

# Chapter 5

## Linear Transformations and Orthogonal Transformations

In this chapter, we explore linear transformations through the lens of matrix operations. We will focus on the form  $\mathbf{AX} = \mathbf{B}$ , where  $\mathbf{A}$  is called the transformation matrix, and examine special cases such as orthogonal transformations.

### 5.1 Linear Transformations

#### Definition and Properties of Linear Transformations

**Definition 5.1.** A linear transformation is a mapping between matrices that can be represented by the equation  $\mathbf{AX} = \mathbf{B}$ , where  $\mathbf{A} \in \mathbb{R}^{m \times n}$  is called the transformation matrix.

The matrix  $\mathbf{A}$  completely determines the behavior of the transformation. Depending on the properties of  $\mathbf{A}$ , we classify transformations as follows:

#### Classification of Linear Transformations

Let  $\mathbf{A}$  be a square matrix of order  $n$ . Then:

1. If  $\det(\mathbf{A}) = 0$ , then  $\mathbf{A}$  is called *singular*, *irregular*, or *non-invertible*.
2. If  $\det(\mathbf{A}) \neq 0$ , then  $\mathbf{A}$  is called *regular*, *non-singular*, or *invertible*.

#### Example: Regular and Singular Transformation Matrices

Consider the following transformation matrices:

$$\mathbf{A}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\mathbf{A}_2 = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

For  $\mathbf{A}_1$ , we have  $\det(\mathbf{A}_1) = 2 \cdot 3 - 1 \cdot 1 = 6 - 1 = 5 \neq 0$ , so  $\mathbf{A}_1$  is regular (invertible).

For  $\mathbf{A}_2$ , we have  $\det(\mathbf{A}_2) = 2 \cdot 2 - 4 \cdot 1 = 4 - 4 = 0$ , so  $\mathbf{A}_2$  is singular (non-invertible).

The transformation  $\mathbf{A}_1\mathbf{X} = \mathbf{B}$  has a unique solution for any  $\mathbf{B}$ , while the transformation  $\mathbf{A}_2\mathbf{X} = \mathbf{B}$  either has no solution or infinitely many solutions, depending on  $\mathbf{B}$ .

#### Matrix Representation of Linear Transformations

When we work with linear transformations in the form  $\mathbf{AX} = \mathbf{B}$ , the matrix  $\mathbf{A}$  completely encapsulates the transformation's behavior. For square matrices, several important properties

determine the transformation's characteristics:

**Theorem 5.2.** *Let  $\mathbf{A}$  be a square matrix of order  $n$  representing a linear transformation. Then:*

1. *The transformation is invertible if and only if  $\det(\mathbf{A}) \neq 0$ .*
2. *If  $\mathbf{A}$  is invertible, then the inverse transformation is represented by  $\mathbf{A}^{-1}$ .*
3. *The composition of two transformations with matrices  $\mathbf{A}$  and  $\mathbf{B}$  is represented by the product matrix  $\mathbf{BA}$ .*
4. *The identity transformation is represented by the identity matrix  $\mathbf{I}$ .*

### Example: Composition of Transformations

Consider two transformation matrices:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The composition of these transformations (applying  $\mathbf{A}$  followed by  $\mathbf{B}$ ) is represented by the matrix product:

$$\begin{aligned} \mathbf{BA} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

This new matrix represents the composite transformation.

### Key Properties of Transformation Matrices

For a square matrix  $\mathbf{A}$  of order  $n$ :

1. **Rank:** The rank of  $\mathbf{A}$  determines the dimension of the image (range) of the transformation.
2. **Nullity:** The nullity of  $\mathbf{A}$  (dimension of the null space) determines the dimension of the kernel of the transformation.
3. **Rank-Nullity Theorem:**  $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$
4. **Eigenvalues and Eigenvectors:** If  $\mathbf{Av} = \lambda\mathbf{v}$  for some non-zero vector  $\mathbf{v}$  and scalar  $\lambda$ , then  $\mathbf{v}$  is an eigenvector with eigenvalue  $\lambda$ . These determine the directions that are only scaled (not rotated) by the transformation.

## 5.2 Orthogonal Transformations

### Definition and Geometric Interpretation of Orthogonal Transformations

**Definition 5.3.** *A linear transformation represented by a matrix  $\mathbf{A}$  is called an orthogonal transformation if  $\mathbf{A}$  is an orthogonal matrix, i.e.,  $\mathbf{A}^T\mathbf{A} = \mathbf{AA}^T = \mathbf{I}$ .*

An orthogonal transformation preserves the Euclidean length of vectors and the angles between them. This means that if we apply an orthogonal transformation to a geometric figure, its shape and size remain unchanged—it may only be rotated, reflected, or repositioned.

### Example: Identifying Orthogonal Transformations

Consider the following matrices:

$$\mathbf{A}_1 = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Let's check if these represent orthogonal transformations:

For  $\mathbf{A}_1$ :

$$\begin{aligned} \mathbf{A}_1^T \mathbf{A}_1 &= \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \\ &= \begin{pmatrix} \frac{9}{25} + \frac{16}{25} & \frac{12}{25} - \frac{12}{25} \\ \frac{12}{25} - \frac{12}{25} & \frac{16}{25} + \frac{9}{25} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \end{aligned}$$

Therefore,  $\mathbf{A}_1$  represents an orthogonal transformation.

For  $\mathbf{A}_2$ :

$$\begin{aligned} \mathbf{A}_2^T \mathbf{A}_2 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq \mathbf{I} \end{aligned}$$

Therefore,  $\mathbf{A}_2$  does not represent an orthogonal transformation.

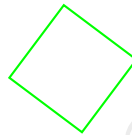
Looking at the geometric interpretation,  $\mathbf{A}_1$  represents rotation by an angle  $\theta$  where  $\cos \theta = \frac{3}{5}$  and  $\sin \theta = \frac{4}{5}$ , which is approximately 53.13°. This preserves the shapes and sizes of objects.

On the other hand,  $\mathbf{A}_2$  represents a shear transformation that distorts shapes.

Original Square



After Shear Transformation



After Orthogonal Transformation

Figure 5.1: Effects of orthogonal vs. non-orthogonal transformations on a square

### Properties of Orthogonal Matrices

An orthogonal transformation is characterized by its orthogonal matrix  $\mathbf{A}$ , which satisfies  $\mathbf{A}^T \mathbf{A} = \mathbf{A} \mathbf{A}^T = \mathbf{I}$ .

### Properties of Orthogonal Matrices

Let  $\mathbf{A}$  be an orthogonal matrix. Then:

1.  $\mathbf{A}^T = \mathbf{A}^{-1}$  (The transpose equals the inverse)
2.  $\det(\mathbf{A}) = \pm 1$
3. The columns of  $\mathbf{A}$  form an orthonormal set
4. The rows of  $\mathbf{A}$  form an orthonormal set
5.  $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|$  for any vector  $\mathbf{x}$  (preserves length)
6.  $(\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{y}) = \mathbf{x}^T\mathbf{y}$  for any vectors  $\mathbf{x}, \mathbf{y}$  (preserves dot product)
7. The eigenvalues of  $\mathbf{A}$  have magnitude 1
8. If  $\det(\mathbf{A}) = 1$ , then  $\mathbf{A}$  represents a proper rotation
9. If  $\det(\mathbf{A}) = -1$ , then  $\mathbf{A}$  represents an improper rotation (rotation + reflection)

### Example: Common Orthogonal Transformations

Here are some common orthogonal transformation matrices in  $\mathbb{R}^2$ :

1. **Rotation by angle  $\theta$ :**

$$\mathbf{R}_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

2. **Reflection across the x-axis:**

$$\mathbf{F}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3. **Reflection across the y-axis:**

$$\mathbf{F}_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. **Reflection across the line  $y = x$ :**

$$\mathbf{F}_{y=x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We can verify that all these matrices satisfy  $\mathbf{A}^T \mathbf{A} = \mathbf{I}$ .

