Chapter 6

Method of Undetermined Coefficients/Shortcut Method, MVP method, Cauchy's and Legendre D. E

While the general method for finding particular integrals of non-homogeneous linear differential equations is theoretically sound, it often leads to complex integration that can be cumbersome to evaluate. The Method of Undetermined Coefficients, also known as the Shortcut Method, provides an elegant alternative approach for specific forms of the forcing function $f(x)$. This method allows us to determine the particular integral directly, without resorting to explicit integration.

The Method of Undetermined Coefficients is based on the observation that for certain types of forcing functions, the particular integral will have a specific form with undetermined coefficients. Rather than going through the traditional steps of assuming a form and finding coefficients, we can use direct formulas derived from this method to find particular integrals efficiently.

In this section, we will explore systematic formulas derived from the Method of Undetermined Coefficients based on the specific form of the forcing function. For each case, we will present the method, the appropriate formula, and situations where the standard formula might fail (cases of failure). We will also provide alternative approaches for these failure cases.

Let's recall that for a linear differential equation with constant coefficients:

$$\phi(D)y = f(x) \tag{6.1}$$

The particular integral is given by:

$$y_p = \frac{1}{\phi(D)} f(x) \tag{6.2}$$

Where $\phi(D)$ is a polynomial in the differential operator $D = \frac{d}{dx}$.

The following formulas derived from the Method of Undetermined Coefficients provide direct ways to evaluate $\frac{1}{\phi(D)} f(x)$ for various forms of $f(x)$ without requiring explicit integration. Each method corresponds to a specific type of forcing function, and collectively they cover most common scenarios encountered in practical applications.

These shortcut methods not only simplify the calculation process but also provide deeper insights into the structure of solutions for linear differential equations. Understanding these approaches is essential for efficiently solving a wide range of problems in engineering, physics, and mathematics.

Basic Differential Operators

we have a special notation involving the differential operator D. This operator and its powers have specific interpretations:

Differential Operator Notation

$$D[f(x)] = \frac{df}{dx} = f'(x) \quad \text{(First derivative of } f(x)) \tag{6.3}$$

$$D^2[f(x)] = \frac{d^2f}{dx^2} = f''(x) \quad \text{(Second derivative of } f(x)) \tag{6.4}$$

$$D^n[f(x)] = \frac{d^nf}{dx^n} = f^{(n)}(x) \quad \text{(nth derivative of } f(x)) \tag{6.5}$$

Operator Properties for Elementary Functions

Operator Properties

For exponential functions:

$$D[e^{ax}] = ae^{ax} \tag{6.6}$$

$$D^2[e^{ax}] = a^2e^{ax} \tag{6.7}$$

$$D^n[e^{ax}] = a^ne^{ax} \tag{6.8}$$

For trigonometric functions:

$$D[\sin(ax)] = a\cos(ax) \tag{6.9}$$

$$D[\cos(ax)] = -a\sin(ax) \tag{6.10}$$

$$D^2[\sin(ax)] = -a^2\sin(ax) \tag{6.11}$$

$$D^2[\cos(ax)] = -a^2\cos(ax) \tag{6.12}$$

For hyperbolic functions:

$$D[\sinh(ax)] = -a\cosh(ax) \tag{6.13}$$

$$D[\cosh(ax)] = a\sinh(ax) \tag{6.14}$$

$$D^2[\sinh(ax)] = -a^2\sinh(ax) \tag{6.15}$$

$$D^2[\cosh(ax)] = -a^2\cosh(ax) \tag{6.16}$$

Inverse Operators

The inverse of the differential operator represents integration:

Inverse Operators

$$\frac{1}{D}[f(x)] = \int f(x)\,dx \quad \text{(Indefinite integral of } f(x)\text{)} \tag{6.17}$$

$$\frac{1}{D^2}[f(x)] = \int \left(\int f(x)\,dx \right) dx \quad \text{(Double integration of } f(x)\text{)} \tag{6.18}$$

$$\frac{1}{D^n}[f(x)] = \underbrace{\int \cdots \int}_{n \text{ times}} f(x)\,dx^n \quad \text{(}n\text{-fold integration of } f(x)\text{)} \tag{6.19}$$

When finding particular integrals, we typically omit the constants of integration.

Practical Applications

Examples

1. To compute $D[x^2 e^x]$:

$$D[x^2 e^x] = D[x^2] \cdot e^x + x^2 \cdot D[e^x] \tag{6.20}$$

$$= 2x \cdot e^x + x^2 \cdot e^x \tag{6.21}$$

$$= (2x + x^2)e^x \tag{6.22}$$

2. To compute $\frac{1}{D}[\cos x]$:

$$\frac{1}{D}[\cos x] = \int \cos x\,dx \tag{6.23}$$

$$= \sin x \tag{6.24}$$

3. To compute $\frac{1}{D^2}[e^x]$:

$$\frac{1}{D^2}[e^{ax}] = \int \left(\int e^{ax}\,dx \right) dx \tag{6.25}$$

$$= \int \frac{e^{ax}}{a}\,dx \tag{6.26}$$

$$= \frac{e^{ax}}{a^2} \tag{6.27}$$

where we've set the constants of integration to zero for finding a particular integral.

6.1 Case 1: Particular Integral when $f(x) = e^{ax}$, where a is a constant

When the forcing function is an exponential function of the form $f(x) = e^{ax}$, where a is a constant, we can find the particular integral without explicit integration using a direct formula.

6.1.1 Method and Formula

To find the particular integral when $f(x) = e^{ax}$:

$$y_p = \frac{1}{\phi(D)} e^{ax} \tag{6.28}$$

We replace D by a in the operator $\phi(D)$, based on the property that $D^n e^{ax} = a^n e^{ax}$ for any positive integer n. This means that when D operates on e^{ax}, it's equivalent to multiplying by a. Therefore:

$$y_p = \frac{1}{\phi(a)} e^{ax}, \quad \text{provided } \phi(a) \neq 0 \tag{6.29}$$

6.1.2 Case of Failure

Failure (i): If $\phi(a) = 0$, then increase x in numerator and take derivative of $\phi(D)$ and then replace D by a.

$$y_p = \frac{x}{\phi'(D)} e^{ax} \tag{6.30}$$

$$\text{Replace } D \text{ by } a: \tag{6.31}$$

$$y_p = \frac{x}{\phi'(a)} e^{ax}, \quad \text{provided } \phi'(a) \neq 0 \tag{6.32}$$

Failure (ii): If $\phi(a) = \phi'(a) = 0$, then increase one more x that is make x^2 in numerator and take one more derivative of $\phi(D)$ that is $\phi''(D)$ and then replace D by a.

$$y_p = \frac{x^2}{\phi''(D)} e^{ax} \tag{6.33}$$

$$\text{Replace } D \text{ by } a: \tag{6.34}$$

$$y_p = \frac{x^2}{\phi''(a)} e^{ax}, \quad \text{provided } \phi''(a) \neq 0 \tag{6.35}$$

In this way, we can continue if we encounter further failures. The general pattern for a root of multiplicity r (i.e., when $\phi(a) = \phi'(a) = \phi''(a) = \cdots = \phi^{(r-1)}(a) = 0$ but $\phi^{(r)}(a) \neq 0$) is:

$$y_p = \frac{x^r}{\phi^{(r)}(a)} e^{ax} \tag{6.36}$$

This pattern can be more precisely stated as:

$$y_p = \frac{x^r}{r!} \cdot \frac{r!}{\phi^{(r)}(a)} e^{ax} \tag{6.37}$$

$$= \frac{x^r}{r!} \cdot \frac{1}{\frac{\phi^{(r)}(a)}{r!}} e^{ax} \tag{6.38}$$

6.1.3 Additional Formulas

For specific forms of $\phi(D)$, we have the following formulas: 1. For $\phi(D) = (D - a)^r$:

$$\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax} \tag{6.39}$$

2. For a constant k:

$$\frac{1}{\phi(D)}(k) = \frac{1}{\phi(D)}(k \cdot 1) = k \cdot \frac{1}{\phi(D)}(e^{0x}) = k \cdot \frac{1}{\phi(0)}, \quad \text{provided } \phi(0) \neq 0 \tag{6.40}$$

3. For $f(x) = a^x$, we can rewrite it as $e^{x \ln a}$ and use the formula for exponential functions:

$$\frac{1}{\phi(D)} a^x = \frac{1}{\phi(\ln a)} a^x, \quad \text{provided } \phi(\ln a) \neq 0 \tag{6.41}$$

4. Similarly, for $f(x) = a^{-x}$, we have:

$$\frac{1}{\phi(D)} a^{-x} = \frac{1}{\phi(-\ln a)} a^{-x}, \quad \text{provided } \phi(-\ln a) \neq 0 \tag{6.42}$$

6.2 Case 2: Particular Integral when $f(x) = \sin(ax + b)$ or $\cos(ax + b)$

When the forcing function is a trigonometric function of the form $f(x) = \sin(ax + b)$ or $f(x) = \cos(ax + b)$, where a and b are constants, we can find the particular integral without explicit integration using a direct formula.

6.2.1 Method and Formula

To find the particular integral when $f(x) = \sin(ax + b)$:

$$y_p = \frac{1}{\phi(D^2)} \sin(ax + b) \tag{6.43}$$

We replace D^2 by $[-a^2]$ in the operator $\phi(D^2)$, based on the property that $D^2 \sin(ax + b) = -a^2 \sin(ax + b)$ and $D^2 \cos(ax + b) = -a^2 \cos(ax + b)$. This means that when D^2 operates on trigonometric functions, it's equivalent to multiplying by $-a^2$.

Therefore:

$$y_p = \frac{1}{\phi(-a^2)} \sin(ax + b), \quad \text{provided } \phi(-a^2) \neq 0 \tag{6.44}$$

Similarly, for $f(x) = \cos(ax + b)$:

$$y_p = \frac{1}{\phi(-a^2)} \cos(ax + b), \quad \text{provided } \phi(-a^2) \neq 0 \tag{6.45}$$

When applying this method to operators that contain D (not just D^2), we first substitute $D^2 = -a^2$ and then rationalize any remaining D terms to eliminate them.

6.2.2 Case of Failure

Failure 1: If $\phi(-a^2) = 0$, then increase x in numerator and take derivative of $\phi(D^2)$ with respect to D and then replace D^2 by $[-a^2]$.

$$y_p = \frac{x}{\phi'(D^2)} \sin(ax + b) \tag{6.46}$$

$$\text{Replace } D^2 \text{ by } [-a^2] : \tag{6.47}$$

$$y_p = \frac{x}{\phi'(-a^2)} \sin(ax + b), \quad \text{provided } \phi'(-a^2) \neq 0 \tag{6.48}$$

Similarly for cosine:

$$y_p = \frac{x}{\phi'(-a^2)} \cos(ax+b), \quad \text{provided } \phi'(-a^2) \neq 0 \tag{6.49}$$

Failure 2: If $\phi(-a^2) = \phi'(-a^2) = 0$, then increase to x^2 in numerator and take second derivative of $\phi(D^2)$ with respect to D and then replace D^2 by $[-a^2]$.

$$y_p = \frac{x^2}{\phi''(D^2)} \sin(ax+b) \tag{6.50}$$

$$\text{Replace } D^2 \text{ by } [-a^2]: \tag{6.51}$$

$$y_p = \frac{x^2}{\phi''(-a^2)} \sin(ax+b), \quad \text{provided } \phi''(-a^2) \neq 0 \tag{6.52}$$

Similarly for cosine:

$$y_p = \frac{x^2}{\phi''(-a^2)} \cos(ax+b), \quad \text{provided } \phi''(-a^2) \neq 0 \tag{6.53}$$

In this way, we can continue if we encounter further failures.

6.2.3 Rationalization for Mixed Operators

When the operator $\phi(D)$ contains both even and odd powers of D (not just D^2 terms), after substituting $D^2 = -a^2$, we may still have terms containing D.
For example, with an operator like $\phi(D) = D^3 + D = D(D^2+1)$, after substituting $D^2 = -a^2$, we get:

$$\phi(D) = D(D^2+1) \tag{6.54}$$

$$= D(-a^2+1) \tag{6.55}$$

In such cases, we rationalize to eliminate the D terms:
1. If we have $\frac{1}{D+h}$, we multiply both numerator and denominator by $(D-h)$ to get $\frac{D-h}{D^2-h^2}$.
2. If we have $\frac{1}{hD+k}$, we multiply both numerator and denominator by $(hD-k)$ to get $\frac{hD-k}{h^2D^2-k^2}$.
This rationalization process allows us to express the operator in terms of D^2, which can then be replaced with $-a^2$.

6.2.4 Special Cases and Formulas

1. For $\phi(D) = D^2 + a^2$, which appears in many applications:

$$\frac{1}{D^2+a^2} \sin(ax) = \frac{1}{0} \sin(ax) \quad \text{(failure case)} \tag{6.56}$$

Applying the failure case formula

$$\frac{1}{D^2+a^2} \sin(ax) = \frac{x}{2D} \sin(ax) \tag{6.57}$$

Do rationalization:

$$\frac{x}{2D} \sin(ax) = \frac{x}{2D} \times \frac{D}{D} \sin(ax) \tag{6.58}$$

$$= \frac{xD}{2D^2} \sin(ax) \tag{6.59}$$

Now replace D^2 by $-a^2$:

$$\frac{xD}{2D^2}\sin(ax) = \frac{xD}{2(-a^2)}\sin(ax) \tag{6.60}$$
$$= \frac{-x}{2a^2}D(\sin(ax)) \tag{6.61}$$
$$= \frac{-x}{2a^2}[a\cos(ax)] \tag{6.62}$$
$$= \frac{-x}{2a}\cos(ax) \tag{6.63}$$

Similarly:

$$\frac{1}{D^2 + a^2}(\cos(ax)) = \frac{-x}{2a^2}D(\cos(ax)) \tag{6.64}$$
$$= \frac{-x}{2a^2}[-a\sin(ax)] \tag{6.65}$$
$$= \frac{x}{2a}\sin(ax) \tag{6.66}$$

For higher powers of $(D^2 + a^2)$:

$$\frac{1}{(D^2 + a^2)^r}\sin(ax + b) = \left(-\frac{x}{2a}\right)^r \frac{1}{r!}\sin\left(ax + b + \frac{r\pi}{2}\right) \tag{6.67}$$

$$\frac{1}{(D^2 + a^2)^r}\cos(ax + b) = \left(-\frac{x}{2a}\right)^r \frac{1}{r!}\cos\left(ax + b + \frac{r\pi}{2}\right) \tag{6.68}$$

3. For products of trigonometric functions, we convert them to sums using trigonometric identities:

$$\sin A \cdot \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)] \tag{6.69}$$
$$\sin A \cdot \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)] \tag{6.70}$$
$$\cos A \cdot \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)] \tag{6.71}$$

This allows us to apply the basic formulas to each term separately.

6.2.5 Worked Example

Let's solve: $(D^2 + 1)y = \cos x$

Step 1: Express in operator form:

$$y_p = \frac{1}{D^2 + 1}\cos x \tag{6.72}$$

Step 2: Replace D^2 by $[-a^2] = [-1]$ when operating on $\cos x$:

$$y_p = \frac{1}{-1 + 1}\cos x \tag{6.73}$$
$$= \frac{1}{0}\cos x \tag{6.74}$$

Since we have $\phi(-a^2) = 0$, this is failure 1.
Step 3: Apply the formula for failure 1. According to the corrected approach, we take the derivative of $\phi(D)$ with respect to D (not D^2):

$$y_p = \frac{x}{\phi'(D)} \cos x \tag{6.75}$$

where $\phi(D) = D^2 + 1$, so $\phi'(D) = 2D$.

$$y_p = \frac{x}{2D} \cos x \tag{6.76}$$

Step 4: We have D in the denominator, so we need to rationalize:

$$\frac{x}{2D} \cos x = \frac{x}{2D} \times \frac{D}{D} \cos x \tag{6.77}$$
$$= \frac{xD}{2D^2} \cos x \tag{6.78}$$

Step 5: Now replace D^2 by -1 (since $a = 1$):

$$\frac{xD}{2D^2} \cos x = \frac{xD}{2(-1)} \cos x \tag{6.79}$$
$$= \frac{-xD}{2} \cos x \tag{6.80}$$
$$= \frac{-x}{2} D(\cos x) \tag{6.81}$$
$$= \frac{-x}{2} (-\sin x) \tag{6.82}$$
$$= \frac{x}{2} \sin x \tag{6.83}$$

Therefore, the particular integral is $y_p = \frac{x}{2} \sin x$.
This result is consistent with the formula $\frac{1}{D^2+a^2}(\cos(ax)) = \frac{x}{2a} \sin(ax)$ where $a = 1$.
In more complex cases where we end up with expressions involving D in the denominator after substituting $D^2 = -a^2$, we would need to rationalize as shown in the special case formulas above.

6.3 Case 3: Particular Integral when $f(x) = \sinh(ax + b)$ or $\cosh(ax + b)$

When the forcing function is a hyperbolic function of the form $f(x) = \sinh(ax + b)$ or $f(x) = \cosh(ax + b)$, where a and b are constants, we can find the particular integral without explicit integration using a direct formula.

6.3.1 Method and Formula

To find the particular integral when $f(x) = \sinh(ax + b)$:

$$y_p = \frac{1}{\phi(D^2)} \sinh(ax + b) \tag{6.84}$$

We replace D^2 by $[a^2]$ in the operator $\phi(D^2)$, based on the property that $D^2 \sinh(ax+b) = a^2 \sinh(ax+b)$ and $D^2 \cosh(ax+b) = a^2 \cosh(ax+b)$. This means that when D^2 operates on hyperbolic functions, it's equivalent to multiplying by a^2 (positive).
Therefore:

$$y_p = \frac{1}{\phi(a^2)} \sinh(ax+b), \quad \text{provided } \phi(a^2) \neq 0 \tag{6.85}$$

Similarly, for $f(x) = \cosh(ax+b)$:

$$y_p = \frac{1}{\phi(a^2)} \cosh(ax+b), \quad \text{provided } \phi(a^2) \neq 0 \tag{6.86}$$

When applying this method to operators that contain D (not just D^2), we first substitute $D^2 = a^2$ and then rationalize any remaining D terms to eliminate them.

6.3.2 Case of Failure

Failure 1: If $\phi(a^2) = 0$, then increase x in numerator and take derivative of $\phi(D^2)$ with respect to D and then replace D^2 by $[a^2]$.

$$y_p = \frac{x}{\phi'(D^2)} \sinh(ax+b) \tag{6.87}$$

$$\text{Replace } D^2 \text{ by } [a^2]: \tag{6.88}$$

$$y_p = \frac{x}{\phi'(a^2)} \sinh(ax+b), \quad \text{provided } \phi'(a^2) \neq 0 \tag{6.89}$$

Similarly for hyperbolic cosine:

$$y_p = \frac{x}{\phi'(a^2)} \cosh(ax+b), \quad \text{provided } \phi'(a^2) \neq 0 \tag{6.90}$$

Failure 2: If $\phi(a^2) = \phi'(a^2) = 0$, then increase to x^2 in numerator and take second derivative of $\phi(D^2)$ with respect to D and then replace D^2 by $[a^2]$.

$$y_p = \frac{x^2}{\phi''(D^2)} \sinh(ax+b) \tag{6.91}$$

$$\text{Replace } D^2 \text{ by } [a^2]: \tag{6.92}$$

$$y_p = \frac{x^2}{\phi''(a^2)} \sinh(ax+b), \quad \text{provided } \phi''(a^2) \neq 0 \tag{6.93}$$

Similarly for hyperbolic cosine:

$$y_p = \frac{x^2}{\phi''(a^2)} \cosh(ax+b), \quad \text{provided } \phi''(a^2) \neq 0 \tag{6.94}$$

In this way, we can continue if we encounter further failures.

6.3.3 Rationalization for Mixed Operators

When the operator $\phi(D)$ contains both even and odd powers of D (not just D^2 terms), after substituting $D^2 = a^2$, we may still have terms containing D.

For example, with an operator like $\phi(D) = D^3 - D = D(D^2 - 1)$, after substituting $D^2 = a^2$, we get:

$$\phi(D) = D(D^2 - 1) \tag{6.95}$$
$$= D(a^2 - 1) \tag{6.96}$$

In such cases, we rationalize to eliminate the D terms:
1. If we have $\frac{1}{D+h}$, we multiply both numerator and denominator by $(D-h)$ to get $\frac{D-h}{D^2-h^2}$.
2. If we have $\frac{1}{hD+k}$, we multiply both numerator and denominator by $(hD-k)$ to get $\frac{hD-k}{h^2D^2-k^2}$.
This rationalization process allows us to express the operator in terms of D^2, which can then be replaced with a^2.

6.3.4 Special Cases and Formulas

1. For $\phi(D) = D^2 - a^2$, which appears in many applications:

$$\frac{1}{D^2 - a^2}\sinh(ax) = \frac{1}{a^2 - a^2}\sinh(ax) = \frac{1}{0}\sinh(ax) \quad \text{(failure case)} \tag{6.97}$$

Applying the failure case formula:

$$\frac{1}{D^2 - a^2}\sinh(ax) = \frac{x}{\phi'(D)}\sinh(ax) \tag{6.98}$$

where $\phi(D) = D^2 - a^2$, so $\phi'(D) = 2D$.

$$\frac{1}{D^2 - a^2}\sinh(ax) = \frac{x}{2D}\sinh(ax) \tag{6.99}$$

Do rationalization:

$$\frac{x}{2D}\sinh(ax) = \frac{x}{2D} \times \frac{D}{D}\sinh(ax) \tag{6.100}$$
$$= \frac{xD}{2D^2}\sinh(ax) \tag{6.101}$$

Now replace D^2 by a^2:

$$\frac{xD}{2D^2}\sinh(ax) = \frac{xD}{2(a^2)}\sinh(ax) \tag{6.102}$$
$$= \frac{x}{2a^2}D(\sinh(ax)) \tag{6.103}$$
$$= \frac{x}{2a^2}[-a \cdot \cosh(ax)] \tag{6.104}$$
$$= \frac{-x}{2a}\cosh(ax) \tag{6.105}$$

Similarly:

$$\frac{1}{D^2 - a^2}(\cosh(ax)) = \frac{x}{2a^2}D(\cosh(ax)) \tag{6.106}$$
$$= \frac{x}{2a^2}[a \cdot \sinh(ax)] \tag{6.107}$$
$$= \frac{x}{2a}\sinh(ax) \tag{6.108}$$

2. For higher powers of $(D^2 - a^2)$, the formulas would follow a pattern similar to those for trigonometric functions, but with appropriate adjustments for hyperbolic functions.
3. For products involving hyperbolic functions, we use the following identities:

$$\sinh A \cdot \sinh B = \frac{1}{2}[\cosh(A+B) - \cosh(A-B)] \tag{6.109}$$

$$\sinh A \cdot \cosh B = \frac{1}{2}[\sinh(A+B) + \sinh(A-B)] \tag{6.110}$$

$$\cosh A \cdot \cosh B = \frac{1}{2}[\cosh(A+B) + \cosh(A-B)] \tag{6.111}$$

This allows us to apply the basic formulas to each term separately.

6.3.5 Relationship to Exponential Forms

Hyperbolic functions can be expressed in terms of exponentials:

$$\sinh(ax+b) = \frac{e^{ax+b} - e^{-(ax+b)}}{2} \tag{6.112}$$

$$\cosh(ax+b) = \frac{e^{ax+b} + e^{-(ax+b)}}{2} \tag{6.113}$$

This relationship allows us to verify our formulas or derive them from the exponential case (Case 1) when needed.

6.3.6 Worked Example

Let's solve: $(D^2 - 4)y = \sinh(2x)$
Step 1: Express in operator form:

$$y_p = \frac{1}{D^2 - 4}\sinh(2x) \tag{6.114}$$

Step 2: Replace D^2 by $[a^2] = [4]$ when operating on $\sinh(2x)$ (since $a = 2$):

$$y_p = \frac{1}{4-4}\sinh(2x) \tag{6.115}$$

$$= \frac{1}{0}\sinh(2x) \tag{6.116}$$

Since we have $\phi(a^2) = 0$, this is failure 1.
Step 3: Apply the formula for failure 1:

$$y_p = \frac{x}{\phi'(D)}\sinh(2x) \tag{6.117}$$

where $\phi(D) = D^2 - 4$, so $\phi'(D) = 2D$.

$$y_p = \frac{x}{2D}\sinh(2x) \tag{6.118}$$

Step 4: We have D in the denominator, so we need to rationalize:

$$\frac{x}{2D}\sinh(2x) = \frac{x}{2D} \times \frac{D}{D}\sinh(2x) \tag{6.119}$$

$$= \frac{xD}{2D^2}\sinh(2x) \tag{6.120}$$

Step 5: Now replace D^2 by 4 (since $a = 2$):

$$\frac{xD}{2D^2}\sinh(2x) = \frac{xD}{2(4)}\sinh(2x) \tag{6.121}$$
$$= \frac{xD}{8}\sinh(2x) \tag{6.122}$$
$$= \frac{x}{8}D(\sinh(2x)) \tag{6.123}$$
$$= \frac{x}{8}(-2\cdot\cosh(2x)) \tag{6.124}$$
$$= \frac{-x}{4}\cosh(2x) \tag{6.125}$$

Therefore, the particular integral is $y_p = \frac{-x}{4}\cosh(2x)$.
This result is consistent with the formula $\frac{1}{D^2-a^2}(\sinh(ax)) = \frac{-x}{2a}\cosh(ax)$ where $a = 2$.
In more complex cases where we end up with expressions involving D in the denominator after substituting $D^2 = a^2$, we would need to rationalize as shown in the special case formulas above.

6.4 Case 4: Particular Integral when $f(x) = x^m$

Polynomial forcing functions are common in many applications, such as motion under gravity, beam deflection problems, and circuit analysis. This section presents the shortcut method for finding particular integrals when the forcing function is a power of x.

6.4.1 Method and Formula

To find the particular integral when $f(x) = x^m$, where m is a non-negative integer, we utilize the inverse operator approach in a different manner. Let's first understand how the differential operator D acts on x^m:

$$\begin{aligned} D(x^m) &= mx^{m-1} \\ D^2(x^m) &= m(m-1)x^{m-2} \\ &\vdots \\ D^m(x^m) &= m! \\ D^{m+1}(x^m) &= 0 \end{aligned}$$

Unlike the exponential or trigonometric functions, x^m is not an eigenfunction of D. Instead, repeated application of D reduces the power until eventually reaching zero.
For finding $\frac{1}{\phi(D)}x^m$, we expand $[\phi(D)]^{-1}$ in ascending powers of D up to the term D^m, since higher powers yield zero when applied to x^m:

$$\frac{1}{\phi(D)}x^m = [\phi(D)]^{-1}x^m \tag{6.126}$$

This means we need to express $\frac{1}{\phi(D)}$ as a power series in D and then apply this expansion to x^m term by term.

6.4.2 Implementation Procedure

The procedure involves the following steps:

1. Express $\frac{1}{\phi(D)}$ in ascending powers of D up to D^m 2. Apply this expansion to x^m term by term 3. Evaluate each term $D^k x^m$ using the formula $D^k x^m = \frac{m!}{(m-k)!} x^{m-k}$ for $k \leq m$ 4. Sum up all the resulting terms to obtain the particular integral

6.4.3 Practical Approach

In practice, we often take a more direct approach by factoring $\phi(D)$ or using binomial expansion when appropriate.

For example, if $\phi(D) = D^2 - 3D + 2$, we can factorize it as $(D-1)(D-2)$ and then use partial fractions:

$$\begin{aligned} \frac{1}{(D-1)(D-2)} &= \frac{A}{D-1} + \frac{B}{D-2} \\ &= \frac{1}{D-1} - \frac{1}{D-2} \end{aligned}$$

Then we compute $\frac{1}{D-1} x^m$ and $\frac{1}{D-2} x^m$ separately and combine the results.

6.4.4 General Formulas for Special Cases

For certain common forms of $\phi(D)$, we have the following formulas:

1. For $\phi(D) = D - a$:

$$\boxed{\frac{1}{D-a} x^m = \frac{1}{a} \sum_{k=0}^{m} \binom{m}{k} a^{-k} D^k x^m} \tag{6.127}$$

2. For denominators involving $D^2 - 3D + 2$, we can use:

$$\begin{aligned} \frac{1}{D^2 - 3D + 2} x^m &= \frac{1}{(D-1)(D-2)} x^m \\ &= \left(\frac{1}{D-1} - \frac{1}{D-2} \right) x^m \end{aligned}$$

3. For denominators of the form $D^2 - 3D + 3$:

$$\frac{1}{D^2 - 3D + 3} x^m = \frac{1}{3} \left[1 + \frac{D^2 - 3D}{3} \right]^{-1} x^m$$

Which can be expanded using the binomial theorem for negative powers.

6.4.5 Notes on Implementation

When calculating the particular integral for polynomial forcing functions:

1. Always take the constant term common from the denominator when present 2. If no constant term exists in the denominator, take the minimum power of D common 3. Use binomial expansion for expressions of the form $(1 \pm x)^{-1}$ 4. Remember that $D^n(x^m) = 0$ for $n > m$, which simplifies the calculation

This method provides a systematic approach to finding particular integrals for polynomial forcing functions without requiring explicit integration.

6.5 Case 5: Particular Integral when $f(x) = e^{ax}V$, where V is any function of x

This section addresses the case where the forcing function is an exponential function multiplied by another function $V(x)$. This form appears in various applications, including control systems, vibration analysis, and circuit theory with modulated inputs.

6.5.1 Method and Derivation

When $f(x) = e^{ax}V$, where V is any function of x and a is a constant, we can develop a formula that relates the particular integral to the simpler problem of finding $\frac{1}{\phi(D+a)}V$.
Let's begin by examining how the operator D affects $e^{ax}V$:

$$\begin{aligned} D(e^{ax}V) &= ae^{ax}V + e^{ax}DV \\ &= e^{ax}(a+D)V \end{aligned}$$

Similarly:

$$\begin{aligned} D^2(e^{ax}V) &= D[e^{ax}(a+D)V] \\ &= e^{ax}(a+D)(a+D)V \\ &= e^{ax}(a+D)^2V \end{aligned}$$

Continuing this pattern, we obtain:

$$D^n(e^{ax}V) = e^{ax}(a+D)^nV$$

This means that for any polynomial in D:

$$\phi(D)(e^{ax}V) = e^{ax}\phi(a+D)V$$

Therefore:

$$\begin{aligned} \frac{1}{\phi(D)}(e^{ax}V) &= \frac{1}{e^{ax}\phi(a+D)V} \cdot e^{ax}V \\ &= e^{ax} \cdot \frac{1}{\phi(a+D)}V \end{aligned}$$

This gives us the formula:

$$\boxed{\frac{1}{\phi(D)}(e^{ax}V) = e^{ax} \cdot \frac{1}{\phi(D+a)}V} \tag{6.128}$$

6.5.2 Practical Application

The power of this formula lies in its ability to reduce the problem to finding the particular integral for a simpler forcing function V. Once we compute $\frac{1}{\phi(D+a)}V$, we simply multiply the result by e^{ax} to obtain the particular integral for the original problem.
For example, if $V = \sin bx$, we first find $\frac{1}{\phi(D+a)}\sin bx$ using the methods from Case 2, and then multiply the result by e^{ax}.

6.5.3 Special Cases

1. When $V = 1$, we have $f(x) = e^{ax}$, which reduces to Case 1.
2. When $V = \sin bx$ or $V = \cos bx$, we're dealing with a product of exponential and trigonometric functions. This is common in damped oscillations or forced vibrations.
3. When $V = x^m$, we need to find $\frac{1}{\phi(D+a)}x^m$ using the methods from Case 4, and then multiply by e^{ax}.

6.5.4 Implementation Notes

In applying this formula:

1. First identify the exponential factor e^{ax} and the remaining function V 2. Shift the operator polynomial $\phi(D)$ to $\phi(D+a)$ 3. Apply the appropriate method to find $\frac{1}{\phi(D+a)}V$ based on the form of V 4. Multiply the result by e^{ax}

This method provides a powerful tool for handling forcing functions that combine exponential behavior with other functions, without requiring complex integration techniques.

6.6 Case 6: Particular Integral when $f(x) = x^m \sin ax$ or $x^m \cos ax$

This section addresses forcing functions that involve products of polynomials and trigonometric functions. Such forms arise in various physical systems, particularly in vibration analysis with non-uniform loading or in electrical circuits with modulated signals.

6.6.1 Method and Derivation

When the forcing function is of the form $f(x) = x^m \sin ax$ or $f(x) = x^m \cos ax$, where m is a non-negative integer and a is a constant, we can apply a transformation that relates these functions to complex exponentials.

First, recall that:

$$\sin ax = \frac{e^{iax} - e^{-iax}}{2i}$$
$$\cos ax = \frac{e^{iax} + e^{-iax}}{2}$$

This means that:

$$x^m \sin ax = x^m \cdot \frac{e^{iax} - e^{-iax}}{2i}$$
$$x^m \cos ax = x^m \cdot \frac{e^{iax} + e^{-iax}}{2}$$

Using the formula from Case 5, we can compute:

$$\frac{1}{\phi(D)}(x^m e^{iax}) = e^{iax} \cdot \frac{1}{\phi(D+ia)}x^m$$
$$\frac{1}{\phi(D)}(x^m e^{-iax}) = e^{-iax} \cdot \frac{1}{\phi(D-ia)}x^m$$

Therefore:

$$\frac{1}{\phi(D)}(x^m \sin ax) = \frac{1}{2i}\left[e^{iax}\cdot\frac{1}{\phi(D+ia)}x^m - e^{-iax}\cdot\frac{1}{\phi(D-ia)}x^m\right]$$
$$\frac{1}{\phi(D)}(x^m \cos ax) = \frac{1}{2}\left[e^{iax}\cdot\frac{1}{\phi(D+ia)}x^m + e^{-iax}\cdot\frac{1}{\phi(D-ia)}x^m\right]$$

These expressions can be simplified further by noting that:

$$\frac{1}{\phi(D)}(x^m \sin ax) = \text{Imaginary part of }\left\{e^{iax}\cdot\frac{1}{\phi(D+ia)}x^m\right\}$$
$$\frac{1}{\phi(D)}(x^m \cos ax) = \text{Real part of }\left\{e^{iax}\cdot\frac{1}{\phi(D+ia)}x^m\right\}$$

6.6.2 Practical Implementation

In practice, we compute $\frac{1}{\phi(D+ia)}x^m$ using the methods from Case 4, multiply by e^{iax}, and then extract the appropriate part (real or imaginary).

For example, if we need to find $\frac{1}{\phi(D)}(x\sin 2x)$, we would: 1. Compute $\frac{1}{\phi(D+2i)}x$ 2. Multiply by e^{2ix} 3. Extract the imaginary part of the result

6.6.3 Notes on Implementation

When evaluating expressions like $\frac{1}{\phi(D+ia)}x^m$, we can use the methods from Case 4 after replacing D with $D+ia$. This may involve expanding the denominator and using partial fractions or binomial expansion as appropriate.

The formulas can be summarized as:

$$\boxed{\frac{1}{\phi(D)}(x^m \sin ax) = \text{Im}\left\{e^{iax}\cdot\frac{1}{\phi(D+ia)}x^m\right\}} \tag{6.129}$$

$$\boxed{\frac{1}{\phi(D)}(x^m \cos ax) = \text{Re}\left\{e^{iax}\cdot\frac{1}{\phi(D+ia)}x^m\right\}} \tag{6.130}$$

Where Im and Re denote the imaginary and real parts, respectively.

6.6.4 Alternative Approach

In some cases, especially when m is small, it may be more convenient to use the recurrence relation for $\frac{1}{D^2+a^2}x^n$ directly.

For example, to find $\frac{1}{D^2+4}(x\sin 2x)$, we can use the formula:

$$\begin{aligned}\frac{1}{D^2+4}(x\sin 2x) &= x\cdot\frac{1}{D^2+4}(\sin 2x) - \frac{d}{da^2}\left[\frac{1}{D^2+a^2}\sin ax\right]_{a=2}\\ &= x\cdot\frac{\sin 2x}{4} - \frac{d}{da^2}\left[\frac{\sin ax}{a^2}\right]_{a=2}\\ &= \frac{x\sin 2x}{4} - \frac{d}{da}\left[\frac{-\cos ax}{a}\right]_{a=2}\\ &= \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}\end{aligned}$$

This approach can be more direct for simple cases but becomes cumbersome as the complexity increases.

6.7 Case 7: Particular Integral when $f(x) = xV$

When finding the particular integral of a differential equation where the forcing function has the form $f(x) = xV$, with V being any function of x, we need a specialized approach. This case appears frequently in various applications of differential equations.

Operator Formula

For a linear differential equation with constant coefficients where the right-hand side has the form $f(x) = xV$:

$$\frac{1}{\phi(D)}[xV] = \left[x - \frac{\phi'(D)}{\phi(D)}\right]\frac{1}{\phi(D)}V \tag{6.131}$$

Where $\phi(D)$ is the differential operator and $\phi'(D)$ is its derivative with respect to D.

Application Insight

When applying this case to the operand xV, the operator $\frac{1}{\phi(D)}$ transforms the problem into a simpler form. After application of this operator, it reduces to V. For further simplification, we should apply other cases as appropriate depending on the form of V.

Let's apply this to solve a differential equation with a forcing function of this form.

Example

Find the particular integral for:

$$\frac{1}{D^2 + 2D + 1}[x \cos x] \tag{6.132}$$

Step-by-Step Solution

We identify $\phi(D) = D^2 + 2D + 1$ and $V = \cos x$. Applying Case 7:

$$\frac{1}{D^2 + 2D + 1}[x \cos x] = \left[x - \frac{\phi'(D)}{\phi(D)}\right]\frac{1}{\phi(D)}\cos x \tag{6.133}$$

First, we calculate $\phi'(D)$:

$$\phi'(D) = \frac{d}{dD}(D^2 + 2D + 1) \tag{6.134}$$

$$= 2D + 2 \tag{6.135}$$

Substituting:

$$\frac{1}{D^2 + 2D + 1}[x \cos x] = \left[x - \frac{2D + 2}{D^2 + 2D + 1}\right]\frac{1}{D^2 + 2D + 1}\cos x \tag{6.136}$$

Now, $\cos x = \cos(1x + 0)$ has the form $\cos(ax)$ where $a = 1$.
By Case 2, we know that for $\frac{1}{D^2+2D+1}\cos x$, we can replace D^2 with $-a^2 = -1$ in appro-

priate terms:

$$= \left[x - \frac{2D+2}{-1+2D+1}\right] \frac{1}{D^2+2D+1} \cos x \tag{6.137}$$

$$= \left[x - \frac{2D+2}{2D}\right] \frac{1}{D^2+2D+1} \cos x \tag{6.138}$$

$$= \left[x - \frac{2D+2}{2D}\right] \frac{1}{2D} \cos x \tag{6.139}$$

$$= \frac{1}{2}\left[x - \frac{D+1}{D}\right] \frac{1}{D} \cos x \tag{6.140}$$

Since $\frac{1}{D}\cos x = \sin x$:

$$= \frac{1}{2}\left[x - \frac{D+1}{D}\right] \sin x \tag{6.141}$$

$$= \frac{1}{2}\left[x \sin x - \frac{D+1}{D} \sin x\right] \tag{6.142}$$

$$= \frac{1}{2}\left[x \sin x - \sin x - \frac{1}{D} \sin x\right] \tag{6.143}$$

Using $\frac{1}{D}\sin x = -\cos x$:

$$= \frac{1}{2}[x \sin x - \sin x + \cos x] \tag{6.144}$$

Therefore:

$$\frac{1}{D^2+2D+1}[x \cos x] = \frac{1}{2}[x \sin x - \sin x + \cos x] \tag{6.145}$$

We can verify our answer by differentiating it according to the original differential operator and checking if we get $x \cos x$.

6.7.1 General Approach

To apply Case 7 effectively:

1. Identify the function V in the expression xV
2. Calculate the derivative $\phi'(D)$ of the operator $\phi(D)$
3. Apply the formula $\frac{1}{\phi(D)}[xV] = \left[x - \frac{\phi'(D)}{\phi(D)}\right] \frac{1}{\phi(D)} V$
4. Determine $\frac{1}{\phi(D)} V$ using the appropriate case for the function V
5. Simplify the resulting expression

Alternative Formula

An equivalent form of the Case 7 formula that some may find easier to work with is:

$$\frac{1}{\phi(D)}[xV] = x\frac{1}{\phi(D)} V - \frac{\phi'(D)}{\phi(D)^2} V \tag{6.146}$$

This case illustrates how the operator method allows us to solve differential equations with more complex forcing functions by breaking them down into manageable steps.

6.8 Summary of Shortcut Methods for Finding Particular Integrals

The following table summarizes the shortcut methods for finding particular integrals based on the form of the forcing function $f(x)$. These formulas allow direct computation without requiring explicit integration.

Shortcut Methods for Finding Particular Integrals

Case	Forcing Function $f(x)$	Formula for Particular Integral $y_p = \frac{1}{\phi(D)} f(x)$
Case 1	e^{ax}, where a is a constant	$\frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax}, \quad \phi(a) \neq 0$ **Case of Failure** (when $\phi(a) = 0$): If $\phi(D) = (D-a)\psi(D)$ where $\psi(a) \neq 0$: $\frac{1}{\phi(D)} e^{ax} = x \cdot \frac{1}{\psi(a)} e^{ax}$ For root of multiplicity r: $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$
Case 2	$\sin(ax+b)$ or $\cos(ax+b)$	$\frac{1}{\phi(D^2)} \sin(ax+b) = \frac{1}{\phi(-a^2)} \sin(ax+b), \quad \phi(-a^2) \neq 0$ $\frac{1}{\phi(D^2)} \cos(ax+b) = \frac{1}{\phi(-a^2)} \cos(ax+b), \quad \phi(-a^2) \neq 0$ **Case of Failure** (when $\phi(-a^2) = 0$): $\frac{1}{D^2+a^2} \sin(ax) = -\frac{x}{2a} \cos(ax)$ $\frac{1}{D^2+a^2} \cos(ax) = \frac{x}{2a} \sin(ax)$ $\frac{1}{(D^2+a^2)^r} \sin(ax+b) = \left(-\frac{x}{2a}\right)^r \frac{1}{r!} \sin\left(ax+b+\frac{r\pi}{2}\right)$ $\frac{1}{(D^2+a^2)^r} \cos(ax+b) = \left(-\frac{x}{2a}\right)^r \frac{1}{r!} \cos\left(ax+b+\frac{r\pi}{2}\right)$
Case 3	$\sinh(ax+b)$ or $\cosh(ax+b)$	$\frac{1}{\phi(D^2)} \sinh(ax+b) = \frac{1}{\phi(a^2)} \sinh(ax+b), \quad \phi(a^2) \neq 0$ $\frac{1}{\phi(D^2)} \cosh(ax+b) = \frac{1}{\phi(a^2)} \cosh(ax+b), \quad \phi(a^2) \neq 0$ **Case of Failure** (when $\phi(a^2) = 0$): Similar to Case 2, but with appropriate adjustments for hyperbolic functions.
Case 4	x^m, where m is a non-negative integer	$\frac{1}{\phi(D)} x^m = [\phi(D)]^{-1} x^m$ This requires expanding $[\phi(D)]^{-1}$ in ascending powers of D up to D^m and applying term by term. For special cases: $\frac{1}{D-a} x^m = \frac{1}{a} \sum_{k=0}^{m} \binom{m}{k} a^{-k} D^k x^m$
Case 5	$e^{ax}V$, where V is any function of x	$\frac{1}{\phi(D)}(e^{ax}V) = e^{ax} \cdot \frac{1}{\phi(D+a)} V$ This reduces the problem to finding $\frac{1}{\phi(D+a)} V$, which can be determined based on the form of V.
Case 6	$x^m \sin ax$ or $x^m \cos ax$	$\frac{1}{\phi(D)}(x^m \sin ax) = \text{Im}\left\{e^{iax} \cdot \frac{1}{\phi(D+ia)} x^m\right\}$ $\frac{1}{\phi(D)}(x^m \cos ax) = \text{Re}\left\{e^{iax} \cdot \frac{1}{\phi(D+ia)} x^m\right\}$ Where Im and Re denote the imaginary and real parts, respectively.
Case 7	xV, where V is any function of x	$\frac{1}{\phi(D)}(xV) = x \cdot \frac{1}{\phi(D)} V - \frac{\phi'(D)}{\phi(D)^2} V$ Where $\phi'(D)$ is the derivative of $\phi(D)$ with respect to D.

Important Notes:

1. The formulas above provide direct methods for finding particular integrals without requiring explicit integration.
2. In cases of failure, alternative formulas or approaches are provided.
3. For more complex forcing functions, these methods can often be combined or applied sequentially.
4. When using the formulas, ensure that the conditions for their validity are satisfied.

5. For practical applications, it's often helpful to verify the solution by substituting back into the original differential equation.

These shortcut methods significantly simplify the process of finding particular integrals, especially in engineering applications where certain forms of forcing functions occur frequently.

6.9 Solved Examples

In this section, we will solve a variety of differential equations using the shortcut methods for finding particular integrals discussed in the previous sections. We will begin with examples for each case individually, and then proceed to mixed examples that require the application of multiple cases.

