# Chapter 4

# Vector Differential Calculus III: Divergence and Curl

This chapter explores two fundamental differential operations on vector fields: divergence and curl. These operations, along with the gradient studied in the previous chapter, form the cornerstone of vector calculus and have profound applications in physics and engineering.

## 4.1 Divergence of a Vector Field

### 4.1.1 Definition and Properties

The divergence of a vector field measures the "outflowing" or "expansion" rate of the vector field at a point.

**Definition 4.1** (Divergence)**.** *The divergence of a vector field* $\bar{\mathbf{F}}(x,y,z) = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}}$ *is a scalar field defined as:*

$$div\ \bar{\mathbf{F}} = \nabla \cdot \bar{\mathbf{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{4.1}$$

**Properties of Divergence**

For vector fields $\bar{\mathbf{F}}$ and $\bar{\mathbf{G}}$, scalar field $\phi$, and constant $c$:
1. $\nabla \cdot (\bar{\mathbf{F}} + \bar{\mathbf{G}}) = \nabla \cdot \bar{\mathbf{F}} + \nabla \cdot \bar{\mathbf{G}}$ (Linearity)
2. $\nabla \cdot (c\bar{\mathbf{F}}) = c(\nabla \cdot \bar{\mathbf{F}})$ (Scalar multiplication)
3. $\nabla \cdot (\phi\bar{\mathbf{F}}) = \phi(\nabla \cdot \bar{\mathbf{F}}) + \nabla\phi \cdot \bar{\mathbf{F}}$ (Product rule)
4. $\nabla \cdot (\bar{\mathbf{F}} \times \bar{\mathbf{G}}) = \bar{\mathbf{G}} \cdot (\nabla \times \bar{\mathbf{F}}) - \bar{\mathbf{F}} \cdot (\nabla \times \bar{\mathbf{G}})$

**Computing Divergence**

Calculate the divergence of $\bar{\mathbf{F}}(x,y,z) = x^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$.
Applying the definition:

$$\nabla \cdot \bar{\mathbf{F}} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(z^2) \tag{4.2}$$
$$= 2x + x + 2z \tag{4.3}$$
$$= 3x + 2z \tag{4.4}$$

## 4.2 Physical Interpretation of Divergence

### 4.2.1 Flux Density and Source/Sink Behavior

The divergence has a rich physical interpretation related to the concept of flux density.

**Definition 4.2** (Flux Density)**.** *The divergence $\nabla \cdot \bar{\mathbf{F}}$ at a point represents the net outward flux of the vector field per unit volume as the volume around the point approaches zero.*

This interpretation leads to the following characterizations:

- $\nabla \cdot \bar{\mathbf{F}} > 0$: The point acts as a source (outflow exceeds inflow)
- $\nabla \cdot \bar{\mathbf{F}} < 0$: The point acts as a sink (inflow exceeds outflow)
- $\nabla \cdot \bar{\mathbf{F}} = 0$: The field is locally conservative (outflow equals inflow)

$\nabla \cdot \bar{\mathbf{F}} > 0$ (Source) $\quad$ $\nabla \cdot \bar{\mathbf{F}} < 0$ (Sink)

**Figure 4.1:** Source and sink behavior in vector fields

### 4.2.2 Applications

#### Fluid Dynamics

In fluid dynamics, if $\bar{\mathbf{v}}$ represents the velocity field of a fluid:

- $\nabla \cdot \bar{\mathbf{v}}$ represents the rate of expansion or contraction of the fluid
- $\nabla \cdot \bar{\mathbf{v}} = 0$ indicates an incompressible fluid

#### Electromagnetism

In electromagnetism, according to Gauss's law:

$$\nabla \cdot \bar{\mathbf{E}} = \frac{\rho}{\epsilon_0} \tag{4.5}$$

where $\bar{\mathbf{E}}$ is the electric field, $\rho$ is the charge density, and $\epsilon_0$ is the permittivity of free space. This shows that:

- Electric field diverges from positive charges (sources)
- Electric field converges to negative charges (sinks)
- In charge-free regions, $\nabla \cdot \bar{\mathbf{E}} = 0$

For the magnetic field $\bar{\mathbf{B}}$, Gauss's law for magnetism states:

$$\nabla \cdot \bar{\mathbf{B}} = 0 \tag{4.6}$$

indicating that magnetic monopoles do not exist in classical electromagnetism.

#### Heat Conduction

In heat transfer, if $\bar{\mathbf{q}}$ is the heat flux vector:

$$\nabla \cdot \bar{\mathbf{q}} = -c\rho\frac{\partial T}{\partial t} \tag{4.7}$$

where $c$ is the specific heat capacity, $\rho$ is the density, and $T$ is the temperature. This equation indicates that divergence of heat flux is related to the rate of temperature change.