### Applications of Orthogonal Transformations

Orthogonal transformations are particularly useful in:

1. **Computer Graphics:** Rotation, reflection, and other rigid transformations are represented by orthogonal matrices.
2. **Physics:** Coordinate transformations between different reference frames often use orthogonal matrices.
3. **Engineering:** Rigid body mechanics relies on orthogonal transformations to describe motion.
4. **Data Analysis:** Techniques like Principal Component Analysis (PCA) use orthogonal transformations to find uncorrelated features.
5. **Quantum Mechanics:** Unitary transformations (complex analogues of orthogonal transformations) represent quantum operations.

## Self-Assessment Problems

1. Prove that if  $\mathbf{A}$  and  $\mathbf{B}$  are orthogonal matrices, then  $\mathbf{AB}$  is also orthogonal.
2. Determine whether the following matrix represents an orthogonal transformation:

$$\mathbf{C} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

3. Find the matrix representing a reflection across the line  $y = -x$  in  $\mathbb{R}^2$ .
4. Show that an orthogonal matrix in  $\mathbb{R}^3$  with determinant 1 represents a rotation around some axis.
5. If  $\mathbf{A}$  is a  $3 \times 3$  orthogonal matrix with determinant -1, what type of transformation does it represent?

## Solutions

1. To prove that  $\mathbf{AB}$  is orthogonal when  $\mathbf{A}$  and  $\mathbf{B}$  are orthogonal, we need to show that  $(\mathbf{AB})^T(\mathbf{AB}) = \mathbf{I}$ :

$$\begin{aligned} (\mathbf{AB})^T(\mathbf{AB}) &= \mathbf{B}^T \mathbf{A}^T \mathbf{AB} \\ &= \mathbf{B}^T \mathbf{IB} \quad (\text{since } \mathbf{A} \text{ is orthogonal, } \mathbf{A}^T \mathbf{A} = \mathbf{I}) \\ &= \mathbf{B}^T \mathbf{B} \\ &= \mathbf{I} \quad (\text{since } \mathbf{B} \text{ is orthogonal, } \mathbf{B}^T \mathbf{B} = \mathbf{I}) \end{aligned}$$

Therefore,  $\mathbf{AB}$  is orthogonal.

2. For matrix  $\mathbf{C}$ , we check if  $\mathbf{C}^T \mathbf{C} = \mathbf{I}$ :

$$\begin{aligned} \mathbf{C}^T \mathbf{C} &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4} + \frac{3}{4} & \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} & \frac{3}{4} + \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \end{aligned}$$

Thus,  $\mathbf{C}$  represents an orthogonal transformation. In fact, it represents a rotation by  $60^\circ$  since  $\cos(60^\circ) = \frac{1}{2}$  and  $\sin(60^\circ) = \frac{\sqrt{3}}{2}$ .

3. For a reflection across the line  $y = -x$ , we need to find where a point  $(x, y)$  gets mapped. The reflection of  $(x, y)$  across the line  $y = -x$  is  $(-y, -x)$ . This transformation is represented by the matrix:

$$\mathbf{F}_{y=-x} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

We can verify that this is orthogonal:

$$\mathbf{F}_{y=-x}^T \mathbf{F}_{y=-x} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

4. By Euler's rotation theorem, any orthogonal matrix in  $\mathbb{R}^3$  with determinant 1 represents a rotation around some axis. To prove this:

Let  $\mathbf{A}$  be a  $3 \times 3$  orthogonal matrix with  $\det(\mathbf{A}) = 1$ . Since  $\mathbf{A}$  is orthogonal, its eigenvalues have magnitude 1, meaning they are of the form  $e^{i\theta}$  for some angle  $\theta$ .

Since  $\mathbf{A}$  is real, complex eigenvalues come in conjugate pairs. The determinant is the product of eigenvalues, and with  $\det(\mathbf{A}) = 1$ , we know that  $\mathbf{A}$  must have at least one real eigenvalue, which must be either 1 or -1.

If  $\det(\mathbf{A}) = 1$ , then the real eigenvalue must be 1, and there exists a non-zero vector  $\mathbf{v}$  such that  $\mathbf{A}\mathbf{v} = \mathbf{v}$ . This vector  $\mathbf{v}$  represents the axis of rotation.

5. If  $\mathbf{A}$  is a  $3 \times 3$  orthogonal matrix with  $\det(\mathbf{A}) = -1$ , it represents an improper rotation, which is a rotation followed by a reflection. Specifically, it is a rotation about some axis followed by a reflection in a plane perpendicular to that axis.

### 5.3 Additional Solved Examples

#### Example 1: Finding Preimage Coordinates in a Linear Transformation

Given the transformation

$$\mathbf{Y} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{A}\mathbf{X}$$

Find the coordinates  $(x_1, x_2, x_3)$  in  $\mathbf{X}$  corresponding to  $(1, 2, -1)$  in  $\mathbf{Y}$ .

**Solution:**

Let's solve the system of linear equations directly:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 2 \\ x_1 + 0x_2 - 2x_3 &= -1 \end{aligned}$$

From the third equation:

$$\begin{aligned} x_1 - 2x_3 &= -1 \\ \Rightarrow x_1 &= -1 + 2x_3 \end{aligned}$$

Substituting this into the first equation:

$$\begin{aligned} 2(-1 + 2x_3) + x_2 + x_3 &= 1 \\ -2 + 4x_3 + x_2 + x_3 &= 1 \\ x_2 + 5x_3 &= 3 \end{aligned}$$

Substituting  $x_1 = -1 + 2x_3$  into the second equation:

$$\begin{aligned} (-1 + 2x_3) + x_2 + 2x_3 &= 2 \\ -1 + 2x_3 + x_2 + 2x_3 &= 2 \\ x_2 + 4x_3 &= 3 \end{aligned}$$

Now we have two equations:

$$\begin{aligned} x_2 + 5x_3 &= 3 \quad (1) \\ x_2 + 4x_3 &= 3 \quad (2) \end{aligned}$$

Subtracting (2) from (1):

$$x_3 = 0$$

Substituting back into (2):

$$x_2 + 4(0) = 3$$

$$x_2 = 3$$

And substituting back to find  $x_1$ :

$$x_1 = -1 + 2x_3$$

$$= -1 + 2(0)$$

$$= -1$$

Therefore, the coordinates are  $(x_1, x_2, x_3) = (-1, 3, 0)$ .

Let's verify this solution:

$$\begin{aligned} \mathbf{A}\mathbf{X} &= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \cdot (-1) + 1 \cdot 3 + 1 \cdot 0 \\ 1 \cdot (-1) + 1 \cdot 3 + 2 \cdot 0 \\ 1 \cdot (-1) + 0 \cdot 3 + (-2) \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 + 3 + 0 \\ -1 + 3 + 0 \\ -1 + 0 + 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \end{aligned}$$

This confirms that the coordinates  $(x_1, x_2, x_3) = (-1, 3, 0)$  in  $\mathbf{X}$  correspond to  $(1, 2, -1)$  in  $\mathbf{Y}$ .

### Example 2: Regular Transformation and Its Inverse

Show that the transformation

$$y_1 = 2x_1 + x_2 + x_3$$

$$y_2 = x_1 + x_2 + 2x_3$$

$$y_3 = x_1 - 2x_3$$

is regular. Write down the inverse transformation.

**Solution:**

**Step 1:** First, let's write the transformation in matrix form:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Let's denote the transformation matrix as  $\mathbf{A}$ .