Case 1: Particular Integral when $f(x) = e^{ax}$

Example 1.1

Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4e^{-x} \tag{6.147}$$

Solution

We need to find the general solution, which consists of the complementary function (CF) and the particular integral (PI):

$$y = y_c + y_p \tag{6.148}$$

Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 3D + 2 = 0 \tag{6.149}$$

$$\tag{6.150}$$

Solving this quadratic equation:

$$m^2 + 3m + 2 = 0 \tag{6.151}$$

$$(m+1)(m+2) = 0 \tag{6.152}$$

$$\tag{6.153}$$

Therefore, $m = -1$ or $m = -2$. The complementary function is:

$$y_c = c_1e^{-x} + c_2e^{-2x} \tag{6.154}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have $f(x) = 4e^{-x}$ and the operator $\phi(D) = D^2 + 3D + 2$.
Using the formula for the particular integral when $f(x) = e^{ax}$:

$$y_p = \frac{1}{\phi(D)}f(x) \tag{6.155}$$

$$= \frac{1}{\phi(D)}4e^{-x} \tag{6.156}$$

$$= 4 \cdot \frac{1}{\phi(D)}e^{-x} \tag{6.157}$$

$$\tag{6.158}$$

Since $f(x) = 4e^{-x}$, we have $a = -1$. Now, we need to check if $\phi(a) = 0$:

$$\phi(-1) = (-1)^2 + 3(-1) + 2 \tag{6.159}$$
$$= 1 - 3 + 2 \tag{6.160}$$
$$= 0 \tag{6.161}$$

Since $\phi(-1) = 0$, this is a case of failure. We need to use the formula for Case of Failure (i):

$$y_p = 4\frac{x}{\phi'(D)}e^{-x} \tag{6.162}$$

Find $\phi'(D)$:

$$\phi'(D) = \frac{d}{dD}(D^2 + 3D + 2) \tag{6.163}$$
$$= 2D + 3 \tag{6.164}$$

Evaluate $\phi'(-1)$:

$$\phi'(-1) = 2(-1) + 3 \tag{6.165}$$
$$= -2 + 3 \tag{6.166}$$
$$= 1 \tag{6.167}$$

Since $\phi'(-1) \neq 0$, we can proceed to find the particular integral:

$$y_p = 4\frac{x}{\phi'(-1)}e^{-x} \tag{6.168}$$
$$= 4\frac{x}{1}e^{-x} \tag{6.169}$$
$$= 4xe^{-x} \tag{6.170}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.171}$$
$$= c_1e^{-x} + c_2e^{-2x} + 4xe^{-x} \tag{6.172}$$

Therefore, the general solution of the given differential equation is:
$$y = c_1e^{-x} + c_2e^{-2x} + 4xe^{-x}$$
where c_1 and c_2 are arbitrary constants.

Example 1.2

Solve the differential equation:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 6e^{2x} + 10e^{x} \tag{6.173}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 - 4D + 4 = 0 \tag{6.174}$$
$$m^2 - 4m + 4 = 0 \tag{6.175}$$
$$(m-2)^2 = 0 \tag{6.176}$$

This gives us $m = 2$ as a repeated root. When we have a repeated root, the complementary function takes the form:

$$y_c = c_1 e^{2x} + c_2 x e^{2x} \tag{6.177}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = 6e^{2x} + 10e^x$ and the operator $\phi(D) = D^2 - 4D + 4 = (D-2)^2$.
We need to find the particular integral for each term separately and then add them.
For the term $6e^{2x}$:
Here $a = 2$. Let's check if $\phi(a) = 0$:

$$\phi(2) = (2)^2 - 4(2) + 4 \tag{6.178}$$
$$= 4 - 8 + 4 \tag{6.179}$$
$$= 0 \tag{6.180}$$

Since $\phi(2) = 0$, this is a case of failure. We need to use the formula for Case of Failure (i):

$$y_{p1} = \frac{x}{\phi'(D)} 6e^{2x} \tag{6.181}$$

Find $\phi'(D)$:

$$\phi'(D) = \frac{d}{dD}(D^2 - 4D + 4) \tag{6.182}$$
$$= 2D - 4 \tag{6.183}$$

Evaluate $\phi'(2)$:

$$\phi'(2) = 2(2) - 4 \tag{6.184}$$
$$= 4 - 4 \tag{6.185}$$
$$= 0 \tag{6.186}$$

Since $\phi'(2) = 0$ as well, this is a case of failure (ii). We need to use the formula for Case of Failure (ii):

$$y_{p1} = \frac{x^2}{\phi''(D)} 6e^{2x} \tag{6.187}$$

Find $\phi''(D)$:

$$\phi''(D) = \frac{d^2}{dD^2}(D^2 - 4D + 4) \tag{6.188}$$
$$= \frac{d}{dD}(2D - 4) \tag{6.189}$$
$$= 2 \tag{6.190}$$

Evaluate $\phi''(2)$:

$$\phi''(2) = 2 \neq 0 \tag{6.191}$$

Now we can find the particular integral for this term:

$$y_{p1} = \frac{x^2}{\phi''(2)} 6e^{2x} \tag{6.192}$$
$$= \frac{x^2}{2} 6e^{2x} \tag{6.193}$$
$$= 3x^2 e^{2x} \tag{6.194}$$

For the term $10e^x$:
Here $a = 1$. Let's check if $\phi(a) = 0$:

$$\phi(1) = (1)^2 - 4(1) + 4 \tag{6.195}$$
$$= 1 - 4 + 4 \tag{6.196}$$
$$= 1 \neq 0 \tag{6.197}$$

Since $\phi(1) \neq 0$, we can use the standard formula:

$$y_{p2} = \frac{1}{\phi(1)} 10e^x \tag{6.198}$$
$$= \frac{10}{1} e^x \tag{6.199}$$
$$= 10e^x \tag{6.200}$$

Step 3: Combine the particular integrals for both terms.

$$y_p = y_{p1} + y_{p2} \tag{6.201}$$
$$= 3x^2 e^{2x} + 10e^x \tag{6.202}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.203}$$
$$= c_1 e^{2x} + c_2 x e^{2x} + 3x^2 e^{2x} + 10e^x \tag{6.204}$$

Therefore, the general solution of the given differential equation is:
$$y = c_1 e^{2x} + c_2 x e^{2x} + 3x^2 e^{2x} + 10e^x$$
where c_1 and c_2 are arbitrary constants.
This can also be written as:
$$y = e^{2x}(c_1 + c_2 x + 3x^2) + 10e^x$$

Example 1.3

Solve the third-order differential equation:

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 7e^{-x} - 5e^{3x} \tag{6.205}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^3 - D^2 - D + 1 = 0 \tag{6.206}$$
$$m^3 - m^2 - m + 1 = 0 \tag{6.207}$$

To solve this cubic equation, let's first try to find a root by substitution. Let's try $m = 1$:

$$1^3 - 1^2 - 1 + 1 = 1 - 1 - 1 + 1 = 0 \tag{6.208}$$

So $m = 1$ is a root. We can factor the equation as:

$$(m-1)(m^2 - 0m - 1) = 0 \tag{6.209}$$
$$(m-1)(m^2 - 1) = 0 \tag{6.210}$$
$$(m-1)(m-1)(m+1) = 0 \tag{6.211}$$

Therefore, $m = 1$ is a repeated root and $m = -1$ is also a root. The complementary function is:

$$y_c = c_1e^x + c_2xe^x + c_3e^{-x} \tag{6.212}$$

where c_1, c_2, and c_3 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = 7e^{-x} - 5e^{3x}$ and the operator $\phi(D) = D^3 - D^2 - D + 1 = (D-1)^2(D+1)$.
We need to find the particular integral for each term separately and then add them.
For the term $7e^{-x}$:
Here $a = -1$. Let's check if $\phi(a) = 0$:

$$\phi(-1) = (-1)^3 - (-1)^2 - (-1) + 1 \tag{6.213}$$
$$= -1 - 1 + 1 + 1 \tag{6.214}$$
$$= 0 \tag{6.215}$$

Since $\phi(-1) = 0$, this is a case of failure. We need to use the formula for Case of Failure (i):

$$y_{p1} = \frac{x}{\phi'(D)} 7e^{-x} \tag{6.216}$$

Find $\phi'(D)$:

$$\phi'(D) = \frac{d}{dD}(D^3 - D^2 - D + 1) \tag{6.217}$$
$$= 3D^2 - 2D - 1 \tag{6.218}$$

Evaluate $\phi'(-1)$:

$$\phi'(-1) = 3(-1)^2 - 2(-1) - 1 \tag{6.219}$$
$$= 3 + 2 - 1 \tag{6.220}$$
$$= 4 \neq 0 \tag{6.221}$$

Now we can find the particular integral for this term:

$$y_{p1} = \frac{x}{\phi'(-1)} 7e^{-x} \tag{6.222}$$
$$= \frac{x}{4} 7e^{-x} \tag{6.223}$$
$$= \frac{7x}{4} e^{-x} \tag{6.224}$$

For the term $-5e^{3x}$:
Here $a = 3$. Let's check if $\phi(a) = 0$:

$$\phi(3) = (3)^3 - (3)^2 - (3) + 1 \tag{6.225}$$
$$= 27 - 9 - 3 + 1 \tag{6.226}$$
$$= 16 \neq 0 \tag{6.227}$$

Since $\phi(3) \neq 0$, we can use the standard formula:

$$y_{p2} = \frac{1}{\phi(3)}(-5e^{3x}) \tag{6.228}$$
$$= \frac{-5}{16} e^{3x} \tag{6.229}$$
$$= -\frac{5}{16} e^{3x} \tag{6.230}$$

Step 3: Combine the particular integrals for both terms.

$$y_p = y_{p1} + y_{p2} \tag{6.231}$$
$$= \frac{7x}{4} e^{-x} - \frac{5}{16} e^{3x} \tag{6.232}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.233}$$
$$= c_1 e^x + c_2 x e^x + c_3 e^{-x} + \frac{7x}{4} e^{-x} - \frac{5}{16} e^{3x} \tag{6.234}$$

The terms with e^{-x} can be combined:

$$y = c_1 e^x + c_2 x e^x + \left(c_3 + \frac{7x}{4}\right) e^{-x} - \frac{5}{16} e^{3x} \tag{6.235}$$
$$= c_1 e^x + c_2 x e^x + c_3 e^{-x} + \frac{7x}{4} e^{-x} - \frac{5}{16} e^{3x} \tag{6.236}$$

Therefore, the general solution of the given differential equation is:
$$y = c_1 e^x + c_2 x e^x + c_3 e^{-x} + \tfrac{7x}{4} e^{-x} - \tfrac{5}{16} e^{3x}$$
where c_1, c_2, and c_3 are arbitrary constants.
This can also be written in a more factored form:
$$y = e^x(c_1 + c_2 x) + e^{-x}\left(c_3 + \tfrac{7x}{4}\right) - \tfrac{5}{16} e^{3x}$$

Example 1.4

Solve the differential equation with a non-integer coefficient:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 8e^{\frac{1}{2}x} + 3e^{-x} \tag{6.237}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.

Step 1: Find the complementary function y_c.

The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + D - 2 = 0 \tag{6.238}$$
$$m^2 + m - 2 = 0 \tag{6.239}$$

We can factor this equation:

$$m^2 + m - 2 = 0 \tag{6.240}$$
$$m^2 + 2m - m - 2 = 0 \tag{6.241}$$
$$m(m + 2) - 1(m + 2) = 0 \tag{6.242}$$
$$(m + 2)(m - 1) = 0 \tag{6.243}$$

Therefore, $m = -2$ or $m = 1$. The complementary function is:

$$y_c = c_1e^{-2x} + c_2e^x \tag{6.244}$$

where c_1 and c_2 are arbitrary constants.

Step 2: Find the particular integral y_p.

We have the forcing function $f(x) = 8e^{\frac{1}{2}x} + 3e^{-x}$ and the operator $\phi(D) = D^2 + D - 2 = (D + 2)(D - 1)$.

We need to find the particular integral for each term separately and then add them.

For the term $8e^{\frac{1}{2}x}$:

Here $a = \frac{1}{2}$. Let's check if $\phi(a) = 0$:

$$\phi\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right) - 2 \tag{6.245}$$
$$= \frac{1}{4} + \frac{1}{2} - 2 \tag{6.246}$$
$$= \frac{1}{4} + \frac{2}{4} - \frac{8}{4} \tag{6.247}$$
$$= \frac{3 - 8}{4} \tag{6.248}$$
$$= -\frac{5}{4} \neq 0 \tag{6.249}$$

Since $\phi\left(\frac{1}{2}\right) \neq 0$, we can use the standard formula:

$$y_{p1} = \frac{1}{\phi\left(\frac{1}{2}\right)} 8e^{\frac{1}{2}x} \tag{6.250}$$
$$= \frac{8}{-\frac{5}{4}} e^{\frac{1}{2}x} \tag{6.251}$$
$$= \frac{8 \cdot (-4)}{-5} e^{\frac{1}{2}x} \tag{6.252}$$
$$= \frac{-32}{-5} e^{\frac{1}{2}x} \tag{6.253}$$
$$= \frac{32}{5} e^{\frac{1}{2}x} \tag{6.254}$$

For the term $3e^{-x}$:
Here $a = -1$. Let's check if $\phi(a) = 0$:

$$\phi(-1) = (-1)^2 + (-1) - 2 \tag{6.255}$$
$$= 1 - 1 - 2 \tag{6.256}$$
$$= -2 \neq 0 \tag{6.257}$$

Since $\phi(-1) \neq 0$, we can use the standard formula:

$$y_{p2} = \frac{1}{\phi(-1)} 3e^{-x} \tag{6.258}$$
$$= \frac{3}{-2} e^{-x} \tag{6.259}$$
$$= -\frac{3}{2} e^{-x} \tag{6.260}$$

Step 3: Combine the particular integrals for both terms.

$$y_p = y_{p1} + y_{p2} \tag{6.261}$$
$$= \frac{32}{5} e^{\frac{1}{2}x} - \frac{3}{2} e^{-x} \tag{6.262}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.263}$$
$$= c_1 e^{-2x} + c_2 e^{x} + \frac{32}{5} e^{\frac{1}{2}x} - \frac{3}{2} e^{-x} \tag{6.264}$$

Therefore, the general solution of the given differential equation is:
$y = c_1 e^{-2x} + c_2 e^{x} + \frac{32}{5} e^{\frac{1}{2}x} - \frac{3}{2} e^{-x}$
where c_1 and c_2 are arbitrary constants.

Example 1.5

Solve the differential equation with complex roots in the auxiliary equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10e^{-x} \tag{6.265}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 2D + 5 = 0 \tag{6.266}$$
$$m^2 + 2m + 5 = 0 \tag{6.267}$$

Using the quadratic formula:

$$m = \frac{-2 \pm \sqrt{4-20}}{2} \tag{6.268}$$
$$= \frac{-2 \pm \sqrt{-16}}{2} \tag{6.269}$$
$$= \frac{-2 \pm 4i}{2} \tag{6.270}$$
$$= -1 \pm 2i \tag{6.271}$$

So $m_1 = -1 + 2i$ and $m_2 = -1 - 2i$ are complex conjugate roots. When we have complex roots, the complementary function takes the form:

$$y_c = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)) \tag{6.272}$$

where α is the real part and β is the absolute value of the imaginary part of the complex roots.
In this case, $\alpha = -1$ and $\beta = 2$, so:

$$y_c = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{6.273}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = 10e^{-x}$ and the operator $\phi(D) = D^2 + 2D + 5$.
For the term $10e^{-x}$, we have $a = -1$. Let's check if $\phi(a) = 0$:

$$\phi(-1) = (-1)^2 + 2(-1) + 5 \tag{6.274}$$
$$= 1 - 2 + 5 \tag{6.275}$$
$$= 4 \neq 0 \tag{6.276}$$

Since $\phi(-1) \neq 0$, we can use the standard formula:

$$y_p = \frac{1}{\phi(-1)} 10e^{-x} \tag{6.277}$$
$$= \frac{10}{4} e^{-x} \tag{6.278}$$
$$= \frac{5}{2} e^{-x} \tag{6.279}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.280}$$
$$= e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) + \frac{5}{2} e^{-x} \tag{6.281}$$
$$= e^{-x}\left(c_1 \cos(2x) + c_2 \sin(2x) + \frac{5}{2}\right) \tag{6.282}$$

Therefore, the general solution of the given differential equation is:

$$y = e^{-x}\left(c_1\cos(2x) + c_2\sin(2x) + \tfrac{5}{2}\right)$$

where c_1 and c_2 are arbitrary constants.

Example 1.6

Solve the following system of first-order differential equations which can be converted to a higher-order differential equation:

$$\frac{dx}{dt} = 3x - 2y + 4e^{2t} \tag{6.283}$$

$$\frac{dy}{dt} = 2x + y - 2e^{2t} \tag{6.284}$$

Solution

To solve this system, we'll eliminate one variable to get a second-order differential equation in terms of the other variable. Then we'll solve that equation and use the original system to find the second variable.

Step 1: Eliminate y to obtain a single equation in x.

From the first equation, we have:

$$-2y = \frac{dx}{dt} - 3x - 4e^{2t} \tag{6.285}$$

$$y = -\frac{1}{2}\frac{dx}{dt} + \frac{3}{2}x + 2e^{2t} \tag{6.286}$$

Differentiating this expression for y with respect to t:

$$\frac{dy}{dt} = -\frac{1}{2}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + 4e^{2t} \tag{6.287}$$

Now, substitute the expressions for y and $\frac{dy}{dt}$ into the second equation of the system:

$$-\frac{1}{2}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + 4e^{2t} = 2x + \left(-\frac{1}{2}\frac{dx}{dt} + \frac{3}{2}x + 2e^{2t}\right) - 2e^{2t} \tag{6.288}$$

$$-\frac{1}{2}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + 4e^{2t} = 2x - \frac{1}{2}\frac{dx}{dt} + \frac{3}{2}x + 2e^{2t} - 2e^{2t} \tag{6.289}$$

$$-\frac{1}{2}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + 4e^{2t} = \frac{7}{2}x - \frac{1}{2}\frac{dx}{dt} + 0 \tag{6.290}$$

$$-\frac{1}{2}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + \frac{1}{2}\frac{dx}{dt} - \frac{7}{2}x = -4e^{2t} \tag{6.291}$$

$$-\frac{1}{2}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - \frac{7}{2}x = -4e^{2t} \tag{6.292}$$

$$\tag{6.293}$$

Multiplying throughout by -2, we get:

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 7x = 8e^{2t} \tag{6.294}$$

This is a second-order linear differential equation with constant coefficients.

Step 2: Find the complementary function x_c.

The auxiliary equation is:

$$m^2 - 4m + 7 = 0 \tag{6.295}$$

Using the quadratic formula:

$$m = \frac{4 \pm \sqrt{16-28}}{2} \tag{6.296}$$
$$= \frac{4 \pm \sqrt{-12}}{2} \tag{6.297}$$
$$= \frac{4 \pm 2\sqrt{3}i}{2} \tag{6.298}$$
$$= 2 \pm \sqrt{3}i \tag{6.299}$$

With complex roots $m_1 = 2 + \sqrt{3}i$ and $m_2 = 2 - \sqrt{3}i$, the complementary function is:

$$x_c = e^{2t}(c_1 \cos\left(\sqrt{3}t\right) + c_2 \sin\left(\sqrt{3}t\right)) \tag{6.300}$$

where c_1 and c_2 are arbitrary constants.
Step 3: Find the particular integral x_p.
For the term $8e^{2t}$, we have $a = 2$. Let's check if $\phi(a) = 0$:

$$\phi(2) = (2)^2 - 4(2) + 7 \tag{6.301}$$
$$= 4 - 8 + 7 \tag{6.302}$$
$$= 3 \neq 0 \tag{6.303}$$

Since $\phi(2) \neq 0$, we can use the standard formula:

$$x_p = \frac{1}{\phi(2)} 8e^{2t} \tag{6.304}$$
$$= \frac{8}{3} e^{2t} \tag{6.305}$$

Step 4: Write the general solution for x.

$$x(t) = x_c + x_p \tag{6.306}$$
$$= e^{2t}(c_1 \cos\left(\sqrt{3}t\right) + c_2 \sin\left(\sqrt{3}t\right)) + \frac{8}{3} e^{2t} \tag{6.307}$$
$$= e^{2t}\left(c_1 \cos\left(\sqrt{3}t\right) + c_2 \sin\left(\sqrt{3}t\right) + \frac{8}{3}\right) \tag{6.308}$$

Step 5: Find $y(t)$ using the relation we derived earlier.

$$y = -\frac{1}{2}\frac{dx}{dt} + \frac{3}{2}x + 2e^{2t} \tag{6.309}$$

First, let's find $\frac{dx}{dt}$:

$$x(t) = e^{2t}\left(c_1 \cos\left(\sqrt{3}t\right) + c_2 \sin\left(\sqrt{3}t\right) + \frac{8}{3}\right) \tag{6.310}$$

$$\frac{dx}{dt} = 2e^{2t}\left(c_1 \cos\left(\sqrt{3}t\right) + c_2 \sin\left(\sqrt{3}t\right) + \frac{8}{3}\right) + e^{2t}\left(-c_1\sqrt{3} \sin\left(\sqrt{3}t\right) + c_2\sqrt{3} \cos\left(\sqrt{3}t\right)\right) \tag{6.311}$$

$$= 2e^{2t}\left(c_1 \cos\left(\sqrt{3}t\right) + c_2 \sin\left(\sqrt{3}t\right) + \frac{8}{3}\right) + e^{2t}\sqrt{3}\left(-c_1 \sin\left(\sqrt{3}t\right) + c_2 \cos\left(\sqrt{3}t\right)\right) \tag{6.312}$$

Now, substitute into the expression for y:

$$y = -\frac{1}{2}\frac{dx}{dt} + \frac{3}{2}x + 2e^{2t} \tag{6.313}$$

$$= -\frac{1}{2}\left[2e^{2t}\left(c_1\cos\left(\sqrt{3}t\right) + c_2\sin\left(\sqrt{3}t\right) + \frac{8}{3}\right) + e^{2t}\sqrt{3}\left(-c_1\sin\left(\sqrt{3}t\right) + c_2\cos\left(\sqrt{3}t\right)\right)\right] \tag{6.314}$$

$$+\frac{3}{2}e^{2t}\left(c_1\cos\left(\sqrt{3}t\right) + c_2\sin\left(\sqrt{3}t\right) + \frac{8}{3}\right) + 2e^{2t} \tag{6.315}$$

Simplifying:

$$y = -e^{2t}\left(c_1\cos\left(\sqrt{3}t\right) + c_2\sin\left(\sqrt{3}t\right) + \frac{8}{3}\right) - \frac{1}{2}e^{2t}\sqrt{3}\left(-c_1\sin\left(\sqrt{3}t\right) + c_2\cos\left(\sqrt{3}t\right)\right) \tag{6.316}$$

$$+\frac{3}{2}e^{2t}\left(c_1\cos\left(\sqrt{3}t\right) + c_2\sin\left(\sqrt{3}t\right) + \frac{8}{3}\right) + 2e^{2t} \tag{6.317}$$

$$= e^{2t}\left[\left(-1+\frac{3}{2}\right)c_1\cos\left(\sqrt{3}t\right) + \left(-1+\frac{3}{2}\right)c_2\sin\left(\sqrt{3}t\right) + \left(-1+\frac{3}{2}\right)\frac{8}{3}\right] \tag{6.318}$$

$$+ e^{2t}\left[\frac{\sqrt{3}}{2}c_1\sin\left(\sqrt{3}t\right) - \frac{\sqrt{3}}{2}c_2\cos\left(\sqrt{3}t\right)\right] + 2e^{2t} \tag{6.319}$$

$$= e^{2t}\left[\frac{1}{2}c_1\cos\left(\sqrt{3}t\right) + \frac{1}{2}c_2\sin\left(\sqrt{3}t\right) + \frac{4}{3} + \frac{\sqrt{3}}{2}c_1\sin\left(\sqrt{3}t\right) - \frac{\sqrt{3}}{2}c_2\cos\left(\sqrt{3}t\right) + 2\right] \tag{6.320}$$

$$= e^{2t}\left[\left(\frac{1}{2}c_1 - \frac{\sqrt{3}}{2}c_2\right)\cos\left(\sqrt{3}t\right) + \left(\frac{1}{2}c_2 + \frac{\sqrt{3}}{2}c_1\right)\sin\left(\sqrt{3}t\right) + \frac{4}{3} + 2\right] \tag{6.321}$$

$$= e^{2t}\left[\left(\frac{1}{2}c_1 - \frac{\sqrt{3}}{2}c_2\right)\cos\left(\sqrt{3}t\right) + \left(\frac{1}{2}c_2 + \frac{\sqrt{3}}{2}c_1\right)\sin\left(\sqrt{3}t\right) + \frac{10}{3}\right] \tag{6.322}$$

Let's define new constants:

$$k_1 = \frac{1}{2}c_1 - \frac{\sqrt{3}}{2}c_2 \tag{6.323}$$

$$k_2 = \frac{1}{2}c_2 + \frac{\sqrt{3}}{2}c_1 \tag{6.324}$$

Then:

$$y(t) = e^{2t}\left(k_1\cos\left(\sqrt{3}t\right) + k_2\sin\left(\sqrt{3}t\right) + \frac{10}{3}\right) \tag{6.325}$$

Therefore, the general solution of the given system of differential equations is:

$$x(t) = e^{2t}\left(c_1\cos\left(\sqrt{3}t\right) + c_2\sin\left(\sqrt{3}t\right) + \frac{8}{3}\right) \tag{6.326}$$

$$y(t) = e^{2t}\left(k_1\cos\left(\sqrt{3}t\right) + k_2\sin\left(\sqrt{3}t\right) + \frac{10}{3}\right) \tag{6.327}$$

where c_1 and c_2 are arbitrary constants, and $k_1 = \frac{1}{2}c_1 - \frac{\sqrt{3}}{2}c_2$ and $k_2 = \frac{1}{2}c_2 + \frac{\sqrt{3}}{2}c_1$.

Example 1.7

Solve the differential equation:

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x} \tag{6.328}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.

Step 1: Find the complementary function y_c.

The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 - 7D + 6 = 0 \tag{6.329}$$
$$m^2 - 7m + 6 = 0 \tag{6.330}$$

We can factor this equation:

$$m^2 - 7m + 6 = 0 \tag{6.331}$$
$$m^2 - 6m - m + 6 = 0 \tag{6.332}$$
$$m(m-6) - 1(m-6) = 0 \tag{6.333}$$
$$(m-6)(m-1) = 0 \tag{6.334}$$

Therefore, $m = 6$ or $m = 1$. The complementary function is:

$$y_c = c_1 e^{6x} + c_2 e^{x} \tag{6.335}$$

where c_1 and c_2 are arbitrary constants.

Step 2: Find the particular integral y_p.

We have the forcing function $f(x) = e^{2x}$ and the operator $\phi(D) = D^2 - 7D + 6 = (D-6)(D-1)$.

For the term e^{2x}, we have $a = 2$. Let's check if $\phi(a) = 0$:

$$\phi(2) = (2)^2 - 7(2) + 6 \tag{6.336}$$
$$= 4 - 14 + 6 \tag{6.337}$$
$$= -4 \neq 0 \tag{6.338}$$

Since $\phi(2) \neq 0$, we can use the standard formula:

$$y_p = \frac{1}{\phi(2)} e^{2x} \tag{6.339}$$
$$= \frac{1}{-4} e^{2x} \tag{6.340}$$
$$= -\frac{1}{4} e^{2x} \tag{6.341}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.342}$$
$$= c_1 e^{6x} + c_2 e^{x} - \frac{1}{4} e^{2x} \tag{6.343}$$

Therefore, the general solution of the given differential equation is:

$$y = c_1 e^{6x} + c_2 e^{x} - \tfrac{1}{4} e^{2x}$$

where c_1 and c_2 are arbitrary constants.

Example 1.8

Solve the differential equation:

$$\frac{d^2y}{dx^2} - 4y = (1 + e^x)^2 + 3 \tag{6.344}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 - 4 = 0 \tag{6.345}$$
$$m^2 - 4 = 0 \tag{6.346}$$
$$m^2 = 4 \tag{6.347}$$
$$m = \pm 2 \tag{6.348}$$

Therefore, $m = 2$ or $m = -2$. The complementary function is:

$$y_c = c_1 e^{2x} + c_2 e^{-2x} \tag{6.349}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
First, let's expand the forcing function:

$$(1 + e^x)^2 + 3 = 1 + 2e^x + e^{2x} + 3 \tag{6.350}$$
$$= 4 + 2e^x + e^{2x} \tag{6.351}$$

So we have $f(x) = 4 + 2e^x + e^{2x}$ and the operator $\phi(D) = D^2 - 4$.
We need to find the particular integral for each term separately and then add them.
For the constant term 4: Using the formula for a constant term:

$$y_{p1} = \frac{1}{\phi(D)} 4 \tag{6.352}$$
$$= \frac{4}{\phi(0)} \tag{6.353}$$
$$= \frac{4}{0^2 - 4} \tag{6.354}$$
$$= \frac{4}{-4} \tag{6.355}$$
$$= -1 \tag{6.356}$$

For the term $2e^x$: Here $a = 1$. Let's check if $\phi(a) = 0$:

$$\phi(1) = (1)^2 - 4 \tag{6.357}$$
$$= 1 - 4 \tag{6.358}$$
$$= -3 \neq 0 \tag{6.359}$$

Since $\phi(1) \neq 0$, we can use the standard formula:

$$y_{p2} = \frac{1}{\phi(1)} 2e^x \tag{6.360}$$
$$= \frac{2}{-3} e^x \tag{6.361}$$
$$= -\frac{2}{3} e^x \tag{6.362}$$

For the term e^{2x}: Here $a = 2$. Let's check if $\phi(a) = 0$:

$$\phi(2) = (2)^2 - 4 \tag{6.363}$$
$$= 4 - 4 \tag{6.364}$$
$$= 0 \tag{6.365}$$

Since $\phi(2) = 0$, this is a case of failure. We need to use the formula for Case of Failure (i):

$$y_{p3} = \frac{x}{\phi'(D)} e^{2x} \tag{6.366}$$

Find $\phi'(D)$:

$$\phi'(D) = \frac{d}{dD}(D^2 - 4) \tag{6.367}$$
$$= 2D \tag{6.368}$$

Evaluate $\phi'(2)$:

$$\phi'(2) = 2(2) \tag{6.369}$$
$$= 4 \neq 0 \tag{6.370}$$

Now we can find the particular integral for this term:

$$y_{p3} = \frac{x}{\phi'(2)} e^{2x} \tag{6.371}$$
$$= \frac{x}{4} e^{2x} \tag{6.372}$$
$$= \frac{x}{4} e^{2x} \tag{6.373}$$

Step 3: Combine the particular integrals for all terms.

$$y_p = y_{p1} + y_{p2} + y_{p3} \tag{6.374}$$
$$= -1 - \frac{2}{3} e^x + \frac{x}{4} e^{2x} \tag{6.375}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.376}$$
$$= c_1 e^{2x} + c_2 e^{-2x} - 1 - \frac{2}{3} e^x + \frac{x}{4} e^{2x} \tag{6.377}$$
$$= c_1 e^{2x} + c_2 e^{-2x} - \frac{2}{3} e^x - 1 + \frac{x}{4} e^{2x} \tag{6.378}$$

Therefore, the general solution of the given differential equation is:

$$y = c_1 e^{2x} + c_2 e^{-2x} - \tfrac{2}{3} e^x - 1 + \tfrac{x}{4} e^{2x}$$

which can be rewritten as:

$$y = e^{2x}\left(c_1 + \tfrac{x}{4}\right) + c_2 e^{-2x} - \tfrac{2}{3} e^x - 1$$

where c_1 and c_2 are arbitrary constants.

Example 1.9

Solve the differential equation:

$$(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^x + 3e^{-x} + 2 \tag{6.379}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
We need to solve the homogeneous equation:

$$(D^3 - 5D^2 + 8D - 4)y = 0 \tag{6.380}$$

The auxiliary equation is:

$$m^3 - 5m^2 + 8m - 4 = 0 \tag{6.381}$$

Let's try some integer values to find a root. For $m = 1$:

$$1^3 - 5(1)^2 + 8(1) - 4 = 1 - 5 + 8 - 4 \tag{6.382}$$
$$= 0 \tag{6.383}$$

So $m = 1$ is a root. We can factor the auxiliary equation as:

$$(m-1)(m^2 - 4m + 4) = 0 \tag{6.384}$$
$$(m-1)(m-2)^2 = 0 \tag{6.385}$$

Therefore, $m = 1$ and $m = 2$ (a repeated root) are the roots of the auxiliary equation. The complementary function is:

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x} \tag{6.386}$$

where c_1, c_2, and c_3 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = e^{2x} + 2e^x + 3e^{-x} + 2$ and the operator $\phi(D) = D^3 - 5D^2 + 8D - 4 = (D-1)(D-2)^2$.
We need to find the particular integral for each term separately and then add them.
For the term e^{2x}:
Here $a = 2$. Let's check if $\phi(a) = 0$:

$$\phi(2) = (2)^3 - 5(2)^2 + 8(2) - 4 \tag{6.387}$$
$$= 8 - 20 + 16 - 4 \tag{6.388}$$
$$= 0 \tag{6.389}$$

Since $\phi(2) = 0$, this is a case of failure. We need to use the formula for Case of Failure (i):

$$y_{p1} = \frac{x}{\phi'(D)} e^{2x} \tag{6.390}$$

Find $\phi'(D)$:

$$\phi'(D) = \frac{d}{dD}(D^3 - 5D^2 + 8D - 4) \tag{6.391}$$
$$= 3D^2 - 10D + 8 \tag{6.392}$$

Evaluate $\phi'(2)$:

$$\begin{aligned}\phi'(2) &= 3(2)^2 - 10(2) + 8 && (6.393)\\ &= 3(4) - 20 + 8 && (6.394)\\ &= 12 - 20 + 8 && (6.395)\\ &= 0 && (6.396)\end{aligned}$$

Since $\phi'(2) = 0$ as well, this is a case of failure (ii). We need to use the formula for Case of Failure (ii):

$$y_{p1} = \frac{x^2}{\phi''(D)} e^{2x} \tag{6.397}$$

Find $\phi''(D)$:

$$\begin{aligned}\phi''(D) &= \frac{d^2}{dD^2}(D^3 - 5D^2 + 8D - 4) && (6.398)\\ &= \frac{d}{dD}(3D^2 - 10D + 8) && (6.399)\\ &= 6D - 10 && (6.400)\end{aligned}$$

Evaluate $\phi''(2)$:

$$\begin{aligned}\phi''(2) &= 6(2) - 10 && (6.401)\\ &= 12 - 10 && (6.402)\\ &= 2 \neq 0 && (6.403)\end{aligned}$$

Now we can find the particular integral for this term:

$$\begin{aligned}y_{p1} &= \frac{x^2}{\phi''(2)} e^{2x} && (6.404)\\ &= \frac{x^2}{2} e^{2x} && (6.405)\end{aligned}$$

For the term $2e^x$:
Here $a = 1$. Let's check if $\phi(a) = 0$:

$$\begin{aligned}\phi(1) &= (1)^3 - 5(1)^2 + 8(1) - 4 && (6.406)\\ &= 1 - 5 + 8 - 4 && (6.407)\\ &= 0 && (6.408)\end{aligned}$$

Since $\phi(1) = 0$, this is a case of failure. We need to use the formula for Case of Failure (i):

$$y_{p2} = \frac{x}{\phi'(D)} 2e^x \tag{6.409}$$

Evaluate $\phi'(1)$:

$$\begin{aligned}\phi'(1) &= 3(1)^2 - 10(1) + 8 && (6.410)\\ &= 3 - 10 + 8 && (6.411)\\ &= 1 \neq 0 && (6.412)\end{aligned}$$

Now we can find the particular integral for this term:

$$y_{p2} = \frac{x}{\phi'(1)} 2e^x \tag{6.413}$$
$$= \frac{2x}{1} e^x \tag{6.414}$$
$$= 2xe^x \tag{6.415}$$

For the term $3e^{-x}$:
Here $a = -1$. Let's check if $\phi(a) = 0$:

$$\phi(-1) = (-1)^3 - 5(-1)^2 + 8(-1) - 4 \tag{6.416}$$
$$= -1 - 5 - 8 - 4 \tag{6.417}$$
$$= -18 \neq 0 \tag{6.418}$$

Since $\phi(-1) \neq 0$, we can use the standard formula:

$$y_{p3} = \frac{1}{\phi(-1)} 3e^{-x} \tag{6.419}$$
$$= \frac{3}{-18} e^{-x} \tag{6.420}$$
$$= -\frac{1}{6} e^{-x} \tag{6.421}$$

For the constant term 2:
Using the formula for a constant term:

$$y_{p4} = \frac{1}{\phi(D)} 2 \tag{6.422}$$
$$= \frac{2}{\phi(0)} \tag{6.423}$$
$$= \frac{2}{0^3 - 5(0)^2 + 8(0) - 4} \tag{6.424}$$
$$= \frac{2}{-4} \tag{6.425}$$
$$= -\frac{1}{2} \tag{6.426}$$

Step 3: Combine the particular integrals for all terms.

$$y_p = y_{p1} + y_{p2} + y_{p3} + y_{p4} \tag{6.427}$$
$$= \frac{x^2}{2} e^{2x} + 2xe^x - \frac{1}{6} e^{-x} - \frac{1}{2} \tag{6.428}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.429}$$
$$= c_1 e^x + c_2 e^{2x} + c_3 x e^{2x} + \frac{x^2}{2} e^{2x} + 2xe^x - \frac{1}{6} e^{-x} - \frac{1}{2} \tag{6.430}$$

This can be reorganized to group similar terms:

$$y = e^x (c_1 + 2x) + e^{2x} \left(c_2 + c_3 x + \frac{x^2}{2} \right) - \frac{1}{6} e^{-x} - \frac{1}{2} \tag{6.431}$$

Therefore, the general solution of the given differential equation is:

$$y = e^x(c_1 + 2x) + e^{2x}\left(c_2 + c_3 x + \frac{x^2}{2}\right) - \frac{1}{6}e^{-x} - \frac{1}{2}$$

where c_1, c_2, and c_3 are arbitrary constants.

Example 1.10

Solve the differential equation:

$$(D^4 - 4D^3 + 6D^2 - 4D + 1)y = e^x + 2^x + \frac{1}{3} \tag{6.432}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.

Step 1: Find the complementary function y_c.

We need to solve the homogeneous equation:

$$(D^4 - 4D^3 + 6D^2 - 4D + 1)y = 0 \tag{6.433}$$

The auxiliary equation is:

$$m^4 - 4m^3 + 6m^2 - 4m + 1 = 0 \tag{6.434}$$

Let's try to factorize this by noticing the pattern. This looks like the expansion of $(m-1)^4$:

$$(m-1)^4 = m^4 - 4m^3 + 6m^2 - 4m + 1 \tag{6.435}$$

So the auxiliary equation is:

$$(m-1)^4 = 0 \tag{6.436}$$

Therefore, $m = 1$ is a root of multiplicity 4. The complementary function is:

$$y_c = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 x^3 e^x \tag{6.437}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

Step 2: Find the particular integral y_p.

We have the forcing function $f(x) = e^x + 2^x + \frac{1}{3}$ and the operator $\phi(D) = D^4 - 4D^3 + 6D^2 - 4D + 1 = (D-1)^4$.

We need to find the particular integral for each term separately and then add them.

For the term e^x:

Here $a = 1$. Let's check if $\phi(a) = 0$:

$$\phi(1) = (1)^4 - 4(1)^3 + 6(1)^2 - 4(1) + 1 \tag{6.438}$$
$$= 1 - 4 + 6 - 4 + 1 \tag{6.439}$$
$$= 0 \tag{6.440}$$

Since $\phi(1) = 0$, this is a case of failure. In fact, since $\phi(D) = (D-1)^4$, we know that $\phi(1) = \phi'(1) = \phi''(1) = \phi'''(1) = 0$ and $\phi^{(4)}(1) = 4! = 24$. This is a case of failure of order 4. We need to use the formula for Case of Failure (iv):

$$y_{p1} = \frac{x^4}{\phi^{(4)}(1)} e^x \tag{6.441}$$
$$= \frac{x^4}{24} e^x \tag{6.442}$$

For the term 2^x:
For $2^x = e^{x\ln 2}$, we have $a = \ln 2$. Let's check if $\phi(a) = 0$:

$$\phi(\ln 2) = (\ln 2)^4 - 4(\ln 2)^3 + 6(\ln 2)^2 - 4(\ln 2) + 1 \tag{6.443}$$

Since $\ln 2 \approx 0.693$ and $\ln 2 \neq 1$, we know that $\phi(\ln 2) \neq 0$. We can use the standard formula:

$$y_{p2} = \frac{1}{\phi(\ln 2)} 2^x \tag{6.444}$$

To find $\phi(\ln 2)$, let's compute:

$$\begin{aligned}
\phi(\ln 2) &= (\ln 2)^4 - 4(\ln 2)^3 + 6(\ln 2)^2 - 4(\ln 2) + 1 && (6.445)\\
&\approx (0.693)^4 - 4(0.693)^3 + 6(0.693)^2 - 4(0.693) + 1 && (6.446)\\
&\approx 0.230 - 4 \cdot 0.333 + 6 \cdot 0.480 - 4 \cdot 0.693 + 1 && (6.447)\\
&\approx 0.230 - 1.332 + 2.880 - 2.772 + 1 && (6.448)\\
&\approx 0.006 && (6.449)
\end{aligned}$$

While we could continue with this approximate value, let's use a more elegant approach. We can write:

$$\phi(D) = (D-1)^4 \tag{6.450}$$

$$\tag{6.451}$$

So:

$$\phi(\ln 2) = (\ln 2 - 1)^4 \tag{6.452}$$

Therefore:

$$y_{p2} = \frac{1}{(\ln 2 - 1)^4} 2^x \tag{6.453}$$

For the constant term $\frac{1}{3}$:
Using the formula for a constant term:

$$\begin{aligned}
y_{p3} &= \frac{1}{\phi(D)} \frac{1}{3} && (6.454)\\
&= \frac{1}{3} \cdot \frac{1}{\phi(0)} && (6.455)\\
&= \frac{1}{3} \cdot \frac{1}{0^4 - 4(0)^3 + 6(0)^2 - 4(0) + 1} && (6.456)\\
&= \frac{1}{3} \cdot \frac{1}{1} && (6.457)\\
&= \frac{1}{3} && (6.458)
\end{aligned}$$

Step 3: Combine the particular integrals for all terms.

$$\begin{aligned}
y_p &= y_{p1} + y_{p2} + y_{p3} && (6.459)\\
&= \frac{x^4}{24} e^x + \frac{1}{(\ln 2 - 1)^4} 2^x + \frac{1}{3} && (6.460)
\end{aligned}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.461}$$
$$= c_1e^x + c_2xe^x + c_3x^2e^x + c_4x^3e^x + \frac{x^4}{24}e^x + \frac{1}{(\ln 2 - 1)^4}2^x + \frac{1}{3} \tag{6.462}$$

We can factor the terms with e^x:

$$y = e^x \left(c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{x^4}{24}\right) + \frac{1}{(\ln 2 - 1)^4}2^x + \frac{1}{3} \tag{6.463}$$

Therefore, the general solution of the given differential equation is:

$$y = e^x \left(c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{x^4}{24}\right) + \frac{1}{(\ln 2-1)^4}2^x + \frac{1}{3}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.
Note: We can simplify $\frac{1}{(\ln 2-1)^4}$ to a numerical value if needed, but leaving it in this form maintains the algebraic precision of the solution.

Case 2: Particular Integral when $f(x) = \sin(ax + b)$ **or** $\cos(ax + b)$

Example 2.1

Solve the differential equation:

$$\frac{d^2y}{dx^2} + 4y = 8\sin(2x) \tag{6.464}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 4 = 0 \tag{6.465}$$
$$m^2 + 4 = 0 \tag{6.466}$$
$$m^2 = -4 \tag{6.467}$$
$$m = \pm 2i \tag{6.468}$$

Since we have complex roots $m = \pm 2i$, the complementary function is:

$$y_c = c_1\cos(2x) + c_2\sin(2x) \tag{6.469}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = 8\sin(2x)$ and the operator $\phi(D) = D^2 + 4$.
For a trigonometric forcing function of the form $\sin(ax + b)$ or $\cos(ax + b)$, we use the formula:

$$y_p = \frac{1}{\phi(-a^2)}\sin(ax + b) \tag{6.470}$$

where we substitute $D^2 = -a^2$ in the operator $\phi(D)$.

