

Chapter 4

Vector Differential Calculus III: Divergence and Curl

This chapter explores two fundamental differential operations on vector fields: divergence and curl. These operations, along with the gradient studied in the previous chapter, form the cornerstone of vector calculus and have profound applications in physics and engineering.

4.1 Divergence of a Vector Field

4.1.1 Definition and Properties

The divergence of a vector field measures the "outflowing" or "expansion" rate of the vector field at a point.

Definition 4.1 (Divergence). *The divergence of a vector field $\bar{\mathbf{F}}(x, y, z) = F_1(x, y, z)\hat{\mathbf{i}} + F_2(x, y, z)\hat{\mathbf{j}} + F_3(x, y, z)\hat{\mathbf{k}}$ is a scalar field defined as:*

$$\operatorname{div} \bar{\mathbf{F}} = \nabla \cdot \bar{\mathbf{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad (4.1)$$

Properties of Divergence

For vector fields $\bar{\mathbf{F}}$ and $\bar{\mathbf{G}}$, scalar field ϕ , and constant c :

1. $\nabla \cdot (\bar{\mathbf{F}} + \bar{\mathbf{G}}) = \nabla \cdot \bar{\mathbf{F}} + \nabla \cdot \bar{\mathbf{G}}$ (Linearity)
2. $\nabla \cdot (c\bar{\mathbf{F}}) = c(\nabla \cdot \bar{\mathbf{F}})$ (Scalar multiplication)
3. $\nabla \cdot (\phi\bar{\mathbf{F}}) = \phi(\nabla \cdot \bar{\mathbf{F}}) + \nabla\phi \cdot \bar{\mathbf{F}}$ (Product rule)
4. $\nabla \cdot (\bar{\mathbf{F}} \times \bar{\mathbf{G}}) = \bar{\mathbf{G}} \cdot (\nabla \times \bar{\mathbf{F}}) - \bar{\mathbf{F}} \cdot (\nabla \times \bar{\mathbf{G}})$

Computing Divergence

Calculate the divergence of $\bar{\mathbf{F}}(x, y, z) = x^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$.

Applying the definition:

$$\nabla \cdot \bar{\mathbf{F}} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(z^2) \quad (4.2)$$

$$= 2x + x + 2z \quad (4.3)$$

$$= 3x + 2z \quad (4.4)$$

4.2 Physical Interpretation of Divergence

4.2.1 Flux Density and Source/Sink Behavior

The divergence has a rich physical interpretation related to the concept of flux density.

Definition 4.2 (Flux Density). *The divergence $\nabla \cdot \vec{F}$ at a point represents the net outward flux of the vector field per unit volume as the volume around the point approaches zero.*

This interpretation leads to the following characterizations:

- $\nabla \cdot \vec{F} > 0$: The point acts as a source (outflow exceeds inflow)
- $\nabla \cdot \vec{F} < 0$: The point acts as a sink (inflow exceeds outflow)
- $\nabla \cdot \vec{F} = 0$: The field is locally conservative (outflow equals inflow)

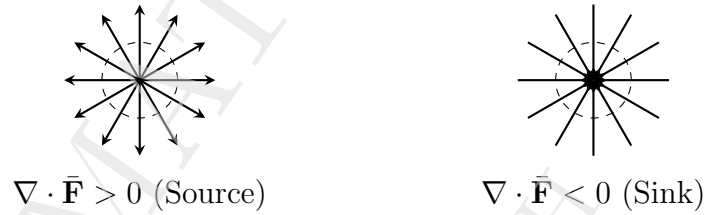


Figure 4.1: Source and sink behavior in vector fields

4.2.2 Applications

Fluid Dynamics

In fluid dynamics, if \vec{v} represents the velocity field of a fluid:

- $\nabla \cdot \vec{v}$ represents the rate of expansion or contraction of the fluid
- $\nabla \cdot \vec{v} = 0$ indicates an incompressible fluid

Electromagnetism

In electromagnetism, according to Gauss's law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (4.5)$$

where \vec{E} is the electric field, ρ is the charge density, and ϵ_0 is the permittivity of free space. This shows that:

- Electric field diverges from positive charges (sources)
- Electric field converges to negative charges (sinks)
- In charge-free regions, $\nabla \cdot \vec{E} = 0$

For the magnetic field \vec{B} , Gauss's law for magnetism states:

$$\nabla \cdot \vec{B} = 0 \quad (4.6)$$

indicating that magnetic monopoles do not exist in classical electromagnetism.

Heat Conduction

In heat transfer, if \vec{q} is the heat flux vector:

$$\nabla \cdot \vec{q} = -c\rho \frac{\partial T}{\partial t} \quad (4.7)$$

where c is the specific heat capacity, ρ is the density, and T is the temperature. This equation indicates that divergence of heat flux is related to the rate of temperature change.

4.3 Curl of a Vector Field

4.3.1 Definition and Properties

The curl of a vector field measures the rotational or "spinning" tendency of the field at a point.

