Chapter 4

Homogeneous Functions and Euler's Theorem

4.1 Homogeneous Functions

Definition of Homogeneous Functions

A function $f(x, y)$ is called **homogeneous of degree** n if for any $t \neq 0$:

$$f(tx, ty) = t^n f(x, y) \tag{4.1}$$

In other words, when all variables are multiplied by the same factor t, the function's value is multiplied by t^n.

Standard Form of Homogeneous Function

An expression of the form:

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_n y^n \tag{4.2}$$

where all terms have the same total degree n, is a homogeneous function of degree n.

Alternative Forms of Homogeneous Functions

A homogeneous function $f(x, y)$ of degree n can always be written in the following equivalent forms:
Form 1: Using x as a factor

$$f(x, y) = x^n \phi\left(\frac{y}{x}\right) \tag{4.3}$$

where $\phi(u) = f(1, u)$ with $u = \frac{y}{x}$.
Form 2: Using y as a factor

$$f(x, y) = y^n \psi\left(\frac{x}{y}\right) \tag{4.4}$$

where $\psi(v) = f(v, 1)$ with $v = \frac{x}{y}$.

Verification: Setting $t = \frac{1}{x}$ in the homogeneity condition $f(tx, ty) = t^n f(x, y)$:

$$f(1, \frac{y}{x}) = \left(\frac{1}{x}\right)^n f(x, y) \tag{4.5}$$

$$\Rightarrow f(x, y) = x^n f\left(1, \frac{y}{x}\right) = x^n \phi\left(\frac{y}{x}\right) \tag{4.6}$$

Similarly, setting $t = \frac{1}{y}$ gives the second form.

Testing for Homogeneity Using Substitution

For any homogeneous function $f(x, y)$ of degree n, replacing x with tx and y with ty yields:

$$f(tx, ty) = a_0(tx)^n + a_1(tx)^{n-1}(ty) + \ldots + a_n(ty)^n \tag{4.7}$$

$$= t^n[a_0x^n + a_1x^{n-1}y + \ldots + a_ny^n] \tag{4.8}$$

$$= t^n f(x, y) \tag{4.9}$$

This property provides a quick test for checking if a function is homogeneous.

Examples of Homogeneous Functions

Let's examine some functions to understand homogeneity:
Example 1: $f(x, y) = x^3y^2 + 2xy^4$
If we substitute tx for x and ty for y:

$$f(tx, ty) = (tx)^3(ty)^2 + 2(tx)(ty)^4 \tag{4.10}$$

$$= t^3x^3 \cdot t^2y^2 + 2tx \cdot t^4y^4 \tag{4.11}$$

$$= t^5(x^3y^2 + 2xy^4) \tag{4.12}$$

$$= t^5 f(x, y) \tag{4.13}$$

Therefore, $f(x, y)$ is homogeneous of degree 5.
Example 2: $f(x, y) = \frac{x^2+y^2}{xy}$
Substituting tx for x and ty for y:

$$f(tx, ty) = \frac{(tx)^2 + (ty)^2}{(tx)(ty)} \tag{4.14}$$

$$= \frac{t^2x^2 + t^2y^2}{t^2xy} \tag{4.15}$$

$$= \frac{t^2(x^2 + y^2)}{t^2(xy)} \tag{4.16}$$

$$= \frac{x^2 + y^2}{xy} \tag{4.17}$$

$$= f(x, y) \tag{4.18}$$

Therefore, $f(x, y)$ is homogeneous of degree 0.
Example 3: $f(x, y) = x^2 + y$ is not homogeneous as the terms have different degrees.

Functions of Homogeneous Expressions

Not all functions containing homogeneous expressions are themselves homogeneous:
Example 1: Consider $f(x, y) = \sin\left(\frac{x^2+y^2}{x+y}\right)$
The expression $\frac{x^2+y^2}{x+y}$ is homogeneous of degree 1, since:

$$\frac{(tx)^2 + (ty)^2}{(tx) + (ty)} = \frac{t^2(x^2 + y^2)}{t(x + y)} = t\frac{x^2 + y^2}{x + y} \tag{4.19}$$

However, $f(x, y)$ itself is not homogeneous because the sine function disrupts the scaling property.
Example 2: The function $g(x, y) = \log\left(\frac{y}{x}\right)$ is homogeneous of degree 0:

$$g(tx, ty) = \log\left(\frac{ty}{tx}\right) \tag{4.20}$$

$$= \log\left(\frac{y}{x}\right) \tag{4.21}$$

$$= g(x, y) \tag{4.22}$$

4.2 Euler's Theorem on Homogeneous Functions

Euler's Theorem for Two Variables

If $f(x, y)$ is a homogeneous function of degree n, then:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y) \tag{4.23}$$

Proof of Euler's Theorem

Let $f(x, y)$ be a homogeneous function of degree n. Then by definition:

$$f(tx, ty) = t^n f(x, y) \tag{4.24}$$

Differentiating both sides with respect to t:

$$\frac{\partial f}{\partial x} \cdot x + \frac{\partial f}{\partial y} \cdot y = n \cdot t^{n-1} \cdot f(x, y) \tag{4.25}$$

Setting $t = 1$:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n \cdot f(x, y) \tag{4.26}$$

This proves Euler's theorem.

Application: Verifying Homogeneity

Euler's theorem provides an alternative method to verify whether a function is homogeneous.
For example, given $f(x, y) = x^2y + xy^2$:

1. Calculate the partial derivatives:

$$\frac{\partial f}{\partial x} = 2xy + y^2 \tag{4.27}$$

$$\frac{\partial f}{\partial y} = x^2 + 2xy \tag{4.28}$$

2. Compute $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x(2xy + y^2) + y(x^2 + 2xy) \tag{4.29}$$

$$= 2x^2y + xy^2 + x^2y + 2xy^2 \tag{4.30}$$

$$= 3x^2y + 3xy^2 \tag{4.31}$$

$$= 3(x^2y + xy^2) \tag{4.32}$$

$$= 3f(x, y) \tag{4.33}$$

Since $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f(x, y)$, we confirm that $f(x, y)$ is homogeneous of degree 3.

Euler's Theorem for n Variables

If $f(x_1, x_2, \ldots, x_n)$ is homogeneous of degree k, then:

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \ldots + x_n\frac{\partial f}{\partial x_n} = k \cdot f(x_1, x_2, \ldots, x_n) \tag{4.34}$$

4.3 Applications and Extensions of Euler's Theorem

Deduction 1: Second-Order Homogeneous Functions

If $z = f(x, y)$ is a homogeneous function of degree n, then:

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z \tag{4.35}$$

Proof of Deduction 1

From Euler's theorem, we have:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \tag{1}$$

Differentiating equation (1) with respect to x:

$$\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial y \partial x} = n\frac{\partial z}{\partial x} \tag{4.36}$$

Rearranging:

$$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial y \partial x} = (n-1)\frac{\partial z}{\partial x} \tag{2}$$

Similarly, differentiating equation (1) with respect to y:

$$x\frac{\partial^2 z}{\partial x \partial y} + y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y} \tag{3}$$

Multiplying equation (2) by x and equation (3) by y, and adding them:

$$x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial y \partial x} + xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) \tag{4.37}$$

(4.38)

Using the equality of mixed partial derivatives $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$ and substituting from equation (1):

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)(nz) \tag{4.39}$$

$$= n(n-1)z \tag{4.40}$$

Thus, the result is proved.

Deduction 2: Homogeneous Functions and Transformations

If $z = f(u)$ where $u = g(x, y)$ is a homogeneous function of degree n, then:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{g(u)}{f'(u)} \tag{4.41}$$

Further, if we define $h(u) = n\frac{g(u)}{f'(u)}$, then:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = h(u)[h'(u) - 1] \tag{4.42}$$

Applying Euler's Theorem

Consider the function $f(x, y) = \frac{x^3y^2}{x^2+y^2}$.
To determine if it's homogeneous and find its degree:
1. Test for homogeneity:

$$f(tx, ty) = \frac{(tx)^3(ty)^2}{(tx)^2 + (ty)^2} \tag{4.43}$$

$$= \frac{t^3x^3 \cdot t^2y^2}{t^2(x^2 + y^2)} \tag{4.44}$$

$$= \frac{t^5x^3y^2}{t^2(x^2 + y^2)} \tag{4.45}$$

$$= t^3\frac{x^3y^2}{x^2 + y^2} \tag{4.46}$$

$$= t^3 f(x, y) \tag{4.47}$$

So $f(x, y)$ is homogeneous of degree 3.

2. Verify using Euler's theorem:

$$\frac{\partial f}{\partial x} = \frac{3x^2y^2(x^2+y^2) - x^3y^2 \cdot 2x}{(x^2+y^2)^2} \tag{4.48}$$

$$= \frac{3x^2y^2(x^2+y^2) - 2x^4y^2}{(x^2+y^2)^2} \tag{4.49}$$

$$\frac{\partial f}{\partial y} = \frac{x^3 \cdot 2y(x^2+y^2) - x^3y^2 \cdot 2y}{(x^2+y^2)^2} \tag{4.50}$$

$$= \frac{2x^3y(x^2+y^2) - 2x^3y^3}{(x^2+y^2)^2} \tag{4.51}$$

$$\tag{4.52}$$

Computing $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ and simplifying (steps omitted for brevity) should yield $3f(x,y)$, confirming that the function is homogeneous of degree 3.