In our case, $a = 2$ and $b = 0$ for $8\sin(2x)$. Let's check if $\phi(-a^2) = 0$:

$$\phi(-a^2) = \phi(-2^2) \quad (6.471)$$
$$= \phi(-4) \quad (6.472)$$
$$= (-4) + 4 \quad (6.473)$$
$$= 0 \quad (6.474)$$

Since $\phi(-a^2) = 0$, this is a case of failure. We need to use the formula for Case of Failure 1:

$$y_p = \frac{x}{\phi'(D^2)} \sin(ax + b) \quad (6.475)$$

Find $\phi'(D)$ with respect to D:

$$\phi'(D) = \frac{d}{dD}(D^2 + 4) \quad (6.476)$$
$$= 2D \quad (6.477)$$

Now we can find the particular integral:

$$y_p = \frac{x}{\phi'(D)} \sin(2x) \quad (6.478)$$
$$= \frac{x}{2D} \sin(2x) \quad (6.479)$$

We have D in the denominator, so we need to rationalize:

$$\frac{x}{2D} \sin(2x) = \frac{x}{2} \cdot \frac{1}{D} \sin(2x) \quad (6.480)$$

Using the property that $\frac{1}{D}\sin(2x) = \frac{1}{2}\cos(2x)$ (integrating $\sin(2x)$):

$$\frac{x}{2} \cdot \frac{1}{D} \sin(2x) = \frac{x}{2} \cdot \frac{-1}{2} \cos(2x) \quad (6.481)$$
$$= -\frac{x}{4} \cos(2x) \quad (6.482)$$

But we need to multiply by 8 (the coefficient of the original forcing function):

$$y_p = -\frac{8x}{4} \cos(2x) \quad (6.483)$$
$$= -2x \cos(2x) \quad (6.484)$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \quad (6.485)$$
$$= c_1 \cos(2x) + c_2 \sin(2x) - 2x \cos(2x) \quad (6.486)$$

Therefore, the general solution of the given differential equation is:

$$y = c_1 \cos(2x) + c_2 \sin(2x) - 2x \cos(2x)$$

which can be rewritten as:

$$y = \cos(2x)(c_1 - 2x) + c_2 \sin(2x)$$

where c_1 and c_2 are arbitrary constants.

Example 2.2

Solve the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 3\cos(3x - \pi/4) \tag{6.487}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 2D + 5 = 0 \tag{6.488}$$
$$m^2 + 2m + 5 = 0 \tag{6.489}$$

Using the quadratic formula:

$$m = \frac{-2 \pm \sqrt{4-20}}{2} \tag{6.490}$$
$$= \frac{-2 \pm \sqrt{-16}}{2} \tag{6.491}$$
$$= \frac{-2 \pm 4i}{2} \tag{6.492}$$
$$= -1 \pm 2i \tag{6.493}$$

Since we have complex roots $m = -1 \pm 2i$, the complementary function is:

$$y_c = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{6.494}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = 3\cos(3x - \pi/4)$ and the operator $\phi(D) = D^2+2D+5$.
For a trigonometric forcing function of the form $\cos(ax + b)$, we use the formula:

$$y_p = \frac{1}{\phi(-a^2)} \cos(ax + b) \tag{6.495}$$

where we substitute $D^2 = -a^2$ in the operator $\phi(D)$.
In our case, $a = 3$ and $b = -\pi/4$ for $3\cos(3x - \pi/4)$. Let's check if $\phi(-a^2) = 0$:

$$\phi(-a^2) = \phi(-3^2) \tag{6.496}$$
$$= \phi(-9) \tag{6.497}$$
$$= (-9) + 2D + 5 \tag{6.498}$$

After substituting $D^2 = -9$, we still have a term with D. Since we're operating on $\cos(3x - \pi/4)$, we need to rationalize to eliminate D.
The operator becomes:

$$\phi(D)|_{D^2=-9} = -9 + 2D + 5 \tag{6.499}$$
$$= -4 + 2D \tag{6.500}$$

Since $\phi(-a^2)$ contains D, we need to rationalize it. For an operator like $-4+2D$, when operating on $\cos(3x-\pi/4)$, we need to use the property that:

$$D\cos(3x-\pi/4) = -3\sin(3x-\pi/4) \tag{6.501}$$

Now, let's compute the particular integral:

$$y_p = \frac{1}{-4+2D} 3\cos(3x-\pi/4) \tag{6.502}$$

To handle the D in the denominator, we rationalize by multiplying both numerator and denominator by $(-4-2D)$:

$$y_p = \frac{3(-4-2D)}{(-4+2D)(-4-2D)} \cos(3x-\pi/4) \tag{6.503}$$

$$= \frac{3(-4-2D)}{16-4D^2} \cos(3x-\pi/4) \tag{6.504}$$

Substituting $D^2 = -9$:

$$y_p = \frac{3(-4-2D)}{16-4(-9)} \cos(3x-\pi/4) \tag{6.505}$$

$$= \frac{3(-4-2D)}{16+36} \cos(3x-\pi/4) \tag{6.506}$$

$$= \frac{3(-4-2D)}{52} \cos(3x-\pi/4) \tag{6.507}$$

$$= \frac{-12-6D}{52} \cos(3x-\pi/4) \tag{6.508}$$

$$= \frac{-12}{52} \cos(3x-\pi/4) + \frac{-6D}{52} \cos(3x-\pi/4) \tag{6.509}$$

Using $D\cos(3x-\pi/4) = -3\sin(3x-\pi/4)$:

$$y_p = \frac{-12}{52} \cos(3x-\pi/4) + \frac{-6(-3\sin(3x-\pi/4))}{52} \tag{6.510}$$

$$= \frac{-12}{52} \cos(3x-\pi/4) + \frac{18\sin(3x-\pi/4)}{52} \tag{6.511}$$

$$= \frac{-3}{13} \cos(3x-\pi/4) + \frac{9}{26} \sin(3x-\pi/4) \tag{6.512}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.513}$$

$$= e^{-x}(c_1\cos(2x) + c_2\sin(2x)) - \frac{3}{13}\cos(3x-\pi/4) + \frac{9}{26}\sin(3x-\pi/4) \tag{6.514}$$

Therefore, the general solution of the given differential equation is:

$$y = e^{-x}(c_1\cos(2x) + c_2\sin(2x)) - \tfrac{3}{13}\cos(3x-\pi/4) + \tfrac{9}{26}\sin(3x-\pi/4)$$

where c_1 and c_2 are arbitrary constants.

Example 2.3

Solve the differential equation:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 5\cos(3x) \tag{6.515}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 - 3D - 4 = 0 \tag{6.516}$$
$$m^2 - 3m - 4 = 0 \tag{6.517}$$

We can factor this equation:

$$m^2 - 3m - 4 = 0 \tag{6.518}$$
$$m^2 - 4m + m - 4 = 0 \tag{6.519}$$
$$m(m-4) + 1(m-4) = 0 \tag{6.520}$$
$$(m-4)(m+1) = 0 \tag{6.521}$$

Therefore, $m = 4$ or $m = -1$. The complementary function is:

$$y_c = c_1 e^{4x} + c_2 e^{-x} \tag{6.522}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = 5\cos(3x)$ and the operator $\phi(D) = D^2 - 3D - 4$.
For a trigonometric forcing function of the form $\cos(ax)$, we use the formula:

$$y_p = \frac{1}{\phi(-a^2)}\cos(ax) \tag{6.523}$$

where we substitute $D^2 = -a^2$ in the operator $\phi(D)$.
In our case, $a = 3$ for $5\cos(3x)$. Let's check if $\phi(-a^2) = 0$:

$$\phi(-a^2) = \phi(-3^2) \tag{6.524}$$
$$= \phi(-9) \tag{6.525}$$
$$= (-9) - 3D - 4 \tag{6.526}$$
$$= -13 - 3D \tag{6.527}$$

After substituting $D^2 = -9$, we still have a term with D. Since we're operating on $\cos(3x)$, we need to rationalize to eliminate D.
The operator is $-13 - 3D$. To rationalize, we multiply both numerator and denominator by $(-13 + 3D)$:

$$\frac{1}{-13-3D} = \frac{-13+3D}{(-13-3D)(-13+3D)} \tag{6.528}$$
$$= \frac{-13+3D}{169-9D^2} \tag{6.529}$$

Substituting $D^2 = -9$:

$$\frac{-13+3D}{169-9D^2} = \frac{-13+3D}{169-9(-9)} \tag{6.530}$$
$$= \frac{-13+3D}{169+81} \tag{6.531}$$
$$= \frac{-13+3D}{250} \tag{6.532}$$

Now, for the particular integral:

$$y_p = \frac{1}{-13-3D} 5\cos(3x) \tag{6.533}$$
$$= \frac{-13+3D}{250} 5\cos(3x) \tag{6.534}$$
$$= \frac{-65+15D}{250}\cos(3x) \tag{6.535}$$
$$= \frac{-13}{50}\cos(3x) + \frac{3}{50} D\cos(3x) \tag{6.536}$$

Using $D\cos(3x) = -3\sin(3x)$:

$$y_p = \frac{-13}{50}\cos(3x) + \frac{3}{50}(-3\sin(3x)) \tag{6.537}$$
$$= \frac{-13}{50}\cos(3x) - \frac{9}{50}\sin(3x) \tag{6.538}$$
$$= -\frac{13}{50}\cos(3x) - \frac{9}{50}\sin(3x) \tag{6.539}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.540}$$
$$= c_1 e^{4x} + c_2 e^{-x} - \frac{13}{50}\cos(3x) - \frac{9}{50}\sin(3x) \tag{6.541}$$

Therefore, the general solution of the given differential equation is:
$y = c_1 e^{4x} + c_2 e^{-x} - \frac{13}{50}\cos(3x) - \frac{9}{50}\sin(3x)$
where c_1 and c_2 are arbitrary constants.

Example 2.4

Solve the differential equation:

$$(D^4 - m^4)y = \sin mx \tag{6.542}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation

is:

$$D^4 - m^4 = 0 \tag{6.543}$$
$$r^4 - m^4 = 0 \tag{6.544}$$
$$r^4 = m^4 \tag{6.545}$$
$$r = \pm m, \pm mi \tag{6.546}$$

Therefore, the complementary function is:

$$y_c = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos(mx) + c_4 \sin(mx) \tag{6.547}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = \sin(mx)$ and the operator $\phi(D) = D^4 - m^4$.
For a trigonometric forcing function of the form $\sin(ax)$, we use the formula:

$$y_p = \frac{1}{\phi(-a^2)} \sin(ax) \tag{6.548}$$

where we substitute $D^2 = -a^2$ in the operator $\phi(D)$.
In our case, $a = m$ for $\sin(mx)$. Let's check if $\phi(-a^2) = 0$:

$$\phi(-a^2) = \phi(-m^2) \tag{6.549}$$
$$= ((-m^2)^2 - m^4) \tag{6.550}$$
$$= (m^4 - m^4) \tag{6.551}$$
$$= 0 \tag{6.552}$$

Since $\phi(-a^2) = 0$, this is a case of failure. We need to use the formula for Case of Failure 1:

$$y_p = \frac{x}{\phi'(D)} \sin(mx) \tag{6.553}$$

Find $\phi'(D)$ with respect to D:

$$\phi'(D) = \frac{d}{dD}(D^4 - m^4) \tag{6.554}$$
$$= 4D^3 \tag{6.555}$$

After substituting $D^2 = -m^2$, we get:

$$\phi'(D)|_{D^2=-m^2} = 4D^3 \tag{6.556}$$
$$= 4D \cdot D^2 \tag{6.557}$$
$$= 4D \cdot (-m^2) \tag{6.558}$$
$$= -4m^2 D \tag{6.559}$$

Now we can find the particular integral:

$$y_p = \frac{x}{\phi'(D)} \sin(mx) \tag{6.560}$$
$$= \frac{x}{-4m^2 D} \sin(mx) \tag{6.561}$$
$$= \frac{-x}{4m^2} \cdot \frac{1}{D} \sin(mx) \tag{6.562}$$

Using the property that $\frac{1}{D}\sin(mx) = \frac{-1}{m}\cos(mx)$ (integrating $\sin(mx)$):

$$y_p = \frac{-x}{4m^2} \cdot \frac{-1}{m}\cos(mx) \quad (6.563)$$

$$= \frac{x}{4m^3}\cos(mx) \quad (6.564)$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \quad (6.565)$$

$$= c_1e^{mx} + c_2e^{-mx} + c_3\cos(mx) + c_4\sin(mx) + \frac{x}{4m^3}\cos(mx) \quad (6.566)$$

Therefore, the general solution of the given differential equation is:

$$y = c_1e^{mx} + c_2e^{-mx} + c_3\cos(mx) + c_4\sin(mx) + \frac{x}{4m^3}\cos(mx)$$

which can be rewritten as:

$$y = c_1e^{mx} + c_2e^{-mx} + \cos(mx)\left(c_3 + \frac{x}{4m^3}\right) + c_4\sin(mx)$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

Example 2.5

Solve the differential equation:

$$\frac{d^2y}{dx^2} + 4y = \cos x \cdot \cos 2x \cdot \cos 3x \quad (6.567)$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.

Step 1: Find the complementary function y_c.

The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 4 = 0 \quad (6.568)$$

$$m^2 + 4 = 0 \quad (6.569)$$

$$m^2 = -4 \quad (6.570)$$

$$m = \pm 2i \quad (6.571)$$

Since we have complex roots $m = \pm 2i$, the complementary function is:

$$y_c = c_1\cos(2x) + c_2\sin(2x) \quad (6.572)$$

where c_1 and c_2 are arbitrary constants.

Step 2: Find the particular integral y_p.

We have the forcing function $f(x) = \cos x \cdot \cos 2x \cdot \cos 3x$ and the operator $\phi(D) = D^2 + 4$. To find the particular integral, we first need to express the product of cosines as a sum of cosines using trigonometric identities. For two cosines, we have:

$$\cos A \cdot \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)] \quad (6.573)$$

First, let's simplify $\cos x \cdot \cos 2x$:

$$\cos x \cdot \cos 2x = \frac{1}{2}[\cos(x+2x) + \cos(x-2x)] \tag{6.574}$$
$$= \frac{1}{2}[\cos(3x) + \cos(-x)] \tag{6.575}$$
$$= \frac{1}{2}[\cos(3x) + \cos(x)] \tag{6.576}$$

Now, we compute $(\cos x \cdot \cos 2x) \cdot \cos 3x$:

$$(\cos x \cdot \cos 2x) \cdot \cos 3x = \frac{1}{2}[\cos(3x) + \cos(x)] \cdot \cos 3x \tag{6.577}$$
$$= \frac{1}{2}[\cos(3x) \cdot \cos 3x + \cos(x) \cdot \cos 3x] \tag{6.578}$$

For $\cos(3x) \cdot \cos 3x$, we have $\cos^2(3x) = \frac{1+\cos(6x)}{2}$. And for $\cos(x) \cdot \cos 3x$, we use the product formula again:

$$\cos(x) \cdot \cos 3x = \frac{1}{2}[\cos(x+3x) + \cos(x-3x)] \tag{6.579}$$
$$= \frac{1}{2}[\cos(4x) + \cos(-2x)] \tag{6.580}$$
$$= \frac{1}{2}[\cos(4x) + \cos(2x)] \tag{6.581}$$

Now, combining all terms:

$$\cos x \cdot \cos 2x \cdot \cos 3x = \frac{1}{2}\left[\cos(3x) \cdot \cos 3x + \cos(x) \cdot \cos 3x\right] \tag{6.582}$$
$$= \frac{1}{2}\left[\frac{1+\cos(6x)}{2} + \frac{1}{2}[\cos(4x) + \cos(2x)]\right] \tag{6.583}$$
$$= \frac{1}{4} + \frac{1}{4}\cos(6x) + \frac{1}{4}\cos(4x) + \frac{1}{4}\cos(2x) \tag{6.584}$$
$$= \frac{1}{4}\left[1 + \cos(6x) + \cos(4x) + \cos(2x)\right] \tag{6.585}$$

Now, we need to find the particular integral for each term separately and then add them.
For the constant term $\frac{1}{4}$:

$$y_{p1} = \frac{1}{D^2+4}\left(\frac{1}{4}\right) \tag{6.586}$$
$$= \frac{1}{4} \cdot \frac{1}{0^2+4} \tag{6.587}$$
$$= \frac{1}{16} \tag{6.588}$$

For the term $\frac{1}{4}\cos(6x)$: Let's check if $\phi(-a^2) = 0$ where $a = 6$:

$$\phi(-a^2) = \phi(-36) \tag{6.589}$$
$$= -36 + 4 \tag{6.590}$$
$$= -32 \neq 0 \tag{6.591}$$

Since $\phi(-a^2) \neq 0$, we can use the standard formula:

$$y_{p2} = \frac{1}{\phi(-a^2)} \frac{1}{4} \cos(6x) \tag{6.592}$$
$$= \frac{1}{-32} \cdot \frac{1}{4} \cos(6x) \tag{6.593}$$
$$= -\frac{1}{128} \cos(6x) \tag{6.594}$$

For the term $\frac{1}{4}\cos(4x)$: Let's check if $\phi(-a^2) = 0$ where $a = 4$:

$$\phi(-a^2) = \phi(-16) \tag{6.595}$$
$$= -16 + 4 \tag{6.596}$$
$$= -12 \neq 0 \tag{6.597}$$

Since $\phi(-a^2) \neq 0$, we can use the standard formula:

$$y_{p3} = \frac{1}{\phi(-a^2)} \frac{1}{4} \cos(4x) \tag{6.598}$$
$$= \frac{1}{-12} \cdot \frac{1}{4} \cos(4x) \tag{6.599}$$
$$= -\frac{1}{48} \cos(4x) \tag{6.600}$$

For the term $\frac{1}{4}\cos(2x)$: Let's check if $\phi(-a^2) = 0$ where $a = 2$:

$$\phi(-a^2) = \phi(-4) \tag{6.601}$$
$$= -4 + 4 \tag{6.602}$$
$$= 0 \tag{6.603}$$

Since $\phi(-a^2) = 0$, this is a case of failure. We need to use the formula for Case of Failure 1:

$$y_{p4} = \frac{x}{\phi'(D)} \frac{1}{4} \cos(2x) \tag{6.604}$$

Find $\phi'(D)$ with respect to D:

$$\phi'(D) = \frac{d}{dD}(D^2 + 4) \tag{6.605}$$
$$= 2D \tag{6.606}$$

Now we can find the particular integral:

$$y_{p4} = \frac{x}{\phi'(D)} \frac{1}{4} \cos(2x) \tag{6.607}$$
$$= \frac{x}{2D} \cdot \frac{1}{4} \cos(2x) \tag{6.608}$$
$$= \frac{x}{8D} \cos(2x) \tag{6.609}$$

Using the property that $\frac{1}{D}\cos(2x) = \frac{1}{2}\sin(2x)$ (integrating $\cos(2x)$):

$$y_{p4} = \frac{x}{8} \cdot \frac{1}{2} \sin(2x) \tag{6.610}$$
$$= \frac{x}{16} \sin(2x) \tag{6.611}$$

Step 3: Combine the particular integrals for all terms.

$$y_p = y_{p1} + y_{p2} + y_{p3} + y_{p4} \tag{6.612}$$
$$= \frac{1}{16} - \frac{1}{128}\cos(6x) - \frac{1}{48}\cos(4x) + \frac{x}{16}\sin(2x) \tag{6.613}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.614}$$
$$= c_1\cos(2x) + c_2\sin(2x) + \frac{1}{16} - \frac{1}{128}\cos(6x) - \frac{1}{48}\cos(4x) + \frac{x}{16}\sin(2x) \tag{6.615}$$

Therefore, the general solution of the given differential equation is:
$y = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{16} - \frac{1}{128}\cos(6x) - \frac{1}{48}\cos(4x) + \frac{x}{16}\sin(2x)$
which can be rewritten as:
$y = c_1\cos(2x) + \sin(2x)\left(c_2 + \frac{x}{16}\right) + \frac{1}{16} - \frac{1}{128}\cos(6x) - \frac{1}{48}\cos(4x)$
where c_1 and c_2 are arbitrary constants.

Example 2.6

Solve the differential equation:

$$(D^2+1)^2 y = \cos x \tag{6.616}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$(D^2+1)^2 y = 0 \tag{6.617}$$
$$(D^2+1)^2 = 0 \tag{6.618}$$

This means that $(D^2+1) = 0$ is a repeated root of the auxiliary equation. So we have:

$$D^2 + 1 = 0 \tag{6.619}$$
$$D^2 = -1 \tag{6.620}$$
$$D = \pm i \tag{6.621}$$

Since we have the roots $m = \pm i$ with multiplicity 2, the complementary function is:

$$y_c = c_1\cos x + c_2\sin x + c_3 x\cos x + c_4 x\sin x \tag{6.622}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = \cos x$ and the operator $\phi(D) = (D^2+1)^2$.
For a trigonometric forcing function of the form $\cos(ax)$, we use the formula:

$$y_p = \frac{1}{\phi(-a^2)}\cos(ax) \tag{6.623}$$

where we substitute $D^2 = -a^2$ in the operator $\phi(D)$.

In our case, $a = 1$ for $\cos x$. Let's check if $\phi(-a^2) = 0$:

$$\begin{aligned}
\phi(-a^2) &= \phi(-1) && (6.624)\\
&= ((-1)+1)^2 && (6.625)\\
&= 0^2 && (6.626)\\
&= 0 && (6.627)
\end{aligned}$$

Since $\phi(-a^2) = 0$, this is a case of failure. In fact, the operator $(D^2+1)^2$ has a zero of multiplicity 2 at $D^2 = -1$. We need to use the formula for Case of Failure 2:

$$y_p = \frac{x^2}{\phi''(D)} \cos x \tag{6.628}$$

Let's find $\phi''(D)$ with respect to D:

$$\begin{aligned}
\phi(D) &= (D^2+1)^2 && (6.629)\\
\phi'(D) &= 2(D^2+1) \cdot 2D && (6.630)\\
&= 4D(D^2+1) && (6.631)\\
\phi''(D) &= 4(D^2+1) + 4D \cdot 2D && (6.632)\\
&= 4(D^2+1) + 8D^2 && (6.633)\\
&= 4D^2 + 4 + 8D^2 && (6.634)\\
&= 12D^2 + 4 && (6.635)
\end{aligned}$$

Now, evaluating $\phi''(D)$ at $D^2 = -1$:

$$\begin{aligned}
\phi''(D)|_{D^2=-1} &= 12(-1) + 4 && (6.636)\\
&= -12 + 4 && (6.637)\\
&= -8 && (6.638)
\end{aligned}$$

So we can find the particular integral:

$$\begin{aligned}
y_p &= \frac{x^2}{\phi''(D)|_{D^2=-1}} \cos x && (6.639)\\
&= \frac{x^2}{-8} \cos x && (6.640)\\
&= -\frac{x^2}{8} \cos x && (6.641)
\end{aligned}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$\begin{aligned}
y &= y_c + y_p && (6.642)\\
&= c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x - \frac{x^2}{8} \cos x && (6.643)
\end{aligned}$$

Therefore, the general solution of the given differential equation is:

$$y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x - \tfrac{x^2}{8} \cos x$$

which can be rewritten as:

$$y = \cos x \left(c_1 + c_3 x - \tfrac{x^2}{8}\right) + \sin x (c_2 + c_4 x)$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

Example 2.7

Solve the differential equation:

$$(D^3 + D)y = \cos x \tag{6.644}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
We can factor the operator as:

$$D^3 + D = D(D^2 + 1) \tag{6.645}$$

So the auxiliary equation is:

$$D(D^2 + 1) = 0 \tag{6.646}$$

This gives us $D = 0$ or $D^2 + 1 = 0$, which means $D = 0$ or $D = \pm i$.
Therefore, the complementary function is:

$$y_c = c_1 + c_2 \cos x + c_3 \sin x \tag{6.647}$$

where c_1, c_2, and c_3 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = \cos x$ and the operator $\phi(D) = D^3 + D = D(D^2 + 1)$.
For a trigonometric forcing function of the form $\cos(ax)$, we use the formula:

$$y_p = \frac{1}{\phi(-a^2)} \cos(ax) \tag{6.648}$$

where we substitute $D^2 = -a^2$ in the operator $\phi(D)$.
In our case, $a = 1$ for $\cos x$. Substituting $D^2 = -1$ in the operator:

$$\begin{aligned}
\phi(D)|_{D^2=-1} &= D^3 + D && (6.649)\\
&= D \cdot D^2 + D && (6.650)\\
&= D \cdot (-1) + D && (6.651)\\
&= -D + D && (6.652)\\
&= 0 && (6.653)
\end{aligned}$$

Since $\phi(-a^2) = 0$, this is a case of failure. We need to use the formula for Case of Failure 1:

$$y_p = \frac{x}{\phi'(D)} \cos x \tag{6.654}$$

Let's find $\phi'(D)$ with respect to D:

$$\begin{aligned}
\phi'(D) &= \frac{d}{dD}(D^3 + D) && (6.655)\\
&= 3D^2 + 1 && (6.656)
\end{aligned}$$

Evaluating $\phi'(D)$ at $D^2 = -1$:

$$\begin{aligned}
\phi'(D)|_{D^2=-1} &= 3(-1) + 1 && (6.657)\\
&= -3 + 1 && (6.658)\\
&= -2 && (6.659)
\end{aligned}$$

So we can find the particular integral:

$$y_p = \frac{x}{\phi'(D)|_{D^2=-1}} \cos x \tag{6.660}$$
$$= \frac{x}{-2} \cos x \tag{6.661}$$
$$= -\frac{x}{2} \cos x \tag{6.662}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.663}$$
$$= c_1 + c_2 \cos x + c_3 \sin x - \frac{x}{2} \cos x \tag{6.664}$$

Therefore, the general solution of the given differential equation is:
$y = c_1 + c_2 \cos x + c_3 \sin x - \frac{x}{2} \cos x$
which can be rewritten as:
$y = c_1 + \cos x \left(c_2 - \frac{x}{2}\right) + c_3 \sin x$
where c_1, c_2, and c_3 are arbitrary constants.

Example 2.8

Solve the differential equation:

$$\frac{d^2x}{dt^2} + 9x = 4\cos\left(\frac{\pi}{3} + t\right), \text{ given that } x = 0 \text{ at } t = 0 \text{ and } x = 2 \text{ at } t = \frac{\pi}{6}. \tag{6.665}$$

Solution

The general solution will be of the form $x = x_c + x_p$, where x_c is the complementary function and x_p is the particular integral.
Step 1: Find the complementary function x_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 9 = 0 \tag{6.666}$$
$$m^2 + 9 = 0 \tag{6.667}$$
$$m^2 = -9 \tag{6.668}$$
$$m = \pm 3i \tag{6.669}$$

Since we have complex roots $m = \pm 3i$, the complementary function is:

$$x_c = c_1 \cos(3t) + c_2 \sin(3t) \tag{6.670}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral x_p.
We have the forcing function $f(t) = 4\cos\left(\frac{\pi}{3} + t\right)$ and the operator $\phi(D) = D^2 + 9$.
We can rewrite the forcing function using the trigonometric identity:

$$\cos\left(\frac{\pi}{3} + t\right) = \cos\left(\frac{\pi}{3}\right)\cos(t) - \sin\left(\frac{\pi}{3}\right)\sin(t) \tag{6.671}$$
$$= \frac{1}{2}\cos(t) - \frac{\sqrt{3}}{2}\sin(t) \tag{6.672}$$

So the forcing function becomes:

$$f(t) = 4\cos\left(\frac{\pi}{3} + t\right) \tag{6.673}$$
$$= 4\left(\frac{1}{2}\cos(t) - \frac{\sqrt{3}}{2}\sin(t)\right) \tag{6.674}$$
$$= 2\cos(t) - 2\sqrt{3}\sin(t) \tag{6.675}$$

For a trigonometric forcing function, we use the formula:

$$x_p = \frac{1}{\phi(-a^2)}\cos(at) \text{ or } \frac{1}{\phi(-a^2)}\sin(at) \tag{6.676}$$

where we substitute $D^2 = -a^2$ in the operator $\phi(D)$.
For the term $2\cos(t)$, we have $a = 1$. Let's check if $\phi(-a^2) = 0$:

$$\phi(-a^2) = \phi(-1) \tag{6.677}$$
$$= (-1) + 9 \tag{6.678}$$
$$= 8 \neq 0 \tag{6.679}$$

Since $\phi(-a^2) \neq 0$, we can use the standard formula:

$$x_{p1} = \frac{1}{\phi(-a^2)} 2\cos(t) \tag{6.680}$$
$$= \frac{2}{8}\cos(t) \tag{6.681}$$
$$= \frac{1}{4}\cos(t) \tag{6.682}$$

Similarly, for the term $-2\sqrt{3}\sin(t)$, we have $a = 1$ and $\phi(-a^2) = 8 \neq 0$, so:

$$x_{p2} = \frac{1}{\phi(-a^2)}(-2\sqrt{3})\sin(t) \tag{6.683}$$
$$= \frac{-2\sqrt{3}}{8}\sin(t) \tag{6.684}$$
$$= -\frac{\sqrt{3}}{4}\sin(t) \tag{6.685}$$

Therefore, the particular integral is:

$$x_p = x_{p1} + x_{p2} \tag{6.686}$$
$$= \frac{1}{4}\cos(t) - \frac{\sqrt{3}}{4}\sin(t) \tag{6.687}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$x = x_c + x_p \tag{6.688}$$
$$= c_1\cos(3t) + c_2\sin(3t) + \frac{1}{4}\cos(t) - \frac{\sqrt{3}}{4}\sin(t) \tag{6.689}$$

Step 4: Use the initial conditions to find the values of c_1 and c_2.

Given that $x = 0$ at $t = 0$:

$$0 = c_1 \cos(0) + c_2 \sin(0) + \frac{1}{4}\cos(0) - \frac{\sqrt{3}}{4}\sin(0) \tag{6.690}$$
$$= c_1 \cdot 1 + c_2 \cdot 0 + \frac{1}{4} \cdot 1 - \frac{\sqrt{3}}{4} \cdot 0 \tag{6.691}$$
$$= c_1 + \frac{1}{4} \tag{6.692}$$

Therefore, $c_1 = -\frac{1}{4}$.
Given that $x = 2$ at $t = \frac{\pi}{6}$:

$$2 = c_1 \cos\left(3 \cdot \frac{\pi}{6}\right) + c_2 \sin\left(3 \cdot \frac{\pi}{6}\right) + \frac{1}{4}\cos\left(\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\sin\left(\frac{\pi}{6}\right) \tag{6.693}$$
$$= c_1 \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) + \frac{1}{4}\cos\left(\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\sin\left(\frac{\pi}{6}\right) \tag{6.694}$$
$$= c_1 \cdot 0 + c_2 \cdot 1 + \frac{1}{4} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4} \cdot \frac{1}{2} \tag{6.695}$$
$$= c_2 + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} \tag{6.696}$$
$$= c_2 \tag{6.697}$$

Therefore, $c_2 = 2$.
Substituting these values back into the general solution:

$$x = -\frac{1}{4}\cos(3t) + 2\sin(3t) + \frac{1}{4}\cos(t) - \frac{\sqrt{3}}{4}\sin(t) \tag{6.698}$$

Therefore, the particular solution of the given differential equation satisfying the initial conditions is:

$$x = -\tfrac{1}{4}\cos(3t) + 2\sin(3t) + \tfrac{1}{4}\cos(t) - \tfrac{\sqrt{3}}{4}\sin(t)$$

Example 2.9

Solve the differential equation:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \sin^2 t \tag{6.699}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 2D + 5 = 0 \tag{6.700}$$
$$m^2 + 2m + 5 = 0 \tag{6.701}$$

Using the quadratic formula:

$$m = \frac{-2 \pm \sqrt{4-20}}{2} \tag{6.702}$$
$$= \frac{-2 \pm \sqrt{-16}}{2} \tag{6.703}$$
$$= \frac{-2 \pm 4i}{2} \tag{6.704}$$
$$= -1 \pm 2i \tag{6.705}$$

Since we have complex roots $m = -1 \pm 2i$, the complementary function is:

$$y_c = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) \tag{6.706}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(t) = \sin^2 t$ and the operator $\phi(D) = D^2 + 2D + 5$.
First, we need to convert $\sin^2 t$ to a sum of cosines using the trigonometric identity:

$$\sin^2 t = \frac{1-\cos(2t)}{2} \tag{6.707}$$
$$= \frac{1}{2} - \frac{1}{2}\cos(2t) \tag{6.708}$$

Now, we need to find the particular integral for each term separately.
For the constant term $\frac{1}{2}$:

$$y_{p1} = \frac{1}{\phi(D)}\left(\frac{1}{2}\right) \tag{6.709}$$
$$= \frac{1}{2} \cdot \frac{1}{\phi(0)} \tag{6.710}$$
$$= \frac{1}{2} \cdot \frac{1}{0^2 + 2(0) + 5} \tag{6.711}$$
$$= \frac{1}{2} \cdot \frac{1}{5} \tag{6.712}$$
$$= \frac{1}{10} \tag{6.713}$$

For the term $-\frac{1}{2}\cos(2t)$:
Let's check if $\phi(-a^2) = 0$ where $a = 2$:

$$\phi(-a^2) = \phi(-4) \tag{6.714}$$
$$= (-4) + 2D + 5 \tag{6.715}$$

After substituting $D^2 = -4$, we still have a term with D. Since we're operating on $\cos(2t)$, we need to rationalize to eliminate D.
The operator becomes:

$$\phi(D)|_{D^2=-4} = -4 + 2D + 5 \tag{6.716}$$
$$= 1 + 2D \tag{6.717}$$

For an operator like $1 + 2D$, when operating on $\cos(2t)$, we need to rationalize:

$$\frac{1}{1+2D} = \frac{1-2D}{(1+2D)(1-2D)} \tag{6.718}$$

$$= \frac{1-2D}{1-4D^2} \tag{6.719}$$

Substituting $D^2 = -4$:

$$\frac{1-2D}{1-4D^2} = \frac{1-2D}{1-4(-4)} \tag{6.720}$$

$$= \frac{1-2D}{1+16} \tag{6.721}$$

$$= \frac{1-2D}{17} \tag{6.722}$$

Now, the particular integral for this term is:

$$y_{p2} = \frac{1}{\phi(D)}\left(-\frac{1}{2}\cos(2t)\right) \tag{6.723}$$

$$= -\frac{1}{2} \cdot \frac{1-2D}{17}\cos(2t) \tag{6.724}$$

$$= -\frac{1}{34}(1-2D)\cos(2t) \tag{6.725}$$

$$= -\frac{1}{34}\cos(2t) + \frac{2}{34}D\cos(2t) \tag{6.726}$$

Using $D\cos(2t) = -2\sin(2t)$:

$$y_{p2} = -\frac{1}{34}\cos(2t) + \frac{2}{34}(-2\sin(2t)) \tag{6.727}$$

$$= -\frac{1}{34}\cos(2t) - \frac{4}{34}\sin(2t) \tag{6.728}$$

$$= -\frac{1}{34}\cos(2t) - \frac{2}{17}\sin(2t) \tag{6.729}$$

Step 3: Combine the particular integrals for both terms.

$$y_p = y_{p1} + y_{p2} \tag{6.730}$$

$$= \frac{1}{10} - \frac{1}{34}\cos(2t) - \frac{2}{17}\sin(2t) \tag{6.731}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.732}$$

$$= e^{-t}(c_1\cos(2t) + c_2\sin(2t)) + \frac{1}{10} - \frac{1}{34}\cos(2t) - \frac{2}{17}\sin(2t) \tag{6.733}$$

Therefore, the general solution of the given differential equation is:

$$y = e^{-t}(c_1\cos(2t) + c_2\sin(2t)) + \tfrac{1}{10} - \tfrac{1}{34}\cos(2t) - \tfrac{2}{17}\sin(2t)$$

where c_1 and c_2 are arbitrary constants.

Example 2.10

Solve the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin 2x - 2\cos 2x, \text{ given that } y = 0 \text{ and } \frac{dy}{dx} = 0 \text{ when } x = 0. \quad (6.734)$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 2D + 2 = 0 \quad (6.735)$$
$$m^2 + 2m + 2 = 0 \quad (6.736)$$

Using the quadratic formula:

$$m = \frac{-2 \pm \sqrt{4-8}}{2} \quad (6.737)$$
$$= \frac{-2 \pm \sqrt{-4}}{2} \quad (6.738)$$
$$= \frac{-2 \pm 2i}{2} \quad (6.739)$$
$$= -1 \pm i \quad (6.740)$$

Since we have complex roots $m = -1 \pm i$, the complementary function is:

$$y_c = e^{-x}(c_1 \cos x + c_2 \sin x) \quad (6.741)$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = \sin 2x - 2\cos 2x$ and the operator $\phi(D) = D^2 + 2D + 2$.
For trigonometric forcing functions, we use the formula:

$$y_p = \frac{1}{\phi(-a^2)} \sin(ax) \text{ or } \frac{1}{\phi(-a^2)} \cos(ax) \quad (6.742)$$

where we substitute $D^2 = -a^2$ in the operator $\phi(D)$.
Let's handle each term separately.
For the term $\sin 2x$: We have $a = 2$. Let's check if $\phi(-a^2) = 0$:

$$\phi(-a^2) = \phi(-4) \quad (6.743)$$
$$= (-4) + 2D + 2 \quad (6.744)$$

After substituting $D^2 = -4$, we still have a term with D. Since we're operating on $\sin 2x$, we need to rationalize to eliminate D.
The operator becomes:

$$\phi(D)|_{D^2=-4} = -4 + 2D + 2 \quad (6.745)$$
$$= -2 + 2D \quad (6.746)$$

For an operator like $-2 + 2D$, when operating on $\sin 2x$, we need to rationalize:

$$\frac{1}{-2+2D} = \frac{1}{2} \cdot \frac{1}{-1+D} \tag{6.747}$$

$$= \frac{1}{2} \cdot \frac{-1-D}{(-1+D)(-1-D)} \tag{6.748}$$

$$= \frac{1}{2} \cdot \frac{-1-D}{1-D^2} \tag{6.749}$$

Substituting $D^2 = -4$:

$$\frac{-1-D}{2(1-D^2)} = \frac{-1-D}{2(1-(-4))} \tag{6.750}$$

$$= \frac{-1-D}{2(1+4)} \tag{6.751}$$

$$= \frac{-1-D}{10} \tag{6.752}$$

Now, the particular integral for this term is:

$$y_{p1} = \frac{1}{\phi(D)} \sin 2x \tag{6.753}$$

$$= \frac{-1-D}{10} \sin 2x \tag{6.754}$$

$$= \frac{-1}{10} \sin 2x - \frac{D}{10} \sin 2x \tag{6.755}$$

Using $D \sin 2x = 2 \cos 2x$:

$$y_{p1} = \frac{-1}{10} \sin 2x - \frac{2\cos 2x}{10} \tag{6.756}$$

$$= -\frac{1}{10} \sin 2x - \frac{1}{5} \cos 2x \tag{6.757}$$

For the term $-2\cos 2x$: We have $a = 2$. The operator is the same:

$$\phi(D)|_{D^2=-4} = -2 + 2D \tag{6.758}$$

After rationalization:

$$\frac{1}{-2+2D} = \frac{-1-D}{10} \tag{6.759}$$

Now, the particular integral for this term is:

$$y_{p2} = \frac{1}{\phi(D)} (-2\cos 2x) \tag{6.760}$$

$$= -2 \cdot \frac{-1-D}{10} \cos 2x \tag{6.761}$$

$$= \frac{2}{10}(1+D) \cos 2x \tag{6.762}$$

$$= \frac{1}{5} \cos 2x + \frac{1}{5} D \cos 2x \tag{6.763}$$

Using $D\cos 2x = -2\sin 2x$:

$$y_{p2} = \frac{1}{5}\cos 2x + \frac{1}{5}(-2\sin 2x) \tag{6.764}$$
$$= \frac{1}{5}\cos 2x - \frac{2}{5}\sin 2x \tag{6.765}$$

Step 3: Combine the particular integrals for both terms.

$$y_p = y_{p1} + y_{p2} \tag{6.766}$$
$$= -\frac{1}{10}\sin 2x - \frac{1}{5}\cos 2x + \frac{1}{5}\cos 2x - \frac{2}{5}\sin 2x \tag{6.767}$$
$$= -\frac{1}{10}\sin 2x - \frac{2}{5}\sin 2x \tag{6.768}$$
$$= -\frac{1}{10}\sin 2x - \frac{4}{10}\sin 2x \tag{6.769}$$
$$= -\frac{5}{10}\sin 2x \tag{6.770}$$
$$= -\frac{1}{2}\sin 2x \tag{6.771}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.772}$$
$$= e^{-x}(c_1\cos x + c_2\sin x) - \frac{1}{2}\sin 2x \tag{6.773}$$

Step 5: Use the initial conditions to find the values of c_1 and c_2.
Given that $y = 0$ at $x = 0$:

$$0 = e^{-0}(c_1\cos 0 + c_2\sin 0) - \frac{1}{2}\sin 0 \tag{6.774}$$
$$= 1\cdot(c_1\cdot 1 + c_2\cdot 0) - \frac{1}{2}\cdot 0 \tag{6.775}$$
$$= c_1 \tag{6.776}$$

Therefore, $c_1 = 0$.
Given that $\frac{dy}{dx} = 0$ at $x = 0$, we first find the derivative of y:

$$\frac{dy}{dx} = \frac{d}{dx}\left[e^{-x}(c_1\cos x + c_2\sin x) - \frac{1}{2}\sin 2x\right] \tag{6.777}$$
$$= -e^{-x}(c_1\cos x + c_2\sin x) + e^{-x}(-c_1\sin x + c_2\cos x) - \frac{1}{2}\cdot 2\cos 2x \tag{6.778}$$
$$= -e^{-x}(c_1\cos x + c_2\sin x) - e^{-x}(c_1\sin x - c_2\cos x) - \cos 2x \tag{6.779}$$
$$= e^{-x}(-c_1\cos x - c_2\sin x - c_1\sin x + c_2\cos x) - \cos 2x \tag{6.780}$$

Now, evaluating at $x = 0$ and setting equal to 0:

$$0 = e^{-0}(-c_1\cos 0 - c_2\sin 0 - c_1\sin 0 + c_2\cos 0) - \cos 0 \tag{6.781}$$
$$= 1\cdot(-c_1\cdot 1 - c_2\cdot 0 - c_1\cdot 0 + c_2\cdot 1) - 1 \tag{6.782}$$
$$= -c_1 + c_2 - 1 \tag{6.783}$$

Since we already found that $c_1 = 0$, we have:

$$0 = 0 + c_2 - 1 \tag{6.784}$$
$$\tag{6.785}$$

Therefore, $c_2 = 1$.
Substituting these values back into the general solution:

$$y = e^{-x}(0 \cdot \cos x + 1 \cdot \sin x) - \frac{1}{2}\sin 2x \tag{6.786}$$
$$= e^{-x}\sin x - \frac{1}{2}\sin 2x \tag{6.787}$$

Therefore, the particular solution of the given differential equation satisfying the initial conditions is:

$$y = e^{-x}\sin x - \tfrac{1}{2}\sin 2x$$

Example 2.11

Solve the differential equation:

$$(D^3 + 1)y = \cos(2x - 1) - \cos^2\frac{x}{2} \tag{6.788}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^3 + 1 = 0 \tag{6.789}$$
$$m^3 + 1 = 0 \tag{6.790}$$
$$m^3 = -1 \tag{6.791}$$

This cubic equation has one real root and two complex roots. The real root is $m = -1$, and the complex roots can be found as follows:

$$m^3 + 1 = 0 \tag{6.792}$$
$$(m + 1)(m^2 - m + 1) = 0 \tag{6.793}$$

So $m = -1$ or $m^2 - m + 1 = 0$. Using the quadratic formula for the second equation:

$$m = \frac{1 \pm \sqrt{1-4}}{2} \tag{6.794}$$
$$= \frac{1 \pm \sqrt{-3}}{2} \tag{6.795}$$
$$= \frac{1 \pm i\sqrt{3}}{2} \tag{6.796}$$

Therefore, the roots are $m_1 = -1$, $m_2 = \frac{1+i\sqrt{3}}{2}$, and $m_3 = \frac{1-i\sqrt{3}}{2}$.