## 4.3 Curl of a Vector Field

### 4.3.1 Definition and Properties

The curl of a vector field measures the rotational or "spinning" tendency of the field at a point.
**Definition 4.3** (Curl). *The curl of a vector field* $\bar{\mathbf{F}}(x, y, z) = F_1(x, y, z)\hat{\mathbf{i}} + F_2(x, y, z)\hat{\mathbf{j}} + F_3(x, y, z)\hat{\mathbf{k}}$ *is a vector field defined as:*

$$curl\ \bar{\mathbf{F}} = \nabla \times \bar{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{4.8}$$

*This expands to:*

$$\nabla \times \bar{\mathbf{F}} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}} \tag{4.9}$$

**Properties of Curl**

For vector fields $\bar{\mathbf{F}}$ and $\bar{\mathbf{G}}$, scalar field $\phi$, and constant $c$:
1. $\nabla \times (\bar{\mathbf{F}} + \bar{\mathbf{G}}) = \nabla \times \bar{\mathbf{F}} + \nabla \times \bar{\mathbf{G}}$ (Linearity)
2. $\nabla \times (c\bar{\mathbf{F}}) = c(\nabla \times \bar{\mathbf{F}})$ (Scalar multiplication)
3. $\nabla \times (\phi\bar{\mathbf{F}}) = \phi(\nabla \times \bar{\mathbf{F}}) + (\nabla\phi) \times \bar{\mathbf{F}}$ (Product rule)
4. $\nabla \times (\nabla\phi) = \mathbf{0}$ (Curl of a gradient is zero)
5. $\nabla \cdot (\nabla \times \bar{\mathbf{F}}) = 0$ (Divergence of a curl is zero)

**Computing Curl**

Calculate the curl of $\bar{\mathbf{F}}(x, y, z) = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{\mathbf{k}}$.
Applying the definition:

$$\nabla \times \bar{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \tag{4.10}$$

$$= \left(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial y}{\partial z} - \frac{\partial x}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial y}\right)\hat{\mathbf{k}} \tag{4.11}$$

$$= (0 - 1)\hat{\mathbf{i}} + (0 - 1)\hat{\mathbf{j}} + (0 - 0)\hat{\mathbf{k}} \tag{4.12}$$

$$= -\hat{\mathbf{i}} - \hat{\mathbf{j}} \tag{4.13}$$

## 4.4 Physical Interpretation of Curl

### 4.4.1 Rotation and Angular Velocity

The curl of a vector field at a point represents the rotation or "vorticity" of the field around that point.

**Definition 4.4** (Circulation Density)**.** *The curl $\nabla \times \bar{\mathbf{F}}$ at a point represents the circulation per unit area in the plane perpendicular to the curl vector, as the area approaches zero.*
The magnitude of the curl $|\nabla \times \bar{\mathbf{F}}|$ gives the strength of rotation, while the direction of the curl vector gives the axis of rotation following the right-hand rule.

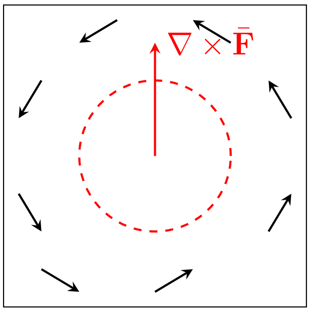


**Figure 4.2:** Circulation and curl in a vector field

### 4.4.2 Applications

#### Fluid Dynamics

In fluid dynamics, if $\bar{\mathbf{v}}$ represents the velocity field of a fluid:

- $\nabla \times \bar{\mathbf{v}}$ represents the vorticity or local rotation of the fluid
- $\nabla \times \bar{\mathbf{v}} = \mathbf{0}$ indicates irrotational flow

**Vorticity in Fluid Flow**

Consider a fluid rotating like a rigid body with angular velocity $\omega$ around the $z$-axis. The velocity field is:

$$\bar{\mathbf{v}} = -\omega y\hat{\mathbf{i}} + \omega x\hat{\mathbf{j}} \tag{4.14}$$

The curl of this velocity field is:

$$\nabla \times \bar{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} \tag{4.15}$$

$$= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(\omega x)\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}(-\omega y) - \frac{\partial}{\partial x}(0)\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right)\hat{\mathbf{k}} \tag{4.16}$$

$$= (0-0)\hat{\mathbf{i}} + (0-0)\hat{\mathbf{j}} + (\omega + \omega)\hat{\mathbf{k}} \tag{4.17}$$

$$= 2\omega\hat{\mathbf{k}} \tag{4.18}$$

This shows that the vorticity is uniform throughout the fluid and is twice the angular velocity, pointing along the axis of rotation.

#### Electromagnetism

In electromagnetism, Maxwell's equations relate the curl of the electric and magnetic fields:

$$\nabla \times \bar{\mathbf{E}} = -\frac{\partial \bar{\mathbf{B}}}{\partial t} \tag{4.19}$$

$$\nabla \times \bar{\mathbf{B}} = \mu_0 \bar{\mathbf{J}} + \mu_0 \epsilon_0 \frac{\partial \bar{\mathbf{E}}}{\partial t} \tag{4.20}$$

where $\bar{\mathbf{E}}$ is the electric field, $\bar{\mathbf{B}}$ is the magnetic field, $\bar{\mathbf{J}}$ is the current density, $\mu_0$ is the permeability of free space, and $\epsilon_0$ is the permittivity of free space.