Exercise

Determine whether the following functions are homogeneous. If so, find their degree and verify using Euler's theorem:
1. $f(x,y) = x^4 - 2x^2y^2 + y^4$ 2. $g(x,y) = \frac{x-y}{x+y}$ 3. $h(x,y) = x^3 + y^2$ 4. $k(x,y) = e^{x/y}$ 5. $m(x,y,z) = \frac{xyz}{x+y+z}$

Answers to Exercise

1. $f(x,y) = x^4 - 2x^2y^2 + y^4$
Testing for homogeneity:

$$f(tx,ty) = (tx)^4 - 2(tx)^2(ty)^2 + (ty)^4 \tag{4.53}$$

$$= t^4x^4 - 2t^4x^2y^2 + t^4y^4 \tag{4.54}$$

$$= t^4(x^4 - 2x^2y^2 + y^4) \tag{4.55}$$

$$= t^4 f(x,y) \tag{4.56}$$

Therefore, $f(x,y)$ is homogeneous of degree 4.
2. $g(x,y) = \frac{x-y}{x+y}$
Testing for homogeneity:

$$g(tx,ty) = \frac{tx - ty}{tx + ty} \tag{4.57}$$

$$= \frac{t(x-y)}{t(x+y)} \tag{4.58}$$

$$= \frac{x-y}{x+y} \tag{4.59}$$

$$= g(x,y) \tag{4.60}$$

Therefore, $g(x,y)$ is homogeneous of degree 0.
3. $h(x,y) = x^3 + y^2$
The terms have different degrees (3 and 2), so $h(x,y)$ is not homogeneous.
4. $k(x,y) = e^{x/y}$

Testing for homogeneity:

$$k(tx,ty) = e^{tx/ty} \tag{4.61}$$
$$= e^{x/y} \tag{4.62}$$
$$= k(x,y) \tag{4.63}$$

Therefore, $k(x,y)$ is homogeneous of degree 0.
5. $m(x,y,z) = \frac{xyz}{x+y+z}$
Testing for homogeneity:

$$m(tx,ty,tz) = \frac{(tx)(ty)(tz)}{tx+ty+tz} \tag{4.64}$$
$$= \frac{t^3xyz}{t(x+y+z)} \tag{4.65}$$
$$= t^2\frac{xyz}{x+y+z} \tag{4.66}$$
$$= t^2m(x,y,z) \tag{4.67}$$

Therefore, $m(x,y,z)$ is homogeneous of degree 2.

4.4 Solved Examples

Example: Homogeneous Function and Euler's Theorem

If $u = \frac{\sqrt{x^7+y^7}}{4\sqrt{x^4+y^4}} + \cos\left[\frac{xy+y^2}{4xy}\right] + \log\left(\frac{x}{y}\right)$
Then find the value of: $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Detailed Solution

Step 1: Let's analyze the function u term by term to identify whether it's homogeneous and its degree.
We have $u = u_1 + u_2 + u_3$, where:

$$u_1 = \frac{\sqrt{x^7+y^7}}{4\sqrt{x^4+y^4}} \tag{4.68}$$
$$u_2 = \cos\left[\frac{xy+y^2}{4xy}\right] \tag{4.69}$$
$$u_3 = \log\left(\frac{x}{y}\right) \tag{4.70}$$

For u_1, let's check if it's homogeneous by substituting (tx, ty) for (x, y):

$$u_1(tx, ty) = \frac{\sqrt{(tx)^7 + (ty)^7}}{4\sqrt{(tx)^4 + (ty)^4}} \tag{4.71}$$
$$= \frac{\sqrt{t^7x^7 + t^7y^7}}{4\sqrt{t^4x^4 + t^4y^4}} \tag{4.72}$$
$$= \frac{\sqrt{t^7(x^7 + y^7)}}{4\sqrt{t^4(x^4 + y^4)}} \tag{4.73}$$
$$= \frac{t^{7/2}\sqrt{x^7 + y^7}}{4 \cdot t^{4/2}\sqrt{x^4 + y^4}} \tag{4.74}$$
$$= \frac{t^{3.5}}{t^2} \cdot \frac{\sqrt{x^7 + y^7}}{4\sqrt{x^4 + y^4}} \tag{4.75}$$
$$= t^{1.5} \cdot u_1(x, y) \tag{4.76}$$

So u_1 is homogeneous of degree $\frac{3}{2}$.
For u_2, let's examine the argument inside the cosine:

$$\frac{(tx)(ty) + (ty)^2}{4(tx)(ty)} = \frac{t^2xy + t^2y^2}{4t^2xy} \tag{4.77}$$
$$= \frac{t^2(xy + y^2)}{4t^2xy} \tag{4.78}$$
$$= \frac{xy + y^2}{4xy} \tag{4.79}$$

Since the argument doesn't change with t, and cosine is a function of this argument, we have:

$$u_2(tx, ty) = u_2(x, y) \tag{4.80}$$

This means u_2 is homogeneous of degree 0.
For u_3:

$$u_3(tx, ty) = \log\left(\frac{tx}{ty}\right) \tag{4.81}$$
$$= \log\left(\frac{x}{y}\right) \tag{4.82}$$
$$= u_3(x, y) \tag{4.83}$$

So u_3 is also homogeneous of degree 0.
Step 2: Let's identify the expression we need to evaluate.
The given expression is:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \tag{4.84}$$

Notice that the first three terms constitute the left side of the extension of Euler's theorem for second derivatives:

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z \tag{4.85}$$

where z is a homogeneous function of degree n.
The last two terms constitute the left side of Euler's theorem:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \tag{4.86}$$

Step 3: Apply Euler's theorem to each component of u.
For u_1 (homogeneous of degree $\frac{3}{2}$):

$$x^2\frac{\partial^2 u_1}{\partial x^2} + 2xy\frac{\partial^2 u_1}{\partial x \partial y} + y^2\frac{\partial^2 u_1}{\partial y^2} = \frac{3}{2}\left(\frac{3}{2} - 1\right) u_1 \tag{4.87}$$

$$= \frac{3}{2} \cdot \frac{1}{2} \cdot u_1 \tag{4.88}$$

$$= \frac{3}{4} u_1 \tag{4.89}$$

And:

$$x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} = \frac{3}{2} u_1 \tag{4.90}$$

For u_2 (homogeneous of degree 0):

$$x^2\frac{\partial^2 u_2}{\partial x^2} + 2xy\frac{\partial^2 u_2}{\partial x \partial y} + y^2\frac{\partial^2 u_2}{\partial y^2} = 0 \cdot (0 - 1) \cdot u_2 \tag{4.91}$$

$$= 0 \tag{4.92}$$

And:

$$x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} = 0 \cdot u_2 \tag{4.93}$$

$$= 0 \tag{4.94}$$

For u_3 (homogeneous of degree 0):

$$x^2\frac{\partial^2 u_3}{\partial x^2} + 2xy\frac{\partial^2 u_3}{\partial x \partial y} + y^2\frac{\partial^2 u_3}{\partial y^2} = 0 \cdot (0 - 1) \cdot u_3 \tag{4.95}$$

$$= 0 \tag{4.96}$$

And:

$$x\frac{\partial u_3}{\partial x} + y\frac{\partial u_3}{\partial y} = 0 \cdot u_3 \tag{4.97}$$

$$= 0 \tag{4.98}$$

Step 4: Combine the results for the complete function u.
Since $u = u_1 + u_2 + u_3$, and the differential operators are linear:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{3}{4} u_1 + 0 + 0 \tag{4.99}$$

$$= \frac{3}{4} u_1 \tag{4.100}$$

And:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u_1 + 0 + 0 \tag{4.101}$$

$$= \frac{3}{2}u_1 \tag{4.102}$$

Therefore, the value of the given expression is:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{4}u_1 + \frac{3}{2}u_1 \tag{4.103}$$

$$= \left(\frac{3}{4} + \frac{3}{2}\right)u_1 \tag{4.104}$$

$$= \frac{3}{4} + \frac{6}{4}u_1 \tag{4.105}$$

$$= \frac{9}{4}u_1 \tag{4.106}$$

$$= \frac{9}{4} \cdot \frac{\sqrt{x^7 + y^7}}{4\sqrt{x^4 + y^4}} \tag{4.107}$$

$$= \frac{9\sqrt{x^7 + y^7}}{16\sqrt{x^4 + y^4}} \tag{4.108}$$

Final Answer: The value of the given expression is $\boxed{\dfrac{9\sqrt{x^7 + y^7}}{16\sqrt{x^4 + y^4}}}$

Example 2: Euler's Theorem Application

If $T = \sin\left(\frac{xy}{x^2+y^2}\right) + \sqrt{x^2 + y^2} + \frac{x^2 y}{x+y}$,
find the value of $x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y}$.