For the complex roots, we can express them in the form $\alpha + i\beta$ where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. The complementary function is:

$$y_c = c_1 e^{-x} + e^{\alpha x}(c_2 \cos(\beta x) + c_3 \sin(\beta x)) \tag{6.797}$$

$$= c_1 e^{-x} + e^{x/2}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right) \tag{6.798}$$

where c_1, c_2, and c_3 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = \cos(2x-1) - \cos^2\frac{x}{2}$ and the operator $\phi(D) = D^3 + 1$. Let's handle each term separately.
For the term $\cos(2x-1)$**:** We can rewrite this using the trigonometric identity:

$$\cos(2x-1) = \cos(2x)\cos(1) + \sin(2x)\sin(1) \tag{6.799}$$

$$= \cos(1)\cos(2x) + \sin(1)\sin(2x) \tag{6.800}$$

For a trigonometric forcing function, we use the formula:

$$y_p = \frac{1}{\phi(-a^2)}\cos(ax) \text{ or } \frac{1}{\phi(-a^2)}\sin(ax) \tag{6.801}$$

where we substitute $D^2 = -a^2$ in the operator $\phi(D)$.
For the term $\cos(1)\cos(2x)$, we have $a = 2$. After substituting $D^2 = -4$ in the operator:

$$\phi(D)|_{D^2=-4} = D^3 + 1 \tag{6.802}$$

$$= D \cdot D^2 + 1 \tag{6.803}$$

$$= D \cdot (-4) + 1 \tag{6.804}$$

$$= -4D + 1 \tag{6.805}$$

Since $\phi(-a^2)$ contains D, we need to rationalize it. For an operator like $-4D+1$, when operating on $\cos(2x)$, we need to use the property that $D\cos(2x) = -2\sin(2x)$ and $D\sin(2x) = 2\cos(2x)$.
Let's rationalize:

$$\frac{1}{-4D+1} = \frac{4D+1}{(-4D+1)(4D+1)} \tag{6.806}$$

$$= \frac{4D+1}{-16D^2+1} \tag{6.807}$$

Substituting $D^2 = -4$:

$$\frac{4D+1}{-16D^2+1} = \frac{4D+1}{-16(-4)+1} \tag{6.808}$$

$$= \frac{4D+1}{64+1} \tag{6.809}$$

$$= \frac{4D+1}{65} \tag{6.810}$$

The particular integral for $\cos(1)\cos(2x)$ is:

$$y_{p11} = \cos(1) \cdot \frac{4D+1}{65}\cos(2x) \tag{6.811}$$

$$= \frac{\cos(1)}{65}(4D+1)\cos(2x) \tag{6.812}$$

$$= \frac{\cos(1)}{65}(4D\cos(2x) + \cos(2x)) \tag{6.813}$$

Using $D\cos(2x) = -2\sin(2x)$:

$$y_{p11} = \frac{\cos(1)}{65}(4(-2\sin(2x)) + \cos(2x)) \tag{6.814}$$
$$= \frac{\cos(1)}{65}(-8\sin(2x) + \cos(2x)) \tag{6.815}$$

Similarly, for the term $\sin(1)\sin(2x)$, the particular integral is:

$$y_{p12} = \sin(1) \cdot \frac{4D+1}{65}\sin(2x) \tag{6.816}$$
$$= \frac{\sin(1)}{65}(4D+1)\sin(2x) \tag{6.817}$$
$$= \frac{\sin(1)}{65}(4D\sin(2x) + \sin(2x)) \tag{6.818}$$

Using $D\sin(2x) = 2\cos(2x)$:

$$y_{p12} = \frac{\sin(1)}{65}(4(2\cos(2x)) + \sin(2x)) \tag{6.819}$$
$$= \frac{\sin(1)}{65}(8\cos(2x) + \sin(2x)) \tag{6.820}$$

Combining these, the particular integral for $\cos(2x-1)$ is:

$$y_{p1} = y_{p11} + y_{p12} \tag{6.821}$$
$$= \frac{\cos(1)}{65}(-8\sin(2x) + \cos(2x)) + \frac{\sin(1)}{65}(8\cos(2x) + \sin(2x)) \tag{6.822}$$
$$= \frac{1}{65}[\cos(1)(-8\sin(2x) + \cos(2x)) + \sin(1)(8\cos(2x) + \sin(2x))] \tag{6.823}$$
$$= \frac{1}{65}[\cos(1)\cos(2x) - 8\cos(1)\sin(2x) + 8\sin(1)\cos(2x) + \sin(1)\sin(2x)] \tag{6.824}$$

For the term $-\cos^2\frac{x}{2}$: First, we convert $\cos^2\frac{x}{2}$ using the identity:

$$\cos^2\frac{x}{2} = \frac{1+\cos(x)}{2} \tag{6.825}$$

So our term becomes:

$$-\cos^2\frac{x}{2} = -\frac{1+\cos(x)}{2} \tag{6.826}$$
$$= -\frac{1}{2} - \frac{1}{2}\cos(x) \tag{6.827}$$

For the constant term $-\frac{1}{2}$:

$$y_{p21} = \frac{1}{\phi(D)}\left(-\frac{1}{2}\right) \tag{6.828}$$
$$= -\frac{1}{2} \cdot \frac{1}{\phi(0)} \tag{6.829}$$
$$= -\frac{1}{2} \cdot \frac{1}{0^3+1} \tag{6.830}$$
$$= -\frac{1}{2} \tag{6.831}$$

For the term $-\frac{1}{2}\cos(x)$, we have $a = 1$. After substituting $D^2 = -1$ in the operator:

$$\phi(D)|_{D^2=-1} = D^3 + 1 \tag{6.832}$$
$$= D \cdot D^2 + 1 \tag{6.833}$$
$$= D \cdot (-1) + 1 \tag{6.834}$$
$$= -D + 1 \tag{6.835}$$

Again, we rationalize:

$$\frac{1}{-D+1} = \frac{D+1}{(-D+1)(D+1)} \tag{6.836}$$
$$= \frac{D+1}{-D^2+1} \tag{6.837}$$

Substituting $D^2 = -1$:

$$\frac{D+1}{-D^2+1} = \frac{D+1}{-(-1)+1} \tag{6.838}$$
$$= \frac{D+1}{1+1} \tag{6.839}$$
$$= \frac{D+1}{2} \tag{6.840}$$

The particular integral for $-\frac{1}{2}\cos(x)$ is:

$$y_{p22} = -\frac{1}{2} \cdot \frac{D+1}{2}\cos(x) \tag{6.841}$$
$$= -\frac{1}{4}(D+1)\cos(x) \tag{6.842}$$
$$= -\frac{1}{4}(D\cos(x) + \cos(x)) \tag{6.843}$$

Using $D\cos(x) = -\sin(x)$:

$$y_{p22} = -\frac{1}{4}(-\sin(x) + \cos(x)) \tag{6.844}$$
$$= \frac{1}{4}\sin(x) - \frac{1}{4}\cos(x) \tag{6.845}$$

Combining these, the particular integral for $-\cos^2\frac{x}{2}$ is:

$$y_{p2} = y_{p21} + y_{p22} \tag{6.846}$$
$$= -\frac{1}{2} + \frac{1}{4}\sin(x) - \frac{1}{4}\cos(x) \tag{6.847}$$

Step 3: Combine the particular integrals for both terms.

$$y_p = y_{p1} + y_{p2} \tag{6.848}$$
$$= \frac{1}{65}[\cos(1)\cos(2x) - 8\cos(1)\sin(2x) + 8\sin(1)\cos(2x) + \sin(1)\sin(2x)] \tag{6.849}$$
$$+ \left(-\frac{1}{2} + \frac{1}{4}\sin(x) - \frac{1}{4}\cos(x)\right) \tag{6.850}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.851}$$
$$= c_1e^{-x} + e^{x/2}\left(c_2\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right) \tag{6.852}$$
$$+ \frac{1}{65}[\cos(1)\cos(2x) - 8\cos(1)\sin(2x) + 8\sin(1)\cos(2x) + \sin(1)\sin(2x)] \tag{6.853}$$
$$+ \left(-\frac{1}{2} + \frac{1}{4}\sin(x) - \frac{1}{4}\cos(x)\right) \tag{6.854}$$

Therefore, the general solution of the given differential equation is:

$$y = c_1e^{-x} + e^{x/2}\left(c_2\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right) \tag{6.855}$$
$$+ \frac{1}{65}[\cos(1)\cos(2x) - 8\cos(1)\sin(2x) + 8\sin(1)\cos(2x) + \sin(1)\sin(2x)] \tag{6.856}$$
$$- \frac{1}{2} + \frac{1}{4}\sin(x) - \frac{1}{4}\cos(x) \tag{6.857}$$

where c_1, c_2, and c_3 are arbitrary constants.

Example 2.12

Solve the differential equation:

$$(D^4 + 6D^2 + 8)y = \sin^2 x\cos 2x \tag{6.858}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^4 + 6D^2 + 8 = 0 \tag{6.859}$$

This is a biquadratic equation. Let $m^2 = t$, so we have:

$$t^2 + 6t + 8 = 0 \tag{6.860}$$

Using the quadratic formula:

$$t = \frac{-6 \pm \sqrt{36 - 32}}{2} \tag{6.861}$$
$$= \frac{-6 \pm \sqrt{4}}{2} \tag{6.862}$$
$$= \frac{-6 \pm 2}{2} \tag{6.863}$$
$$= -3 \pm 1 \tag{6.864}$$

Therefore, $t = -2$ or $t = -4$, which means $m^2 = -2$ or $m^2 = -4$, leading to $m = \pm\sqrt{2}i$ or $m = \pm 2i$.
So the roots of the auxiliary equation are $m_1 = \sqrt{2}i$, $m_2 = -\sqrt{2}i$, $m_3 = 2i$, and $m_4 = -2i$.
The complementary function is:

$$y_c = c_1 \cos\left(\sqrt{2}x\right) + c_2 \sin\left(\sqrt{2}x\right) + c_3 \cos(2x) + c_4 \sin(2x) \tag{6.865}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = \sin^2 x \cos 2x$ and the operator $\phi(D) = D^4 + 6D^2 + 8$.
First, we need to convert the product of trigonometric functions to a sum. We start by expressing $\sin^2 x$ in terms of cosines:

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \tag{6.866}$$

So our forcing function becomes:

$$\sin^2 x \cos 2x = \frac{1 - \cos(2x)}{2} \cos 2x \tag{6.867}$$

$$= \frac{\cos 2x - \cos^2 2x}{2} \tag{6.868}$$

Now, we can further simplify $\cos^2 2x$:

$$\cos^2 2x = \frac{1 + \cos(4x)}{2} \tag{6.869}$$

Substituting back:

$$\sin^2 x \cos 2x = \frac{\cos 2x - \frac{1+\cos(4x)}{2}}{2} \tag{6.870}$$

$$= \frac{\cos 2x - \frac{1}{2} - \frac{\cos(4x)}{2}}{2} \tag{6.871}$$

$$= \frac{2\cos 2x - 1 - \cos(4x)}{4} \tag{6.872}$$

$$= \frac{1}{4}(2\cos 2x - 1 - \cos 4x) \tag{6.873}$$

$$= \frac{1}{2}\cos 2x - \frac{1}{4} - \frac{1}{4}\cos 4x \tag{6.874}$$

Now, we need to find the particular integral for each term separately.
For the term $\frac{1}{2}\cos 2x$: We have $a = 2$. Let's check if $\phi(-a^2) = 0$:

$$\phi(-a^2) = \phi(-4) \tag{6.875}$$

$$= (-4)^2 + 6(-4) + 8 \tag{6.876}$$

$$= 16 - 24 + 8 \tag{6.877}$$

$$= 0 \tag{6.878}$$

Since $\phi(-a^2) = 0$, this is a case of failure. We need to use the formula for Case of Failure 1:

$$y_{p1} = \frac{x}{\phi'(D)} \frac{1}{2} \cos(2x) \tag{6.879}$$

Find $\phi'(D)$ with respect to D:

$$\phi'(D) = \frac{d}{dD}(D^4 + 6D^2 + 8) \tag{6.880}$$
$$= 4D^3 + 12D \tag{6.881}$$

Now we need to evaluate $\phi'(D)$ when $D^2 = -4$, which means we substitute $D^2 = -4$ while keeping D in the expression:

$$\phi'(D)|_{D^2=-4} = 4D^3 + 12D \tag{6.882}$$
$$= 4D \cdot D^2 + 12D \tag{6.883}$$
$$= 4D \cdot (-4) + 12D \tag{6.884}$$
$$= -16D + 12D \tag{6.885}$$
$$= -4D \tag{6.886}$$

Since we're operating on $\cos(2x)$, we know that $D\cos(2x) = -2\sin(2x)$. We need to account for this when we complete the particular integral. But first, let's express our intermediate result:

$$y_{p1} = \frac{x}{\phi'(D)|_{D^2=-4}} \frac{1}{2} \cos(2x) \tag{6.887}$$
$$= \frac{x}{-4D} \frac{1}{2} \cos(2x) \tag{6.888}$$
$$= -\frac{x}{8D} \cos(2x) \tag{6.889}$$

Now, to handle the D in the denominator, we use the relation $\frac{1}{D}\cos(2x) = \frac{1}{2}\sin(2x)$ (integration):

$$y_{p1} = -\frac{x}{8D} \cos(2x) \tag{6.890}$$
$$= -\frac{x}{8} \cdot \frac{1}{D} \cos(2x) \tag{6.891}$$
$$= -\frac{x}{8} \cdot \frac{1}{2} \sin(2x) \tag{6.892}$$
$$= -\frac{x}{16} \sin(2x) \tag{6.893}$$

For the constant term $-\frac{1}{4}$:

$$y_{p2} = \frac{1}{\phi(D)} \left(-\frac{1}{4}\right) \tag{6.894}$$
$$= -\frac{1}{4} \cdot \frac{1}{\phi(0)} \tag{6.895}$$
$$= -\frac{1}{4} \cdot \frac{1}{0^4 + 6(0)^2 + 8} \tag{6.896}$$
$$= -\frac{1}{4} \cdot \frac{1}{8} \tag{6.897}$$
$$= -\frac{1}{32} \tag{6.898}$$

For the term $-\frac{1}{4}\cos 4x$: We have $a = 4$. Let's check if $\phi(-a^2) = 0$:

$$\phi(-a^2) = \phi(-16) \tag{6.899}$$
$$= (-16)^2 + 6(-16) + 8 \tag{6.900}$$
$$= 256 - 96 + 8 \tag{6.901}$$
$$= 168 \neq 0 \tag{6.902}$$

Since $\phi(-a^2) \neq 0$, we can use the standard formula:

$$y_{p3} = \frac{1}{\phi(-a^2)}\left(-\frac{1}{4}\cos(4x)\right) \tag{6.903}$$
$$= -\frac{1}{4}\cdot\frac{1}{168}\cos(4x) \tag{6.904}$$
$$= -\frac{1}{672}\cos(4x) \tag{6.905}$$

Step 3: Combine the particular integrals for all terms.

$$y_p = y_{p1} + y_{p2} + y_{p3} \tag{6.906}$$
$$= -\frac{x}{16}\sin(2x) - \frac{1}{32} - \frac{1}{672}\cos(4x) \tag{6.907}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.908}$$
$$= c_1\cos\left(\sqrt{2}x\right) + c_2\sin\left(\sqrt{2}x\right) + c_3\cos(2x) + c_4\sin(2x) \tag{6.909}$$
$$- \frac{x}{16}\sin(2x) - \frac{1}{32} - \frac{1}{672}\cos(4x) \tag{6.910}$$

Therefore, the general solution of the given differential equation is:
$y = c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x) + c_3\cos(2x) + c_4\sin(2x) - \frac{x}{16}\sin(2x) - \frac{1}{32} - \frac{1}{672}\cos(4x)$
which can be rearranged as:
$y = c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x) + c_3\cos(2x) + \sin(2x)\left(c_4 - \frac{x}{16}\right) - \frac{1}{32} - \frac{1}{672}\cos(4x)$
where c_1, c_2, c_3, and c_4 are arbitrary constants.

Solved Examples on Case 3

Example 3.1

Solve the differential equation:

$$(D^3 + 3D)y = \cosh 2x \sinh 3x \tag{6.911}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
We can factor the operator:

$$D^3 + 3D = D(D^2 + 3) \tag{6.912}$$

So the auxiliary equation is:

$$D(D^2+3)=0 \tag{6.913}$$

This gives us $D=0$ or $D^2+3=0$, which means $D=0$ or $D=\pm i\sqrt{3}$.
Therefore, the complementary function is:

$$y_c = c_1 + c_2\cos\left(\sqrt{3}x\right) + c_3\sin\left(\sqrt{3}x\right) \tag{6.914}$$

where c_1, c_2, and c_3 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x)=\cosh 2x\sinh 3x$ and the operator $\phi(D)=D^3+3D=D(D^2+3)$.
First, we need to convert the product of hyperbolic functions to a sum using the identity:

$$\cosh A\sinh B = \frac{1}{2}[\sinh(A+B)+\sinh(B-A)] \tag{6.915}$$

Applying this identity with $A=2x$ and $B=3x$:

$$\begin{aligned}
\cosh 2x\sinh 3x &= \frac{1}{2}[\sinh(2x+3x)+\sinh(3x-2x)] && (6.916)\\
&= \frac{1}{2}[\sinh(5x)+\sinh(x)] && (6.917)\\
&= \frac{1}{2}\sinh(5x)+\frac{1}{2}\sinh(x) && (6.918)
\end{aligned}$$

So the forcing function becomes:

$$\begin{aligned}
f(x) &= \cosh 2x\sinh 3x && (6.919)\\
&= \frac{1}{2}\sinh(5x)+\frac{1}{2}\sinh(x) && (6.920)
\end{aligned}$$

For a hyperbolic forcing function, we use the formula:

$$y_p = \frac{1}{\phi(a^2)}\sinh(ax) \tag{6.921}$$

where we substitute $D^2=a^2$ in the operator $\phi(D)$.
For the term $\frac{1}{2}\sinh(5x)$: We have $a=5$. After substituting $D^2=25$ in the operator:

$$\begin{aligned}
\phi(D)|_{D^2=25} &= D^3+3D && (6.922)\\
&= D\cdot D^2+3D && (6.923)\\
&= D\cdot 25+3D && (6.924)\\
&= 25D+3D && (6.925)\\
&= 28D && (6.926)
\end{aligned}$$

So the particular integral for this term is:

$$\begin{aligned}
y_{p1} &= \frac{1}{\phi(D)|_{D^2=25}}\frac{1}{2}\sinh(5x) && (6.927)\\
&= \frac{1}{28D}\frac{1}{2}\sinh(5x) && (6.928)\\
&= \frac{1}{56D}\sinh(5x) && (6.929)
\end{aligned}$$

We have D in the denominator, so we need to rationalize. For a hyperbolic sine function, we know that $D\sinh(5x) = 5\cosh(5x)$, so:

$$\frac{1}{56D}\sinh(5x) = \frac{1}{56}\cdot\frac{1}{D}\sinh(5x) \tag{6.930}$$
$$= \frac{1}{56}\cdot\frac{1}{5}\cosh(5x) \tag{6.931}$$
$$= \frac{1}{280}\cosh(5x) \tag{6.932}$$

For the term $\frac{1}{2}\sinh(x)$: We have $a = 1$. After substituting $D^2 = 1$ in the operator:

$$\phi(D)|_{D^2=1} = D^3 + 3D \tag{6.933}$$
$$= D\cdot D^2 + 3D \tag{6.934}$$
$$= D\cdot 1 + 3D \tag{6.935}$$
$$= D + 3D \tag{6.936}$$
$$= 4D \tag{6.937}$$

So the particular integral for this term is:

$$y_{p2} = \frac{1}{\phi(D)|_{D^2=1}}\frac{1}{2}\sinh(x) \tag{6.938}$$
$$= \frac{1}{4D}\frac{1}{2}\sinh(x) \tag{6.939}$$
$$= \frac{1}{8D}\sinh(x) \tag{6.940}$$

Again, we have D in the denominator, so we need to rationalize. For a hyperbolic sine function, we know that $D\sinh(x) = \cosh(x)$, so:

$$\frac{1}{8D}\sinh(x) = \frac{1}{8}\cdot\frac{1}{D}\sinh(x) \tag{6.941}$$
$$= \frac{1}{8}\cdot\cosh(x) \tag{6.942}$$
$$= \frac{1}{8}\cosh(x) \tag{6.943}$$

Step 3: Combine the particular integrals for both terms.

$$y_p = y_{p1} + y_{p2} \tag{6.944}$$
$$= \frac{1}{280}\cosh(5x) + \frac{1}{8}\cosh(x) \tag{6.945}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.946}$$
$$= c_1 + c_2\cos\left(\sqrt{3}x\right) + c_3\sin\left(\sqrt{3}x\right) + \frac{1}{280}\cosh(5x) + \frac{1}{8}\cosh(x) \tag{6.947}$$

Therefore, the general solution of the given differential equation is:

$$y = c_1 + c_2\cos\left(\sqrt{3}x\right) + c_3\sin\left(\sqrt{3}x\right) + \tfrac{1}{280}\cosh(5x) + \tfrac{1}{8}\cosh(x)$$

where c_1, c_2, and c_3 are arbitrary constants.

Example 3.2

Solve the differential equation:

$$(D^4 - 1)y = \cosh x \sinh x \tag{6.948}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
We can factor the operator:

$$\begin{aligned} D^4 - 1 &= (D^2 - 1)(D^2 + 1) & (6.949)\\ &= (D - 1)(D + 1)(D^2 + 1) & (6.950)\end{aligned}$$

So the auxiliary equation is:

$$(D - 1)(D + 1)(D^2 + 1) = 0 \tag{6.951}$$

This gives us $D = 1$ or $D = -1$ or $D^2 + 1 = 0$, which means $D = 1$ or $D = -1$ or $D = \pm i$.
Therefore, the complementary function is:

$$y_c = c_1 e^x + c_2 e^{-x} + c_3 \cos(x) + c_4 \sin(x) \tag{6.952}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = \cosh x \sinh x$ and the operator $\phi(D) = D^4 - 1 = (D^2 - 1)(D^2 + 1)$.
First, we can simplify the hyperbolic product using the identity:

$$\cosh x \sinh x = \frac{1}{2} \sinh(2x) \tag{6.953}$$

So our forcing function becomes:

$$\begin{aligned} f(x) &= \cosh x \sinh x & (6.954)\\ &= \frac{1}{2} \sinh(2x) & (6.955)\end{aligned}$$

For a hyperbolic forcing function, we use the formula:

$$y_p = \frac{1}{\phi(a^2)} \sinh(ax) \tag{6.956}$$

where we substitute $D^2 = a^2$ in the operator $\phi(D)$.
For the term $\frac{1}{2}\sinh(2x)$, we have $a = 2$. After substituting $D^2 = 4$ in the operator:

$$\begin{aligned} \phi(D)|_{D^2=4} &= D^4 - 1 & (6.957)\\ &= (D^2)^2 - 1 & (6.958)\\ &= 4^2 - 1 & (6.959)\\ &= 16 - 1 & (6.960)\\ &= 15 & (6.961)\end{aligned}$$

Since $\phi(a^2) \neq 0$, we can use the standard formula:

$$y_p = \frac{1}{\phi(a^2)} \frac{1}{2} \sinh(2x) \tag{6.962}$$
$$= \frac{1}{15} \cdot \frac{1}{2} \sinh(2x) \tag{6.963}$$
$$= \frac{1}{30} \sinh(2x) \tag{6.964}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.965}$$
$$= c_1 e^{x} + c_2 e^{-x} + c_3 \cos(x) + c_4 \sin(x) + \frac{1}{30} \sinh(2x) \tag{6.966}$$

Therefore, the general solution of the given differential equation is:
$y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos(x) + c_4 \sin(x) + \frac{1}{30} \sinh(2x)$
where c_1, c_2, c_3, and c_4 are arbitrary constants.

Example 3.3

Solve the differential equation:

$$(D^2 + 13D + 36)y = e^{-4x} + \sinh x \tag{6.967}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 13D + 36 = 0 \tag{6.968}$$
$$m^2 + 13m + 36 = 0 \tag{6.969}$$

We can factor this equation:

$$m^2 + 13m + 36 = 0 \tag{6.970}$$
$$m^2 + 9m + 4m + 36 = 0 \tag{6.971}$$
$$m(m+9) + 4(m+9) = 0 \tag{6.972}$$
$$(m+9)(m+4) = 0 \tag{6.973}$$

Therefore, $m = -9$ or $m = -4$. The complementary function is:

$$y_c = c_1 e^{-9x} + c_2 e^{-4x} \tag{6.974}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = e^{-4x} + \sinh x$ and the operator $\phi(D) = D^2 + 13D + 36$. We need to find the particular integral for each term separately and then add them.

For the term e^{-4x}: Using Case 1 methodology for exponential functions, we have $a = -4$. Let's check if $\phi(a) = 0$:

$$\phi(-4) = (-4)^2 + 13(-4) + 36 \tag{6.975}$$
$$= 16 - 52 + 36 \tag{6.976}$$
$$= 0 \tag{6.977}$$

Since $\phi(-4) = 0$, this is a case of failure. We need to use the formula for Case of Failure 1:

$$y_{p1} = \frac{x}{\phi'(D)} e^{-4x} \tag{6.978}$$

Find $\phi'(D)$ with respect to D:

$$\phi'(D) = \frac{d}{dD}(D^2 + 13D + 36) \tag{6.979}$$
$$= 2D + 13 \tag{6.980}$$

Evaluating $\phi'(D)$ at $D = -4$:

$$\phi'(-4) = 2(-4) + 13 \tag{6.981}$$
$$= -8 + 13 \tag{6.982}$$
$$= 5 \tag{6.983}$$

So the particular integral for this term is:

$$y_{p1} = \frac{x}{\phi'(-4)} e^{-4x} \tag{6.984}$$
$$= \frac{x}{5} e^{-4x} \tag{6.985}$$

For the term $\sinh x$: Using Case 3 methodology for hyperbolic functions, we have $a = 1$. After substituting $D^2 = 1$ in the operator:

$$\phi(D)|_{D^2=1} = D^2 + 13D + 36 \tag{6.986}$$
$$= 1 + 13D + 36 \tag{6.987}$$
$$= 37 + 13D \tag{6.988}$$

Since $\phi(D)|_{D^2=1}$ contains D, we need to rationalize. For the operator $37 + 13D$, when operating on $\sinh x$, we need to use the property that $D \sinh x = \cosh x$.
Let's rationalize:

$$\frac{1}{37 + 13D} = \frac{37 - 13D}{(37 + 13D)(37 - 13D)} \tag{6.989}$$
$$= \frac{37 - 13D}{37^2 - (13D)^2} \tag{6.990}$$
$$= \frac{37 - 13D}{1369 - 169D^2} \tag{6.991}$$

Substituting $D^2 = 1$:

$$\frac{37 - 13D}{1369 - 169D^2} = \frac{37 - 13D}{1369 - 169 \cdot 1} \tag{6.992}$$
$$= \frac{37 - 13D}{1369 - 169} \tag{6.993}$$
$$= \frac{37 - 13D}{1200} \tag{6.994}$$

So the particular integral for this term is:

$$y_{p2} = \frac{1}{\phi(D)|_{D^2=1}} \sinh x \quad (6.995)$$
$$= \frac{37 - 13D}{1200} \sinh x \quad (6.996)$$
$$= \frac{37}{1200} \sinh x - \frac{13D}{1200} \sinh x \quad (6.997)$$

Using $D \sinh x = \cosh x$:

$$y_{p2} = \frac{37}{1200} \sinh x - \frac{13}{1200} \cosh x \quad (6.998)$$
$$= \frac{37}{1200} \sinh x - \frac{13}{1200} \cosh x \quad (6.999)$$

Step 3: Combine the particular integrals for both terms.

$$y_p = y_{p1} + y_{p2} \quad (6.1000)$$
$$= \frac{x}{5} e^{-4x} + \frac{37}{1200} \sinh x - \frac{13}{1200} \cosh x \quad (6.1001)$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \quad (6.1002)$$
$$= c_1 e^{-9x} + c_2 e^{-4x} + \frac{x}{5} e^{-4x} + \frac{37}{1200} \sinh x - \frac{13}{1200} \cosh x \quad (6.1003)$$

The term $c_2 e^{-4x}$ can be combined with $\frac{x}{5} e^{-4x}$:

$$y = c_1 e^{-9x} + e^{-4x} \left(c_2 + \frac{x}{5} \right) + \frac{37}{1200} \sinh x - \frac{13}{1200} \cosh x \quad (6.1004)$$

Therefore, the general solution of the given differential equation is:

$$y = c_1 e^{-9x} + e^{-4x} \left(c_2 + \frac{x}{5} \right) + \frac{37}{1200} \sinh x - \frac{13}{1200} \cosh x$$

where c_1 and c_2 are arbitrary constants.

Example 3.4

Solve the differential equation:

$$(D^3 + 1)y = \sin(2x + 3) + e^{-x} + 2^x \quad (6.1005)$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.

Step 1: Find the complementary function y_c.

The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^3 + 1 = 0 \quad (6.1006)$$
$$m^3 + 1 = 0 \quad (6.1007)$$

This is a cubic equation. We can factor it as:

$$m^3 + 1 = 0 \tag{6.1008}$$
$$(m+1)(m^2 - m + 1) = 0 \tag{6.1009}$$

So $m = -1$ or $m^2 - m + 1 = 0$. Using the quadratic formula for the second equation:

$$m = \frac{1 \pm \sqrt{1-4}}{2} \tag{6.1010}$$
$$= \frac{1 \pm \sqrt{-3}}{2} \tag{6.1011}$$
$$= \frac{1 \pm i\sqrt{3}}{2} \tag{6.1012}$$

Therefore, the roots are $m_1 = -1$, $m_2 = \frac{1+i\sqrt{3}}{2}$, and $m_3 = \frac{1-i\sqrt{3}}{2}$.
For the complex roots, we can write:

$$\frac{1 \pm i\sqrt{3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \tag{6.1013}$$

So we have $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. The complementary function is:

$$y_c = c_1 e^{-x} + e^{\alpha x}(c_2 \cos(\beta x) + c_3 \sin(\beta x)) \tag{6.1014}$$
$$= c_1 e^{-x} + e^{x/2}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right) \tag{6.1015}$$

where c_1, c_2, and c_3 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = \sin(2x+3) + e^{-x} + 2^x$ and the operator $\phi(D) = D^3 + 1$.
We'll find the particular integral for each term separately.
For the term $\sin(2x+3)$: Using Case 2 methodology for trigonometric functions, we have $a = 2$ and $b = 3$. We substitute $D^2 = -a^2 = -4$ in the operator:

$$\phi(D)|_{D^2=-4} = D^3 + 1 \tag{6.1016}$$
$$= D \cdot D^2 + 1 \tag{6.1017}$$
$$= D \cdot (-4) + 1 \tag{6.1018}$$
$$= -4D + 1 \tag{6.1019}$$

Since $\phi(D)|_{D^2=-4}$ contains D, we need to rationalize. For the operator $-4D+1$, when operating on $\sin(2x+3)$, we need to use the property that $D\sin(2x+3) = 2\cos(2x+3)$.
Let's rationalize:

$$\frac{1}{-4D+1} = \frac{4D+1}{(-4D+1)(4D+1)} \tag{6.1020}$$
$$= \frac{4D+1}{-16D^2+1} \tag{6.1021}$$

Substituting $D^2 = -4$:

$$\frac{4D+1}{-16D^2+1} = \frac{4D+1}{-16(-4)+1} \tag{6.1022}$$
$$= \frac{4D+1}{64+1} \tag{6.1023}$$
$$= \frac{4D+1}{65} \tag{6.1024}$$

So the particular integral for this term is:

$$y_{p1} = \frac{1}{\phi(D)|_{D^2=-4}} \sin(2x+3) \tag{6.1025}$$
$$= \frac{4D+1}{65} \sin(2x+3) \tag{6.1026}$$
$$= \frac{4}{65} D \sin(2x+3) + \frac{1}{65} \sin(2x+3) \tag{6.1027}$$

Using $D \sin(2x+3) = 2\cos(2x+3)$:

$$y_{p1} = \frac{4}{65} \cdot 2\cos(2x+3) + \frac{1}{65} \sin(2x+3) \tag{6.1028}$$
$$= \frac{8}{65} \cos(2x+3) + \frac{1}{65} \sin(2x+3) \tag{6.1029}$$

For the term e^{-x}: Using Case 1 methodology for exponential functions, we have $a = -1$. Let's check if $\phi(a) = 0$:

$$\phi(-1) = (-1)^3 + 1 \tag{6.1030}$$
$$= -1 + 1 \tag{6.1031}$$
$$= 0 \tag{6.1032}$$

Since $\phi(-1) = 0$, this is a case of failure. We need to use the formula for Case of Failure 1:

$$y_{p2} = \frac{x}{\phi'(D)} e^{-x} \tag{6.1033}$$

Find $\phi'(D)$ with respect to D:

$$\phi'(D) = \frac{d}{dD}(D^3 + 1) \tag{6.1034}$$
$$= 3D^2 \tag{6.1035}$$

Evaluating $\phi'(D)$ at $D = -1$:

$$\phi'(-1) = 3(-1)^2 \tag{6.1036}$$
$$= 3 \cdot 1 \tag{6.1037}$$
$$= 3 \tag{6.1038}$$

So the particular integral for this term is:

$$y_{p2} = \frac{x}{\phi'(-1)} e^{-x} \tag{6.1039}$$
$$= \frac{x}{3} e^{-x} \tag{6.1040}$$

For the term 2^x: We can rewrite $2^x = e^{x \ln 2}$, so we have $a = \ln 2$. Let's check if $\phi(a) = 0$:

$$\phi(\ln 2) = (\ln 2)^3 + 1 \tag{6.1041}$$

Since $\ln 2 \approx 0.693$ and $\ln 2 \neq -1$, we have $\phi(\ln 2) \neq 0$. We can use the standard formula:

$$y_{p3} = \frac{1}{\phi(\ln 2)} 2^x \tag{6.1042}$$
$$= \frac{1}{(\ln 2)^3 + 1} 2^x \tag{6.1043}$$

Since $(\ln 2)^3 + 1$ is a numerical value, we can compute it:

$$(\ln 2)^3 + 1 \approx (0.693)^3 + 1 \tag{6.1044}$$
$$\approx 0.333 + 1 \tag{6.1045}$$
$$\approx 1.333 \tag{6.1046}$$

So the particular integral for this term is:

$$y_{p3} = \frac{1}{1.333} 2^x \tag{6.1047}$$
$$\approx \frac{3}{4} 2^x \tag{6.1048}$$

Step 3: Combine the particular integrals for all terms.

$$y_p = y_{p1} + y_{p2} + y_{p3} \tag{6.1049}$$
$$= \frac{8}{65}\cos(2x+3) + \frac{1}{65}\sin(2x+3) + \frac{x}{3}e^{-x} + \frac{3}{4}2^x \tag{6.1050}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1051}$$
$$= c_1 e^{-x} + e^{x/2}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right) \tag{6.1052}$$
$$+ \frac{8}{65}\cos(2x+3) + \frac{1}{65}\sin(2x+3) + \frac{x}{3}e^{-x} + \frac{3}{4}2^x \tag{6.1053}$$

The terms $c_1 e^{-x}$ and $\frac{x}{3}e^{-x}$ can be combined:

$$y = e^{-x}\left(c_1 + \frac{x}{3}\right) + e^{x/2}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right) \tag{6.1054}$$
$$+ \frac{8}{65}\cos(2x+3) + \frac{1}{65}\sin(2x+3) + \frac{3}{4}2^x \tag{6.1055}$$

Therefore, the general solution of the given differential equation is:
$y = e^{-x}\left(c_1 + \frac{x}{3}\right) + e^{x/2}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right) + \frac{8}{65}\cos(2x+3) + \frac{1}{65}\sin(2x+3) + \frac{3}{4}2^x$
where c_1, c_2, and c_3 are arbitrary constants.

Solved Examples on Case 4

Example 4.1

Solve the differential equation:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 50x \quad \text{with} \quad y = 0, \frac{dy}{dx} = 1 \quad \text{at} \quad x = 0 \tag{6.1056}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.

Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 6D + 10 = 0 \tag{6.1057}$$
$$m^2 + 6m + 10 = 0 \tag{6.1058}$$

Using the quadratic formula:

$$m = \frac{-6 \pm \sqrt{36 - 40}}{2} \tag{6.1059}$$
$$= \frac{-6 \pm \sqrt{-4}}{2} \tag{6.1060}$$
$$= \frac{-6 \pm 2i}{2} \tag{6.1061}$$
$$= -3 \pm i \tag{6.1062}$$

Since we have complex roots $m = -3 \pm i$, the complementary function is:

$$y_c = e^{-3x}(c_1 \cos x + c_2 \sin x) \tag{6.1063}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = 50x$ and the operator $\phi(D) = D^2 + 6D + 10$.
Since $f(x) = 50x$ is a polynomial with $m = 1$, we'll use the Case 4 methodology. We need to express $\frac{1}{\phi(D)}$ in ascending powers of D up to D^1 (since $m = 1$).
First, let's factor out the constant term from $\phi(D)$:

$$\phi(D) = D^2 + 6D + 10 \tag{6.1064}$$
$$= 10\left(\frac{D^2}{10} + \frac{6D}{10} + 1\right) \tag{6.1065}$$
$$= 10\left(1 + \frac{6D}{10} + \frac{D^2}{10}\right) \tag{6.1066}$$

Now, we can use the binomial expansion for $(1 + x)^{-1} = 1 - x + x^2 - \ldots$:

$$\frac{1}{\phi(D)} = \frac{1}{10}\left(1 + \frac{6D}{10} + \frac{D^2}{10}\right)^{-1} \tag{6.1067}$$
$$= \frac{1}{10}\left[1 - \left(\frac{6D}{10} + \frac{D^2}{10}\right) + \left(\frac{6D}{10} + \frac{D^2}{10}\right)^2 - \ldots\right] \tag{6.1068}$$

Since we only need terms up to D^1 when applying to x^1, we can simplify this to:

$$\frac{1}{\phi(D)} = \frac{1}{10}\left[1 - \frac{6D}{10} + \ldots\right] \tag{6.1069}$$
$$= \frac{1}{10} - \frac{6}{100}D + \ldots \tag{6.1070}$$

Now, we apply this expansion to $50x$:

$$y_p = \frac{1}{\phi(D)}(50x) \tag{6.1071}$$
$$= \left(\frac{1}{10} - \frac{6}{100}D + \ldots\right)(50x) \tag{6.1072}$$
$$= \frac{50}{10}x - \frac{6 \cdot 50}{100}Dx + \ldots \tag{6.1073}$$

We know that $Dx = 1$, so:

$$y_p = 5x - \frac{6 \cdot 50}{100} \cdot 1 + \dots \quad (6.1074)$$
$$= 5x - 3 + \dots \quad (6.1075)$$

Since the higher-order terms in the expansion (D^2 and above) will result in zero when applied to x, we can stop here:

$$y_p = 5x - 3 \quad (6.1076)$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \quad (6.1077)$$
$$= e^{-3x}(c_1 \cos x + c_2 \sin x) + 5x - 3 \quad (6.1078)$$

Step 4: Apply the initial conditions to find the values of c_1 and c_2.
Given that $y = 0$ at $x = 0$:

$$0 = e^{-3 \cdot 0}(c_1 \cos 0 + c_2 \sin 0) + 5 \cdot 0 - 3 \quad (6.1079)$$
$$0 = 1 \cdot (c_1 \cdot 1 + c_2 \cdot 0) + 0 - 3 \quad (6.1080)$$
$$0 = c_1 - 3 \quad (6.1081)$$

Therefore, $c_1 = 3$.
Given that $\frac{dy}{dx} = 1$ at $x = 0$, we first find the derivative of y:

$$\frac{dy}{dx} = -3e^{-3x}(c_1 \cos x + c_2 \sin x) + e^{-3x}(-c_1 \sin x + c_2 \cos x) + 5 \quad (6.1082)$$
$$= e^{-3x}[(-3c_1) \cos x + (-3c_2) \sin x + (-c_1) \sin x + c_2 \cos x] + 5 \quad (6.1083)$$
$$= e^{-3x}[(-3c_1 + c_2) \cos x + (-3c_2 - c_1) \sin x] + 5 \quad (6.1084)$$

Now, evaluating at $x = 0$ and setting equal to 1:

$$1 = e^{-3 \cdot 0}[(-3c_1 + c_2) \cos 0 + (-3c_2 - c_1) \sin 0] + 5 \quad (6.1085)$$
$$1 = 1 \cdot [(-3c_1 + c_2) \cdot 1 + (-3c_2 - c_1) \cdot 0] + 5 \quad (6.1086)$$
$$1 = -3c_1 + c_2 + 5 \quad (6.1087)$$

Substituting $c_1 = 3$:

$$1 = -3 \cdot 3 + c_2 + 5 \quad (6.1088)$$
$$1 = -9 + c_2 + 5 \quad (6.1089)$$
$$1 = c_2 - 4 \quad (6.1090)$$
$$c_2 = 5 \quad (6.1091)$$

Therefore, $c_1 = 3$ and $c_2 = 5$.
Substituting these values back into the general solution:

$$y = e^{-3x}(3 \cos x + 5 \sin x) + 5x - 3 \quad (6.1092)$$

Therefore, the particular solution of the given differential equation satisfying the initial conditions is:

$$y = e^{-3x}(3 \cos x + 5 \sin x) + 5x - 3$$

Example 4.2

Solve the differential equation:

$$(D^2 - 2D + 5)y = 25x^2 \tag{6.1093}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 - 2D + 5 = 0 \tag{6.1094}$$
$$m^2 - 2m + 5 = 0 \tag{6.1095}$$

Using the quadratic formula:

$$m = \frac{2 \pm \sqrt{4-20}}{2} \tag{6.1096}$$
$$= \frac{2 \pm \sqrt{-16}}{2} \tag{6.1097}$$
$$= \frac{2 \pm 4i}{2} \tag{6.1098}$$
$$= 1 \pm 2i \tag{6.1099}$$

Since we have complex roots $m = 1 \pm 2i$, the complementary function is:

$$y_c = e^x(c_1 \cos(2x) + c_2 \sin(2x)) \tag{6.1100}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = 25x^2$ and the operator $\phi(D) = D^2 - 2D + 5$.
Since $f(x) = 25x^2$ is a polynomial with $m = 2$, we'll use the Case 4 methodology. We need to express $\frac{1}{\phi(D)}$ in ascending powers of D up to D^2 (since $m = 2$).
First, let's factor out the constant term from $\phi(D)$:

$$\phi(D) = D^2 - 2D + 5 \tag{6.1101}$$
$$= 5\left(\frac{D^2}{5} - \frac{2D}{5} + 1\right) \tag{6.1102}$$
$$= 5\left(1 - \frac{2D}{5} + \frac{D^2}{5}\right) \tag{6.1103}$$

Now, we can use the binomial expansion for $(1+x)^{-1} = 1 - x + x^2 - \ldots$:

$$\frac{1}{\phi(D)} = \frac{1}{5}\left(1 - \frac{2D}{5} + \frac{D^2}{5}\right)^{-1} \tag{6.1104}$$
$$= \frac{1}{5}\left[1 - \left(-\frac{2D}{5} + \frac{D^2}{5}\right) + \left(-\frac{2D}{5} + \frac{D^2}{5}\right)^2 - \ldots\right] \tag{6.1105}$$
$$= \frac{1}{5}\left[1 + \frac{2D}{5} - \frac{D^2}{5} + \left(\frac{2D}{5} - \frac{D^2}{5}\right)^2 - \ldots\right] \tag{6.1106}$$

Let's calculate the squared term:

$$\left(\frac{2D}{5} - \frac{D^2}{5}\right)^2 = \left(\frac{2D}{5}\right)^2 - 2 \cdot \frac{2D}{5} \cdot \frac{D^2}{5} + \left(\frac{D^2}{5}\right)^2 \tag{6.1107}$$

$$= \frac{4D^2}{25} - \frac{4D^3}{25} + \frac{D^4}{25} \tag{6.1108}$$

Since we only need terms up to D^2 when applying to x^2, we can simplify this to:

$$\frac{1}{\phi(D)} = \frac{1}{5}\left[1 + \frac{2D}{5} - \frac{D^2}{5} + \frac{4D^2}{25} + \dots\right] \tag{6.1109}$$

$$= \frac{1}{5} + \frac{2}{25}D + \left(-\frac{1}{25} + \frac{4}{125}\right)D^2 + \dots \tag{6.1110}$$

$$= \frac{1}{5} + \frac{2}{25}D + \left(\frac{-5+4}{125}\right)D^2 + \dots \tag{6.1111}$$

$$= \frac{1}{5} + \frac{2}{25}D - \frac{1}{125}D^2 + \dots \tag{6.1112}$$

Now, we apply this expansion to $25x^2$:

$$y_p = \frac{1}{\phi(D)}(25x^2) \tag{6.1113}$$

$$= \left(\frac{1}{5} + \frac{2}{25}D - \frac{1}{125}D^2 + \dots\right)(25x^2) \tag{6.1114}$$

$$= \frac{25}{5}x^2 + \frac{2 \cdot 25}{25}Dx^2 - \frac{25}{125}D^2x^2 + \dots \tag{6.1115}$$

We know that:

$$Dx^2 = 2x \tag{6.1116}$$

$$D^2x^2 = D(2x) = 2 \tag{6.1117}$$

So:

$$y_p = 5x^2 + 2 \cdot 2x - \frac{25}{125} \cdot 2 + \dots \tag{6.1118}$$

$$= 5x^2 + 4x - \frac{50}{125} + \dots \tag{6.1119}$$

$$= 5x^2 + 4x - \frac{2}{5} + \dots \tag{6.1120}$$

Since the higher-order terms in the expansion (D^3 and above) will result in zero when applied to x^2, we can stop here:

$$y_p = 5x^2 + 4x - \frac{2}{5} \tag{6.1121}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1122}$$

$$= e^x(c_1\cos(2x) + c_2\sin(2x)) + 5x^2 + 4x - \frac{2}{5} \tag{6.1123}$$

Therefore, the general solution of the given differential equation is:

$$y = e^x(c_1\cos(2x) + c_2\sin(2x)) + 5x^2 + 4x - \tfrac{2}{5}$$

where c_1 and c_2 are arbitrary constants.

Example 4.3

Solve the differential equation:

$$(D^4 + D^2 + 1)y = 53x^2 + 17 \tag{6.1124}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.

Step 1: Find the complementary function y_c.

The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^4 + D^2 + 1 = 0 \tag{6.1125}$$

This is a biquadratic equation. Let $m^2 = t$, so we have:

$$t^2 + t + 1 = 0 \tag{6.1126}$$

Using the quadratic formula:

$$t = \frac{-1 \pm \sqrt{1-4}}{2} \tag{6.1127}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} \tag{6.1128}$$

$$= \frac{-1 \pm i\sqrt{3}}{2} \tag{6.1129}$$

Therefore, $t = \frac{-1+i\sqrt{3}}{2}$ or $t = \frac{-1-i\sqrt{3}}{2}$, which means $m^2 = \frac{-1\pm i\sqrt{3}}{2}$.

Taking the square root of complex numbers, we get four roots for m:

$$m = \pm\sqrt{\frac{-1+i\sqrt{3}}{2}}, \pm\sqrt{\frac{-1-i\sqrt{3}}{2}} \tag{6.1130}$$

These roots can be expressed in the form $m = \alpha \pm i\beta$ for appropriate values of α and β.

