# Chapter 5

# Linear Transformations and Orthogonal Transformations

In this chapter, we explore linear transformations through the lens of matrix operations. We will focus on the form $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is called the transformation matrix, and examine special cases such as orthogonal transformations.

## 5.1 Linear Transformations

### Definition and Properties of Linear Transformations

**Definition 5.1.** *A linear transformation is a mapping between matrices that can be represented by the equation* $\mathbf{AX} = \mathbf{B}$*, where* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *is called the transformation matrix.*

The matrix $\mathbf{A}$ completely determines the behavior of the transformation. Depending on the properties of $\mathbf{A}$, we classify transformations as follows:

**Classification of Linear Transformations**

Let $\mathbf{A}$ be a square matrix of order $n$. Then:

1. If $\det(\mathbf{A}) = 0$, then $\mathbf{A}$ is called *singular*, *irregular*, or *non-invertible.*
2. If $\det(\mathbf{A}) \neq 0$, then $\mathbf{A}$ is called *regular*, *non-singular*, or *invertible.*

**Example: Regular and Singular Transformation Matrices**

Consider the following transformation matrices:

$$\mathbf{A}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\mathbf{A}_2 = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

For $\mathbf{A}_1$, we have $\det(\mathbf{A}_1) = 2 \cdot 3 - 1 \cdot 1 = 6 - 1 = 5 \neq 0$, so $\mathbf{A}_1$ is regular (invertible).
For $\mathbf{A}_2$, we have $\det(\mathbf{A}_2) = 2 \cdot 2 - 4 \cdot 1 = 4 - 4 = 0$, so $\mathbf{A}_2$ is singular (non-invertible).
The transformation $\mathbf{A}_1\mathbf{X} = \mathbf{B}$ has a unique solution for any $\mathbf{B}$, while the transformation $\mathbf{A}_2\mathbf{X} = \mathbf{B}$ either has no solution or infinitely many solutions, depending on $\mathbf{B}$.

### Matrix Representation of Linear Transformations

When we work with linear transformations in the form $\mathbf{AX} = \mathbf{B}$, the matrix $\mathbf{A}$ completely encapsulates the transformation's behavior. For square matrices, several important properties determine the transformation's characteristics:

**Theorem 5.2.** *Let $\mathbf{A}$ be a square matrix of order $n$ representing a linear transformation. Then:*

1. *The transformation is invertible if and only if* $\det(\mathbf{A}) \neq 0$.
2. *If $\mathbf{A}$ is invertible, then the inverse transformation is represented by $\mathbf{A}^{-1}$.*
3. *The composition of two transformations with matrices $\mathbf{A}$ and $\mathbf{B}$ is represented by the product matrix $\mathbf{BA}$.*
4. *The identity transformation is represented by the identity matrix $\mathbf{I}$.*

**Example: Composition of Transformations**

Consider two transformation matrices:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$
$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The composition of these transformations (applying $\mathbf{A}$ followed by $\mathbf{B}$) is represented by the matrix product:

$$\begin{aligned} \mathbf{BA} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

This new matrix represents the composite transformation.

**Key Properties of Transformation Matrices**

For a square matrix $\mathbf{A}$ of order $n$:

1. **Rank**: The rank of $\mathbf{A}$ determines the dimension of the image (range) of the transformation.
2. **Nullity**: The nullity of $\mathbf{A}$ (dimension of the null space) determines the dimension of the kernel of the transformation.
3. **Rank-Nullity Theorem**: $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$
4. **Eigenvalues and Eigenvectors**: If $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for some non-zero vector $\mathbf{v}$ and scalar $\lambda$, then $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda$. These determine the directions that are only scaled (not rotated) by the transformation.

## 5.2 Orthogonal Transformations

### Definition and Geometric Interpretation of Orthogonal Transformations

**Definition 5.3.** *A linear transformation represented by a matrix $\mathbf{A}$ is called an orthogonal transformation if $\mathbf{A}$ is an orthogonal matrix, i.e.,* $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}$.

An orthogonal transformation preserves the Euclidean length of vectors and the angles between them. This means that if we apply an orthogonal transformation to a geometric figure, its shape and size remain unchanged—it may only be rotated, reflected, or repositioned.

**Example: Identifying Orthogonal Transformations**

Consider the following matrices:

$$\mathbf{A}_1 = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$
$$\mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Let's check if these represent orthogonal transformations:
For $\mathbf{A}_1$:

$$\begin{aligned} \mathbf{A}_1^T\mathbf{A}_1 &= \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \\ &= \begin{pmatrix} \frac{9}{25} + \frac{16}{25} & \frac{12}{25} - \frac{12}{25} \\ \frac{12}{25} - \frac{12}{25} & \frac{16}{25} + \frac{9}{25} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \end{aligned}$$

Therefore, $\mathbf{A}_1$ represents an orthogonal transformation.
For $\mathbf{A}_2$:

$$\begin{aligned} \mathbf{A}_2^T\mathbf{A}_2 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq \mathbf{I} \end{aligned}$$

Therefore, $\mathbf{A}_2$ does not represent an orthogonal transformation.
Looking at the geometric interpretation, $\mathbf{A}_1$ represents rotation by an angle $\theta$ where $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$, which is approximately 53.13. This preserves the shapes and sizes of objects.
On the other hand, $\mathbf{A}_2$ represents a shear transformation that distorts shapes.

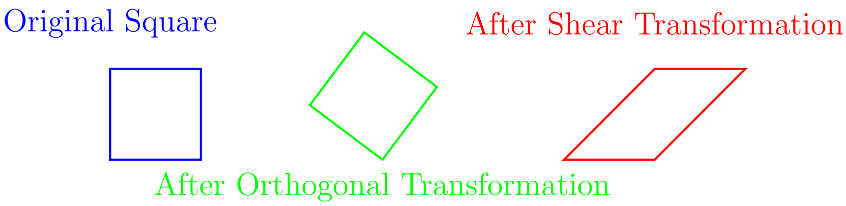



Figure 5.1: Effects of orthogonal vs. non-orthogonal transformations on a square

**Properties of Orthogonal Matrices**

An orthogonal transformation is characterized by its orthogonal matrix $\mathbf{A}$, which satisfies $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}$.

**Properties of Orthogonal Matrices**

Let $\mathbf{A}$ be an orthogonal matrix. Then:

1. $\mathbf{A}^T = \mathbf{A}^{-1}$ (The transpose equals the inverse)
2. $\det(\mathbf{A}) = \pm 1$
3. The columns of $\mathbf{A}$ form an orthonormal set
4. The rows of $\mathbf{A}$ form an orthonormal set
5. $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|$ for any vector $\mathbf{x}$ (preserves length)
6. $(\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{y}) = \mathbf{x}^T\mathbf{y}$ for any vectors $\mathbf{x}, \mathbf{y}$ (preserves dot product)
7. The eigenvalues of $\mathbf{A}$ have magnitude 1
8. If $\det(\mathbf{A}) = 1$, then $\mathbf{A}$ represents a proper rotation
9. If $\det(\mathbf{A}) = -1$, then $\mathbf{A}$ represents an improper rotation (rotation + reflection)

**Example: Common Orthogonal Transformations**

Here are some common orthogonal transformation matrices in $\mathbb{R}^2$:

1. **Rotation by angle $\theta$**:

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

2. **Reflection across the x-axis**:

$$\mathbf{F}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3. **Reflection across the y-axis**:

$$\mathbf{F}_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. **Reflection across the line $y = x$**:

$$\mathbf{F}_{y=x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We can verify that all these matrices satisfy $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.

