Chapter 1

Introduction to Vector Calculus

Vector calculus stands at the intersection of mathematics and physics, providing the fundamental tools for understanding and analyzing phenomena that involve direction and magnitude. This powerful mathematical framework enables us to describe physical quantities such as forces, velocities, and electromagnetic fields that cannot be adequately represented by scalar values alone. As we embark on this journey through vector calculus, we will see how its principles form the backbone of many scientific and engineering disciplines.

In this chapter, we begin by reviewing the essential concepts of vector algebra, establishing the groundwork for the more advanced topics to follow.

1.1 Review of Vector Algebra

1.1.1 Vectors and Scalars

At the most fundamental level, physical quantities can be classified into two categories: scalars and vectors. A scalar is fully described by its magnitude (a single number), while a vector requires both magnitude and direction for complete characterization.

Definition 1.1 (Scalar). *A scalar is a physical quantity that is completely specified by a single real number and appropriate units.*

Examples of scalar quantities include temperature, mass, time, energy, and electric potential. Regardless of the coordinate system or reference frame chosen, scalars remain invariant in their representation—they are simply numbers.

Definition 1.2 (Vector). *A vector is a mathematical object with both magnitude and direction, typically represented as an ordered set of components in a given coordinate system.*

In three-dimensional space, we can represent a vector $\mathbf{v}$ in Cartesian coordinates as:

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} \tag{1.1}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit vectors in the x, y, and z directions, respectively.

Key Properties of Vectors

- A vector has both magnitude and direction
- A vector's representation depends on the chosen coordinate system
- Vectors transform in specific ways when changing coordinate systems
- The zero vector $\mathbf{0}$ has zero magnitude and an undefined direction

1.1.2 Vector Operations

The algebraic manipulation of vectors forms the foundation of vector calculus. Let's examine the fundamental operations that can be performed on vectors.

Vector Addition and Subtraction

Given two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, their sum $\mathbf{a}+\mathbf{b}$ and difference $\mathbf{a}-\mathbf{b}$ are defined component-wise:

$$\mathbf{a}+\mathbf{b} = (a_x+b_x)\hat{\mathbf{i}} + (a_y+b_y)\hat{\mathbf{j}} + (a_z+b_z)\hat{\mathbf{k}} \tag{1.2}$$

$$\mathbf{a}-\mathbf{b} = (a_x-b_x)\hat{\mathbf{i}} + (a_y-b_y)\hat{\mathbf{j}} + (a_z-b_z)\hat{\mathbf{k}} \tag{1.3}$$

Vector addition satisfies the commutative and associative properties:

$$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a} \quad \text{(Commutative property)} \tag{1.4}$$
$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c}) \quad \text{(Associative property)} \tag{1.5}$$

Vector Addition

Consider the vectors $\mathbf{a} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = -\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. Their sum is:

$$\mathbf{a}+\mathbf{b} = (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) + (-\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \tag{1.6}$$
$$= (3-1)\hat{\mathbf{i}} + (2+4)\hat{\mathbf{j}} + (-1+2)\hat{\mathbf{k}} \tag{1.7}$$
$$= 2\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + \hat{\mathbf{k}} \tag{1.8}$$

The geometric interpretation of vector addition follows the parallelogram law: if vectors $\mathbf{a}$ and $\mathbf{b}$ are represented as arrows from the same initial point, their sum $\mathbf{a}+\mathbf{b}$ is the diagonal of the parallelogram formed by the two vectors.

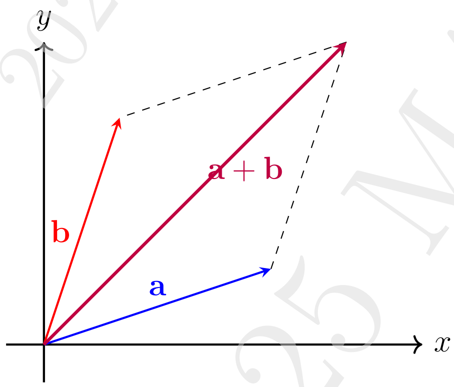

Figure 1.1: Geometric interpretation of vector addition using the parallelogram law.