The complementary function is:

$$y_c = e^{\alpha_1 x}(c_1 \cos(\beta_1 x) + c_2 \sin(\beta_1 x)) + e^{\alpha_2 x}(c_3 \cos(\beta_2 x) + c_4 \sin(\beta_2 x)) \tag{6.1131}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

For our purposes, we can express the complementary function more generally as:

$$y_c = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 \tag{6.1132}$$

where y_1, y_2, y_3, and y_4 are the linearly independent solutions corresponding to the four roots of the auxiliary equation.

Step 2: Find the particular integral y_p.

We have the forcing function $f(x) = 53x^2 + 17$ and the operator $\phi(D) = D^4 + D^2 + 1$.

We'll find the particular integral for each term separately.

For the constant term 17: Since this is a polynomial with $m = 0$, we need to express $\frac{1}{\phi(D)}$ up to D^0. For a constant, we only need the constant term in the expansion:

$$y_{p1} = \frac{1}{\phi(D)}(17) \tag{6.1133}$$
$$= \frac{17}{\phi(0)} \tag{6.1134}$$
$$= \frac{17}{0^4 + 0^2 + 1} \tag{6.1135}$$
$$= \frac{17}{1} \tag{6.1136}$$
$$= 17 \tag{6.1137}$$

For the term $53x^2$: Since this is a polynomial with $m = 2$, we need to express $\frac{1}{\phi(D)}$ in ascending powers of D up to D^2.
First, let's factor out the constant term from $\phi(D)$:

$$\phi(D) = D^4 + D^2 + 1 \tag{6.1138}$$
$$= 1\left(D^4 + D^2 + 1\right) \tag{6.1139}$$

Now, we can use the binomial expansion for $(1 + x)^{-1} = 1 - x + x^2 - \ldots$:

$$\frac{1}{\phi(D)} = \frac{1}{1}\left(D^4 + D^2 + 1\right)^{-1} \tag{6.1140}$$
$$= \left(1 + D^2 + D^4\right)^{-1} \tag{6.1141}$$
$$= 1 - (D^2 + D^4) + (D^2 + D^4)^2 - \ldots \tag{6.1142}$$

Since we only need terms up to D^2 when applying to x^2, we can simplify this to:

$$\frac{1}{\phi(D)} = 1 - D^2 + \ldots \tag{6.1143}$$

Now, we apply this expansion to $53x^2$:

$$y_{p2} = \frac{1}{\phi(D)}(53x^2) \tag{6.1144}$$
$$= \left(1 - D^2 + \ldots\right)(53x^2) \tag{6.1145}$$
$$= 53x^2 - 53D^2x^2 + \ldots \tag{6.1146}$$

We know that $D^2x^2 = D(Dx^2) = D(2x) = 2$, so:

$$y_{p2} = 53x^2 - 53 \cdot 2 + \ldots \tag{6.1147}$$
$$= 53x^2 - 106 + \ldots \tag{6.1148}$$

Since the higher-order terms in the expansion (D^3 and above) will result in zero when applied to x^2, we can stop here:

$$y_{p2} = 53x^2 - 106 \tag{6.1149}$$

Step 3: Combine the particular integrals for both terms.

$$y_p = y_{p1} + y_{p2} \tag{6.1150}$$
$$= 17 + 53x^2 - 106 \tag{6.1151}$$
$$= 53x^2 - 89 \tag{6.1152}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1153}$$
$$= c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 + 53x^2 - 89 \tag{6.1154}$$

Therefore, the general solution of the given differential equation is:
$$y = c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 + 53x^2 - 89$$
where c_1, c_2, c_3, and c_4 are arbitrary constants, and y_1, y_2, y_3, and y_4 are the linearly independent solutions corresponding to the four roots of the auxiliary equation.

Example 4.4

Solve the differential equation:

$$(D^2 + 5D + 4)y = x^2 + 7x + 9 \tag{6.1155}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 5D + 4 = 0 \tag{6.1156}$$
$$m^2 + 5m + 4 = 0 \tag{6.1157}$$

We can factor this equation:

$$m^2 + 5m + 4 = 0 \tag{6.1158}$$
$$m^2 + 4m + m + 4 = 0 \tag{6.1159}$$
$$m(m + 4) + 1(m + 4) = 0 \tag{6.1160}$$
$$(m + 4)(m + 1) = 0 \tag{6.1161}$$

Therefore, $m = -4$ or $m = -1$. The complementary function is:

$$y_c = c_1e^{-4x} + c_2e^{-x} \tag{6.1162}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = x^2 + 7x + 9$ and the operator $\phi(D) = D^2 + 5D + 4$.
We'll find the particular integral for each term separately.
For the term x^2: Since this is a polynomial with $m = 2$, we need to express $\frac{1}{\phi(D)}$ in ascending powers of D up to D^2.
First, let's factor out the constant term from $\phi(D)$:

$$\phi(D) = D^2 + 5D + 4 \tag{6.1163}$$
$$= 4\left(\frac{D^2}{4} + \frac{5D}{4} + 1\right) \tag{6.1164}$$
$$= 4\left(1 + \frac{5D}{4} + \frac{D^2}{4}\right) \tag{6.1165}$$

Now, we can use the binomial expansion for $(1+x)^{-1} = 1 - x + x^2 - \ldots$:

$$\frac{1}{\phi(D)} = \frac{1}{4}\left(1 + \frac{5D}{4} + \frac{D^2}{4}\right)^{-1} \tag{6.1166}$$

$$= \frac{1}{4}\left[1 - \left(\frac{5D}{4} + \frac{D^2}{4}\right) + \left(\frac{5D}{4} + \frac{D^2}{4}\right)^2 - \ldots\right] \tag{6.1167}$$

Let's calculate the squared term:

$$\left(\frac{5D}{4} + \frac{D^2}{4}\right)^2 = \left(\frac{5D}{4}\right)^2 + 2 \cdot \frac{5D}{4} \cdot \frac{D^2}{4} + \left(\frac{D^2}{4}\right)^2 \tag{6.1168}$$

$$= \frac{25D^2}{16} + \frac{10D^3}{16} + \frac{D^4}{16} \tag{6.1169}$$

Since we only need terms up to D^2 when applying to x^2, we can simplify this to:

$$\frac{1}{\phi(D)} = \frac{1}{4}\left[1 - \left(\frac{5D}{4} + \frac{D^2}{4}\right) + \frac{25D^2}{16} + \ldots\right] \tag{6.1170}$$

$$= \frac{1}{4} - \frac{5}{16}D - \frac{1}{16}D^2 + \frac{25}{64}D^2 + \ldots \tag{6.1171}$$

$$= \frac{1}{4} - \frac{5}{16}D + \left(\frac{-1 + 25/4}{16}\right)D^2 + \ldots \tag{6.1172}$$

$$= \frac{1}{4} - \frac{5}{16}D + \frac{21}{64}D^2 + \ldots \tag{6.1173}$$

Now, we apply this expansion to x^2:

$$y_{p1} = \frac{1}{\phi(D)}(x^2) \tag{6.1174}$$

$$= \left(\frac{1}{4} - \frac{5}{16}D + \frac{21}{64}D^2 + \ldots\right)(x^2) \tag{6.1175}$$

$$= \frac{1}{4}x^2 - \frac{5}{16}Dx^2 + \frac{21}{64}D^2x^2 + \ldots \tag{6.1176}$$

We know that:

$$Dx^2 = 2x \tag{6.1177}$$

$$D^2x^2 = D(2x) = 2 \tag{6.1178}$$

So:

$$y_{p1} = \frac{1}{4}x^2 - \frac{5}{16} \cdot 2x + \frac{21}{64} \cdot 2 + \ldots \tag{6.1179}$$

$$= \frac{1}{4}x^2 - \frac{5}{8}x + \frac{21}{32} + \ldots \tag{6.1180}$$

Since the higher-order terms in the expansion (D^3 and above) will result in zero when applied to x^2, we can stop here:

$$y_{p1} = \frac{1}{4}x^2 - \frac{5}{8}x + \frac{21}{32} \tag{6.1181}$$

For the term $7x$: Since this is a polynomial with $m = 1$, we need to express $\frac{1}{\phi(D)}$ in ascending powers of D up to D^1.
Using the expansion we already derived:

$$\frac{1}{\phi(D)} = \frac{1}{4} - \frac{5}{16}D + \dots \tag{6.1182}$$

Now, we apply this expansion to $7x$:

$$y_{p2} = \frac{1}{\phi(D)}(7x) \tag{6.1183}$$
$$= \left(\frac{1}{4} - \frac{5}{16}D + \dots\right)(7x) \tag{6.1184}$$
$$= \frac{7}{4}x - \frac{5}{16}D(7x) + \dots \tag{6.1185}$$

We know that $Dx = 1$, so:

$$y_{p2} = \frac{7}{4}x - \frac{5}{16} \cdot 7 \cdot 1 + \dots \tag{6.1186}$$
$$= \frac{7}{4}x - \frac{35}{16} + \dots \tag{6.1187}$$
$$= \frac{7}{4}x - \frac{35}{16} \tag{6.1188}$$

For the constant term 9: Since this is a polynomial with $m = 0$, we only need the constant term in the expansion:

$$y_{p3} = \frac{1}{\phi(D)}(9) \tag{6.1189}$$
$$= \frac{1}{4} \cdot 9 \tag{6.1190}$$
$$= \frac{9}{4} \tag{6.1191}$$

Step 3: Combine the particular integrals for all terms.

$$y_p = y_{p1} + y_{p2} + y_{p3} \tag{6.1192}$$
$$= \frac{1}{4}x^2 - \frac{5}{8}x + \frac{21}{32} + \frac{7}{4}x - \frac{35}{16} + \frac{9}{4} \tag{6.1193}$$
$$= \frac{1}{4}x^2 + \left(-\frac{5}{8} + \frac{7}{4}\right)x + \left(\frac{21}{32} - \frac{35}{16} + \frac{9}{4}\right) \tag{6.1194}$$
$$= \frac{1}{4}x^2 + \frac{9}{8}x + \left(\frac{21}{32} - \frac{70}{32} + \frac{72}{32}\right) \tag{6.1195}$$
$$= \frac{1}{4}x^2 + \frac{9}{8}x + \frac{23}{32} \tag{6.1196}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1197}$$
$$= c_1e^{-4x} + c_2e^{-x} + \frac{1}{4}x^2 + \frac{9}{8}x + \frac{23}{32} \tag{6.1198}$$

Therefore, the general solution of the given differential equation is:
$$y = c_1e^{-4x} + c_2e^{-x} + \tfrac{1}{4}x^2 + \tfrac{9}{8}x + \tfrac{23}{32}$$
where c_1 and c_2 are arbitrary constants.

Example 4.5

Solve the differential equation:

$$(D^3 - 2D + 4)y = 3x^2 - 5x + 2 \tag{6.1199}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^3 - 2D + 4 = 0 \tag{6.1200}$$
$$m^3 - 2m + 4 = 0 \tag{6.1201}$$

We need to find the roots of this cubic equation. Let's try some values. For $m = 0$:

$$0^3 - 2 \cdot 0 + 4 = 0 + 0 + 4 = 4 \neq 0 \tag{6.1202}$$

For $m = -2$:

$$(-2)^3 - 2(-2) + 4 = -8 + 4 + 4 = 0 \tag{6.1203}$$

So $m = -2$ is a root. We can factor the cubic equation as:

$$m^3 - 2m + 4 = 0 \tag{6.1204}$$
$$(m + 2)(m^2 - 2m + 2) = 0 \tag{6.1205}$$

Now we need to find the roots of $m^2 - 2m + 2 = 0$. Using the quadratic formula:

$$m = \frac{2 \pm \sqrt{4 - 8}}{2} \tag{6.1206}$$
$$= \frac{2 \pm \sqrt{-4}}{2} \tag{6.1207}$$
$$= \frac{2 \pm 2i}{2} \tag{6.1208}$$
$$= 1 \pm i \tag{6.1209}$$

Therefore, the roots are $m_1 = -2$, $m_2 = 1 + i$, and $m_3 = 1 - i$. The complementary function is:

$$y_c = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x) \tag{6.1210}$$

where c_1, c_2, and c_3 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = 3x^2 - 5x + 2$ and the operator $\phi(D) = D^3 - 2D + 4$.
We'll find the particular integral for each term separately.
For the term $3x^2$: Since this is a polynomial with $m = 2$, we need to express $\frac{1}{\phi(D)}$ in ascending powers of D up to D^2.

First, let's factor out the constant term from $\phi(D)$:

$$\phi(D) = D^3 - 2D + 4 \tag{6.1211}$$
$$= 4\left(\frac{D^3}{4} - \frac{2D}{4} + 1\right) \tag{6.1212}$$
$$= 4\left(1 - \frac{2D}{4} + \frac{D^3}{4}\right) \tag{6.1213}$$

Now, we can use the binomial expansion for $(1+x)^{-1} = 1 - x + x^2 - \ldots$:

$$\frac{1}{\phi(D)} = \frac{1}{4}\left(1 - \frac{2D}{4} + \frac{D^3}{4}\right)^{-1} \tag{6.1214}$$
$$= \frac{1}{4}\left[1 - \left(-\frac{2D}{4} + \frac{D^3}{4}\right) + \left(-\frac{2D}{4} + \frac{D^3}{4}\right)^2 - \ldots\right] \tag{6.1215}$$
$$= \frac{1}{4}\left[1 + \frac{2D}{4} - \frac{D^3}{4} + \left(\frac{2D}{4} - \frac{D^3}{4}\right)^2 - \ldots\right] \tag{6.1216}$$

For our purposes, since we need terms up to D^2 and there is no D^2 term in the original operator, we can simplify this to:

$$\frac{1}{\phi(D)} = \frac{1}{4} + \frac{1}{8}D + 0 \cdot D^2 + \ldots \tag{6.1217}$$

Note that we only need to consider the squared term $\left(\frac{2D}{4}\right)^2 = \frac{D^2}{4}$ from the expansion of $\left(\frac{2D}{4} - \frac{D^3}{4}\right)^2$, as the other terms will involve D^3 or higher powers.
Now, we apply this expansion to $3x^2$:

$$y_{p1} = \frac{1}{\phi(D)}(3x^2) \tag{6.1218}$$
$$= \left(\frac{1}{4} + \frac{1}{8}D + \frac{1}{16}D^2 + \ldots\right)(3x^2) \tag{6.1219}$$
$$= \frac{3}{4}x^2 + \frac{3}{8}Dx^2 + \frac{3}{16}D^2x^2 + \ldots \tag{6.1220}$$

We know that:

$$Dx^2 = 2x \tag{6.1221}$$
$$D^2x^2 = D(2x) = 2 \tag{6.1222}$$

So:

$$y_{p1} = \frac{3}{4}x^2 + \frac{3}{8} \cdot 2x + \frac{3}{16} \cdot 2 + \ldots \tag{6.1223}$$
$$= \frac{3}{4}x^2 + \frac{6}{8}x + \frac{6}{16} + \ldots \tag{6.1224}$$
$$= \frac{3}{4}x^2 + \frac{3}{4}x + \frac{3}{8} + \ldots \tag{6.1225}$$

Since the higher-order terms in the expansion (D^3 and above) will result in zero when applied to x^2, we can stop here:

$$y_{p1} = \frac{3}{4}x^2 + \frac{3}{4}x + \frac{3}{8} \tag{6.1226}$$

For the term $-5x$: Since this is a polynomial with $m = 1$, we need to express $\frac{1}{\phi(D)}$ in ascending powers of D up to D^1.
Using the expansion we already derived:

$$\frac{1}{\phi(D)} = \frac{1}{4} + \frac{1}{8}D + \dots \tag{6.1227}$$

Now, we apply this expansion to $-5x$:

$$y_{p2} = \frac{1}{\phi(D)}(-5x) \tag{6.1228}$$
$$= \left(\frac{1}{4} + \frac{1}{8}D + \dots\right)(-5x) \tag{6.1229}$$
$$= -\frac{5}{4}x - \frac{5}{8}Dx + \dots \tag{6.1230}$$

We know that $Dx = 1$, so:

$$y_{p2} = -\frac{5}{4}x - \frac{5}{8} \cdot 1 + \dots \tag{6.1231}$$
$$= -\frac{5}{4}x - \frac{5}{8} + \dots \tag{6.1232}$$
$$= -\frac{5}{4}x - \frac{5}{8} \tag{6.1233}$$

For the constant term 2: Since this is a polynomial with $m = 0$, we only need the constant term in the expansion:

$$y_{p3} = \frac{1}{\phi(D)}(2) \tag{6.1234}$$
$$= \frac{1}{4} \cdot 2 \tag{6.1235}$$
$$= \frac{1}{2} \tag{6.1236}$$

Step 3: Combine the particular integrals for all terms.

$$y_p = y_{p1} + y_{p2} + y_{p3} \tag{6.1237}$$
$$= \frac{3}{4}x^2 + \frac{3}{4}x + \frac{3}{8} - \frac{5}{4}x - \frac{5}{8} + \frac{1}{2} \tag{6.1238}$$
$$= \frac{3}{4}x^2 + \left(\frac{3}{4} - \frac{5}{4}\right)x + \left(\frac{3}{8} - \frac{5}{8} + \frac{1}{2}\right) \tag{6.1239}$$
$$= \frac{3}{4}x^2 - \frac{1}{2}x + \left(\frac{3}{8} - \frac{5}{8} + \frac{4}{8}\right) \tag{6.1240}$$
$$= \frac{3}{4}x^2 - \frac{1}{2}x + \frac{2}{8} \tag{6.1241}$$
$$= \frac{3}{4}x^2 - \frac{1}{2}x + \frac{1}{4} \tag{6.1242}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1243}$$
$$= c_1e^{-2x} + e^x(c_2\cos x + c_3\sin x) + \frac{3}{4}x^2 - \frac{1}{2}x + \frac{1}{4} \tag{6.1244}$$

Therefore, the general solution of the given differential equation is:

$$y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x) + \tfrac{3}{4}x^2 - \tfrac{1}{2}x + \tfrac{1}{4}$$
where c_1, c_2, and c_3 are arbitrary constants.

Example 4.6

Solve the differential equation:

$$(D^2 - 4D + 4)y = 8(e^{2x} + \sin 2x + x^2) \tag{6.1245}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 - 4D + 4 = 0 \tag{6.1246}$$
$$m^2 - 4m + 4 = 0 \tag{6.1247}$$

We can factor this equation:

$$m^2 - 4m + 4 = 0 \tag{6.1248}$$
$$(m-2)^2 = 0 \tag{6.1249}$$

Therefore, $m = 2$ is a repeated root. The complementary function is:

$$y_c = c_1 e^{2x} + c_2 x e^{2x} \tag{6.1250}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = 8(e^{2x} + \sin 2x + x^2)$ and the operator $\phi(D) = D^2 - 4D + 4 = (D-2)^2$.
We'll find the particular integral for each term separately.
For the term $8e^{2x}$: Using Case 1 methodology for exponential functions, we have $a = 2$. Let's check if $\phi(a) = 0$:

$$\phi(2) = (2)^2 - 4(2) + 4 \tag{6.1251}$$
$$= 4 - 8 + 4 \tag{6.1252}$$
$$= 0 \tag{6.1253}$$

Since $\phi(2) = 0$, this is a case of failure. In fact, since $\phi(D) = (D-2)^2$, we have a failure of order 2. We need to use the formula for Case of Failure 2:

$$y_{p1} = \frac{x^2}{\phi''(D)} 8e^{2x} \tag{6.1254}$$

Find $\phi''(D)$ with respect to D:

$$\phi(D) = (D-2)^2 \tag{6.1255}$$
$$\phi'(D) = 2(D-2) \tag{6.1256}$$
$$\phi''(D) = 2 \tag{6.1257}$$

So the particular integral for this term is:

$$y_{p1} = \frac{x^2}{\phi''(2)} 8e^{2x} \tag{6.1258}$$
$$= \frac{x^2}{2} 8e^{2x} \tag{6.1259}$$
$$= 4x^2 e^{2x} \tag{6.1260}$$

For the term $8 \sin 2x$**:** Using Case 2 methodology for trigonometric functions, we have $a = 2$. After substituting $D^2 = -4$ in the operator:

$$\phi(D)|_{D^2=-4} = D^2 - 4D + 4 \tag{6.1261}$$
$$= -4 - 4D + 4 \tag{6.1262}$$
$$= -4D \tag{6.1263}$$

So the particular integral for this term is:

$$y_{p2} = \frac{1}{\phi(D)|_{D^2=-4}} 8 \sin 2x \tag{6.1264}$$
$$= \frac{1}{-4D} 8 \sin 2x \tag{6.1265}$$
$$= \frac{-2}{D} \sin 2x \tag{6.1266}$$

We have D in the denominator, so we need to rationalize. For a sine function, we know that $\frac{1}{D} \sin 2x = -\frac{1}{2} \cos 2x$, so:

$$y_{p2} = \frac{-2}{D} \sin 2x \tag{6.1267}$$
$$= -2 \cdot \left(-\frac{1}{2} \cos 2x\right) \tag{6.1268}$$
$$= \cos 2x \tag{6.1269}$$

For the term $8x^2$**:** Using Case 4 methodology for polynomial functions, we have $m = 2$. We need to express $\frac{1}{\phi(D)}$ in ascending powers of D up to D^2.
First, let's express our operator in a different form for the binomial expansion:

$$\phi(D) = (D - 2)^2 \tag{6.1270}$$

Alternatively, we can express the operator in terms of powers of D and use the binomial expansion directly:

$$\phi(D) = D^2 - 4D + 4 \tag{6.1271}$$
$$= 4(1 - \frac{4D}{4} + \frac{D^2}{4}) \tag{6.1272}$$
$$= 4(1 + \frac{D^2 - 4D}{4}) \tag{6.1273}$$

Using the binomial expansion for $(1 + x)^{-1}$:

$$\frac{1}{\phi(D)} = \frac{1}{4}(1 + \frac{D^2 - 4D}{4})^{-1} \tag{6.1274}$$
$$= \frac{1}{4}[1 - \frac{D^2 - 4D}{4} + (\frac{D^2 - 4D}{4})^2 - \ldots] \tag{6.1275}$$

For our purposes, we need terms up to D^2:

$$\frac{1}{\phi(D)} = \frac{1}{4} - \frac{1}{16}(D^2 - 4D) + \dots \tag{6.1276}$$
$$= \frac{1}{4} - \frac{1}{16}D^2 + \frac{1}{4}D + \dots \tag{6.1277}$$

Now, we apply this expansion to $8x^2$:

$$y_{p3} = \frac{1}{\phi(D)}8x^2 \tag{6.1278}$$
$$= (\frac{1}{4} - \frac{1}{16}D^2 + \frac{1}{4}D + \dots)(8x^2) \tag{6.1279}$$
$$= 2x^2 - \frac{1}{2}D^2x^2 + 2Dx^2 + \dots \tag{6.1280}$$

We know that:

$$D^2x^2 = 2 \tag{6.1281}$$
$$Dx^2 = 2x \tag{6.1282}$$

So:

$$y_{p3} = 2x^2 - \frac{1}{2} \cdot 2 + 2 \cdot 2x + \dots \tag{6.1283}$$
$$= 2x^2 - 1 + 4x + \dots \tag{6.1284}$$
$$= 2x^2 + 4x - 1 + \dots \tag{6.1285}$$

Therefore, the particular integral for this term is:

$$y_{p3} = 2x^2 + 4x - 1 \tag{6.1286}$$

Step 3: Combine the particular integrals for all terms.

$$y_p = y_{p1} + y_{p2} + y_{p3} \tag{6.1287}$$
$$= 4x^2e^{2x} + \cos 2x + 2x^2 + 4x - 1 \tag{6.1288}$$

Step 4: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1289}$$
$$= c_1e^{2x} + c_2xe^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x - 1 \tag{6.1290}$$

We can combine the terms with e^{2x}:

$$y = e^{2x}(c_1 + c_2x + 4x^2) + \cos 2x + 2x^2 + 4x - 1 \tag{6.1291}$$

Therefore, the general solution of the given differential equation is:
$$y = e^{2x}(c_1 + c_2x + 4x^2) + \cos 2x + 2x^2 + 4x - 1$$
where c_1 and c_2 are arbitrary constants.

Solved Examples on Case 5

Example 5.1

Solve the differential equation:

$$(D^2 - 4)y = e^{3x}x^2 \tag{6.1292}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 - 4 = 0 \tag{6.1293}$$
$$m^2 - 4 = 0 \tag{6.1294}$$
$$m^2 = 4 \tag{6.1295}$$
$$m = \pm 2 \tag{6.1296}$$

Therefore, $m = 2$ or $m = -2$. The complementary function is:

$$y_c = c_1 e^{2x} + c_2 e^{-2x} \tag{6.1297}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = e^{3x}x^2$, which is in the form $e^{ax}V$ where $a = 3$ and $V = x^2$.
According to Case 5, we can use the formula:

$$\frac{1}{\phi(D)}(e^{ax}V) = e^{ax} \cdot \frac{1}{\phi(D+a)}V \tag{6.1298}$$

So for our problem:

$$y_p = \frac{1}{D^2 - 4}(e^{3x}x^2) \tag{6.1299}$$
$$= e^{3x} \cdot \frac{1}{(D+3)^2 - 4}x^2 \tag{6.1300}$$
$$= e^{3x} \cdot \frac{1}{D^2 + 6D + 9 - 4}x^2 \tag{6.1301}$$
$$= e^{3x} \cdot \frac{1}{D^2 + 6D + 5}x^2 \tag{6.1302}$$

Now we need to find $\frac{1}{D^2+6D+5}x^2$ using Case 4 methodology for polynomial forcing functions. Using Case 4, we need to express $\frac{1}{D^2+6D+5}$ in ascending powers of D up to D^2 (since $V = x^2$ is a polynomial with $m = 2$).
First, let's factor out the constant term from $D^2 + 6D + 5$:

$$D^2 + 6D + 5 = 5\left(\frac{D^2}{5} + \frac{6D}{5} + 1\right) \tag{6.1303}$$
$$= 5\left(1 + \frac{6D}{5} + \frac{D^2}{5}\right) \tag{6.1304}$$

Now, we can use the binomial expansion for $(1+x)^{-1} = 1 - x + x^2 - \ldots$:

$$\frac{1}{D^2+6D+5} = \frac{1}{5}\left(1+\frac{6D}{5}+\frac{D^2}{5}\right)^{-1} \tag{6.1305}$$
$$= \frac{1}{5}\left[1-\left(\frac{6D}{5}+\frac{D^2}{5}\right)+\left(\frac{6D}{5}+\frac{D^2}{5}\right)^2-\ldots\right] \tag{6.1306}$$

Let's calculate the squared term:

$$\left(\frac{6D}{5}+\frac{D^2}{5}\right)^2 = \left(\frac{6D}{5}\right)^2 + 2\cdot\frac{6D}{5}\cdot\frac{D^2}{5}+\left(\frac{D^2}{5}\right)^2 \tag{6.1307}$$
$$= \frac{36D^2}{25}+\frac{12D^3}{25}+\frac{D^4}{25} \tag{6.1308}$$

Since we only need terms up to D^2 when applying to x^2, we can simplify the expansion to:

$$\frac{1}{D^2+6D+5} = \frac{1}{5}\left[1-\left(\frac{6D}{5}+\frac{D^2}{5}\right)+\frac{36D^2}{25}+\ldots\right] \tag{6.1309}$$
$$= \frac{1}{5}-\frac{6}{25}D-\frac{1}{25}D^2+\frac{36}{125}D^2+\ldots \tag{6.1310}$$
$$= \frac{1}{5}-\frac{6}{25}D+\left(\frac{-1+36/5}{25}\right)D^2+\ldots \tag{6.1311}$$
$$= \frac{1}{5}-\frac{6}{25}D+\frac{31}{125}D^2+\ldots \tag{6.1312}$$

Now, we apply this expansion to x^2:

$$\frac{1}{D^2+6D+5}x^2 = \left(\frac{1}{5}-\frac{6}{25}D+\frac{31}{125}D^2+\ldots\right)x^2 \tag{6.1313}$$
$$= \frac{1}{5}x^2-\frac{6}{25}Dx^2+\frac{31}{125}D^2x^2+\ldots \tag{6.1314}$$

We know that:

$$Dx^2 = 2x \tag{6.1315}$$
$$D^2x^2 = D(2x) = 2 \tag{6.1316}$$

So:

$$\frac{1}{D^2+6D+5}x^2 = \frac{1}{5}x^2-\frac{6}{25}\cdot 2x+\frac{31}{125}\cdot 2+\ldots \tag{6.1317}$$
$$= \frac{1}{5}x^2-\frac{12}{25}x+\frac{62}{125}+\ldots \tag{6.1318}$$

Since the higher-order terms in the expansion (D^3 and above) will result in zero when applied to x^2, we can stop here.
Therefore, using Case 5:

$$y_p = e^{3x}\cdot\frac{1}{D^2+6D+5}x^2 \tag{6.1319}$$
$$= e^{3x}\cdot\left(\frac{1}{5}x^2-\frac{12}{25}x+\frac{62}{125}\right) \tag{6.1320}$$
$$= \frac{1}{5}e^{3x}x^2-\frac{12}{25}e^{3x}x+\frac{62}{125}e^{3x} \tag{6.1321}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1322}$$
$$= c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{5}e^{3x}x^2 - \frac{12}{25}e^{3x}x + \frac{62}{125}e^{3x} \tag{6.1323}$$

Therefore, the general solution of the given differential equation is:
$$y = c_1 e^{2x} + c_2 e^{-2x} + e^{3x}\left(\tfrac{1}{5}x^2 - \tfrac{12}{25}x + \tfrac{62}{125}\right)$$
where c_1 and c_2 are arbitrary constants.

Example 5.2

Solve the differential equation:

$$(D^2 + 2D + 1)y = \frac{e^{-x}}{x+2} \tag{6.1324}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 2D + 1 = 0 \tag{6.1325}$$
$$m^2 + 2m + 1 = 0 \tag{6.1326}$$

We can factor this equation:

$$m^2 + 2m + 1 = 0 \tag{6.1327}$$
$$(m+1)^2 = 0 \tag{6.1328}$$

Therefore, $m = -1$ is a repeated root. The complementary function is:

$$y_c = c_1 e^{-x} + c_2 x e^{-x} \tag{6.1329}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = \frac{e^{-x}}{x+2}$ and the operator $\phi(D) = D^2+2D+1 = (D+1)^2$. This forcing function can be viewed through the lens of Case 5, where $f(x) = e^{-x}V(x)$ with $V(x) = \frac{1}{x+2}$.
According to Case 5, we can use the formula:

$$\frac{1}{\phi(D)}(e^{ax}V) = e^{ax} \cdot \frac{1}{\phi(D+a)}V \tag{6.1330}$$

So for our problem:

$$y_p = \frac{1}{D^2+2D+1}\left(\frac{e^{-x}}{x+2}\right) \tag{6.1331}$$
$$= e^{-x}\cdot\frac{1}{(D-1)^2+2(D-1)+1}\left(\frac{1}{x+2}\right) \tag{6.1332}$$
$$= e^{-x}\cdot\frac{1}{D^2-2D+1+2D-2+1}\left(\frac{1}{x+2}\right) \tag{6.1333}$$
$$= e^{-x}\cdot\frac{1}{D^2+0D+0}\left(\frac{1}{x+2}\right) \tag{6.1334}$$
$$= e^{-x}\cdot\frac{1}{D^2}\left(\frac{1}{x+2}\right) \tag{6.1335}$$

Now we need to find $\frac{1}{D^2}\left(\frac{1}{x+2}\right)$.
We know that $\frac{1}{D}$ is equivalent to integration, so $\frac{1}{D^2}$ is equivalent to applying integration twice. Let's compute this step by step.
First integration:

$$\frac{1}{D}\left(\frac{1}{x+2}\right) = \int\frac{1}{x+2}dx \tag{6.1336}$$
$$= \ln|x+2| + C_1 \tag{6.1337}$$

Second integration:

$$\frac{1}{D^2}\left(\frac{1}{x+2}\right) = \frac{1}{D}[\ln|x+2| + C_1] \tag{6.1338}$$
$$= \int[\ln|x+2| + C_1]dx \tag{6.1339}$$
$$= (x+2)\ln|x+2| - (x+2) + C_1x + C_2 \tag{6.1340}$$
$$= (x+2)\ln|x+2| - x - 2 + C_1x + C_2 \tag{6.1341}$$

For the particular integral, we typically set the integration constants to zero, giving us:

$$\frac{1}{D^2}\left(\frac{1}{x+2}\right) = (x+2)\ln|x+2| - x - 2 \tag{6.1342}$$
$$= (x+2)\ln(x+2) - x - 2 \tag{6.1343}$$

where we've removed the absolute value signs since $x+2>0$ for the domain of interest.
Therefore, using Case 5:

$$y_p = e^{-x}\cdot\frac{1}{D^2}\left(\frac{1}{x+2}\right) \tag{6.1344}$$
$$= e^{-x}\cdot[(x+2)\ln(x+2) - x - 2] \tag{6.1345}$$
$$= e^{-x}(x+2)\ln(x+2) - e^{-x}x - 2e^{-x} \tag{6.1346}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1347}$$
$$= c_1e^{-x} + c_2xe^{-x} + e^{-x}(x+2)\ln(x+2) - e^{-x}x - 2e^{-x} \tag{6.1348}$$
$$\tag{6.1349}$$

Factoring out e^{-x}:

$$y = e^{-x}[c_1 + c_2 x + (x+2)\ln(x+2) - x - 2] \quad (6.1350)$$
$$= e^{-x}[c_1 + (c_2 - 1)x + (x+2)\ln(x+2) - 2] \quad (6.1351)$$

Therefore, the general solution of the given differential equation is:
$$y = e^{-x}[c_1 + (c_2 - 1)x + (x+2)\ln(x+2) - 2]$$
where c_1 and c_2 are arbitrary constants.
We can also redefine our constants if needed. Let $C_1 = c_1 - 2$ and $C_2 = c_2 - 1$, then:
$$y = e^{-x}[C_1 + C_2 x + (x+2)\ln(x+2)]$$

Example 5.3

Solve the differential equation:

$$(D^2 + 6D + 9)y = \frac{1}{x^3} e^{-3x} \quad (6.1352)$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 6D + 9 = 0 \quad (6.1353)$$
$$m^2 + 6m + 9 = 0 \quad (6.1354)$$

We can factor this equation:

$$m^2 + 6m + 9 = 0 \quad (6.1355)$$
$$(m+3)^2 = 0 \quad (6.1356)$$

Therefore, $m = -3$ is a repeated root. The complementary function is:

$$y_c = c_1 e^{-3x} + c_2 x e^{-3x} \quad (6.1357)$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = \frac{1}{x^3} e^{-3x}$ and the operator $\phi(D) = D^2 + 6D + 9 = (D+3)^2$.
This forcing function can be viewed through the lens of Case 5, where $f(x) = e^{-3x} V(x)$ with $V(x) = \frac{1}{x^3}$.
According to Case 5, we can use the formula:

$$\frac{1}{\phi(D)}(e^{ax} V) = e^{ax} \cdot \frac{1}{\phi(D+a)} V \quad (6.1358)$$

So for our problem:

$$y_p = \frac{1}{D^2+6D+9}\left(\frac{1}{x^3}e^{-3x}\right) \tag{6.1359}$$

$$= e^{-3x}\cdot\frac{1}{(D-3)^2+6(D-3)+9}\left(\frac{1}{x^3}\right) \tag{6.1360}$$

$$= e^{-3x}\cdot\frac{1}{D^2-6D+9+6D-18+9}\left(\frac{1}{x^3}\right) \tag{6.1361}$$

$$= e^{-3x}\cdot\frac{1}{D^2+0D+0}\left(\frac{1}{x^3}\right) \tag{6.1362}$$

$$= e^{-3x}\cdot\frac{1}{D^2}\left(\frac{1}{x^3}\right) \tag{6.1363}$$

Now we need to find $\frac{1}{D^2}\left(\frac{1}{x^3}\right)$.
We know that $\frac{1}{D}$ is equivalent to integration, so $\frac{1}{D^2}$ is equivalent to applying integration twice. Let's compute this step by step.
First integration:

$$\frac{1}{D}\left(\frac{1}{x^3}\right) = \int \frac{1}{x^3}dx \tag{6.1364}$$

$$= \int x^{-3}dx \tag{6.1365}$$

$$= \frac{x^{-2}}{-2} + C_1 \tag{6.1366}$$

$$= -\frac{1}{2x^2} + C_1 \tag{6.1367}$$

Second integration:

$$\frac{1}{D^2}\left(\frac{1}{x^3}\right) = \frac{1}{D}\left[-\frac{1}{2x^2}+C_1\right] \tag{6.1368}$$

$$= \int\left[-\frac{1}{2x^2}+C_1\right]dx \tag{6.1369}$$

$$= -\frac{1}{2}\int\frac{1}{x^2}dx + C_1\int dx \tag{6.1370}$$

$$= -\frac{1}{2}\cdot\frac{-1}{x} + C_1x + C_2 \tag{6.1371}$$

$$= \frac{1}{2x} + C_1x + C_2 \tag{6.1372}$$

For the particular integral, we typically set the integration constants to zero, giving us:

$$\frac{1}{D^2}\left(\frac{1}{x^3}\right) = \frac{1}{2x} \tag{6.1373}$$

Therefore, using Case 5:

$$y_p = e^{-3x}\cdot\frac{1}{D^2}\left(\frac{1}{x^3}\right) \tag{6.1374}$$

$$= e^{-3x}\cdot\frac{1}{2x} \tag{6.1375}$$

$$= \frac{e^{-3x}}{2x} \tag{6.1376}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1377}$$
$$= c_1 e^{-3x} + c_2 x e^{-3x} + \frac{e^{-3x}}{2x} \tag{6.1378}$$

Factoring out e^{-3x}:

$$y = e^{-3x}\left[c_1 + c_2 x + \frac{1}{2x}\right] \tag{6.1379}$$

Therefore, the general solution of the given differential equation is:
$y = e^{-3x}\left[c_1 + c_2 x + \frac{1}{2x}\right]$
where c_1 and c_2 are arbitrary constants.

Example 5.4

Solve the differential equation:

$$(D^2 + 2D + 1)y = e^{-x}\log x \tag{6.1380}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 2D + 1 = 0 \tag{6.1381}$$
$$m^2 + 2m + 1 = 0 \tag{6.1382}$$

We can factor this equation:

$$m^2 + 2m + 1 = 0 \tag{6.1383}$$
$$(m+1)^2 = 0 \tag{6.1384}$$

Therefore, $m = -1$ is a repeated root. The complementary function is:

$$y_c = c_1 e^{-x} + c_2 x e^{-x} \tag{6.1385}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = e^{-x}\log x$ and the operator $\phi(D) = D^2 + 2D + 1 = (D+1)^2$.
This forcing function can be viewed through the lens of Case 5, where $f(x) = e^{-x}V(x)$ with $V(x) = \log x$.
According to Case 5, we can use the formula:

$$\frac{1}{\phi(D)}(e^{ax}V) = e^{ax} \cdot \frac{1}{\phi(D+a)}V \tag{6.1386}$$

So for our problem:

$$y_p = \frac{1}{D^2+2D+1}(e^{-x}\log x) \tag{6.1387}$$
$$= e^{-x}\cdot\frac{1}{(D-1)^2+2(D-1)+1}(\log x) \tag{6.1388}$$
$$= e^{-x}\cdot\frac{1}{D^2-2D+1+2D-2+1}(\log x) \tag{6.1389}$$
$$= e^{-x}\cdot\frac{1}{D^2+0D+0}(\log x) \tag{6.1390}$$
$$= e^{-x}\cdot\frac{1}{D^2}(\log x) \tag{6.1391}$$

Now we need to find $\frac{1}{D^2}(\log x)$.
We know that $\frac{1}{D}$ is equivalent to integration, so $\frac{1}{D^2}$ is equivalent to applying integration twice. Let's compute this step by step.
First integration:

$$\frac{1}{D}(\log x) = \int \log x\,dx \tag{6.1392}$$
$$= x\log x - x + C_1 \tag{6.1393}$$

Second integration:

$$\frac{1}{D^2}(\log x) = \frac{1}{D}[x\log x - x + C_1] \tag{6.1394}$$
$$= \int [x\log x - x + C_1]\,dx \tag{6.1395}$$
$$= \int x\log x\,dx - \int x\,dx + C_1\int dx \tag{6.1396}$$

To evaluate $\int x\log x\,dx$, we can use integration by parts with $u = \log x$ and $dv = x\,dx$, giving $du = \frac{1}{x}\,dx$ and $v = \frac{x^2}{2}$:

$$\int x\log x\,dx = \frac{x^2}{2}\log x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx \tag{6.1397}$$
$$= \frac{x^2}{2}\log x - \frac{1}{2}\int x\,dx \tag{6.1398}$$
$$= \frac{x^2}{2}\log x - \frac{1}{2}\cdot\frac{x^2}{2} + C \tag{6.1399}$$
$$= \frac{x^2}{2}\log x - \frac{x^2}{4} + C \tag{6.1400}$$

Continuing with the original calculation:

$$\frac{1}{D^2}(\log x) = \frac{x^2}{2}\log x - \frac{x^2}{4} - \frac{x^2}{2} + C_1x + C_2 \tag{6.1401}$$
$$= \frac{x^2}{2}\log x - \frac{x^2}{4} - \frac{2x^2}{4} + C_1x + C_2 \tag{6.1402}$$
$$= \frac{x^2}{2}\log x - \frac{3x^2}{4} + C_1x + C_2 \tag{6.1403}$$

For the particular integral, we typically set the integration constants to zero, giving us:

$$\frac{1}{D^2}(\log x) = \frac{x^2}{2}\log x - \frac{3x^2}{4} \tag{6.1404}$$

Therefore, using Case 5:

$$y_p = e^{-x} \cdot \frac{1}{D^2}(\log x) \tag{6.1405}$$

$$= e^{-x} \cdot \left(\frac{x^2}{2}\log x - \frac{3x^2}{4}\right) \tag{6.1406}$$

$$= \frac{e^{-x}x^2}{2}\log x - \frac{3e^{-x}x^2}{4} \tag{6.1407}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1408}$$

$$= c_1e^{-x} + c_2xe^{-x} + \frac{e^{-x}x^2}{2}\log x - \frac{3e^{-x}x^2}{4} \tag{6.1409}$$

Factoring out e^{-x}:

$$y = e^{-x}\left[c_1 + c_2x + \frac{x^2}{2}\log x - \frac{3x^2}{4}\right] \tag{6.1410}$$

Therefore, the general solution of the given differential equation is:
$y = e^{-x}\left[c_1 + c_2x + \frac{x^2}{2}\log x - \frac{3x^2}{4}\right]$
where c_1 and c_2 are arbitrary constants.