These equations show that:

- A time-varying magnetic field generates a curling electric field
- Electric currents and time-varying electric fields generate a curling magnetic field

## 4.5 Solenoidal Vector Fields

### 4.5.1 Definition and Properties

**Definition 4.5** (Solenoidal Vector Field). *A vector field $\bar{\mathbf{F}}$ is solenoidal if its divergence is zero throughout the domain:*

$$\nabla \cdot \bar{\mathbf{F}} = 0 \tag{4.21}$$

Solenoidal fields are also called "divergence-free," "incompressible," or "source-free" fields.

**Properties of Solenoidal Fields**

1. The curl of any vector field is solenoidal: $\nabla \cdot (\nabla \times \bar{\mathbf{G}}) = 0$ for any vector field $\bar{\mathbf{G}}$
2. A solenoidal field $\bar{\mathbf{F}}$ can be represented as the curl of another vector field $\bar{\mathbf{A}}$ (called the vector potential):

$$\bar{\mathbf{F}} = \nabla \times \bar{\mathbf{A}} \tag{4.22}$$

### 4.5.2 Examples in Physics

#### Magnetic Fields

The magnetic field $\bar{\mathbf{B}}$ is always solenoidal:

$$\nabla \cdot \bar{\mathbf{B}} = 0 \tag{4.23}$$

This reflects the absence of magnetic monopoles in classical electromagnetism. As a result, the magnetic field can be expressed as the curl of a vector potential $\bar{\mathbf{A}}$:

$$\bar{\mathbf{B}} = \nabla \times \bar{\mathbf{A}} \tag{4.24}$$

#### Incompressible Fluid Flow

For an incompressible fluid with velocity field $\bar{\mathbf{v}}$:

$$\nabla \cdot \bar{\mathbf{v}} = 0 \tag{4.25}$$

This indicates that the fluid density remains constant as it flows. In such cases, the velocity field can sometimes be expressed as the curl of a vector potential known as the stream function.

## 4.6 Irrotational Vector Fields

### 4.6.1 Definition and Properties

**Definition 4.6** (Irrotational Vector Field). *A vector field $\bar{\mathbf{F}}$ is irrotational if its curl is zero throughout the domain:*

$$\nabla \times \bar{\mathbf{F}} = \mathbf{0} \tag{4.26}$$

Irrotational fields are also called "curl-free" fields.

**Properties of Irrotational Fields**

1. The gradient of any scalar field is irrotational: $\nabla \times (\nabla \phi) = \mathbf{0}$ for any scalar field $\phi$
2. An irrotational field $\bar{\mathbf{F}}$ can be represented as the gradient of a scalar potential $\phi$:

$$\bar{\mathbf{F}} = \nabla \phi \tag{4.27}$$

3. Line integrals of irrotational fields are path-independent (conservative property)

**Testing for Irrotational Fields**

Determine if the vector field $\bar{\mathbf{F}}(x, y, z) = (y^2 + z^2)\hat{\mathbf{i}} + (x^2 + z^2)\hat{\mathbf{j}} + (x^2 + y^2)\hat{\mathbf{k}}$ is irrotational. Calculate the curl:

$$\nabla \times \bar{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^2 & x^2 + z^2 & x^2 + y^2 \end{vmatrix} \tag{4.28}$$

$$= \left( \frac{\partial}{\partial y}(x^2 + y^2) - \frac{\partial}{\partial z}(x^2 + z^2) \right) \hat{\mathbf{i}} \tag{4.29}$$

$$+ \left( \frac{\partial}{\partial z}(y^2 + z^2) - \frac{\partial}{\partial x}(x^2 + y^2) \right) \hat{\mathbf{j}} \tag{4.30}$$

$$+ \left( \frac{\partial}{\partial x}(x^2 + z^2) - \frac{\partial}{\partial y}(y^2 + z^2) \right) \hat{\mathbf{k}} \tag{4.31}$$

$$= (2y - 2z)\hat{\mathbf{i}} + (2z - 2x)\hat{\mathbf{j}} + (2x - 2y)\hat{\mathbf{k}} \tag{4.32}$$

Since $\nabla \times \bar{\mathbf{F}} \neq \mathbf{0}$, the field is not irrotational.

### 4.6.2 Examples in Physics

#### Electrostatic Fields

The electric field $\bar{\mathbf{E}}$ in electrostatics is irrotational:

$$\nabla \times \bar{\mathbf{E}} = \mathbf{0} \tag{4.33}$$

This allows the electric field to be expressed as the gradient of the electric potential $V$:

$$\bar{\mathbf{E}} = -\nabla V \tag{4.34}$$

The negative sign indicates that the electric field points from higher to lower potential.