Definition 4.3 (Curl). *The curl of a vector field $\bar{\mathbf{F}}(x, y, z) = F_1(x, y, z)\hat{\mathbf{i}} + F_2(x, y, z)\hat{\mathbf{j}} + F_3(x, y, z)\hat{\mathbf{k}}$ is a vector field defined as:*

$$\text{curl } \bar{\mathbf{F}} = \nabla \times \bar{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \quad (4.8)$$

This expands to:

$$\nabla \times \bar{\mathbf{F}} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{\mathbf{k}} \quad (4.9)$$

Properties of Curl

For vector fields $\bar{\mathbf{F}}$ and $\bar{\mathbf{G}}$, scalar field ϕ , and constant c :

1. $\nabla \times (\bar{\mathbf{F}} + \bar{\mathbf{G}}) = \nabla \times \bar{\mathbf{F}} + \nabla \times \bar{\mathbf{G}}$ (Linearity)
2. $\nabla \times (c\bar{\mathbf{F}}) = c(\nabla \times \bar{\mathbf{F}})$ (Scalar multiplication)
3. $\nabla \times (\phi\bar{\mathbf{F}}) = \phi(\nabla \times \bar{\mathbf{F}}) + (\nabla\phi) \times \bar{\mathbf{F}}$ (Product rule)
4. $\nabla \times (\nabla\phi) = \mathbf{0}$ (Curl of a gradient is zero)
5. $\nabla \cdot (\nabla \times \bar{\mathbf{F}}) = 0$ (Divergence of a curl is zero)

Computing Curl

Calculate the curl of $\bar{\mathbf{F}}(x, y, z) = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}$.

Applying the definition:

$$\nabla \times \bar{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \quad (4.10)$$

$$= \left(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial y}{\partial z} - \frac{\partial x}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial y} \right) \hat{\mathbf{k}} \quad (4.11)$$

$$= (0 - 1)\hat{\mathbf{i}} + (0 - 1)\hat{\mathbf{j}} + (0 - 0)\hat{\mathbf{k}} \quad (4.12)$$

$$= -\hat{\mathbf{i}} - \hat{\mathbf{j}} \quad (4.13)$$

4.4 Physical Interpretation of Curl

4.4.1 Rotation and Angular Velocity

The curl of a vector field at a point represents the rotation or "vorticity" of the field around that point.

Definition 4.4 (Circulation Density). The curl $\nabla \times \bar{\mathbf{F}}$ at a point represents the circulation per unit area in the plane perpendicular to the curl vector, as the area approaches zero.

The magnitude of the curl $|\nabla \times \bar{\mathbf{F}}|$ gives the strength of rotation, while the direction of the curl vector gives the axis of rotation following the right-hand rule.

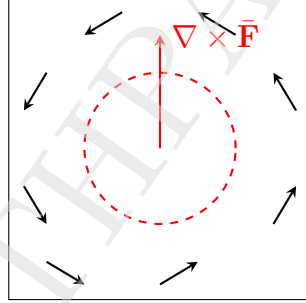


Figure 4.2: Circulation and curl in a vector field

4.4.2 Applications

Fluid Dynamics

In fluid dynamics, if $\bar{\mathbf{v}}$ represents the velocity field of a fluid:

- $\nabla \times \bar{\mathbf{v}}$ represents the vorticity or local rotation of the fluid
- $\nabla \times \bar{\mathbf{v}} = \mathbf{0}$ indicates irrotational flow

Vorticity in Fluid Flow

Consider a fluid rotating like a rigid body with angular velocity ω around the z -axis. The velocity field is:

$$\bar{\mathbf{v}} = -\omega y \hat{\mathbf{i}} + \omega x \hat{\mathbf{j}} \quad (4.14)$$

The curl of this velocity field is:

$$\nabla \times \bar{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} \quad (4.15)$$

$$= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(\omega x) \right) \hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}(-\omega y) - \frac{\partial}{\partial x}(0) \right) \hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y) \right) \hat{\mathbf{k}} \quad (4.16)$$

$$= (0 - 0) \hat{\mathbf{i}} + (0 - 0) \hat{\mathbf{j}} + (\omega + \omega) \hat{\mathbf{k}} \quad (4.17)$$

$$= 2\omega \hat{\mathbf{k}} \quad (4.18)$$

This shows that the vorticity is uniform throughout the fluid and is twice the angular velocity, pointing along the axis of rotation.

Electromagnetism

In electromagnetism, Maxwell's equations relate the curl of the electric and magnetic fields:

$$\nabla \times \bar{\mathbf{E}} = -\frac{\partial \bar{\mathbf{B}}}{\partial t} \quad (4.19)$$

$$\nabla \times \bar{\mathbf{B}} = \mu_0 \bar{\mathbf{J}} + \mu_0 \epsilon_0 \frac{\partial \bar{\mathbf{E}}}{\partial t} \quad (4.20)$$

where $\bar{\mathbf{E}}$ is the electric field, $\bar{\mathbf{B}}$ is the magnetic field, $\bar{\mathbf{J}}$ is the current density, μ_0 is the permeability of free space, and ϵ_0 is the permittivity of free space.

