Chapter 7

Vector Space

7.1 Vector Spaces and Solved Examples on Vector Spaces

Vector spaces form the fundamental mathematical structure upon which linear algebra is built. They provide a rigorous framework for studying vectors, which arise naturally in various mathematical and physical contexts. This section introduces the formal definition of vector spaces and explores key examples that illustrate their versatility.

Definition and Axioms of a Vector Space

Definition 7.1. *A **vector space** over a field F (typically $\mathbb{R}$ or $\mathbb{C}$) is a set V equipped with two operations:*

- ***Vector addition**: $+ : V \times V \to V$*
- ***Scalar multiplication**: $\cdot : F \times V \to V$*

such that the following axioms are satisfied for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $a, b \in F$:

Vector Space Axioms

1. **Closure under addition**: $\mathbf{u} + \mathbf{v} \in V$
2. **Commutativity of addition**: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. **Associativity of addition**: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. **Additive identity**: There exists a zero vector $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$
5. **Additive inverse**: For each $\mathbf{v} \in V$, there exists a vector $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
6. **Closure under scalar multiplication**: $a \cdot \mathbf{v} \in V$
7. **Distributivity of scalar multiplication over vector addition**: $a \cdot (\mathbf{u} + \mathbf{v}) = a \cdot \mathbf{u} + a \cdot \mathbf{v}$
8. **Distributivity of scalar multiplication over field addition**: $(a + b) \cdot \mathbf{v} = a \cdot \mathbf{v} + b \cdot \mathbf{v}$
9. **Compatibility of scalar multiplication with field multiplication**: $a \cdot (b \cdot \mathbf{v}) = (ab) \cdot \mathbf{v}$
10. **Scalar identity**: $1 \cdot \mathbf{v} = \mathbf{v}$ where 1 is the multiplicative identity in F

Remark 7.2. *These axioms formalize our intuitive understanding of vectors as objects that can be added together and scaled by numbers. The first five axioms deal with vector addition, while the remaining five concern scalar multiplication and its interaction with addition.*

Examples of Vector Spaces

Let us explore some common vector spaces that appear throughout mathematics and applications.

Example 7.3 ($\mathbb{R}^n$ - Euclidean Space)**.** *The set $\mathbb{R}^n$ of all ordered n-tuples of real numbers forms a vector space over $\mathbb{R}$ with the operations:*

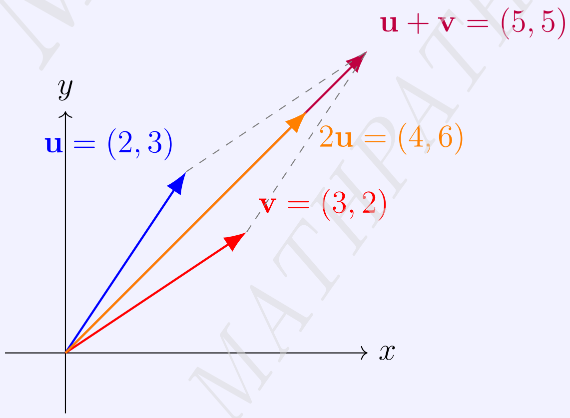
$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$
$$a \cdot (x_1, x_2, \dots, x_n) = (ax_1, ax_2, \dots, ax_n)$$

This is the most familiar vector space, representing geometric vectors in n-dimensional space. When $n = 2$ or $n = 3$, we can visualize these vectors as arrows in a plane or in three-dimensional space.

Particular Example: *In $\mathbb{R}^3$, let's verify a few vector space axioms with specific vectors:*

- ***Vector addition:*** $(1, 2, 3) + (4, 5, 6) = (5, 7, 9)$
- ***Scalar multiplication:*** $3 \cdot (1, 2, 3) = (3, 6, 9)$
- ***Distributivity:*** $2 \cdot ((1, 2, 3) + (4, 5, 6)) = 2 \cdot (5, 7, 9) = (10, 14, 18)$
  $2 \cdot (1, 2, 3) + 2 \cdot (4, 5, 6) = (2, 4, 6) + (8, 10, 12) = (10, 14, 18)$

Visualization of $\mathbb{R}^2$



Example 7.4 ($\mathcal{P}_n$ - Polynomial Space)**.** *The set $\mathcal{P}_n$ of all polynomials with real coefficients of degree at most n forms a vector space over $\mathbb{R}$. For polynomials $p(x) = a_0 + a_1x + \cdots + a_nx^n$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$:*

$$(p + q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$
$$(c \cdot p)(x) = ca_0 + ca_1x + \cdots + ca_nx^n$$

This vector space has dimension $n + 1$, with basis $\{1, x, x^2, \dots, x^n\}$.