Detailed Solution

Step 1: Let's analyze each term of T to check if it's homogeneous and determine its degree.
We have $T = T_1 + T_2 + T_3$, where:

$$T_1 = \sin\left(\frac{xy}{x^2 + y^2}\right) \tag{4.109}$$

$$T_2 = \sqrt{x^2 + y^2} \tag{4.110}$$

$$T_3 = \frac{x^2 y}{x + y} \tag{4.111}$$

For T_1, let's examine the argument of the sine function:

$$\frac{(tx)(ty)}{(tx)^2 + (ty)^2} = \frac{t^2 xy}{t^2(x^2 + y^2)} \tag{4.112}$$

$$= \frac{xy}{x^2 + y^2} \tag{4.113}$$

Since the argument doesn't change when we substitute (tx, ty) for (x, y), and sine is a function of this argument, we have:

$$T_1(tx, ty) = T_1(x, y) \tag{4.114}$$

So T_1 is homogeneous of degree 0.
For T_2:

$$T_2(tx, ty) = \sqrt{(tx)^2 + (ty)^2} \tag{4.115}$$
$$= \sqrt{t^2(x^2 + y^2)} \tag{4.116}$$
$$= |t|\sqrt{x^2 + y^2} \tag{4.117}$$

Since $t > 0$ in our context (scaling factor), T_2 is homogeneous of degree 1.
For T_3:

$$T_3(tx, ty) = \frac{(tx)^2(ty)}{(tx) + (ty)} \tag{4.118}$$
$$= \frac{t^3x^2y}{t(x + y)} \tag{4.119}$$
$$= t^2\frac{x^2y}{x + y} \tag{4.120}$$
$$= t^2T_3(x, y) \tag{4.121}$$

So T_3 is homogeneous of degree 2.
Step 2: Apply Euler's theorem to each component.
Recall Euler's theorem: If $f(x, y)$ is homogeneous of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$.
For T_1 (homogeneous of degree 0):

$$x\frac{\partial T_1}{\partial x} + y\frac{\partial T_1}{\partial y} = 0 \cdot T_1 = 0 \tag{4.122}$$

For T_2 (homogeneous of degree 1):

$$x\frac{\partial T_2}{\partial x} + y\frac{\partial T_2}{\partial y} = 1 \cdot T_2 = \sqrt{x^2 + y^2} \tag{4.123}$$

For T_3 (homogeneous of degree 2):

$$x\frac{\partial T_3}{\partial x} + y\frac{\partial T_3}{\partial y} = 2 \cdot T_3 = 2 \cdot \frac{x^2y}{x + y} \tag{4.124}$$

Step 3: Combine the results for the complete function T.
Since $T = T_1 + T_2 + T_3$, and the differential operator is linear:

$$x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y} = x\frac{\partial T_1}{\partial x} + y\frac{\partial T_1}{\partial y} + x\frac{\partial T_2}{\partial x} + y\frac{\partial T_2}{\partial y} + x\frac{\partial T_3}{\partial x} + y\frac{\partial T_3}{\partial y} \tag{4.125}$$
$$= 0 + \sqrt{x^2 + y^2} + 2\frac{x^2y}{x + y} \tag{4.126}$$
$$= \sqrt{x^2 + y^2} + 2\frac{x^2y}{x + y} \tag{4.127}$$

Final Answer: The value of $x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y}$ is $\boxed{\sqrt{x^2 + y^2} + 2\frac{x^2y}{x + y}}$

Example 3: Euler's Theorem for Three Variables

If $u = \frac{xyz}{2x+y+z} + \log\left(\frac{x^2+y^2+z^2}{xy+yz}\right)$, find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Detailed Solution

Step 1: Let's analyze each term of u to determine if it's homogeneous and its degree. We have $u = u_1 + u_2$, where:

$$u_1 = \frac{xyz}{2x+y+z} \tag{4.128}$$

$$u_2 = \log\left(\frac{x^2+y^2+z^2}{xy+yz}\right) \tag{4.129}$$

For u_1, let's substitute (tx, ty, tz) for (x, y, z):

$$u_1(tx,ty,tz) = \frac{(tx)(ty)(tz)}{2(tx)+(ty)+(tz)} \tag{4.130}$$

$$= \frac{t^3xyz}{t(2x+y+z)} \tag{4.131}$$

$$= t^2\frac{xyz}{2x+y+z} \tag{4.132}$$

$$= t^2u_1(x,y,z) \tag{4.133}$$

So u_1 is homogeneous of degree 2.
For u_2, let's examine the argument of the logarithm:

$$\frac{(tx)^2+(ty)^2+(tz)^2}{(tx)(ty)+(ty)(tz)} = \frac{t^2(x^2+y^2+z^2)}{t^2(xy+yz)} \tag{4.134}$$

$$= \frac{x^2+y^2+z^2}{xy+yz} \tag{4.135}$$

Since the argument doesn't change when we substitute (tx, ty, tz) for (x, y, z), and logarithm is a function of this argument, we have:

$$u_2(tx,ty,tz) = u_2(x,y,z) \tag{4.136}$$

So u_2 is homogeneous of degree 0.
Step 2: Apply Euler's theorem for three variables to each component.
Recall Euler's theorem for three variables: If $f(x, y, z)$ is homogeneous of degree n, then:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x,y,z) \tag{4.137}$$

For u_1 (homogeneous of degree 2):

$$x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} + z\frac{\partial u_1}{\partial z} = 2u_1 = 2\frac{xyz}{2x+y+z} \tag{4.138}$$

For u_2 (homogeneous of degree 0):

$$x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} + z\frac{\partial u_2}{\partial z} = 0u_2 = 0 \tag{4.139}$$

Step 3: Combine the results for the complete function u.
Since $u = u_1 + u_2$, and the differential operator is linear:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \left(x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} + z\frac{\partial u_1}{\partial z}\right) + \left(x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} + z\frac{\partial u_2}{\partial z}\right) \tag{4.140}$$

$$= 2\frac{xyz}{2x+y+z} + 0 \tag{4.141}$$

$$= 2\frac{xyz}{2x+y+z} \tag{4.142}$$

Final Answer: The value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ is $\boxed{2\frac{xyz}{2x+y+z}}$

Example 4: Verifying a Relation with Euler's Theorem

If $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2+y^2}$, then prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$.

Detailed Solution

Step 1: Let's first determine if the function $f(x, y)$ is homogeneous and its degree.
We have $f = f_1 + f_2 + f_3$, where:

$$f_1 = \frac{1}{x^2} \tag{4.143}$$

$$f_2 = \frac{1}{xy} \tag{4.144}$$

$$f_3 = \frac{\log x - \log y}{x^2 + y^2} \tag{4.145}$$

Let's check each term for homogeneity:
For f_1:

$$f_1(tx, ty) = \frac{1}{(tx)^2} \tag{4.146}$$

$$= \frac{1}{t^2x^2} \tag{4.147}$$

$$= t^{-2}f_1(x, y) \tag{4.148}$$

So f_1 is homogeneous of degree -2.
For f_2:

$$f_2(tx, ty) = \frac{1}{(tx)(ty)} \tag{4.149}$$

$$= \frac{1}{t^2xy} \tag{4.150}$$

$$= t^{-2}f_2(x, y) \tag{4.151}$$

So f_2 is also homogeneous of degree -2.
For f_3, let's examine the numerator and denominator separately:
Numerator: $\log(tx) - \log(ty) = \log x + \log t - (\log y + \log t) = \log x - \log y$
Denominator: $(tx)^2 + (ty)^2 = t^2(x^2 + y^2)$

Therefore:

$$f_3(tx, ty) = \frac{\log(tx) - \log(ty)}{(tx)^2 + (ty)^2} \tag{4.152}$$
$$= \frac{\log x - \log y}{t^2(x^2 + y^2)} \tag{4.153}$$
$$= t^{-2} f_3(x, y) \tag{4.154}$$

So f_3 is also homogeneous of degree -2.
Since all terms are homogeneous of the same degree -2, the entire function $f(x, y)$ is homogeneous of degree -2.
Step 2: Apply Euler's theorem to a homogeneous function of degree -2.
According to Euler's theorem, if $g(x, y)$ is homogeneous of degree n, then:

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = ng(x, y) \tag{4.155}$$

In our case, $f(x, y)$ is homogeneous of degree -2, so:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -2f(x, y) \tag{4.156}$$

Step 3: Rearrange to match the required form.
From the above equation, we get:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -2f(x, y) \tag{4.157}$$
$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f(x, y) = 0 \tag{4.158}$$

Thus, we have proven that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$.

Example 5: Euler's Theorem for Three Variables

If $u = \frac{xyz}{2x+y+z} + \log\left(\frac{x^2+y^2+z^2}{xy+yz}\right)$ then find the value of $xu_x + yu_y + zu_z$.

