Chapter 2

Vector Differential Calculus I: Derivatives and Motion

Vector differential calculus provides the mathematical framework needed to analyze motion, forces, and other physical phenomena involving direction and magnitude. This chapter introduces fundamental concepts of vector derivatives and their physical interpretations.

2.1 Derivatives of Vector Functions

A vector function $\mathbf{r}(t)$ assigns a vector to each value of parameter t:

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} \tag{2.1}$$

2.1.1 Definition of Vector Derivative

The derivative of a vector function is defined as:

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \tag{2.2}$$

If the component functions $x(t)$, $y(t)$, and $z(t)$ are differentiable, then:

$$\mathbf{r}'(t) = x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}} + z'(t)\hat{\mathbf{k}} \tag{2.3}$$

Computing a Vector Derivative

For $\mathbf{r}(t) = t^2\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + e^t\hat{\mathbf{k}}$, the derivative is:

$$\mathbf{r}'(t) = \frac{d}{dt}(t^2)\hat{\mathbf{i}} + \frac{d}{dt}(\sin t)\hat{\mathbf{j}} + \frac{d}{dt}(e^t)\hat{\mathbf{k}} \tag{2.4}$$

$$= 2t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}} + e^t\hat{\mathbf{k}} \tag{2.5}$$

2.1.2 Differentiation Rules

Vector derivatives follow rules similar to scalar derivatives:

Vector Differentiation Rules

For vector functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$, and scalar function $f(t)$:
1. Sum Rule: $\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \mathbf{u}'(t)+\mathbf{v}'(t)$
2. Scalar Multiple: $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$ for constant c
3. Product Rule with Scalar: $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$
4. Dot Product Rule: $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$
5. Cross Product Rule: $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$

Dot Product Differentiation

For $\mathbf{u}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}}$ and $\mathbf{v}(t) = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}$, find $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)]$.
First, find the derivatives:

$$\mathbf{u}'(t) = -\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}} \tag{2.6}$$
$$\mathbf{v}'(t) = \hat{\mathbf{i}} + 2t\hat{\mathbf{j}} \tag{2.7}$$

Using the dot product rule:

$$\begin{aligned}
\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] &= \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t) && (2.8)\\
&= (-\sin t\hat{\mathbf{i}}+\cos t\hat{\mathbf{j}})\cdot(t\hat{\mathbf{i}}+t^2\hat{\mathbf{j}}) + (\cos t\hat{\mathbf{i}}+\sin t\hat{\mathbf{j}})\cdot(\hat{\mathbf{i}}+2t\hat{\mathbf{j}}) && (2.9)\\
&= (-\sin t\cdot t+\cos t\cdot t^2)+(\cos t\cdot 1+\sin t\cdot 2t) && (2.10)\\
&= -t\sin t + t^2\cos t + \cos t + 2t\sin t && (2.11)\\
&= t^2\cos t + \cos t + t\sin t && (2.12)
\end{aligned}$$

2.2 Velocity Vector

For a particle whose position is given by the vector function $\mathbf{r}(t)$, the velocity vector is defined as the derivative of the position vector with respect to time:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}(t)}{dt} \tag{2.13}$$

2.2.1 Physical Interpretation

The velocity vector has several important interpretations:
1. Direction: $\mathbf{v}(t)$ is tangent to the particle's trajectory at time t
2. Magnitude: $|\mathbf{v}(t)|$ is the speed of the particle
3. Components: $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$, $v_z = \frac{dz}{dt}$ represent rates of change in each coordinate direction

Calculating Velocity

A particle moves according to $\mathbf{r}(t) = 2\cos t\hat{\mathbf{i}} + 2\sin t\hat{\mathbf{j}} + t\hat{\mathbf{k}}$. Find its velocity vector at $t = \pi/4$.

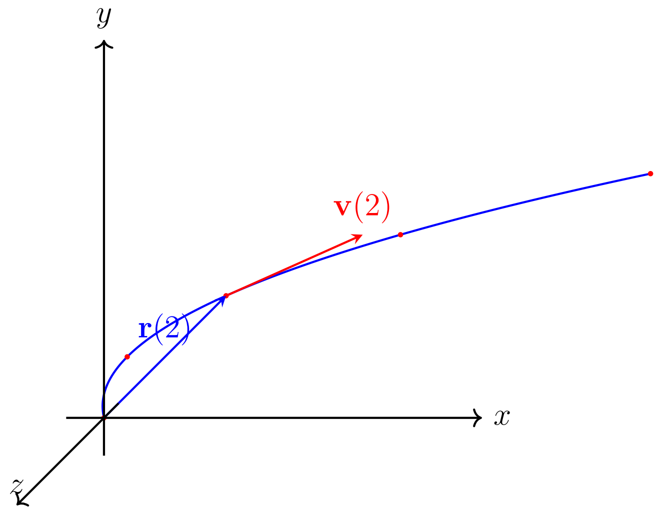

Figure 2.1: Position and velocity vectors for a particle moving along a curved path

