Chapter 5

Linear Differential Equations introduction and General Method of particular Integral

5.1 Introduction to Linear Differential Equations

Differential equations describe the relationship between a function and its derivatives. They appear frequently across many disciplines including physics, engineering, economics, and biology. Linear differential equations form a crucial class of differential equations that are both analytically tractable and widely applicable.

Definition 5.1. *A linear differential equation of order n is a differential equation that can be written in the form:*

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{5.1}$$

where $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $f(x)$ are functions of x only, and $a_n(x) \neq 0$.

The distinguishing feature of linear differential equations is that they involve only:

- Linear terms in the dependent variable y and its derivatives
- No products of y or its derivatives (like $y \cdot y'$)
- No functions of y or its derivatives (like $\sin(y)$ or e^y)

When $f(x) = 0$, we call the equation **homogeneous**; otherwise, it is **non-homogeneous**.

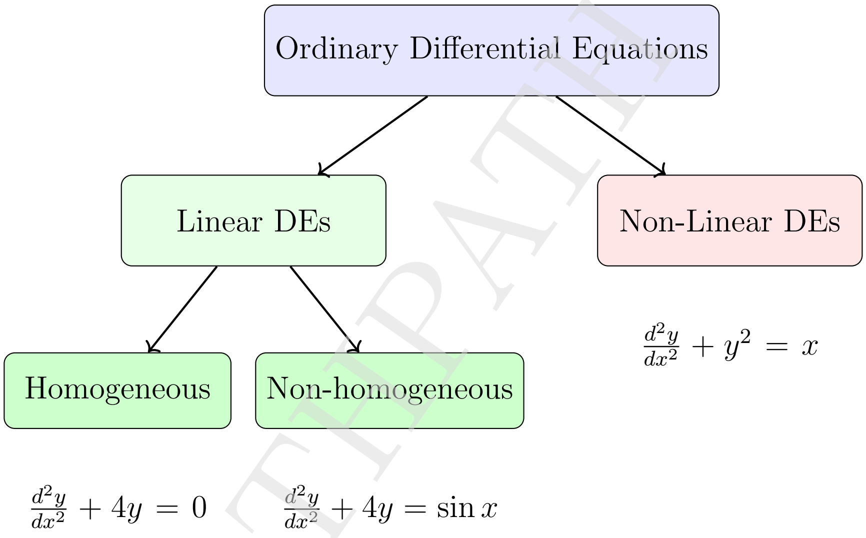

Figure 5.1: Classification of Differential Equations

5.2 Linear Differential Equations with Constant Coefficients

In this book, we will focus primarily on linear differential equations with constant coefficients, which are of the form:

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0y = f(x) \tag{5.2}$$

where $a_n, a_{n-1}, \ldots, a_0$ are constants, with $a_n \neq 0$.

5.2.1 The Differential Operator D

To simplify our notation and analysis, we introduce the differential operator D, defined as:

$$D = \frac{d}{dx} \tag{5.3}$$

Higher powers of D represent higher-order derivatives:

$$Dy = \frac{dy}{dx} \tag{5.4}$$

$$D^2y = \frac{d^2y}{dx^2} \tag{5.5}$$

$$D^3y = \frac{d^3y}{dx^3} \tag{5.6}$$

and so on.

Using this notation, a linear differential equation with constant coefficients can be written in the compact form:

$$(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y = f(x) \tag{5.7}$$

We can represent this as:

$$\phi(D)y = f(x) \tag{5.8}$$

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where $\phi(D) = a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$ is called the differential operator polynomial.

Properties of the Differential Operator D

If y_1 and y_2 are differentiable functions and a is a constant, then:

1. $D^m(D^n y) = D^{m+n} y$
2. $D(y_1 + y_2) = Dy_1 + Dy_2$
3. $D(ay) = a(Dy)$
4. $(D - m_1)(D - m_2)y = D^2 y - (m_1 + m_2)Dy + m_1 m_2 y$

5.2.2 Auxiliary Equation

For a homogeneous linear differential equation with constant coefficients:

$$\phi(D)y = 0 \tag{5.9}$$

We define the auxiliary equation (also called the characteristic equation) as:

$$\phi(m) = 0 \tag{5.10}$$

This is obtained by replacing the operator D with the algebraic variable m.
For example, if we have the differential equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \tag{5.11}$$

Using operator notation, this becomes:

$$(D^2 - 5D + 6)y = 0 \tag{5.12}$$

And the auxiliary equation is:

$$m^2 - 5m + 6 = 0 \tag{5.13}$$

Solving this, we get $m = 2$ or $m = 3$.
The roots of the auxiliary equation are crucial for determining the general solution of the differential equation. The roots of this auxiliary equation guide us to the structure of the general solution.

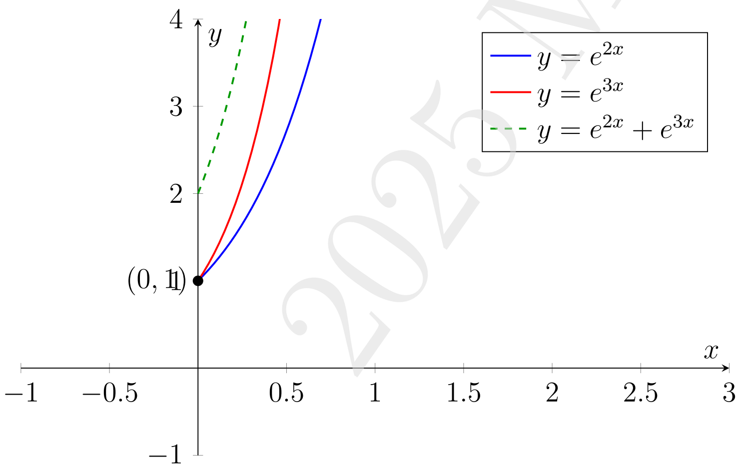

Figure 5.2: Example solutions to a second-order linear homogeneous differential equation $y'' - 5y' + 6y = 0$

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5.3 Homogeneous Linear Differential Equations with Constant Coefficients

We now examine how to find general solutions to homogeneous linear differential equations with constant coefficients based on the roots of the auxiliary equation.

5.3.1 Case 1: Distinct Real Roots

If the auxiliary equation $\phi(m) = 0$ has n distinct real roots $m_1, m_2, \ldots, m_n$, then the general solution is:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} \tag{5.14}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.
This solution works because each $e^{m_i x}$ is a solution to the differential equation, and by the principle of superposition, any linear combination of these solutions is also a solution.