Particular Example: *Consider $\mathcal{P}_2$, the space of polynomials of degree at most 2. Let's work with specific polynomials:*

$$p(x) = 2 + 3x - x^2$$
$$q(x) = 4 - 2x + 5x^2$$

Then:

- ***Vector addition:*** $(p + q)(x) = (2 + 4) + (3 - 2)x + (-1 + 5)x^2 = 6 + x + 4x^2$
- ***Scalar multiplication:*** $(3 \cdot p)(x) = 3 \cdot (2 + 3x - x^2) = 6 + 9x - 3x^2$
- ***Zero vector:*** *The zero polynomial $0 + 0x + 0x^2$ acts as the additive identity*
- ***Additive inverse:*** $(-p)(x) = -2 - 3x + x^2$ *since* $p(x) + (-p)(x) = 0$

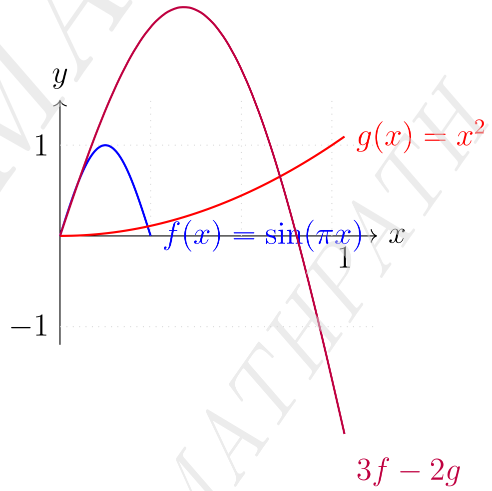
Example 7.5 ($C[a, b)$. $-ContinuousFunctionSpace]ThesetC[a,b]ofallcontinuousfunctionsf$: *[a,b]* $\to \mathbb{R}$ *forms an infinite-dimensional vector space over* $\mathbb{R}$ *with the operations:*

$$(f+g)(x) = f(x) + g(x)$$
$$(c \cdot f)(x) = c \cdot f(x)$$

This is an example of a function space, which is crucial in mathematical analysis and differential equations.

Particular Example: *Consider* $C[0, 1]$ *with the specific functions* $f(x) = \sin(\pi x)$ *and* $g(x) = x^2$. *Then:*

- ***Vector addition:*** $(f + g)(x) = \sin(\pi x) + x^2$
- ***Scalar multiplication:*** $(2 \cdot f)(x) = 2\sin(\pi x)$
- ***Zero vector:*** *The constant function* $h(x) = 0$ *serves as the additive identity*
- ***Linear combination:*** $3f - 2g = 3\sin(\pi x) - 2x^2$ *is also in* $C[0, 1]$



The figure above illustrates these functions over $[0, 1]$, *with domain scaled for better visualization.*

Example 7.6 ($M_{m,n}$ - Matrix Space)**.** *The set* $M_{m,n}$ *of all* $m \times n$ *matrices with real entries forms a vector space over* $\mathbb{R}$ *with the operations of matrix addition and scalar multiplication:*

$$(A + B)_{ij} = A_{ij} + B_{ij}$$
$$(cA)_{ij} = c \cdot A_{ij}$$

This vector space has dimension mn *and connects vector spaces to systems of linear equations and linear transformations.*

Particular Example: *Consider* $M_{2,2}$, *the space of* 2×2 *matrices. Let:*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & -1 \\ 0 & 2 \end{pmatrix}$$

Then:

- ***Vector addition:*** $A + B = \begin{pmatrix} 1+5 & 2+(-1) \\ 3+0 & 4+2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 3 & 6 \end{pmatrix}$
- ***Scalar multiplication:*** $3A = 3 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$
- ***Zero vector:*** *The zero matrix* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *serves as the additive identity*
- ***Linear combination:*** $2A - B = 2 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} - \begin{pmatrix} 5 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 5 \\ 6 & 6 \end{pmatrix}$

$$\overset{A}{\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}} + \overset{B}{\begin{pmatrix}5 & -1\\ 0 & 2\end{pmatrix}} = \overset{A+B}{\begin{pmatrix}6 & 1\\ 3 & 6\end{pmatrix}}$$

$$A \xrightarrow{\times 3} \overset{3A}{\begin{pmatrix}3 & 6\\ 9 & 12\end{pmatrix}}$$

Application: Signal Processing

In signal processing, functions representing signals form vector spaces. For example, audio signals can be represented as functions of time $s(t)$, forming a vector space where addition corresponds to superposition of signals, and scalar multiplication corresponds to amplification or attenuation.
This vector space structure allows decomposition of complex signals into simpler components through techniques like Fourier analysis, which finds a representation of signals in terms of sine and cosine functions.

Non-Examples of Vector Spaces

Not every set with addition and scalar multiplication operations forms a vector space. Understanding when a structure fails to be a vector space helps deepen our comprehension of the concept.