Example 5.5

Solve the differential equation:

$$\frac{d^2y}{dx^2} = y = \cosh x \cos x \tag{6.1411}$$

Solution

Let's rewrite the differential equation in operator form:

$$\frac{d^2y}{dx^2} - y = \cosh x \cos x \tag{6.1412}$$

$$(D^2 - 1)y = \cosh x \cos x \tag{6.1413}$$

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.
Step 1: Find the complementary function y_c.
The auxiliary equation corresponding to the homogeneous part of the differential equation

is:

$$D^2 - 1 = 0 \tag{6.1414}$$
$$m^2 - 1 = 0 \tag{6.1415}$$
$$m^2 = 1 \tag{6.1416}$$
$$m = \pm 1 \tag{6.1417}$$

Therefore, $m = 1$ or $m = -1$. The complementary function is:

$$y_c = c_1 e^x + c_2 e^{-x} \tag{6.1418}$$

where c_1 and c_2 are arbitrary constants.
Step 2: Find the particular integral y_p.
We have the forcing function $f(x) = \cosh x \cos x$ and the operator $\phi(D) = D^2 - 1$.
Let's express $\cosh x$ in its exponential form:

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{6.1419}$$

So our forcing function becomes:

$$\cosh x \cos x = \frac{e^x + e^{-x}}{2} \cos x \tag{6.1420}$$
$$= \frac{e^x \cos x + e^{-x} \cos x}{2} \tag{6.1421}$$

We need to find the particular integral for each term separately.
For the term $\frac{e^x \cos x}{2}$: This is in the form $e^{ax}V$, where $a = 1$ and $V = \frac{\cos x}{2}$. According to Case 5:

$$y_{p1} = \frac{1}{D^2 - 1}\left(\frac{e^x \cos x}{2}\right) \tag{6.1422}$$
$$= e^x \cdot \frac{1}{(D+1)^2 - 1}\left(\frac{\cos x}{2}\right) \tag{6.1423}$$
$$= e^x \cdot \frac{1}{D^2 + 2D + 1 - 1}\left(\frac{\cos x}{2}\right) \tag{6.1424}$$
$$= e^x \cdot \frac{1}{D^2 + 2D}\left(\frac{\cos x}{2}\right) \tag{6.1425}$$
$$= e^x \cdot \frac{1}{D(D+2)}\left(\frac{\cos x}{2}\right) \tag{6.1426}$$

Using partial fractions:

$$\frac{1}{D(D+2)} = \frac{A}{D} + \frac{B}{D+2} \tag{6.1427}$$

Multiplying both sides by $D(D+2)$:

$$1 = A(D+2) + BD \tag{6.1428}$$
$$= AD + 2A + BD \tag{6.1429}$$

Comparing coefficients:

$$A + B = 0 \tag{6.1430}$$
$$2A = 1 \tag{6.1431}$$

Solving, we get $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. So:

$$\frac{1}{D(D+2)} = \frac{1}{2D} - \frac{1}{2(D+2)} \tag{6.1432}$$

Now:

$$y_{p1} = e^x \cdot \left[\frac{1}{2D} - \frac{1}{2(D+2)}\right]\left(\frac{\cos x}{2}\right) \tag{6.1433}$$

$$= e^x \cdot \left[\frac{1}{4D}(\cos x) - \frac{1}{4(D+2)}(\cos x)\right] \tag{6.1434}$$

We know:

$$\frac{1}{D}(\cos x) = \sin x \tag{6.1435}$$

For $\frac{1}{D+2}(\cos x)$, we can use the fact that $\frac{1}{D+a}(\cos bx) = e^{-ax}\int e^{ax}\cos bx\,dx$. With $a = 2$ and $b = 1$:

$$\frac{1}{D+2}(\cos x) = e^{-2x}\int e^{2x}\cos x\,dx \tag{6.1436}$$

Using integration by parts, let $u = \cos x$ and $dv = e^{2x}\,dx$, so $du = -\sin x\,dx$ and $v = \frac{e^{2x}}{2}$:

$$\int e^{2x}\cos x\,dx = \frac{e^{2x}}{2}\cos x - \int \frac{e^{2x}}{2}(-\sin x)\,dx \tag{6.1437}$$

$$= \frac{e^{2x}}{2}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx \tag{6.1438}$$

For the second integral, let $u = \sin x$ and $dv = e^{2x}\,dx$, so $du = \cos x\,dx$ and $v = \frac{e^{2x}}{2}$:

$$\int e^{2x}\sin x\,dx = \frac{e^{2x}}{2}\sin x - \int \frac{e^{2x}}{2}\cos x\,dx \tag{6.1439}$$

Substituting back:

$$\int e^{2x}\cos x\,dx = \frac{e^{2x}}{2}\cos x + \frac{1}{2}\left[\frac{e^{2x}}{2}\sin x - \int \frac{e^{2x}}{2}\cos x\,dx\right] \tag{6.1440}$$

$$= \frac{e^{2x}}{2}\cos x + \frac{e^{2x}}{4}\sin x - \frac{1}{4}\int e^{2x}\cos x\,dx \tag{6.1441}$$

Solving for the integral:

$$\frac{5}{4}\int e^{2x}\cos x\,dx = \frac{e^{2x}}{2}\cos x + \frac{e^{2x}}{4}\sin x \tag{6.1442}$$

$$\int e^{2x}\cos x\,dx = \frac{e^{2x}}{5}(2\cos x + \sin x) \tag{6.1443}$$

Therefore:

$$\frac{1}{D+2}(\cos x) = e^{-2x}\cdot\frac{e^{2x}}{5}(2\cos x + \sin x) \tag{6.1444}$$

$$= \frac{1}{5}(2\cos x + \sin x) \tag{6.1445}$$

Now, continuing with y_{p1}:

$$y_{p1} = e^x \cdot \left[\frac{1}{4}\sin x - \frac{1}{4}\cdot\frac{1}{5}(2\cos x + \sin x)\right] \tag{6.1446}$$
$$= e^x \cdot \left[\frac{\sin x}{4} - \frac{2\cos x + \sin x}{20}\right] \tag{6.1447}$$
$$= e^x \cdot \left[\frac{5\sin x}{20} - \frac{2\cos x + \sin x}{20}\right] \tag{6.1448}$$
$$= e^x \cdot \left[\frac{4\sin x}{20} - \frac{2\cos x}{20}\right] \tag{6.1449}$$
$$= e^x \cdot \left[\frac{\sin x}{5} - \frac{\cos x}{10}\right] \tag{6.1450}$$

For the term $\frac{e^{-x}\cos x}{2}$: Similarly, this is in the form $e^{ax}V$, where $a = -1$ and $V = \frac{\cos x}{2}$. According to Case 5:

$$y_{p2} = \frac{1}{D^2 - 1}\left(\frac{e^{-x}\cos x}{2}\right) \tag{6.1451}$$
$$= e^{-x} \cdot \frac{1}{(D-1)^2 - 1}\left(\frac{\cos x}{2}\right) \tag{6.1452}$$
$$= e^{-x} \cdot \frac{1}{D^2 - 2D + 1 - 1}\left(\frac{\cos x}{2}\right) \tag{6.1453}$$
$$= e^{-x} \cdot \frac{1}{D^2 - 2D}\left(\frac{\cos x}{2}\right) \tag{6.1454}$$
$$= e^{-x} \cdot \frac{1}{D(D-2)}\left(\frac{\cos x}{2}\right) \tag{6.1455}$$

Using a similar approach as before and partial fractions, we can find:

$$\frac{1}{D(D-2)} = \frac{1}{2D} - \frac{1}{2(D-2)} \tag{6.1456}$$

For $\frac{1}{D-2}(\cos x)$, we have:

$$\frac{1}{D-2}(\cos x) = e^{2x}\int e^{-2x}\cos x\,dx \tag{6.1457}$$

Using a similar integration process as before, we can find $\frac{1}{D-2}(\cos x)$ and then compute y_{p2}.
Combining the results for y_{p1} and y_{p2}, we get:

$$y_p = y_{p1} + y_{p2} \tag{6.1458}$$
$$= e^x \cdot \left[\frac{\sin x}{5} - \frac{\cos x}{10}\right] + e^{-x} \cdot \left[\frac{\sin x}{5} + \frac{\cos x}{10}\right] \tag{6.1459}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1460}$$
$$= c_1e^x + c_2e^{-x} + e^x \cdot \left[\frac{\sin x}{5} - \frac{\cos x}{10}\right] + e^{-x} \cdot \left[\frac{\sin x}{5} + \frac{\cos x}{10}\right] \tag{6.1461}$$

Regrouping terms:

$$y = \left(c_1 + \frac{\sin x}{5} - \frac{\cos x}{10}\right) e^x + \left(c_2 + \frac{\sin x}{5} + \frac{\cos x}{10}\right) e^{-x} \tag{6.1462}$$

Therefore, the general solution of the given differential equation is:

$$y = \left(c_1 + \frac{\sin x}{5} - \frac{\cos x}{10}\right) e^x + \left(c_2 + \frac{\sin x}{5} + \frac{\cos x}{10}\right) e^{-x}$$

where c_1 and c_2 are arbitrary constants.

Solved Examples on Case 7

Example 7.1

Solve the differential equation:

$$(D^2 + 2D + 1)y = x\cos x \tag{6.1463}$$

Solution

The general solution will be of the form $y = y_c + y_p$, where y_c is the complementary function and y_p is the particular integral.

Step 1: Find the complementary function y_c.

The auxiliary equation corresponding to the homogeneous part of the differential equation is:

$$D^2 + 2D + 1 = 0 \tag{6.1464}$$
$$m^2 + 2m + 1 = 0 \tag{6.1465}$$

We can factor this equation:

$$m^2 + 2m + 1 = 0 \tag{6.1466}$$
$$(m+1)^2 = 0 \tag{6.1467}$$

Therefore, $m = -1$ is a repeated root. The complementary function is:

$$y_c = c_1 e^{-x} + c_2 x e^{-x} \tag{6.1468}$$

where c_1 and c_2 are arbitrary constants.

Step 2: Find the particular integral y_p.

We have the forcing function $f(x) = x\cos x$ and the operator $\phi(D) = D^2 + 2D + 1 = (D+1)^2$.

This is in the form xV, where $V = \cos x$. According to Case 7, we can use the formula:

$$\frac{1}{\phi(D)}(xV) = x \cdot \frac{1}{\phi(D)}V - \frac{\phi'(D)}{\phi(D)^2}V \tag{6.1469}$$

First, we need to compute $\frac{1}{\phi(D)}V = \frac{1}{(D+1)^2}(\cos x)$.

To find this, we'll use Case 2 for trigonometric functions. For $\cos x$, we substitute $D^2 = -1$ in the operator:

$$(D+1)^2|_{D^2=-1} = D^2 + 2D + 1 \tag{6.1470}$$
$$= -1 + 2D + 1 \tag{6.1471}$$
$$= 2D \tag{6.1472}$$

So:

$$\frac{1}{(D+1)^2}(\cos x) = \frac{1}{2D}(\cos x) \tag{6.1473}$$

Using the property that $\frac{1}{D}(\cos x) = \sin x$:

$$\frac{1}{(D+1)^2}(\cos x) = \frac{1}{2} \cdot \frac{1}{D}(\cos x) \tag{6.1474}$$

$$= \frac{1}{2}\sin x \tag{6.1475}$$

Next, we need to compute $\frac{\phi'(D)}{\phi(D)^2}V$. The derivative of $\phi(D) = (D+1)^2$ with respect to D is:

$$\phi'(D) = 2(D+1) \cdot 1 \tag{6.1476}$$

$$= 2(D+1) \tag{6.1477}$$

So:

$$\frac{\phi'(D)}{\phi(D)^2}(\cos x) = \frac{2(D+1)}{(D+1)^4}(\cos x) \tag{6.1478}$$

$$= \frac{2}{(D+1)^3}(\cos x) \tag{6.1479}$$

To find $\frac{1}{(D+1)^3}(\cos x)$, we again use Case 2, substituting $D^2 = -1$:

$$(D+1)^3|_{D^2=-1} = (D+1)\cdot(D+1)^2 \tag{6.1480}$$

$$= (D+1)\cdot 2D \tag{6.1481}$$

$$= 2D^2 + 2D \tag{6.1482}$$

$$= 2(-1) + 2D \tag{6.1483}$$

$$= -2 + 2D \tag{6.1484}$$

So:

$$\frac{1}{(D+1)^3}(\cos x) = \frac{1}{-2+2D}(\cos x) \tag{6.1485}$$

$$= \frac{1}{2}\cdot\frac{1}{-1+D}(\cos x) \tag{6.1486}$$

For $\frac{1}{-1+D}(\cos x) = \frac{1}{D-1}(\cos x)$, we can use the property that:

$$\frac{1}{D-1}(\cos x) = e^x \int e^{-x}\cos x\,dx \tag{6.1487}$$

Using integration by parts, let $u = \cos x$ and $dv = e^{-x}\,dx$, so $du = -\sin x\,dx$ and $v = -e^{-x}$:

$$\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \int -e^{-x}\cdot(-\sin x)\,dx \tag{6.1488}$$

$$= -e^{-x}\cos x - \int e^{-x}\sin x\,dx \tag{6.1489}$$

For the second integral, let $u = \sin x$ and $dv = e^{-x}\,dx$, so $du = \cos x\,dx$ and $v = -e^{-x}$:

$$\int e^{-x}\sin x\,dx = -e^{-x}\sin x - \int -e^{-x}\cdot\cos x\,dx \tag{6.1490}$$

$$= -e^{-x}\sin x + \int e^{-x}\cos x\,dx \tag{6.1491}$$

Substituting back into the original integral:

$$\int e^{-x}\cos x\,dx = -e^{-x}\cos x - [-e^{-x}\sin x + \int e^{-x}\cos x\,dx] \tag{6.1492}$$

$$= -e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x\,dx \tag{6.1493}$$

Solving for the integral:

$$2\int e^{-x}\cos x\,dx = -e^{-x}\cos x + e^{-x}\sin x \tag{6.1494}$$

$$\int e^{-x}\cos x\,dx = \frac{-e^{-x}\cos x + e^{-x}\sin x}{2} \tag{6.1495}$$

$$= \frac{e^{-x}(\sin x - \cos x)}{2} \tag{6.1496}$$

Therefore:

$$\frac{1}{D-1}(\cos x) = e^{x}\cdot\frac{e^{-x}(\sin x - \cos x)}{2} \tag{6.1497}$$

$$= \frac{\sin x - \cos x}{2} \tag{6.1498}$$

And:

$$\frac{1}{(D+1)^3}(\cos x) = \frac{1}{2}\cdot\frac{\sin x - \cos x}{2} \tag{6.1499}$$

$$= \frac{\sin x - \cos x}{4} \tag{6.1500}$$

Going back to our original calculation:

$$\frac{\phi'(D)}{\phi(D)^2}(\cos x) = \frac{2}{(D+1)^3}(\cos x) \tag{6.1501}$$

$$= 2\cdot\frac{\sin x - \cos x}{4} \tag{6.1502}$$

$$= \frac{\sin x - \cos x}{2} \tag{6.1503}$$

Now, using the Case 7 formula:

$$\frac{1}{\phi(D)}(x\cos x) = x\cdot\frac{1}{\phi(D)}(\cos x) - \frac{\phi'(D)}{\phi(D)^2}(\cos x) \tag{6.1504}$$

$$= x\cdot\frac{1}{2}\sin x - \frac{\sin x - \cos x}{2} \tag{6.1505}$$

$$= \frac{x\sin x}{2} - \frac{\sin x - \cos x}{2} \tag{6.1506}$$

$$= \frac{x\sin x - \sin x + \cos x}{2} \tag{6.1507}$$

$$= \frac{(x-1)\sin x + \cos x}{2} \tag{6.1508}$$

Therefore, the particular integral is:

$$y_p = \frac{(x-1)\sin x + \cos x}{2} \tag{6.1509}$$

Step 3: Write the general solution by combining the complementary function and the particular integral.

$$y = y_c + y_p \tag{6.1510}$$

$$= c_1 e^{-x} + c_2 x e^{-x} + \frac{(x-1)\sin x + \cos x}{2} \tag{6.1511}$$

Therefore, the general solution of the given differential equation is:

$y = c_1 e^{-x} + c_2 x e^{-x} + \frac{(x-1)\sin x + \cos x}{2}$

where c_1 and c_2 are arbitrary constants.

6.10 Method of Variation of Parameters

When the Method of Undetermined Coefficients/Shortcut Method fails to yield a particular integral for certain forms of forcing functions, we must employ more general techniques. The Method of Variation of Parameters, developed by the renowned mathematician Lagrange, provides a powerful and elegant approach for finding particular integrals in such cases.

6.10.1 Conceptual Foundation

The Method of Variation of Parameters is based on a creative insight: what if the constants in the complementary function were not actually constants, but functions of the independent variable? This seemingly paradoxical approach transforms the search for a particular integral into finding specific functions that replace the constants in the complementary function.

Consider a second-order linear differential equation:

$$\frac{d^2y}{dx^2} + y = f(x) \tag{6.1512}$$

Let us suppose that the complementary function (solution to the homogeneous equation) is:

$$y_c = A\cos x + B\sin x \tag{6.1513}$$

Where A and B are arbitrary constants.

The method of variation of parameters suggests replacing these constants with functions $A(x)$ and $B(x)$, so that:

$$y = A(x)\cos x + B(x)\sin x \tag{6.1514}$$

This expression must satisfy the original differential equation to be a solution. Additionally, we need to determine the functions $A(x)$ and $B(x)$.

6.10.2 Derivation of Conditions

To find the functions $A(x)$ and $B(x)$, we establish two conditions:

1. The assumed solution must satisfy the original differential equation.
2. We impose an additional constraint to simplify the calculations.

Differentiating our assumed solution:

$$y' = -A(x)\sin x + B(x)\cos x + A'(x)\cos x + B'(x)\sin x \tag{6.1515}$$

To simplify our work, we impose the first constraint:

$$A'(x)\cos x + B'(x)\sin x = 0 \tag{6.1516}$$

This simplifies the first derivative to:

$$y' = -A(x)\sin x + B(x)\cos x \tag{6.1517}$$

Taking the second derivative:

$$y'' = -A(x)\cos x - B(x)\sin x - A'(x)\sin x + B'(x)\cos x \tag{6.1518}$$

$$= -A(x)\cos x - B(x)\sin x - A'(x)\sin x + B'(x)\cos x \tag{6.1519}$$

Substituting into the original differential equation:

$$y'' + y = f(x) \tag{6.1520}$$

$$[-A(x)\cos x - B(x)\sin x - A'(x)\sin x + B'(x)\cos x] + [A(x)\cos x + B(x)\sin x] = f(x) \tag{6.1521}$$

Simplifying:

$$-A'(x)\sin x + B'(x)\cos x = f(x) \tag{6.1522}$$

Now we have two equations for $A'(x)$ and $B'(x)$:

$$A'(x)\cos x + B'(x)\sin x = 0 \tag{6.1523}$$

$$-A'(x)\sin x + B'(x)\cos x = f(x) \tag{6.1524}$$

Solving this system simultaneously:

$$A'(x) = -\frac{\sin x \cdot f(x)}{\cos x \cdot \sin x + \sin x \cdot \cos x} = -\frac{\sin x \cdot f(x)}{\sin 2x/2} = -\frac{\sin^2 x}{(\cos x)(\sin x)} \cdot \frac{f(x)}{\sin x} = -\frac{\sin x}{\cos x} \cdot f(x) \tag{6.1525}$$

$$B'(x) = \frac{\cos x \cdot f(x)}{\sin 2x/2} = \frac{\cos x}{\sin x \cdot \cos x} \cdot f(x) = \frac{1}{\sin x} \cdot f(x) \tag{6.1526}$$

Therefore:

$$A'(x) = -\frac{\sin x}{\cos x} \cdot f(x) \tag{6.1527}$$

$$B'(x) = \frac{f(x)}{\sin x} \tag{6.1528}$$

6.10.3 Determining the Functions $A(x)$ and $B(x)$

Integrating these expressions:

$$A(x) = \int -\frac{\sin x}{\cos x} \cdot f(x)\, dx = \int -\frac{\sin x \cdot f(x)}{\cos x}\, dx = \int -\frac{\cos^2 x - 1}{\cos x}\, dx = \int (\cos x - \sec x)\, dx \tag{6.1529}$$

$$= \sin x - \ln|\sec x + \tan x| \tag{6.1530}$$

And similarly:

$$B(x) = \int \frac{f(x)}{\sin x}\, dx = \int \frac{\tan x}{\sin x}\, dx = -\int \cos x\, dx = -\sin x \tag{6.1531}$$

Note that we do not include constants of integration since we are seeking a particular integral.

6.10.4 Formulating the Particular Integral

Now, we can write the particular integral as:

$$y_p = A(x)\cos x + B(x)\sin x \tag{6.1532}$$
$$= [\sin x - \ln|\sec x + \tan x|]\cos x + [-\sin x]\sin x \tag{6.1533}$$
$$= \sin x\cos x - \ln|\sec x + \tan x|\cos x - \sin^2 x \tag{6.1534}$$
$$= \cos x[\sin x - \ln(\sec x + \tan x)] - \sin x\cos x \tag{6.1535}$$
$$= -\ln|\sec x + \tan x|\cos x \tag{6.1536}$$

Thus, the complete solution to the original differential equation is:

$$y = y_c + y_p \tag{6.1537}$$
$$= c_1\cos x + c_2\sin x - \cos x\ln|\sec x + \tan x| \tag{6.1538}$$

6.10.5 Generalized Method of Variation of Parameters

For a general second-order linear differential equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X(x) \tag{6.1539}$$

where a, b, and c are constants, the Method of Variation of Parameters can be systematized. If the complementary function is $y_c = Ay_1 + By_2$, where y_1 and y_2 are linearly independent solutions of the homogeneous equation, then the particular integral is:

$$y_p = u_1y_1 + u_2y_2 \tag{6.1540}$$

Where:

$$u_1 = \int \frac{-y_2X}{W}\,dx \tag{6.1541}$$
$$u_2 = \int \frac{y_1X}{W}\,dx \tag{6.1542}$$

Here, W is the Wronskian determinant:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \tag{6.1543}$$

This generalized approach extends to higher-order linear differential equations as well, providing a systematic method for finding particular integrals in cases where other methods fail.

6.11 Solved Examples of Variation of Parameters Method

Example 1: Basic Application

Example 1: Solving $y'' + y$

Find the general solution of the differential equation:

$$y'' + y = \tan x \tag{6.1544}$$

Solution

Step 1: Identify the homogeneous equation and find its complementary function.
The homogeneous equation is $y'' + y = 0$, which has the characteristic equation $r^2 + 1 = 0$. Solving:

$$r^2 + 1 = 0 \tag{6.1545}$$
$$r^2 = -1 \tag{6.1546}$$
$$r = \pm i \tag{6.1547}$$

Therefore, the complementary function is:

$$y_c = c_1 \cos x + c_2 \sin x \tag{6.1548}$$

Step 2: Identify the two linearly independent solutions and their derivatives:

$$y_1 = \cos x \qquad y_1' = -\sin x \tag{6.1549}$$
$$y_2 = \sin x \qquad y_2' = \cos x \tag{6.1550}$$

Step 3: Calculate the Wronskian of these solutions:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{6.1551}$$
$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \tag{6.1552}$$
$$= \cos x \cdot \cos x - (-\sin x) \cdot \sin x \tag{6.1553}$$
$$= \cos^2 x + \sin^2 x \tag{6.1554}$$
$$= 1 \tag{6.1555}$$

Step 4: Apply the variation of parameters formula to find the functions $u_1'(x)$ and $u_2'(x)$:

$$u_1'(x) = -\frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} = -\frac{\sin x \cdot \tan x}{1} = -\sin x \cdot \frac{\sin x}{\cos x} = -\frac{\sin^2 x}{\cos x} \tag{6.1556}$$
$$u_2'(x) = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} = \frac{\cos x \cdot \tan x}{1} = \cos x \cdot \frac{\sin x}{\cos x} = \sin x \tag{6.1557}$$

Step 5: Integrate to find $u_1(x)$ and $u_2(x)$.
For $u_2(x)$:

$$u_2(x) = \int \sin x \, dx \tag{6.1558}$$
$$= -\cos x + C_2 \tag{6.1559}$$

For $u_1(x)$:

$$u_1(x) = \int -\frac{\sin^2 x}{\cos x} \, dx \tag{6.1560}$$

We can rewrite this using $\sin^2 x = 1 - \cos^2 x$:

$$u_1(x) = \int -\frac{1 - \cos^2 x}{\cos x} \, dx \tag{6.1561}$$
$$= \int \left(-\frac{1}{\cos x} + \cos x \right) dx \tag{6.1562}$$
$$= \int (-\sec x + \cos x) \, dx \tag{6.1563}$$
$$= -\ln|\sec x + \tan x| + \sin x + C_1 \tag{6.1564}$$

Step 6: Determine the particular solution.
Since we're finding a particular solution, we can set the constants of integration $C_1 = C_2 = 0$:

$$u_1(x) = -\ln|\sec x + \tan x| + \sin x \quad (6.1565)$$
$$u_2(x) = -\cos x \quad (6.1566)$$

Now we can form the particular integral:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \quad (6.1567)$$
$$= [-\ln|\sec x + \tan x| + \sin x]\cos x + [-\cos x]\sin x \quad (6.1568)$$
$$= -\cos x \ln|\sec x + \tan x| + \sin x \cos x - \cos x \sin x \quad (6.1569)$$
$$= -\cos x \ln|\sec x + \tan x| + \sin x \cos x - \sin x \cos x \quad (6.1570)$$
$$= -\cos x \ln|\sec x + \tan x| \quad (6.1571)$$

Step 7: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$y = y_c + y_p \quad (6.1572)$$
$$= c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x| \quad (6.1573)$$

This is the complete general solution to the differential equation $y'' + y = \tan x$.

Example 2: Non-trigonometric Functions

Example 2: Solving $y'' - y$

Find the general solution of the differential equation:

$$y'' - y = e^x \quad (6.1574)$$

Solution

Step 1: Identify the homogeneous equation and find its complementary function.
The homogeneous equation is $y'' - y = 0$, which has the characteristic equation $r^2 - 1 = 0$. Solving:

$$r^2 - 1 = 0 \quad (6.1575)$$
$$r^2 = 1 \quad (6.1576)$$
$$r = \pm 1 \quad (6.1577)$$

Therefore, the complementary function is:

$$y_c = c_1 e^x + c_2 e^{-x} \quad (6.1578)$$

Step 2: Identify the two linearly independent solutions and their derivatives:

$$y_1 = e^x \qquad y_1' = e^x \quad (6.1579)$$
$$y_2 = e^{-x} \qquad y_2' = -e^{-x} \quad (6.1580)$$

Step 3: Calculate the Wronskian of these solutions:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{6.1581}$$
$$= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} \tag{6.1582}$$
$$= e^x \cdot (-e^{-x}) - e^x \cdot e^{-x} \tag{6.1583}$$
$$= -e^{x-x} - e^{x-x} \tag{6.1584}$$
$$= -1 - 1 \tag{6.1585}$$
$$= -2 \tag{6.1586}$$

Step 4: Apply the variation of parameters formula to find the functions $u_1'(x)$ and $u_2'(x)$:

$$u_1'(x) = -\frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} = -\frac{e^{-x} \cdot e^x}{-2} = \frac{e^{-x+x}}{2} = \frac{1}{2} \tag{6.1587}$$
$$u_2'(x) = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} = \frac{e^x \cdot e^x}{-2} = -\frac{e^{2x}}{2} \tag{6.1588}$$

Step 5: Integrate to find $u_1(x)$ and $u_2(x)$.
For $u_1(x)$:

$$u_1(x) = \int \frac{1}{2}\, dx \tag{6.1589}$$
$$= \frac{x}{2} + C_1 \tag{6.1590}$$

For $u_2(x)$:

$$u_2(x) = \int -\frac{e^{2x}}{2}\, dx \tag{6.1591}$$
$$= -\frac{1}{4}e^{2x} + C_2 \tag{6.1592}$$

Step 6: Determine the particular solution.
Since we're finding a particular solution, we can set the constants of integration $C_1 = C_2 = 0$:

$$u_1(x) = \frac{x}{2} \tag{6.1593}$$
$$u_2(x) = -\frac{1}{4}e^{2x} \tag{6.1594}$$

Now we can form the particular integral:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{6.1595}$$
$$= \frac{x}{2} \cdot e^x + \left(-\frac{1}{4}e^{2x}\right) \cdot e^{-x} \tag{6.1596}$$
$$= \frac{x}{2}e^x - \frac{1}{4}e^{2x} \cdot e^{-x} \tag{6.1597}$$
$$= \frac{x}{2}e^x - \frac{1}{4}e^x \tag{6.1598}$$
$$= e^x \left(\frac{x}{2} - \frac{1}{4}\right) \tag{6.1599}$$

Step 7: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$y = y_c + y_p \tag{6.1600}$$
$$= c_1 e^x + c_2 e^{-x} + e^x \left(\frac{x}{2} - \frac{1}{4}\right) \tag{6.1601}$$
$$= \left(c_1 + \frac{x}{2} - \frac{1}{4}\right) e^x + c_2 e^{-x} \tag{6.1602}$$
$$= C_1 e^x + c_2 e^{-x} + \frac{x}{2} e^x \tag{6.1603}$$

Where $C_1 = c_1 - \frac{1}{4}$ is a new arbitrary constant.
Therefore, the general solution to the differential equation $y'' - y = e^x$ is:

$$y = C_1 e^x + C_2 e^{-x} + \frac{x}{2} e^x \tag{6.1604}$$

Example 3: Using the Wronskian Approach

Example 3: Solving $y'' - 4y' + 4y$

Find the general solution of the differential equation:

$$y'' - 4y' + 4y = e^{2x} \ln x, \quad x > 0 \tag{6.1605}$$

Solution

Step 1: Find the complementary function by solving the homogeneous equation.
The homogeneous equation is $y'' - 4y' + 4y = 0$, which has the characteristic equation $r^2 - 4r + 4 = 0$.
Factoring:

$$r^2 - 4r + 4 = 0 \tag{6.1606}$$
$$(r-2)^2 = 0 \tag{6.1607}$$
$$r = 2 \text{ (repeated root)} \tag{6.1608}$$

Therefore, the complementary function is:

$$y_c = c_1 e^{2x} + c_2 x e^{2x} \tag{6.1609}$$

Step 2: Identify the two linearly independent solutions and their derivatives:

$$y_1 = e^{2x} \qquad y_1' = 2e^{2x} \tag{6.1610}$$
$$y_2 = xe^{2x} \qquad y_2' = e^{2x} + 2xe^{2x} = (1+2x)e^{2x} \tag{6.1611}$$

Step 3: Calculate the Wronskian of these solutions:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{6.1612}$$
$$= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix} \tag{6.1613}$$
$$= e^{2x} \cdot (1+2x)e^{2x} - 2e^{2x} \cdot xe^{2x} \tag{6.1614}$$
$$= e^{4x} + 2xe^{4x} - 2xe^{4x} \tag{6.1615}$$
$$= e^{4x} \tag{6.1616}$$

Step 4: Apply the variation of parameters formula to find the functions $u_1'(x)$ and $u_2'(x)$:

$$u_1'(x) = -\frac{y_2(x)\cdot f(x)}{W(y_1,y_2)} = -\frac{xe^{2x}\cdot e^{2x}\ln x}{e^{4x}} = -\frac{xe^{4x}\ln x}{e^{4x}} = -x\ln x \tag{6.1617}$$

$$u_2'(x) = \frac{y_1(x)\cdot f(x)}{W(y_1,y_2)} = \frac{e^{2x}\cdot e^{2x}\ln x}{e^{4x}} = \frac{e^{4x}\ln x}{e^{4x}} = \ln x \tag{6.1618}$$

Step 5: Integrate to find $u_1(x)$ and $u_2(x)$.
For $u_1(x)$:

$$u_1(x) = \int -x\ln x\,dx \tag{6.1619}$$

We can use integration by parts with $u = \ln x$ and $dv = -x\,dx$:

$$u_1(x) = \ln x\cdot\left(-\frac{x^2}{2}\right) - \int\left(-\frac{x^2}{2}\right)\cdot\frac{1}{x}\,dx \tag{6.1620}$$

$$= -\frac{x^2}{2}\ln x + \int\frac{x}{2}\,dx \tag{6.1621}$$

$$= -\frac{x^2}{2}\ln x + \frac{x^2}{4} + C_1 \tag{6.1622}$$

For $u_2(x)$:

$$u_2(x) = \int \ln x\,dx \tag{6.1623}$$

$$= x\ln x - x + C_2 \tag{6.1624}$$

Step 6: Form the particular solution.
Since we're finding a particular solution, we can set $C_1 = C_2 = 0$:

$$u_1(x) = -\frac{x^2}{2}\ln x + \frac{x^2}{4} \tag{6.1625}$$

$$u_2(x) = x\ln x - x \tag{6.1626}$$

Now we can form the particular integral:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{6.1627}$$

$$= \left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right)e^{2x} + (x\ln x - x)\cdot xe^{2x} \tag{6.1628}$$

$$= \left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right)e^{2x} + (x^2\ln x - x^2)e^{2x} \tag{6.1629}$$

$$= e^{2x}\left[-\frac{x^2}{2}\ln x + \frac{x^2}{4} + x^2\ln x - x^2\right] \tag{6.1630}$$

$$= e^{2x}\left[\left(-\frac{x^2}{2} + x^2\right)\ln x + \frac{x^2}{4} - x^2\right] \tag{6.1631}$$

$$= e^{2x}\left[\frac{x^2}{2}\ln x + x^2\left(\frac{1}{4} - 1\right)\right] \tag{6.1632}$$

$$= e^{2x}\left[\frac{x^2}{2}\ln x - \frac{3x^2}{4}\right] \tag{6.1633}$$

Step 7: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$\begin{aligned} y &= y_c + y_p && (6.1634)\\ &= c_1 e^{2x} + c_2 x e^{2x} + e^{2x}\left[\frac{x^2}{2}\ln x - \frac{3x^2}{4}\right] && (6.1635)\\ &= e^{2x}\left[c_1 + c_2 x + \frac{x^2}{2}\ln x - \frac{3x^2}{4}\right] && (6.1636)\end{aligned}$$

Therefore, the general solution to the differential equation $y'' - 4y' + 4y = e^{2x}\ln x$ for $x > 0$ is:

$$y = e^{2x}\left[c_1 + c_2 x + \frac{x^2}{2}\ln x - \frac{3x^2}{4}\right] \quad (6.1637)$$

Example 4: Forced Harmonic Oscillator

Example 4: Solving $\frac{d^2x}{dt^2} + 4x$

Find the general solution of the differential equation that represents a forced harmonic oscillator:

$$\frac{d^2x}{dt^2} + 4x = 3\sin(2t) \quad (6.1638)$$

Solution

Step 1: Find the complementary function.
The homogeneous equation is $\frac{d^2x}{dt^2} + 4x = 0$, which has the characteristic equation $r^2 + 4 = 0$. Solving:

$$\begin{aligned} r^2 + 4 &= 0 && (6.1639)\\ r^2 &= -4 && (6.1640)\\ r &= \pm 2i && (6.1641)\end{aligned}$$

Therefore, the complementary function is:

$$x_c = c_1\cos(2t) + c_2\sin(2t) \quad (6.1642)$$

Step 2: Identify the two linearly independent solutions and their derivatives:

$$\begin{aligned} y_1 &= \cos(2t) & y_1' &= -2\sin(2t) && (6.1643)\\ y_2 &= \sin(2t) & y_2' &= 2\cos(2t) && (6.1644)\end{aligned}$$

Step 3: Calculate the Wronskian of these solutions:

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} && (6.1645)\\ &= \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} && (6.1646)\\ &= \cos(2t)\cdot 2\cos(2t) - (-2\sin(2t))\cdot\sin(2t) && (6.1647)\\ &= 2\cos^2(2t) + 2\sin^2(2t) && (6.1648)\\ &= 2(\cos^2(2t) + \sin^2(2t)) && (6.1649)\\ &= 2 && (6.1650)\end{aligned}$$

Step 4: Apply the variation of parameters formula to find the functions $u_1'(t)$ and $u_2'(t)$:

$$u_1'(t) = -\frac{y_2(t)\cdot f(t)}{W(y_1,y_2)} = -\frac{\sin(2t)\cdot 3\sin(2t)}{2} = -\frac{3\sin^2(2t)}{2} \tag{6.1651}$$

$$u_2'(t) = \frac{y_1(t)\cdot f(t)}{W(y_1,y_2)} = \frac{\cos(2t)\cdot 3\sin(2t)}{2} = \frac{3\cos(2t)\sin(2t)}{2} = \frac{3\sin(4t)}{4} \tag{6.1652}$$

Step 5: Integrate to find $u_1(t)$ and $u_2(t)$.
For $u_1(t)$:

$$u_1(t) = \int -\frac{3\sin^2(2t)}{2}\,dt \tag{6.1653}$$

Using the identity $\sin^2(2t) = \frac{1-\cos(4t)}{2}$:

$$u_1(t) = \int -\frac{3}{2}\cdot\frac{1-\cos(4t)}{2}\,dt \tag{6.1654}$$

$$= \int -\frac{3}{4}(1-\cos(4t))\,dt \tag{6.1655}$$

$$= -\frac{3t}{4} + \frac{3\sin(4t)}{16} + C_1 \tag{6.1656}$$

For $u_2(t)$:

$$u_2(t) = \int \frac{3\sin(4t)}{4}\,dt \tag{6.1657}$$

$$= -\frac{3\cos(4t)}{16} + C_2 \tag{6.1658}$$

Step 6: Determine the particular solution.
Since we're finding a particular solution, we can set $C_1 = C_2 = 0$:

$$u_1(t) = -\frac{3t}{4} + \frac{3\sin(4t)}{16} \tag{6.1659}$$

$$u_2(t) = -\frac{3\cos(4t)}{16} \tag{6.1660}$$

Now we can form the particular integral:

$$x_p = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{6.1661}$$

$$= \left(-\frac{3t}{4} + \frac{3\sin(4t)}{16}\right)\cos(2t) + \left(-\frac{3\cos(4t)}{16}\right)\sin(2t) \tag{6.1662}$$

We can simplify this using trigonometric identities:

$$\sin(4t)\cos(2t) = \frac{\sin(6t)+\sin(2t)}{2} \tag{6.1663}$$

$$\cos(4t)\sin(2t) = \frac{\sin(6t)-\sin(2t)}{2} \tag{6.1664}$$

Substituting:

$$x_p = -\frac{3t}{4}\cos(2t) + \frac{3}{16}\cos(2t)\left(\frac{\sin(6t)+\sin(2t)}{2}\right) - \frac{3}{16}\sin(2t)\left(\frac{\sin(6t)-\sin(2t)}{2}\right) \tag{6.1665}$$

$$= -\frac{3t}{4}\cos(2t) + \frac{3}{32}[\cos(2t)\sin(6t) + \cos(2t)\sin(2t)] - \frac{3}{32}[\sin(2t)\sin(6t) - \sin^2(2t)] \tag{6.1666}$$

After further simplification and using the identity $\sin(A)\cos(B) = \frac{\sin(A+B)+\sin(A-B)}{2}$:

$$x_p = -\frac{3t}{4}\cos(2t) + \frac{3}{16}\sin(2t) \tag{6.1667}$$

Step 7: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$x = x_c + x_p \tag{6.1668}$$
$$= c_1\cos(2t) + c_2\sin(2t) + \left(-\frac{3t}{4}\cos(2t) + \frac{3}{16}\sin(2t)\right) \tag{6.1669}$$
$$= \left(c_1 - \frac{3t}{4}\right)\cos(2t) + \left(c_2 + \frac{3}{16}\right)\sin(2t) \tag{6.1670}$$

This can be rewritten as:

$$x = C_1\cos(2t) + C_2\sin(2t) - \frac{3t}{4}\cos(2t) \tag{6.1671}$$

where $C_1 = c_1$ and $C_2 = c_2 + \frac{3}{16}$ are new arbitrary constants.
The presence of the term $-\frac{3t}{4}\cos(2t)$ indicates resonance behavior, where the amplitude of oscillation grows linearly with time. This occurs because the forcing frequency (2) matches the natural frequency of the system (2).

Additional Examples

Example 5: Solving $y'' + y$

Find the general solution of the differential equation:

$$y'' + y = \sec x \tag{6.1672}$$

Solution

Step 1: Identify the homogeneous equation and find its complementary function.
The homogeneous equation is $y'' + y = 0$, which has the characteristic equation $r^2 + 1 = 0$. Solving:

$$r^2 + 1 = 0 \tag{6.1673}$$
$$r^2 = -1 \tag{6.1674}$$
$$r = \pm i \tag{6.1675}$$

Therefore, the complementary function is:

$$y_c = c_1\cos x + c_2\sin x \tag{6.1676}$$

Step 2: Identify the two linearly independent solutions and their derivatives:

$$y_1 = \cos x \qquad y_1' = -\sin x \tag{6.1677}$$
$$y_2 = \sin x \qquad y_2' = \cos x \tag{6.1678}$$

Step 3: Calculate the Wronskian of these solutions:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{6.1679}$$
$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \tag{6.1680}$$
$$= \cos x \cdot \cos x - (-\sin x) \cdot \sin x \tag{6.1681}$$
$$= \cos^2 x + \sin^2 x \tag{6.1682}$$
$$= 1 \tag{6.1683}$$

Step 4: Apply the variation of parameters formula to find the functions $u_1'(x)$ and $u_2'(x)$:

$$u_1'(x) = -\frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} = -\frac{\sin x \cdot \sec x}{1} = -\sin x \cdot \frac{1}{\cos x} = -\tan x \tag{6.1684}$$
$$u_2'(x) = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} = \frac{\cos x \cdot \sec x}{1} = \cos x \cdot \frac{1}{\cos x} = 1 \tag{6.1685}$$

Step 5: Integrate to find $u_1(x)$ and $u_2(x)$.
For $u_1(x)$:

$$u_1(x) = \int -\tan x \, dx \tag{6.1686}$$
$$= \ln|\cos x| + C_1 \tag{6.1687}$$

For $u_2(x)$:

$$u_2(x) = \int 1 \, dx \tag{6.1688}$$
$$= x + C_2 \tag{6.1689}$$

Step 6: Determine the particular solution.
Since we're finding a particular solution, we can set the constants of integration $C_1 = C_2 = 0$:

$$u_1(x) = \ln|\cos x| \tag{6.1690}$$
$$u_2(x) = x \tag{6.1691}$$

Now we can form the particular integral:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{6.1692}$$
$$= \ln|\cos x| \cdot \cos x + x \cdot \sin x \tag{6.1693}$$

Step 7: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$y = y_c + y_p \tag{6.1694}$$
$$= c_1 \cos x + c_2 \sin x + \ln|\cos x| \cdot \cos x + x \cdot \sin x \tag{6.1695}$$

Therefore, the general solution to the differential equation $y'' + y = \sec x$ is:

$$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x \tag{6.1696}$$

Example 6: Solving $\frac{d^2y}{dx^2} + y$

Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + y = x \sin x \tag{6.1697}$$

Solution

Step 1: Identify the homogeneous equation and find its complementary function.
The homogeneous equation is $\frac{d^2y}{dx^2} + y = 0$, which has the characteristic equation $r^2 + 1 = 0$.
Solving:

$$r^2 + 1 = 0 \tag{6.1698}$$
$$r^2 = -1 \tag{6.1699}$$
$$r = \pm i \tag{6.1700}$$

Therefore, the complementary function is:

$$y_c = c_1 \cos x + c_2 \sin x \tag{6.1701}$$

Step 2: Identify the two linearly independent solutions and their derivatives:

$$y_1 = \cos x \qquad y_1' = -\sin x \tag{6.1702}$$
$$y_2 = \sin x \qquad y_2' = \cos x \tag{6.1703}$$

Step 3: Calculate the Wronskian of these solutions:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{6.1704}$$
$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \tag{6.1705}$$
$$= \cos x \cdot \cos x - (-\sin x) \cdot \sin x \tag{6.1706}$$
$$= \cos^2 x + \sin^2 x \tag{6.1707}$$
$$= 1 \tag{6.1708}$$

Step 4: Apply the variation of parameters formula to find the functions $u_1'(x)$ and $u_2'(x)$:

$$u_1'(x) = -\frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} = -\frac{\sin x \cdot x \sin x}{1} = -x \sin^2 x \tag{6.1709}$$
$$u_2'(x) = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} = \frac{\cos x \cdot x \sin x}{1} = x \cos x \sin x \tag{6.1710}$$

Step 5: Integrate to find $u_1(x)$ and $u_2(x)$.
For $u_1(x)$, we use the identity $\sin^2 x = \frac{1-\cos(2x)}{2}$:

$$u_1(x) = \int -x \sin^2 x \, dx \tag{6.1711}$$
$$= \int -x \cdot \frac{1 - \cos(2x)}{2} \, dx \tag{6.1712}$$
$$= \int \left(-\frac{x}{2} + \frac{x \cos(2x)}{2} \right) dx \tag{6.1713}$$

For the first term:

$$\int -\frac{x}{2}\,dx = -\frac{x^2}{4} \tag{6.1714}$$

For the second term, we use integration by parts with $u = x$ and $dv = \cos(2x)\,dx$:

$$\begin{aligned}
\int \frac{x\cos(2x)}{2}\,dx &= \frac{1}{2}\int x\cos(2x)\,dx && (6.1715)\\
&= \frac{1}{2}\left(x\cdot\frac{\sin(2x)}{2} - \int \frac{\sin(2x)}{2}\,dx\right) && (6.1716)\\
&= \frac{x\sin(2x)}{4} - \frac{1}{2}\int \frac{\sin(2x)}{2}\,dx && (6.1717)\\
&= \frac{x\sin(2x)}{4} - \frac{1}{2}\cdot\frac{-\cos(2x)}{4} + C_1 && (6.1718)\\
&= \frac{x\sin(2x)}{4} + \frac{\cos(2x)}{8} + C_1 && (6.1719)
\end{aligned}$$

Combining the terms:

$$u_1(x) = -\frac{x^2}{4} + \frac{x\sin(2x)}{4} + \frac{\cos(2x)}{8} + C_1 \tag{6.1720}$$

For $u_2(x)$, we use the identity $\cos x \sin x = \frac{\sin(2x)}{2}$:

$$\begin{aligned}
u_2(x) &= \int x\cos x \sin x\,dx && (6.1721)\\
&= \int x\cdot\frac{\sin(2x)}{2}\,dx && (6.1722)\\
&= \frac{1}{2}\int x\sin(2x)\,dx && (6.1723)
\end{aligned}$$

Using integration by parts with $u = x$ and $dv = \sin(2x)\,dx$:

$$\begin{aligned}
u_2(x) &= \frac{1}{2}\left(x\cdot\frac{-\cos(2x)}{2} - \int \frac{-\cos(2x)}{2}\,dx\right) && (6.1724)\\
&= -\frac{x\cos(2x)}{4} + \frac{1}{2}\int \frac{\cos(2x)}{2}\,dx && (6.1725)\\
&= -\frac{x\cos(2x)}{4} + \frac{1}{2}\cdot\frac{\sin(2x)}{4} + C_2 && (6.1726)\\
&= -\frac{x\cos(2x)}{4} + \frac{\sin(2x)}{8} + C_2 && (6.1727)
\end{aligned}$$

Step 6: Determine the particular solution.
Since we're finding a particular solution, we can set the constants of integration $C_1 = C_2 = 0$:

$$\begin{aligned}
u_1(x) &= -\frac{x^2}{4} + \frac{x\sin(2x)}{4} + \frac{\cos(2x)}{8} && (6.1728)\\
u_2(x) &= -\frac{x\cos(2x)}{4} + \frac{\sin(2x)}{8} && (6.1729)
\end{aligned}$$

Now we can form the particular integral:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{6.1730}$$
$$= \left(-\frac{x^2}{4} + \frac{x\sin(2x)}{4} + \frac{\cos(2x)}{8}\right)\cos x + \left(-\frac{x\cos(2x)}{4} + \frac{\sin(2x)}{8}\right)\sin x \tag{6.1731}$$

We can simplify using trigonometric identities. Note that:

$$\sin(2x)\cos x = \sin(2x+x) + \sin(2x-x) = \sin(3x) + \sin(x) \tag{6.1732}$$
$$\cos(2x)\cos x = \frac{\cos(2x+x)+\cos(2x-x)}{2} = \frac{\cos(3x)+\cos(x)}{2} \tag{6.1733}$$
$$\cos(2x)\sin x = \frac{\sin(2x+x)-\sin(2x-x)}{2} = \frac{\sin(3x)-\sin(x)}{2} \tag{6.1734}$$
$$\sin(2x)\sin x = \frac{\cos(2x-x)-\cos(2x+x)}{2} = \frac{\cos(x)-\cos(3x)}{2} \tag{6.1735}$$

After a significant amount of algebraic manipulation and simplification, we arrive at:

$$y_p = -\frac{x^2\cos x}{4} + \frac{x\sin x}{2} \tag{6.1736}$$

Step 7: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$y = y_c + y_p \tag{6.1737}$$
$$= c_1\cos x + c_2\sin x - \frac{x^2\cos x}{4} + \frac{x\sin x}{2} \tag{6.1738}$$

Therefore, the general solution to the differential equation $\frac{d^2y}{dx^2} + y = x\sin x$ is:

$$y = c_1\cos x + c_2\sin x - \frac{x^2\cos x}{4} + \frac{x\sin x}{2} \tag{6.1739}$$

Example 7: Solving $(D^2+3D+2)y$

Find the general solution of the differential equation:

$$(D^2+3D+2)y = \sin e^x \tag{6.1740}$$

where $D = \frac{d}{dx}$ is the differential operator.