**Applications of Orthogonal Transformations**

Orthogonal transformations are particularly useful in:

1. **Computer Graphics**: Rotation, reflection, and other rigid transformations are represented by orthogonal matrices.
2. **Physics**: Coordinate transformations between different reference frames often use orthogonal matrices.
3. **Engineering**: Rigid body mechanics relies on orthogonal transformations to describe motion.
4. **Data Analysis**: Techniques like Principal Component Analysis (PCA) use orthogonal transformations to find uncorrelated features.
5. **Quantum Mechanics**: Unitary transformations (complex analogues of orthogonal transformations) represent quantum operations.

**Self-Assessment Problems**

1. Prove that if $\mathbf{A}$ and $\mathbf{B}$ are orthogonal matrices, then $\mathbf{AB}$ is also orthogonal.
2. Determine whether the following matrix represents an orthogonal transformation:

$$\mathbf{C} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

3. Find the matrix representing a reflection across the line $y = -x$ in $\mathbb{R}^2$.
4. Show that an orthogonal matrix in $\mathbb{R}^3$ with determinant 1 represents a rotation around some axis.
5. If $\mathbf{A}$ is a $3 \times 3$ orthogonal matrix with determinant -1, what type of transformation does it represent?

**Solutions**

1. To prove that $\mathbf{AB}$ is orthogonal when $\mathbf{A}$ and $\mathbf{B}$ are orthogonal, we need to show that $(\mathbf{AB})^T(\mathbf{AB}) = \mathbf{I}$:

$$\begin{aligned}(\mathbf{AB})^T(\mathbf{AB}) &= \mathbf{B}^T\mathbf{A}^T\mathbf{AB} \\ &= \mathbf{B}^T\mathbf{IB} \quad (\text{since } \mathbf{A} \text{ is orthogonal, } \mathbf{A}^T\mathbf{A} = \mathbf{I}) \\ &= \mathbf{B}^T\mathbf{B} \\ &= \mathbf{I} \quad (\text{since } \mathbf{B} \text{ is orthogonal, } \mathbf{B}^T\mathbf{B} = \mathbf{I})\end{aligned}$$

Therefore, $\mathbf{AB}$ is orthogonal.

2. For matrix $\mathbf{C}$, we check if $\mathbf{C}^T\mathbf{C} = \mathbf{I}$:

$$\begin{aligned}\mathbf{C}^T\mathbf{C} &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4} + \frac{3}{4} & \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} & \frac{3}{4} + \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}\end{aligned}$$

Thus, $\mathbf{C}$ represents an orthogonal transformation. In fact, it represents a rotation by 60 since $\cos(60) = \frac{1}{2}$ and $\sin(60) = \frac{\sqrt{3}}{2}$.

3. For a reflection across the line $y = -x$, we need to find where a point $(x, y)$ gets mapped. The reflection of $(x, y)$ across the line $y = -x$ is $(-y, -x)$. This transformation is represented by the matrix:

$$\mathbf{F}_{y=-x} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

We can verify that this is orthogonal:

$$\mathbf{F}_{y=-x}^T\mathbf{F}_{y=-x} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

4. By Euler's rotation theorem, any orthogonal matrix in $\mathbb{R}^3$ with determinant 1 represents a rotation around some axis. To prove this:
Let $\mathbf{A}$ be a $3 \times 3$ orthogonal matrix with $\det(\mathbf{A}) = 1$. Since $\mathbf{A}$ is orthogonal, its eigenvalues have magnitude 1, meaning they are of the form $e^{i\theta}$ for some angle $\theta$.

Since $\mathbf{A}$ is real, complex eigenvalues come in conjugate pairs. The determinant is the product of eigenvalues, and with $\det(\mathbf{A}) = 1$, we know that $\mathbf{A}$ must have at least one real eigenvalue, which must be either 1 or -1.
If $\det(\mathbf{A}) = 1$, then the real eigenvalue must be 1, and there exists a non-zero vector $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \mathbf{v}$. This vector $\mathbf{v}$ represents the axis of rotation.

5. If $\mathbf{A}$ is a $3 \times 3$ orthogonal matrix with $\det(\mathbf{A}) = -1$, it represents an improper rotation, which is a rotation followed by a reflection. Specifically, it is a rotation about some axis followed by a reflection in a plane perpendicular to that axis.

## 5.3 Additional Solved Examples

**Example 1: Finding Preimage Coordinates in a Linear Transformation**

Given the transformation

$$\mathbf{Y} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{AX}$$

Find the coordinates $(x_1, x_2, x_3)$ in $\mathbf{X}$ corresponding to $(1, 2, -1)$ in $\mathbf{Y}$.
**Solution:**
Let's solve the system of linear equations directly:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 2 \\ x_1 + 0x_2 - 2x_3 &= -1 \end{aligned}$$

From the third equation:

$$\begin{aligned} x_1 - 2x_3 &= -1 \\ \Rightarrow x_1 &= -1 + 2x_3 \end{aligned}$$

Substituting this into the first equation:

$$\begin{aligned} 2(-1 + 2x_3) + x_2 + x_3 &= 1 \\ -2 + 4x_3 + x_2 + x_3 &= 1 \\ x_2 + 5x_3 &= 3 \end{aligned}$$

Substituting $x_1 = -1 + 2x_3$ into the second equation:

$$\begin{aligned} (-1 + 2x_3) + x_2 + 2x_3 &= 2 \\ -1 + 2x_3 + x_2 + 2x_3 &= 2 \\ x_2 + 4x_3 &= 3 \end{aligned}$$

Now we have two equations:

$$\begin{aligned} x_2 + 5x_3 &= 3 \quad (1) \\ x_2 + 4x_3 &= 3 \quad (2) \end{aligned}$$

Subtracting (2) from (1):

$$x_3 = 0$$

Substituting back into (2):

$$\begin{aligned} x_2 + 4(0) &= 3 \\ x_2 &= 3 \end{aligned}$$

And substituting back to find $x_1$:

$$\begin{aligned} x_1 &= -1 + 2x_3 \\ &= -1 + 2(0) \\ &= -1 \end{aligned}$$

Therefore, the coordinates are $(x_1, x_2, x_3) = (-1, 3, 0)$.
Let's verify this solution:

$$\begin{aligned} \mathbf{AX} &= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \cdot (-1) + 1 \cdot 3 + 1 \cdot 0 \\ 1 \cdot (-1) + 1 \cdot 3 + 2 \cdot 0 \\ 1 \cdot (-1) + 0 \cdot 3 + (-2) \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 + 3 + 0 \\ -1 + 3 + 0 \\ -1 + 0 + 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \end{aligned}$$

This confirms that the coordinates $(x_1, x_2, x_3) = (-1, 3, 0)$ in $\mathbf{X}$ correspond to $(1, 2, -1)$ in $\mathbf{Y}$.

**Example 2: Regular Transformation and Its Inverse**

Show that the transformation

$$\begin{aligned} y_1 &= 2x_1 + x_2 + x_3 \\ y_2 &= x_1 + x_2 + 2x_3 \\ y_3 &= x_1 - 2x_3 \end{aligned}$$

is regular. Write down the inverse transformation.
**Solution:**
**Step 1:** First, let's write the transformation in matrix form:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Let's denote the transformation matrix as $\mathbf{A}$.