These equations show that:

- A time-varying magnetic field generates a curling electric field
- Electric currents and time-varying electric fields generate a curling magnetic field

4.5 Solenoidal Vector Fields

4.5.1 Definition and Properties

Definition 4.5 (Solenoidal Vector Field). *A vector field $\bar{\mathbf{F}}$ is solenoidal if its divergence is zero throughout the domain:*

$$\nabla \cdot \bar{\mathbf{F}} = 0 \quad (4.21)$$

Solenoidal fields are also called "divergence-free," "incompressible," or "source-free" fields.

Properties of Solenoidal Fields

1. The curl of any vector field is solenoidal: $\nabla \cdot (\nabla \times \bar{\mathbf{G}}) = 0$ for any vector field $\bar{\mathbf{G}}$
2. A solenoidal field $\bar{\mathbf{F}}$ can be represented as the curl of another vector field $\bar{\mathbf{A}}$ (called the vector potential):

$$\bar{\mathbf{F}} = \nabla \times \bar{\mathbf{A}} \quad (4.22)$$

4.5.2 Examples in Physics

Magnetic Fields

The magnetic field $\bar{\mathbf{B}}$ is always solenoidal:

$$\nabla \cdot \bar{\mathbf{B}} = 0 \quad (4.23)$$

This reflects the absence of magnetic monopoles in classical electromagnetism. As a result, the magnetic field can be expressed as the curl of a vector potential $\bar{\mathbf{A}}$:

$$\bar{\mathbf{B}} = \nabla \times \bar{\mathbf{A}} \quad (4.24)$$

Incompressible Fluid Flow

For an incompressible fluid with velocity field $\bar{\mathbf{v}}$:

$$\nabla \cdot \bar{\mathbf{v}} = 0 \quad (4.25)$$

This indicates that the fluid density remains constant as it flows. In such cases, the velocity field can sometimes be expressed as the curl of a vector potential known as the stream function.

4.6 Irrotational Vector Fields

4.6.1 Definition and Properties

Definition 4.6 (Irrotational Vector Field). A vector field $\bar{\mathbf{F}}$ is irrotational if its curl is zero throughout the domain:

$$\nabla \times \bar{\mathbf{F}} = \mathbf{0} \quad (4.26)$$

Irrotational fields are also called "curl-free" fields.

Properties of Irrotational Fields

1. The gradient of any scalar field is irrotational: $\nabla \times (\nabla \phi) = \mathbf{0}$ for any scalar field ϕ
2. An irrotational field $\bar{\mathbf{F}}$ can be represented as the gradient of a scalar potential ϕ :

$$\bar{\mathbf{F}} = \nabla \phi \quad (4.27)$$

3. Line integrals of irrotational fields are path-independent (conservative property)

Testing for Irrotational Fields

Determine if the vector field $\bar{\mathbf{F}}(x, y, z) = (y^2 + z^2)\hat{\mathbf{i}} + (x^2 + z^2)\hat{\mathbf{j}} + (x^2 + y^2)\hat{\mathbf{k}}$ is irrotational. Calculate the curl:

$$\nabla \times \bar{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^2 & x^2 + z^2 & x^2 + y^2 \end{vmatrix} \quad (4.28)$$

$$= \left(\frac{\partial}{\partial y}(x^2 + y^2) - \frac{\partial}{\partial z}(x^2 + z^2) \right) \hat{\mathbf{i}} \quad (4.29)$$

$$+ \left(\frac{\partial}{\partial z}(y^2 + z^2) - \frac{\partial}{\partial x}(x^2 + y^2) \right) \hat{\mathbf{j}} \quad (4.30)$$

$$+ \left(\frac{\partial}{\partial x}(x^2 + z^2) - \frac{\partial}{\partial y}(y^2 + z^2) \right) \hat{\mathbf{k}} \quad (4.31)$$

$$= (2y - 2z)\hat{\mathbf{i}} + (2z - 2x)\hat{\mathbf{j}} + (2x - 2y)\hat{\mathbf{k}} \quad (4.32)$$

Since $\nabla \times \bar{\mathbf{F}} \neq \mathbf{0}$, the field is not irrotational.

4.6.2 Examples in Physics

Electrostatic Fields

The electric field $\bar{\mathbf{E}}$ in electrostatics is irrotational:

$$\nabla \times \bar{\mathbf{E}} = \mathbf{0} \quad (4.33)$$

This allows the electric field to be expressed as the gradient of the electric potential V :

$$\bar{\mathbf{E}} = -\nabla V \quad (4.34)$$

The negative sign indicates that the electric field points from higher to lower potential.