Solution

Step 1: Rewrite the equation in standard form.
The differential equation $(D^2+3D+2)y = \sin e^x$ can be written as:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin e^x \tag{6.1741}$$

Step 2: Find the complementary function by solving the homogeneous equation.
The homogeneous equation is $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$, which has the characteristic equation $r^2 + 3r + 2 = 0$.

Factoring:

$$r^2 + 3r + 2 = 0 \tag{6.1742}$$
$$(r+1)(r+2) = 0 \tag{6.1743}$$

Therefore, $r = -1$ or $r = -2$, and the complementary function is:

$$y_c = c_1 e^{-x} + c_2 e^{-2x} \tag{6.1744}$$

Step 3: Identify the two linearly independent solutions and their derivatives:

$$y_1 = e^{-x} \qquad y_1' = -e^{-x} \tag{6.1745}$$
$$y_2 = e^{-2x} \qquad y_2' = -2e^{-2x} \tag{6.1746}$$

Step 4: Calculate the Wronskian of these solutions:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{6.1747}$$
$$= \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} \tag{6.1748}$$
$$= e^{-x} \cdot (-2e^{-2x}) - (-e^{-x}) \cdot e^{-2x} \tag{6.1749}$$
$$= -2e^{-3x} + e^{-3x} \tag{6.1750}$$
$$= -e^{-3x} \tag{6.1751}$$

Step 5: Apply the variation of parameters formula to find the functions $u_1'(x)$ and $u_2'(x)$:

$$u_1'(x) = -\frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} = -\frac{e^{-2x} \cdot \sin e^x}{-e^{-3x}} = \frac{e^{-2x} \sin e^x}{e^{-3x}} = e^x \sin e^x \tag{6.1752}$$
$$u_2'(x) = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} = \frac{e^{-x} \cdot \sin e^x}{-e^{-3x}} = -\frac{e^{-x} \sin e^x}{e^{-3x}} = -e^{2x} \sin e^x \tag{6.1753}$$

Step 6: Integrate to find $u_1(x)$ and $u_2(x)$.
For $u_1(x)$, we need to evaluate:

$$u_1(x) = \int e^x \sin e^x \, dx \tag{6.1754}$$

Let's make the substitution $u = e^x$, which gives $du = e^x dx$, or $dx = \frac{du}{e^x}$. Then:

$$u_1(x) = \int e^x \sin e^x \, dx \tag{6.1755}$$
$$= \int \sin u \, du \tag{6.1756}$$
$$= -\cos u + C_1 \tag{6.1757}$$
$$= -\cos e^x + C_1 \tag{6.1758}$$

For $u_2(x)$, we need to evaluate:

$$u_2(x) = \int -e^{2x} \sin e^x \, dx \tag{6.1759}$$

This is a more challenging integral. Using the substitution $u = e^x$, we get $du = e^x dx$, or $dx = \frac{du}{e^x}$. Then:

$$u_2(x) = \int -e^{2x} \sin e^x \, dx \tag{6.1760}$$
$$= \int -e^{2x} \sin u \cdot \frac{du}{e^x} \tag{6.1761}$$
$$= \int -e^x \sin u \, du \tag{6.1762}$$

But we still have a factor of $e^x = u$, so:

$$u_2(x) = \int -u \sin u \, du \tag{6.1763}$$

Using integration by parts with $v = -u$ and $dw = \sin u \, du$:

$$u_2(x) = \int -u \sin u \, du \tag{6.1764}$$
$$= -u(-\cos u) - \int -\cos u \, du \tag{6.1765}$$
$$= u \cos u - \int -\cos u \, du \tag{6.1766}$$
$$= u \cos u + \int \cos u \, du \tag{6.1767}$$
$$= u \cos u + \sin u + C_2 \tag{6.1768}$$
$$= e^x \cos e^x + \sin e^x + C_2 \tag{6.1769}$$

Step 7: Determine the particular solution.
Since we're finding a particular solution, we can set the constants of integration $C_1 = C_2 = 0$:

$$u_1(x) = -\cos e^x \tag{6.1770}$$
$$u_2(x) = e^x \cos e^x + \sin e^x \tag{6.1771}$$

Now we can form the particular integral:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{6.1772}$$
$$= (-\cos e^x) \cdot e^{-x} + (e^x \cos e^x + \sin e^x) \cdot e^{-2x} \tag{6.1773}$$
$$= -e^{-x} \cos e^x + e^{-x} \cos e^x + e^{-2x} \sin e^x \tag{6.1774}$$
$$= e^{-2x} \sin e^x \tag{6.1775}$$

Step 8: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$y = y_c + y_p \tag{6.1776}$$
$$= c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \sin e^x \tag{6.1777}$$

Therefore, the general solution to the differential equation $(D^2 + 3D + 2)y = \sin e^x$ is:

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \sin e^x \tag{6.1778}$$

Example 8: Solving $\frac{d^2y}{dx^2} - 2\frac{dy}{dx}$

Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x \tag{6.1779}$$

Solution

Step 1: Rewrite the equation in standard form.
The differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$ can be rewritten as:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 0y = e^x \sin x \tag{6.1780}$$

Step 2: Find the complementary function by solving the homogeneous equation.
The homogeneous equation is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$, which has the characteristic equation $r^2 - 2r = 0$.
Factoring:

$$r^2 - 2r = 0 \tag{6.1781}$$
$$r(r-2) = 0 \tag{6.1782}$$

Therefore, $r = 0$ or $r = 2$, and the complementary function is:

$$y_c = c_1 + c_2 e^{2x} \tag{6.1783}$$

Step 3: Identify the two linearly independent solutions and their derivatives:

$$y_1 = 1 \qquad y_1' = 0 \tag{6.1784}$$
$$y_2 = e^{2x} \qquad y_2' = 2e^{2x} \tag{6.1785}$$

Step 4: Calculate the Wronskian of these solutions:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{6.1786}$$
$$= \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} \tag{6.1787}$$
$$= 1 \cdot 2e^{2x} - 0 \cdot e^{2x} \tag{6.1788}$$
$$= 2e^{2x} \tag{6.1789}$$

Step 5: Apply the variation of parameters formula to find the functions $u_1'(x)$ and $u_2'(x)$:

$$u_1'(x) = -\frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} = -\frac{e^{2x} \cdot e^x \sin x}{2e^{2x}} = -\frac{e^{3x} \sin x}{2e^{2x}} = -\frac{e^x \sin x}{2} \tag{6.1790}$$
$$u_2'(x) = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} = \frac{1 \cdot e^x \sin x}{2e^{2x}} = \frac{e^x \sin x}{2e^{2x}} = \frac{e^{-x} \sin x}{2} \tag{6.1791}$$

Step 6: Integrate to find $u_1(x)$ and $u_2(x)$.
For $u_1(x)$, we need to evaluate:

$$u_1(x) = \int -\frac{e^x \sin x}{2}\, dx \tag{6.1792}$$

For integrals of the form $\int e^{ax} \sin bx\, dx$ or $\int e^{ax} \cos bx\, dx$, we can use the formula:

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C \tag{6.1793}$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} + C \tag{6.1794}$$

Applying this formula with $a = 1$ and $b = 1$:

$$u_1(x) = \int -\frac{e^x \sin x}{2}\, dx \tag{6.1795}$$

$$= -\frac{1}{2} \int e^x \sin x\, dx \tag{6.1796}$$

$$= -\frac{1}{2} \cdot \frac{e^x(\sin x - \cos x)}{2} + C_1 \tag{6.1797}$$

$$= -\frac{e^x(\sin x - \cos x)}{4} + C_1 \tag{6.1798}$$

For $u_2(x)$, we need to evaluate:

$$u_2(x) = \int \frac{e^{-x} \sin x}{2}\, dx \tag{6.1799}$$

Using the same formula with $a = -1$ and $b = 1$:

$$u_2(x) = \frac{1}{2} \int e^{-x} \sin x\, dx \tag{6.1800}$$

$$= \frac{1}{2} \cdot \frac{e^{-x}(-\sin x - \cos x)}{1 + 1} + C_2 \tag{6.1801}$$

$$= \frac{e^{-x}(-\sin x - \cos x)}{4} + C_2 \tag{6.1802}$$

$$= -\frac{e^{-x}(\sin x + \cos x)}{4} + C_2 \tag{6.1803}$$

Step 7: Determine the particular solution.
Since we're finding a particular solution, we can set the constants of integration $C_1 = C_2 = 0$:

$$u_1(x) = -\frac{e^x(\sin x - \cos x)}{4} \tag{6.1804}$$

$$u_2(x) = -\frac{e^{-x}(\sin x + \cos x)}{4} \tag{6.1805}$$

Now we can form the particular integral:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{6.1806}$$
$$= \left(-\frac{e^x(\sin x - \cos x)}{4}\right) \cdot 1 + \left(-\frac{e^{-x}(\sin x + \cos x)}{4}\right) \cdot e^{2x} \tag{6.1807}$$
$$= -\frac{e^x(\sin x - \cos x)}{4} + \left(-\frac{e^{-x}(\sin x + \cos x)}{4}\right) \cdot e^{2x} \tag{6.1808}$$
$$= -\frac{e^x(\sin x - \cos x)}{4} - \frac{e^{2x-x}(\sin x + \cos x)}{4} \tag{6.1809}$$
$$= -\frac{e^x(\sin x - \cos x)}{4} - \frac{e^x(\sin x + \cos x)}{4} \tag{6.1810}$$
$$= -\frac{e^x(\sin x - \cos x + \sin x + \cos x)}{4} \tag{6.1811}$$
$$= -\frac{e^x \cdot 2\sin x}{4} \tag{6.1812}$$
$$= -\frac{e^x \sin x}{2} \tag{6.1813}$$

Step 8: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$y = y_c + y_p \tag{6.1814}$$
$$= c_1 + c_2 e^{2x} - \frac{e^x \sin x}{2} \tag{6.1815}$$

Therefore, the general solution to the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$ is:

$$y = c_1 + c_2 e^{2x} - \frac{e^x \sin x}{2} \tag{6.1816}$$

Example 9: Solving $(D^2 + 4)y$

Find the general solution of the differential equation:

$$(D^2 + 4)y = 4\sec^2 2x \tag{6.1817}$$

where $D = \frac{d}{dx}$ is the differential operator.

Solution

Step 1: Rewrite the equation in standard form.
The differential equation $(D^2 + 4)y = 4\sec^2 2x$ can be written as:

$$\frac{d^2y}{dx^2} + 4y = 4\sec^2 2x \tag{6.1818}$$

Step 2: Find the complementary function by solving the homogeneous equation.
The homogeneous equation is $\frac{d^2y}{dx^2} + 4y = 0$, which has the characteristic equation $r^2 + 4 = 0$.
Solving:

$$r^2 + 4 = 0 \tag{6.1819}$$
$$r^2 = -4 \tag{6.1820}$$
$$r = \pm 2i \tag{6.1821}$$

Therefore, the complementary function is:

$$y_c = c_1 \cos 2x + c_2 \sin 2x \tag{6.1822}$$

Step 3: Identify the two linearly independent solutions and their derivatives:

$$y_1 = \cos 2x \qquad y_1' = -2\sin 2x \tag{6.1823}$$

$$y_2 = \sin 2x \qquad y_2' = 2\cos 2x \tag{6.1824}$$

Step 4: Calculate the Wronskian of these solutions:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{6.1825}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} \tag{6.1826}$$

$$= \cos 2x \cdot 2\cos 2x - (-2\sin 2x) \cdot \sin 2x \tag{6.1827}$$

$$= 2\cos^2 2x + 2\sin^2 2x \tag{6.1828}$$

$$= 2(\cos^2 2x + \sin^2 2x) \tag{6.1829}$$

$$= 2 \tag{6.1830}$$

Step 5: Apply the variation of parameters formula to find the functions $u_1'(x)$ and $u_2'(x)$:

$$u_1'(x) = -\frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} = -\frac{\sin 2x \cdot 4\sec^2 2x}{2} \tag{6.1831}$$

$$= -2\sin 2x \cdot \sec^2 2x = -2\sin 2x \cdot \frac{1}{\cos^2 2x} = -\frac{2\sin 2x}{\cos^2 2x} = -2\tan 2x \sec 2x \tag{6.1832}$$

$$u_2'(x) = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} = \frac{\cos 2x \cdot 4\sec^2 2x}{2} \tag{6.1833}$$

$$= 2\cos 2x \cdot \sec^2 2x = 2\cos 2x \cdot \frac{1}{\cos^2 2x} = \frac{2\cos 2x}{\cos^2 2x} = \frac{2}{\cos 2x} = 2\sec 2x \tag{6.1834}$$

Step 6: Integrate to find $u_1(x)$ and $u_2(x)$.
For $u_1(x)$, we need to evaluate:

$$u_1(x) = \int -2\tan 2x \sec 2x \, dx \tag{6.1835}$$

Using the substitution $u = 2x$, we get $du = 2\,dx$, or $dx = \frac{du}{2}$. Then:

$$u_1(x) = \int -2\tan 2x \sec 2x \, dx \tag{6.1836}$$

$$= \int -2\tan u \sec u \cdot \frac{du}{2} \tag{6.1837}$$

$$= -\int \tan u \sec u \, du \tag{6.1838}$$

We know that $\frac{d}{du}(\sec u) = \sec u \tan u$, so $\int \tan u \sec u \, du = \sec u + C$. Therefore:

$$u_1(x) = -\int \tan u \sec u \, du \tag{6.1839}$$

$$= -\sec u + C_1 \tag{6.1840}$$

$$= -\sec 2x + C_1 \tag{6.1841}$$

For $u_2(x)$, we need to evaluate:

$$u_2(x) = \int 2\sec 2x \, dx \tag{6.1842}$$

Using the substitution $u = 2x$, we get $dx = \frac{du}{2}$. Then:

$$u_2(x) = \int 2\sec 2x \, dx \tag{6.1843}$$
$$= \int 2\sec u \cdot \frac{du}{2} \tag{6.1844}$$
$$= \int \sec u \, du \tag{6.1845}$$

The integral of the secant function is:

$$\int \sec u \, du = \ln|\sec u + \tan u| + C \tag{6.1846}$$

Therefore:

$$u_2(x) = \ln|\sec u + \tan u| + C_2 \tag{6.1847}$$
$$= \ln|\sec 2x + \tan 2x| + C_2 \tag{6.1848}$$

Step 7: Determine the particular solution.
Since we're finding a particular solution, we can set the constants of integration $C_1 = C_2 = 0$:

$$u_1(x) = -\sec 2x \tag{6.1849}$$
$$u_2(x) = \ln|\sec 2x + \tan 2x| \tag{6.1850}$$

Now we can form the particular integral:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{6.1851}$$
$$= (-\sec 2x) \cdot \cos 2x + \ln|\sec 2x + \tan 2x| \cdot \sin 2x \tag{6.1852}$$
$$= -\sec 2x \cdot \cos 2x + \ln|\sec 2x + \tan 2x| \cdot \sin 2x \tag{6.1853}$$
$$= -1 + \ln|\sec 2x + \tan 2x| \cdot \sin 2x \tag{6.1854}$$

Step 8: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$y = y_c + y_p \tag{6.1855}$$
$$= c_1 \cos 2x + c_2 \sin 2x + (-1 + \ln|\sec 2x + \tan 2x| \cdot \sin 2x) \tag{6.1856}$$
$$= c_1 \cos 2x + c_2 \sin 2x - 1 + \ln|\sec 2x + \tan 2x| \cdot \sin 2x \tag{6.1857}$$
$$= c_1 \cos 2x + (c_2 + \ln|\sec 2x + \tan 2x|) \sin 2x - 1 \tag{6.1858}$$

Redefining the constant c_2 to include the constant term -1, we can write the general solution as:

$$y = c_1 \cos 2x + c_2 \sin 2x + \ln|\sec 2x + \tan 2x| \cdot \sin 2x - 1 \tag{6.1859}$$

Therefore, the general solution to the differential equation $(D^2 + 4)y = 4\sec^2 2x$ is:

$$y = c_1 \cos 2x + c_2 \sin 2x + \ln|\sec 2x + \tan 2x| \cdot \sin 2x - 1 \tag{6.1860}$$

Example 10: Solving $(D^2 + D)y$

Find the general solution of the differential equation:

$$(D^2 + D)y = (1 + e^x)^{-1} \tag{6.1861}$$

where $D = \frac{d}{dx}$ is the differential operator.

Solution

Step 1: Rewrite the equation in standard form.
The differential equation $(D^2 + D)y = (1 + e^x)^{-1}$ can be written as:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{1 + e^x} \tag{6.1862}$$

Step 2: Find the complementary function by solving the homogeneous equation.
The homogeneous equation is $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$, which has the characteristic equation $r^2 + r = 0$. Factoring:

$$r^2 + r = 0 \tag{6.1863}$$

$$r(r + 1) = 0 \tag{6.1864}$$

Therefore, $r = 0$ or $r = -1$, and the complementary function is:

$$y_c = c_1 + c_2 e^{-x} \tag{6.1865}$$

Step 3: Identify the two linearly independent solutions and their derivatives:

$$y_1 = 1 \qquad y_1' = 0 \tag{6.1866}$$

$$y_2 = e^{-x} \qquad y_2' = -e^{-x} \tag{6.1867}$$

Step 4: Calculate the Wronskian of these solutions:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{6.1868}$$

$$= \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} \tag{6.1869}$$

$$= 1 \cdot (-e^{-x}) - 0 \cdot e^{-x} \tag{6.1870}$$

$$= -e^{-x} \tag{6.1871}$$

Step 5: Apply the variation of parameters formula to find the functions $u_1'(x)$ and $u_2'(x)$:

$$u_1'(x) = -\frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} = -\frac{e^{-x} \cdot \frac{1}{1+e^x}}{-e^{-x}} = \frac{e^{-x} \cdot \frac{1}{1+e^x}}{e^{-x}} = \frac{1}{1 + e^x} \tag{6.1872}$$

$$u_2'(x) = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} = \frac{1 \cdot \frac{1}{1+e^x}}{-e^{-x}} = -\frac{e^x}{1 + e^x} \tag{6.1873}$$

Step 6: Integrate to find $u_1(x)$ and $u_2(x)$.
For $u_1(x)$, we need to evaluate:

$$u_1(x) = \int \frac{1}{1 + e^x}\, dx \tag{6.1874}$$

Using the substitution $u = e^x$, we get $du = e^x\,dx$, or $dx = \frac{du}{u}$. Then:

$$u_1(x) = \int \frac{1}{1+e^x}\,dx \tag{6.1875}$$

$$= \int \frac{1}{1+u} \cdot \frac{du}{u} \tag{6.1876}$$

$$= \int \frac{1}{u(1+u)}\,du \tag{6.1877}$$

Using partial fractions:

$$\frac{1}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u} \tag{6.1878}$$

$$1 = A(1+u) + Bu \tag{6.1879}$$

$$1 = A + Au + Bu \tag{6.1880}$$

Equating coefficients:

$$A = 1 \tag{6.1881}$$

$$A + B = 0 \tag{6.1882}$$

$$\Rightarrow B = -1 \tag{6.1883}$$

Therefore:

$$u_1(x) = \int \left(\frac{1}{u} - \frac{1}{1+u}\right) du \tag{6.1884}$$

$$= \ln|u| - \ln|1+u| + C_1 \tag{6.1885}$$

$$= \ln\left|\frac{u}{1+u}\right| + C_1 \tag{6.1886}$$

$$= \ln\left|\frac{e^x}{1+e^x}\right| + C_1 \tag{6.1887}$$

$$= \ln\left|\frac{e^x}{1+e^x}\right| + C_1 \tag{6.1888}$$

Since $e^x > 0$ for all x, we can simplify:

$$u_1(x) = \ln\left(\frac{e^x}{1+e^x}\right) + C_1 \tag{6.1889}$$

$$= \ln(e^x) - \ln(1+e^x) + C_1 \tag{6.1890}$$

$$= x - \ln(1+e^x) + C_1 \tag{6.1891}$$

For $u_2(x)$, we need to evaluate:

$$u_2(x) = \int -\frac{e^x}{1+e^x}\,dx \tag{6.1892}$$

Using the substitution $u = e^x$, we get $dx = \frac{du}{u}$. Then:

$$u_2(x) = \int -\frac{e^x}{1+e^x}\,dx \tag{6.1893}$$
$$= \int -\frac{u}{1+u} \cdot \frac{du}{u} \tag{6.1894}$$
$$= -\int \frac{1}{1+u}\,du \tag{6.1895}$$
$$= -\ln|1+u| + C_2 \tag{6.1896}$$
$$= -\ln|1+e^x| + C_2 \tag{6.1897}$$

Since $1 + e^x > 0$ for all x, we can simplify:

$$u_2(x) = -\ln(1+e^x) + C_2 \tag{6.1898}$$

Step 7: Determine the particular solution.
Since we're finding a particular solution, we can set the constants of integration $C_1 = C_2 = 0$:

$$u_1(x) = x - \ln(1+e^x) \tag{6.1899}$$
$$u_2(x) = -\ln(1+e^x) \tag{6.1900}$$

Now we can form the particular integral:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{6.1901}$$
$$= [x - \ln(1+e^x)] \cdot 1 + [-\ln(1+e^x)] \cdot e^{-x} \tag{6.1902}$$
$$= x - \ln(1+e^x) - \ln(1+e^x) \cdot e^{-x} \tag{6.1903}$$

Step 8: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$y = y_c + y_p \tag{6.1904}$$
$$= c_1 + c_2e^{-x} + x - \ln(1+e^x) - \ln(1+e^x) \cdot e^{-x} \tag{6.1905}$$
$$= c_1 + c_2e^{-x} + x - \ln(1+e^x)[1+e^{-x}] \tag{6.1906}$$

Therefore, the general solution to the differential equation $(D^2 + D)y = (1+e^x)^{-1}$ is:

$$y = c_1 + c_2e^{-x} + x - \ln(1+e^x)[1+e^{-x}] \tag{6.1907}$$

Example 11: Solving $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y$

Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x} \tag{6.1908}$$

Solution

Step 1: Find the complementary function by solving the homogeneous equation.
The homogeneous equation is $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$, which has the characteristic equation $r^2 + 3r + 2 = 0$.

Factoring:

$$r^2 + 3r + 2 = 0 \tag{6.1909}$$
$$(r+1)(r+2) = 0 \tag{6.1910}$$

Therefore, $r = -1$ or $r = -2$, and the complementary function is:

$$y_c = c_1 e^{-x} + c_2 e^{-2x} \tag{6.1911}$$

Step 2: Identify the two linearly independent solutions and their derivatives:

$$y_1 = e^{-x} \qquad y_1' = -e^{-x} \tag{6.1912}$$
$$y_2 = e^{-2x} \qquad y_2' = -2e^{-2x} \tag{6.1913}$$

Step 3: Calculate the Wronskian of these solutions:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{6.1914}$$
$$= \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} \tag{6.1915}$$
$$= e^{-x} \cdot (-2e^{-2x}) - (-e^{-x}) \cdot e^{-2x} \tag{6.1916}$$
$$= -2e^{-3x} + e^{-3x} \tag{6.1917}$$
$$= -e^{-3x} \tag{6.1918}$$

Step 4: Apply the variation of parameters formula to find the functions $u_1'(x)$ and $u_2'(x)$:

$$u_1'(x) = -\frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} = -\frac{e^{-2x} \cdot e^{e^x}}{-e^{-3x}} = \frac{e^{-2x} \cdot e^{e^x}}{e^{-3x}} = e^x \cdot e^{e^x} \tag{6.1919}$$
$$u_2'(x) = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} = \frac{e^{-x} \cdot e^{e^x}}{-e^{-3x}} = -\frac{e^{-x} \cdot e^{e^x}}{e^{-3x}} = -e^{2x} \cdot e^{e^x} \tag{6.1920}$$

Step 5: Integrate to find $u_1(x)$ and $u_2(x)$.
For $u_1(x)$, we need to evaluate:

$$u_1(x) = \int e^x \cdot e^{e^x} \, dx \tag{6.1921}$$

Using substitution, let $u = e^x$, which gives $du = e^x \, dx$. Then:

$$u_1(x) = \int e^x \cdot e^{e^x} \, dx \tag{6.1922}$$
$$= \int e^u \, du \tag{6.1923}$$
$$= e^u + C_1 \tag{6.1924}$$
$$= e^{e^x} + C_1 \tag{6.1925}$$

For $u_2(x)$, we need to evaluate:

$$u_2(x) = \int -e^{2x} \cdot e^{e^x} \, dx \tag{6.1926}$$

This is a more challenging integral. We can't directly use the same substitution as before. However, we can approach it differently.
Note that e^{e^x} is the derivative of e^{e^x} with respect to e^x. Using this insight, we can try the substitution $v = e^x$, which gives $dv = e^x\,dx$, or $dx = \frac{dv}{v}$. Then:

$$u_2(x) = \int -e^{2x} \cdot e^{e^x}\,dx \tag{6.1927}$$
$$= \int -v^2 \cdot e^v \cdot \frac{dv}{v} \tag{6.1928}$$
$$= -\int v \cdot e^v\,dv \tag{6.1929}$$

Using integration by parts with $u = v$ and $dw = e^v\,dv$, we get $du = dv$ and $w = e^v$:

$$-\int v \cdot e^v\,dv = -[v \cdot e^v - \int e^v\,dv] \tag{6.1930}$$
$$= -v \cdot e^v + \int e^v\,dv \tag{6.1931}$$
$$= -v \cdot e^v + e^v + C_2 \tag{6.1932}$$
$$= -e^x \cdot e^{e^x} + e^{e^x} + C_2 \tag{6.1933}$$
$$= e^{e^x}(1 - e^x) + C_2 \tag{6.1934}$$

Step 6: Determine the particular solution.
Since we're finding a particular solution, we can set the constants of integration $C_1 = C_2 = 0$:

$$u_1(x) = e^{e^x} \tag{6.1935}$$
$$u_2(x) = e^{e^x}(1 - e^x) \tag{6.1936}$$

Now we can form the particular integral:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{6.1937}$$
$$= e^{e^x} \cdot e^{-x} + e^{e^x}(1 - e^x) \cdot e^{-2x} \tag{6.1938}$$
$$= e^{e^x} \cdot e^{-x} + e^{e^x}(1 - e^x) \cdot e^{-2x} \tag{6.1939}$$
$$= e^{e^x - x} + e^{e^x} \cdot e^{-2x} - e^{e^x} \cdot e^x \cdot e^{-2x} \tag{6.1940}$$
$$= e^{e^x - x} + e^{e^x - 2x} - e^{e^x + x - 2x} \tag{6.1941}$$
$$= e^{e^x - x} + e^{e^x - 2x} - e^{e^x - x} \tag{6.1942}$$
$$= e^{e^x - 2x} \tag{6.1943}$$

Step 7: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$y = y_c + y_p \tag{6.1944}$$
$$= c_1 e^{-x} + c_2 e^{-2x} + e^{e^x - 2x} \tag{6.1945}$$

Therefore, the general solution to the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$ is:

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{e^x - 2x} \tag{6.1946}$$

Example 12: Solving $(D^2 - 4D + 4)y$

Find the general solution of the differential equation:

$$(D^2 - 4D + 4)y = e^{2x}\sec^2 x \tag{6.1947}$$

where $D = \frac{d}{dx}$ is the differential operator.

Solution

Step 1: Rewrite the equation in standard form.
The differential equation $(D^2 - 4D + 4)y = e^{2x}\sec^2 x$ can be written as:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}\sec^2 x \tag{6.1948}$$

Step 2: Find the complementary function by solving the homogeneous equation.
The homogeneous equation is $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$, which has the characteristic equation $r^2 - 4r + 4 = 0$.
Factoring:

$$r^2 - 4r + 4 = 0 \tag{6.1949}$$
$$(r-2)^2 = 0 \tag{6.1950}$$

Therefore, $r = 2$ (repeated root), and the complementary function is:

$$y_c = c_1 e^{2x} + c_2 x e^{2x} \tag{6.1951}$$

Step 3: Identify the two linearly independent solutions and their derivatives:

$$y_1 = e^{2x} \qquad y_1' = 2e^{2x} \tag{6.1952}$$
$$y_2 = xe^{2x} \qquad y_2' = e^{2x} + 2xe^{2x} = e^{2x}(1+2x) \tag{6.1953}$$

Step 4: Calculate the Wronskian of these solutions:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{6.1954}$$
$$= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}(1+2x) \end{vmatrix} \tag{6.1955}$$
$$= e^{2x} \cdot e^{2x}(1+2x) - 2e^{2x} \cdot xe^{2x} \tag{6.1956}$$
$$= e^{4x}(1+2x) - 2xe^{4x} \tag{6.1957}$$
$$= e^{4x}(1+2x-2x) \tag{6.1958}$$
$$= e^{4x} \tag{6.1959}$$

Step 5: Apply the variation of parameters formula to find the functions $u_1'(x)$ and $u_2'(x)$:

$$u_1'(x) = -\frac{y_2(x)\cdot f(x)}{W(y_1,y_2)} = -\frac{xe^{2x}\cdot e^{2x}\sec^2 x}{e^{4x}} = -\frac{xe^{4x}\sec^2 x}{e^{4x}} = -x\sec^2 x \tag{6.1960}$$
$$u_2'(x) = \frac{y_1(x)\cdot f(x)}{W(y_1,y_2)} = \frac{e^{2x}\cdot e^{2x}\sec^2 x}{e^{4x}} = \frac{e^{4x}\sec^2 x}{e^{4x}} = \sec^2 x \tag{6.1961}$$

Step 6: Integrate to find $u_1(x)$ and $u_2(x)$.

For $u_1(x)$, we need to evaluate:

$$u_1(x) = \int -x \sec^2 x \, dx \tag{6.1962}$$

Using integration by parts with $u = -x$ and $dv = \sec^2 x \, dx$, we get $du = -dx$ and $v = \tan x$:

$$\begin{aligned} u_1(x) &= \int -x \sec^2 x \, dx && (6.1963) \\ &= -x \tan x - \int -1 \cdot \tan x \, dx && (6.1964) \\ &= -x \tan x + \int \tan x \, dx && (6.1965) \\ &= -x \tan x - \ln|\cos x| + C_1 && (6.1966) \end{aligned}$$

For $u_2(x)$, we need to evaluate:

$$\begin{aligned} u_2(x) &= \int \sec^2 x \, dx && (6.1967) \\ &= \tan x + C_2 && (6.1968) \end{aligned}$$

Step 7: Determine the particular solution.
Since we're finding a particular solution, we can set the constants of integration $C_1 = C_2 = 0$:

$$\begin{aligned} u_1(x) &= -x \tan x - \ln|\cos x| && (6.1969) \\ u_2(x) &= \tan x && (6.1970) \end{aligned}$$

Now we can form the particular integral:

$$\begin{aligned} y_p &= u_1(x)y_1(x) + u_2(x)y_2(x) && (6.1971) \\ &= (-x \tan x - \ln|\cos x|) \cdot e^{2x} + \tan x \cdot xe^{2x} && (6.1972) \\ &= -x \tan x \cdot e^{2x} - \ln|\cos x| \cdot e^{2x} + \tan x \cdot xe^{2x} && (6.1973) \\ &= -x \tan x \cdot e^{2x} - \ln|\cos x| \cdot e^{2x} + x \tan x \cdot e^{2x} && (6.1974) \\ &= -\ln|\cos x| \cdot e^{2x} && (6.1975) \end{aligned}$$

Step 8: Write the general solution.
The general solution is the sum of the complementary function and the particular integral:

$$\begin{aligned} y &= y_c + y_p && (6.1976) \\ &= c_1 e^{2x} + c_2 x e^{2x} - \ln|\cos x| \cdot e^{2x} && (6.1977) \\ &= e^{2x}(c_1 + c_2 x - \ln|\cos x|) && (6.1978) \end{aligned}$$

Therefore, the general solution to the differential equation $(D^2 - 4D + 4)y = e^{2x} \sec^2 x$ is:

$$y = e^{2x}(c_1 + c_2 x - \ln|\cos x|) \tag{6.1979}$$

6.12 Key Insights and Applications

The Wronskian Method: Summary

The Method of Variation of Parameters using the Wronskian approach follows these systematic steps:

1. Find the complementary function $y_c = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$
2. Calculate the Wronskian $W(y_1, y_2, \ldots, y_n)$
3. For a second-order equation, determine u_1' and u_2' using:

$$u_1'(x) = -\frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} \tag{6.1980}$$

$$u_2'(x) = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} \tag{6.1981}$$

4. For higher-order equations, use Cramer's rule with Wronskians
5. Integrate to find $u_1(x)$, $u_2(x)$, etc.
6. Form the particular solution $y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \ldots + u_n(x)y_n(x)$
7. Write the general solution $y = y_c + y_p$

When to Use the Wronskian Approach

The Wronskian method for variation of parameters is particularly useful in the following scenarios:

1. When the forcing function has a complex form that doesn't easily fit standard patterns
2. When the Method of Undetermined Coefficients is not applicable
3. When the forcing function includes terms that appear in the complementary function (resonance cases)
4. When dealing with variable coefficient differential equations
5. For higher-order differential equations with complex forcing functions
6. When working with special functions like Bessel functions

Common Pitfalls and Tips

When applying the Wronskian method, be careful about:

1. Computing the Wronskian correctly - a small error can lead to incorrect results
2. Ensuring the formula for $u_i'(x)$ has the correct sign
3. Remembering that for a second-order equation, $u_1'(x)$ involves y_2 and vice versa
4. Being methodical in the integration step, especially with complex forcing functions
5. Setting the constants of integration to zero when finding the particular solution
6. Checking your answer by substituting back into the original equation
7. Recognizing that for some special equations (like Euler-Cauchy), there may be more efficient methods

The Wronskian: Properties and Applications

Key properties of the Wronskian that are useful in the method of variation of parameters:

1. The Wronskian W is identically zero if and only if the functions are linearly dependent
2. For linear homogeneous equations with constant coefficients, the Wronskian is either identically zero or never zero

3. For a second-order linear homogeneous equation $y'' + p(x)y' + q(x)y = 0$, the Wronskian of two linearly independent solutions satisfies Abel's identity:

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p(t)\,dt\right) \tag{6.1982}$$

4. For equations in the form $y'' + p(x)y' + q(x)y = f(x)$, if W is the Wronskian of two linearly independent solutions of the homogeneous equation, then the particular solution is:

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx \tag{6.1983}$$

Historical Note: Lagrange's Contribution

The Method of Variation of Parameters was developed by Joseph-Louis Lagrange (1736-1813), one of the greatest mathematicians of the 18th century. Lagrange developed this method while studying the three-body problem in celestial mechanics.
The method represents a fundamental advancement in the theory of differential equations, providing a systematic approach to finding particular solutions when other methods fail. Lagrange's insight was to allow the "constants" in the complementary function to vary, leading to a method that is applicable to a wide range of differential equations.
The Wronskian, named after the Polish mathematician Józef Hoene-Wroński (1776-1853), plays a central role in the implementation of Lagrange's method, providing a elegant way to determine the functions needed for the variation of parameters.

6.13 Equations Reducible to Linear Differential Equations with Constant Coefficients

Certain types of linear differential equations with variable coefficients can be transformed into linear differential equations with constant coefficients through appropriate substitutions. This section examines two important classes of such equations: Cauchy's homogeneous linear differential equation and Legendre's linear equation.

6.13.1 Cauchy's Homogeneous Linear Differential Equation

Cauchy's homogeneous linear differential equation, sometimes attributed to Euler, is characterized by coefficients that are proportional to powers of the independent variable. It takes the form:

$$a_0x^nD^ny + a_1x^{n-1}D^{n-1}y + \cdots + a_{n-1}xDy + a_ny = F(x) \tag{6.1984}$$

Where $a_0, a_1, \ldots, a_n$ are constants. This can be written more explicitly as:

$$a_0x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_ny = F(x) \tag{6.1985}$$

The distinctive feature of this equation is that the coefficient of each derivative term contains x raised to a power equal to the order of the derivative.

Transformation to Constant Coefficients

The key insight in solving Cauchy's equation is to transform the independent variable using the substitution:

$$x = e^z \quad \text{or equivalently} \quad z = \ln x \tag{6.1986}$$

This transformation requires us to express derivatives with respect to x in terms of derivatives with respect to z. First, we observe:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz} \tag{6.1987}$$

This gives us:

$$x\frac{dy}{dx} = \frac{dy}{dz} = Dy \tag{6.1988}$$

Where $D = \frac{d}{dz}$ is the differential operator with respect to z.
For the second derivative, we need to apply the chain rule carefully:

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x} \cdot \frac{dy}{dz}\right) \\
&= -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d}{dx}\left(\frac{dy}{dz}\right) \\
&= -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{dz}{dx} \\
&= -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{1}{x} \\
&= -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x^2} \cdot \frac{d^2y}{dz^2}
\end{aligned}$$

Multiplying both sides by x^2:

$$x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2} = -Dy + D^2y = D(D-1)y \tag{6.1989}$$

Similarly, we can derive:

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y \tag{6.1990}$$

And in general:

$$x^r\frac{d^ry}{dx^r} = D(D-1)(D-2)\cdots(D-r+1)y \tag{6.1991}$$

Transformed Equation

Substituting these expressions into Cauchy's equation yields a linear differential equation with constant coefficients in terms of z. The transformed equation can be solved using standard methods for linear differential equations with constant coefficients, and then we can revert to the original variable x to obtain the solution to Cauchy's equation.

6.13.2 Legendre's Linear Equation

Legendre's linear equation is another important class of differential equations with variable coefficients that can be reduced to an equation with constant coefficients. It has the form:

$$a_0(ax+b)^n\frac{d^ny}{dx^n}+a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_ny=F(x) \tag{6.1992}$$

Where $a_0, a_1, \ldots, a_n$ are constants, and a and b are constants appearing in the coefficient expressions.

Transformation to Constant Coefficients

For Legendre's equation, we introduce the substitution:

$$ax+b=e^t \quad \text{or equivalently} \quad t=\ln(ax+b) \tag{6.1993}$$

Under this transformation:

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{a}{ax+b}\cdot\frac{dy}{dt} \tag{6.1994}$$

This gives us:

$$(ax+b)\frac{dy}{dx}=a\frac{dy}{dt}=aDy \tag{6.1995}$$

Where $D=\frac{d}{dt}$ is the differential operator with respect to t.
For the second derivative, following a derivation similar to the one for Cauchy's equation:

$$\begin{aligned}
\frac{d^2y}{dx^2}&=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{a}{ax+b}\cdot\frac{dy}{dt}\right)\\
&=-\frac{a^2}{(ax+b)^2}\cdot\frac{dy}{dt}+\frac{a}{ax+b}\cdot\frac{d}{dx}\left(\frac{dy}{dt}\right)\\
&=-\frac{a^2}{(ax+b)^2}\cdot\frac{dy}{dt}+\frac{a}{ax+b}\cdot\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}\\
&=-\frac{a^2}{(ax+b)^2}\cdot\frac{dy}{dt}+\frac{a}{ax+b}\cdot\frac{d^2y}{dt^2}\cdot\frac{a}{ax+b}\\
&=-\frac{a^2}{(ax+b)^2}\cdot\frac{dy}{dt}+\frac{a^2}{(ax+b)^2}\cdot\frac{d^2y}{dt^2}
\end{aligned}$$

Multiplying both sides by $(ax+b)^2$:

$$(ax+b)^2\frac{d^2y}{dx^2}=-a^2\frac{dy}{dt}+a^2\frac{d^2y}{dt^2}=a^2(-D+D^2)y=a^2D(D-1)y \tag{6.1996}$$

Similarly, we can derive:

$$(ax+b)^3\frac{d^3y}{dx^3}=a^3D(D-1)(D-2)y \tag{6.1997}$$

And in general:

$$(ax+b)^r \frac{d^r y}{dx^r} = a^r D(D-1)(D-2)\cdots(D-r+1)y \tag{6.1998}$$

Transformed Equation

Substituting these expressions into Legendre's equation yields a linear differential equation with constant coefficients in terms of t. This transformed equation can be solved using standard methods, and then we can revert to the original variable x to obtain the solution to Legendre's equation.

6.13.3 Significance and Applications

Cauchy's and Legendre's equations appear in numerous theoretical and applied contexts. The ability to transform these variable-coefficient equations into constant-coefficient equations significantly simplifies their solution. Once transformed, the full arsenal of techniques for solving linear differential equations with constant coefficients becomes available, including the methods discussed in previous sections.

6.14 Solved Examples

Example 1: Solving a Cauchy-Euler (Equidimensional) Differential Equation

Solve the differential equation:

$$x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5 \tag{6.1999}$$

Step-by-Step Solution

This is a Cauchy-Euler (also called Cauchy's homogeneous linear) differential equation with a non-homogeneous term x^5. We'll solve it using the substitution $x = e^z$ or equivalently $z = \ln x$.

Step 1: Make the substitution $x = e^z$ and transform the equation.

Recall from our derivation that:

$$x\frac{dy}{dx} = \frac{dy}{dz} = Dy \tag{6.2000}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y \tag{6.2001}$$

Substituting these into our original equation:

$$x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5 \tag{6.2002}$$

$$D(D-1)y - 4Dy + 6y = e^{5z} \tag{6.2003}$$

Expanding the first term:

$$D^2y - Dy - 4Dy + 6y = e^{5z} \tag{6.2004}$$

$$D^2y - 5Dy + 6y = e^{5z} \tag{6.2005}$$

Step 2: Solve the homogeneous part of the transformed equation.

The characteristic equation is:

$$m^2 - 5m + 6 = 0 \tag{6.2006}$$
$$(m-2)(m-3) = 0 \tag{6.2007}$$

This gives us roots $m = 2$ and $m = 3$.
Therefore, the complementary function (general solution of the homogeneous equation) is:

$$y_c = c_1 e^{2z} + c_2 e^{3z} \tag{6.2008}$$

Step 3: Find a particular integral for the non-homogeneous part.
We need to find a particular solution to:

$$D^2 y - 5Dy + 6y = e^{5z} \tag{6.2009}$$

For the right side of the form e^{5z}, we try a particular solution of the form $y_p = Ae^{5z}$.
Substituting this into the left side:

$$D^2(Ae^{5z}) - 5D(Ae^{5z}) + 6(Ae^{5z}) = e^{5z} \tag{6.2010}$$
$$A(5)^2 e^{5z} - 5A(5)e^{5z} + 6Ae^{5z} = e^{5z} \tag{6.2011}$$
$$25Ae^{5z} - 25Ae^{5z} + 6Ae^{5z} = e^{5z} \tag{6.2012}$$
$$6Ae^{5z} = e^{5z} \tag{6.2013}$$

Therefore, $A = \frac{1}{6}$ and our particular solution is:

$$y_p = \frac{1}{6}e^{5z} \tag{6.2014}$$

Step 4: Write the general solution in terms of z.
The general solution in terms of z is:

$$y(z) = c_1 e^{2z} + c_2 e^{3z} + \frac{1}{6}e^{5z} \tag{6.2015}$$

Step 5: Convert back to the original variable x using $z = \ln x$.

$$y(x) = c_1 e^{2\ln x} + c_2 e^{3\ln x} + \frac{1}{6}e^{5\ln x} \tag{6.2016}$$
$$= c_1 x^2 + c_2 x^3 + \frac{1}{6}x^5 \tag{6.2017}$$

Therefore, the general solution to the original Cauchy-Euler equation is:

$$y(x) = c_1 x^2 + c_2 x^3 + \frac{1}{6}x^5 \tag{6.2018}$$

Verification: Let's verify our solution by substituting it back into the original equation.
For the particular solution $y_p = \frac{1}{6}x^5$, we have:

$$\frac{dy_p}{dx} = \frac{5}{6}x^4 \tag{6.2019}$$
$$\frac{d^2y_p}{dx^2} = \frac{20}{6}x^3 = \frac{10}{3}x^3 \tag{6.2020}$$

Substituting into the left side of the original equation:

$$x^2\frac{d^2y_p}{dx^2} - 4x\frac{dy_p}{dx} + 6y_p = x^2 \cdot \frac{10}{3}x^3 - 4x \cdot \frac{5}{6}x^4 + 6 \cdot \frac{1}{6}x^5 \tag{6.2021}$$
$$= \frac{10}{3}x^5 - \frac{20}{6}x^5 + x^5 \tag{6.2022}$$
$$= \frac{10}{3}x^5 - \frac{10}{3}x^5 + x^5 \tag{6.2023}$$
$$= x^5 \tag{6.2024}$$

This equals the right side of the original equation, confirming that our particular solution is correct.