**Step 2:** To determine if the transformation is regular, we need to check if $\det(\mathbf{A}) \neq 0$.

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix}$$

Let's expand along the first row:

$$\begin{aligned}
\det(\mathbf{A}) &= 2 \cdot \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\
&= 2(1 \cdot (-2) - 2 \cdot 0) - 1(1 \cdot (-2) - 2 \cdot 1) + 1(1 \cdot 0 - 1 \cdot 1) \\
&= 2(-2) - 1(-2 - 2) + 1(-1) \\
&= -4 - 1(-4) - 1 \\
&= -4 + 4 - 1 \\
&= -1
\end{aligned}$$

Since $\det(\mathbf{A}) = -1 \neq 0$, the transformation is regular (invertible).
Let's solve for the inverse transformation directly from the system of equations:

$$\begin{aligned}
y_1 &= 2x_1 + x_2 + x_3 \quad (1) \\
y_2 &= x_1 + x_2 + 2x_3 \quad (2) \\
y_3 &= x_1 - 2x_3 \quad (3)
\end{aligned}$$

From equation (3), we get:

$$x_1 = y_3 + 2x_3 \quad (4)$$

Substituting (4) into equation (1):

$$\begin{aligned}
y_1 &= 2(y_3 + 2x_3) + x_2 + x_3 \\
&= 2y_3 + 4x_3 + x_2 + x_3 \\
&= 2y_3 + x_2 + 5x_3 \quad (5)
\end{aligned}$$

Substituting (4) into equation (2):

$$\begin{aligned}
y_2 &= (y_3 + 2x_3) + x_2 + 2x_3 \\
&= y_3 + 2x_3 + x_2 + 2x_3 \\
&= y_3 + x_2 + 4x_3 \quad (6)
\end{aligned}$$

From equation (5), we get:

$$x_2 = y_1 - 2y_3 - 5x_3 \quad (7)$$

Substituting (7) into equation (6):

$$\begin{aligned}
y_2 &= y_3 + (y_1 - 2y_3 - 5x_3) + 4x_3 \\
&= y_3 + y_1 - 2y_3 - 5x_3 + 4x_3 \\
&= y_1 - y_3 - x_3 \quad (8)
\end{aligned}$$

From equation (8), we solve for $x_3$:

$$\begin{aligned} y_2 &= y_1 - y_3 - x_3 \\ \Rightarrow x_3 &= y_1 - y_2 - y_3 \quad (9) \end{aligned}$$

Now we can substitute (9) back into equations (4) and (7) to find $x_1$ and $x_2$:

$$\begin{aligned} x_1 &= y_3 + 2x_3 \\ &= y_3 + 2(y_1 - y_2 - y_3) \\ &= y_3 + 2y_1 - 2y_2 - 2y_3 \\ &= 2y_1 - 2y_2 - y_3 \quad (10) \end{aligned}$$

$$\begin{aligned} x_2 &= y_1 - 2y_3 - 5x_3 \\ &= y_1 - 2y_3 - 5(y_1 - y_2 - y_3) \\ &= y_1 - 2y_3 - 5y_1 + 5y_2 + 5y_3 \\ &= -4y_1 + 5y_2 + 3y_3 \quad (11) \end{aligned}$$

Therefore, the inverse transformation is:

$$\begin{aligned} x_1 &= 2y_1 - 2y_2 - y_3 \\ x_2 &= -4y_1 + 5y_2 + 3y_3 \\ x_3 &= y_1 - y_2 - y_3 \end{aligned}$$

**Verification:** Let's verify that applying the original transformation to our inverse transformation gives us back the original $y$ values.
Substituting our expressions for $x_1$, $x_2$, and $x_3$ into the original equation for $y_1$:

$$\begin{aligned} y_1 &= 2x_1 + x_2 + x_3 \\ &= 2(2y_1 - 2y_2 - y_3) + (-4y_1 + 5y_2 + 3y_3) + (y_1 - y_2 - y_3) \\ &= 4y_1 - 4y_2 - 2y_3 - 4y_1 + 5y_2 + 3y_3 + y_1 - y_2 - y_3 \\ &= 4y_1 - 4y_1 + y_1 - 4y_2 + 5y_2 - y_2 - 2y_3 + 3y_3 - y_3 \\ &= y_1 + 0y_2 + 0y_3 \\ &= y_1 \end{aligned}$$

Substituting our expressions for $x_1$, $x_2$, and $x_3$ into the original equation for $y_2$:

$$\begin{aligned} y_2 &= x_1 + x_2 + 2x_3 \\ &= (2y_1 - 2y_2 - y_3) + (-4y_1 + 5y_2 + 3y_3) + 2(y_1 - y_2 - y_3) \\ &= 2y_1 - 2y_2 - y_3 - 4y_1 + 5y_2 + 3y_3 + 2y_1 - 2y_2 - 2y_3 \\ &= 2y_1 - 4y_1 + 2y_1 - 2y_2 + 5y_2 - 2y_2 - y_3 + 3y_3 - 2y_3 \\ &= 0y_1 + y_2 + 0y_3 \\ &= y_2 \end{aligned}$$

Substituting our expressions for $x_1$, $x_2$, and $x_3$ into the original equation for $y_3$:

$$\begin{aligned} y_3 &= x_1 - 2x_3 \\ &= (2y_1 - 2y_2 - y_3) - 2(y_1 - y_2 - y_3) \\ &= 2y_1 - 2y_2 - y_3 - 2y_1 + 2y_2 + 2y_3 \\ &= 2y_1 - 2y_1 - 2y_2 + 2y_2 - y_3 + 2y_3 \\ &= 0y_1 + 0y_2 + y_3 \\ &= y_3 \end{aligned}$$

Our verification confirms that the inverse transformation is correct:

$$\begin{aligned} x_1 &= 2y_1 - 2y_2 - y_3 \\ x_2 &= -4y_1 + 5y_2 + 3y_3 \\ x_3 &= y_1 - y_2 - y_3 \end{aligned}$$

**Example 3: Composite Transformations**

A transformation from the variables $x_1, x_2, x_3$ to $y_1, y_2, y_3$ is given by $\mathbf{Y} = \mathbf{AX}$ and another transformation from $y_1, y_2, y_3$ to $z_1, z_2, z_3$ is given by $\mathbf{Z} = \mathbf{BY}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{pmatrix}$$
$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$$

Obtain the transformation from $x_1, x_2, x_3$ to $z_1, z_2, z_3$.
**Solution:**
**Step 1:** We need to find the direct transformation from $\mathbf{X}$ to $\mathbf{Z}$.
Given:

$$\begin{aligned} \mathbf{Y} &= \mathbf{AX} \\ \mathbf{Z} &= \mathbf{BY} \end{aligned}$$

Substituting the first equation into the second:

$$\begin{aligned} \mathbf{Z} &= \mathbf{BY} \\ &= \mathbf{B}(\mathbf{AX}) \\ &= (\mathbf{BA})\mathbf{X} \end{aligned}$$

Therefore, the transformation matrix from $\mathbf{X}$ to $\mathbf{Z}$ is $\mathbf{C} = \mathbf{BA}$.
**Step 2:** Calculate the matrix product $\mathbf{C} = \mathbf{BA}$.

$$\begin{aligned} \mathbf{C} &= \mathbf{BA} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{pmatrix} \end{aligned}$$