Gravitational Fields

The gravitational field $\bar{\mathbf{g}}$ is irrotational:

$$\nabla \times \bar{\mathbf{g}} = \mathbf{0} \quad (4.35)$$

It can be expressed as the gradient of the gravitational potential ϕ_G :

$$\bar{\mathbf{g}} = -\nabla \phi_G \quad (4.36)$$

Irrotational Fluid Flow

In fluid dynamics, an irrotational flow has no vorticity:

$$\nabla \times \bar{\mathbf{v}} = \mathbf{0} \quad (4.37)$$

This allows the velocity field to be expressed as the gradient of a scalar velocity potential ϕ_v :

$$\bar{\mathbf{v}} = \nabla \phi_v \quad (4.38)$$

Key Vector Identities

For scalar field ϕ and vector field $\bar{\mathbf{F}}$:

1. $\nabla \times (\nabla \phi) = \mathbf{0}$ (Curl of a gradient is zero)
2. $\nabla \cdot (\nabla \times \bar{\mathbf{F}}) = 0$ (Divergence of a curl is zero)
3. $\nabla \times (\nabla \times \bar{\mathbf{F}}) = \nabla(\nabla \cdot \bar{\mathbf{F}}) - \nabla^2 \bar{\mathbf{F}}$ (Vector Laplacian identity)
4. $\nabla \cdot (\phi \bar{\mathbf{F}}) = \phi(\nabla \cdot \bar{\mathbf{F}}) + \nabla \phi \cdot \bar{\mathbf{F}}$ (Product rule for divergence)
5. $\nabla \times (\phi \bar{\mathbf{F}}) = \phi(\nabla \times \bar{\mathbf{F}}) + (\nabla \phi) \times \bar{\mathbf{F}}$ (Product rule for curl)

4.6.3 Classification of Vector Fields

Based on divergence and curl properties, vector fields can be classified into four categories:

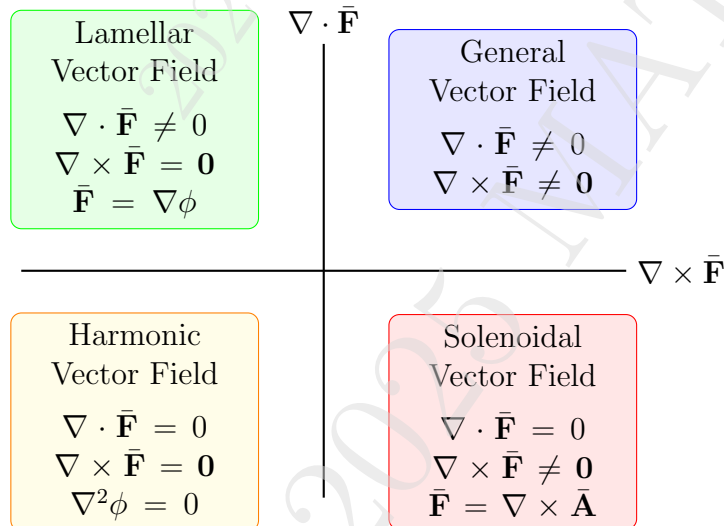


Figure 4.3: Classification of vector fields based on divergence and curl properties

4.7 Solved Examples on Solenoidal and Irrotational

Example: Proving Vector is Irrotational and Solenoidal

Prove that the vector $\vec{F} = \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}$ is irrotational and solenoidal vector.

Solution

Given: $\vec{F} = \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}$

A vector field is: - Irrotational if its curl ($\nabla \times \vec{F}$) is zero - Solenoidal if its divergence ($\nabla \cdot \vec{F}$) is zero

Identifying the components:

$$F_1 = \frac{x}{x^2 + y^2}$$

$$F_2 = \frac{y}{x^2 + y^2}$$

$$F_3 = 0$$

Part 1: Proving \vec{F} is Irrotational (curl = 0)

Step 1: Calculate the curl of \vec{F} .

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Expanding the determinant:

$$\begin{aligned} \nabla \times \vec{F} &= \hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} - \hat{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} + \hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \\ &= \hat{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \hat{j} \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \end{aligned}$$

Since $F_3 = 0$ (our vector field has only x and y components):

$$\begin{aligned} \nabla \times \vec{F} &= \hat{i} (0 - 0) - \hat{j} (0 - 0) + \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &= 0\hat{i} + 0\hat{j} + \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \end{aligned}$$

The only non-zero component of curl for 2D fields is:

$$(\nabla \times \vec{F})_k = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Step 2: Find $\frac{\partial F_2}{\partial x}$.

$$\begin{aligned} \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{y}{x^2 + y^2} \right) \\ &= y \cdot \frac{\partial}{\partial x} \left(\frac{1}{x^2 + y^2} \right) \\ &= y \cdot \left(-\frac{1}{(x^2 + y^2)^2} \cdot 2x \right) \\ &= -\frac{2xy}{(x^2 + y^2)^2} \end{aligned}$$

Step 3: Find $\frac{\partial F_1}{\partial y}$.