Solving a Second-Order DE with Distinct Real Roots

Consider the equation: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$
Writing this using the operator D: $(D^2 - 5D + 6)y = 0$
The auxiliary equation is: $m^2 - 5m + 6 = 0$
Factoring this equation: $(m-2)(m-3) = 0$
The roots are: $m_1 = 2$ and $m_2 = 3$
Therefore, the general solution is: $y = c_1 e^{2x} + c_2 e^{3x}$

5.3.2 Case 2: Repeated Real Roots

If the auxiliary equation has a repeated root m with multiplicity k, then the corresponding part of the general solution is:

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{mx} \tag{5.15}$$

For instance, if m is a double root (appears twice), the corresponding part of the solution is:

$$(c_1 + c_2 x) e^{mx} \tag{5.16}$$

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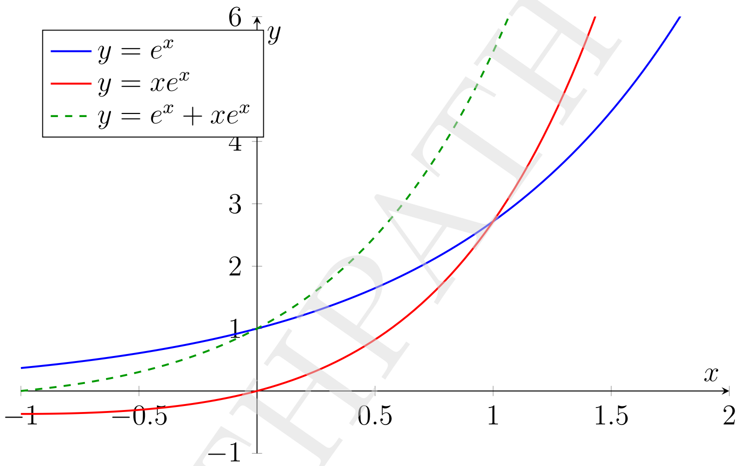

Figure 5.3: Example solutions for a differential equation with a repeated real root $y'' - 2y' + y = 0$ (Double root $m = 1$)

Solving a Second-Order DE with Repeated Real Roots

Consider the equation: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$
Using operator notation: $(D^2 - 2D + 1)y = 0$
The auxiliary equation is: $m^2 - 2m + 1 = 0$
This can be written as: $(m - 1)^2 = 0$
The equation has a repeated root $m = 1$ with multiplicity 2.
Therefore, the general solution is: $y = (c_1 + c_2x)e^x$

5.3.3 Case 3: Complex Conjugate Roots

If the auxiliary equation has complex conjugate roots $\alpha \pm i\beta$ (where $\beta \neq 0$), then the corresponding part of the general solution is:

$$e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)) \tag{5.17}$$

This representation uses Euler's formula and keeps the solution in terms of real functions, which is more practical for applications.

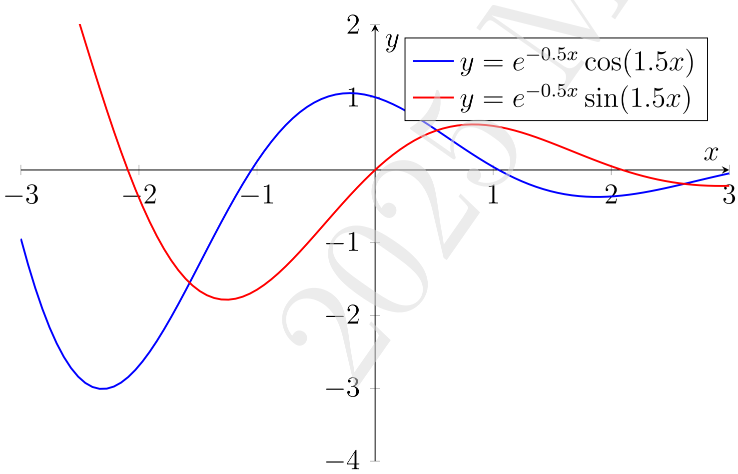

Figure 5.4: Example solutions from complex conjugate roots $m = -0.5 \pm 1.5i$

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Solving a Second-Order DE with Complex Conjugate Roots

Consider the equation: $\frac{d^2y}{dx^2} + y = 0$
Using operator notation: $(D^2 + 1)y = 0$
The auxiliary equation is: $m^2 + 1 = 0$
The roots are: $m = \pm i$
Therefore, the general solution is: $y = c_1 \cos(x) + c_2 \sin(x)$
Note that this is a special case where $\alpha = 0$, so the exponential factor becomes $e^{0x} = 1$.

5.3.4 Special Case: $D^2 - m^2 = 0$

When the auxiliary equation has the form $m^2 - a^2 = 0$ with roots $m = \pm a$, we can express the solution using hyperbolic functions:

$$y = c_1 \cosh(mx) + c_2 \sinh(mx) \tag{5.18}$$

This is equivalent to $y = c_1 e^{mx} + c_2 e^{-mx}$ but can be more convenient in certain applications.

5.3.5 Summary of Homogeneous Solutions

General Solution of Homogeneous Linear DEs with Constant Coefficients

For the equation $\phi(D)y = 0$, where $\phi(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$:

1. **Case 1 - Distinct Real Roots**: If the auxiliary equation $\phi(m) = 0$ has distinct real roots $m_1, m_2, \ldots, m_n$, then:
$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} \tag{5.19}$$
2. **Case 2 - Repeated Real Roots**: If a real root m_j has multiplicity k, then:
$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{m_j x} \tag{5.20}$$
3. **Case 3 - Complex Conjugate Roots**: If $\alpha \pm i\beta$ are complex conjugate roots, then:
$$e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)) \tag{5.21}$$
4. **Case 4 - Repeated Complex Roots**: If the complex conjugate roots $\alpha \pm i\beta$ each appear with multiplicity k, then:
$$\sum_{j=0}^{k-1} e^{\alpha x}(c_{1j} x^j \cos(\beta x) + c_{2j} x^j \sin(\beta x)) \tag{5.22}$$

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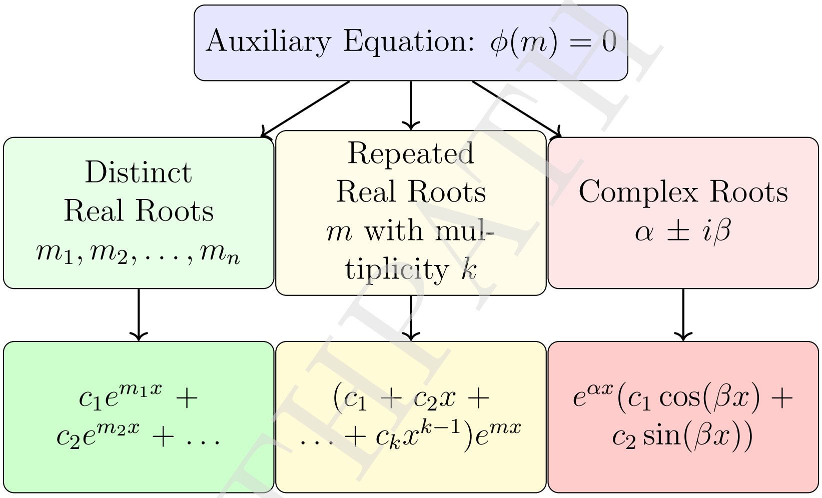

Figure 5.5: Flowchart for solving homogeneous linear differential equations with constant coefficients

5.4 Non-homogeneous Linear Differential Equations

A non-homogeneous linear differential equation with constant coefficients has the form:

$$\phi(D)y = f(x) \tag{5.23}$$

where $f(x) \neq 0$.