Additional Examples of Vector Spaces

Before exploring non-examples, let's examine some other important vector spaces:
1. Solution Space of a Homogeneous System: The set of all solutions to a homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ forms a vector space.
Particular Example: Consider the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 0 \end{aligned}$$

The solutions can be parametrized as $x_3 = t$, $x_2 = \frac{-3t-3x_1}{7}$, x_1 free. If we set $x_1 = s$, this gives:

$$\mathbf{x} = \begin{pmatrix} s \\ \frac{-3t-3s}{7} \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ \frac{-3}{7} \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ \frac{-3}{7} \\ 1 \end{pmatrix}$$

This forms a 2-dimensional vector space.
2. Vector Space of Sequences: The set ℓ^2 of all infinite sequences $(a_1, a_2, a_3, \ldots)$ of real numbers such that $\sum_{i=1}^{\infty} a_i^2 < \infty$ forms a vector space over $\mathbb{R}$.
Particular Example: The sequences $\mathbf{a} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ and $\mathbf{b} = (0, 1, 0, \frac{1}{4}, 0, \ldots)$ are both in ℓ^2 since:

$$\sum_{i=1}^{\infty} a_i^2 = 1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \cdots = 1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{4/3}{1} < \infty$$

$$\sum_{i=1}^{\infty} b_i^2 = 0^2 + 1^2 + 0^2 + \left(\frac{1}{4}\right)^2 + \cdots = 1 + \frac{1}{16} + \cdots < \infty$$

Their sum $\mathbf{a} + \mathbf{b} = (1, \frac{3}{2}, \frac{1}{4}, \frac{3}{8}, \ldots)$ is also in ℓ^2.

Non-Example: Positive Real Numbers

The set $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ of positive real numbers with the standard addition and multiplication fails to be a vector space over $\mathbb{R}$.

Analysis

While $\mathbb{R}^+$ is closed under addition and positive scalar multiplication, it violates several vector space axioms:

- **No additive identity**: There is no element $0 \in \mathbb{R}^+$ since positive numbers exclude zero.
- **No additive inverses**: For any $x \in \mathbb{R}^+$, we have $-x < 0$, so $-x \notin \mathbb{R}^+$.
- **Not closed under scalar multiplication**: For any $x \in \mathbb{R}^+$ and scalar $c = -1$, we have $cx = -x \notin \mathbb{R}^+$.

Non-Example: $\mathbb{R}^2$ with Component-wise Multiplication

Consider $\mathbb{R}^2$ with component-wise addition $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ but with scalar multiplication defined as component-wise multiplication: $c \cdot (x_1, x_2) = (c \cdot x_1 \cdot x_2, x_2)$.

Analysis

This structure fails to be a vector space because it violates the distributivity axiom:

$$\begin{aligned} c \cdot ((x_1, x_2) + (y_1, y_2)) &= c \cdot (x_1 + y_1, x_2 + y_2) \\ &= (c \cdot (x_1 + y_1) \cdot (x_2 + y_2), x_2 + y_2) \end{aligned}$$

But:

$$\begin{aligned} c \cdot (x_1, x_2) + c \cdot (y_1, y_2) &= (c \cdot x_1 \cdot x_2, x_2) + (c \cdot y_1 \cdot y_2, y_2) \\ &= (c \cdot x_1 \cdot x_2 + c \cdot y_1 \cdot y_2, x_2 + y_2) \end{aligned}$$

These are generally not equal, so the distributivity axiom fails.

Non-Example: $\mathbb{Z}$ - Integers

The set of integers $\mathbb{Z}$ with standard addition and multiplication by real numbers.

Analysis

While $\mathbb{Z}$ is closed under addition and has an additive identity (0) and inverses, it is not closed under scalar multiplication by arbitrary real numbers. For instance, if $z \in \mathbb{Z}$ and $c = \frac{1}{2} \in \mathbb{R}$, then $c \cdot z = \frac{z}{2}$ which is not an integer unless z is even.
However, $\mathbb{Z}$ does form a vector space over the field $\mathbb{Q}$ of rational numbers for the same reason.

Key Insight

A critical observation from these non-examples is that the definition of a vector space is highly dependent on:

1. The set of vectors V
2. The operations defined on the set

3. The field F over which the vector space is defined

Theorem 7.7 (Consequences of Vector Space Axioms)**.** *The following properties hold in any vector space V over a field F:*

1. *The zero vector $\mathbf{0}$ is unique.*
2. *For each vector $\mathbf{v} \in V$, the additive inverse $-\mathbf{v}$ is unique.*
3. *For any vector $\mathbf{v} \in V$, $0 \cdot \mathbf{v} = \mathbf{0}$, where 0 is the additive identity in F.*
4. *For any vector $\mathbf{v} \in V$, $(-1) \cdot \mathbf{v} = -\mathbf{v}$.*
5. *For any scalars $a \in F$ and any vector $\mathbf{v} \in V$, if $a \cdot \mathbf{v} = \mathbf{0}$, then either $a = 0$ or $\mathbf{v} = \mathbf{0}$.*

Remark 7.8. *The concept of a vector space provides a powerful abstraction that unifies many different mathematical objects. By recognizing when a mathematical structure forms a vector space, we can immediately apply the rich theory of linear algebra to analyze it.*

Additional Solved Examples

Example 1: $\mathbb{R}$ as a Vector Space

Problem: Show that the set of real numbers $\mathbb{R}$ itself is a vector space with respect to usual addition and scalar multiplication of real numbers.