Detailed Solution

Step 1: Let's analyze each term of u to determine if it's homogeneous and its degree.
We have $u = u_1 + u_2$, where:

$$u_1 = \frac{xyz}{2x + y + z} \tag{4.159}$$
$$u_2 = \log\left(\frac{x^2 + y^2 + z^2}{xy + yz}\right) \tag{4.160}$$

For u_1, let's substitute (tx, ty, tz) for (x, y, z):

$$u_1(tx, ty, tz) = \frac{(tx)(ty)(tz)}{2(tx) + (ty) + (tz)} \tag{4.161}$$

$$= \frac{t^3 xyz}{t(2x + y + z)} \tag{4.162}$$

$$= t^2 \frac{xyz}{2x + y + z} \tag{4.163}$$

$$= t^2 u_1(x, y, z) \tag{4.164}$$

So u_1 is homogeneous of degree 2.
For u_2, let's examine the argument of the logarithm:

$$\frac{(tx)^2 + (ty)^2 + (tz)^2}{(tx)(ty) + (ty)(tz)} = \frac{t^2(x^2 + y^2 + z^2)}{t^2(xy + yz)} \tag{4.165}$$

$$= \frac{x^2 + y^2 + z^2}{xy + yz} \tag{4.166}$$

Since the argument doesn't change when we substitute (tx, ty, tz) for (x, y, z), and logarithm is a function of this argument, we have:

$$u_2(tx, ty, tz) = u_2(x, y, z) \tag{4.167}$$

So u_2 is homogeneous of degree 0.
Step 2: Apply Euler's theorem for three variables to each component.
According to Euler's theorem for three variables, if $f(x, y, z)$ is homogeneous of degree n, then:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x, y, z) \tag{4.168}$$

For u_1 (homogeneous of degree 2):

$$x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} + z\frac{\partial u_1}{\partial z} = 2u_1 = 2\frac{xyz}{2x + y + z} \tag{4.169}$$

For u_2 (homogeneous of degree 0):

$$x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} + z\frac{\partial u_2}{\partial z} = 0u_2 = 0 \tag{4.170}$$

Step 3: Combine the results for the complete function u.
Since $u = u_1 + u_2$, and the differential operator is linear:

$$xu_x + yu_y + zu_z = \left(x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} + z\frac{\partial u_1}{\partial z}\right) + \left(x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} + z\frac{\partial u_2}{\partial z}\right) \tag{4.171}$$

$$= 2\frac{xyz}{2x + y + z} + 0 \tag{4.172}$$

$$= 2\frac{xyz}{2x + y + z} \tag{4.173}$$

Final Answer: The value of $xu_x + yu_y + zu_z$ is $\boxed{2\frac{xyz}{2x + y + z}}$

Example 6: Verifying a Specific Relation Using Euler's Theorem

If $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$ then show that $\frac{\partial u}{\partial x} + \frac{y}{x}\frac{\partial u}{\partial y} = 0$.

Detailed Solution

Step 1: First, let's check if the given function is homogeneous and determine its degree.
Consider $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$.
Let's substitute (tx, ty) for (x, y) and examine the argument of $\sin^{-1}$:

$$\frac{\sqrt{tx}-\sqrt{ty}}{\sqrt{tx}+\sqrt{ty}} = \frac{\sqrt{t}\sqrt{x}-\sqrt{t}\sqrt{y}}{\sqrt{t}\sqrt{x}+\sqrt{t}\sqrt{y}} \tag{4.174}$$

$$= \frac{\sqrt{t}(\sqrt{x}-\sqrt{y})}{\sqrt{t}(\sqrt{x}+\sqrt{y})} \tag{4.175}$$

$$= \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} \tag{4.176}$$

Since the argument remains unchanged when we replace (x, y) with (tx, ty), and $\sin^{-1}$ is a function of this argument, we have:

$$u(tx, ty) = u(x, y) \tag{4.177}$$

This means u is homogeneous of degree 0.
Step 2: Apply Euler's theorem for homogeneous functions.
For a homogeneous function $f(x, y)$ of degree n, Euler's theorem states:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y) \tag{4.178}$$

Since u is homogeneous of degree 0, we have:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \cdot u = 0 \tag{4.179}$$

Step 3: Rearrange to match the required form.
From the equation above:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \tag{4.180}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{y}{x}\frac{\partial u}{\partial y} = 0 \tag{4.181}$$

Final Answer: We have shown that $\frac{\partial u}{\partial x} + \frac{y}{x}\frac{\partial u}{\partial y} = 0$ using Euler's theorem for homogeneous functions of degree 0.

Example 7: Second-Order Extension of Euler's Theorem

If $z = x^n f\left(\frac{y}{x}\right) + y^{-n}\phi\left(\frac{x}{y}\right)$ then, prove that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$

Detailed Solution

Step 1: Let's first determine if z is homogeneous and find its degree.
The function z consists of two terms:

$$z_1 = x^n f\left(\frac{y}{x}\right) \tag{4.182}$$

$$z_2 = y^{-n}\phi\left(\frac{x}{y}\right) \tag{4.183}$$

Let's check if z_1 is homogeneous by substituting (tx, ty) for (x, y):

$$z_1(tx, ty) = (tx)^n f\left(\frac{ty}{tx}\right) \tag{4.184}$$

$$= t^n x^n f\left(\frac{y}{x}\right) \tag{4.185}$$

$$= t^n z_1(x, y) \tag{4.186}$$

So z_1 is homogeneous of degree n.
For z_2:

$$z_2(tx, ty) = (ty)^{-n}\phi\left(\frac{tx}{ty}\right) \tag{4.187}$$

$$= t^{-n}y^{-n}\phi\left(\frac{x}{y}\right) \tag{4.188}$$

$$= t^{-n}z_2(x, y) \tag{4.189}$$

So z_2 is homogeneous of degree $-n$.
Since the two terms have different degrees (n and $-n$), the function z is not homogeneous in the traditional sense. However, we can still apply Euler's theorem to each homogeneous component separately.
Step 2: Apply Euler's theorem to each component.
For a homogeneous function $g(x, y)$ of degree m, Euler's theorem states:

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = mg(x, y) \tag{4.190}$$

For z_1 (homogeneous of degree n):

$$x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y} = nz_1 \tag{4.191}$$

For z_2 (homogeneous of degree $-n$):

$$x\frac{\partial z_2}{\partial x} + y\frac{\partial z_2}{\partial y} = -nz_2 \tag{4.192}$$

Therefore, for the complete function $z = z_1 + z_2$:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y} + x\frac{\partial z_2}{\partial x} + y\frac{\partial z_2}{\partial y} \tag{4.193}$$

$$= nz_1 + (-n)z_2 \tag{4.194}$$

$$= n(z_1 - z_2) \tag{4.195}$$

Step 3: Apply the extension of Euler's theorem for second derivatives.
For a homogeneous function $g(x, y)$ of degree m, the second-order extension of Euler's theorem states:

$$x^2\frac{\partial^2 g}{\partial x^2} + 2xy\frac{\partial^2 g}{\partial x\partial y} + y^2\frac{\partial^2 g}{\partial y^2} = m(m-1)g \tag{4.196}$$

For z_1 (homogeneous of degree n):

$$x^2\frac{\partial^2 z_1}{\partial x^2} + 2xy\frac{\partial^2 z_1}{\partial x\partial y} + y^2\frac{\partial^2 z_1}{\partial y^2} = n(n-1)z_1 \tag{4.197}$$

For z_2 (homogeneous of degree $-n$):

$$x^2\frac{\partial^2 z_2}{\partial x^2} + 2xy\frac{\partial^2 z_2}{\partial x\partial y} + y^2\frac{\partial^2 z_2}{\partial y^2} = (-n)(-n-1)z_2 = n(n+1)z_2 \tag{4.198}$$

Therefore, for the complete function $z = z_1 + z_2$:

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = x^2\frac{\partial^2 z_1}{\partial x^2} + 2xy\frac{\partial^2 z_1}{\partial x\partial y} + y^2\frac{\partial^2 z_1}{\partial y^2} + x^2\frac{\partial^2 z_2}{\partial x^2} + 2xy\frac{\partial^2 z_2}{\partial x\partial y} + y^2\frac{\partial^2 z_2}{\partial y^2} \tag{4.199}$$

$$= n(n-1)z_1 + n(n+1)z_2 \tag{4.200}$$

$$= n^2 z_1 - nz_1 + n^2 z_2 + nz_2 \tag{4.201}$$

$$= n^2(z_1 + z_2) + n(z_2 - z_1) \tag{4.202}$$

$$= n^2 z - n(z_1 - z_2) \tag{4.203}$$

Step 4: Combine the results to prove the required relation.
Now we need to evaluate:

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \tag{4.204}$$

From steps 2 and 3, we have:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n(z_1 - z_2) \tag{4.205}$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n^2 z - n(z_1 - z_2) \tag{4.206}$$

Combining these:

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z - n(z_1 - z_2) + n(z_1 - z_2) \tag{4.207}$$

$$= n^2 z - n(z_1 - z_2) + n(z_1 - z_2) \tag{4.208}$$

$$= n^2 z \tag{4.209}$$

Final Answer: We have proven that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$

Example 8: Second-Order Derivatives and Euler's Theorem

Find $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$ if $u = \log(x^3 + y^3 - x^2y - xy^2)$.

Simplified Solution Using Euler's Deduction

Step 1: Let's introduce a substitution to simplify the problem.
Let $z = e^u = x^3 + y^3 - x^2y - xy^2$
First, we verify that z is homogeneous:

$$z(tx, ty) = (tx)^3 + (ty)^3 - (tx)^2(ty) - (tx)(ty)^2 \tag{4.210}$$
$$= t^3x^3 + t^3y^3 - t^3x^2y - t^3xy^2 \tag{4.211}$$
$$= t^3(x^3 + y^3 - x^2y - xy^2) \tag{4.212}$$
$$= t^3z(x, y) \tag{4.213}$$

So z is homogeneous of degree 3.
Step 2: Apply the deduction from Euler's theorem for homogeneous functions.
If $z = f(u)$ is a homogeneous function of degree n, then:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \tag{4.214}$$

where $g(u) = n\frac{f(u)}{f'(u)}$.
In our case, $z = e^u$ is homogeneous of degree 3, so:

$$g(u) = 3\frac{e^u}{e^u} \tag{4.215}$$
$$= 3 \tag{4.216}$$

Therefore:

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = g(u)[g'(u) - 1] \tag{4.217}$$
$$= 3[0 - 1] \tag{4.218}$$
$$= -3 \tag{4.219}$$

Final Answer: The value of $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$ is $\boxed{-3}$.