$$\begin{aligned}\frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) \\ &= x \cdot \frac{\partial}{\partial y} \left(\frac{1}{x^2 + y^2} \right) \\ &= x \cdot \left(-\frac{1}{(x^2 + y^2)^2} \cdot 2y \right) \\ &= -\frac{2xy}{(x^2 + y^2)^2}\end{aligned}$$

Step 4: Calculate the curl.

$$\begin{aligned}(\nabla \times \vec{F})_k &= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \\ &= -\frac{2xy}{(x^2 + y^2)^2} - \left(-\frac{2xy}{(x^2 + y^2)^2} \right) \\ &= -\frac{2xy}{(x^2 + y^2)^2} + \frac{2xy}{(x^2 + y^2)^2} \\ &= 0\end{aligned}$$

Since $\nabla \times \vec{F} = 0$, the vector field is irrotational.

Part 2: Proving \vec{F} is Solenoidal (divergence = 0)

Step 5: Calculate the divergence of \vec{F} .

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Step 6: Find $\frac{\partial F_1}{\partial x}$.

$$\begin{aligned}\frac{\partial F_1}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) \\ &= \frac{(x^2 + y^2)(1) - x(2x)}{(x^2 + y^2)^2} \\ &= \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2}\end{aligned}$$

Step 7: Find $\frac{\partial F_2}{\partial y}$.

$$\begin{aligned}\frac{\partial F_2}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) \\ &= \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2} \\ &= \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} \\ &= \frac{x^2 - y^2}{(x^2 + y^2)^2}\end{aligned}$$

Step 8: Calculate the divergence.

$$\begin{aligned}
 \nabla \cdot \vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\
 &= \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} + 0 \\
 &= \frac{y^2 - x^2 + x^2 - y^2}{(x^2 + y^2)^2} \\
 &= \frac{0}{(x^2 + y^2)^2} \\
 &= 0
 \end{aligned}$$

Since $\nabla \cdot \vec{F} = 0$, the vector field is solenoidal.

Conclusion: The vector field $\vec{F} = \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}$ is both irrotational (curl = 0) and solenoidal (divergence = 0).

Example: Proving Cross Product is Solenoidal

If $\vec{F}_1 = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and $\vec{F}_2 = (\vec{a} \cdot \vec{r})\vec{a}$ then Show that $\vec{F}_1 \times \vec{F}_2$ is solenoidal

Solution

Given:

$$\vec{F}_1 = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$\vec{F}_2 = (\vec{a} \cdot \vec{r})\vec{a}$$

Where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and \vec{a} is a constant vector.

Step 1: Calculate $\vec{a} \cdot \vec{r}$.

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

$$\begin{aligned}
 \vec{a} \cdot \vec{r} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) \\
 &= a_1x + a_2y + a_3z
 \end{aligned}$$

Therefore:

$$\vec{F}_2 = (a_1x + a_2y + a_3z)\vec{a}$$

Step 2: Calculate $\vec{F}_1 \times \vec{F}_2$.

$$\vec{F}_1 \times \vec{F}_2 = (yz\hat{i} + zx\hat{j} + xy\hat{k}) \times [(a_1x + a_2y + a_3z)\vec{a}]$$

Since scalar quantities can be factored out of cross products:

$$\vec{F}_1 \times \vec{F}_2 = (a_1x + a_2y + a_3z)(\vec{F}_1 \times \vec{a})$$

Step 3: Calculate $\vec{F}_1 \times \vec{a}$.

$$\begin{aligned}
 \vec{F}_1 \times \vec{a} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ yz & zx & xy \\ a_1 & a_2 & a_3 \end{vmatrix} \\
 &= \hat{i}(zx \cdot a_3 - xy \cdot a_2) - \hat{j}(yz \cdot a_3 - xy \cdot a_1) + \hat{k}(yz \cdot a_2 - zx \cdot a_1) \\
 &= \hat{i}(zxa_3 - xya_2) - \hat{j}(yza_3 - xya_1) + \hat{k}(yza_2 - zxa_1) \\
 &= (zxa_3 - xya_2)\hat{i} - (yza_3 - xya_1)\hat{j} + (yza_2 - zxa_1)\hat{k}
 \end{aligned}$$

Step 4: Find the divergence of $\vec{F}_1 \times \vec{F}_2$.

From Step 2:

$$\vec{F}_1 \times \vec{F}_2 = (a_1x + a_2y + a_3z)\vec{G}$$

Where $\vec{G} = \vec{F}_1 \times \vec{a}$ is the vector field from Step 3.

$$\begin{aligned}\nabla \cdot (\vec{F}_1 \times \vec{F}_2) &= \nabla \cdot [(a_1x + a_2y + a_3z)\vec{G}] \\ &= \nabla(a_1x + a_2y + a_3z) \cdot \vec{G} + (a_1x + a_2y + a_3z)(\nabla \cdot \vec{G})\end{aligned}$$

Step 5: Calculate $\nabla(a_1x + a_2y + a_3z)$.

$$\begin{aligned}\nabla(a_1x + a_2y + a_3z) &= \frac{\partial}{\partial x}(a_1x + a_2y + a_3z)\hat{i} + \frac{\partial}{\partial y}(a_1x + a_2y + a_3z)\hat{j} + \frac{\partial}{\partial z}(a_1x + a_2y + a_3z)\hat{k} \\ &= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \\ &= \vec{a}\end{aligned}$$

Step 6: Calculate $\nabla \cdot \vec{G}$.