The velocity vector is:

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) && (2.14)\\
&= \frac{d}{dt}(2\cos t)\hat{\mathbf{i}} + \frac{d}{dt}(2\sin t)\hat{\mathbf{j}} + \frac{d}{dt}(t)\hat{\mathbf{k}} && (2.15)\\
&= -2\sin t\hat{\mathbf{i}} + 2\cos t\hat{\mathbf{j}} + \hat{\mathbf{k}} && (2.16)
\end{aligned}$$

At $t = \pi/4$:

$$\begin{aligned}
\mathbf{v}(\pi/4) &= -2\sin(\pi/4)\hat{\mathbf{i}} + 2\cos(\pi/4)\hat{\mathbf{j}} + \hat{\mathbf{k}} && (2.17)\\
&= -2\cdot\frac{\sqrt{2}}{2}\hat{\mathbf{i}} + 2\cdot\frac{\sqrt{2}}{2}\hat{\mathbf{j}} + \hat{\mathbf{k}} && (2.18)\\
&= -\sqrt{2}\hat{\mathbf{i}} + \sqrt{2}\hat{\mathbf{j}} + \hat{\mathbf{k}} && (2.19)
\end{aligned}$$

The speed is:

$$\begin{aligned}
|\mathbf{v}(\pi/4)| &= \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2 + 1^2} && (2.20)\\
&= \sqrt{2+2+1} = \sqrt{5} && (2.21)
\end{aligned}$$

2.2.2 Unit Tangent Vector

The unit tangent vector $\hat{\mathbf{T}}$ points in the direction of motion and is defined as:

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \tag{2.22}$$

This vector gives the direction of motion, regardless of speed.

2.3 Acceleration Vector

The acceleration vector is the derivative of the velocity vector with respect to time:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \frac{d^2\mathbf{r}(t)}{dt^2} \tag{2.23}$$

2.3.1 Physical Interpretation

Acceleration represents the rate of change of velocity and has several important characteristics:
1. Direction: Unlike velocity, acceleration is generally not tangent to the path
2. Components: $a_x = \frac{d^2x}{dt^2}$, $a_y = \frac{d^2y}{dt^2}$, $a_z = \frac{d^2z}{dt^2}$
3. Physical meaning: Acceleration indicates both changes in speed and changes in direction

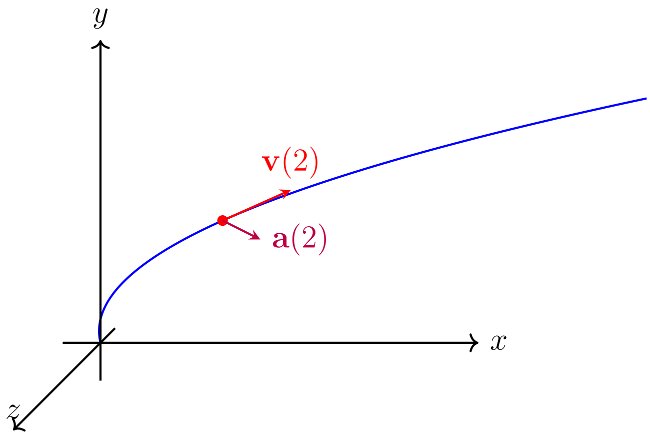

Figure 2.2: Velocity and acceleration vectors for a particle moving along a curved path. Notice that the acceleration is not tangent to the path.

Calculating Acceleration

For a particle with position $\mathbf{r}(t) = t^2\hat{\mathbf{i}} + t\hat{\mathbf{j}} + \sin t\hat{\mathbf{k}}$, find the acceleration at $t = \pi/2$.
First, find the velocity:

$$\mathbf{v}(t) = \mathbf{r}'(t) \tag{2.24}$$
$$= 2t\hat{\mathbf{i}} + \hat{\mathbf{j}} + \cos t\hat{\mathbf{k}} \tag{2.25}$$

Then, find the acceleration:

$$\mathbf{a}(t) = \mathbf{v}'(t) \tag{2.26}$$
$$= 2\hat{\mathbf{i}} + 0\hat{\mathbf{j}} - \sin t\hat{\mathbf{k}} \tag{2.27}$$
$$= 2\hat{\mathbf{i}} - \sin t\hat{\mathbf{k}} \tag{2.28}$$

At $t = \pi/2$:

$$\mathbf{a}(\pi/2) = 2\hat{\mathbf{i}} - \sin(\pi/2)\hat{\mathbf{k}} \tag{2.29}$$
$$= 2\hat{\mathbf{i}} - \hat{\mathbf{k}} \tag{2.30}$$

2.4 Tangential and Normal Components of Acceleration

The acceleration vector can be decomposed into two components: tangential and normal.