**Step 2:** To determine if the transformation is regular, we need to check if  $\det(\mathbf{A}) \neq 0$ .

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix}$$

Let's expand along the first row:

$$\begin{aligned} \det(\mathbf{A}) &= 2 \cdot \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\ &= 2(1 \cdot (-2) - 2 \cdot 0) - 1(1 \cdot (-2) - 2 \cdot 1) + 1(1 \cdot 0 - 1 \cdot 1) \\ &= 2(-2) - 1(-2 - 2) + 1(-1) \\ &= -4 - 1(-4) - 1 \\ &= -4 + 4 - 1 \\ &= -1 \end{aligned}$$

Since  $\det(\mathbf{A}) = -1 \neq 0$ , the transformation is regular (invertible).

Let's solve for the inverse transformation directly from the system of equations:

$$y_1 = 2x_1 + x_2 + x_3 \quad (1)$$

$$y_2 = x_1 + x_2 + 2x_3 \quad (2)$$

$$y_3 = x_1 - 2x_3 \quad (3)$$

From equation (3), we get:

$$x_1 = y_3 + 2x_3 \quad (4)$$

Substituting (4) into equation (1):

$$\begin{aligned} y_1 &= 2(y_3 + 2x_3) + x_2 + x_3 \\ &= 2y_3 + 4x_3 + x_2 + x_3 \\ &= 2y_3 + x_2 + 5x_3 \quad (5) \end{aligned}$$

Substituting (4) into equation (2):

$$\begin{aligned} y_2 &= (y_3 + 2x_3) + x_2 + 2x_3 \\ &= y_3 + 2x_3 + x_2 + 2x_3 \\ &= y_3 + x_2 + 4x_3 \quad (6) \end{aligned}$$

From equation (5), we get:

$$x_2 = y_1 - 2y_3 - 5x_3 \quad (7)$$

Substituting (7) into equation (6):

$$\begin{aligned} y_2 &= y_3 + (y_1 - 2y_3 - 5x_3) + 4x_3 \\ &= y_3 + y_1 - 2y_3 - 5x_3 + 4x_3 \\ &= y_1 - y_3 - x_3 \quad (8) \end{aligned}$$



From equation (8), we solve for  $x_3$ :

$$\begin{aligned} y_2 &= y_1 - y_3 - x_3 \\ \Rightarrow x_3 &= y_1 - y_2 - y_3 \quad (9) \end{aligned}$$

Now we can substitute (9) back into equations (4) and (7) to find  $x_1$  and  $x_2$ :

$$\begin{aligned} x_1 &= y_3 + 2x_3 \\ &= y_3 + 2(y_1 - y_2 - y_3) \\ &= y_3 + 2y_1 - 2y_2 - 2y_3 \\ &= 2y_1 - 2y_2 - y_3 \quad (10) \end{aligned}$$

$$\begin{aligned} x_2 &= y_1 - 2y_3 - 5x_3 \\ &= y_1 - 2y_3 - 5(y_1 - y_2 - y_3) \\ &= y_1 - 2y_3 - 5y_1 + 5y_2 + 5y_3 \\ &= -4y_1 + 5y_2 + 3y_3 \quad (11) \end{aligned}$$

Therefore, the inverse transformation is:

$$\begin{aligned} x_1 &= 2y_1 - 2y_2 - y_3 \\ x_2 &= -4y_1 + 5y_2 + 3y_3 \\ x_3 &= y_1 - y_2 - y_3 \end{aligned}$$

**Verification:** Let's verify that applying the original transformation to our inverse transformation gives us back the original  $y$  values.

Substituting our expressions for  $x_1$ ,  $x_2$ , and  $x_3$  into the original equation for  $y_1$ :

$$\begin{aligned} y_1 &= 2x_1 + x_2 + x_3 \\ &= 2(2y_1 - 2y_2 - y_3) + (-4y_1 + 5y_2 + 3y_3) + (y_1 - y_2 - y_3) \\ &= 4y_1 - 4y_2 - 2y_3 - 4y_1 + 5y_2 + 3y_3 + y_1 - y_2 - y_3 \\ &= 4y_1 - 4y_1 + y_1 - 4y_2 + 5y_2 - y_2 - 2y_3 + 3y_3 - y_3 \\ &= y_1 + 0y_2 + 0y_3 \\ &= y_1 \end{aligned}$$

Substituting our expressions for  $x_1$ ,  $x_2$ , and  $x_3$  into the original equation for  $y_2$ :

$$\begin{aligned} y_2 &= x_1 + x_2 + 2x_3 \\ &= (2y_1 - 2y_2 - y_3) + (-4y_1 + 5y_2 + 3y_3) + 2(y_1 - y_2 - y_3) \\ &= 2y_1 - 2y_2 - y_3 - 4y_1 + 5y_2 + 3y_3 + 2y_1 - 2y_2 - 2y_3 \\ &= 2y_1 - 4y_1 + 2y_1 - 2y_2 + 5y_2 - 2y_2 - y_3 + 3y_3 - 2y_3 \\ &= 0y_1 + y_2 + 0y_3 \\ &= y_2 \end{aligned}$$

Substituting our expressions for  $x_1$ ,  $x_2$ , and  $x_3$  into the original equation for  $y_3$ :

$$\begin{aligned}
 y_3 &= x_1 - 2x_3 \\
 &= (2y_1 - 2y_2 - y_3) - 2(y_1 - y_2 - y_3) \\
 &= 2y_1 - 2y_2 - y_3 - 2y_1 + 2y_2 + 2y_3 \\
 &= 2y_1 - 2y_1 - 2y_2 + 2y_2 - y_3 + 2y_3 \\
 &= 0y_1 + 0y_2 + y_3 \\
 &= y_3
 \end{aligned}$$

Our verification confirms that the inverse transformation is correct:

$$\begin{aligned}
 x_1 &= 2y_1 - 2y_2 - y_3 \\
 x_2 &= -4y_1 + 5y_2 + 3y_3 \\
 x_3 &= y_1 - y_2 - y_3
 \end{aligned}$$

### Example 3: Composite Transformations

A transformation from the variables  $x_1, x_2, x_3$  to  $y_1, y_2, y_3$  is given by  $\mathbf{Y} = \mathbf{A}\mathbf{X}$  and another transformation from  $y_1, y_2, y_3$  to  $z_1, z_2, z_3$  is given by  $\mathbf{Z} = \mathbf{B}\mathbf{Y}$ , where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$$

Obtain the transformation from  $x_1, x_2, x_3$  to  $z_1, z_2, z_3$ .

**Solution:**

**Step 1:** We need to find the direct transformation from  $\mathbf{X}$  to  $\mathbf{Z}$ .

Given:

$$\begin{aligned}
 \mathbf{Y} &= \mathbf{A}\mathbf{X} \\
 \mathbf{Z} &= \mathbf{B}\mathbf{Y}
 \end{aligned}$$

Substituting the first equation into the second:

$$\begin{aligned}
 \mathbf{Z} &= \mathbf{B}\mathbf{Y} \\
 &= \mathbf{B}(\mathbf{A}\mathbf{X}) \\
 &= (\mathbf{B}\mathbf{A})\mathbf{X}
 \end{aligned}$$

Therefore, the transformation matrix from  $\mathbf{X}$  to  $\mathbf{Z}$  is  $\mathbf{C} = \mathbf{B}\mathbf{A}$ .