Scalar Multiplication

Multiplication of a vector $\mathbf{a}$ by a scalar λ scales the magnitude of the vector by $|\lambda|$ and:

- Preserves the direction if $\lambda > 0$
- Reverses the direction if $\lambda < 0$
- Results in the zero vector if $\lambda = 0$

Mathematically, scalar multiplication is defined as:

$$\lambda \mathbf{a} = \lambda a_x \hat{\mathbf{i}} + \lambda a_y \hat{\mathbf{j}} + \lambda a_z \hat{\mathbf{k}} \tag{1.9}$$

Scalar multiplication satisfies the following properties:

$$\lambda(\mathbf{a} + \mathbf{b}) = \lambda \mathbf{a} + \lambda \mathbf{b} \quad \text{(Distributive property)} \tag{1.10}$$
$$(\lambda + \mu)\mathbf{a} = \lambda \mathbf{a} + \mu \mathbf{a} \quad \text{(Distributive property)} \tag{1.11}$$
$$\lambda(\mu \mathbf{a}) = (\lambda\mu)\mathbf{a} \quad \text{(Associative property)} \tag{1.12}$$

Scalar Multiplication

Given $\mathbf{v} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\lambda = -2$, we have:

$$\lambda \mathbf{v} = -2(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) \tag{1.13}$$
$$= -4\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}} \tag{1.14}$$

Note that the direction of the resulting vector is opposite to that of $\mathbf{v}$ since $\lambda < 0$.

Vector Magnitude

The magnitude or length of a vector $\mathbf{a}$, denoted by $|\mathbf{a}|$ or $\|\mathbf{a}\|$, is defined as:

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{1.15}$$

This definition follows from the Pythagorean theorem in three dimensions.

Vector Magnitude

For the vector $\mathbf{v} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$, the magnitude is:

$$|\mathbf{v}| = \sqrt{3^2 + (-4)^2 + 12^2} \tag{1.16}$$
$$= \sqrt{9 + 16 + 144} \tag{1.17}$$
$$= \sqrt{169} \tag{1.18}$$
$$= 13 \tag{1.19}$$

A unit vector, denoted by $\hat{\mathbf{a}}$, is a vector with a magnitude of 1. Any non-zero vector $\mathbf{a}$ can be converted to a unit vector in the same direction by dividing by its magnitude:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} \tag{1.20}$$

Unit Vector

To find the unit vector in the direction of $\mathbf{v} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} \tag{1.21}$$
$$= \frac{3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 12\hat{\mathbf{k}}}{13} \tag{1.22}$$
$$= \frac{3}{13}\hat{\mathbf{i}} - \frac{4}{13}\hat{\mathbf{j}} + \frac{12}{13}\hat{\mathbf{k}} \tag{1.23}$$

Dot Product

The dot product (or scalar product) of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \cdot \mathbf{b}$, is defined as:

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z \tag{1.24}$$

Geometrically, the dot product can also be expressed as:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta \tag{1.25}$$

where θ is the angle between the two vectors.

Properties of the Dot Product

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (Commutative property)
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (Distributive property)
- $\lambda(\mathbf{a} \cdot \mathbf{b}) = (\lambda\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda\mathbf{b})$ (Scalar multiplication)
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ (Square of the magnitude)
- $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (perpendicular)

Dot Product

Calculate the dot product of $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$.

$$\mathbf{a} \cdot \mathbf{b} = (2)(4) + (3)(-2) + (-1)(5) \tag{1.26}$$
$$= 8 - 6 - 5 \tag{1.27}$$
$$= -3 \tag{1.28}$$

To find the angle between these vectors:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \tag{1.29}$$
$$= \frac{-3}{\sqrt{2^2 + 3^2 + (-1)^2}\sqrt{4^2 + (-2)^2 + 5^2}} \tag{1.30}$$
$$= \frac{-3}{\sqrt{14}\sqrt{45}} \tag{1.31}$$
$$= \frac{-3}{\sqrt{630}} \tag{1.32}$$
$$\approx -0.12 \tag{1.33}$$

Therefore, $\theta \approx 97°$, indicating that these vectors are slightly more than perpendicular to each other.

Work in Physics

In physics, the work W done by a constant force $\mathbf{F}$ acting on an object that undergoes a displacement $\mathbf{d}$ is given by the dot product:

$$W = \mathbf{F} \cdot \mathbf{d} = |\mathbf{F}||\mathbf{d}| \cos\theta \tag{1.34}$$

where θ is the angle between the force and displacement vectors. This formulation captures the fact that only the component of force in the direction of motion contributes to work.