5.4.1 Structure of the General Solution

General Solution of Non-homogeneous Linear DEs

The general solution of a non-homogeneous linear differential equation $\phi(D)y = f(x)$ is:

$$y = y_c + y_p \tag{5.24}$$

where:

- y_c is the complementary function (general solution of the homogeneous equation $\phi(D)y = 0$)
- y_p is a particular integral (any specific solution of the full equation $\phi(D)y = f(x)$)

The complementary function y_c involves arbitrary constants and captures the homogeneous part of the solution. The particular integral y_p contains no arbitrary constants and accounts for the non-homogeneous term $f(x)$.

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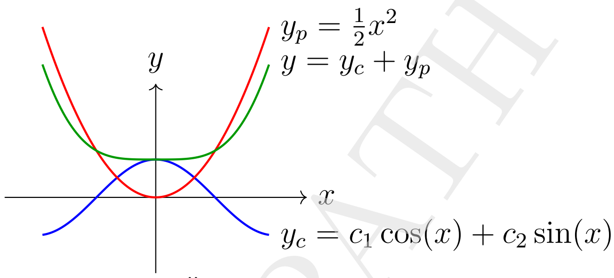

Solutions of $y'' + y = x$

Figure 5.6: Example of complementary function, particular integral, and general solution

5.4.2 Methods for Finding the Particular Integral

The particular integral y_p can be found using several methods:

1. The General Method (using integration)
2. The Method of Undetermined Coefficients (for specific forms of $f(x)$)
3. The Method of Variation of Parameters

Let's begin with the General Method.

5.4.3 The General Method for Finding the Particular Integral

The particular integral can be represented symbolically as:

$$y_p = \frac{1}{\phi(D)} f(x) \tag{5.25}$$

This notation means that we need to find a function that, when operated on by $\phi(D)$, gives $f(x)$.

Case 1: Simple Linear Factor $(D - m)$

For a differential equation $(D - m)y = f(x)$, the particular integral is:

$$y_p = \frac{1}{D - m} f(x) = e^{mx} \int e^{-mx} f(x) dx \tag{5.26}$$

Finding Particular Integral with a Simple Linear Factor

Consider the equation: $(D - 2)y = e^x$
The particular integral is:

$$\begin{aligned} y_p = \frac{1}{D-2} e^x &= e^{2x} \int e^{-2x} e^x dx && (5.27) \\ &= e^{2x} \int e^{-x} dx && (5.28) \\ &= e^{2x} \left(-e^{-x}\right) + C && (5.29) \\ &= -e^x + Ce^{2x} && (5.30) \end{aligned}$$

Since the particular integral should not contain arbitrary constants, we set $C = 0$.
Therefore, $y_p = -e^x$

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Case 2: Linear Factors $(D - m_1)(D - m_2)$

For a differential equation $(D - m_1)(D - m_2)y = f(x)$, where $m_1 \neq m_2$, we can use partial fractions:

$$\frac{1}{(D - m_1)(D - m_2)} = \frac{1}{m_1 - m_2}\left(\frac{1}{D - m_1} - \frac{1}{D - m_2}\right) \tag{5.31}$$

This leads to:

$$y_p = \frac{1}{m_1 - m_2}\left(e^{m_1 x}\int e^{-m_1 x} f(x)dx - e^{m_2 x}\int e^{-m_2 x} f(x)dx\right) \tag{5.32}$$

Case 3: Higher Order Operators

For higher-order operators, we can use repeated integration:

$$\frac{1}{D} f(x) = \int f(x)dx \tag{5.33}$$

$$\frac{1}{D^2} f(x) = \int \left(\int f(x)dx\right) dx \tag{5.34}$$

and so on.

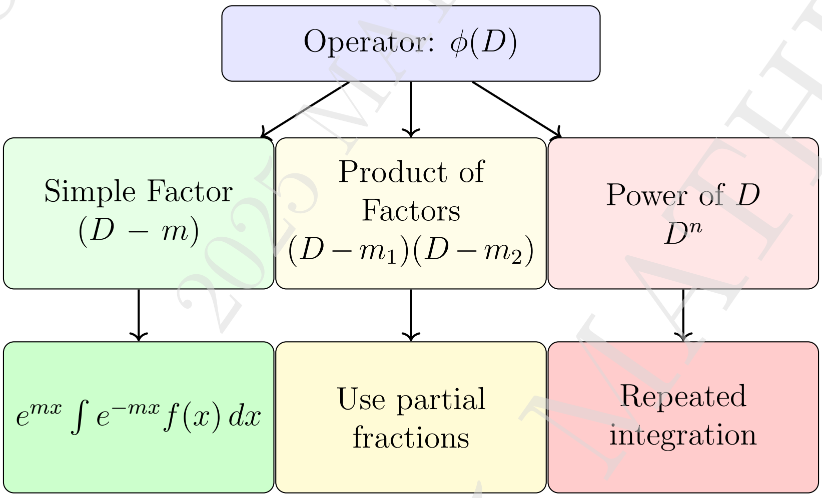

Figure 5.7: General method for finding particular integrals

5.5 Illustrative Examples

Let's look at some examples that demonstrate the process of finding the general solution of linear differential equations with constant coefficients.