Using the components of \vec{G} from Step 3:

$$\begin{aligned}\nabla \cdot \vec{G} &= \frac{\partial}{\partial x}(za_3 - xya_2) + \frac{\partial}{\partial y}(-(yz a_3 - xya_1)) + \frac{\partial}{\partial z}(yz a_2 - zx a_1) \\ &= a_3 \cdot z - a_2 \cdot y - (a_3 \cdot z - a_1 \cdot x) + a_2 \cdot y - a_1 \cdot x \\ &= a_3z - a_2y - a_3z + a_1x + a_2y - a_1x \\ &= 0\end{aligned}$$

Therefore: $\nabla \cdot \vec{G} = 0$

Step 7: Complete the divergence calculation.

From Step 4:

$$\begin{aligned}\nabla \cdot (\vec{F}_1 \times \vec{F}_2) &= \nabla(a_1x + a_2y + a_3z) \cdot \vec{G} + (a_1x + a_2y + a_3z)(\nabla \cdot \vec{G}) \\ &= \vec{a} \cdot \vec{G} + (a_1x + a_2y + a_3z) \cdot 0 \\ &= \vec{a} \cdot \vec{G} + 0 \\ &= \vec{a} \cdot (\vec{F}_1 \times \vec{a})\end{aligned}$$

Step 8: Use the scalar triple product identity.

The scalar triple product identity states that for any three vectors:

$$\vec{a} \cdot (\vec{F}_1 \times \vec{a}) = 0$$

This is because the scalar triple product of a vector with itself crossed with another vector is always zero.

Therefore:

$$\nabla \cdot (\vec{F}_1 \times \vec{F}_2) = \vec{a} \cdot (\vec{F}_1 \times \vec{a}) = 0$$

Hence, $\vec{F}_1 \times \vec{F}_2$ is solenoidal.

Example: Verifying Irrotational Vectors and Finding Scalar Potentials

Verify whether the following vectors are irrotational, if so, find the scalar potential $\phi(x, y, z)$.

- i) $\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$
 ii) $\vec{F} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$
 iii) $\vec{F} = (4xy + z^3)\hat{i} + (2x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$

Solution

To verify if a vector field is irrotational, we check if its curl is zero: $\nabla \times \vec{F} = \vec{0}$.
 For finding the scalar potential ϕ , we use:

$$\phi = \int F_1 dx + \int F_2 dy + \int F_3 dz$$

where: - Treating y, z constant when integrating F_1 w.r.t. x - Treating z constant and take terms free from x when integrating F_2 w.r.t. y - Taking terms free from x and y when integrating F_3 w.r.t. z

Part (i): $\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$

Step 1: Check if \vec{F} is irrotational.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

For i-component:

$$\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z) = 0 - 1 - 3x^2 + 1 = -1 - (-1) = 0$$

For j-component:

$$\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3) = 3z^2 - 0 - 0 - 3z^2 = 3z^2 - 3z^2 = 0$$

For k-component:

$$\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3) = 6x - 0 - 6y + 0 = 6x - 6x = 0$$

Therefore:

$$\nabla \times \vec{F} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

Since $\nabla \times \vec{F} = \vec{0}$, the vector field is irrotational.

Step 2: Find the scalar potential ϕ using the formula.

From $F_1 = 6xy + z^3$: - Integration w.r.t. x : $\int (6xy + z^3) dx = 3x^2y + xz^3$

From $F_2 = 3x^2 - z$: - Terms free from x : $-z$ - Integration w.r.t. y : $\int (-z) dy = -zy$

From $F_3 = 3xz^2 - y$: - Terms free from x and y : None

Therefore:

$$\phi = 3x^2y + xz^3 - zy + C$$

Verification: Check if $\nabla\phi = \vec{F}$

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \frac{\partial}{\partial x}(3x^2y + xz^3 - zy + C) = 6xy + z^3 = F_1 \checkmark \\ \frac{\partial\phi}{\partial y} &= \frac{\partial}{\partial y}(3x^2y + xz^3 - zy + C) = 3x^2 - z = F_2 \checkmark \\ \frac{\partial\phi}{\partial z} &= \frac{\partial}{\partial z}(3x^2y + xz^3 - zy + C) = 3xz^2 - y = F_3 \checkmark\end{aligned}$$

Part (ii): $\vec{F} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (x\cos z + y^2)\hat{k}$

Step 1: Check if \vec{F} is irrotational.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & x\cos z + y^2 \end{vmatrix}$$