Solution: To prove that $\mathbb{R}$ is a vector space over itself, we need to verify all ten vector space axioms. Here, our set $V = \mathbb{R}$ and the field is also $F = \mathbb{R}$. The operations are:

- Vector addition: $u + v$ for $u, v \in \mathbb{R}$ (standard addition of real numbers)
- Scalar multiplication: $c \cdot v$ for $c, v \in \mathbb{R}$ (standard multiplication of real numbers)

Let's verify each axiom systematically:

1. Closure under addition: For any $u, v \in \mathbb{R}$, we need to show that $u + v \in \mathbb{R}$.
We know that the sum of any two real numbers is a real number, so $u + v \in \mathbb{R}$. For instance, $3 + 5 = 8 \in \mathbb{R}$ and $\pi + \sqrt{2} \in \mathbb{R}$.

2. Commutativity of addition: For any $u, v \in \mathbb{R}$, we need to show that $u + v = v + u$.
This follows directly from the commutative property of real number addition. For example, $3 + 5 = 5 + 3 = 8$.

3. Associativity of addition: For any $u, v, w \in \mathbb{R}$, we need to show that $(u + v) + w = u + (v + w)$.
This follows from the associative property of real number addition. For example, $(2+3)+4 = 5 + 4 = 9$ and $2 + (3 + 4) = 2 + 7 = 9$.

4. Additive identity: There exists a zero element $0 \in \mathbb{R}$ such that $v + 0 = v$ for all $v \in \mathbb{R}$.
The number 0 serves as the additive identity in $\mathbb{R}$ since $v + 0 = v$ for any real number v. For example, $7 + 0 = 7$ and $-3.5 + 0 = -3.5$.

5. Additive inverse: For each $v \in \mathbb{R}$, there exists an element $-v \in \mathbb{R}$ such that $v + (-v) = 0$.
For any real number v, its negative $-v$ is also a real number, and $v + (-v) = 0$. For instance, $6 + (-6) = 0$ and $\pi + (-\pi) = 0$.

6. Closure under scalar multiplication: For any scalar $c \in \mathbb{R}$ and any vector $v \in \mathbb{R}$, we need to show that $c \cdot v \in \mathbb{R}$.
Since the product of any two real numbers is a real number, $c \cdot v \in \mathbb{R}$. For example, $3 \cdot 4 = 12 \in \mathbb{R}$ and $\sqrt{2} \cdot \pi \in \mathbb{R}$.

7. Distributivity of scalar multiplication over vector addition: For any scalar $c \in \mathbb{R}$ and any vectors $u, v \in \mathbb{R}$, we need to show that $c \cdot (u + v) = c \cdot u + c \cdot v$.
This follows from the distributive property of real number multiplication over addition. For example, $3 \cdot (4 + 5) = 3 \cdot 9 = 27$ and $3 \cdot 4 + 3 \cdot 5 = 12 + 15 = 27$.

8. Distributivity of scalar multiplication over field addition: For any scalars

$a, b \in \mathbb{R}$ and any vector $v \in \mathbb{R}$, we need to show that $(a+b) \cdot v = a \cdot v + b \cdot v$.
This also follows from the distributive property of real numbers. For example, $(2+3) \cdot 4 = 5 \cdot 4 = 20$ and $2 \cdot 4 + 3 \cdot 4 = 8 + 12 = 20$.
9. Compatibility of scalar multiplication with field multiplication: For any scalars $a, b \in \mathbb{R}$ and any vector $v \in \mathbb{R}$, we need to show that $a \cdot (b \cdot v) = (a \cdot b) \cdot v$.
This follows from the associative property of real number multiplication. For example, $2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24$ and $(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24$.
10. Scalar identity: For any vector $v \in \mathbb{R}$, we need to show that $1 \cdot v = v$.
This follows from the identity property of multiplication in $\mathbb{R}$. For any real number v, $1 \cdot v = v$. For example, $1 \cdot 5 = 5$ and $1 \cdot (-3.7) = -3.7$.
Conclusion: Since all ten vector space axioms are satisfied, we have proven that $\mathbb{R}$ with the standard operations of addition and multiplication is indeed a vector space over $\mathbb{R}$.
Key insight: This example shows that the field $\mathbb{R}$ itself can be viewed as a one-dimensional vector space over $\mathbb{R}$. Any real number v can be expressed as $v = v \cdot 1$, where 1 serves as a basis for this vector space.