**Step 2:** Calculate the matrix product  $\mathbf{C} = \mathbf{B}\mathbf{A}$ .

$$\begin{aligned}
 \mathbf{C} &= \mathbf{B}\mathbf{A} \\
 &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{pmatrix}
 \end{aligned}$$

Let's calculate each element of  $\mathbf{C}$ :

Row 1, Column 1:

$$\begin{aligned}c_{11} &= 1 \cdot 2 + 1 \cdot 0 + 1 \cdot (-1) \\&= 2 + 0 - 1 \\&= 1\end{aligned}$$

Row 1, Column 2:

$$\begin{aligned}c_{12} &= 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 \\&= 1 + 1 + 2 \\&= 4\end{aligned}$$

Row 1, Column 3:

$$\begin{aligned}c_{13} &= 1 \cdot 0 + 1 \cdot (-2) + 1 \cdot 1 \\&= 0 - 2 + 1 \\&= -1\end{aligned}$$

Row 2, Column 1:

$$\begin{aligned}c_{21} &= 1 \cdot 2 + 2 \cdot 0 + 3 \cdot (-1) \\&= 2 + 0 - 3 \\&= -1\end{aligned}$$

Row 2, Column 2:

$$\begin{aligned}c_{22} &= 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 2 \\&= 1 + 2 + 6 \\&= 9\end{aligned}$$

Row 2, Column 3:

$$\begin{aligned}c_{23} &= 1 \cdot 0 + 2 \cdot (-2) + 3 \cdot 1 \\&= 0 - 4 + 3 \\&= -1\end{aligned}$$

Row 3, Column 1:

$$\begin{aligned}c_{31} &= 1 \cdot 2 + 3 \cdot 0 + 5 \cdot (-1) \\&= 2 + 0 - 5 \\&= -3\end{aligned}$$

Row 3, Column 2:

$$\begin{aligned}c_{32} &= 1 \cdot 1 + 3 \cdot 1 + 5 \cdot 2 \\&= 1 + 3 + 10 \\&= 14\end{aligned}$$

Row 3, Column 3:

$$\begin{aligned} c_{33} &= 1 \cdot 0 + 3 \cdot (-2) + 5 \cdot 1 \\ &= 0 - 6 + 5 \\ &= -1 \end{aligned}$$

Therefore:

$$\begin{aligned} \mathbf{C} &= \mathbf{BA} \\ &= \begin{pmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{pmatrix} \end{aligned}$$

**Step 3:** Write the transformation from  $x_1, x_2, x_3$  to  $z_1, z_2, z_3$  in matrix form.

$$\begin{aligned} \mathbf{Z} &= \mathbf{CX} \\ &= \begin{pmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

**Step 4:** Write the transformation equations explicitly.

$$\begin{aligned} z_1 &= 1 \cdot x_1 + 4 \cdot x_2 + (-1) \cdot x_3 \\ &= x_1 + 4x_2 - x_3 \end{aligned}$$

$$\begin{aligned} z_2 &= (-1) \cdot x_1 + 9 \cdot x_2 + (-1) \cdot x_3 \\ &= -x_1 + 9x_2 - x_3 \end{aligned}$$

$$\begin{aligned} z_3 &= (-3) \cdot x_1 + 14 \cdot x_2 + (-1) \cdot x_3 \\ &= -3x_1 + 14x_2 - x_3 \end{aligned}$$

Therefore, the transformation from  $x_1, x_2, x_3$  to  $z_1, z_2, z_3$  is:

$$\begin{aligned} z_1 &= x_1 + 4x_2 - x_3 \\ z_2 &= -x_1 + 9x_2 - x_3 \\ z_3 &= -3x_1 + 14x_2 - x_3 \end{aligned}$$

#### Example 4: Matrix Representation and Composite Transformations

Express each of the transformations  $x_1 = 3y_1 + 2y_2$ ;  $y_1 = z_1 + 2z_2$ ;  $x_2 = -y_1 + 4y_2$ ;  $y_2 = 3z_1$  by the use of matrices and the composite transformation which express  $x_1, x_2$  in terms of  $z_1, z_2$ .

**Solution:**

**Step 1:** Organize the given transformations.

We have two transformations: one from  $(y_1, y_2)$  to  $(x_1, x_2)$  and another from  $(z_1, z_2)$  to  $(y_1, y_2)$ .

First transformation (from  $Y$  to  $X$ ):

$$\begin{aligned}x_1 &= 3y_1 + 2y_2 \\x_2 &= -y_1 + 4y_2\end{aligned}$$

Second transformation (from  $Z$  to  $Y$ ):

$$\begin{aligned}y_1 &= z_1 + 2z_2 \\y_2 &= 3z_1\end{aligned}$$

**Step 2:** Express these transformations in matrix form.

For the transformation from  $Y$  to  $X$ :

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Let's denote this transformation matrix as  $\mathbf{A}$ :

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$$

For the transformation from  $Z$  to  $Y$ :

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Let's denote this transformation matrix as  $\mathbf{B}$ :

$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

**Step 3:** Find the composite transformation from  $Z$  to  $X$ .

We know that  $\mathbf{X} = \mathbf{A} \cdot \mathbf{Y}$  and  $\mathbf{Y} = \mathbf{B} \cdot \mathbf{Z}$ . Substituting the second equation into the first:

$$\begin{aligned}\mathbf{X} &= \mathbf{A} \cdot \mathbf{Y} \\ &= \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{Z}) \\ &= (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{Z}\end{aligned}$$

So, the composite transformation matrix is  $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ .

$$\begin{aligned}\mathbf{C} &= \mathbf{A} \cdot \mathbf{B} \\ &= \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\end{aligned}$$

Let's calculate each element of  $\mathbf{C}$ :

$$c_{11} = 3 \cdot 1 + 2 \cdot 3 = 3 + 6 = 9$$

$$c_{12} = 3 \cdot 2 + 2 \cdot 0 = 6 + 0 = 6$$

$$c_{21} = (-1) \cdot 1 + 4 \cdot 3 = -1 + 12 = 11$$

$$c_{22} = (-1) \cdot 2 + 4 \cdot 0 = -2 + 0 = -2$$

Therefore:

$$\mathbf{C} = \begin{pmatrix} 9 & 6 \\ 11 & -2 \end{pmatrix}$$

**Step 4:** Express the composite transformation in equation form.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 11 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

This gives us:

$$\begin{aligned} x_1 &= 9z_1 + 6z_2 \\ x_2 &= 11z_1 - 2z_2 \end{aligned}$$

**Verification:** Let's verify our result by substituting the equations for  $y_1$  and  $y_2$  into the equations for  $x_1$  and  $x_2$ .

$$\begin{aligned} x_1 &= 3y_1 + 2y_2 \\ &= 3(z_1 + 2z_2) + 2(3z_1) \\ &= 3z_1 + 6z_2 + 6z_1 \\ &= 9z_1 + 6z_2 \end{aligned}$$

$$\begin{aligned} x_2 &= -y_1 + 4y_2 \\ &= -(z_1 + 2z_2) + 4(3z_1) \\ &= -z_1 - 2z_2 + 12z_1 \\ &= 11z_1 - 2z_2 \end{aligned}$$

This confirms our matrix calculation.

Therefore, the composite transformation from  $(z_1, z_2)$  to  $(x_1, x_2)$  is:

$$\begin{aligned} x_1 &= 9z_1 + 6z_2 \\ x_2 &= 11z_1 - 2z_2 \end{aligned}$$

### Example A: Orthogonal Matrices and Transformations

Define orthogonal matrix. Show that the following transformations are orthogonal:

- (a)  $x_1 \cos \theta + x_2 \sin \theta$  ;  $-x_1 \sin \theta + x_2 \cos \theta$
- (b)  $x \cos \theta + z \sin \theta$  ;  $y$  ;  $-x \sin \theta + z \cos \theta$

**Solution:**

**Definition of an Orthogonal Matrix:** A matrix  $\mathbf{A}$  is orthogonal if and only if  $\mathbf{A}^T \mathbf{A} = \mathbf{A} \mathbf{A}^T = \mathbf{I}$ , where  $\mathbf{A}^T$  is the transpose of  $\mathbf{A}$  and  $\mathbf{I}$  is the identity matrix.