Complete Solution of a Second-Order Linear DE

Consider the differential equation: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$
Step 1: Write using operator notation: $(D^2 + 3D + 2)y = e^x$

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Step 2: Find the complementary function by solving $(D^2 + 3D + 2)y = 0$

$$m^2 + 3m + 2 = 0 \tag{5.35}$$
$$(m+1)(m+2) = 0 \tag{5.36}$$

The roots are $m_1 = -1$ and $m_2 = -2$, so:

$$y_c = c_1 e^{-x} + c_2 e^{-2x} \tag{5.37}$$

Step 3: Find the particular integral for $(D^2 + 3D + 2)y = e^x$

$$y_p = \frac{1}{D^2 + 3D + 2} e^x = \frac{1}{(D+1)(D+2)} e^x \tag{5.38}$$

Using partial fractions:

$$\frac{1}{(D+1)(D+2)} = \frac{1}{(D+1) - (D+2)} \left(\frac{1}{D+1} - \frac{1}{D+2} \right) \tag{5.39}$$
$$= \frac{1}{-1} \left(\frac{1}{D+1} - \frac{1}{D+2} \right) \tag{5.40}$$
$$= -\frac{1}{D+1} + \frac{1}{D+2} \tag{5.41}$$

Therefore:

$$y_p = -\frac{1}{D+1} e^x + \frac{1}{D+2} e^x \tag{5.42}$$
$$= -e^{-x} \int e^{-(-x)} e^x dx + e^{-2x} \int e^{-(-2x)} e^x dx \tag{5.43}$$
$$= -e^{-x} \int e^{2x} dx + e^{-2x} \int e^{3x} dx \tag{5.44}$$
$$= -e^{-x} \frac{e^{2x}}{2} + e^{-2x} \frac{e^{3x}}{3} \tag{5.45}$$
$$= -\frac{e^x}{2} + \frac{e^x}{3} \tag{5.46}$$
$$= \frac{-3+2}{6} e^x \tag{5.47}$$
$$= -\frac{1}{6} e^x \tag{5.48}$$

Step 4: Write the general solution as $y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{6} e^x$

Note that the complementary function represents the "natural response" of the system, while the particular integral represents the "forced response" due to the input function $f(x)$.

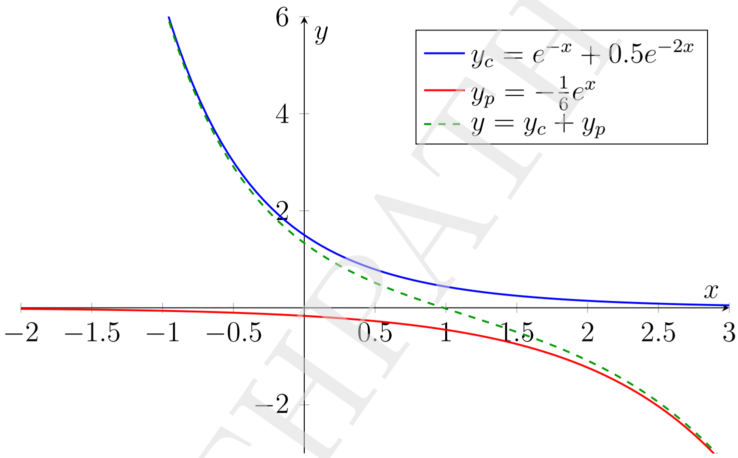

Figure 5.8: Visualization of the complementary function, particular integral, and general solution of $y'' + 3y' + 2y = e^x$

5.5.1 Special Cases in the General Method

When using the general method, we need to be careful about certain special cases:

1. If $f(x) = e^{ax}$ and a is a root of the auxiliary equation, then the formula $\frac{1}{\phi(D)}e^{ax} = \frac{e^{ax}}{\phi(a)}$ is not applicable.
2. In such cases, if a is a root with multiplicity r, then:

$$\frac{1}{\phi(D)}e^{ax} = \frac{x^r e^{ax}}{r!\phi^{(r)}(a)} \tag{5.49}$$

where $\phi^{(r)}(a)$ is the rth derivative of $\phi(m)$ with respect to m, evaluated at $m = a$.

5.6 Method of Undetermined Coefficients

The method of undetermined coefficients is a more direct approach for finding particular integrals when $f(x)$ has specific forms such as polynomials, exponentials, sines, cosines, or products of these.

The basic idea is to assume a particular form for y_p based on the form of $f(x)$, and then determine the coefficients by substituting into the original differential equation.

Method of Undetermined Coefficients

For a differential equation $\phi(D)y = f(x)$, assume a form for y_p based on $f(x)$ according to this table:

Form of $f(x)$	Form of y_p
$P_n(x)$ (polynomial of degree n)	$Q_n(x)$ (polynomial of degree n)
e^{ax}	Ae^{ax}
$\sin(bx)$ or $\cos(bx)$	$A\sin(bx) + B\cos(bx)$
$e^{ax}P_n(x)$	$e^{ax}Q_n(x)$
$e^{ax}\sin(bx)$ or $e^{ax}\cos(bx)$	$e^{ax}(A\sin(bx) + B\cos(bx))$

If any term in the assumed form of y_p is already a solution of the homogeneous equation $\phi(D)y = 0$, multiply that term by x^k where k is the smallest positive integer that makes the resulting term not a solution of the homogeneous equation.

Using the Method of Undetermined Coefficients

Consider the differential equation: $\frac{d^2y}{dx^2} + y = x^2$
Step 1: Find the complementary function by solving $y'' + y = 0$

$$m^2 + 1 = 0 \tag{5.50}$$
$$m = \pm i \tag{5.51}$$

So the complementary function is: $y_c = c_1 \cos(x) + c_2 \sin(x)$
Step 2: For the particular integral, since $f(x) = x^2$ is a polynomial of degree 2, we assume:

$$y_p = Ax^2 + Bx + C \tag{5.52}$$

Step 3: Substitute into the original equation:

$$\frac{d^2}{dx^2}(Ax^2 + Bx + C) + (Ax^2 + Bx + C) = x^2 \tag{5.53}$$
$$2A + (Ax^2 + Bx + C) = x^2 \tag{5.54}$$
$$Ax^2 + Bx + (C + 2A) = x^2 \tag{5.55}$$

Step 4: Equate coefficients:

$$A = 1 \tag{5.56}$$
$$B = 0 \tag{5.57}$$
$$C + 2A = 0 \implies C = -2 \tag{5.58}$$

Therefore, the particular integral is: $y_p = x^2 - 2$
Step 5: The general solution is:

$$y = y_c + y_p = c_1 \cos(x) + c_2 \sin(x) + x^2 - 2 \tag{5.59}$$

5.6.1 Further Examples

Dealing with Terms in the Complementary Function

Consider the differential equation: $\frac{d^2y}{dx^2} - 4y = 2e^{2x}$
Step 1: Find the complementary function by solving $y'' - 4y = 0$

$$m^2 - 4 = 0 \tag{5.60}$$
$$m = \pm 2 \tag{5.61}$$

So the complementary function is: $y_c = c_1e^{2x} + c_2e^{-2x}$
Step 2: For the particular integral, since $f(x) = 2e^{2x}$, we would normally assume $y_p = Ae^{2x}$.
However, e^{2x} is already part of the complementary function, so we need to multiply by x:

$$y_p = Axe^{2x} \tag{5.62}$$

Step 3: Substitute into the original equation:

$$\frac{d^2}{dx^2}(Axe^{2x}) - 4(Axe^{2x}) = 2e^{2x} \tag{5.63}$$
$$A(4xe^{2x} + 4e^{2x}) - 4Axe^{2x} = 2e^{2x} \tag{5.64}$$
$$4Ae^{2x} = 2e^{2x} \tag{5.65}$$
$$4A = 2 \tag{5.66}$$
$$A = \frac{1}{2} \tag{5.67}$$