Example 9: Application of Euler's Theorem Deduction

If $u = \operatorname{cosec}^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ then, show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$

Smart Solution Using Euler's Theorem Deduction

Step 1: Let $z = \operatorname{cosec}(u) = \sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

First, verify that z is homogeneous:

$$z(tx, ty) = \sqrt{\frac{(tx)^{1/2} + (ty)^{1/2}}{(tx)^{1/3} + (ty)^{1/3}}} \tag{4.220}$$

$$= \sqrt{\frac{t^{1/2}(x^{1/2} + y^{1/2})}{t^{1/3}(x^{1/3} + y^{1/3})}} \tag{4.221}$$

$$= t^{(1/2-1/3)/2} \cdot z(x, y) \tag{4.222}$$

$$= t^{1/12} \cdot z(x, y) \tag{4.223}$$

So z is homogeneous of degree $\frac{1}{12}$.

Step 2: Apply the deduction from Euler's theorem.

From the deduction of Euler's theorem, if $z = f(u)$ is homogeneous of degree n, then:

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \tag{4.224}$$

where $g(u) = n\frac{f(u)}{f'(u)}$

Since $z = \text{cosec}(u)$, we have $f(u) = \text{cosec}(u)$ and $f'(u) = -\text{cosec}(u)\cot(u)$

Therefore:

$$g(u) = \frac{1}{12} \cdot \frac{\text{cosec}(u)}{-\text{cosec}(u)\cot(u)} \tag{4.225}$$

$$= \frac{1}{12} \cdot \frac{1}{-\cot(u)} \tag{4.226}$$

$$= -\frac{1}{12} \cdot \frac{1}{\frac{\cos(u)}{\sin(u)}} \tag{4.227}$$

$$= -\frac{1}{12} \cdot \frac{\sin(u)}{\cos(u)} \tag{4.228}$$

$$= -\frac{\tan(u)}{12} \tag{4.229}$$

Now, calculate $g'(u)$:

$$g'(u) = -\frac{1}{12} \cdot \sec^2(u) \tag{4.230}$$

$$= -\frac{1}{12}(1 + \tan^2(u)) \tag{4.231}$$

$$= -\frac{1}{12} - \frac{\tan^2(u)}{12} \tag{4.232}$$

Step 3: Calculate the required expression.

$$g(u)[g'(u) - 1] = -\frac{\tan(u)}{12}\left[-\frac{1}{12} - \frac{\tan^2(u)}{12} - 1\right] \tag{4.233}$$

$$= -\frac{\tan(u)}{12}\left[-\frac{1 + \tan^2(u) + 12}{12}\right] \tag{4.234}$$

$$= -\frac{\tan(u)}{12}\left[-\frac{13 + \tan^2(u)}{12}\right] \tag{4.235}$$

$$= \frac{\tan(u)}{12} \cdot \frac{13 + \tan^2(u)}{12} \tag{4.236}$$

$$= \frac{\tan(u)}{12}\left(\frac{13}{12} + \frac{\tan^2(u)}{12}\right) \tag{4.237}$$

Final Answer: We have shown that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$

Example 10: Deduction of Euler's Theorem

If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then prove that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-\sin u \cos 2u}{4\cos^3 u}$.

Solution Using Euler's Theorem Deduction

Step 1: Let $z = \sin(u) = \frac{x+y}{\sqrt{x}+\sqrt{y}}$
First, let's verify that z is homogeneous by substituting (tx, ty) for (x, y):

$$z(tx, ty) = \frac{tx + ty}{\sqrt{tx} + \sqrt{ty}} \tag{4.238}$$

$$= \frac{t(x + y)}{t^{1/2}(\sqrt{x} + \sqrt{y})} \tag{4.239}$$

$$= t^{1/2}\frac{x + y}{\sqrt{x} + \sqrt{y}} \tag{4.240}$$

$$= t^{1/2}z(x, y) \tag{4.241}$$

So z is homogeneous of degree $\frac{1}{2}$.
Step 2: Apply the deduction from Euler's theorem.
From the deduction of Euler's theorem, if $z = f(u)$ is homogeneous of degree n, then:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \tag{4.242}$$

where $g(u) = n\frac{f(u)}{f'(u)}$
Since $z = \sin(u)$, we have $f(u) = \sin(u)$ and $f'(u) = \cos(u)$
Therefore:

$$g(u) = \frac{1}{2} \cdot \frac{\sin(u)}{\cos(u)} \tag{4.243}$$

$$= \frac{\sin(u)}{2\cos(u)} \tag{4.244}$$

$$= \frac{\tan(u)}{2} \tag{4.245}$$

Now, calculate $g'(u)$:

$$g'(u) = \frac{1}{2} \cdot \sec^2(u) \tag{4.246}$$

$$= \frac{1}{2} \cdot \frac{1}{\cos^2(u)} \tag{4.247}$$

$$= \frac{1}{2\cos^2(u)} \tag{4.248}$$

Step 3: Calculate the required expression.

$$g(u)[g'(u)-1] = \frac{\tan(u)}{2}\left[\frac{1}{2\cos^2(u)}-1\right] \tag{4.249}$$

$$= \frac{\sin(u)}{2\cos(u)}\left[\frac{1}{2\cos^2(u)}-1\right] \tag{4.250}$$

$$= \frac{\sin(u)}{2\cos(u)}\left[\frac{1-2\cos^2(u)}{2\cos^2(u)}\right] \tag{4.251}$$

$$= \frac{\sin(u)}{2\cos(u)}\cdot\frac{1-2\cos^2(u)}{2\cos^2(u)} \tag{4.252}$$

$$= \frac{\sin(u)}{4\cos^3(u)}(1-2\cos^2(u)) \tag{4.253}$$

Now, we need to convert this to the required form. Note that $1-2\cos^2(u) = -\cos(2u)$ since $\cos(2u) = 2\cos^2(u)-1$
Therefore:

$$g(u)[g'(u)-1] = \frac{\sin(u)}{4\cos^3(u)}(-\cos(2u)) \tag{4.254}$$

$$= \frac{-\sin(u)\cos(2u)}{4\cos^3(u)} \tag{4.255}$$

Final Answer: We have proven that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-\sin u \cos 2u}{4\cos^3 u}$

Example 11: Deduction of Euler's Theorem

If $u = \sin^{-1}\sqrt{x^2+y^2}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan^3 u$.

Solution Using Euler's Theorem Deduction

Step 1: Let $z = \sin(u) = \sqrt{x^2+y^2}$
First, verify that z is homogeneous by substituting (tx, ty) for (x, y):

$$z(tx,ty) = \sqrt{(tx)^2+(ty)^2} \tag{4.256}$$

$$= \sqrt{t^2(x^2+y^2)} \tag{4.257}$$

$$= |t|\sqrt{x^2+y^2} \tag{4.258}$$

Since $t > 0$ in our context (scaling factor), z is homogeneous of degree 1.
Step 2: Apply the deduction from Euler's theorem.
From the deduction of Euler's theorem, if $z = f(u)$ is homogeneous of degree n, then:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1] \tag{4.259}$$

where $g(u) = n\frac{f(u)}{f'(u)}$
Since $z = \sin(u)$, we have $f(u) = \sin(u)$ and $f'(u) = \cos(u)$
Therefore:

$$g(u) = 1\cdot\frac{\sin(u)}{\cos(u)} \tag{4.260}$$

$$= \tan(u) \tag{4.261}$$

Now, calculate $g'(u)$:

$$g'(u) = \sec^2(u) \tag{4.262}$$
$$= \frac{1}{\cos^2(u)} \tag{4.263}$$

Step 3: Calculate the required expression.

$$g(u)[g'(u) - 1] = \tan(u)\left[\frac{1}{\cos^2(u)} - 1\right] \tag{4.264}$$
$$= \tan(u)\left[\frac{1 - \cos^2(u)}{\cos^2(u)}\right] \tag{4.265}$$
$$= \tan(u)\left[\frac{\sin^2(u)}{\cos^2(u)}\right] \tag{4.266}$$
$$= \tan(u) \cdot \tan^2(u) \tag{4.267}$$
$$= \tan^3(u) \tag{4.268}$$

Final Answer: We have proven that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan^3 u$

Example 12: Deduction of Euler's Theorem

If $u = \sin^{-1}(x^2 + y^2)^{1/5}$, then prove that: $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = \frac{2}{5}\tan u\left[\frac{2}{5}\tan^2 u - \frac{3}{5}\right]$.