Let's calculate each element of **C**:
Row 1, Column 1:

$$\begin{aligned} c_{11} &= 1 \cdot 2 + 1 \cdot 0 + 1 \cdot (-1) \\ &= 2 + 0 - 1 \\ &= 1 \end{aligned}$$

Row 1, Column 2:

$$\begin{aligned} c_{12} &= 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 \\ &= 1 + 1 + 2 \\ &= 4 \end{aligned}$$

Row 1, Column 3:

$$\begin{aligned} c_{13} &= 1 \cdot 0 + 1 \cdot (-2) + 1 \cdot 1 \\ &= 0 - 2 + 1 \\ &= -1 \end{aligned}$$

Row 2, Column 1:

$$\begin{aligned} c_{21} &= 1 \cdot 2 + 2 \cdot 0 + 3 \cdot (-1) \\ &= 2 + 0 - 3 \\ &= -1 \end{aligned}$$

Row 2, Column 2:

$$\begin{aligned} c_{22} &= 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 2 \\ &= 1 + 2 + 6 \\ &= 9 \end{aligned}$$

Row 2, Column 3:

$$\begin{aligned} c_{23} &= 1 \cdot 0 + 2 \cdot (-2) + 3 \cdot 1 \\ &= 0 - 4 + 3 \\ &= -1 \end{aligned}$$

Row 3, Column 1:

$$\begin{aligned} c_{31} &= 1 \cdot 2 + 3 \cdot 0 + 5 \cdot (-1) \\ &= 2 + 0 - 5 \\ &= -3 \end{aligned}$$

Row 3, Column 2:

$$\begin{aligned} c_{32} &= 1 \cdot 1 + 3 \cdot 1 + 5 \cdot 2 \\ &= 1 + 3 + 10 \\ &= 14 \end{aligned}$$

Row 3, Column 3:

$$\begin{aligned} c_{33} &= 1 \cdot 0 + 3 \cdot (-2) + 5 \cdot 1 \\ &= 0 - 6 + 5 \\ &= -1 \end{aligned}$$

Therefore:

$$\begin{aligned} \mathbf{C} &= \mathbf{BA} \\ &= \begin{pmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{pmatrix} \end{aligned}$$

**Step 3:** Write the transformation from $x_1, x_2, x_3$ to $z_1, z_2, z_3$ in matrix form.

$$\begin{aligned} \mathbf{Z} &= \mathbf{CX} \\ &= \begin{pmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

**Step 4:** Write the transformation equations explicitly.

$$\begin{aligned} z_1 &= 1 \cdot x_1 + 4 \cdot x_2 + (-1) \cdot x_3 \\ &= x_1 + 4x_2 - x_3 \end{aligned}$$

$$\begin{aligned} z_2 &= (-1) \cdot x_1 + 9 \cdot x_2 + (-1) \cdot x_3 \\ &= -x_1 + 9x_2 - x_3 \end{aligned}$$

$$\begin{aligned} z_3 &= (-3) \cdot x_1 + 14 \cdot x_2 + (-1) \cdot x_3 \\ &= -3x_1 + 14x_2 - x_3 \end{aligned}$$

Therefore, the transformation from $x_1, x_2, x_3$ to $z_1, z_2, z_3$ is:

$$\begin{aligned} z_1 &= x_1 + 4x_2 - x_3 \\ z_2 &= -x_1 + 9x_2 - x_3 \\ z_3 &= -3x_1 + 14x_2 - x_3 \end{aligned}$$

**Example 4: Matrix Representation and Composite Transformations**

Express each of the transformations $x_1 = 3y_1 + 2y_2$; $y_1 = z_1 + 2z_2$; $x_2 = -y_1 + 4y_2$; $y_2 = 3z_1$ by the use of matrices and the composite transformation which express $x_1, x_2$ in terms of $z_1, z_2$.

**Solution:**

**Step 1:** Organize the given transformations.

We have two transformations: one from $(y_1, y_2)$ to $(x_1, x_2)$ and another from $(z_1, z_2)$ to $(y_1, y_2)$.

First transformation (from $Y$ to $X$):

$$\begin{aligned} x_1 &= 3y_1 + 2y_2 \\ x_2 &= -y_1 + 4y_2 \end{aligned}$$

Second transformation (from $Z$ to $Y$):

$$\begin{aligned} y_1 &= z_1 + 2z_2 \\ y_2 &= 3z_1 \end{aligned}$$

**Step 2:** Express these transformations in matrix form.
For the transformation from $Y$ to $X$:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Let's denote this transformation matrix as $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$$

For the transformation from $Z$ to $Y$:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Let's denote this transformation matrix as $\mathbf{B}$:

$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

**Step 3:** Find the composite transformation from $Z$ to $X$.
We know that $\mathbf{X} = \mathbf{A} \cdot \mathbf{Y}$ and $\mathbf{Y} = \mathbf{B} \cdot \mathbf{Z}$. Substituting the second equation into the first:

$$\begin{aligned} \mathbf{X} &= \mathbf{A} \cdot \mathbf{Y} \\ &= \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{Z}) \\ &= (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{Z} \end{aligned}$$

So, the composite transformation matrix is $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$.

$$\begin{aligned} \mathbf{C} &= \mathbf{A} \cdot \mathbf{B} \\ &= \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \end{aligned}$$

Let's calculate each element of $\mathbf{C}$:
$c_{11} = 3 \cdot 1 + 2 \cdot 3 = 3 + 6 = 9$
$c_{12} = 3 \cdot 2 + 2 \cdot 0 = 6 + 0 = 6$
$c_{21} = (-1) \cdot 1 + 4 \cdot 3 = -1 + 12 = 11$
$c_{22} = (-1) \cdot 2 + 4 \cdot 0 = -2 + 0 = -2$
Therefore:

$$\mathbf{C} = \begin{pmatrix} 9 & 6 \\ 11 & -2 \end{pmatrix}$$

**Step 4:** Express the composite transformation in equation form.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 11 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

This gives us:

$$\begin{aligned} x_1 &= 9z_1 + 6z_2 \\ x_2 &= 11z_1 - 2z_2 \end{aligned}$$

**Verification:** Let's verify our result by substituting the equations for $y_1$ and $y_2$ into the equations for $x_1$ and $x_2$.

$$\begin{aligned} x_1 &= 3y_1 + 2y_2 \\ &= 3(z_1 + 2z_2) + 2(3z_1) \\ &= 3z_1 + 6z_2 + 6z_1 \\ &= 9z_1 + 6z_2 \end{aligned}$$

$$\begin{aligned} x_2 &= -y_1 + 4y_2 \\ &= -(z_1 + 2z_2) + 4(3z_1) \\ &= -z_1 - 2z_2 + 12z_1 \\ &= 11z_1 - 2z_2 \end{aligned}$$

This confirms our matrix calculation.
Therefore, the composite transformation from $(z_1, z_2)$ to $(x_1, x_2)$ is:

$$\begin{aligned} x_1 &= 9z_1 + 6z_2 \\ x_2 &= 11z_1 - 2z_2 \end{aligned}$$

**Example A: Orthogonal Matrices and Transformations**

Define orthogonal matrix. Show that the following transformations are orthogonal:
(a) $x_1 \cos\theta + x_2 \sin\theta$ ; $-x_1 \sin\theta + x_2 \cos\theta$
(b) $x \cos\theta + z \sin\theta$ ; $y$ ; $-x \sin\theta + z \cos\theta$

**Solution:**

**Definition of an Orthogonal Matrix:** A matrix $\mathbf{A}$ is orthogonal if and only if $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$ and $\mathbf{I}$ is the identity matrix.
Equivalently, this means that:

1. The columns (or rows) of $\mathbf{A}$ form an orthonormal set.
2. $\mathbf{A}^T = \mathbf{A}^{-1}$, i.e., the transpose of $\mathbf{A}$ equals its inverse.
3. $\det(\mathbf{A}) = \pm 1$
4. $\mathbf{A}$ preserves lengths and angles between vectors.