Cross Product

The cross product (or vector product) of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$, produces a third vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. It is defined as:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1.35}$$

Expanding this determinant, we get:

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\hat{\mathbf{i}} + (a_z b_x - a_x b_z)\hat{\mathbf{j}} + (a_x b_y - a_y b_x)\hat{\mathbf{k}} \tag{1.36}$$

The magnitude of the cross product is:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta \tag{1.37}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Properties of the Cross Product

- $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ (Anti-commutative property)
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (Distributive property)
- $\lambda(\mathbf{a} \times \mathbf{b}) = (\lambda\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda\mathbf{b})$ (Scalar multiplication)
- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ (Cross product of a vector with itself is the zero vector)
- $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel (or one of them is the zero vector)
- $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$, $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$ (Right-hand rule for standard basis vectors)

The direction of the cross product follows the right-hand rule: if the fingers of the right hand curl from the first vector toward the second vector, the thumb points in the direction of the cross product.

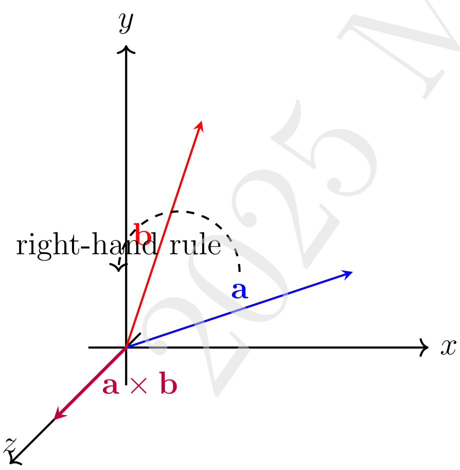

Figure 1.2: Geometric interpretation of the cross product and the right-hand rule.

Cross Product

Calculate the cross product of $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix} \tag{1.38}$$

$$= \hat{\mathbf{i}} \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \tag{1.39}$$

$$= \hat{\mathbf{i}}(1 \cdot 1 - 3 \cdot (-2)) - \hat{\mathbf{j}}(2 \cdot 1 - 3 \cdot 1) + \hat{\mathbf{k}}(2 \cdot (-2) - 1 \cdot 1) \tag{1.40}$$

$$= \hat{\mathbf{i}}(1 + 6) - \hat{\mathbf{j}}(2 - 3) + \hat{\mathbf{k}}(-4 - 1) \tag{1.41}$$

$$= 7\hat{\mathbf{i}} + \hat{\mathbf{j}} - 5\hat{\mathbf{k}} \tag{1.42}$$

Torque in Physics

In physics, the torque $\vec{\tau}$ produced by a force $\vec{F}$ acting at a position $\vec{r}$ relative to a pivot point is given by:

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{1.43}$$

The cross product formulation captures both the magnitude of the torque (which depends on the perpendicular distance from the line of action of the force to the pivot) and its direction (perpendicular to both the position and force vectors).

Triple Products

Two important operations involving three vectors are the scalar triple product and the vector triple product.

The scalar triple product, denoted by $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, gives the volume of the parallelepiped formed by the three vectors:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{1.44}$$

Scalar Triple Product Identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \tag{1.45}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0 \text{ if and only if } \mathbf{a}, \mathbf{b}, \text{ and } \mathbf{c} \text{ are coplanar} \tag{1.46}$$

The vector triple product, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, can be expanded using the BAC-CAB formula:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{1.47}$$

Vector Triple Product Identity (BAC-CAB)

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{1.48}$$

This identity is often remembered by the mnemonic "BAC minus CAB."

1.1.3 Vector Spaces and Linear Independence

A vector space is a mathematical structure formed by a collection of vectors that can be added together and multiplied by scalars while satisfying certain axioms.

Definition 1.3 (Vector Space)**.** *A vector space V over a field F (typically $\mathbb{R}$ or $\mathbb{C}$) is a set equipped with two operations: vector addition and scalar multiplication, satisfying the following axioms:*