Equivalently, this means that:

1. The columns (or rows) of  $\mathbf{A}$  form an orthonormal set.
2.  $\mathbf{A}^T = \mathbf{A}^{-1}$ , i.e., the transpose of  $\mathbf{A}$  equals its inverse.
3.  $\det(\mathbf{A}) = \pm 1$
4.  $\mathbf{A}$  preserves lengths and angles between vectors.

A transformation represented by an orthogonal matrix is called an orthogonal transformation.

**Part (a):**  $x_1 \cos \theta + x_2 \sin \theta$  ;  $-x_1 \sin \theta + x_2 \cos \theta$

**Step 1:** First, let's understand what this transformation means and express it in matrix

form.

Let's denote the transformed coordinates as  $y_1$  and  $y_2$ :

$$\begin{aligned}y_1 &= x_1 \cos \theta + x_2 \sin \theta \\y_2 &= -x_1 \sin \theta + x_2 \cos \theta\end{aligned}$$

In matrix form, this becomes:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let's call this transformation matrix  $\mathbf{A}$ :

$$\mathbf{A} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

**Step 2:** To verify that this is an orthogonal matrix, we'll check if  $\mathbf{A}^T \mathbf{A} = \mathbf{I}$ .  
First, let's calculate  $\mathbf{A}^T$ :

$$\mathbf{A}^T = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Now, let's compute  $\mathbf{A}^T \mathbf{A}$ :

$$\mathbf{A}^T \mathbf{A} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

Let's calculate each element:

For element (1,1):

$$\begin{aligned}(\mathbf{A}^T \mathbf{A})_{11} &= \cos \theta \cdot \cos \theta + (-\sin \theta) \cdot (-\sin \theta) \\&= \cos^2 \theta + \sin^2 \theta \\&= 1 \quad (\text{using the Pythagorean identity})\end{aligned}$$

For element (1,2):

$$\begin{aligned}(\mathbf{A}^T \mathbf{A})_{12} &= \cos \theta \cdot \sin \theta + (-\sin \theta) \cdot \cos \theta \\&= \cos \theta \sin \theta - \sin \theta \cos \theta \\&= 0\end{aligned}$$

For element (2,1):

$$\begin{aligned}(\mathbf{A}^T \mathbf{A})_{21} &= \sin \theta \cdot \cos \theta + \cos \theta \cdot (-\sin \theta) \\&= \sin \theta \cos \theta - \cos \theta \sin \theta \\&= 0\end{aligned}$$

For element (2,2):

$$\begin{aligned}(\mathbf{A}^T \mathbf{A})_{22} &= \sin \theta \cdot \sin \theta + \cos \theta \cdot \cos \theta \\&= \sin^2 \theta + \cos^2 \theta \\&= 1 \quad (\text{using the Pythagorean identity})\end{aligned}$$

Therefore:

$$\mathbf{A}^T \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

Similarly, we can verify that  $\mathbf{A}\mathbf{A}^T = \mathbf{I}$ .

Also, we can calculate the determinant:

$$\begin{aligned} \det(\mathbf{A}) &= \cos \theta \cdot \cos \theta - \sin \theta \cdot (-\sin \theta) \\ &= \cos^2 \theta + \sin^2 \theta \\ &= 1 \end{aligned}$$

Therefore, the transformation in part (a) is indeed orthogonal.

Geometrically, this transformation represents a rotation of the coordinate system by angle  $\theta$  in the counterclockwise direction.

**Part (b):**  $x \cos \theta + z \sin \theta$  ;  $y$  ;  $-x \sin \theta + z \cos \theta$

**Step 1:** Let's understand this transformation and express it in matrix form.

Let's denote the transformed coordinates as  $x'$ ,  $y'$ , and  $z'$ :

$$\begin{aligned} x' &= x \cos \theta + z \sin \theta \\ y' &= y \\ z' &= -x \sin \theta + z \cos \theta \end{aligned}$$

In matrix form, this becomes:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Let's call this transformation matrix  $\mathbf{B}$ :

$$\mathbf{B} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

**Step 2:** To verify that this is an orthogonal matrix, we'll check if  $\mathbf{B}^T \mathbf{B} = \mathbf{I}$ .

First, let's calculate  $\mathbf{B}^T$ :

$$\mathbf{B}^T = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}$$

Now, let's compute  $\mathbf{B}^T \mathbf{B}$ :

$$\mathbf{B}^T \mathbf{B} = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

Let's calculate each element (focusing on non-trivial calculations):

For element  $(1, 1)$ :

$$\begin{aligned} (\mathbf{B}^T \mathbf{B})_{11} &= \cos \theta \cdot \cos \theta + 0 \cdot 0 + (-\sin \theta) \cdot (-\sin \theta) \\ &= \cos^2 \theta + \sin^2 \theta \\ &= 1 \end{aligned}$$



For element (1, 3):

$$\begin{aligned}(\mathbf{B}^T \mathbf{B})_{13} &= \cos \theta \cdot \sin \theta + 0 \cdot 0 + (-\sin \theta) \cdot \cos \theta \\&= \cos \theta \sin \theta - \sin \theta \cos \theta \\&= 0\end{aligned}$$

For element (2, 2):

$$\begin{aligned}(\mathbf{B}^T \mathbf{B})_{22} &= 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 \\&= 1\end{aligned}$$

For element (3, 1):

$$\begin{aligned}(\mathbf{B}^T \mathbf{B})_{31} &= \sin \theta \cdot \cos \theta + 0 \cdot 0 + \cos \theta \cdot (-\sin \theta) \\&= \sin \theta \cos \theta - \cos \theta \sin \theta \\&= 0\end{aligned}$$

For element (3, 3):

$$\begin{aligned}(\mathbf{B}^T \mathbf{B})_{33} &= \sin \theta \cdot \sin \theta + 0 \cdot 0 + \cos \theta \cdot \cos \theta \\&= \sin^2 \theta + \cos^2 \theta \\&= 1\end{aligned}$$

All other elements will be 0 due to the structure of the matrices. Therefore:

$$\mathbf{B}^T \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Similarly, we can verify that  $\mathbf{B}\mathbf{B}^T = \mathbf{I}$ .

Also, we can calculate the determinant:

$$\begin{aligned}\det(\mathbf{B}) &= \cos \theta \cdot 1 \cdot \cos \theta + 0 + 0 - \sin \theta \cdot 1 \cdot \sin \theta - 0 - 0 \\&= \cos^2 \theta - \sin^2 \theta \\&= \cos^2 \theta + \sin^2 \theta \\&= 1\end{aligned}$$

Therefore, the transformation in part (b) is also orthogonal.

Geometrically, this transformation represents a rotation around the y-axis by angle  $\theta$ .

### Example 6: Finding Values to Make Matrices Orthogonal

Determine the value of  $a, b, c$  when the following matrices are orthogonal:

(a)  $\begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$

(b)  $\frac{1}{3} \begin{pmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{pmatrix}$

$$(c) \begin{pmatrix} 1 & \frac{2}{3} & a \\ 0 & \frac{1}{3} & b \\ 0 & -\frac{2}{3} & c \end{pmatrix}$$

**Solution:**

A matrix  $\mathbf{A}$  is orthogonal if and only if  $\mathbf{A}\mathbf{A}^T = \mathbf{I}$ , where  $\mathbf{I}$  is the identity matrix. We'll use this definition directly to find the values of  $a$ ,  $b$ , and  $c$ .