### Gravitational Fields

The gravitational field $\bar{\mathbf{g}}$ is irrotational:

$$\nabla \times \bar{\mathbf{g}} = \mathbf{0} \tag{4.35}$$

It can be expressed as the gradient of the gravitational potential $\phi_G$:

$$\bar{\mathbf{g}} = -\nabla \phi_G \tag{4.36}$$

### Irrotational Fluid Flow

In fluid dynamics, an irrotational flow has no vorticity:

$$\nabla \times \bar{\mathbf{v}} = \mathbf{0} \tag{4.37}$$

This allows the velocity field to be expressed as the gradient of a scalar velocity potential $\phi_v$:

$$\bar{\mathbf{v}} = \nabla \phi_v \tag{4.38}$$

**Key Vector Identities**

For scalar field $\phi$ and vector field $\bar{\mathbf{F}}$:
1. $\nabla \times (\nabla \phi) = \mathbf{0}$ (Curl of a gradient is zero)
2. $\nabla \cdot (\nabla \times \bar{\mathbf{F}}) = 0$ (Divergence of a curl is zero)
3. $\nabla \times (\nabla \times \bar{\mathbf{F}}) = \nabla(\nabla \cdot \bar{\mathbf{F}}) - \nabla^2 \bar{\mathbf{F}}$ (Vector Laplacian identity)
4. $\nabla \cdot (\phi \bar{\mathbf{F}}) = \phi(\nabla \cdot \bar{\mathbf{F}}) + \nabla \phi \cdot \bar{\mathbf{F}}$ (Product rule for divergence)
5. $\nabla \times (\phi \bar{\mathbf{F}}) = \phi(\nabla \times \bar{\mathbf{F}}) + (\nabla \phi) \times \bar{\mathbf{F}}$ (Product rule for curl)

### 4.6.3 Classification of Vector Fields

Based on divergence and curl properties, vector fields can be classified into four categories:

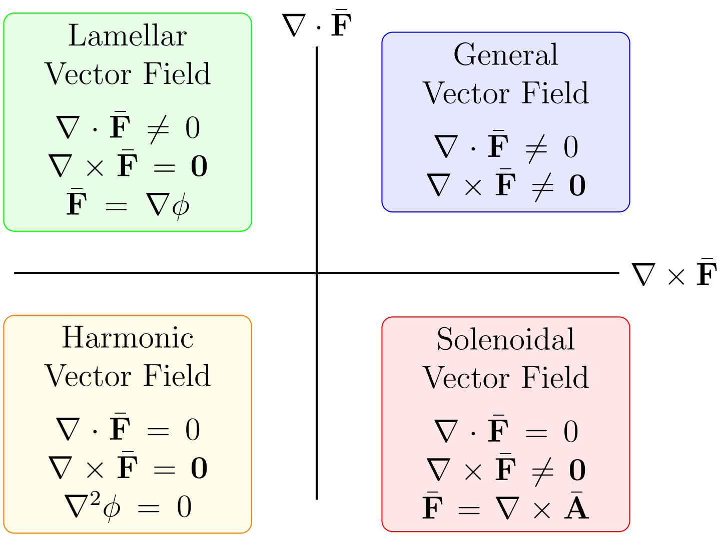


**Figure 4.3:** Classification of vector fields based on divergence and curl properties

## 4.7 Solved Examples on Solenoidal and Irrotational

**Example: Proving Vector is Irrotational and Solenoidal**

Prove that the vector $\vec{F} = \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}$ is irrotational and solenoidal vector.

**Solution**

Given: $\vec{F} = \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}$

A vector field is: - Irrotational if its curl ($\nabla \times \vec{F}$) is zero - Solenoidal if its divergence ($\nabla \cdot \vec{F}$) is zero

Identifying the components:

$$F_1 = \frac{x}{x^2+y^2}$$
$$F_2 = \frac{y}{x^2+y^2}$$
$$F_3 = 0$$

**Part 1: Proving $\vec{F}$ is Irrotational (curl = 0)**

**Step 1:** Calculate the curl of $\vec{F}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Expanding the determinant:

$$\begin{aligned}
\nabla \times \vec{F} &= \hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} - \hat{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} + \hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \\
&= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)
\end{aligned}$$

Since $F_3 = 0$ (our vector field has only x and y components):

$$\begin{aligned}
\nabla \times \vec{F} &= \hat{i}\,(0-0) - \hat{j}\,(0-0) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
&= 0\hat{i} + 0\hat{j} + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)
\end{aligned}$$

The only non-zero component of curl for 2D fields is:

$$(\nabla \times \vec{F})_k = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

**Step 2:** Find $\frac{\partial F_2}{\partial x}$.

$$\begin{aligned}
\frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) \\
&= y \cdot \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right) \\
&= y \cdot \left(-\frac{1}{(x^2+y^2)^2} \cdot 2x\right) \\
&= -\frac{2xy}{(x^2+y^2)^2}
\end{aligned}$$

**Step 3:** Find $\frac{\partial F_1}{\partial y}$.

$$\begin{aligned}\frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) \\ &= x \cdot \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right) \\ &= x \cdot \left(-\frac{1}{(x^2+y^2)^2} \cdot 2y\right) \\ &= -\frac{2xy}{(x^2+y^2)^2}\end{aligned}$$

**Step 4:** Calculate the curl.

$$\begin{aligned}(\nabla \times \vec{F})_k &= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \\ &= -\frac{2xy}{(x^2+y^2)^2} - \left(-\frac{2xy}{(x^2+y^2)^2}\right) \\ &= -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} \\ &= 0\end{aligned}$$

Since $\nabla \times \vec{F} = 0$, the vector field is irrotational.