Solution Using Euler's Theorem Deduction

Step 1: Let $z = \sin(u) = (x^2 + y^2)^{1/5}$
First, verify that z is homogeneous by substituting (tx, ty) for (x, y):

$$z(tx, ty) = ((tx)^2 + (ty)^2)^{1/5} \tag{4.269}$$
$$= (t^2(x^2 + y^2))^{1/5} \tag{4.270}$$
$$= t^{2/5}(x^2 + y^2)^{1/5} \tag{4.271}$$
$$= t^{2/5} z(x, y) \tag{4.272}$$

So z is homogeneous of degree $\frac{2}{5}$.
Step 2: Apply the deduction from Euler's theorem.
From the deduction of Euler's theorem, if $z = f(u)$ is homogeneous of degree n, then:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \tag{4.273}$$

where $g(u) = n\frac{f(u)}{f'(u)}$
Since $z = \sin(u)$, we have $f(u) = \sin(u)$ and $f'(u) = \cos(u)$
Therefore:

$$g(u) = \frac{2}{5} \cdot \frac{\sin(u)}{\cos(u)} \tag{4.274}$$
$$= \frac{2}{5}\tan(u) \tag{4.275}$$

Now, calculate $g'(u)$:

$$g'(u) = \frac{2}{5} \cdot \sec^2(u) \tag{4.276}$$
$$= \frac{2}{5} \cdot \frac{1}{\cos^2(u)} \tag{4.277}$$
$$= \frac{2}{5\cos^2(u)} \tag{4.278}$$

Step 3: Calculate the required expression.

$$g(u)[g'(u) - 1] = \frac{2}{5}\tan(u)\left[\frac{2}{5\cos^2(u)} - 1\right] \tag{4.279}$$
$$= \frac{2}{5}\tan(u)\left[\frac{2}{5\cos^2(u)} - \frac{5}{5}\right] \tag{4.280}$$
$$= \frac{2}{5}\tan(u)\left[\frac{2 - 5\cos^2(u)}{5\cos^2(u)}\right] \tag{4.281}$$
$$\tag{4.282}$$

Let's expand the numerator:

$$2 - 5\cos^2(u) = 2 - 5(1 - \sin^2(u)) \tag{4.283}$$
$$= 2 - 5 + 5\sin^2(u) \tag{4.284}$$
$$= -3 + 5\sin^2(u) \tag{4.285}$$

Continuing with our calculation:

$$g(u)[g'(u) - 1] = \frac{2}{5}\tan(u)\left[\frac{-3 + 5\sin^2(u)}{5\cos^2(u)}\right] \tag{4.286}$$
$$= \frac{2}{5}\tan(u)\left[\frac{-3}{5\cos^2(u)} + \frac{5\sin^2(u)}{5\cos^2(u)}\right] \tag{4.287}$$
$$= \frac{2}{5}\tan(u)\left[-\frac{3}{5\cos^2(u)} + \frac{\sin^2(u)}{\cos^2(u)}\right] \tag{4.288}$$
$$= \frac{2}{5}\tan(u)\left[-\frac{3}{5\cos^2(u)} + \tan^2(u)\right] \tag{4.289}$$
$$\tag{4.290}$$

Since $\frac{1}{\cos^2(u)} = 1 + \tan^2(u)$, we can write:

$$\frac{-3}{5\cos^2(u)} = \frac{-3}{5}(1 + \tan^2(u)) \tag{4.291}$$
$$= -\frac{3}{5} - \frac{3}{5}\tan^2(u) \tag{4.292}$$

Finally:

$$g(u)[g'(u)-1] = \frac{2}{5}\tan(u)\left[-\frac{3}{5}-\frac{3}{5}\tan^2(u)+\tan^2(u)\right] \tag{4.293}$$
$$= \frac{2}{5}\tan(u)\left[-\frac{3}{5}+\tan^2(u)-\frac{3}{5}\tan^2(u)\right] \tag{4.294}$$
$$= \frac{2}{5}\tan(u)\left[-\frac{3}{5}+\tan^2(u)(1-\frac{3}{5})\right] \tag{4.295}$$
$$= \frac{2}{5}\tan(u)\left[-\frac{3}{5}+\tan^2(u)\cdot\frac{2}{5}\right] \tag{4.296}$$
$$= \frac{2}{5}\tan(u)\left[\frac{2}{5}\tan^2(u)-\frac{3}{5}\right] \tag{4.297}$$
$$\tag{4.298}$$

Final Answer: We have proven that $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\frac{2}{5}\tan u\left[\frac{2}{5}\tan^2 u-\frac{3}{5}\right]$

Example 13: Deduction of Euler's Theorem

If $u=\sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]^{\frac{1}{2}}$ then show that $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\frac{1}{4}\tan u\times[\tan^2 u-1]$.

Solution Using Euler's Theorem Deduction

Step 1: Let $z=\sin(u)=\left[\frac{x^2+y^2}{x+y}\right]^{\frac{1}{2}}$
First, verify that z is homogeneous by substituting (tx,ty) for (x,y):

$$z(tx,ty) = \left[\frac{(tx)^2+(ty)^2}{(tx)+(ty)}\right]^{\frac{1}{2}} \tag{4.299}$$
$$= \left[\frac{t^2(x^2+y^2)}{t(x+y)}\right]^{\frac{1}{2}} \tag{4.300}$$
$$= \left[t\cdot\frac{x^2+y^2}{x+y}\right]^{\frac{1}{2}} \tag{4.301}$$
$$= t^{\frac{1}{2}}\cdot\left[\frac{x^2+y^2}{x+y}\right]^{\frac{1}{2}} \tag{4.302}$$
$$= t^{\frac{1}{2}}\cdot z(x,y) \tag{4.303}$$

So z is homogeneous of degree $\frac{1}{2}$.
Step 2: Apply the deduction from Euler's theorem.
From the deduction of Euler's theorem, if $z=f(u)$ is homogeneous of degree n, then:

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1] \tag{4.304}$$

where $g(u)=n\frac{f(u)}{f'(u)}$
Since $z=\sin(u)$, we have $f(u)=\sin(u)$ and $f'(u)=\cos(u)$

Therefore:

$$g(u) = \frac{1}{2} \cdot \frac{\sin(u)}{\cos(u)} \tag{4.305}$$
$$= \frac{1}{2}\tan(u) \tag{4.306}$$

Now, calculate $g'(u)$:

$$g'(u) = \frac{1}{2} \cdot \sec^2(u) \tag{4.307}$$
$$= \frac{1}{2} \cdot \frac{1}{\cos^2(u)} \tag{4.308}$$
$$= \frac{1}{2\cos^2(u)} \tag{4.309}$$

Step 3: Calculate the required expression.

$$g(u)[g'(u) - 1] = \frac{1}{2}\tan(u)\left[\frac{1}{2\cos^2(u)} - 1\right] \tag{4.310}$$
$$= \frac{1}{2}\tan(u)\left[\frac{1 - 2\cos^2(u)}{2\cos^2(u)}\right] \tag{4.311}$$
$$\tag{4.312}$$

We can simplify the numerator:

$$1 - 2\cos^2(u) = 1 - 2(1 - \sin^2(u)) \tag{4.313}$$
$$= 1 - 2 + 2\sin^2(u) \tag{4.314}$$
$$= -1 + 2\sin^2(u) \tag{4.315}$$

Continuing with our calculation:

$$g(u)[g'(u) - 1] = \frac{1}{2}\tan(u)\left[\frac{-1 + 2\sin^2(u)}{2\cos^2(u)}\right] \tag{4.316}$$
$$= \frac{1}{2}\tan(u)\left[\frac{-1}{2\cos^2(u)} + \frac{2\sin^2(u)}{2\cos^2(u)}\right] \tag{4.317}$$
$$= \frac{1}{2}\tan(u)\left[\frac{-1}{2\cos^2(u)} + \frac{\sin^2(u)}{\cos^2(u)}\right] \tag{4.318}$$
$$= \frac{1}{2}\tan(u)\left[\frac{-1}{2\cos^2(u)} + \tan^2(u)\right] \tag{4.319}$$
$$\tag{4.320}$$

Since $\frac{1}{\cos^2(u)} = 1 + \tan^2(u)$, we have:

$$\frac{-1}{2\cos^2(u)} = \frac{-1}{2}(1 + \tan^2(u)) \tag{4.321}$$
$$= -\frac{1}{2} - \frac{1}{2}\tan^2(u) \tag{4.322}$$

Substituting back:

$$g(u)[g'(u)-1] = \frac{1}{2}\tan(u)\left[-\frac{1}{2}-\frac{1}{2}\tan^2(u)+\tan^2(u)\right] \tag{4.323}$$
$$= \frac{1}{2}\tan(u)\left[-\frac{1}{2}+\tan^2(u)-\frac{1}{2}\tan^2(u)\right] \tag{4.324}$$
$$= \frac{1}{2}\tan(u)\left[-\frac{1}{2}+\frac{1}{2}\tan^2(u)\right] \tag{4.325}$$
$$= \frac{1}{2}\tan(u)\cdot\frac{1}{2}[\tan^2(u)-1] \tag{4.326}$$
$$= \frac{1}{4}\tan(u)[\tan^2(u)-1] \tag{4.327}$$

Final Answer: We have shown that $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\frac{1}{4}\tan u\times[\tan^2 u-1]$

Example 14: Deduction of Euler's Theorem

If $u=\tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$ then prove that $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\sin 2u[1-4\sin^2 u]$.