Therefore, the particular integral is: $y_p = \frac{1}{2}xe^{2x}$
Step 4: The general solution is:

$$y = y_c + y_p = c_1e^{2x} + c_2e^{-2x} + \frac{1}{2}xe^{2x} \tag{5.68}$$

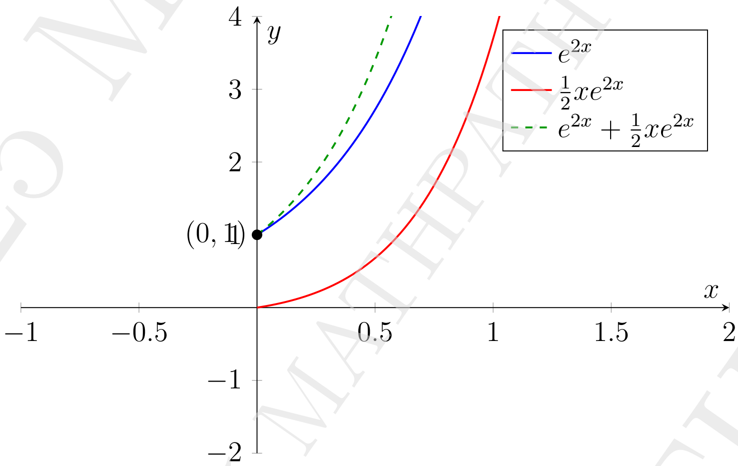

Figure 5.9: Example showing how a particular solution can grow faster than the complementary function for $y'' - 4y = 2e^{2x}$

5.7 Summary of Methods for Solving Linear Differential Equations

1. **Step 1**: Identify the differential equation as linear with constant coefficients: $\phi(D)y = f(x)$.
2. **Step 2**: Find the complementary function y_c by solving the homogeneous equation $\phi(D)y = 0$:
   - Find the roots of the auxiliary equation $\phi(m) = 0$
   - Construct y_c based on the nature of these roots (distinct real, repeated real, or complex conjugate)
3. **Step 3**: Find a particular integral y_p using either:
   - The General Method (using the inverse operator $\frac{1}{\phi(D)}$)
   - The Method of Undetermined Coefficients
4. **Step 4**: Combine to get the general solution: $y = y_c + y_p$

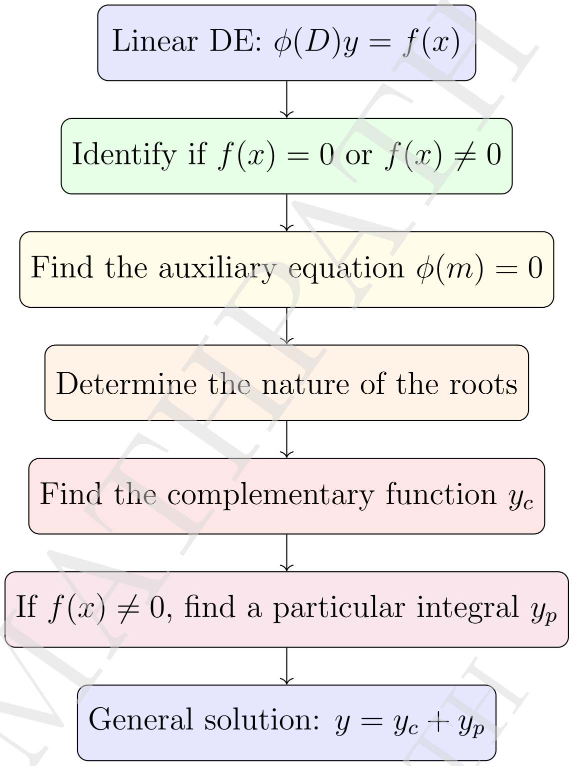

Figure 5.10: General procedure for solving linear differential equations with constant coefficients

5.8 Conclusion

In this chapter, we have covered the fundamental theory and methods for solving linear differential equations with constant coefficients. We learned how to:

- Identify and classify linear differential equations
- Use the differential operator D to simplify our notation
- Find the complementary function by solving the auxiliary equation
- Determine the particular integral using the general method
- Apply the method of undetermined coefficients for specific forms of $f(x)$

5.9 Solved Examples on General Method

These techniques form the foundation for solving a wide range of physical, engineering, and scientific problems that involve rates of change and dynamic systems. In subsequent chapters, we will explore more advanced methods and applications of differential equations.

Example 1: Solving a Non-homogeneous Linear Differential Equation

Solve the differential equation: $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^x$

Step 1: Express the equation using the differential operator D.
$(D^2 + 3D + 2)y = e^x$
Step 2: Find the complementary function by solving the homogeneous equation $(D^2 + 3D + 2)y = 0$.
First, we write the auxiliary equation: $m^2 + 3m + 2 = 0$
Factoring this equation: $(m + 1)(m + 2) = 0$

The roots are $m_1 = -1$ and $m_2 = -2$.
Therefore, the complementary function is: $y_c = c_1 e^{-x} + c_2 e^{-2x}$
Step 3: Find the particular integral for $(D^2 + 3D + 2)y = e^x$ using the general method.
We need to evaluate $y_p = \dfrac{1}{D^2 + 3D + 2} e^x = \dfrac{1}{(D+1)(D+2)} e^x$
Using partial fractions:

$$\frac{1}{(D+1)(D+2)} = \frac{A}{D+1} + \frac{B}{D+2}$$

Multiplying both sides by $(D+1)(D+2)$:

$$\begin{aligned} 1 &= A(D+2) + B(D+1) \\ (0)D + 1 &= (A+B)D + (2A+B) \end{aligned}$$

Comparing coefficients:

$$\begin{aligned} \text{From coefficient of } D&: A + B = 0 \\ \text{From constant term}&: 2A + B = 1 \end{aligned}$$