For i-component:

$$\frac{\partial}{\partial y}(x\cos z + y^2) - \frac{\partial}{\partial z}(x\sin z + 2yz) = x\cos z + 2y - (x\cos z + 2y) = x\cos z + 2y - x\cos z - 2y = 0$$

For j-component:

$$\frac{\partial}{\partial x}(x\cos z + y^2) - \frac{\partial}{\partial z}(y\sin z - \sin x) = y\cos z - 0 - y\cos z - 0 = y\cos z - y\cos z = 0$$

For k-component:

$$\frac{\partial}{\partial x}(x\sin z + 2yz) - \frac{\partial}{\partial y}(y\sin z - \sin x) = \sin z + 0 - \sin z - 0 = \sin z - \sin z = 0$$

Therefore:

$$\nabla \times \vec{F} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

Since $\nabla \times \vec{F} = \vec{0}$, the vector field is irrotational.

Step 2: Find the scalar potential ϕ using the formula.

From $F_1 = y\sin z - \sin x$: - Integration w.r.t. x : $\int (y\sin z - \sin x)dx = x y \sin z + \cos x$

From $F_2 = x\sin z + 2yz$: - Terms free from x : $2yz$ - Integration w.r.t. y : $\int 2yz dy = y^2 z$

From $F_3 = x\cos z + y^2$: - Terms free from x and y : None

Therefore:

$$\phi = x y \sin z + \cos x + y^2 z + C$$

Verification: Check if $\nabla\phi = \vec{F}$

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \frac{\partial}{\partial x}(x y \sin z + \cos x + y^2 z + C) = y \sin z - \sin x = F_1 \checkmark \\ \frac{\partial\phi}{\partial y} &= \frac{\partial}{\partial y}(x y \sin z + \cos x + y^2 z + C) = x \sin z + 2yz = F_2 \checkmark \\ \frac{\partial\phi}{\partial z} &= \frac{\partial}{\partial z}(x y \sin z + \cos x + y^2 z + C) = x y \cos z + y^2 = F_3 \checkmark\end{aligned}$$

Part (iii): $\vec{F} = (4xy + z^3)\hat{i} + (2x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$

Step 1: Check if \vec{F} is irrotational.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy + z^3 & 2x^2 - z & 3xz^2 - y \end{vmatrix}$$

For i-component:

$$\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(2x^2 - z) = 0 - 1 - 4x^2 + 1 = -1 - (-1) = 0$$

For j-component:

$$\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(4xy + z^3) = 3z^2 - 0 - 0 - 3z^2 = 3z^2 - 3z^2 = 0$$

For k-component:

$$\frac{\partial}{\partial x}(2x^2 - z) - \frac{\partial}{\partial y}(4xy + z^3) = 4x - 0 - 4y + 0 = 4x - 4x = 0$$

Therefore:

$$\nabla \times \vec{F} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

Since $\nabla \times \vec{F} = \vec{0}$, the vector field is irrotational.

Step 2: Find the scalar potential ϕ using the formula.

From $F_1 = 4xy + z^3$: - Integration w.r.t. x : $\int (4xy + z^3)dx = 2x^2y + xz^3$

From $F_2 = 2x^2 - z$: - Terms free from x : $-z$ - Integration w.r.t. y : $\int (-z)dy = -zy$

From $F_3 = 3xz^2 - y$: - Terms free from x and y : None

Therefore:

$$\phi = 2x^2y + xz^3 - zy + C$$

Example: Finding Scalar Potential for Irrotational Vector Field

Show that $\vec{F} = (ye^{xy}\cos z)\hat{i} + (xe^{xy}\cos z)\hat{j} + (-e^{xy}\sin z)\hat{k}$ is irrotational. Find corresponding scalar potential ϕ s.t. $\vec{F} = \nabla\phi$.

Solution

Step 1: Verify that the vector field is irrotational by checking if $\nabla \times \vec{F} = \vec{0}$.

Given: $\vec{F} = (ye^{xy}\cos z)\hat{i} + (xe^{xy}\cos z)\hat{j} + (-e^{xy}\sin z)\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{xy}\cos z & xe^{xy}\cos z & -e^{xy}\sin z \end{vmatrix}$$

For i-component:

$$\begin{aligned}\frac{\partial}{\partial y}(-e^{xy}\sin z) - \frac{\partial}{\partial z}(xe^{xy}\cos z) &= -e^{xy} \cdot x \cdot \sin z - xe^{xy}(-\sin z) \\ &= -xe^{xy}\sin z + xe^{xy}\sin z \\ &= 0\end{aligned}$$

For j-component:

$$\begin{aligned}\frac{\partial}{\partial z}(ye^{xy}\cos z) - \frac{\partial}{\partial x}(-e^{xy}\sin z) &= ye^{xy}(-\sin z) - (-e^{xy}y \cdot \sin z) \\ &= -ye^{xy}\sin z + ye^{xy}\sin z \\ &= 0\end{aligned}$$

For k-component:

$$\begin{aligned}\frac{\partial}{\partial x}(xe^{xy}\cos z) - \frac{\partial}{\partial y}(ye^{xy}\cos z) &= [e^{xy}\cos z + xe^{xy}y \cdot \cos z] - [e^{xy}\cos z + ye^{xy}x \cdot \cos z] \\ &= e^{xy}\cos z + xye^{xy}\cos z - e^{xy}\cos z - xye^{xy}\cos z \\ &= 0\end{aligned}$$

Since all components are zero, $\nabla \times \vec{F} = \vec{0}$, confirming that the vector field is irrotational.