Solution Using Euler's Theorem Deduction

Step 1: Let $z=\tan(u)=\frac{x^3+y^3}{x+y}$
First, verify that z is homogeneous by substituting (tx,ty) for (x,y):

$$z(tx,ty)=\frac{(tx)^3+(ty)^3}{(tx)+(ty)} \tag{4.328}$$
$$=\frac{t^3(x^3+y^3)}{t(x+y)} \tag{4.329}$$
$$=t^2\cdot\frac{x^3+y^3}{x+y} \tag{4.330}$$
$$=t^2\cdot z(x,y) \tag{4.331}$$

So z is homogeneous of degree 2.
Step 2: Apply the deduction from Euler's theorem.
From the deduction of Euler's theorem, if $z=f(u)$ is homogeneous of degree n, then:

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1] \tag{4.332}$$

where $g(u)=n\frac{f(u)}{f'(u)}$
Since $z=\tan(u)$, we have $f(u)=\tan(u)$ and $f'(u)=\sec^2(u)$

Therefore:

$$g(u) = 2 \cdot \frac{\tan(u)}{\sec^2(u)} \tag{4.333}$$
$$= 2 \cdot \frac{\tan(u)}{1 + \tan^2(u)} \tag{4.334}$$
$$= 2 \cdot \frac{\sin(u)/\cos(u)}{1/\cos^2(u)} \tag{4.335}$$
$$= 2 \cdot \frac{\sin(u)/\cos(u) \cdot \cos^2(u)}{1} \tag{4.336}$$
$$= 2 \cdot \sin(u)\cos(u) \tag{4.337}$$
$$= \sin(2u) \tag{4.338}$$

Now, calculate $g'(u)$:

$$g'(u) = \frac{d}{du}[\sin(2u)] \tag{4.339}$$
$$= 2\cos(2u) \tag{4.340}$$

Step 3: Calculate the required expression.

$$g(u)[g'(u) - 1] = \sin(2u)[2\cos(2u) - 1] \tag{4.341}$$
$$\tag{4.342}$$

We need to express $\cos(2u)$ in terms of $\sin(u)$:

$$\cos(2u) = \cos^2(u) - \sin^2(u) \tag{4.343}$$
$$= 1 - \sin^2(u) - \sin^2(u) \tag{4.344}$$
$$= 1 - 2\sin^2(u) \tag{4.345}$$

Substituting back:

$$g(u)[g'(u) - 1] = \sin(2u)[2(1 - 2\sin^2(u)) - 1] \tag{4.346}$$
$$= \sin(2u)[2 - 4\sin^2(u) - 1] \tag{4.347}$$
$$= \sin(2u)[1 - 4\sin^2(u)] \tag{4.348}$$

Final Answer: We have proven that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \sin 2u[1 - 4\sin^2 u]$

Example 15: Deduction of Euler's Theorem

If $u = \tan^{-1}\left[\frac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}\right]$, then prove that: $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u$.

Solution Using Euler's Theorem Deduction

Step 1: Let $z = \tan(u) = \frac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}$

First, verify that z is homogeneous by substituting (tx, ty) for (x, y):

$$z(tx, ty) = \frac{\sqrt{(tx)^3 + (ty)^3}}{\sqrt{tx} + \sqrt{ty}} \tag{4.349}$$
$$= \frac{\sqrt{t^3(x^3 + y^3)}}{t^{1/2}(\sqrt{x} + \sqrt{y})} \tag{4.350}$$
$$= \frac{t^{3/2}\sqrt{x^3 + y^3}}{t^{1/2}(\sqrt{x} + \sqrt{y})} \tag{4.351}$$
$$= t^1 \cdot \frac{\sqrt{x^3 + y^3}}{\sqrt{x} + \sqrt{y}} \tag{4.352}$$
$$= t \cdot z(x, y) \tag{4.353}$$

So z is homogeneous of degree 1.
Step 2: Apply the deduction from Euler's theorem.
From the deduction of Euler's theorem, if $z = f(u)$ is homogeneous of degree n, then:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \tag{4.354}$$

where $g(u) = n\frac{f(u)}{f'(u)}$
Since $z = \tan(u)$, we have $f(u) = \tan(u)$ and $f'(u) = \sec^2(u)$
Therefore:

$$g(u) = 1 \cdot \frac{\tan(u)}{\sec^2(u)} \tag{4.355}$$
$$= \frac{\tan(u)}{1 + \tan^2(u)} \tag{4.356}$$
$$= \frac{\sin(u)/\cos(u)}{1/\cos^2(u)} \tag{4.357}$$
$$= \frac{\sin(u)/\cos(u) \cdot \cos^2(u)}{1} \tag{4.358}$$
$$= \sin(u)\cos(u) \tag{4.359}$$
$$= \frac{\sin(2u)}{2} \tag{4.360}$$

Now, calculate $g'(u)$:

$$g'(u) = \frac{d}{du}\left[\frac{\sin(2u)}{2}\right] \tag{4.361}$$
$$= \frac{2\cos(2u)}{2} \tag{4.362}$$
$$= \cos(2u) \tag{4.363}$$

Step 3: Calculate the required expression.

$$g(u)[g'(u) - 1] = \frac{\sin(2u)}{2}[\cos(2u) - 1] \tag{4.364}$$
$$\tag{4.365}$$

Using the identity $\cos(2u) - 1 = -2\sin^2(u)$:

$$g(u)[g'(u) - 1] = \frac{\sin(2u)}{2}[-2\sin^2(u)] \tag{4.366}$$

$$= \frac{\sin(2u)}{2} \cdot (-2\sin^2(u)) \tag{4.367}$$

$$= -\sin(2u)\sin^2(u) \tag{4.368}$$

Final Answer: We have proven that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u$

Example 16: Deduction of Euler's Theorem

If $u = \sec^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ then prove that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = -\frac{1}{4}\cot u[3 + \cot^2 u]$.

Solution Using Euler's Theorem Deduction

Step 1: Let $z = \sec(u) = \frac{x+y}{\sqrt{x}+\sqrt{y}}$
First, verify that z is homogeneous by substituting (tx, ty) for (x, y):

$$z(tx, ty) = \frac{tx + ty}{\sqrt{tx} + \sqrt{ty}} \tag{4.369}$$

$$= \frac{t(x+y)}{t^{1/2}(\sqrt{x} + \sqrt{y})} \tag{4.370}$$

$$= t^{1/2} \cdot \frac{x+y}{\sqrt{x}+\sqrt{y}} \tag{4.371}$$

$$= t^{1/2} \cdot z(x, y) \tag{4.372}$$

So z is homogeneous of degree $\frac{1}{2}$.
Step 2: Apply the deduction from Euler's theorem.
From the deduction of Euler's theorem, if $z = f(u)$ is homogeneous of degree n, then:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \tag{4.373}$$

where $g(u) = n\frac{f(u)}{f'(u)}$
Since $z = \sec(u)$, we have $f(u) = \sec(u)$ and $f'(u) = \sec(u)\tan(u)$
Therefore:

$$g(u) = \frac{1}{2} \cdot \frac{\sec(u)}{\sec(u)\tan(u)} \tag{4.374}$$

$$= \frac{1}{2} \cdot \frac{1}{\tan(u)} \tag{4.375}$$

$$= \frac{1}{2} \cdot \cot(u) \tag{4.376}$$

Now, calculate $g'(u)$:

$$g'(u) = \frac{1}{2} \cdot \frac{d}{du}[\cot(u)] \tag{4.377}$$

$$= \frac{1}{2} \cdot (-\csc^2(u)) \tag{4.378}$$

$$= -\frac{1}{2\sin^2(u)} \tag{4.379}$$

Step 3: Calculate the required expression.

$$g(u)[g'(u)-1] = \frac{\cot(u)}{2}\left[-\frac{1}{2\sin^2(u)} - 1\right] \tag{4.380}$$
$$= \frac{\cot(u)}{2}\left[-\frac{1}{2\sin^2(u)} - \frac{2\sin^2(u)}{2\sin^2(u)}\right] \tag{4.381}$$
$$= \frac{\cot(u)}{2}\left[-\frac{1+2\sin^2(u)}{2\sin^2(u)}\right] \tag{4.382}$$
$$= -\frac{\cot(u)}{4}\cdot\frac{1+2\sin^2(u)}{\sin^2(u)} \tag{4.383}$$
$$\tag{4.384}$$

Let's simplify the fraction:

$$\frac{1+2\sin^2(u)}{\sin^2(u)} = \frac{1}{\sin^2(u)} + \frac{2\sin^2(u)}{\sin^2(u)} \tag{4.385}$$
$$= \csc^2(u) + 2 \tag{4.386}$$
$$= 1 + \cot^2(u) + 2 \tag{4.387}$$
$$= 3 + \cot^2(u) \tag{4.388}$$

Therefore:

$$g(u)[g'(u)-1] = -\frac{\cot(u)}{4}\cdot(3+\cot^2(u)) \tag{4.389}$$
$$= -\frac{1}{4}\cot(u)[3+\cot^2(u)] \tag{4.390}$$

Final Answer: We have proven that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = -\frac{1}{4}\cot u[3+\cot^2 u]$

Example 17: Application of Euler's Theorem

If $u = \cos\left(\frac{xy}{x^2+y^2}\right) + \sqrt{x^2+y^2} + \frac{xy^2}{x+y}$, find the value of $xu_x + yu_y$ at the point $(3,4)$.