2.4.1 Tangential Acceleration

The tangential component of acceleration acts along the direction of motion and is responsible for changes in speed:

$$a_T = \mathbf{a} \cdot \hat{\mathbf{T}} = \frac{d|\mathbf{v}|}{dt} \tag{2.31}$$

2.4.2 Normal Acceleration

The normal component of acceleration is perpendicular to the trajectory and is responsible for changes in direction:

$$a_N = |\mathbf{a} \times \hat{\mathbf{T}}| = \frac{|\mathbf{v}|^2}{\rho} \tag{2.32}$$

where ρ is the radius of curvature.

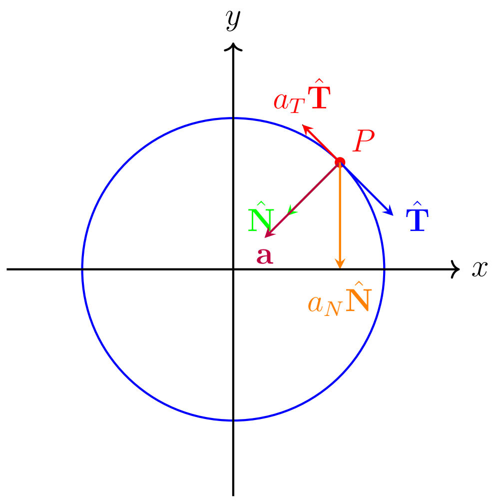

Figure 2.3: Decomposition of acceleration into tangential and normal components

Acceleration Decomposition

The acceleration vector can be written as:

$$\mathbf{a} = a_T\hat{\mathbf{T}} + a_N\hat{\mathbf{N}} \tag{2.33}$$

where $\hat{\mathbf{T}}$ is the unit tangent vector and $\hat{\mathbf{N}}$ is the unit normal vector pointing toward the center of curvature.

Tangential and Normal Acceleration

A particle moves in a circle of radius 2 meters according to $\mathbf{r}(t) = 2\cos t\hat{\mathbf{i}} + 2\sin t\hat{\mathbf{j}}$. Find the tangential and normal components of acceleration at $t = \pi/4$.
First, find the velocity:

$$\mathbf{v}(t) = \mathbf{r}'(t) \tag{2.34}$$
$$= -2\sin t\hat{\mathbf{i}} + 2\cos t\hat{\mathbf{j}} \tag{2.35}$$

The speed is:

$$|\mathbf{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} \tag{2.36}$$
$$= \sqrt{4\sin^2 t + 4\cos^2 t} \tag{2.37}$$
$$= \sqrt{4(\sin^2 t + \cos^2 t)} \tag{2.38}$$
$$= \sqrt{4 \cdot 1} = 2 \tag{2.39}$$

The unit tangent vector is:

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} \tag{2.40}$$
$$= \frac{-2\sin t\hat{\mathbf{i}} + 2\cos t\hat{\mathbf{j}}}{2} \tag{2.41}$$
$$= -\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}} \tag{2.42}$$

The acceleration is:

$$\mathbf{a}(t) = \mathbf{v}'(t) \tag{2.43}$$
$$= -2\cos t\hat{\mathbf{i}} - 2\sin t\hat{\mathbf{j}} \tag{2.44}$$

The tangential component is:

$$a_T = \mathbf{a}(t) \cdot \hat{\mathbf{T}}(t) \tag{2.45}$$
$$= (-2\cos t\hat{\mathbf{i}} - 2\sin t\hat{\mathbf{j}}) \cdot (-\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}}) \tag{2.46}$$
$$= -2\cos t \cdot (-\sin t) - 2\sin t \cdot \cos t \tag{2.47}$$
$$= 2\cos t \sin t - 2\sin t \cos t \tag{2.48}$$
$$= 0 \tag{2.49}$$

This makes sense because the particle moves with constant speed around the circle.
The normal component is:

$$a_N = \sqrt{|\mathbf{a}(t)|^2 - a_T^2} \tag{2.50}$$
$$= \sqrt{|-2\cos t\hat{\mathbf{i}} - 2\sin t\hat{\mathbf{j}}|^2 - 0^2} \tag{2.51}$$
$$= \sqrt{4\cos^2 t + 4\sin^2 t} \tag{2.52}$$
$$= \sqrt{4} = 2 \tag{2.53}$$

Alternatively, $a_N = \frac{|\mathbf{v}(t)|^2}{\rho} = \frac{2^2}{2} = 2$.
At $t = \pi/4$, both components remain the same since $a_T = 0$ for all t and $a_N = 2$ for all t.