Example 2: Cauchy-Euler Equation Using the Shortcut Method

Solve the differential equation:

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x \tag{6.2025}$$

Step-by-Step Solution

This is a Cauchy-Euler equation with non-homogeneous terms. We'll first transform it to a linear differential equation with constant coefficients using the substitution $x = e^z$ (or $z = \ln x$), then apply the Shortcut Method of Undetermined Coefficients.

Step 1: Make the substitution $x = e^z$ and transform the equation.

From our derivation, we know:

$$x\frac{dy}{dx} = \frac{dy}{dz} = Dy \tag{6.2026}$$
$$x^2\frac{d^2y}{dx^2} = D(D-1)y \tag{6.2027}$$

Also, we need to transform the right-hand side:

$$x^2 = (e^z)^2 = e^{2z} \tag{6.2028}$$
$$2\log x = 2z \tag{6.2029}$$

Substituting these into our original equation:

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x \tag{6.2030}$$
$$D(D-1)y - 2Dy - 4y = e^{2z} + 2z \tag{6.2031}$$

Expanding the first term:

$$D^2y - Dy - 2Dy - 4y = e^{2z} + 2z \tag{6.2032}$$
$$D^2y - 3Dy - 4y = e^{2z} + 2z \tag{6.2033}$$

Let's denote our differential operator as $\phi(D) = D^2 - 3D - 4$.

Step 2: Solve the homogeneous part of the transformed equation.

The characteristic equation is:

$$m^2 - 3m - 4 = 0 \tag{6.2034}$$
$$(m-4)(m+1) = 0 \tag{6.2035}$$

This gives us roots $m = 4$ and $m = -1$.
Therefore, the complementary function (general solution of the homogeneous equation) is:

$$y_c = c_1 e^{4z} + c_2 e^{-z} \tag{6.2036}$$

Step 3: Find a particular integral for the non-homogeneous part using the Shortcut Method.
The right side has two terms: e^{2z} and $2z$. We'll find the particular integral for each term separately.
(a) For e^{2z}: This corresponds to Case 1 from the Shortcut Method where $f(x) = e^{ax}$ with $a = 2$ here.
The formula states:

$$\frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax}, \quad \text{provided } \phi(a) \neq 0 \tag{6.2037}$$

In our case:

$$\frac{1}{\phi(D)} e^{2z} = \frac{1}{\phi(2)} e^{2z} \tag{6.2038}$$

$$= \frac{1}{2^2 - 3(2) - 4} e^{2z} \tag{6.2039}$$

$$= \frac{1}{4 - 6 - 4} e^{2z} \tag{6.2040}$$

$$= \frac{1}{-6} e^{2z} \tag{6.2041}$$

$$= -\frac{1}{6} e^{2z} \tag{6.2042}$$

So, $y_{p1} = -\frac{1}{6} e^{2z}$.
(b) For $2z$: This corresponds to Case 4 from the Shortcut Method where $f(x) = x^m$ with $m = 1$ (since z is equivalent to x^1 in the transformed space).
For this type, we need to expand $\frac{1}{\phi(D)}$ in ascending powers of D up to D^m. Since $m = 1$, we need terms up to D^1.
Let's express $\frac{1}{\phi(D)}$ as a power series in D:

$$\frac{1}{\phi(D)} = \frac{1}{D^2 - 3D - 4} \tag{6.2043}$$

$$= \frac{1}{-4} \cdot \frac{1}{1 - \frac{D^2 - 3D}{4}} \tag{6.2044}$$

Using the binomial expansion for $(1 - x)^{-1} = 1 + x + x^2 + \ldots$, with $x = \frac{D^2 - 3D}{4}$:

$$\frac{1}{\phi(D)} = -\frac{1}{4}\left(1 + \frac{D^2 - 3D}{4} + \ldots\right) \tag{6.2045}$$

$$= -\frac{1}{4} - \frac{D^2 - 3D}{16} - \ldots \tag{6.2046}$$

Since we only need terms up to D^1 when applying to z, and $D^2 z = 0$, we can simplify:

$$\frac{1}{\phi(D)}(2z) = 2\left(-\frac{1}{4} - \frac{D^2 - 3D}{16} - \dots\right) z \tag{6.2047}$$

$$= 2\left(-\frac{1}{4}z + \frac{3D}{16}z + \dots\right) \tag{6.2048}$$

$$= 2\left(-\frac{1}{4}z + \frac{3}{16} + \dots\right) \tag{6.2049}$$

$$= -\frac{1}{2}z + \frac{3}{8} \tag{6.2050}$$

Here, I've used $Dz = 1$ and dropped higher-order terms since they vanish when applied to z.
So, $y_{p2} = -\frac{1}{2}z + \frac{3}{8}$.
The complete particular solution is:

$$y_p = y_{p1} + y_{p2} = -\frac{1}{6}e^{2z} - \frac{1}{2}z + \frac{3}{8} \tag{6.2051}$$

Step 4: Write the general solution in terms of z.
The general solution in terms of z is:

$$y(z) = c_1 e^{4z} + c_2 e^{-z} - \frac{1}{6}e^{2z} - \frac{1}{2}z + \frac{3}{8} \tag{6.2052}$$

Step 5: Convert back to the original variable x using $z = \ln x$.

$$y(x) = c_1 e^{4\ln x} + c_2 e^{-\ln x} - \frac{1}{6}e^{2\ln x} - \frac{1}{2}\ln x + \frac{3}{8} \tag{6.2053}$$

$$= c_1 x^4 + c_2 x^{-1} - \frac{1}{6}x^2 - \frac{1}{2}\ln x + \frac{3}{8} \tag{6.2054}$$

$$= c_1 x^4 + \frac{c_2}{x} - \frac{1}{6}x^2 - \frac{1}{2}\ln x + \frac{3}{8} \tag{6.2055}$$

Therefore, the general solution to the original Cauchy-Euler equation is:

$$y(x) = c_1 x^4 + \frac{c_2}{x} - \frac{1}{6}x^2 - \frac{1}{2}\ln x + \frac{3}{8} \tag{6.2056}$$

Verification: Let's verify our particular solution by substituting it into the original equation.
For $y_p = -\frac{1}{6}x^2 - \frac{1}{2}\ln x + \frac{3}{8}$, we have:

$$\frac{dy_p}{dx} = -\frac{1}{3}x - \frac{1}{2x} \tag{6.2057}$$

$$\frac{d^2y_p}{dx^2} = -\frac{1}{3} + \frac{1}{2x^2} \tag{6.2058}$$

Substituting into the left side of the original equation:

$$x^2\frac{d^2y_p}{dx^2} - 2x\frac{dy_p}{dx} - 4y_p = x^2\left(-\frac{1}{3} + \frac{1}{2x^2}\right) - 2x\left(-\frac{1}{3}x - \frac{1}{2x}\right) - 4\left(-\frac{1}{6}x^2 - \frac{1}{2}\ln x + \frac{3}{8}\right) \tag{6.2059}$$

$$= -\frac{1}{3}x^2 + \frac{1}{2} + \frac{2}{3}x^2 + 1 + \frac{2}{3}x^2 + 2\ln x - \frac{3}{2} \tag{6.2060}$$

$$= x^2 + 2\ln x \tag{6.2061}$$

This matches the right side of the original equation, confirming that our particular solution is correct.

Example 3: Third-Order Cauchy-Euler Equation

Solve the differential equation:

$$x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} - 2y = x^2 + x^{-3} \tag{6.2062}$$

Step-by-Step Solution

This is a third-order Cauchy-Euler equation with non-homogeneous terms. We'll transform it to a linear differential equation with constant coefficients using the substitution $x = e^z$ (or $z = \ln x$), then apply the Shortcut Method.

Step 1: Make the substitution $x = e^z$ and transform the equation.

From the theory of Cauchy-Euler equations, we know:

$$x\frac{dy}{dx} = \frac{dy}{dz} = Dy \tag{6.2063}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y \tag{6.2064}$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y \tag{6.2065}$$

Also, we need to transform the right-hand side:

$$x^2 = (e^z)^2 = e^{2z} \tag{6.2066}$$

$$x^{-3} = (e^z)^{-3} = e^{-3z} \tag{6.2067}$$

Substituting these into our original equation:

$$x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} - 2y = x^2 + x^{-3} \tag{6.2068}$$

$$D(D-1)(D-2)y + D(D-1)y - 2y = e^{2z} + e^{-3z} \tag{6.2069}$$

Expanding the differential operators:

$$(D^3 - 3D^2 + 2D) + (D^2 - D) - 2 = e^{2z} + e^{-3z} \tag{6.2070}$$

$$D^3 - 3D^2 + 2D + D^2 - D - 2 = e^{2z} + e^{-3z} \tag{6.2071}$$

$$D^3 - 2D^2 + D - 2 = e^{2z} + e^{-3z} \tag{6.2072}$$

So our transformed equation is:

$$D^3 - 2D^2 + D - 2y = e^{2z} + e^{-3z} \tag{6.2073}$$

$$(D^3 - 2D^2 + D - 2)y = e^{2z} + e^{-3z} \tag{6.2074}$$

Let's denote our differential operator as $\phi(D) = D^3 - 2D^2 + D - 2$.

Step 2: Solve the homogeneous part of the transformed equation.

The characteristic equation is:

$$m^3 - 2m^2 + m - 2 = 0 \tag{6.2075}$$

To find the roots, we can try some values. Let's try $m = 1$:

$$1^3 - 2(1)^2 + 1 - 2 = 1 - 2 + 1 - 2 = -2 \neq 0 \tag{6.2076}$$

Let's try $m = 2$:

$$2^3 - 2(2)^2 + 2 - 2 = 8 - 8 + 2 - 2 = 0 \tag{6.2077}$$

So $m = 2$ is a root. We can factorize the characteristic equation:

$$m^3 - 2m^2 + m - 2 = (m-2)(m^2 + 0m + 1) \tag{6.2078}$$
$$= (m-2)(m^2+1) \tag{6.2079}$$

The quadratic factor $m^2 + 1$ has roots $m = \pm i$. So the characteristic equation has roots $m = 2$, $m = i$, and $m = -i$.
Therefore, the complementary function (general solution of the homogeneous equation) is:

$$y_c = c_1 e^{2z} + c_2 \cos z + c_3 \sin z \tag{6.2080}$$

Step 3: Find a particular integral for the non-homogeneous part using the Shortcut Method.
The right side has two terms: e^{2z} and e^{-3z}. We'll find the particular integral for each term separately using Case 1 from the Shortcut Method.
(a) For e^{2z}: This corresponds to Case 1 where $f(x) = e^{ax}$ with $a = 2$ here.
The formula states:

$$\frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax}, \quad \text{provided } \phi(a) \neq 0 \tag{6.2081}$$

However, we note that $m = 2$ is a root of the characteristic equation, so $\phi(2) = 0$. This is a failure case (i) in Case 1, and we must use:

$$\frac{1}{\phi(D)} e^{ax} = \frac{x}{\phi'(a)} e^{ax}, \quad \text{provided } \phi'(a) \neq 0 \tag{6.2082}$$

In our case,

$$\phi'(D) = 3D^2 - 4D + 1 \tag{6.2083}$$
$$\phi'(2) = 3(2)^2 - 4(2) + 1 \tag{6.2084}$$
$$= 12 - 8 + 1 \tag{6.2085}$$
$$= 5 \tag{6.2086}$$

Therefore:

$$\frac{1}{\phi(D)} e^{2z} = \frac{z}{\phi'(2)} e^{2z} \tag{6.2087}$$
$$= \frac{z}{5} e^{2z} \tag{6.2088}$$

So, $y_{p1} = \frac{z}{5} e^{2z}$.
(b) For e^{-3z}: This again corresponds to Case 1 where $f(x) = e^{ax}$ with $a = -3$ here.
Since $m = -3$ is not a root of the characteristic equation, we can directly use:

$$\frac{1}{\phi(D)} e^{-3z} = \frac{1}{\phi(-3)} e^{-3z} \tag{6.2089}$$

Let's calculate $\phi(-3)$:

$$\phi(-3) = (-3)^3 - 2(-3)^2 + (-3) - 2 \tag{6.2090}$$
$$= -27 - 2(9) - 3 - 2 \tag{6.2091}$$
$$= -27 - 18 - 3 - 2 \tag{6.2092}$$
$$= -50 \tag{6.2093}$$

Therefore:

$$\frac{1}{\phi(D)}e^{-3z} = \frac{1}{-50}e^{-3z} \tag{6.2094}$$
$$= -\frac{1}{50}e^{-3z} \tag{6.2095}$$

So, $y_{p2} = -\frac{1}{50}e^{-3z}$.
The complete particular solution is:

$$y_p = y_{p1} + y_{p2} = \frac{z}{5}e^{2z} - \frac{1}{50}e^{-3z} \tag{6.2096}$$

Step 4: Write the general solution in terms of z.
The general solution in terms of z is:

$$y(z) = c_1 e^{2z} + c_2 \cos z + c_3 \sin z + \frac{z}{5}e^{2z} - \frac{1}{50}e^{-3z} \tag{6.2097}$$

Step 5: Convert back to the original variable x using $z = \ln x$.

$$y(x) = c_1 e^{2\ln x} + c_2 \cos(\ln x) + c_3 \sin(\ln x) + \frac{\ln x}{5}e^{2\ln x} - \frac{1}{50}e^{-3\ln x} \tag{6.2098}$$
$$= c_1 x^2 + c_2 \cos(\ln x) + c_3 \sin(\ln x) + \frac{\ln x}{5}x^2 - \frac{1}{50}x^{-3} \tag{6.2099}$$
$$= c_1 x^2 + c_2 \cos(\ln x) + c_3 \sin(\ln x) + \frac{x^2 \ln x}{5} - \frac{1}{50x^3} \tag{6.2100}$$

Therefore, the general solution to the original Cauchy-Euler equation is:

$$y(x) = c_1 x^2 + c_2 \cos(\ln x) + c_3 \sin(\ln x) + \frac{x^2 \ln x}{5} - \frac{1}{50x^3} \tag{6.2101}$$

Verification: Let's verify our particular solution by testing a few terms.
For $y_{p1} = \frac{x^2 \ln x}{5}$:

First, let's calculate the derivatives:

$$y_{p1} = \frac{x^2 \ln x}{5} \tag{6.2102}$$
$$\frac{dy_{p1}}{dx} = \frac{1}{5}\left(2x\ln x + x^2 \cdot \frac{1}{x}\right) \tag{6.2103}$$
$$= \frac{1}{5}(2x\ln x + x) \tag{6.2104}$$
$$\frac{d^2y_{p1}}{dx^2} = \frac{1}{5}\left(2\ln x + 2x \cdot \frac{1}{x} + 1\right) \tag{6.2105}$$
$$= \frac{1}{5}(2\ln x + 2 + 1) \tag{6.2106}$$
$$= \frac{1}{5}(2\ln x + 3) \tag{6.2107}$$
$$\frac{d^3y_{p1}}{dx^3} = \frac{1}{5} \cdot 2 \cdot \frac{1}{x} \tag{6.2108}$$
$$= \frac{2}{5x} \tag{6.2109}$$

Now, substituting into the left side of the original equation:

$$x^3\frac{d^3y_{p1}}{dx^3} + x^2\frac{d^2y_{p1}}{dx^2} - 2y_{p1} = x^3 \cdot \frac{2}{5x} + x^2 \cdot \frac{1}{5}(2\ln x + 3) - 2 \cdot \frac{x^2\ln x}{5} \tag{6.2110}$$
$$= \frac{2x^2}{5} + \frac{x^2}{5}(2\ln x + 3) - \frac{2x^2\ln x}{5} \tag{6.2111}$$
$$= \frac{2x^2}{5} + \frac{2x^2\ln x + 3x^2}{5} - \frac{2x^2\ln x}{5} \tag{6.2112}$$
$$= \frac{2x^2 + 3x^2}{5} \tag{6.2113}$$
$$= \frac{5x^2}{5} \tag{6.2114}$$
$$= x^2 \tag{6.2115}$$

For $y_{p2} = -\frac{1}{50x^3}$:
Let's calculate the derivatives:

$$y_{p2} = -\frac{1}{50x^3} \tag{6.2116}$$
$$\frac{dy_{p2}}{dx} = -\frac{1}{50} \cdot (-3)x^{-4} \tag{6.2117}$$
$$= \frac{3}{50x^4} \tag{6.2118}$$
$$\frac{d^2y_{p2}}{dx^2} = \frac{3}{50} \cdot (-4)x^{-5} \tag{6.2119}$$
$$= -\frac{12}{50x^5} \tag{6.2120}$$
$$\frac{d^3y_{p2}}{dx^3} = -\frac{12}{50} \cdot (-5)x^{-6} \tag{6.2121}$$
$$= \frac{60}{50x^6} \tag{6.2122}$$
$$= \frac{6}{5x^6} \tag{6.2123}$$

Substituting into the left side of the original equation:

$$x^3\frac{d^3y_{p2}}{dx^3} + x^2\frac{d^2y_{p2}}{dx^2} - 2y_{p2} = x^3 \cdot \frac{6}{5x^6} + x^2 \cdot \left(-\frac{12}{50x^5}\right) - 2 \cdot \left(-\frac{1}{50x^3}\right) \tag{6.2124}$$

$$= \frac{6x^3}{5x^6} - \frac{12x^2}{50x^5} + \frac{2}{50x^3} \tag{6.2125}$$

$$= \frac{6}{5x^3} - \frac{12}{50x^3} + \frac{2}{50x^3} \tag{6.2126}$$

$$= \frac{6}{5x^3} - \frac{10}{50x^3} \tag{6.2127}$$

$$= \frac{6}{5x^3} - \frac{1}{5x^3} \tag{6.2128}$$

$$= \frac{5}{5x^3} \tag{6.2129}$$

$$= \frac{1}{x^3} \tag{6.2130}$$

Adding the two results:

$$x^2 + \frac{1}{x^3} = x^2 + x^{-3} \tag{6.2131}$$

This matches the right side of the original equation, confirming that our particular solution is correct.

Example 4: Cauchy-Euler Equation with Logarithmic Term

Solve the differential equation:

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \log x \tag{6.2132}$$

Step-by-Step Solution

This is a Cauchy-Euler equation with a non-homogeneous term involving $\log x$. We'll transform it using the substitution $x = e^z$ (or $z = \ln x$).
Step 1: Make the substitution $x = e^z$ and transform the equation.
With this substitution, we have:

$$x\frac{dy}{dx} = \frac{dy}{dz} = Dy \tag{6.2133}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y \tag{6.2134}$$

The right-hand side becomes:

$$x^2 \log x = e^{2z} \cdot z \tag{6.2135}$$

Substituting these into our original equation:

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \log x \tag{6.2136}$$

$$D(D-1)y - 3Dy + 5y = e^{2z} \cdot z \tag{6.2137}$$

Expanding the first term:

$$D^2y - Dy - 3Dy + 5y = e^{2z} \cdot z \tag{6.2138}$$
$$D^2y - 4Dy + 5y = e^{2z} \cdot z \tag{6.2139}$$

Let's denote our differential operator as $\phi(D) = D^2 - 4D + 5$.
Step 2: Solve the homogeneous part of the transformed equation.
The characteristic equation is:

$$m^2 - 4m + 5 = 0 \tag{6.2140}$$

Using the quadratic formula:

$$m = \frac{4 \pm \sqrt{16 - 20}}{2} \tag{6.2141}$$
$$= \frac{4 \pm \sqrt{-4}}{2} \tag{6.2142}$$
$$= \frac{4 \pm 2i}{2} \tag{6.2143}$$
$$= 2 \pm i \tag{6.2144}$$

So the roots are $m = 2 + i$ and $m = 2 - i$.
Therefore, the complementary function is:

$$y_c = e^{2z}(c_1 \cos z + c_2 \sin z) \tag{6.2145}$$

Step 3: Find a particular integral for the non-homogeneous part.
The right side is $e^{2z} \cdot z$, which falls under Case 5 of the Method of Undetermined Coefficients, where $f(x) = e^{ax}V$ with $a = 2$ and $V = z$.
According to Case 5, we have:

$$\frac{1}{\phi(D)}(e^{ax}V) = e^{ax} \cdot \frac{1}{\phi(D + a)}V \tag{6.2146}$$

In our problem, this becomes:

$$\frac{1}{\phi(D)}(e^{2z} \cdot z) = e^{2z} \cdot \frac{1}{\phi(D + 2)}z \tag{6.2147}$$
$$= e^{2z} \cdot \frac{1}{(D + 2)^2 - 4(D + 2) + 5}z \tag{6.2148}$$
$$= e^{2z} \cdot \frac{1}{D^2 + 4D + 4 - 4D - 8 + 5}z \tag{6.2149}$$
$$= e^{2z} \cdot \frac{1}{D^2 + 1}z \tag{6.2150}$$

To evaluate $\frac{1}{D^2+1}z$, we can use Case 2 of the Shortcut Method. Since $D^2z = 0$ (as z is a first-degree polynomial), we have:

$$\frac{1}{D^2 + 1}z = z \tag{6.2151}$$

Therefore, the particular solution is:

$$y_p = e^{2z} \cdot z = e^{2z}z \tag{6.2152}$$

Step 4: Write the general solution in terms of z.
The general solution in terms of z is:

$$y(z) = e^{2z}(c_1 \cos z + c_2 \sin z) + e^{2z} z \tag{6.2153}$$

Step 5: Convert back to the original variable x using $z = \ln x$.

$$y(x) = e^{2\ln x}[c_1 \cos(\ln x) + c_2 \sin(\ln x)] + e^{2\ln x} \ln x \tag{6.2154}$$

$$= x^2[c_1 \cos(\ln x) + c_2 \sin(\ln x)] + x^2 \ln x \tag{6.2155}$$

Therefore, the general solution to the original Cauchy-Euler equation is:

$$y(x) = x^2[c_1 \cos(\ln x) + c_2 \sin(\ln x)] + x^2 \ln x \tag{6.2156}$$

Example 5: Cauchy-Euler Equation with Trigonometric Term

Solve the differential equation:

$$x^3 \frac{d^2y}{dx^2} + 3x^2 \frac{dy}{dx} + xy = \sin(\log x) \tag{6.2157}$$

Step-by-Step Solution

This is a Cauchy-Euler equation with a trigonometric term involving $\log x$. We'll first divide the equation by x to put it in standard form, then transform it using the substitution $x = e^z$ (or $z = \ln x$).
Step 1: Divide the equation by x to get:

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{\sin(\log x)}{x} \tag{6.2158}$$

Step 2: Make the substitution $x = e^z$ and transform the equation.
From our derivation of Cauchy-Euler equations, we know:

$$x \frac{dy}{dx} = \frac{dy}{dz} = Dy \tag{6.2159}$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y \tag{6.2160}$$

Also, we need to transform the right-hand side:

$$\frac{\sin(\log x)}{x} = \frac{\sin(z)}{e^z} \tag{6.2161}$$

$$= e^{-z} \sin(z) \tag{6.2162}$$

Substituting these into our equation:

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{\sin(\log x)}{x} \tag{6.2163}$$

$$D(D-1)y + 3Dy + y = e^{-z} \sin(z) \tag{6.2164}$$

Expanding the first term:

$$D^2y - Dy + 3Dy + y = e^{-z} \sin(z) \tag{6.2165}$$

$$D^2y + 2Dy + y = e^{-z} \sin(z) \tag{6.2166}$$

Let's denote our differential operator as $\phi(D) = D^2 + 2D + 1 = (D+1)^2$.
Step 3: Solve the homogeneous part of the transformed equation.
The characteristic equation is:

$$m^2 + 2m + 1 = 0 \tag{6.2167}$$

$$(m+1)^2 = 0 \tag{6.2168}$$

This gives us a repeated root $m = -1$ with multiplicity 2.
Therefore, the complementary function (general solution of the homogeneous equation) is:

$$y_c = (c_1 + c_2 z)e^{-z} \tag{6.2169}$$

Step 4: Find a particular integral for the non-homogeneous part.
The right side is $e^{-z}\sin(z)$, which falls under Case 5 of the Method of Undetermined Coefficients/Shortcut Method, where $f(x) = e^{ax}V$ with $a = -1$ and $V = \sin(z)$.
According to Case 5, we have:

$$\frac{1}{\phi(D)}(e^{ax}V) = e^{ax} \cdot \frac{1}{\phi(D+a)}V \tag{6.2170}$$

In our problem, this becomes:

$$\frac{1}{\phi(D)}(e^{-z}\sin(z)) = e^{-z} \cdot \frac{1}{\phi(D-1)}\sin(z) \tag{6.2171}$$

$$= e^{-z} \cdot \frac{1}{(D-1)^2 + 2(D-1) + 1}\sin(z) \tag{6.2172}$$

$$= e^{-z} \cdot \frac{1}{D^2 - 2D + 1 + 2D - 2 + 1}\sin(z) \tag{6.2173}$$

$$= e^{-z} \cdot \frac{1}{D^2 + 0D + 0}\sin(z) \tag{6.2174}$$

$$= e^{-z} \cdot \frac{1}{D^2}\sin(z) \tag{6.2175}$$

Now we need to find $\frac{1}{D^2}\sin(z)$.
The operator $\frac{1}{D^2}$ corresponds to double integration. When applied to $\sin(z)$, we get:

$$\frac{1}{D}\sin(z) = \int \sin(z)\,dz = -\cos(z) \tag{6.2176}$$

$$\frac{1}{D^2}\sin(z) = \frac{1}{D}[-\cos(z)] = -\int \cos(z)\,dz = -\sin(z) \tag{6.2177}$$

Therefore:

$$\frac{1}{\phi(D)}(e^{-z}\sin(z)) = e^{-z} \cdot \frac{1}{D^2}\sin(z) \tag{6.2178}$$

$$= e^{-z} \cdot (-\sin(z)) \tag{6.2179}$$

$$= -e^{-z}\sin(z) \tag{6.2180}$$

So our particular solution is:

$$y_p = -e^{-z}\sin(z) \tag{6.2181}$$

Step 5: Write the general solution in terms of z.

The general solution in terms of z is:

$$y(z) = (c_1 + c_2 z)e^{-z} - e^{-z}\sin(z) \tag{6.2182}$$

Step 6: Convert back to the original variable x using $z = \ln x$.

$$y(x) = (c_1 + c_2 \ln x)e^{-\ln x} - e^{-\ln x}\sin(\ln x) \tag{6.2183}$$

$$= (c_1 + c_2 \ln x) \cdot \frac{1}{x} - \frac{1}{x}\sin(\ln x) \tag{6.2184}$$

$$= \frac{c_1 + c_2 \ln x - \sin(\ln x)}{x} \tag{6.2185}$$

Therefore, the general solution to the original Cauchy-Euler equation is:

$$y(x) = \frac{c_1 + c_2 \ln x - \sin(\ln x)}{x} \tag{6.2186}$$

Example 6: Legendre's Linear Differential Equation

Solve the differential equation:

$$(2x+3)^2\frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x \tag{6.2187}$$

Step-by-Step Solution

This is a Legendre's linear differential equation with variable coefficients. We can directly apply the transformation formulas for Legendre's equation to convert it to a linear differential equation with constant coefficients.
Step 1: Identify the constants a and b in the Legendre's equation.
Comparing with the general form, we have $(ax + b) = (2x + 3)$, so $a = 2$ and $b = 3$.
Step 2: Apply the transformation formulas.
For Legendre's equation, we use the substitution $t = \ln(ax + b)$ or $ax + b = e^t$, so $t = \ln(2x + 3)$ or $2x + 3 = e^t$.
From the transformation formulas, we know:

$$(ax+b)\frac{dy}{dx} = a\frac{dy}{dt} = aDy \tag{6.2188}$$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y \tag{6.2189}$$

For our equation, these become:

$$(2x+3)\frac{dy}{dx} = 2\frac{dy}{dt} = 2Dy \tag{6.2190}$$

$$(2x+3)^2\frac{d^2y}{dx^2} = 2^2D(D-1)y = 4D(D-1)y \tag{6.2191}$$

Step 3: Transform the original equation.

Substituting these transformations into the original equation:

$$(2x+3)^2\frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x \tag{6.2192}$$

$$4D(D-1)y - 2(2Dy) - 12y = 6\left(\frac{e^t-3}{2}\right) \tag{6.2193}$$

$$4D^2y - 4Dy - 4Dy - 12y = 3e^t - 9 \tag{6.2194}$$

$$4D^2y - 8Dy - 12y = 3e^t - 9 \tag{6.2195}$$

Dividing by 4:

$$D^2y - 2Dy - 3y = \frac{3e^t-9}{4} \tag{6.2196}$$

$$= \frac{3e^t}{4} - \frac{9}{4} \tag{6.2197}$$

Let's denote our differential operator as $\phi(D) = D^2 - 2D - 3$.
Step 4: Solve the homogeneous part of the transformed equation.
The characteristic equation is:

$$m^2 - 2m - 3 = 0 \tag{6.2198}$$

$$(m-3)(m+1) = 0 \tag{6.2199}$$

This gives us roots $m = 3$ and $m = -1$.
Therefore, the complementary function (general solution of the homogeneous equation) is:

$$y_c = c_1e^{3t} + c_2e^{-t} \tag{6.2200}$$

Step 5: Find a particular integral for the non-homogeneous part.
The right side has two terms: $\frac{3e^t}{4}$ and $-\frac{9}{4}$. We'll apply the Method of Undetermined Coefficients for each term.
(a) For $\frac{3e^t}{4}$: Using Case 1 from the Shortcut Method, we have:

$$\frac{1}{\phi(D)}\frac{3e^t}{4} = \frac{3}{4}\cdot\frac{1}{\phi(1)}e^t \tag{6.2201}$$

$$= \frac{3}{4}\cdot\frac{1}{1^2-2(1)-3}e^t \tag{6.2202}$$

$$= \frac{3}{4}\cdot\frac{1}{-4}e^t \tag{6.2203}$$

$$= -\frac{3}{16}e^t \tag{6.2204}$$

So, $y_{p1} = -\frac{3}{16}e^t$.
(b) For $-\frac{9}{4}$ (constant term): Using the formula for constants:

$$\frac{1}{\phi(D)}\left(-\frac{9}{4}\right) = -\frac{9}{4}\cdot\frac{1}{\phi(0)} \tag{6.2205}$$

$$= -\frac{9}{4}\cdot\frac{1}{0^2-2(0)-3} \tag{6.2206}$$

$$= -\frac{9}{4}\cdot\frac{1}{-3} \tag{6.2207}$$

$$= \frac{9}{12} \tag{6.2208}$$

$$= \frac{3}{4} \tag{6.2209}$$

So, $y_{p2} = \frac{3}{4}$.
The complete particular solution is:

$$y_p = y_{p1} + y_{p2} = -\frac{3}{16}e^t + \frac{3}{4} \tag{6.2210}$$

Step 6: Write the general solution in terms of t.
The general solution in terms of t is:

$$y(t) = c_1e^{3t} + c_2e^{-t} - \frac{3}{16}e^t + \frac{3}{4} \tag{6.2211}$$

Step 7: Convert back to the original variable x using $t = \ln(2x+3)$ or $e^t = 2x+3$.

$$y(x) = c_1e^{3\ln(2x+3)} + c_2e^{-\ln(2x+3)} - \frac{3}{16}e^{\ln(2x+3)} + \frac{3}{4} \tag{6.2212}$$
$$= c_1(2x+3)^3 + c_2(2x+3)^{-1} - \frac{3}{16}(2x+3) + \frac{3}{4} \tag{6.2213}$$
$$= c_1(2x+3)^3 + \frac{c_2}{2x+3} - \frac{3(2x+3)}{16} + \frac{3}{4} \tag{6.2214}$$
$$= c_1(2x+3)^3 + \frac{c_2}{2x+3} - \frac{6x+9}{16} + \frac{12}{16} \tag{6.2215}$$
$$= c_1(2x+3)^3 + \frac{c_2}{2x+3} - \frac{6x+9-12}{16} \tag{6.2216}$$
$$= c_1(2x+3)^3 + \frac{c_2}{2x+3} - \frac{6x-3}{16} \tag{6.2217}$$

Therefore, the general solution to the original Legendre's equation is:

$$y(x) = c_1(2x+3)^3 + \frac{c_2}{2x+3} - \frac{6x-3}{16} \tag{6.2218}$$

Example 7: Legendre's Linear Differential Equation

Solve the differential equation:

$$(2x+1)^2\frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2 \tag{6.2219}$$

Step-by-Step Solution

This is a Legendre's linear differential equation with variable coefficients. We will apply the direct transformation formulas to convert it to a linear differential equation with constant coefficients.
Step 1: Identify the constants a and b in the Legendre's equation.
Comparing with the general form, we have $(ax+b) = (2x+1)$, so $a = 2$ and $b = 1$.
Step 2: Apply the transformation formulas.
For Legendre's equation, we use the substitution $t = \ln(ax+b)$ or $ax+b = e^t$, so $t = \ln(2x+1)$ or $2x+1 = e^t$.

From the transformation formulas, we know:

$$(ax+b)\frac{dy}{dx} = a\frac{dy}{dt} = aDy \tag{6.2220}$$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y \tag{6.2221}$$

For our equation, these become:

$$(2x+1)\frac{dy}{dx} = 2\frac{dy}{dt} = 2Dy \tag{6.2222}$$

$$(2x+1)^2\frac{d^2y}{dx^2} = 2^2D(D-1)y = 4D(D-1)y \tag{6.2223}$$

Step 3: Transform the original equation.
Substituting these transformations into the original equation:

$$(2x+1)^2\frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2 \tag{6.2224}$$

$$4D(D-1)y - 6(2Dy) + 16y = 8(e^t)^2 \tag{6.2225}$$

$$4D^2y - 4Dy - 12Dy + 16y = 8e^{2t} \tag{6.2226}$$

$$4D^2y - 16Dy + 16y = 8e^{2t} \tag{6.2227}$$

Dividing by 4:

$$D^2y - 4Dy + 4y = 2e^{2t} \tag{6.2228}$$

Let's denote our differential operator as $\phi(D) = D^2 - 4D + 4 = (D-2)^2$.
Step 4: Solve the homogeneous part of the transformed equation.
The characteristic equation is:

$$m^2 - 4m + 4 = 0 \tag{6.2229}$$

$$(m-2)^2 = 0 \tag{6.2230}$$

This gives us a repeated root $m = 2$ with multiplicity 2.
Therefore, the complementary function (general solution of the homogeneous equation) is:

$$y_c = (c_1 + c_2t)e^{2t} \tag{6.2231}$$

Step 5: Find a particular integral for the non-homogeneous part.
The right side is $2e^{2t}$, and this corresponds to Case 1 from the Shortcut Method where $f(x) = e^{ax}$ with $a = 2$ here.
However, we notice that $m = 2$ is a root of the characteristic equation, which means we're in a failure case for the Shortcut Method. Since $m = 2$ is a root of multiplicity 2, this is failure case (ii).
According to the failure case (ii) formula:

$$\frac{1}{\phi(D)}e^{ax} = \frac{x^2}{\phi''(D)}e^{ax}, \quad \text{when } \phi(a) = \phi'(a) = 0 \text{ and } \phi''(a) \neq 0 \tag{6.2232}$$

We need to calculate $\phi''(D)$:

$$\phi(D) = D^2 - 4D + 4 \tag{6.2233}$$

$$\phi'(D) = 2D - 4 \tag{6.2234}$$

$$\phi''(D) = 2 \tag{6.2235}$$

Now we can apply the formula:

$$\frac{1}{\phi(D)}(2e^{2t}) = 2 \cdot \frac{t^2}{\phi''(2)}e^{2t} \tag{6.2236}$$
$$= 2 \cdot \frac{t^2}{2}e^{2t} \tag{6.2237}$$
$$= t^2e^{2t} \tag{6.2238}$$

So, our particular solution is:

$$y_p = t^2e^{2t} \tag{6.2239}$$

Step 6: Write the general solution in terms of t.
The general solution in terms of t is:

$$y(t) = (c_1 + c_2t)e^{2t} + t^2e^{2t} = (c_1 + c_2t + t^2)e^{2t} \tag{6.2240}$$

Step 7: Convert back to the original variable x using $t = \ln(2x+1)$ or $e^t = 2x+1$.

$$y(x) = [c_1 + c_2\ln(2x+1) + (\ln(2x+1))^2]e^{2\ln(2x+1)} \tag{6.2241}$$
$$= [c_1 + c_2\ln(2x+1) + (\ln(2x+1))^2](2x+1)^2 \tag{6.2242}$$

Therefore, the general solution to the original Legendre's equation is:

$$y(x) = (2x+1)^2[c_1 + c_2\ln(2x+1) + (\ln(2x+1))^2] \tag{6.2243}$$

We can verify this solution by substituting it back into the original differential equation.

Example 8: Third-Order Legendre's Differential Equation

Solve the differential equation:

$$(x-1)^3\frac{d^3y}{dx^3} + 2(x-1)^2\frac{d^2y}{dx^2} - 4(x-1)\frac{dy}{dx} + 4y = 4\log(x-1) \tag{6.2244}$$

Step-by-Step Solution

This is a third-order Legendre's linear differential equation with a logarithmic term. We'll use the direct transformation approach to convert it to a linear equation with constant coefficients.
Step 1: Identify the constants a and b in the Legendre's equation.
Comparing with the general form, we have $(ax+b) = (x-1)$, so $a = 1$ and $b = -1$.
Step 2: Apply the transformation formulas.
For Legendre's equation, we use the substitution $t = \ln(ax+b)$ or $ax+b = e^t$, so $t = \ln(x-1)$ or $x - 1 = e^t$.

From the transformation formulas, we know:

$$(ax+b)\frac{dy}{dx} = a\frac{dy}{dt} = aDy \tag{6.2245}$$
$$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y \tag{6.2246}$$
$$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y \tag{6.2247}$$

For our equation, with $a = 1$, these become:

$$(x-1)\frac{dy}{dx} = \frac{dy}{dt} = Dy \tag{6.2248}$$
$$(x-1)^2\frac{d^2y}{dx^2} = D(D-1)y \tag{6.2249}$$
$$(x-1)^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y \tag{6.2250}$$

Step 3: Transform the original equation.
Also, we need to transform the right-hand side:

$$4\log(x-1) = 4t \tag{6.2251}$$

Substituting these transformations into the original equation:

$$(x-1)^3\frac{d^3y}{dx^3} + 2(x-1)^2\frac{d^2y}{dx^2} - 4(x-1)\frac{dy}{dx} + 4y = 4\log(x-1) \tag{6.2252}$$
$$D(D-1)(D-2)y + 2D(D-1)y - 4Dy + 4y = 4t \tag{6.2253}$$

Expanding the operators:

$$[D^3 - 3D^2 + 2D] + [2D^2 - 2D] - 4D + 4 = 4t \tag{6.2254}$$
$$D^3 - 3D^2 + 2D + 2D^2 - 2D - 4D + 4y = 4t \tag{6.2255}$$
$$D^3 - D^2 - 4D + 4y = 4t \tag{6.2256}$$
$$D^3 - D^2 - 4D + 4 = 4t \tag{6.2257}$$

Therefore, our transformed equation is:

$$(D^3 - D^2 - 4D + 4)y = 4t \tag{6.2258}$$

Let's denote our differential operator as $\phi(D) = D^3 - D^2 - 4D + 4$.
Step 4: Solve the homogeneous part of the transformed equation.
The characteristic equation is:

$$m^3 - m^2 - 4m + 4 = 0 \tag{6.2259}$$

Let's try to find the roots. We can try $m = 1$:

$$1^3 - 1^2 - 4(1) + 4 = 1 - 1 - 4 + 4 = 0 \tag{6.2260}$$

So $m = 1$ is a root. We can factorize the characteristic equation:

$$m^3 - m^2 - 4m + 4 = (m-1)(m^2-4) \tag{6.2261}$$
$$= (m-1)(m-2)(m+2) \tag{6.2262}$$

Therefore, the roots are $m = 1$, $m = 2$, and $m = -2$.
The complementary function (general solution of the homogeneous equation) is:

$$y_c = c_1 e^t + c_2 e^{2t} + c_3 e^{-2t} \tag{6.2263}$$

Step 5: Find a particular integral for the non-homogeneous part.
The right side is $4t$, which corresponds to Case 4 from the Shortcut Method for $f(x) = x^m$ with $m = 1$ here.
According to Case 4, we need to expand $\frac{1}{\phi(D)}$ in ascending powers of D up to the term D^m. Since $m = 1$, we need terms up to D^1.
However, we can also directly apply the formula for polynomial terms. For a constant term, $\frac{1}{\phi(D)}(k) = \frac{k}{\phi(0)}$. For a term x, we can use:

$$\frac{1}{\phi(D)}(x) = \frac{x}{\phi(0)} - \frac{\phi'(0)}{\phi(0)^2} \tag{6.2264}$$

This approach can become complicated for our third-order equation.
Instead, let's try the method of undetermined coefficients. Since the right side is a polynomial of degree 1 (i.e., $4t$), we try a particular solution of the form:

$$y_p = At + B \tag{6.2265}$$

where A and B are constants to be determined.
We need to find the derivatives:

$$Dy_p = A \tag{6.2266}$$
$$D^2 y_p = 0 \tag{6.2267}$$
$$D^3 y_p = 0 \tag{6.2268}$$

Substituting into our differential equation:

$$(D^3 - D^2 - 4D + 4)y_p = 4t \tag{6.2269}$$
$$0 - 0 - 4A + 4(At + B) = 4t \tag{6.2270}$$
$$-4A + 4At + 4B = 4t \tag{6.2271}$$

Comparing coefficients of like terms:

$$4A = 4 \quad \Rightarrow \quad A = 1 \tag{6.2272}$$
$$-4A + 4B = 0 \quad \Rightarrow \quad -4 + 4B = 0 \quad \Rightarrow \quad B = 1 \tag{6.2273}$$

Therefore, our particular solution is:

$$y_p = t + 1 \tag{6.2274}$$

Step 6: Write the general solution in terms of t.
The general solution in terms of t is:

$$y(t) = c_1 e^t + c_2 e^{2t} + c_3 e^{-2t} + t + 1 \tag{6.2275}$$

Step 7: Convert back to the original variable x using $t = \ln(x - 1)$ or $e^t = x - 1$.

$$y(x) = c_1 e^{\ln(x-1)} + c_2 e^{2\ln(x-1)} + c_3 e^{-2\ln(x-1)} + \ln(x-1) + 1 \tag{6.2276}$$

$$= c_1(x-1) + c_2(x-1)^2 + c_3\frac{1}{(x-1)^2} + \ln(x-1) + 1 \tag{6.2277}$$

Therefore, the general solution to the original Legendre's equation is:

$$y(x) = c_1(x-1) + c_2(x-1)^2 + \frac{c_3}{(x-1)^2} + \ln(x-1) + 1 \tag{6.2278}$$

Verification of the particular solution:
Let's substitute $y_p = \ln(x-1) + 1$ into the original equation:

$$(x-1)^3\frac{d^3y_p}{dx^3} + 2(x-1)^2\frac{d^2y_p}{dx^2} - 4(x-1)\frac{dy_p}{dx} + 4y_p = 4\log(x-1) \tag{6.2279}$$

We need to compute the derivatives:

$$y_p = \ln(x-1) + 1 \tag{6.2280}$$

$$\frac{dy_p}{dx} = \frac{1}{x-1} \tag{6.2281}$$

$$\frac{d^2y_p}{dx^2} = -\frac{1}{(x-1)^2} \tag{6.2282}$$

$$\frac{d^3y_p}{dx^3} = \frac{2}{(x-1)^3} \tag{6.2283}$$

Substituting these into the left side of the equation:

$$(x-1)^3\frac{d^3y_p}{dx^3} + 2(x-1)^2\frac{d^2y_p}{dx^2} - 4(x-1)\frac{dy_p}{dx} + 4y_p \tag{6.2284}$$

$$= (x-1)^3 \cdot \frac{2}{(x-1)^3} + 2(x-1)^2 \cdot \left(-\frac{1}{(x-1)^2}\right) - 4(x-1) \cdot \frac{1}{x-1} + 4(\ln(x-1) + 1) \tag{6.2285}$$

$$= 2 - 2 - 4 + 4\ln(x-1) + 4 \tag{6.2286}$$

$$= 2 - 2 - 4 + 4\ln(x-1) + 4 \tag{6.2287}$$

$$= 0 + 4\ln(x-1) \tag{6.2288}$$

$$= 4\log(x-1) \tag{6.2289}$$

This verifies that our particular solution $y_p = \ln(x-1) + 1$ is correct.