Solution Using Euler's Theorem

Step 1: Let's analyze each term of u to determine if it's homogeneous and its degree. We have $u = u_1 + u_2 + u_3$, where:

$$u_1 = \cos\left(\frac{xy}{x^2+y^2}\right) \tag{4.391}$$
$$u_2 = \sqrt{x^2+y^2} \tag{4.392}$$
$$u_3 = \frac{xy^2}{x+y} \tag{4.393}$$

For u_1, let's examine the argument of cosine:

$$\frac{(tx)(ty)}{(tx)^2+(ty)^2} = \frac{t^2xy}{t^2(x^2+y^2)} \tag{4.394}$$
$$= \frac{xy}{x^2+y^2} \tag{4.395}$$

Since the argument remains unchanged when we substitute (tx, ty) for (x, y), and cosine is a function of this argument, we have:

$$u_1(tx, ty) = u_1(x, y) \tag{4.396}$$

So u_1 is homogeneous of degree 0.
For u_2:

$$u_2(tx, ty) = \sqrt{(tx)^2 + (ty)^2} \tag{4.397}$$
$$= \sqrt{t^2(x^2 + y^2)} \tag{4.398}$$
$$= |t|\sqrt{x^2 + y^2} \tag{4.399}$$

Since $t > 0$ in our context (scaling factor), u_2 is homogeneous of degree 1.
For u_3:

$$u_3(tx, ty) = \frac{(tx)(ty)^2}{(tx) + (ty)} \tag{4.400}$$
$$= \frac{t^3xy^2}{t(x + y)} \tag{4.401}$$
$$= t^2\frac{xy^2}{x + y} \tag{4.402}$$
$$= t^2u_3(x, y) \tag{4.403}$$

So u_3 is homogeneous of degree 2.
Step 2: Apply Euler's theorem to each component.
According to Euler's theorem, if $f(x, y)$ is homogeneous of degree n, then:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y) \tag{4.404}$$

For u_1 (homogeneous of degree 0):

$$x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} = 0 \cdot u_1 = 0 \tag{4.405}$$

For u_2 (homogeneous of degree 1):

$$x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} = 1 \cdot u_2 = \sqrt{x^2 + y^2} \tag{4.406}$$

For u_3 (homogeneous of degree 2):

$$x\frac{\partial u_3}{\partial x} + y\frac{\partial u_3}{\partial y} = 2 \cdot u_3 = 2 \cdot \frac{xy^2}{x + y} \tag{4.407}$$

Step 3: Combine the results and evaluate at the point $(3, 4)$.
Since $u = u_1 + u_2 + u_3$, and the differential operator is linear:

$$xu_x + yu_y = x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} + x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} + x\frac{\partial u_3}{\partial x} + y\frac{\partial u_3}{\partial y} \tag{4.408}$$
$$= 0 + \sqrt{x^2 + y^2} + 2\frac{xy^2}{x + y} \tag{4.409}$$

Now, evaluate at the point $(3,4)$:

$$(xu_x + yu_y)|_{(3,4)} = \sqrt{x^2+y^2}\Big|_{(3,4)} + 2\frac{xy^2}{x+y}\Big|_{(3,4)} \tag{4.410}$$
$$= \sqrt{3^2+4^2} + 2\frac{3\cdot 4^2}{3+4} \tag{4.411}$$
$$= \sqrt{9+16} + 2\frac{3\cdot 16}{7} \tag{4.412}$$
$$= \sqrt{25} + 2\frac{48}{7} \tag{4.413}$$
$$= 5 + \frac{96}{7} \tag{4.414}$$
$$= 5 + \frac{96}{7} \tag{4.415}$$
$$= \frac{35}{7} + \frac{96}{7} \tag{4.416}$$
$$= \frac{131}{7} \tag{4.417}$$

Final Answer: The value of $xu_x + yu_y$ at the point $(3,4)$ is $\boxed{\frac{131}{7}}$.

Example 18: Application of Euler's Theorem to Second Derivatives

If $u = \frac{x^3+y^3}{y\sqrt{x}} + \frac{1}{x}\sin^{-1}\left(\frac{x^2+y^2}{2xy}\right)$, find the value of $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$ at the point $(1,1)$.

Solution Using Euler's Theorem

Step 1: Let's analyze each term of u to determine if it's homogeneous and its degree. We have $u = u_1 + u_2$, where:

$$u_1 = \frac{x^3+y^3}{y\sqrt{x}} \tag{4.418}$$
$$u_2 = \frac{1}{x}\sin^{-1}\left(\frac{x^2+y^2}{2xy}\right) \tag{4.419}$$

For u_1, let's check if it's homogeneous by substituting (tx, ty) for (x,y):

$$u_1(tx,ty) = \frac{(tx)^3+(ty)^3}{(ty)\sqrt{tx}} \tag{4.420}$$
$$= \frac{t^3x^3+t^3y^3}{ty\cdot t^{1/2}\sqrt{x}} \tag{4.421}$$
$$= \frac{t^3(x^3+y^3)}{t^{3/2}y\sqrt{x}} \tag{4.422}$$
$$= t^{3/2}\frac{x^3+y^3}{y\sqrt{x}} \tag{4.423}$$
$$= t^{3/2}u_1(x,y) \tag{4.424}$$

So u_1 is homogeneous of degree $\frac{3}{2}$.

For u_2, let's first examine the argument of $\sin^{-1}$:

$$\frac{(tx)^2 + (ty)^2}{2(tx)(ty)} = \frac{t^2(x^2 + y^2)}{2t^2xy} \tag{4.425}$$

$$= \frac{x^2 + y^2}{2xy} \tag{4.426}$$

The argument doesn't change with t. Now for the entire term u_2:

$$u_2(tx, ty) = \frac{1}{tx}\sin^{-1}\left(\frac{(tx)^2 + (ty)^2}{2(tx)(ty)}\right) \tag{4.427}$$

$$= \frac{1}{tx}\sin^{-1}\left(\frac{x^2 + y^2}{2xy}\right) \tag{4.428}$$

$$= t^{-1} \cdot \frac{1}{x}\sin^{-1}\left(\frac{x^2 + y^2}{2xy}\right) \tag{4.429}$$

$$= t^{-1}u_2(x, y) \tag{4.430}$$

So u_2 is homogeneous of degree -1.
Step 2: Apply the extension of Euler's theorem for second derivatives.
For a homogeneous function $f(x, y)$ of degree n, the second-order extension states:

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f(x, y) \tag{4.431}$$

For u_1 (homogeneous of degree $\frac{3}{2}$):

$$x^2\frac{\partial^2 u_1}{\partial x^2} + 2xy\frac{\partial^2 u_1}{\partial x\partial y} + y^2\frac{\partial^2 u_1}{\partial y^2} = \frac{3}{2}\left(\frac{3}{2} - 1\right)u_1 \tag{4.432}$$

$$= \frac{3}{2} \cdot \frac{1}{2} \cdot u_1 \tag{4.433}$$

$$= \frac{3}{4}u_1 \tag{4.434}$$

$$= \frac{3}{4} \cdot \frac{x^3 + y^3}{y\sqrt{x}} \tag{4.435}$$

For u_2 (homogeneous of degree -1):

$$x^2\frac{\partial^2 u_2}{\partial x^2} + 2xy\frac{\partial^2 u_2}{\partial x\partial y} + y^2\frac{\partial^2 u_2}{\partial y^2} = (-1)(-1-1)u_2 \tag{4.436}$$

$$= (-1)(-2)u_2 \tag{4.437}$$

$$= 2u_2 \tag{4.438}$$

$$= 2 \cdot \frac{1}{x}\sin^{-1}\left(\frac{x^2 + y^2}{2xy}\right) \tag{4.439}$$

Step 3: Combine the results and evaluate at the point $(1, 1)$.
Since $u = u_1 + u_2$, and the differential operator is linear:

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{3}{4} \cdot \frac{x^3 + y^3}{y\sqrt{x}} + 2 \cdot \frac{1}{x}\sin^{-1}\left(\frac{x^2 + y^2}{2xy}\right) \tag{4.440}$$

At the point $(1,1)$:

$$\left.(x^2u_{xx}+2xyu_{xy}+y^2u_{yy})\right|_{(1,1)} = \frac{3}{4}\cdot\left.\frac{x^3+y^3}{y\sqrt{x}}\right|_{(1,1)} + 2\cdot\frac{1}{x}\sin^{-1}\left.\left(\frac{x^2+y^2}{2xy}\right)\right|_{(1,1)} \tag{4.441}$$

$$= \frac{3}{4}\cdot\frac{1^3+1^3}{1\sqrt{1}} + 2\cdot\frac{1}{1}\sin^{-1}\left(\frac{1^2+1^2}{2\cdot 1\cdot 1}\right) \tag{4.442}$$

$$= \frac{3}{4}\cdot\frac{2}{1} + 2\cdot\sin^{-1}\left(\frac{2}{2}\right) \tag{4.443}$$

$$= \frac{3}{4}\cdot 2 + 2\cdot\sin^{-1}(1) \tag{4.444}$$

$$= \frac{6}{4} + 2\cdot\frac{\pi}{2} \tag{4.445}$$

$$= \frac{3}{2} + \pi \tag{4.446}$$

$$= 1.5 + \pi \tag{4.447}$$

Final Answer: The value of $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}$ at the point $(1,1)$ is $\boxed{1.5+\pi}$ or $\boxed{\frac{3}{2}+\pi}$.