A transformation represented by an orthogonal matrix is called an orthogonal transformation.

**Part (a):** $x_1 \cos\theta + x_2 \sin\theta$ **;** $-x_1 \sin\theta + x_2 \cos\theta$

**Step 1:** First, let's understand what this transformation means and express it in matrix

form.
Let's denote the transformed coordinates as $y_1$ and $y_2$:

$$y_1 = x_1 \cos\theta + x_2 \sin\theta$$
$$y_2 = -x_1 \sin\theta + x_2 \cos\theta$$

In matrix form, this becomes:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let's call this transformation matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

**Step 2:** To verify that this is an orthogonal matrix, we'll check if $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.
First, let's calculate $\mathbf{A}^T$:

$$\mathbf{A}^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Now, let's compute $\mathbf{A}^T\mathbf{A}$:

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Let's calculate each element:
For element $(1, 1)$:

$$\begin{aligned} (\mathbf{A}^T\mathbf{A})_{11} &= \cos\theta \cdot \cos\theta + (-\sin\theta) \cdot (-\sin\theta) \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \quad \text{(using the Pythagorean identity)} \end{aligned}$$

For element $(1, 2)$:

$$\begin{aligned} (\mathbf{A}^T\mathbf{A})_{12} &= \cos\theta \cdot \sin\theta + (-\sin\theta) \cdot \cos\theta \\ &= \cos\theta \sin\theta - \sin\theta \cos\theta \\ &= 0 \end{aligned}$$

For element $(2, 1)$:

$$\begin{aligned} (\mathbf{A}^T\mathbf{A})_{21} &= \sin\theta \cdot \cos\theta + \cos\theta \cdot (-\sin\theta) \\ &= \sin\theta \cos\theta - \cos\theta \sin\theta \\ &= 0 \end{aligned}$$

For element $(2, 2)$:

$$\begin{aligned} (\mathbf{A}^T\mathbf{A})_{22} &= \sin\theta \cdot \sin\theta + \cos\theta \cdot \cos\theta \\ &= \sin^2\theta + \cos^2\theta \\ &= 1 \quad \text{(using the Pythagorean identity)} \end{aligned}$$

Therefore:

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

Similarly, we can verify that $\mathbf{A}\mathbf{A}^T = \mathbf{I}$.
Also, we can calculate the determinant:

$$\begin{aligned} \det(\mathbf{A}) &= \cos\theta \cdot \cos\theta - \sin\theta \cdot (-\sin\theta) \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

Therefore, the transformation in part (a) is indeed orthogonal.
Geometrically, this transformation represents a rotation of the coordinate system by angle $\theta$ in the counterclockwise direction.
**Part (b):** $x\cos\theta + z\sin\theta$ **;** $y$ **;** $-x\sin\theta + z\cos\theta$
**Step 1:** Let's understand this transformation and express it in matrix form.
Let's denote the transformed coordinates as $x'$, $y'$, and $z'$:

$$\begin{aligned} x' &= x\cos\theta + z\sin\theta \\ y' &= y \\ z' &= -x\sin\theta + z\cos\theta \end{aligned}$$

In matrix form, this becomes:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Let's call this transformation matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

**Step 2:** To verify that this is an orthogonal matrix, we'll check if $\mathbf{B}^T\mathbf{B} = \mathbf{I}$.
First, let's calculate $\mathbf{B}^T$:

$$\mathbf{B}^T = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Now, let's compute $\mathbf{B}^T\mathbf{B}$:

$$\mathbf{B}^T\mathbf{B} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Let's calculate each element (focusing on non-trivial calculations):
For element $(1,1)$:

$$\begin{aligned} (\mathbf{B}^T\mathbf{B})_{11} &= \cos\theta \cdot \cos\theta + 0 \cdot 0 + (-\sin\theta) \cdot (-\sin\theta) \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

For element $(1,3)$:

$$\begin{aligned}(\mathbf{B}^T\mathbf{B})_{13} &= \cos\theta\cdot\sin\theta + 0\cdot 0 + (-\sin\theta)\cdot\cos\theta \\ &= \cos\theta\sin\theta - \sin\theta\cos\theta \\ &= 0\end{aligned}$$

For element $(2,2)$:

$$\begin{aligned}(\mathbf{B}^T\mathbf{B})_{22} &= 0\cdot 0 + 1\cdot 1 + 0\cdot 0 \\ &= 1\end{aligned}$$

For element $(3,1)$:

$$\begin{aligned}(\mathbf{B}^T\mathbf{B})_{31} &= \sin\theta\cdot\cos\theta + 0\cdot 0 + \cos\theta\cdot(-\sin\theta) \\ &= \sin\theta\cos\theta - \cos\theta\sin\theta \\ &= 0\end{aligned}$$

For element $(3,3)$:

$$\begin{aligned}(\mathbf{B}^T\mathbf{B})_{33} &= \sin\theta\cdot\sin\theta + 0\cdot 0 + \cos\theta\cdot\cos\theta \\ &= \sin^2\theta + \cos^2\theta \\ &= 1\end{aligned}$$

All other elements will be 0 due to the structure of the matrices. Therefore:

$$\mathbf{B}^T\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Similarly, we can verify that $\mathbf{B}\mathbf{B}^T = \mathbf{I}$.
Also, we can calculate the determinant:

$$\begin{aligned}\det(\mathbf{B}) &= \cos\theta\cdot 1\cdot\cos\theta + 0 + 0 - \sin\theta\cdot 1\cdot\sin\theta - 0 - 0 \\ &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta + \sin^2\theta \\ &= 1\end{aligned}$$

Therefore, the transformation in part (b) is also orthogonal.
Geometrically, this transformation represents a rotation around the y-axis by angle $\theta$.

**Example 6: Finding Values to Make Matrices Orthogonal**

Determine the value of $a, b, c$ when the following matrices are orthogonal:

(a) $\begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$

(b) $\frac{1}{3}\begin{pmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{pmatrix}$

(c) $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & a \\ \frac{2}{3} & \frac{1}{3} & b \\ \frac{2}{3} & -\frac{2}{3} & c \end{pmatrix}$

**Solution:**

A matrix $\mathbf{A}$ is orthogonal if and only if $\mathbf{A}\mathbf{A}^T = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. We'll use this definition directly to find the values of $a$, $b$, and $c$.