- $\mathbf{u}+\mathbf{v} \in V$ *for all* $\mathbf{u},\mathbf{v} \in V$ *(Closure under addition)*
- $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ *for all* $\mathbf{u},\mathbf{v} \in V$ *(Commutative addition)*
- $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ *for all* $\mathbf{u},\mathbf{v},\mathbf{w} \in V$ *(Associative addition)*
- *There exists a zero vector* $\mathbf{0} \in V$ *such that* $\mathbf{u}+\mathbf{0} = \mathbf{u}$ *for all* $\mathbf{u} \in V$ *(Additive identity)*
- *For each* $\mathbf{u} \in V$*, there exists* $-\mathbf{u} \in V$ *such that* $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$ *(Additive inverse)*
- $\alpha\mathbf{u} \in V$ *for all* $\alpha \in F$ *and* $\mathbf{u} \in V$ *(Closure under scalar multiplication)*
- $\alpha(\mathbf{u}+\mathbf{v}) = \alpha\mathbf{u}+\alpha\mathbf{v}$ *for all* $\alpha \in F$ *and* $\mathbf{u},\mathbf{v} \in V$ *(Distributive property)*
- $(\alpha+\beta)\mathbf{u} = \alpha\mathbf{u}+\beta\mathbf{u}$ *for all* $\alpha,\beta \in F$ *and* $\mathbf{u} \in V$ *(Distributive property)*
- $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$ *for all* $\alpha,\beta \in F$ *and* $\mathbf{u} \in V$ *(Associative scalar multiplication)*
- $1\mathbf{u} = \mathbf{u}$ *for all* $\mathbf{u} \in V$ *(Scalar identity)*

Linear independence is a crucial concept that characterizes the minimum number of vectors needed to span a vector space.

Definition 1.4 (Linear Independence)**.** *A set of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is linearly independent if the only solution to the equation:*

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0} \tag{1.49}$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

If a set of vectors is not linearly independent, it is linearly dependent, meaning at least one vector can be expressed as a linear combination of the others.

Linear Independence

Check whether the vectors $\mathbf{v}_1 = (1,2,3)$, $\mathbf{v}_2 = (2,1,0)$, and $\mathbf{v}_3 = (5,4,3)$ are linearly independent.

We need to determine whether there exist scalars $\alpha_1, \alpha_2, \alpha_3$ (not all zero) such that:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0} \tag{1.50}$$

Substituting the vectors:

$$\alpha_1(1,2,3) + \alpha_2(2,1,0) + \alpha_3(5,4,3) = (0,0,0) \tag{1.51}$$

$$(\alpha_1 + 2\alpha_2 + 5\alpha_3, 2\alpha_1 + \alpha_2 + 4\alpha_3, 3\alpha_1 + 0\alpha_2 + 3\alpha_3) = (0,0,0) \tag{1.52}$$

This gives us the system of equations:

$$\alpha_1 + 2\alpha_2 + 5\alpha_3 = 0 \tag{1.53}$$

$$2\alpha_1 + \alpha_2 + 4\alpha_3 = 0 \tag{1.54}$$

$$3\alpha_1 + 3\alpha_3 = 0 \tag{1.55}$$

From the third equation, $\alpha_1 = -\alpha_3$. Substituting this into the first two equations:

$$-\alpha_3 + 2\alpha_2 + 5\alpha_3 = 0 \tag{1.56}$$

$$-2\alpha_3 + \alpha_2 + 4\alpha_3 = 0 \tag{1.57}$$

Simplifying:

$$2\alpha_2 + 4\alpha_3 = 0 \tag{1.58}$$
$$\alpha_2 + 2\alpha_3 = 0 \tag{1.59}$$

From the second equation, $\alpha_2 = -2\alpha_3$. Substituting into the first equation:

$$2(-2\alpha_3) + 4\alpha_3 = 0 \tag{1.60}$$
$$-4\alpha_3 + 4\alpha_3 = 0 \tag{1.61}$$
$$0 = 0 \tag{1.62}$$

This is true for any value of α_3. Therefore, if we choose $\alpha_3 = 1$, we get $\alpha_2 = -2$ and $\alpha_1 = -1$, which is a non-trivial solution.
We have found a non-zero solution to the equation $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$, specifically $(-1)\mathbf{v}_1 + (-2)\mathbf{v}_2 + (1)\mathbf{v}_3 = \mathbf{0}$ or $\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2$. Therefore, the vectors are linearly dependent.

Basis of a Vector Space

A basis for a vector space V is a linearly independent set of vectors that spans V. The dimension of V is the number of vectors in any basis of V.
In $\mathbb{R}^3$, the standard basis $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ consists of three mutually orthogonal unit vectors that form a right-handed coordinate system.

1.1.4 Lines and Planes in 3D Space

Vector algebra provides elegant methods for representing and analyzing lines and planes in three-dimensional space.