Solving these equations:

$$\begin{aligned} A + B &= 0 \\ 2A + B &= 1 \end{aligned}$$

Subtracting the first equation from the second:

$$\begin{aligned} A &= 1 \\ \Rightarrow B &= -1 \end{aligned}$$

Therefore:

$$\frac{1}{(D+1)(D+2)} = \frac{1}{D+1} - \frac{1}{D+2}$$

Now we can find the particular integral:

$$\begin{aligned} y_p &= \left(\frac{1}{D+1} - \frac{1}{D+2} \right) e^x \\ &= \frac{1}{D+1} e^x - \frac{1}{D+2} e^x \end{aligned}$$

For $\dfrac{1}{D+1} e^x$:

$$\begin{aligned} \frac{1}{D+1} e^x &= \frac{1}{D+1} e^x \\ &= e^{-1 \cdot x} \int e^{1 \cdot x} \cdot e^x \, dx \\ &= e^{-x} \int e^{2x} \, dx \\ &= e^{-x} \cdot \frac{e^{2x}}{2} \\ &= \frac{e^{-x} \cdot e^{2x}}{2} \\ &= \frac{e^x}{2} \end{aligned}$$

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Similarly, for $\frac{1}{D+2}e^x$:

$$\begin{aligned}\frac{1}{D+2}e^x &= e^{-2x}\int e^{2x}\cdot e^x\,dx\\ &= e^{-2x}\int e^{3x}\,dx\\ &= e^{-2x}\cdot\frac{e^{3x}}{3}\\ &= \frac{e^{-2x}\cdot e^{3x}}{3}\\ &= \frac{e^x}{3}\end{aligned}$$

Therefore:

$$\begin{aligned}y_p &= \frac{e^x}{2}-\frac{e^x}{3}\\ &= e^x\left(\frac{3-2}{6}\right)\\ &= \frac{e^x}{6}\end{aligned}$$

Step 4: Write the general solution as the sum of the complementary function and the particular integral.

$$\begin{aligned}y &= y_c + y_p\\ &= c_1e^{-x}+c_2e^{-2x}+\frac{e^x}{6}\end{aligned}$$

This is the complete general solution to the given differential equation.

Example 2: Solving a Non-homogeneous Linear Differential Equation

Solve the differential equation: $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{e^x}$

Step 1: Express the equation using the differential operator D.
$(D^2+3D+2)y=e^{e^x}$
Step 2: Find the complementary function by solving the homogeneous equation $(D^2+3D+2)y=0$.
First, we write the auxiliary equation: $m^2+3m+2=0$
Factoring this equation: $(m+1)(m+2)=0$
The roots are $m_1=-1$ and $m_2=-2$.
Therefore, the complementary function is: $y_c=c_1e^{-x}+c_2e^{-2x}$
Step 3: Find the particular integral for $(D^2+3D+2)y=e^{e^x}$ using the general method.
We can write:

$$y_p=\frac{1}{D^2+3D+2}e^{e^x}=\frac{1}{(D+1)(D+2)}e^{e^x}$$

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Using partial fractions:

$$\frac{1}{(D+1)(D+2)} = \frac{A}{D+1} + \frac{B}{D+2}$$

To find A and B, we use:

$$\frac{1}{(D+1)(D+2)} = \frac{A(D+2)+B(D+1)}{(D+1)(D+2)}$$

So we need:

$$1 = A(D+2) + B(D+1)$$

For this to be true for all D, we need to compare coefficients:

$$\begin{aligned} \text{Coefficient of } D\text{:} \quad & A + B = 0 \\ \text{Constant term:} \quad & 2A + B = 1 \end{aligned}$$

From the first equation: $B = -A$ Substituting into the second equation:

$$\begin{aligned} 2A - A &= 1 \\ A &= 1 \end{aligned}$$

Therefore, $A = 1$ and $B = -1$, giving:

$$\frac{1}{(D+1)(D+2)} = \frac{1}{D+1} - \frac{1}{D+2}$$

Now we can find the particular integral:

$$y_p = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^x}$$

Let's compute each term separately.
For $\frac{1}{D+1} e^{e^x}$:

$$\begin{aligned} \frac{1}{D+1} e^{e^x} &= e^{-1\cdot x} \int e^{1\cdot x} \cdot e^{e^x}\, dx \\ &= e^{-x} \int e^x \cdot e^{e^x}\, dx \end{aligned}$$

Let's make a substitution: $u = e^x$, so $du = e^x dx$, and when x changes, $u = e^x$ changes.

$$\begin{aligned} e^{-x} \int e^x \cdot e^{e^x}\, dx &= e^{-x} \int e^u\, du \\ &= e^{-x} \cdot e^u + C_1 \\ &= e^{-x} \cdot e^{e^x} + C_1 e^{-x} \end{aligned}$$

Since C_1 contributes to the complementary function, we can omit it for the particular integral.
For $\frac{1}{D+2} e^{e^x}$:

$$\frac{1}{D+2} e^{e^x} = e^{-2x} \int e^{2x} \cdot e^{e^x}\, dx$$

With the substitution $u = e^x$, we have $du = e^x dx$, so $e^x dx = du$ and $e^{2x} dx = e^x \cdot du = u \cdot du$.

$$e^{-2x} \int e^{2x} \cdot e^{e^x} \, dx = e^{-2x} \int u \cdot e^u \, du$$

Integrating by parts with $dv = e^u du$ and $v = e^u$:

$$\begin{aligned} e^{-2x} \int u \cdot e^u \, du &= e^{-2x} \left[u \cdot e^u - \int e^u \, du \right] \\ &= e^{-2x} \left[u \cdot e^u - e^u \right] + C_2 \\ &= e^{-2x} \cdot e^u \cdot (u-1) + C_2 \\ &= e^{-2x} \cdot e^{e^x} \cdot (e^x - 1) + C_2 e^{-2x} \\ &= e^{-2x} \cdot e^{e^x} \cdot e^x - e^{-2x} \cdot e^{e^x} + C_2 e^{-2x} \\ &= e^{-x} \cdot e^{e^x} - e^{-2x} \cdot e^{e^x} + C_2 e^{-2x} \end{aligned}$$

Again, C_2 contributes to the complementary function, so we omit it.
Now, let's combine these results for the particular integral:

$$\begin{aligned} y_p &= \frac{1}{D+1} e^{e^x} - \frac{1}{D+2} e^{e^x} \\ &= e^{-x} \cdot e^{e^x} - \left(e^{-x} \cdot e^{e^x} - e^{-2x} \cdot e^{e^x} \right) \\ &= e^{-x} \cdot e^{e^x} - e^{-x} \cdot e^{e^x} + e^{-2x} \cdot e^{e^x} \\ &= e^{-2x} \cdot e^{e^x} \end{aligned}$$

Step 4: Write the general solution as the sum of the complementary function and the particular integral.

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \cdot e^{e^x} \end{aligned}$$

This is the complete general solution to the differential equation.