**Part 2: Proving $\vec{F}$ is Solenoidal (divergence = 0)**

**Step 5:** Calculate the divergence of $\vec{F}$.

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

**Step 6:** Find $\frac{\partial F_1}{\partial x}$.

$$\begin{aligned}\frac{\partial F_1}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) \\ &= \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} \\ &= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} \\ &= \frac{y^2-x^2}{(x^2+y^2)^2}\end{aligned}$$

**Step 7:** Find $\frac{\partial F_2}{\partial y}$.

$$\begin{aligned}\frac{\partial F_2}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) \\ &= \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} \\ &= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \\ &= \frac{x^2-y^2}{(x^2+y^2)^2}\end{aligned}$$

**Step 8:** Calculate the divergence.

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2} + 0 \\ &= \frac{y^2 - x^2 + x^2 - y^2}{(x^2+y^2)^2} \\ &= \frac{0}{(x^2+y^2)^2} \\ &= 0\end{aligned}$$

Since $\nabla \cdot \vec{F} = 0$, the vector field is solenoidal.
**Conclusion:** The vector field $\vec{F} = \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}$ is both irrotational (curl = 0) and solenoidal (divergence = 0).

**Example: Proving Cross Product is Solenoidal**

If $\vec{F_1} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and $\vec{F_2} = (\vec{a} \cdot \vec{r})\vec{a}$ then Show that $\vec{F_1} \times \vec{F_2}$ is solenoidal

**Solution**

Given:

$$\begin{aligned}\vec{F_1} &= yz\hat{i} + zx\hat{j} + xy\hat{k} \\ \vec{F_2} &= (\vec{a} \cdot \vec{r})\vec{a}\end{aligned}$$

Where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ is a constant vector.
**Step 1:** Calculate $\vec{a} \cdot \vec{r}$.
Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

$$\begin{aligned}\vec{a} \cdot \vec{r} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) \\ &= a_1x + a_2y + a_3z\end{aligned}$$

Therefore:

$$\vec{F_2} = (a_1x + a_2y + a_3z)\vec{a}$$

**Step 2:** Calculate $\vec{F_1} \times \vec{F_2}$.

$$\vec{F_1} \times \vec{F_2} = (yz\hat{i} + zx\hat{j} + xy\hat{k}) \times [(a_1x + a_2y + a_3z)\vec{a}]$$

Since scalar quantities can be factored out of cross products:

$$\vec{F_1} \times \vec{F_2} = (a_1x + a_2y + a_3z)(\vec{F_1} \times \vec{a})$$

**Step 3:** Calculate $\vec{F_1} \times \vec{a}$.

$$\begin{aligned}\vec{F_1} \times \vec{a} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ yz & zx & xy \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= \hat{i}(zx \cdot a_3 - xy \cdot a_2) - \hat{j}(yz \cdot a_3 - xy \cdot a_1) + \hat{k}(yz \cdot a_2 - zx \cdot a_1) \\ &= \hat{i}(zxa_3 - xya_2) - \hat{j}(yza_3 - xya_1) + \hat{k}(yza_2 - zxa_1) \\ &= (zxa_3 - xya_2)\hat{i} - (yza_3 - xya_1)\hat{j} + (yza_2 - zxa_1)\hat{k}\end{aligned}$$

**Step 4:** Find the divergence of $\vec{F_1} \times \vec{F_2}$.
From Step 2:

$$\vec{F_1} \times \vec{F_2} = (a_1x + a_2y + a_3z)\vec{G}$$

Where $\vec{G} = \vec{F_1} \times \vec{a}$ is the vector field from Step 3.

$$\begin{aligned}\nabla \cdot (\vec{F_1} \times \vec{F_2}) &= \nabla \cdot [(a_1x + a_2y + a_3z)\vec{G}] \\ &= \nabla(a_1x + a_2y + a_3z) \cdot \vec{G} + (a_1x + a_2y + a_3z)(\nabla \cdot \vec{G})\end{aligned}$$

**Step 5:** Calculate $\nabla(a_1x + a_2y + a_3z)$.

$$\begin{aligned}\nabla(a_1x + a_2y + a_3z) &= \frac{\partial}{\partial x}(a_1x + a_2y + a_3z)\hat{i} + \frac{\partial}{\partial y}(a_1x + a_2y + a_3z)\hat{j} + \frac{\partial}{\partial z}(a_1x + a_2y + a_3z)\hat{k} \\ &= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \\ &= \vec{a}\end{aligned}$$