Remark 7.9. *The real numbers $\mathbb{R}$ as a vector space over itself has dimension 1. This means that any real number can be uniquely represented as a scalar multiple of the basis element 1. For instance, $\pi = \pi \cdot 1$, where π is the scalar and 1 is the basis vector.*

Example 2: A Special Set of Ordered Pairs

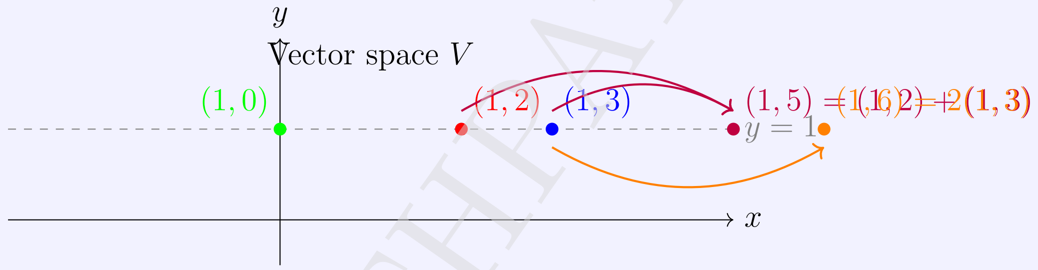
Problem: Show that the set V of all pairs of real numbers of the form $(1, x)$ with the operations defined as $(1, y) + (1, y') = (1, y + y')$ and $k(1, y) = (1, ky)$ is a vector space.
Solution: We need to verify that the set $V = \{(1, x) : x \in \mathbb{R}\}$ with the given operations satisfies all ten vector space axioms. The field here is $\mathbb{R}$.
Preliminary note: It's important to recognize that V is not $\mathbb{R}^2$. The set V consists only of ordered pairs where the first component is always 1. Let's verify each axiom systematically:
1. Closure under addition: For any $(1, y), (1, y') \in V$, we need to show that $(1, y) + (1, y') \in V$.
By the defined operation: $(1, y) + (1, y') = (1, y + y')$
Since $y + y'$ is a real number, $(1, y + y')$ has the form $(1, x)$ where $x \in \mathbb{R}$, so it belongs to V.
For example, $(1, 3) + (1, 4) = (1, 3 + 4) = (1, 7) \in V$.
2. Commutativity of addition: For any $(1, y), (1, y') \in V$, we need to show that $(1, y) + (1, y') = (1, y') + (1, y)$.
$(1, y) + (1, y') = (1, y + y')$ $(1, y') + (1, y) = (1, y' + y)$
Since addition of real numbers is commutative ($y + y' = y' + y$), we have $(1, y + y') = (1, y' + y)$.
For example, $(1, 5) + (1, -2) = (1, 5 + (-2)) = (1, 3)$ and $(1, -2) + (1, 5) = (1, -2 + 5) = (1, 3)$.
3. Associativity of addition: For any $(1, y), (1, y'), (1, y'') \in V$, we need to show that $((1, y) + (1, y')) + (1, y'') = (1, y) + ((1, y') + (1, y''))$.
$((1, y) + (1, y')) + (1, y'') = (1, y + y') + (1, y'') = (1, (y + y') + y'')$ $(1, y) + ((1, y') + (1, y'')) = (1, y) + (1, y' + y'') = (1, y + (y' + y''))$
Since addition of real numbers is associative ($(y + y') + y'' = y + (y' + y'')$), the associativity holds in V.
For example, $((1, 2) + (1, 3)) + (1, 4) = (1, 2 + 3) + (1, 4) = (1, 5) + (1, 4) = (1, 9)$ and $(1, 2) + ((1, 3) + (1, 4)) = (1, 2) + (1, 3 + 4) = (1, 2) + (1, 7) = (1, 9)$.