Example 3: Solving a Non-homogeneous Linear Differential Equation with Combined Forcing

Solve the differential equation: $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{e^x} + \cos e^x$

Step 1: Express the equation using the differential operator D.
$(D^2 + 3D + 2)y = e^{e^x} + \cos e^x$
Step 2: Find the complementary function by solving the homogeneous equation $(D^2 + 3D + 2)y = 0$.
First, we write the auxiliary equation: $m^2 + 3m + 2 = 0$
Factoring this equation: $(m+1)(m+2) = 0$
The roots are $m_1 = -1$ and $m_2 = -2$.
Therefore, the complementary function is: $y_c = c_1 e^{-x} + c_2 e^{-2x}$
Step 3: Find the particular integral using the general method.
Due to the principle of superposition for linear differential equations, we can write:

$$y_p = y_{p1} + y_{p2}$$

Where:

$$y_{p1} = \frac{1}{D^2 + 3D + 2} e^{e^x}$$
$$y_{p2} = \frac{1}{D^2 + 3D + 2} \cos e^x$$

Let's solve for each component separately.
Part A: Finding $y_{p1} = \frac{1}{D^2+3D+2} e^{e^x}$
Using partial fractions:

$$\frac{1}{(D+1)(D+2)} = \frac{A}{D+1} + \frac{B}{D+2}$$

Following the same procedure as in Example 3.2, we get $A = 1$ and $B = -1$, thus:

$$\frac{1}{(D+1)(D+2)} = \frac{1}{D+1} - \frac{1}{D+2}$$

For $\frac{1}{D+1} e^{e^x}$:

$$\frac{1}{D+1} e^{e^x} = e^{-x} \int e^x \cdot e^{e^x}\, dx$$

With substitution $u = e^x$, $du = e^x dx$:

$$\begin{aligned} e^{-x} \int e^x \cdot e^{e^x}\, dx &= e^{-x} \int e^u\, du \\ &= e^{-x} \cdot e^u \\ &= e^{-x} \cdot e^{e^x} \end{aligned}$$

For $\frac{1}{D+2} e^{e^x}$:

$$\frac{1}{D+2} e^{e^x} = e^{-2x} \int e^{2x} \cdot e^{e^x}\, dx$$

With substitution $u = e^x$, $du = e^x dx$, and $e^{2x} dx = e^x \cdot u \cdot du$:

$$e^{-2x} \int e^{2x} \cdot e^{e^x}\, dx = e^{-2x} \int u \cdot e^u\, du$$

Integrating by parts with $dv = e^u du$ and $v = e^u$:

$$\begin{aligned} e^{-2x} \int u \cdot e^u\, du &= e^{-2x} \left[u \cdot e^u - \int e^u\, du \right] \\ &= e^{-2x} \left[u \cdot e^u - e^u \right] \\ &= e^{-2x} \cdot e^u \cdot (u - 1) \\ &= e^{-2x} \cdot e^{e^x} \cdot (e^x - 1) \\ &= e^{-2x} \cdot e^{e^x} \cdot e^x - e^{-2x} \cdot e^{e^x} \\ &= e^{-x} \cdot e^{e^x} - e^{-2x} \cdot e^{e^x} \end{aligned}$$

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Combining the terms for y_{p1}:

$$\begin{aligned} y_{p1} &= \frac{1}{D+1}e^{e^x} - \frac{1}{D+2}e^{e^x} \\ &= e^{-x}\cdot e^{e^x} - \left(e^{-x}\cdot e^{e^x} - e^{-2x}\cdot e^{e^x}\right) \\ &= e^{-x}\cdot e^{e^x} - e^{-x}\cdot e^{e^x} + e^{-2x}\cdot e^{e^x} \\ &= e^{-2x}\cdot e^{e^x} \end{aligned}$$

Part B: Finding $y_{p2} = \frac{1}{D^2+3D+2}\cos e^x$
Using the same partial fraction decomposition:

$$y_{p2} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right)\cos e^x$$

For $\frac{1}{D+1}\cos e^x$:

$$\frac{1}{D+1}\cos e^x = e^{-x}\int e^x\cdot\cos e^x\,dx$$

With substitution $u = e^x$, $du = e^x dx$:

$$\begin{aligned} e^{-x}\int e^x\cdot\cos e^x\,dx &= e^{-x}\int\cos u\,du \\ &= e^{-x}\cdot\sin u \\ &= e^{-x}\cdot\sin e^x \end{aligned}$$

For $\frac{1}{D+2}\cos e^x$:

$$\frac{1}{D+2}\cos e^x = e^{-2x}\int e^{2x}\cdot\cos e^x\,dx$$

With substitution $u = e^x$, $du = e^x dx$, and $e^{2x}dx = e^x\cdot u\cdot du$:

$$e^{-2x}\int e^{2x}\cdot\cos e^x\,dx = e^{-2x}\int u\cdot\cos u\,du$$

Integrating by parts with $dv = \cos u\,du$ and $v = \sin u$:

$$\begin{aligned} e^{-2x}\int u\cdot\cos u\,du &= e^{-2x}\left[u\cdot\sin u - \int\sin u\,du\right] \\ &= e^{-2x}\left[u\cdot\sin u + \cos u\right] \\ &= e^{-2x}\cdot e^x\cdot\sin e^x + e^{-2x}\cdot\cos e^x \\ &= e^{-x}\cdot\sin e^x + e^{-2x}\cdot\cos e^x \end{aligned}$$

Combining the terms for y_{p2}:

$$\begin{aligned} y_{p2} &= \frac{1}{D+1}\cos e^x - \frac{1}{D+2}\cos e^x \\ &= e^{-x}\cdot\sin e^x - \left(e^{-x}\cdot\sin e^x + e^{-2x}\cdot\cos e^x\right) \\ &= e^{-x}\cdot\sin e^x - e^{-x}\cdot\sin e^x - e^{-2x}\cdot\cos e^x \\ &= -e^{-2x}\cdot\cos e^x \end{aligned}$$

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Step 4: The complete particular solution is:

$$\begin{aligned} y_p &= y_{p1} + y_{p2} \\ &= e^{-2x} \cdot e^{e^x} + (-e^{-2x} \cdot \cos e^x) \\ &= e^{-2x} \cdot e^{e^x} - e^{-2x} \cdot \cos e^x \\ &= e^{-2x}(e^{e^x} - \cos e^x) \end{aligned}$$

Step 5: Write the general solution as the sum of the complementary function and the particular integral.

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 e^{-x} + c_2 e^{-2x} + e^{-2x}(e^{e^x} - \cos e^x) \\ &= c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \cdot e^{e^x} - e^{-2x} \cdot \cos e^x \end{aligned}$$

This is the complete general solution to the differential equation.