**Step 6:** Calculate $\nabla \cdot \vec{G}$.
Using the components of $\vec{G}$ from Step 3:

$$\begin{aligned}\nabla \cdot \vec{G} &= \frac{\partial}{\partial x}(zxa_3 - xya_2) + \frac{\partial}{\partial y}(-(yza_3 - xya_1)) + \frac{\partial}{\partial z}(yza_2 - zxa_1) \\ &= a_3 \cdot z - a_2 \cdot y - (a_3 \cdot z - a_1 \cdot x) + a_2 \cdot y - a_1 \cdot x \\ &= a_3z - a_2y - a_3z + a_1x + a_2y - a_1x \\ &= 0\end{aligned}$$

Therefore: $\nabla \cdot \vec{G} = 0$
**Step 7:** Complete the divergence calculation.
From Step 4:

$$\begin{aligned}\nabla \cdot (\vec{F_1} \times \vec{F_2}) &= \nabla(a_1x + a_2y + a_3z) \cdot \vec{G} + (a_1x + a_2y + a_3z)(\nabla \cdot \vec{G}) \\ &= \vec{a} \cdot \vec{G} + (a_1x + a_2y + a_3z) \cdot 0 \\ &= \vec{a} \cdot \vec{G} + 0 \\ &= \vec{a} \cdot (\vec{F_1} \times \vec{a})\end{aligned}$$

**Step 8:** Use the scalar triple product identity.
The scalar triple product identity states that for any three vectors:

$$\vec{a} \cdot (\vec{F_1} \times \vec{a}) = 0$$

This is because the scalar triple product of a vector with itself crossed with another vector is always zero.
Therefore:

$$\nabla \cdot (\vec{F_1} \times \vec{F_2}) = \vec{a} \cdot (\vec{F_1} \times \vec{a}) = 0$$

Hence, $\vec{F_1} \times \vec{F_2}$ is solenoidal.

**Example: Verifying Irrotational Vectors and Finding Scalar Potentials**

Verify whether the following vectors are irrotational, if so, find the scalar potential $\phi(x, y, z)$.
i) $\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$
ii) $\vec{F} = (ysinz - sinx)\hat{i} + (xsinz + 2yz)\hat{j} + (xycosz + y^2)\hat{k}$
iii) $\vec{F} = (4xy + z^3)\hat{i} + (2x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$

**Solution**

To verify if a vector field is irrotational, we check if its curl is zero: $\nabla \times \vec{F} = \vec{0}$.
For finding the scalar potential $\phi$, we use:

$$\phi = \int F_1 dx + \int F_2 dy + \int F_3 dz$$

where: - Treating $y, z$ constant when integrating $F_1$ w.r.t. $x$ - Treating $z$ constant and take terms free from $x$ when integrating $F_2$ w.r.t. $y$ - Taking terms free from $x$ and $y$ when integrating $F_3$ w.r.t. $z$
**Part (i):** $\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$
**Step 1:** Check if $\vec{F}$ is irrotational.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

For i-component:

$$\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z) = 0 - 1 - 3x^2 + 1 = -1 - (-1) = 0$$

For j-component:

$$\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3) = 3z^2 - 0 - 0 - 3z^2 = 3z^2 - 3z^2 = 0$$

For k-component:

$$\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3) = 6x - 0 - 6y + 0 = 6x - 6x = 0$$

Therefore:

$$\nabla \times \vec{F} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

Since $\nabla \times \vec{F} = \vec{0}$, the vector field is irrotational.
**Step 2:** Find the scalar potential $\phi$ using the formula.
From $F_1 = 6xy + z^3$: - Integration w.r.t. $x$: $\int(6xy + z^3)dx = 3x^2y + xz^3$
From $F_2 = 3x^2 - z$: - Terms free from $x$: $-z$ - Integration w.r.t. $y$: $\int(-z)dy = -zy$
From $F_3 = 3xz^2 - y$: - Terms free from $x$ and $y$: None
Therefore:

$$\phi = 3x^2y + xz^3 - zy + C$$

**Verification:** Check if $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(3x^2y + xz^3 - zy + C) = 6xy + z^3 = F_1\checkmark$$
$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(3x^2y + xz^3 - zy + C) = 3x^2 - z = F_2\checkmark$$
$$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}(3x^2y + xz^3 - zy + C) = 3xz^2 - y = F_3\checkmark$$

**Part (ii):** $\vec{F} = (ysinz - sinx)\hat{\imath} + (xsinz + 2yz)\hat{\jmath} + (xycosz + y^2)\hat{k}$
**Step 1:** Check if $\vec{F}$ is irrotational.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ysinz - sinx & xsinz + 2yz & xycosz + y^2 \end{vmatrix}$$