**Part (a):** Let's find the values for matrix  $\mathbf{A} = \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$

**Step 1:** Calculate  $\mathbf{A}^T$ :

$$\mathbf{A}^T = \begin{pmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{pmatrix}$$

**Step 2:** Calculate  $\mathbf{A}\mathbf{A}^T$ :

$$\begin{aligned} \mathbf{A}\mathbf{A}^T &= \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix} \begin{pmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{pmatrix} \\ &= \begin{pmatrix} 0 \cdot 0 + 2b \cdot 2b + c \cdot c & 0 \cdot a + 2b \cdot b + c \cdot (-c) & 0 \cdot a + 2b \cdot (-b) + c \cdot c \\ a \cdot 0 + b \cdot 2b + (-c) \cdot c & a \cdot a + b \cdot b + (-c) \cdot (-c) & a \cdot a + b \cdot (-b) + (-c) \cdot c \\ a \cdot 0 + (-b) \cdot 2b + c \cdot c & a \cdot a + (-b) \cdot b + c \cdot (-c) & a \cdot a + (-b) \cdot (-b) + c \cdot c \end{pmatrix} \\ &= \begin{pmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{pmatrix} \end{aligned}$$

**Step 3:** Set  $\mathbf{A}\mathbf{A}^T = \mathbf{I}$  and solve for  $a$ ,  $b$ , and  $c$ .

From the first diagonal element:

$$4b^2 + c^2 = 1 \quad (1)$$

From the second diagonal element:

$$a^2 + b^2 + c^2 = 1 \quad (2)$$

From the third diagonal element:

$$a^2 + b^2 + c^2 = 1 \quad (3)$$

(This is the same as equation (2), so we effectively have only two diagonal conditions.)

From the off-diagonal elements, which must all equal zero:

$$2b^2 - c^2 = 0 \quad (4)$$

$$-2b^2 + c^2 = 0 \quad (5)$$

$$a^2 - b^2 - c^2 = 0 \quad (6)$$

Note that equations (4) and (5) are the same, so we effectively have:

$$2b^2 - c^2 = 0 \quad (4)$$

$$a^2 - b^2 - c^2 = 0 \quad (6)$$

From equation (4):

$$2b^2 = c^2 \\ \Rightarrow c^2 = 2b^2$$

Substituting this into equation (1):

$$4b^2 + c^2 = 1 \\ 4b^2 + 2b^2 = 1 \\ 6b^2 = 1 \\ \Rightarrow b^2 = \frac{1}{6} \\ \Rightarrow b = \pm \frac{1}{\sqrt{6}}$$

Let's choose  $b = \frac{1}{\sqrt{6}}$  for simplicity.

Now, using  $c^2 = 2b^2$ :

$$c^2 = 2 \cdot \frac{1}{6} = \frac{1}{3} \\ \Rightarrow c = \pm \frac{1}{\sqrt{3}}$$

Let's choose  $c = \frac{1}{\sqrt{3}}$ .

From equation (6):

$$a^2 - b^2 - c^2 = 0 \\ a^2 = b^2 + c^2 \\ a^2 = \frac{1}{6} + \frac{1}{3} = \frac{1}{6} + \frac{2}{6} = \frac{3}{6} = \frac{1}{2} \\ \Rightarrow a = \pm \frac{1}{\sqrt{2}}$$

Let's choose  $a = \frac{1}{\sqrt{2}}$ .

Therefore, the values that make the matrix orthogonal are:

$$a = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ b = \frac{1}{\sqrt{6}} \\ c = \frac{1}{\sqrt{3}}$$

**Part (b):** Let's find the values for matrix  $\mathbf{B} = \frac{1}{3} \begin{pmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{pmatrix}$

**Step 1:** Calculate  $\mathbf{B}^T$ :

$$\mathbf{B}^T = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{pmatrix}$$

**Step 2:** Calculate  $\mathbf{B}\mathbf{B}^T$ :

$$\begin{aligned}\mathbf{B}\mathbf{B}^T &= \frac{1}{3} \begin{pmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{pmatrix} \\ &= \frac{1}{9} \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 + a \cdot a & 1 \cdot 2 + 2 \cdot 1 + a \cdot b & 1 \cdot 2 + 2 \cdot (-2) + a \cdot c \\ 2 \cdot 1 + 1 \cdot 2 + b \cdot a & 2 \cdot 2 + 1 \cdot 1 + b \cdot b & 2 \cdot 2 + 1 \cdot (-2) + b \cdot c \\ 2 \cdot 1 + (-2) \cdot 2 + c \cdot a & 2 \cdot 2 + (-2) \cdot 1 + c \cdot b & 2 \cdot 2 + (-2) \cdot (-2) + c \cdot c \end{pmatrix} \\ &= \frac{1}{9} \begin{pmatrix} 1 + 4 + a^2 & 2 + 2 + ab & 2 - 4 + ac \\ 2 + 2 + ba & 4 + 1 + b^2 & 4 - 2 + bc \\ 2 - 4 + ca & 4 - 2 + cb & 4 + 4 + c^2 \end{pmatrix} \\ &= \frac{1}{9} \begin{pmatrix} 5 + a^2 & 4 + ab & -2 + ac \\ 4 + ab & 5 + b^2 & 2 + bc \\ -2 + ca & 2 + cb & 8 + c^2 \end{pmatrix}\end{aligned}$$

**Step 3:** Set  $\mathbf{B}\mathbf{B}^T = \mathbf{I}$  and solve for  $a$ ,  $b$ , and  $c$ .

From the diagonal elements:

$$\begin{aligned}\frac{1}{9}(5 + a^2) &= 1 \Rightarrow 5 + a^2 = 9 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2 \\ \frac{1}{9}(5 + b^2) &= 1 \Rightarrow 5 + b^2 = 9 \Rightarrow b^2 = 4 \Rightarrow b = \pm 2 \\ \frac{1}{9}(8 + c^2) &= 1 \Rightarrow 8 + c^2 = 9 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1\end{aligned}$$

From the off-diagonal elements (which must equal zero):

$$\begin{aligned}\frac{1}{9}(4 + ab) &= 0 \Rightarrow 4 + ab = 0 \Rightarrow ab = -4 \\ \frac{1}{9}(-2 + ac) &= 0 \Rightarrow -2 + ac = 0 \Rightarrow ac = 2 \\ \frac{1}{9}(2 + bc) &= 0 \Rightarrow 2 + bc = 0 \Rightarrow bc = -2\end{aligned}$$

Let's choose  $c = 1$  for simplicity. Then:

$$\begin{aligned}ac = 2 &\Rightarrow a \cdot 1 = 2 \Rightarrow a = 2 \\ bc = -2 &\Rightarrow b \cdot 1 = -2 \Rightarrow b = -2\end{aligned}$$

We should verify that these values also satisfy  $ab = -4$ :

$$ab = 2 \cdot (-2) = -4 \checkmark$$

Therefore, the values that make the matrix orthogonal are:

$$\begin{aligned}a &= 2 \\ b &= -2 \\ c &= 1\end{aligned}$$

**Part (c):** Let's find the values for matrix  $\mathbf{C} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & a \\ \frac{2}{3} & -\frac{1}{3} & b \\ \frac{2}{3} & \frac{2}{3} & c \end{pmatrix}$

**Step 1:** Calculate  $\mathbf{C}^T$ :

$$\mathbf{C}^T = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ a & b & c \end{pmatrix}$$