2.5 Arc Length and Curvature

2.5.1 Arc Length

The arc length s traversed by a moving particle from time t_1 to time t_2 is:

$$s = \int_{t_1}^{t_2} |\mathbf{v}(t)|\, dt = \int_{t_1}^{t_2} |\mathbf{r}'(t)|\, dt \tag{2.54}$$

If we use arc length s as the parameter instead of time t, we get the arc length parametrization of the curve. In this parametrization, the particle moves with unit speed, meaning $|\mathbf{r}'(s)| = 1$.

Arc Length Calculation

Calculate the arc length of the helix $\mathbf{r}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + t\hat{\mathbf{k}}$ from $t = 0$ to $t = 2\pi$.
The velocity vector is:

$$\mathbf{v}(t) = \mathbf{r}'(t) = -\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}} + \hat{\mathbf{k}} \tag{2.55}$$

The speed is:

$$|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} \tag{2.56}$$
$$= \sqrt{\sin^2 t + \cos^2 t + 1} \tag{2.57}$$
$$= \sqrt{1+1} = \sqrt{2} \tag{2.58}$$

The arc length is:

$$s = \int_0^{2\pi} |\mathbf{v}(t)|\, dt \tag{2.59}$$
$$= \int_0^{2\pi} \sqrt{2}\, dt \tag{2.60}$$
$$= \sqrt{2} \cdot 2\pi \tag{2.61}$$
$$= 2\pi\sqrt{2} \tag{2.62}$$

2.5.2 Curvature

Curvature measures how sharply a curve bends at each point. For a curve given by $\mathbf{r}(t)$, the curvature κ is defined as:

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} \tag{2.63}$$

The radius of curvature ρ is the reciprocal of the curvature:

$$\rho = \frac{1}{\kappa} \tag{2.64}$$

Calculating Curvature

Find the curvature of the curve $\mathbf{r}(t) = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}$ at $t = 1$.
First, find the derivatives:

$$\mathbf{r}'(t) = \hat{\mathbf{i}} + 2t\hat{\mathbf{j}} \tag{2.65}$$
$$\mathbf{r}''(t) = 2\hat{\mathbf{j}} \tag{2.66}$$

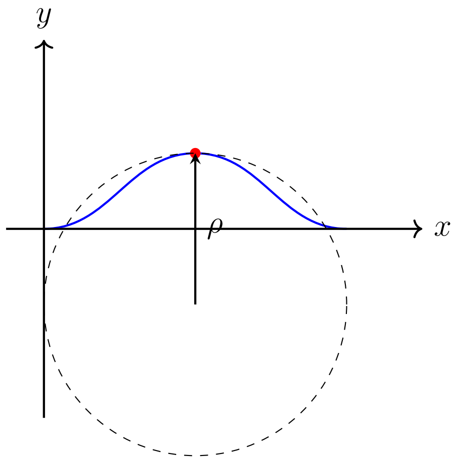

Figure 2.4: Radius of curvature and osculating circle for a curved path

Calculate the cross product:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} \tag{2.67}$$

$$= \hat{\mathbf{i}}(2t \cdot 0 - 0 \cdot 2) - \hat{\mathbf{j}}(1 \cdot 0 - 0 \cdot 0) + \hat{\mathbf{k}}(1 \cdot 2 - 2t \cdot 0) \tag{2.68}$$

$$= 0\hat{\mathbf{i}} - 0\hat{\mathbf{j}} + 2\hat{\mathbf{k}} \tag{2.69}$$

$$= 2\hat{\mathbf{k}} \tag{2.70}$$

The magnitude of this cross product is:

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |2\hat{\mathbf{k}}| = 2 \tag{2.71}$$

The magnitude of the velocity vector is:

$$|\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2} \tag{2.72}$$

$$= \sqrt{1 + 4t^2} \tag{2.73}$$

Therefore, the curvature is:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \tag{2.74}$$

$$= \frac{2}{(1 + 4t^2)^{3/2}} \tag{2.75}$$

At $t = 1$:

$$\kappa(1) = \frac{2}{(1 + 4 \cdot 1)^{3/2}} \tag{2.76}$$

$$= \frac{2}{5^{3/2}} \tag{2.77}$$

$$= \frac{2}{5\sqrt{5}} \tag{2.78}$$

$$= \frac{2}{5^{3/2}} \tag{2.79}$$