Step 2: Find the scalar potential ϕ using:

$$\phi = \int F_1 dx + \int F_2 dy + \int F_3 dz$$

where: - Treating y, z constant when integrating F_1 w.r.t. x - Treating z constant and take terms free from x when integrating F_2 w.r.t. y - Taking terms free from x and y when integrating F_3 w.r.t. z

Step 3: Integrate F_1 with respect to x .

Given: $F_1 = ye^{xy}\cos z$

$$\int F_1 dx = \int ye^{xy}\cos z dx$$

Let $u = xy$, then $du = ydx$, so $dx = \frac{du}{y}$

$$\begin{aligned}\int ye^{xy}\cos z dx &= \cos z \int ye^u \frac{du}{y} \\ &= \cos z \int e^u du \\ &= \cos z \cdot e^u \\ &= \cos z \cdot e^{xy} \\ &= e^{xy}\cos z\end{aligned}$$

Step 4: From F_2 , identify and integrate terms free from x .

Given: $F_2 = xe^{xy}\cos z$

The term F_2 contains x , so there are no terms free from x to integrate.

Step 5: From F_3 , identify and integrate terms free from x and y .

Given: $F_3 = -e^{xy}\sin z$

The term F_3 contains both x and y , so there are no terms free from both x and y to integrate.

Step 6: Combine the results.

Therefore:

$$\phi = e^{xy}\cos z + C$$

Verification: Check if $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(e^{xy}\cos z + C) = e^{xy}y \cdot \cos z = ye^{xy}\cos z = F_1 \checkmark$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(e^{xy}\cos z + C) = e^{xy}x \cdot \cos z = xe^{xy}\cos z = F_2 \checkmark$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}(e^{xy}\cos z + C) = e^{xy}(-\sin z) = -e^{xy}\sin z = F_3 \checkmark$$

All three partial derivatives match the components of \vec{F} , confirming our solution.

Example: Finding Constants and Scalar Potential for Irrotational Vector Field

If the vector field given by $\vec{F} = (ax + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + z)\hat{k}$ is irrotational. Then find a, b, c and also determine scalar potential ϕ .

Solution

Step 1: For the vector field to be irrotational, $\nabla \times \vec{F} = \vec{0}$.

Given: $\vec{F} = (ax + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + z)\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax + 2y + az & bx - 3y - z & 4x + cy + z \end{vmatrix}$$

For i-component:

$$\begin{aligned} \frac{\partial}{\partial y}(4x + cy + z) - \frac{\partial}{\partial z}(bx - 3y - z) &= c - (-1) \\ &= c + 1 \end{aligned}$$

For j-component:

$$\frac{\partial}{\partial z}(ax + 2y + az) - \frac{\partial}{\partial x}(4x + cy + z) = a - 4$$

For k-component:

$$\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(ax + 2y + az) = b - 2$$

For the vector field to be irrotational, all components must equal zero:

$$c + 1 = 0 \Rightarrow c = -1$$

$$a - 4 = 0 \Rightarrow a = 4$$

$$b - 2 = 0 \Rightarrow b = 2$$

Therefore: $a = 4$, $b = 2$, $c = -1$

Step 2: Substitute values back into the vector field.

$$\vec{F} = (4x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x - y + z)\hat{k}$$

Step 3: Find the scalar potential ϕ using:

$$\phi = \int F_1 dx + \int F_2 dy + \int F_3 dz$$

From $F_1 = 4x + 2y + 4z$:

$$\int F_1 dx = \int (4x + 2y + 4z) dx = 2x^2 + 2xy + 4xz$$

From $F_2 = 2x - 3y - z$: - Terms free from x : $-3y - z$

$$\int F_2 dy = \int (-3y - z) dy = -\frac{3y^2}{2} - zy$$

From $F_3 = 4x - y + z$: - Terms free from x and y : z

$$\int F_3 dz = \int z dz = \frac{z^2}{2}$$

Therefore:

$$\phi = 2x^2 + 2xy + 4xz - \frac{3y^2}{2} - zy + \frac{z^2}{2} + C$$

Verification: Check if $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x} = 4x + 2y + 4z = F_1 \checkmark$$

$$\frac{\partial\phi}{\partial y} = 2x - 3y - z = F_2 \checkmark$$

$$\frac{\partial\phi}{\partial z} = 4x - y + z = F_3 \checkmark$$

All three partial derivatives match the components of \vec{F} , confirming our solution.