**Step 2:** Calculate  $\mathbf{C}\mathbf{C}^T$ :

$$\begin{aligned} \mathbf{C}\mathbf{C}^T &= \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & a \\ \frac{2}{3} & \frac{1}{3} & b \\ \frac{2}{3} & -\frac{2}{3} & c \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ a & b & c \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{9} + \frac{4}{9} + a^2 & \frac{2}{9} + \frac{2}{9} + ab & \frac{2}{9} - \frac{4}{9} + ac \\ \frac{2}{9} + \frac{2}{9} + ba & \frac{4}{9} + \frac{1}{9} + b^2 & \frac{4}{9} - \frac{2}{9} + bc \\ \frac{2}{9} - \frac{4}{9} + ca & \frac{4}{9} - \frac{2}{9} + cb & \frac{4}{9} + \frac{4}{9} + c^2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{9} + a^2 & \frac{4}{9} + ab & -\frac{2}{9} + ac \\ \frac{4}{9} + ab & \frac{5}{9} + b^2 & \frac{2}{9} + bc \\ -\frac{2}{9} + ca & \frac{2}{9} + cb & \frac{8}{9} + c^2 \end{pmatrix} \end{aligned}$$

**Step 3:** Set  $\mathbf{C}\mathbf{C}^T = \mathbf{I}$  and solve for  $a$ ,  $b$ , and  $c$ .

From the diagonal elements:

$$\begin{aligned} \frac{5}{9} + a^2 &= 1 \quad \Rightarrow a^2 = 1 - \frac{5}{9} = \frac{9-5}{9} = \frac{4}{9} \quad \Rightarrow a = \pm \frac{2}{3} \\ \frac{5}{9} + b^2 &= 1 \quad \Rightarrow b^2 = \frac{4}{9} \quad \Rightarrow b = \pm \frac{2}{3} \\ \frac{8}{9} + c^2 &= 1 \quad \Rightarrow c^2 = \frac{1}{9} \quad \Rightarrow c = \pm \frac{1}{3} \end{aligned}$$

From the off-diagonal elements (which must equal zero):

$$\begin{aligned} \frac{4}{9} + ab &= 0 \quad \Rightarrow ab = -\frac{4}{9} \\ -\frac{2}{9} + ac &= 0 \quad \Rightarrow ac = \frac{2}{9} \\ \frac{2}{9} + bc &= 0 \quad \Rightarrow bc = -\frac{2}{9} \end{aligned}$$

Let's choose  $c = \frac{1}{3}$  for simplicity. Then:

$$\begin{aligned} ac &= \frac{2}{9} \quad \Rightarrow a \cdot \frac{1}{3} = \frac{2}{9} \quad \Rightarrow a = \frac{2}{3} \\ bc &= -\frac{2}{9} \quad \Rightarrow b \cdot \frac{1}{3} = -\frac{2}{9} \quad \Rightarrow b = -\frac{2}{3} \end{aligned}$$

We should verify that these values also satisfy  $ab = -\frac{4}{9}$ :

$$ab = \frac{2}{3} \cdot \left(-\frac{2}{3}\right) = -\frac{4}{9} \checkmark$$

Therefore, the values that make the matrix orthogonal are:

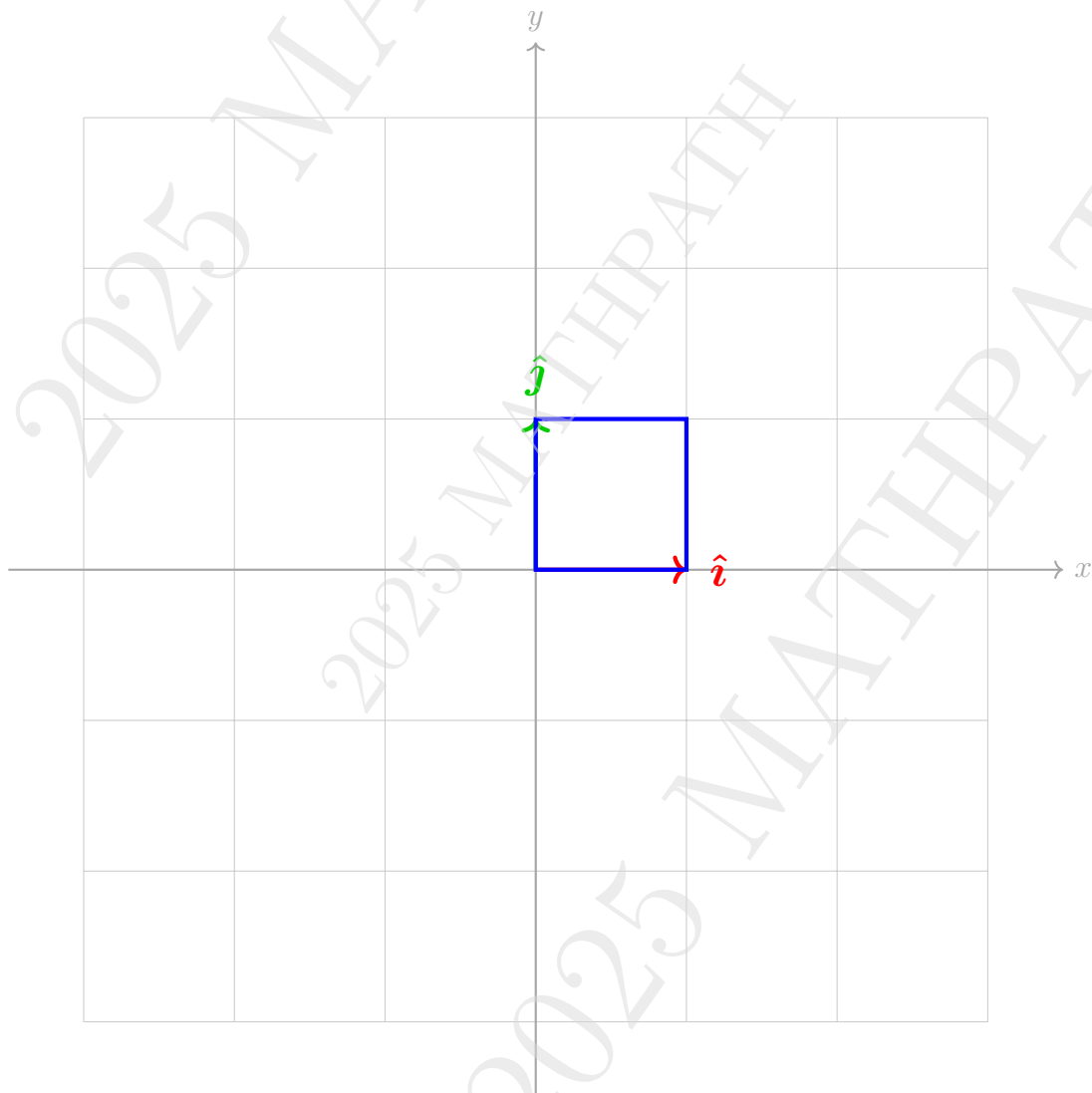
$$\begin{aligned} a &= \frac{2}{3} \\ b &= -\frac{2}{3} \\ c &= \frac{1}{3} \end{aligned}$$

## 5.4 Matrix as a Transformation Visualizations

Matrices can be visualized as transformations in space. When we multiply a vector by a matrix, we transform the vector in some way.