Lines in 3D Space

A line in $\mathbb{R}^3$ can be represented by a point $\mathbf{p}_0$ on the line and a direction vector $\mathbf{v}$ parallel to the line. The set of all points $\mathbf{p}$ on the line can be expressed parametrically as:

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{v} \tag{1.63}$$

where t is a real parameter that varies along the line.

Line in 3D

Find the parametric equation of the line passing through the points $P(1, 2, 3)$ and $Q(4, 0, 5)$.
First, we need a point on the line. We can use either P or Q; let's use P as our reference point $\mathbf{p}_0 = (1, 2, 3)$.
Next, we need a direction vector $\mathbf{v}$, which is the vector from P to Q:

$$\mathbf{v} = \mathbf{q} - \mathbf{p}_0 \tag{1.64}$$
$$= (4, 0, 5) - (1, 2, 3) \tag{1.65}$$
$$= (3, -2, 2) \tag{1.66}$$

Therefore, the parametric equation of the line is:

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{v} \tag{1.67}$$
$$= (1, 2, 3) + t(3, -2, 2) \tag{1.68}$$
$$= (1 + 3t, 2 - 2t, 3 + 2t) \tag{1.69}$$

In component form, this gives us:

$$x = 1 + 3t \tag{1.70}$$
$$y = 2 - 2t \tag{1.71}$$
$$z = 3 + 2t \tag{1.72}$$

where $t \in \mathbb{R}$.

The distance from a point $\mathbf{q}$ to a line passing through point $\mathbf{p}_0$ with direction vector $\mathbf{v}$ is given by:

$$d = \frac{|(\mathbf{q} - \mathbf{p}_0) \times \mathbf{v}|}{|\mathbf{v}|} \tag{1.73}$$

Distance from a Point to a Line

Find the distance from the point $R(2, 4, 1)$ to the line passing through $P(1, 2, 3)$ with direction vector $\mathbf{v} = (3, -2, 2)$.
We use the formula:

$$d = \frac{|(\mathbf{r} - \mathbf{p}_0) \times \mathbf{v}|}{|\mathbf{v}|} \tag{1.74}$$
$$= \frac{|((2, 4, 1) - (1, 2, 3)) \times (3, -2, 2)|}{|(3, -2, 2)|} \tag{1.75}$$
$$= \frac{|(1, 2, -2) \times (3, -2, 2)|}{|(3, -2, 2)|} \tag{1.76}$$

Computing the cross product:

$$(1, 2, -2) \times (3, -2, 2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & -2 \\ 3 & -2 & 2 \end{vmatrix} \tag{1.77}$$
$$= \hat{\mathbf{i}} \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} \tag{1.78}$$
$$= \hat{\mathbf{i}}(2 \cdot 2 - (-2) \cdot (-2)) - \hat{\mathbf{j}}(1 \cdot 2 - (-2) \cdot 3) + \hat{\mathbf{k}}(1 \cdot (-2) - 2 \cdot 3) \tag{1.79}$$
$$= \hat{\mathbf{i}}(4 - 4) - \hat{\mathbf{j}}(2 + 6) + \hat{\mathbf{k}}(-2 - 6) \tag{1.80}$$
$$= \hat{\mathbf{i}}(0) - \hat{\mathbf{j}}(8) + \hat{\mathbf{k}}(-8) \tag{1.81}$$
$$= -8\hat{\mathbf{j}} - 8\hat{\mathbf{k}} \tag{1.82}$$

The magnitude of this cross product is:

$$|(-8\hat{\mathbf{j}} - 8\hat{\mathbf{k}})| = \sqrt{0^2 + (-8)^2 + (-8)^2} \tag{1.83}$$
$$= \sqrt{0 + 64 + 64} \tag{1.84}$$
$$= \sqrt{128} \tag{1.85}$$
$$= 8\sqrt{2} \tag{1.86}$$

The magnitude of the direction vector is:

$$|(3, -2, 2)| = \sqrt{3^2 + (-2)^2 + 2^2} \tag{1.87}$$
$$= \sqrt{9 + 4 + 4} \tag{1.88}$$
$$= \sqrt{17} \tag{1.89}$$

Therefore, the distance is:

$$d = \frac{8\sqrt{2}}{\sqrt{17}} \tag{1.90}$$
$$= \frac{8\sqrt{2}}{\sqrt{17}} \cdot \frac{\sqrt{17}}{\sqrt{17}} \tag{1.91}$$
$$= \frac{8\sqrt{2} \cdot \sqrt{17}}{17} \tag{1.92}$$
$$= \frac{8\sqrt{34}}{17} \tag{1.93}$$
$$\approx 2.74 \text{ units} \tag{1.94}$$