For i-component:

$$\frac{\partial}{\partial y}(xycosz + y^2) - \frac{\partial}{\partial z}(xsinz + 2yz) = xcosz + 2y - (xcosz + 2y) = xcosz + 2y - xcosz - 2y = 0$$

For j-component:

$$\frac{\partial}{\partial x}(xycosz + y^2) - \frac{\partial}{\partial z}(ysinz - sinx) = ycosz - 0 - ycosz - 0 = ycosz - ycosz = 0$$

For k-component:

$$\frac{\partial}{\partial x}(xsinz + 2yz) - \frac{\partial}{\partial y}(ysinz - sinx) = sinz + 0 - sinz - 0 = sinz - sinz = 0$$

Therefore:

$$\nabla \times \vec{F} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = \vec{0}$$

Since $\nabla \times \vec{F} = \vec{0}$, the vector field is irrotational.
**Step 2:** Find the scalar potential $\phi$ using the formula.
From $F_1 = ysinz - sinx$: - Integration w.r.t. $x$: $\int(ysinz - sinx)dx = xysinz + cosx$
From $F_2 = xsinz + 2yz$: - Terms free from $x$: $2yz$ - Integration w.r.t. $y$: $\int 2yzdy = y^2z$
From $F_3 = xycosz + y^2$: - Terms free from $x$ and $y$: None
Therefore:

$$\phi = xysinz + cosx + y^2z + C$$

**Verification:** Check if $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(xysinz + cosx + y^2z + C) = ysinz - sinx = F_1\checkmark$$
$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(xysinz + cosx + y^2z + C) = xsinz + 2yz = F_2\checkmark$$
$$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}(xysinz + cosx + y^2z + C) = xycosz + y^2 = F_3\checkmark$$

**Part (iii):** $\vec{F} = (4xy + z^3)\hat{i} + (2x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$
**Step 1:** Check if $\vec{F}$ is irrotational.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy + z^3 & 2x^2 - z & 3xz^2 - y \end{vmatrix}$$

For i-component:

$$\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(2x^2 - z) = 0 - 1 - 4x^2 + 1 = -1 - (-1) = 0$$

For j-component:

$$\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(4xy + z^3) = 3z^2 - 0 - 0 - 3z^2 = 3z^2 - 3z^2 = 0$$

For k-component:

$$\frac{\partial}{\partial x}(2x^2 - z) - \frac{\partial}{\partial y}(4xy + z^3) = 4x - 0 - 4y + 0 = 4x - 4x = 0$$

Therefore:

$$\nabla \times \vec{F} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

Since $\nabla \times \vec{F} = \vec{0}$, the vector field is irrotational.
**Step 2:** Find the scalar potential $\phi$ using the formula.
From $F_1 = 4xy + z^3$: - Integration w.r.t. $x$: $\int(4xy + z^3)dx = 2x^2y + xz^3$
From $F_2 = 2x^2 - z$: - Terms free from $x$: $-z$ - Integration w.r.t. $y$: $\int(-z)dy = -zy$
From $F_3 = 3xz^2 - y$: - Terms free from $x$ and $y$: None
Therefore:

$$\phi = 2x^2y + xz^3 - zy + C$$

**Example: Finding Scalar Potential for Irrotational Vector Field**

Show that $\vec{F} = (ye^{xy}cosz)\hat{i} + (xe^{xy}cosz)\hat{j} + (-e^{xy}sinz)\hat{k}$ is irrotational. Find corresponding scalar potential $\phi$ s.t. $\vec{F} = \nabla\phi$.

**Solution**

**Step 1:** Verify that the vector field is irrotational by checking if $\nabla \times \vec{F} = \vec{0}$.
Given: $\vec{F} = (ye^{xy}cosz)\hat{i} + (xe^{xy}cosz)\hat{j} + (-e^{xy}sinz)\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{xy}cosz & xe^{xy}cosz & -e^{xy}sinz \end{vmatrix}$$

For i-component:

$$\begin{aligned}
\frac{\partial}{\partial y}(-e^{xy} sinz) - \frac{\partial}{\partial z}(xe^{xy} cosz) &= -e^{xy} \cdot x \cdot sinz - xe^{xy}(-sinz) \\
&= -xe^{xy} sinz + xe^{xy} sinz \\
&= 0
\end{aligned}$$

For j-component:

$$\begin{aligned}
\frac{\partial}{\partial z}(ye^{xy} cosz) - \frac{\partial}{\partial x}(-e^{xy} sinz) &= ye^{xy}(-sinz) - (-e^{xy} y \cdot sinz) \\
&= -ye^{xy} sinz + ye^{xy} sinz \\
&= 0
\end{aligned}$$

For k-component:

$$\begin{aligned}
\frac{\partial}{\partial x}(xe^{xy} cosz) - \frac{\partial}{\partial y}(ye^{xy} cosz) &= [e^{xy} cosz + xe^{xy} y \cdot cosz] - [e^{xy} cosz + ye^{xy} x \cdot cosz] \\
&= e^{xy} cosz + xye^{xy} cosz - e^{xy} cosz - xye^{xy} cosz \\
&= 0
\end{aligned}$$