### Identity Transformation

The identity matrix  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  leaves vectors unchanged:

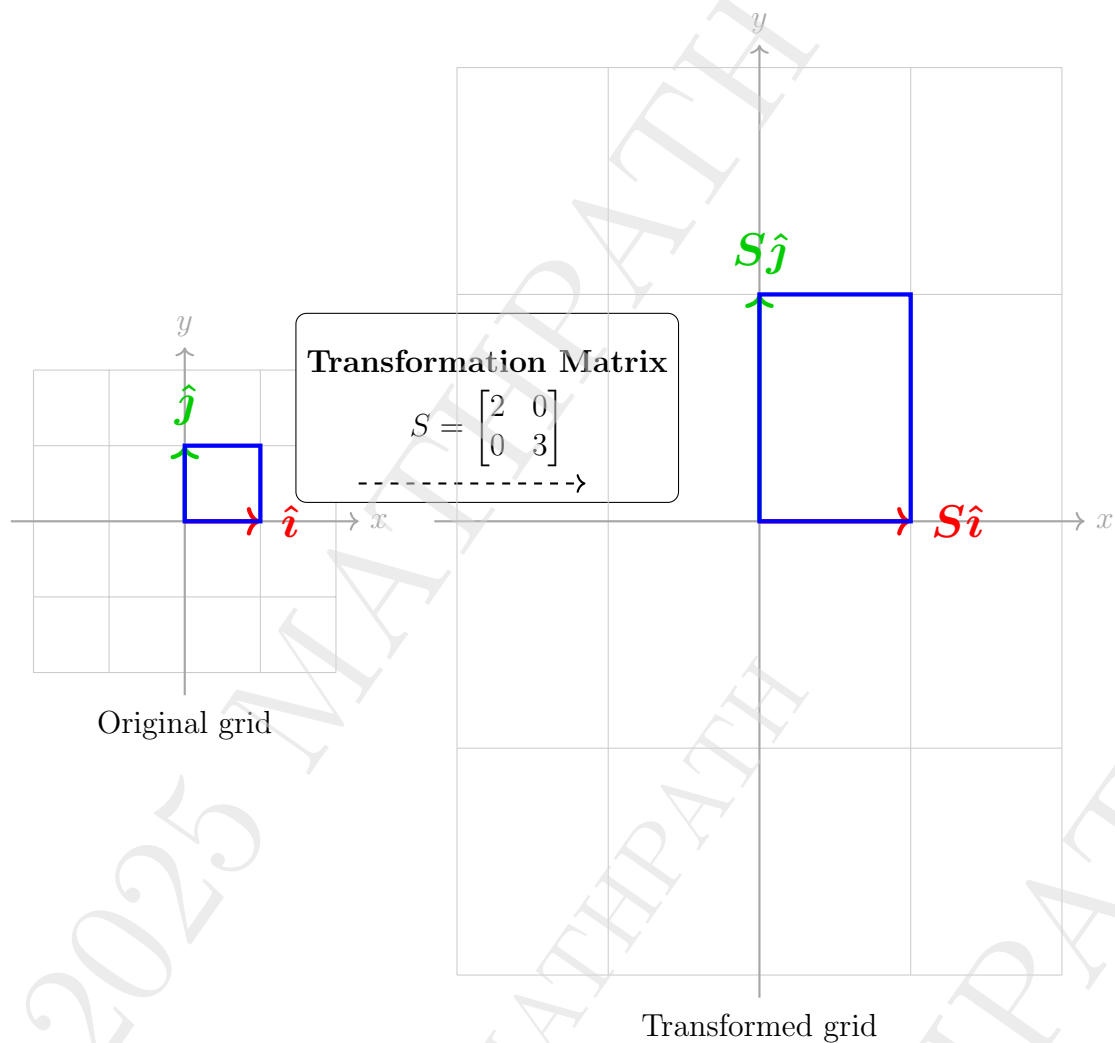


Original and transformed grid (identical)

The grid remains unchanged when multiplied by the identity matrix.

### Scaling Transformation

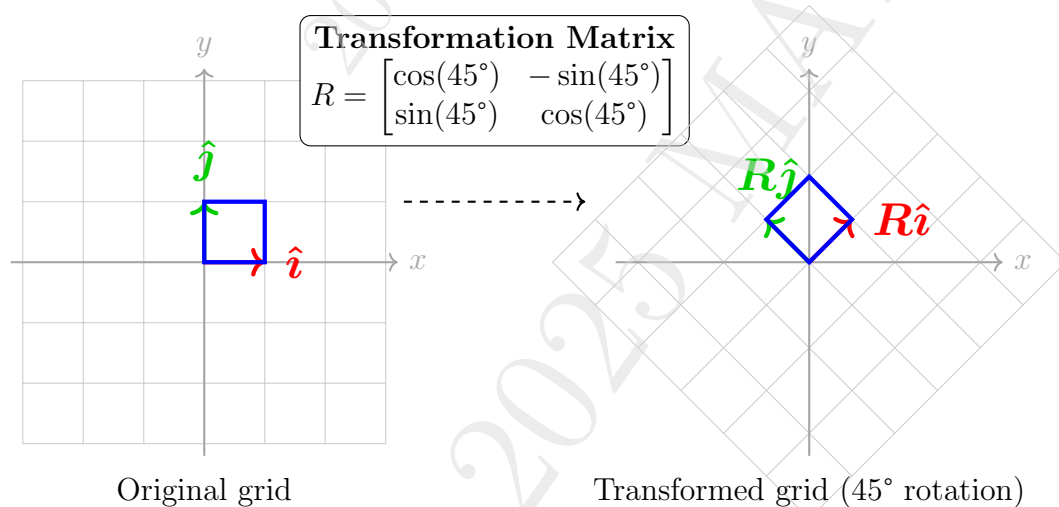
The matrix  $S = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$  scales the x-coordinate by 2 and the y-coordinate by 3:



The grid is stretched horizontally by a factor of 2 and vertically by a factor of 3.

### Rotation Transformation

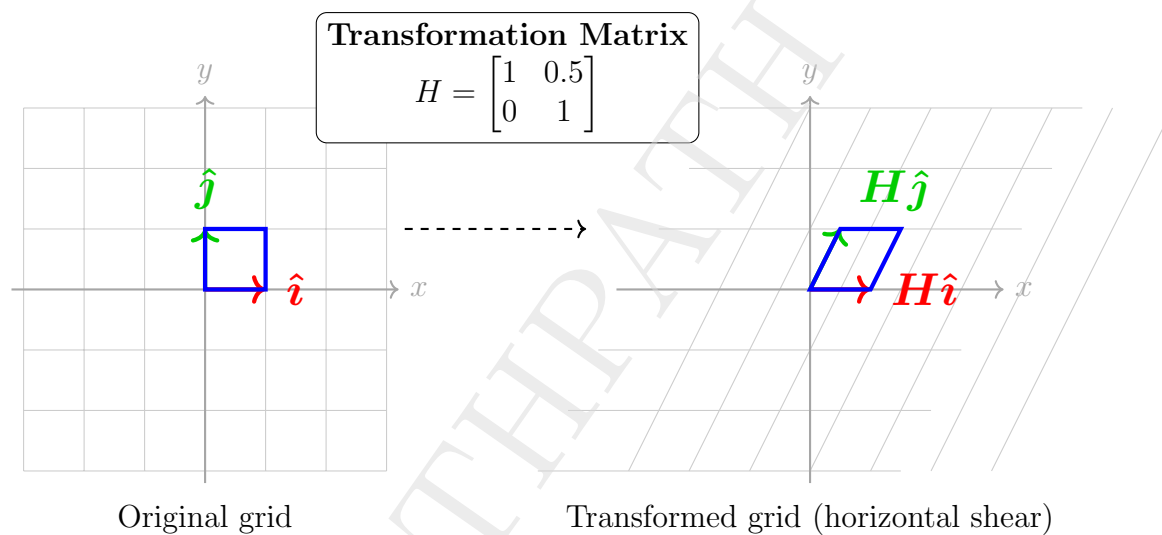
The matrix  $R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  rotates vectors counterclockwise by angle  $\theta$ :



The grid is rotated around the origin.

### Shear Transformation

The matrix  $H = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$  creates a horizontal shear:



The grid is slanted while keeping the y-coordinates unchanged.