Chapter 1

Matrix Fundamentals

1.1 Introduction to Matrices

Definition 1.1. *A **matrix** is a rectangular array of numbers, symbols, or expressions arranged in rows and columns. A matrix with m rows and n columns is called an $m \times n$ matrix (read as "m by n" matrix).*
The general representation of an $m \times n$ matrix A is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{1.1}$$

where a_{ij} represents the element in the i-th row and j-th column.

Matrix Notation

We often denote matrices using capital letters (e.g., A, B, C) and their elements using lowercase letters with subscripts (e.g., a_{ij}, b_{ij}, c_{ij}).
A matrix A of size $m \times n$ can also be denoted as $A = [a_{ij}]_{m \times n}$.

1.2 Basic Matrix Algebra

Matrix Addition and Subtraction

For two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size $m \times n$:

$$A + B = [a_{ij} + b_{ij}]_{m \times n} \tag{1.2}$$

$$A - B = [a_{ij} - b_{ij}]_{m \times n} \tag{1.3}$$

Addition Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \tag{1.4}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} \tag{1.5}$$

Scalar Multiplication

For a scalar c and matrix $A = [a_{ij}]$:

$$cA = [c \cdot a_{ij}]_{m\times n} \tag{1.6}$$

Matrix Multiplication

For matrices A of size $m \times n$ and B of size $n \times p$, their product AB is a matrix of size $m \times p$ defined as:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{1.7}$$

Matrix Multiplication Conditions

For matrix multiplication AB to be defined, the number of columns in A must equal the number of rows in B.

Multiplication Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \tag{1.8}$$

$$AB = \begin{bmatrix} 1\cdot 5 + 2\cdot 7 & 1\cdot 6 + 2\cdot 8 \\ 3\cdot 5 + 4\cdot 7 & 3\cdot 6 + 4\cdot 8 \end{bmatrix} \tag{1.9}$$

$$= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \tag{1.10}$$

Matrix Transposition

The transpose of a matrix $A = [a_{ij}]_{m\times n}$ is denoted by A^T and is an $n \times m$ matrix defined as:

$$(A^T)_{ij} = a_{ji} \tag{1.11}$$

In other words, the rows of A become the columns of A^T and vice versa.

1.3 Types of Matrices

Square Matrix A matrix with the same number of rows and columns ($m = n$).

Diagonal Matrix A square matrix whose non-diagonal elements are all zero:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \tag{1.12}$$

Identity Matrix A diagonal matrix with all diagonal elements equal to 1:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{1.13}$$

Zero Matrix A matrix with all elements equal to zero, denoted as O.

Upper Triangular Matrix A square matrix where all elements below the main diagonal are zero:

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \tag{1.14}$$

Lower Triangular Matrix A square matrix where all elements above the main diagonal are zero:

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \tag{1.15}$$

Symmetric Matrix A square matrix that equals its transpose: $A = A^T$.

Skew-Symmetric Matrix A square matrix that equals the negative of its transpose: $A = -A^T$.

Orthogonal Matrix A square matrix whose transpose equals its inverse: $A^T = A^{-1}$, or equivalently, $AA^T = A^TA = I$.

1.4 Elementary Row and Column Operations

Elementary row and column operations are fundamental transformations that can be applied to matrices. They are crucial for understanding concepts like rank, row echelon form, and matrix inversion.

Elementary Row Operations There are three types of elementary row operations:

1. **Row Swap**: Interchange two rows ($R_i \leftrightarrow R_j$).
2. **Row Scaling**: Multiply a row by a non-zero scalar ($R_i \to cR_i$, where $c \neq 0$).
3. **Row Addition**: Add a scalar multiple of one row to another row ($R_i \to R_i + cR_j$, where $i \neq j$).

Elementary Row Operations

Let's apply elementary row operations to matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{1.16}$$

1. Row Swap ($R_1 \leftrightarrow R_3$):
$$\begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \tag{1.17}$$
2. Row Scaling ($R_2 \to 2R_2$):
$$\begin{bmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{bmatrix} \tag{1.18}$$
3. Row Addition ($R_3 \to R_3 - 2R_1$):
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 4 & 3 \end{bmatrix} \tag{1.19}$$

Elementary Column Operations Similarly, there are three types of elementary column operations:

1. **Column Swap**: Interchange two columns ($C_i \leftrightarrow C_j$).
2. **Column Scaling**: Multiply a column by a non-zero scalar ($C_i \to cC_i$, where $c \neq 0$).
3. **Column Addition**: Add a scalar multiple of one column to another column ($C_i \to C_i + cC_j$, where $i \neq j$).

Elementary Column Operations

Let's apply elementary column operations to matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{1.20}$$

1. Column Swap ($C_1 \leftrightarrow C_2$):
$$\begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{bmatrix} \tag{1.21}$$
2. Column Scaling ($C_3 \to 3C_3$):
$$\begin{bmatrix} 1 & 2 & 9 \\ 4 & 5 & 18 \\ 7 & 8 & 27 \end{bmatrix} \tag{1.22}$$
3. Column Addition ($C_2 \to C_2 - C_1$):
$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 1 & 6 \\ 7 & 1 & 9 \end{bmatrix} \tag{1.23}$$

1.5 Determinants

Definition 1.2. *The* ***determinant*** *of a square matrix A is a scalar value that provides important information about the matrix's properties. It is denoted by* $\det(A)$ *or* $|A|$.