Planes in 3D Space

A plane in $\mathbb{R}^3$ can be defined by a point $\mathbf{p}_0$ on the plane and a normal vector $\mathbf{n}$ perpendicular to the plane. The set of all points $\mathbf{p}$ on the plane satisfies:

$$(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{n} = 0 \tag{1.95}$$

This can be rewritten in the form:

$$\mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{p}_0 \tag{1.96}$$

In scalar form, the equation of a plane can be expressed as:

$$ax + by + cz + d = 0 \tag{1.97}$$

where $\mathbf{n} = (a, b, c)$ is the normal vector, and $d = -ax_0 - by_0 - cz_0$ for a point $\mathbf{p}_0 = (x_0, y_0, z_0)$ on the plane.

Plane in 3D

Find the equation of the plane passing through the point $P(2, -1, 3)$ with normal vector $\mathbf{n} = (4, 5, -2)$.

Using the equation $(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{n} = 0$:

$$(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{n} = 0 \tag{1.98}$$
$$((x, y, z) - (2, -1, 3)) \cdot (4, 5, -2) = 0 \tag{1.99}$$
$$(x - 2, y + 1, z - 3) \cdot (4, 5, -2) = 0 \tag{1.100}$$

Expanding the dot product:

$$4(x - 2) + 5(y + 1) + (-2)(z - 3) = 0 \tag{1.101}$$
$$4x - 8 + 5y + 5 - 2z + 6 = 0 \tag{1.102}$$
$$4x + 5y - 2z + 3 = 0 \tag{1.103}$$

This is the equation of the plane in the form $ax + by + cz + d = 0$.

The distance from a point $\mathbf{q}$ to a plane with normal vector $\mathbf{n}$ and point $\mathbf{p}_0$ is given by:

$$d = \frac{|(\mathbf{q} - \mathbf{p}_0) \cdot \mathbf{n}|}{|\mathbf{n}|} \tag{1.104}$$

Distance from a Point to a Plane

Calculate the distance from the point $Q(1, 0, 4)$ to the plane $4x + 5y - 2z + 3 = 0$.
First, we identify the normal vector of the plane as $\mathbf{n} = (4, 5, -2)$.
Next, we need a point on the plane. We can find one by setting $y = 0$ and $z = 0$:

$$4x + 5(0) - 2(0) + 3 = 0 \tag{1.105}$$
$$4x + 3 = 0 \tag{1.106}$$
$$x = -\frac{3}{4} \tag{1.107}$$

So, $P(-\frac{3}{4}, 0, 0)$ is a point on the plane.
Now, we can calculate the distance:

$$d = \frac{|(\mathbf{q} - \mathbf{p}_0) \cdot \mathbf{n}|}{|\mathbf{n}|} \tag{1.108}$$
$$= \frac{|((1, 0, 4) - (-\frac{3}{4}, 0, 0)) \cdot (4, 5, -2)|}{|(4, 5, -2)|} \tag{1.109}$$
$$= \frac{|(\frac{7}{4}, 0, 4) \cdot (4, 5, -2)|}{|(4, 5, -2)|} \tag{1.110}$$
$$= \frac{|(4 \cdot \frac{7}{4} + 5 \cdot 0 + (-2) \cdot 4)|}{|(4, 5, -2)|} \tag{1.111}$$

Computing the dot product in the numerator:

$$(4 \cdot \frac{7}{4} + 5 \cdot 0 + (-2) \cdot 4) = 7 + 0 - 8 \tag{1.112}$$
$$= -1 \tag{1.113}$$

Computing the magnitude of the normal vector:

$$|(4, 5, -2)| = \sqrt{4^2 + 5^2 + (-2)^2} \tag{1.114}$$
$$= \sqrt{16 + 25 + 4} \tag{1.115}$$
$$= \sqrt{45} \tag{1.116}$$
$$= 3\sqrt{5} \tag{1.117}$$

Therefore, the distance is:

$$d = \frac{|-1|}{3\sqrt{5}} \tag{1.118}$$
$$= \frac{1}{3\sqrt{5}} \tag{1.119}$$
$$= \frac{\sqrt{5}}{15} \tag{1.120}$$
$$\approx 0.149 \text{ units} \tag{1.121}$$