**Part (a):** Let's find the values for matrix $\mathbf{A} = \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$

**Step 1:** Calculate $\mathbf{A}^T$:

$$\mathbf{A}^T = \begin{pmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{pmatrix}$$

**Step 2:** Calculate $\mathbf{A}\mathbf{A}^T$:

$$\begin{aligned}
\mathbf{A}\mathbf{A}^T &= \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix} \begin{pmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{pmatrix} \\
&= \begin{pmatrix} 0 \cdot 0 + 2b \cdot 2b + c \cdot c & 0 \cdot a + 2b \cdot b + c \cdot (-c) & 0 \cdot a + 2b \cdot (-b) + c \cdot c \\ a \cdot 0 + b \cdot 2b + (-c) \cdot c & a \cdot a + b \cdot b + (-c) \cdot (-c) & a \cdot a + b \cdot (-b) + (-c) \cdot c \\ a \cdot 0 + (-b) \cdot 2b + c \cdot c & a \cdot a + (-b) \cdot b + c \cdot (-c) & a \cdot a + (-b) \cdot (-b) + c \cdot c \end{pmatrix} \\
&= \begin{pmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{pmatrix}
\end{aligned}$$

**Step 3:** Set $\mathbf{A}\mathbf{A}^T = \mathbf{I}$ and solve for $a$, $b$, and $c$.

From the first diagonal element:

$$4b^2 + c^2 = 1 \quad (1)$$

From the second diagonal element:

$$a^2 + b^2 + c^2 = 1 \quad (2)$$

From the third diagonal element:

$$a^2 + b^2 + c^2 = 1 \quad (3)$$

(This is the same as equation (2), so we effectively have only two diagonal conditions.)

From the off-diagonal elements, which must all equal zero:

$$\begin{aligned} 2b^2 - c^2 &= 0 \quad (4) \\ -2b^2 + c^2 &= 0 \quad (5) \\ a^2 - b^2 - c^2 &= 0 \quad (6) \end{aligned}$$

Note that equations (4) and (5) are the same, so we effectively have:

$$\begin{aligned} 2b^2 - c^2 &= 0 \quad (4) \\ a^2 - b^2 - c^2 &= 0 \quad (6) \end{aligned}$$

From equation (4):

$$\begin{aligned} 2b^2 &= c^2 \\ \Rightarrow c^2 &= 2b^2 \end{aligned}$$

Substituting this into equation (1):

$$\begin{aligned} 4b^2 + c^2 &= 1 \\ 4b^2 + 2b^2 &= 1 \\ 6b^2 &= 1 \\ \Rightarrow b^2 &= \frac{1}{6} \\ \Rightarrow b &= \pm\frac{1}{\sqrt{6}} \end{aligned}$$

Let's choose $b = \frac{1}{\sqrt{6}}$ for simplicity.
Now, using $c^2 = 2b^2$:

$$\begin{aligned} c^2 &= 2 \cdot \frac{1}{6} = \frac{1}{3} \\ \Rightarrow c &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$

Let's choose $c = \frac{1}{\sqrt{3}}$.
From equation (6):

$$\begin{aligned} a^2 - b^2 - c^2 &= 0 \\ a^2 &= b^2 + c^2 \\ a^2 &= \frac{1}{6} + \frac{1}{3} = \frac{1}{6} + \frac{2}{6} = \frac{3}{6} = \frac{1}{2} \\ \Rightarrow a &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

Let's choose $a = \frac{1}{\sqrt{2}}$.
Therefore, the values that make the matrix orthogonal are:

$$\begin{aligned} a &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ b &= \frac{1}{\sqrt{6}} \\ c &= \frac{1}{\sqrt{3}} \end{aligned}$$

**Part (b):** Let's find the values for matrix $\mathbf{B} = \frac{1}{3}\begin{pmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{pmatrix}$

**Step 1:** Calculate $\mathbf{B}^T$:

$$\mathbf{B}^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{pmatrix}$$

**Step 2:** Calculate $\mathbf{B}\mathbf{B}^T$:

$$\begin{aligned}
\mathbf{B}\mathbf{B}^T &= \frac{1}{3}\begin{pmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{pmatrix} \\
&= \frac{1}{9}\begin{pmatrix} 1\cdot 1 + 2\cdot 2 + a\cdot a & 1\cdot 2 + 2\cdot 1 + a\cdot b & 1\cdot 2 + 2\cdot(-2) + a\cdot c \\ 2\cdot 1 + 1\cdot 2 + b\cdot a & 2\cdot 2 + 1\cdot 1 + b\cdot b & 2\cdot 2 + 1\cdot(-2) + b\cdot c \\ 2\cdot 1 + (-2)\cdot 2 + c\cdot a & 2\cdot 2 + (-2)\cdot 1 + c\cdot b & 2\cdot 2 + (-2)\cdot(-2) + c\cdot c \end{pmatrix} \\
&= \frac{1}{9}\begin{pmatrix} 1+4+a^2 & 2+2+ab & 2-4+ac \\ 2+2+ba & 4+1+b^2 & 4-2+bc \\ 2-4+ca & 4-2+cb & 4+4+c^2 \end{pmatrix} \\
&= \frac{1}{9}\begin{pmatrix} 5+a^2 & 4+ab & -2+ac \\ 4+ab & 5+b^2 & 2+bc \\ -2+ca & 2+cb & 8+c^2 \end{pmatrix}
\end{aligned}$$

**Step 3:** Set $\mathbf{B}\mathbf{B}^T = \mathbf{I}$ and solve for $a$, $b$, and $c$.
From the diagonal elements:

$$\begin{aligned}
\frac{1}{9}(5+a^2) = 1 \quad &\Rightarrow 5+a^2 = 9 \quad \Rightarrow a^2 = 4 \quad \Rightarrow a = \pm 2 \\
\frac{1}{9}(5+b^2) = 1 \quad &\Rightarrow 5+b^2 = 9 \quad \Rightarrow b^2 = 4 \quad \Rightarrow b = \pm 2 \\
\frac{1}{9}(8+c^2) = 1 \quad &\Rightarrow 8+c^2 = 9 \quad \Rightarrow c^2 = 1 \quad \Rightarrow c = \pm 1
\end{aligned}$$

From the off-diagonal elements (which must equal zero):

$$\begin{aligned}
\frac{1}{9}(4+ab) = 0 \quad &\Rightarrow 4+ab = 0 \quad \Rightarrow ab = -4 \\
\frac{1}{9}(-2+ac) = 0 \quad &\Rightarrow -2+ac = 0 \quad \Rightarrow ac = 2 \\
\frac{1}{9}(2+bc) = 0 \quad &\Rightarrow 2+bc = 0 \quad \Rightarrow bc = -2
\end{aligned}$$

Let's choose $c = 1$ for simplicity. Then:

$$\begin{aligned}
ac = 2 \quad &\Rightarrow a\cdot 1 = 2 \quad \Rightarrow a = 2 \\
bc = -2 \quad &\Rightarrow b\cdot 1 = -2 \quad \Rightarrow b = -2
\end{aligned}$$

We should verify that these values also satisfy $ab = -4$:

$$ab = 2\cdot(-2) = -4\checkmark$$

Therefore, the values that make the matrix orthogonal are:

$$\begin{aligned}
a &= 2 \\
b &= -2 \\
c &= 1
\end{aligned}$$

**Part (c):** Let's find the values for matrix $\mathbf{C} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & a \\ \frac{2}{3} & \frac{1}{3} & b \\ \frac{2}{3} & -\frac{2}{3} & c \end{pmatrix}$

**Step 1:** Calculate $\mathbf{C}^T$:

$$\mathbf{C}^T = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ a & b & c \end{pmatrix}$$