Properties of Determinants

1. The determinant of the identity matrix is 1: $\det(I_n) = 1$.
2. If any row or column of a matrix contains only zeros, then its determinant is zero.
3. Interchanging any two rows or columns multiplies the determinant by -1.
4. Multiplying any row or column by a scalar k multiplies the determinant by k.
5. The determinant of a product of matrices equals the product of their determinants: $\det(AB) = \det(A) \cdot \det(B)$.
6. The determinant of a matrix equals the determinant of its transpose: $\det(A) = \det(A^T)$.
7. Adding a multiple of one row (or column) to another row (or column) does not change the determinant.

Calculating Determinants

For 2×2 Matrices

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is:

$$\det(A) = ad - bc \tag{1.24}$$

Determinant of a 2×2 Matrix

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 7 \end{bmatrix} \tag{1.25}$$

$$\det(A) = 3 \cdot 7 - 5 \cdot 2 = 21 - 10 = 11 \tag{1.26}$$

For 3×3 Matrices

For a 3×3 matrix, we can use the method of cofactor expansion.

Definition 1.3. *The **minor** M_{ij} of an element a_{ij} is the determinant of the submatrix obtained by deleting the i-th row and j-th column of the original matrix.*
*The **cofactor** C_{ij} of an element a_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.*
The determinant can be calculated by expanding along any row or column:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{(expansion along the } i\text{-th row)} \tag{1.27}$$

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{(expansion along the } j\text{-th column)} \tag{1.28}$$

Determinant of a 3×3 Matrix

Let's calculate the determinant of:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} \tag{1.29}$$

Expanding along the first row:

$$\begin{aligned}
\det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && (1.30)\\
&= 2 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} + 3 \cdot (-1)^{1+2} \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} + 1 \cdot (-1)^{1+3} \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} && (1.31)\\
&= 2 \cdot (1 \cdot 2 - 3 \cdot 5) + 3 \cdot (-1) \cdot (4 \cdot 2 - 3 \cdot 2) + 1 \cdot (4 \cdot 5 - 1 \cdot 2) && (1.32)\\
&= 2 \cdot (-13) + 3 \cdot (-1) \cdot 2 + 1 \cdot 18 && (1.33)\\
&= -26 - 6 + 18 = -14 && (1.34)
\end{aligned}$$

Sarrus Rule for 3×3 Matrices

For 3×3 matrices, we can also use Sarrus' rule, which is a mnemonic technique:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \tag{1.35}$$

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \tag{1.36}$$

This can be visualized as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

Sum the products along the black arrows and subtract the products along the red arrows.

For Larger Matrices

For larger matrices, we typically use:

1. Cofactor expansion (selecting a row or column with many zeros)
2. Row reduction to upper triangular form, where the determinant equals the product of the diagonal elements
3. Computational methods

1.6 Matrix Inverse

Definition 1.4. *For a square matrix A of size $n \times n$, its **inverse**, if it exists, is a matrix A^{-1} such that:*

$$AA^{-1} = A^{-1}A = I_n \tag{1.37}$$

where I_n is the identity matrix of size $n \times n$.

Invertible Matrix

A square matrix A is invertible (or non-singular) if and only if its determinant is non-zero: $\det(A) \neq 0$.

Finding the Inverse Using Elementary Row Operations

One method to find the inverse of a matrix A is to use elementary row operations:

1. Form the augmented matrix $[A|I_n]$, where I_n is the identity matrix of the same size as A.
2. Apply elementary row operations to transform the left half (matrix A) into the identity matrix I_n.
3. If successful, the right half will be A^{-1}.

Finding Inverse Using Elementary Row Operations

Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
Form the augmented matrix:

$$[A|I_2] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} \tag{1.38}$$

Apply elementary row operations:

$$R_2 \to R_2 - 3R_1 : \tag{1.39}$$
$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix} \tag{1.40}$$
$$R_2 \to -\frac{1}{2}R_2 : \tag{1.41}$$
$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \tag{1.42}$$
$$R_1 \to R_1 - 2R_2 : \tag{1.43}$$
$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \tag{1.44}$$

Therefore:

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \tag{1.45}$$

Verification:

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \tag{1.46}$$
$$= \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{bmatrix} \tag{1.47}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \tag{1.48}$$

Finding the Inverse Using Adjoint Method
Another method to find the inverse of a matrix is using the adjoint:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) \tag{1.49}$$

where $\operatorname{adj}(A)$ is the adjoint of A, which is the transpose of the cofactor matrix of A.
For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{1.50}$$

Finding Inverse Using Adjoint Method

Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ using the adjoint method.
First, calculate the determinant:

$$\det(A) = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2 \tag{1.51}$$

Next, find the adjoint:

$$\operatorname{adj}(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \tag{1.52}$$

Therefore:

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \tag{1.53}$$