4. Additive identity: We need to show there exists a zero element $(1, e) \in V$ such that $(1, y) + (1, e) = (1, y)$ for all $(1, y) \in V$.
For $(1, y) + (1, e) = (1, y)$, we need $(1, y + e) = (1, y)$, which means $y + e = y$. This is satisfied when $e = 0$.
Therefore, $(1, 0)$ is the additive identity in V.
For example, $(1, 7) + (1, 0) = (1, 7 + 0) = (1, 7)$.
5. Additive inverse: For each $(1, y) \in V$, we need to show there exists an element $(1, -y) \in V$ such that $(1, y) + (1, -y) = (1, 0)$.
For $(1, y) + (1, z) = (1, 0)$, we need $(1, y + z) = (1, 0)$, which means $y + z = 0$. This is satisfied when $z = -y$.
Therefore, for any $(1, y) \in V$, its additive inverse is $(1, -y) \in V$.
For example, $(1, 4) + (1, -4) = (1, 4 + (-4)) = (1, 0)$.
6. Closure under scalar multiplication: For any scalar $k \in \mathbb{R}$ and any vector $(1, y) \in V$, we need to show that $k(1, y) \in V$.
By the defined operation: $k(1, y) = (1, ky)$
Since ky is a real number, $(1, ky)$ has the form $(1, x)$ where $x \in \mathbb{R}$, so it belongs to V.
For example, $3(1, 5) = (1, 3 \cdot 5) = (1, 15) \in V$.
7. Distributivity of scalar multiplication over vector addition: For any scalar $k \in \mathbb{R}$ and any vectors $(1, y), (1, y') \in V$, we need to show that $k((1, y) + (1, y')) = k(1, y) + k(1, y')$.
$k((1, y) + (1, y')) = k(1, y + y') = (1, k(y + y'))$ $k(1, y) + k(1, y') = (1, ky) + (1, ky') = (1, ky + ky')$
Since multiplication distributes over addition in $\mathbb{R}$ ($k(y+y') = ky+ky'$), we have $(1, k(y+y')) = (1, ky + ky')$.
For example, $2((1, 3) + (1, 4)) = 2(1, 7) = (1, 14)$ and $2(1, 3) + 2(1, 4) = (1, 6) + (1, 8) = (1, 14)$.
8. Distributivity of scalar multiplication over field addition: For any scalars $k, l \in \mathbb{R}$ and any vector $(1, y) \in V$, we need to show that $(k + l)(1, y) = k(1, y) + l(1, y)$.
$(k + l)(1, y) = (1, (k + l)y)$ $k(1, y) + l(1, y) = (1, ky) + (1, ly) = (1, ky + ly)$
Since multiplication distributes over addition in $\mathbb{R}$ ($(k + l)y = ky + ly$), we have $(1, (k + l)y) = (1, ky + ly)$.
For example, $(2+3)(1, 4) = 5(1, 4) = (1, 20)$ and $2(1, 4)+3(1, 4) = (1, 8)+(1, 12) = (1, 20)$.
9. Compatibility of scalar multiplication with field multiplication: For any scalars $k, l \in \mathbb{R}$ and any vector $(1, y) \in V$, we need to show that $k(l(1, y)) = (kl)(1, y)$.
$k(l(1, y)) = k(1, ly) = (1, k(ly))$ $(kl)(1, y) = (1, (kl)y)$
Since multiplication is associative in $\mathbb{R}$ ($k(ly) = (kl)y$), we have $(1, k(ly)) = (1, (kl)y)$.
For example, $2(3(1, 5)) = 2(1, 15) = (1, 30)$ and $(2 \cdot 3)(1, 5) = 6(1, 5) = (1, 30)$.
10. Scalar identity: For any vector $(1, y) \in V$, we need to show that $1(1, y) = (1, y)$.
$1(1, y) = (1, 1 \cdot y) = (1, y)$
This confirms that the multiplicative identity 1 acts as expected on vectors in V.
For example, $1(1, 8) = (1, 1 \cdot 8) = (1, 8)$.
Conclusion: Since all ten vector space axioms are satisfied, we have proven that $V = \{(1, x) : x \in \mathbb{R}\}$ with the defined operations is indeed a vector space over $\mathbb{R}$.
Key insight: This vector space V is isomorphic to $\mathbb{R}$. There is a one-to-one correspondence between elements of V and $\mathbb{R}$ given by the mapping $(1, x) \mapsto x$. This mapping preserves the vector space operations. Therefore, V is essentially a one-dimensional vector space.

Remark 7.10. *This example illustrates an important concept: a vector space need not be the full space $\mathbb{R}^n$. Here, we have a subset of $\mathbb{R}^2$ (specifically, points lying on the horizontal line $y = 1$)*

that forms a vector space with appropriately defined operations. Note that these operations differ from the standard ones in $\mathbb{R}^2$ *- they're designed to ensure the first component always remains 1.*

Geometric Interpretation



Example 3: Modified Operations on $\mathbb{R}^2$

Problem: Check whether $V = \mathbb{R}^2$ is a vector space with respect to the operations

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1 - 2, u_2 + v_2 - 3)$$
$$\alpha(u_1, u_2) = (\alpha u_1 + 2\alpha - 2, \alpha u_2 - 3\alpha + 3)$$

Solution: To determine if $V = \mathbb{R}^2$ with these operations forms a vector space, we need to verify all ten vector space axioms. Let's examine each one systematically.
1. Closure under addition: For any $(u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$, their sum $(u_1 + v_1 - 2, u_2 + v_2 - 3)$ is also in $\mathbb{R}^2$.
This axiom is satisfied since any pair of real numbers is in $\mathbb{R}^2$.
2. Commutativity of addition: We need to check if $(u_1, u_2)+(v_1, v_2) = (v_1, v_2)+(u_1, u_2)$ for all vectors.
$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1 - 2, u_2 + v_2 - 3)$ $(v_1, v_2) + (u_1, u_2) = (v_1 + u_1 - 2, v_2 + u_2 - 3)$
Since addition of real numbers is commutative, $u_1 + v_1 = v_1 + u_1$ and $u_2 + v_2 = v_2 + u_2$, so this axiom is satisfied.
3. Associativity of addition: We need to verify if $((u_1, u_2) + (v_1, v_2)) + (w_1, w_2) = (u_1, u_2) + ((v_1, v_2) + (w_1, w_2))$.
Let's calculate:

$$\begin{aligned}((u_1, u_2) + (v_1, v_2)) + (w_1, w_2) &= (u_1 + v_1 - 2, u_2 + v_2 - 3) + (w_1, w_2)\\ &= ((u_1 + v_1 - 2) + w_1 - 2, (u_2 + v_2 - 3) + w_2 - 3)\\ &= (u_1 + v_1 + w_1 - 4, u_2 + v_2 + w_2 - 6)\end{aligned}$$