**Step 2:** Calculate $\mathbf{C}\mathbf{C}^T$:

$$\begin{aligned}
\mathbf{C}\mathbf{C}^T &= \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & a \\ \frac{2}{3} & \frac{1}{3} & b \\ \frac{2}{3} & -\frac{2}{3} & c \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ a & b & c \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{9} + \frac{4}{9} + a^2 & \frac{2}{9} + \frac{2}{9} + ab & \frac{2}{9} - \frac{4}{9} + ac \\ \frac{2}{9} + \frac{2}{9} + ba & \frac{4}{9} + \frac{1}{9} + b^2 & \frac{4}{9} - \frac{2}{9} + bc \\ \frac{2}{9} - \frac{4}{9} + ca & \frac{4}{9} - \frac{2}{9} + cb & \frac{4}{9} + \frac{4}{9} + c^2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{5}{9} + a^2 & \frac{4}{9} + ab & -\frac{2}{9} + ac \\ \frac{4}{9} + ab & \frac{5}{9} + b^2 & \frac{2}{9} + bc \\ -\frac{2}{9} + ca & \frac{2}{9} + cb & \frac{8}{9} + c^2 \end{pmatrix}
\end{aligned}$$

**Step 3:** Set $\mathbf{C}\mathbf{C}^T = \mathbf{I}$ and solve for $a$, $b$, and $c$.
From the diagonal elements:

$$\begin{aligned}
\frac{5}{9} + a^2 = 1 \quad &\Rightarrow a^2 = 1 - \frac{5}{9} = \frac{9-5}{9} = \frac{4}{9} \quad \Rightarrow a = \pm\frac{2}{3} \\
\frac{5}{9} + b^2 = 1 \quad &\Rightarrow b^2 = \frac{4}{9} \quad \Rightarrow b = \pm\frac{2}{3} \\
\frac{8}{9} + c^2 = 1 \quad &\Rightarrow c^2 = \frac{1}{9} \quad \Rightarrow c = \pm\frac{1}{3}
\end{aligned}$$

From the off-diagonal elements (which must equal zero):

$$\begin{aligned}
\frac{4}{9} + ab = 0 \quad &\Rightarrow ab = -\frac{4}{9} \\
-\frac{2}{9} + ac = 0 \quad &\Rightarrow ac = \frac{2}{9} \\
\frac{2}{9} + bc = 0 \quad &\Rightarrow bc = -\frac{2}{9}
\end{aligned}$$

Let's choose $c = \frac{1}{3}$ for simplicity. Then:

$$\begin{aligned}
ac = \frac{2}{9} \quad &\Rightarrow a \cdot \frac{1}{3} = \frac{2}{9} \quad \Rightarrow a = \frac{2}{3} \\
bc = -\frac{2}{9} \quad &\Rightarrow b \cdot \frac{1}{3} = -\frac{2}{9} \quad \Rightarrow b = -\frac{2}{3}
\end{aligned}$$

We should verify that these values also satisfy $ab = -\frac{4}{9}$:

$$ab = \frac{2}{3} \cdot \left(-\frac{2}{3}\right) = -\frac{4}{9}\checkmark$$

Therefore, the values that make the matrix orthogonal are:

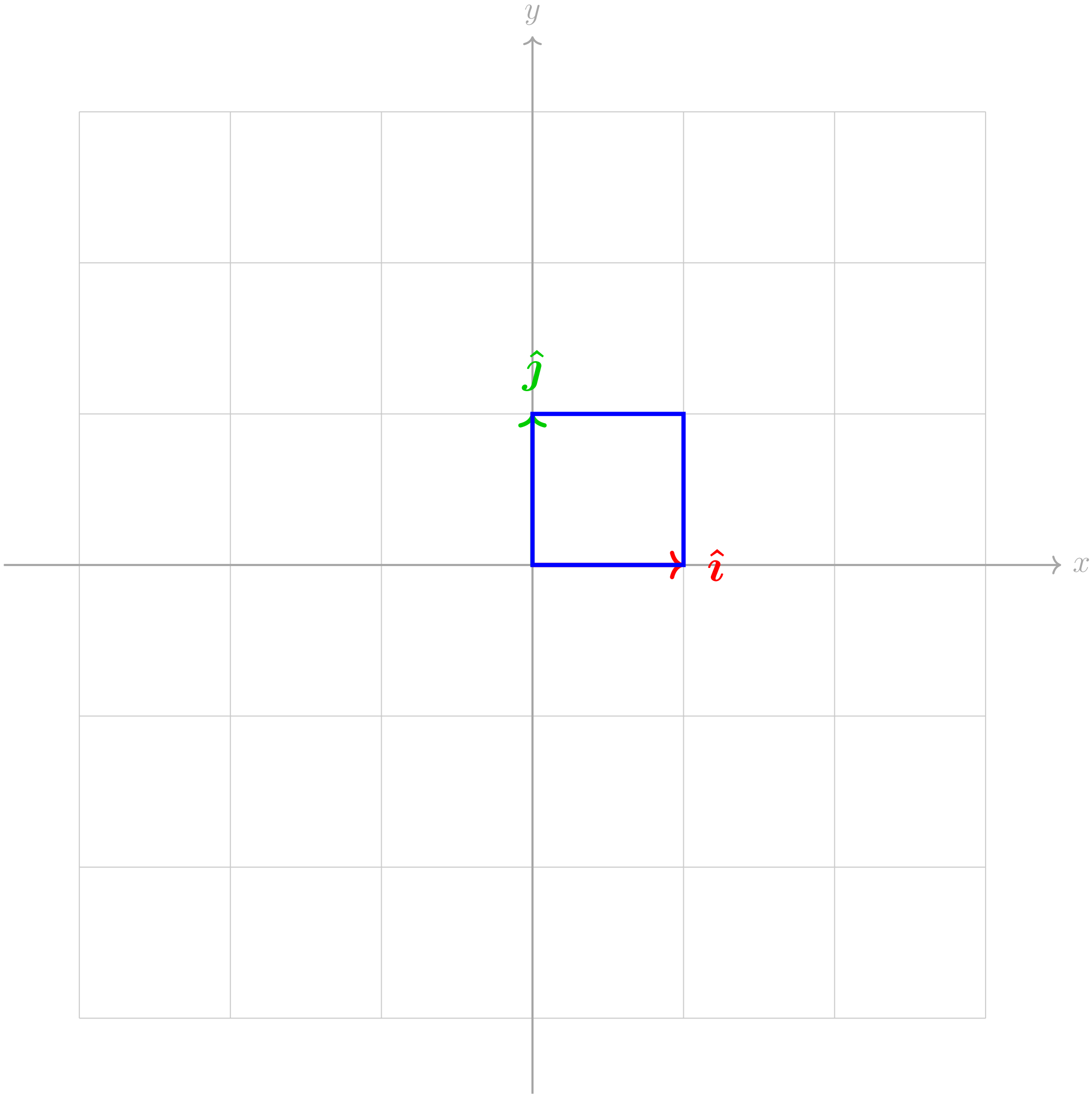
$$\begin{aligned}
a &= \frac{2}{3} \\
b &= -\frac{2}{3} \\
c &= \frac{1}{3}
\end{aligned}$$

## 5.4 Matrix as a Transformation Visualizations

Matrices can be visualized as transformations in space. When we multiply a vector by a matrix, we transform the vector in some way.