Example 4: Solving a Non-homogeneous Linear Differential Equation

Solve the differential equation: $(D^2 + 3D + 2)y = \sin e^x$

Step 1: First identify that this is already in operator form.
$(D^2 + 3D + 2)y = \sin e^x$
Step 2: Find the complementary function by solving the homogeneous equation $(D^2 + 3D + 2)y = 0$.
First, we write the auxiliary equation: $m^2 + 3m + 2 = 0$
Factoring this equation: $(m + 1)(m + 2) = 0$
The roots are $m_1 = -1$ and $m_2 = -2$.
Therefore, the complementary function is: $y_c = c_1 e^{-x} + c_2 e^{-2x}$
Step 3: Find the particular integral using the general method.
We can write:

$$y_p = \frac{1}{D^2 + 3D + 2} \sin e^x = \frac{1}{(D + 1)(D + 2)} \sin e^x$$

Using partial fractions:

$$\frac{1}{(D + 1)(D + 2)} = \frac{A}{D + 1} + \frac{B}{D + 2}$$

To find A and B, we use:

$$\frac{1}{(D + 1)(D + 2)} = \frac{A(D + 2) + B(D + 1)}{(D + 1)(D + 2)}$$

So we need:

$$1 = A(D + 2) + B(D + 1)$$

Comparing coefficients:

$$\begin{aligned} \text{Coefficient of } D\text{:} \quad & A + B = 0 \\ \text{Constant term:} \quad & 2A + B = 1 \end{aligned}$$

From the first equation: $B = -A$ Substituting into the second equation:

$$\begin{aligned} 2A - A &= 1 \\ A &= 1 \end{aligned}$$

Therefore, $A = 1$ and $B = -1$, giving:

$$\frac{1}{(D+1)(D+2)} = \frac{1}{D+1} - \frac{1}{D+2}$$

Now we can find the particular integral:

$$y_p = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) \sin e^x$$

Let's compute each term separately.
For $\frac{1}{D+1} \sin e^x$:

$$\begin{aligned} \frac{1}{D+1} \sin e^x &= e^{-1\cdot x} \int e^{1\cdot x} \cdot \sin e^x \, dx \\ &= e^{-x} \int e^x \cdot \sin e^x \, dx \end{aligned}$$

Let's make a substitution: $u = e^x$, so $du = e^x dx$.

$$\begin{aligned} e^{-x} \int e^x \cdot \sin e^x \, dx &= e^{-x} \int \sin u \, du \\ &= e^{-x} \cdot (-\cos u) + C_1 \\ &= -e^{-x} \cdot \cos e^x + C_1 e^{-x} \end{aligned}$$

Since C_1 contributes to the complementary function, we can omit it for the particular integral.
For $\frac{1}{D+2} \sin e^x$:

$$\frac{1}{D+2} \sin e^x = e^{-2x} \int e^{2x} \cdot \sin e^x \, dx$$

With the substitution $u = e^x$, we have $du = e^x dx$, so $e^{2x} dx = e^x \cdot du \cdot e^x = u \cdot du$.

$$e^{-2x} \int e^{2x} \cdot \sin e^x \, dx = e^{-2x} \int u \cdot \sin u \, du$$

Integrating by parts with $dv = \sin u \, du$ and $v = -\cos u$:

$$\begin{aligned} e^{-2x} \int u \cdot \sin u \, du &= e^{-2x} \left[-u \cdot \cos u - \int (-\cos u) \, du\right] \\ &= e^{-2x} \left[-u \cdot \cos u + \int \cos u \, du\right] \\ &= e^{-2x} \left[-u \cdot \cos u + \sin u\right] \\ &= -e^{-2x} \cdot e^x \cdot \cos e^x + e^{-2x} \cdot \sin e^x \\ &= -e^{-x} \cdot \cos e^x + e^{-2x} \cdot \sin e^x \end{aligned}$$

Now, let's combine these results for the particular integral:

$$\begin{aligned} y_p &= \frac{1}{D+1}\sin e^x - \frac{1}{D+2}\sin e^x \\ &= -e^{-x}\cdot\cos e^x - \left(-e^{-x}\cdot\cos e^x + e^{-2x}\cdot\sin e^x\right) \\ &= -e^{-x}\cdot\cos e^x + e^{-x}\cdot\cos e^x - e^{-2x}\cdot\sin e^x \\ &= -e^{-2x}\cdot\sin e^x \end{aligned}$$

Step 4: Write the general solution as the sum of the complementary function and the particular integral.

$$\begin{aligned} y &= y_c + y_p \\ &= c_1e^{-x} + c_2e^{-2x} - e^{-2x}\cdot\sin e^x \\ &= c_1e^{-x} + c_2e^{-2x} - e^{-2x}\cdot\sin e^x \end{aligned}$$

This is the complete general solution to the differential equation.

Example 5: Solving a Linear Differential Equation

Solve the differential equation: $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x$

Step 1: Express the equation using the differential operator D.
$(D^2+1)y = \operatorname{cosec} x$
Step 2: Find the complementary function by solving the homogeneous equation $(D^2+1)y = 0$.
Auxiliary equation: $D^2+1=0$
Roots: $D = \pm i$
Therefore, the complementary function is: $y_c = c_1\cos x + c_2\sin x$
Step 3: Find the particular integral using the general method.

$$y_p = \frac{1}{D^2+1}(\operatorname{cosec} x)$$

Since $\operatorname{cosec} x = \frac{1}{\sin x}$, this is not in a standard form that we can directly apply formulas to. Let's use the variation of parameters method.
For a second-order differential equation $(D^2+1)y = f(x)$ with complementary function $y_c = c_1\cos x + c_2\sin x$, the particular integral is:

$$\begin{aligned} y_p &= -\cos x\int \sin x\cdot f(x)\,dx + \sin x\int \cos x\cdot f(x)\,dx \\ &= -\cos x\int \sin x\cdot\operatorname{cosec} x\,dx + \sin x\int \cos x\cdot\operatorname{cosec} x\,dx \\ &= -\cos x\int 1\,dx + \sin x\int \cot x\,dx \\ &= -\cos x\cdot x + \sin x\cdot\ln|\sin x| \end{aligned}$$

Therefore, the particular integral is: $y_p = -x\cos x + \ln|\sin x|\sin x$
Step 4: Write the general solution as the sum of the complementary function and the particular integral.

$$
\begin{aligned}
y &= y_c + y_p \\
&= c_1 \cos x + c_2 \sin x - x \cos x + \ln|\sin x| \sin x \\
&= (c_1 - x) \cos x + (c_2 + \ln|\sin x|) \sin x
\end{aligned}
$$

This is the complete general solution to the differential equation.

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