And:

$$\begin{aligned}(u_1, u_2) + ((v_1, v_2) + (w_1, w_2)) &= (u_1, u_2) + (v_1 + w_1 - 2, v_2 + w_2 - 3)\\ &= (u_1 + (v_1 + w_1 - 2) - 2, u_2 + (v_2 + w_2 - 3) - 3)\\ &= (u_1 + v_1 + w_1 - 4, u_2 + v_2 + w_2 - 6)\end{aligned}$$

The results are equal, so associativity holds.
4. Additive identity: We need to find a zero vector $(z_1, z_2) \in \mathbb{R}^2$ such that $(u_1, u_2) + (z_1, z_2) = (u_1, u_2)$ for all vectors (u_1, u_2).
$(u_1, u_2) + (z_1, z_2) = (u_1 + z_1 - 2, u_2 + z_2 - 3)$
For this to equal (u_1, u_2), we need:

$$u_1 + z_1 - 2 = u_1$$
$$u_2 + z_2 - 3 = u_2$$

Solving for z_1 and z_2:

$$\begin{aligned} z_1 &= 2 \\ z_2 &= 3 \end{aligned}$$

So the additive identity is $(2,3)$. Let's verify:

$$\begin{aligned} (u_1,u_2)+(2,3) &= (u_1+2-2, u_2+3-3) \\ &= (u_1,u_2) \end{aligned}$$

This checks out. The zero vector is $(2,3)$.
5. Additive inverse: For each $(u_1,u_2) \in \mathbb{R}^2$, we need to find $(-u_1,-u_2) \in \mathbb{R}^2$ such that $(u_1,u_2)+(-u_1,-u_2) = (2,3)$ (our zero vector).
$(u_1,u_2)+(-u_1,-u_2) = (u_1+(-u_1)-2, u_2+(-u_2)-3)$
For this to equal $(2,3)$, we need:

$$\begin{aligned} u_1+(-u_1)-2 &= 2 \\ u_2+(-u_2)-3 &= 3 \end{aligned}$$

Simplifying:

$$\begin{aligned} (-u_1) &= 4-u_1 \\ (-u_2) &= 6-u_2 \end{aligned}$$

So the additive inverse of (u_1,u_2) is $(4-u_1, 6-u_2)$. Let's verify:

$$\begin{aligned} (u_1,u_2)+(4-u_1,6-u_2) &= (u_1+(4-u_1)-2, u_2+(6-u_2)-3) \\ &= (4-2, 6-3) \\ &= (2,3) \end{aligned}$$

This checks out. For each vector, there exists an additive inverse.
6. Closure under scalar multiplication: For any $\alpha \in \mathbb{R}$ and $(u_1,u_2) \in \mathbb{R}^2$, the product $\alpha(u_1,u_2) = (\alpha u_1+2\alpha-2, \alpha u_2-3\alpha+3)$ is in $\mathbb{R}^2$.
This axiom is satisfied since any pair of real numbers is in $\mathbb{R}^2$.
7. Distributivity of scalar multiplication over vector addition: We need to check if $\alpha((u_1,u_2)+(v_1,v_2)) = \alpha(u_1,u_2)+\alpha(v_1,v_2)$.
Let's calculate:

$$\begin{aligned} \alpha((u_1,u_2)+(v_1,v_2)) &= \alpha(u_1+v_1-2, u_2+v_2-3) \\ &= (\alpha(u_1+v_1-2)+2\alpha-2, \alpha(u_2+v_2-3)-3\alpha+3) \\ &= (\alpha u_1+\alpha v_1-2\alpha+2\alpha-2, \alpha u_2+\alpha v_2-3\alpha-3\alpha+3) \\ &= (\alpha u_1+\alpha v_1-2, \alpha u_2+\alpha v_2-6\alpha+3) \end{aligned}$$

And:

$$\begin{aligned} \alpha(u_1,u_2)+\alpha(v_1,v_2) &= (\alpha u_1+2\alpha-2, \alpha u_2-3\alpha+3)+(\alpha v_1+2\alpha-2, \alpha v_2-3\alpha+3) \\ &= ((\alpha u_1+2\alpha-2)+(\alpha v_1+2\alpha-2)-2, (\alpha u_2-3\alpha+3)+(\alpha v_2-3\alpha+3)-3) \\ &= (\alpha u_1+2\alpha-2+\alpha v_1+2\alpha-2-2, \alpha u_2-3\alpha+3+\alpha v_2-3\alpha+3-3) \\ &= (\alpha u_1+\alpha v_1+4\alpha-6, \alpha u_2+\alpha v_2-6\alpha+3) \end{aligned}$$