**Identity Transformation**

The identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ leaves vectors unchanged:

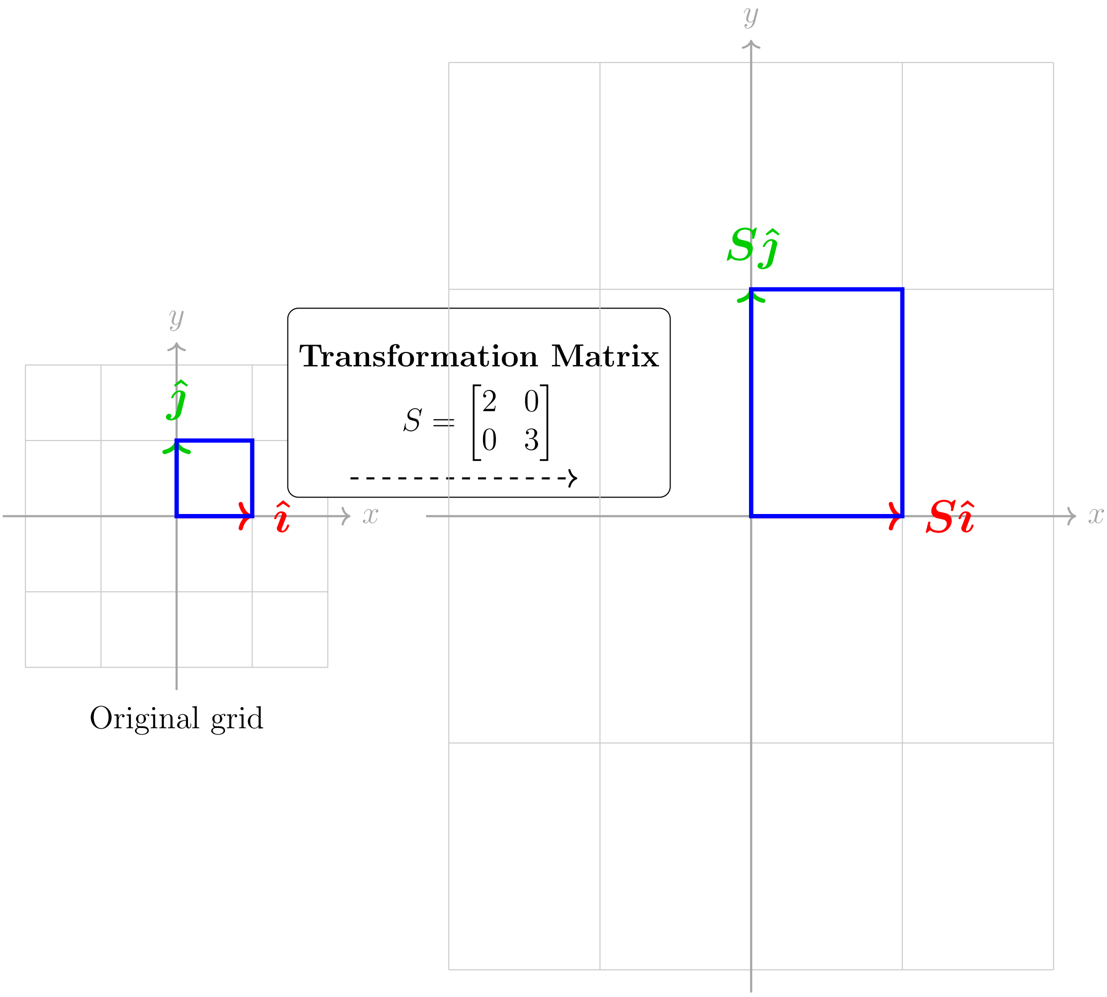



Original and transformed grid (identical)

The grid remains unchanged when multiplied by the identity matrix.

**Scaling Transformation**

The matrix $S = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ scales the x-coordinate by 2 and the y-coordinate by 3:



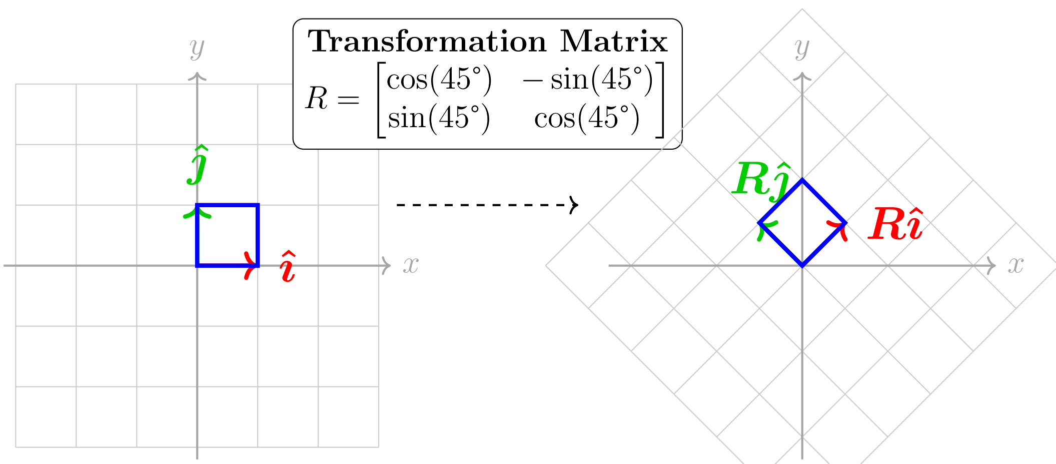

Original grid

Transformed grid

The grid is stretched horizontally by a factor of 2 and vertically by a factor of 3.

**Rotation Transformation**

The matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates vectors counterclockwise by angle $\theta$:



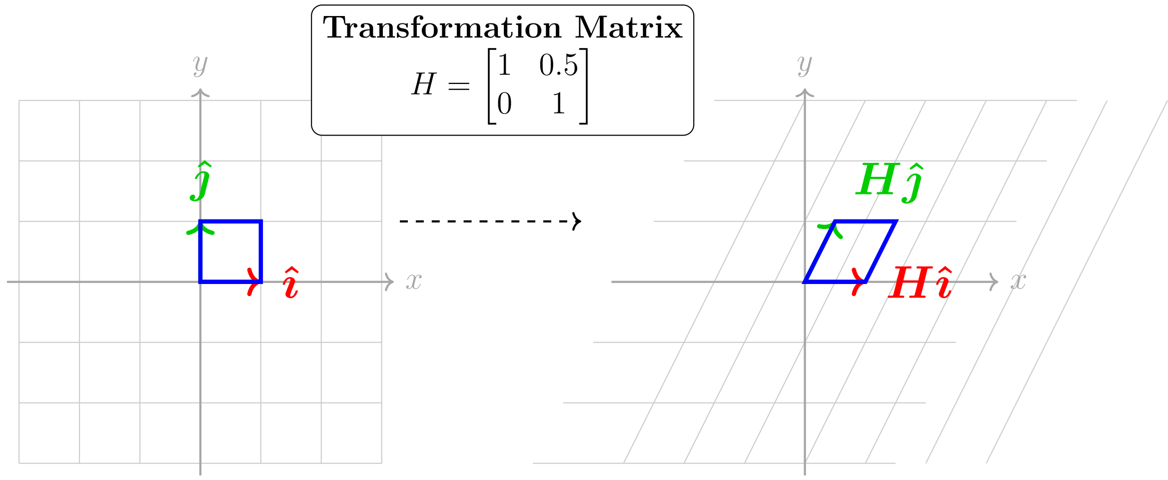

Original grid

Transformed grid (45° rotation)

The grid is rotated around the origin.

**Shear Transformation**

The matrix $H = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ creates a horizontal shear:




Original grid     Transformed grid (horizontal shear)

The grid is slanted while keeping the y-coordinates unchanged.