

Step 5: Evaluate the right side of the equation $xU_x + yU_y$.
Substituting the expressions for the first-order derivatives:

$$
\begin{aligned}
xU_x + yU_y &= x(2xf'(t)) + y(2yf'(t)) && (2.229)\\
&= 2x^2f'(t) + 2y^2f'(t) && (2.230)\\
&= 2(x^2 + y^2)f'(t) && (2.231)\\
&= 2tf'(t) && (2.232)
\end{aligned}
$$

Step 6: Compare both sides.
Since both the left and right sides equal $2tf'(t)$, we have proven that:

$$y^2U_{xx} - 2xyU_{xy} + x^2U_{yy} = xU_x + yU_y$$

Note: This is a special case of Euler's homogeneous differential equation in partial derivatives. The equation we've proven shows that for radially symmetric functions (functions that depend only on the distance from the origin), certain combinations of second derivatives equal a weighted sum of first derivatives.