Since all components are zero, $\nabla \times \vec{F} = \vec{0}$, confirming that the vector field is irrotational.
**Step 2:** Find the scalar potential $\phi$ using:

$$\phi = \int F_1 dx + \int F_2 dy + \int F_3 dz$$

where: - Treating $y, z$ constant when integrating $F_1$ w.r.t. $x$ - Treating $z$ constant and take terms free from $x$ when integrating $F_2$ w.r.t. $y$ - Taking terms free from $x$ and $y$ when integrating $F_3$ w.r.t. $z$
**Step 3:** Integrate $F_1$ with respect to $x$.
Given: $F_1 = ye^{xy} cosz$

$$\int F_1 dx = \int ye^{xy} coszdx$$

Let $u = xy$, then $du = ydx$, so $dx = \frac{du}{y}$

$$\begin{aligned}
\int ye^{xy} coszdx &= cosz \int ye^{u} \frac{du}{y} \\
&= cosz \int e^{u} du \\
&= cosz \cdot e^{u} \\
&= cosz \cdot e^{xy} \\
&= e^{xy} cosz
\end{aligned}$$

**Step 4:** From $F_2$, identify and integrate terms free from $x$.
Given: $F_2 = xe^{xy} cosz$
The term $F_2$ contains $x$, so there are no terms free from $x$ to integrate.

**Step 5:** From $F_3$, identify and integrate terms free from $x$ and $y$.
Given: $F_3 = -e^{xy} sinz$
The term $F_3$ contains both $x$ and $y$, so there are no terms free from both $x$ and $y$ to integrate.
**Step 6:** Combine the results.
Therefore:

$$\phi = e^{xy} cosz + C$$

**Verification:** Check if $\nabla\phi = \vec{F}$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(e^{xy} cosz + C) = e^{xy} y \cdot cosz = ye^{xy} cosz = F_1 \checkmark$$
$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(e^{xy} cosz + C) = e^{xy} x \cdot cosz = xe^{xy} cosz = F_2 \checkmark$$
$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}(e^{xy} cosz + C) = e^{xy}(-sinz) = -e^{xy} sinz = F_3 \checkmark$$

All three partial derivatives match the components of $\vec{F}$, confirming our solution.

**Example: Finding Constants and Scalar Potential for Irrotational Vector Field**

If the vector field given by $\vec{F} = (ax + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + z)\hat{k}$ is irrotational. Then find $a, b, c$ and also determine scalar potential $\phi$.

**Solution**

**Step 1:** For the vector field to be irrotational, $\nabla \times \vec{F} = \vec{0}$.
Given: $\vec{F} = (ax + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + z)\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax + 2y + az & bx - 3y - z & 4x + cy + z \end{vmatrix}$$

For i-component:

$$\begin{aligned} \frac{\partial}{\partial y}(4x + cy + z) - \frac{\partial}{\partial z}(bx - 3y - z) &= c - (-1) \\ &= c + 1 \end{aligned}$$

For j-component:

$$\frac{\partial}{\partial z}(ax + 2y + az) - \frac{\partial}{\partial x}(4x + cy + z) = a - 4$$

For k-component:

$$\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(ax + 2y + az) = b - 2$$

For the vector field to be irrotational, all components must equal zero:

$$c + 1 = 0 \Rightarrow c = -1$$
$$a - 4 = 0 \Rightarrow a = 4$$
$$b - 2 = 0 \Rightarrow b = 2$$

Therefore: $a = 4$, $b = 2$, $c = -1$
**Step 2:** Substitute values back into the vector field.

$$\vec{F} = (4x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x - y + z)\hat{k}$$

**Step 3:** Find the scalar potential $\phi$ using:

$$\phi = \int F_1 dx + \int F_2 dy + \int F_3 dz$$

From $F_1 = 4x + 2y + 4z$:

$$\int F_1 dx = \int (4x + 2y + 4z) dx = 2x^2 + 2xy + 4xz$$

From $F_2 = 2x - 3y - z$: - Terms free from $x$: $-3y - z$

$$\int F_2 dy = \int (-3y - z) dy = -\frac{3y^2}{2} - zy$$

From $F_3 = 4x - y + z$: - Terms free from $x$ and $y$: $z$

$$\int F_3 dz = \int z dz = \frac{z^2}{2}$$

Therefore:

$$\phi = 2x^2 + 2xy + 4xz - \frac{3y^2}{2} - zy + \frac{z^2}{2} + C$$

**Verification:** Check if $\nabla\phi = \vec{F}$

$$\frac{\partial \phi}{\partial x} = 4x + 2y + 4z = F_1 \checkmark$$
$$\frac{\partial \phi}{\partial y} = 2x - 3y - z = F_2 \checkmark$$
$$\frac{\partial \phi}{\partial z} = 4x - y + z = F_3 \checkmark$$

All three partial derivatives match the components of $\vec{F}$, confirming our solution.