Comparing the two results, we have: - First component: $\alpha u_1+\alpha v_1-2$ vs. $\alpha u_1+\alpha v_1+4\alpha-6$ - Second component: $\alpha u_2+\alpha v_2-6\alpha+3$ vs. $\alpha u_2+\alpha v_2-6\alpha+3$
The second components match, but the first components do not. They differ by the term $4\alpha-4$. For these to be equal, we would need $4\alpha-4=0$ for all α, which is only true when $\alpha=1$.
Therefore, this axiom is not satisfied for all scalars.
Conclusion: Since at least one of the vector space axioms (specifically, distributivity of scalar multiplication over vector addition) is not satisfied, $\mathbb{R}^2$ with the given operations is not a vector space.
Additional Verification: To be thorough, we could also check the remaining axioms, but since we've already found one failure, we can conclude that this is not a vector space. For completeness, let's briefly discuss what would happen with the scalar multiplication identity axiom:
10. Scalar identity: We would need to check if $1\cdot(u_1,u_2)=(u_1,u_2)$ for all vectors.
$1\cdot(u_1,u_2)=(1\cdot u_1+2\cdot 1-2, 1\cdot u_2-3\cdot 1+3)=(u_1+2-2, u_2-3+3)=(u_1,u_2)$
Interestingly, the scalar identity property does hold, but this doesn't change our overall conclusion since we've already found a violation of one of the required axioms.

Remark 7.11. *This example illustrates the importance of verifying all vector space axioms. Even though many of the properties held (including closure, commutativity, associativity, additive identity, additive inverse, and scalar identity), the failure of just one axiom (distributivity of scalar multiplication over vector addition) means that the structure is not a vector space.*

The violation of distributivity can be seen geometrically as introducing a "twist" in the way scaling interacts with addition. In a proper vector space, scaling a sum should be the same as adding the scaled vectors, but the given operations don't preserve this fundamental relationship.

Remark 7.12. *If such question is asked in exam, you may just mention the axiom which does not hold and conclude that it is not vector space.*

Example 4: 2×2 Matrices as a Vector Space

Problem: Let $V=M_{2\times 2}(\mathbb{R})$, the set of all 2×2 matrices with real entries. For any $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ in V, define $A+B=\begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix}$ and $kA=\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$. Show that V is a vector space.
Solution: To prove that $V=M_{2\times 2}(\mathbb{R})$ with the given operations forms a vector space, we need to verify all ten vector space axioms.
1. Closure under addition: For any $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ in V, their sum $A+B=\begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix}$ is also in V.
This is true because the sum of any two real numbers is a real number, so all entries in $A+B$ are real numbers, making $A+B$ a 2×2 matrix with real entries.
For example, if $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, then $A+B=\begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix}=\begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}\in V$.
2. Commutativity of addition: For any $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ in V, we need to show that $A+B=B+A$.
$A+B=\begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix}$ and $B+A=\begin{bmatrix} p+a & q+b \\ r+c & s+d \end{bmatrix}$
Since addition of real numbers is commutative ($a+p=p+a$, $b+q=q+b$, etc.), we have

$A+B=B+A$.

3. Associativity of addition: For any $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, $B=\begin{bmatrix}p&q\\r&s\end{bmatrix}$, and $C=\begin{bmatrix}u&v\\w&x\end{bmatrix}$ in V, we need to show that $(A+B)+C=A+(B+C)$.

$(A+B)+C=\begin{bmatrix}a+p&b+q\\c+r&d+s\end{bmatrix}+\begin{bmatrix}u&v\\w&x\end{bmatrix}=\begin{bmatrix}(a+p)+u&(b+q)+v\\(c+r)+w&(d+s)+x\end{bmatrix}$

$A+(B+C)=\begin{bmatrix}a&b\\c&d\end{bmatrix}+\begin{bmatrix}p+u&q+v\\r+w&s+x\end{bmatrix}=\begin{bmatrix}a+(p+u)&b+(q+v)\\c+(r+w)&d+(s+x)\end{bmatrix}$

Since addition of real numbers is associative ($(a+p)+u=a+(p+u)$, etc.), we have $(A+B)+C=A+(B+C)$.

4. Additive identity: We need to show there exists a zero matrix $O\in V$ such that $A+O=A$ for all $A\in V$.

Let $O=\begin{bmatrix}0&0\\0&0\end{bmatrix}$. Then for any $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, we have: $A+O=\begin{bmatrix}a&b\\c&d\end{bmatrix}+\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}a+0&b+0\\c+0&d+0\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}=A$

Therefore, $O=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is the additive identity in V.

5. Additive inverse: For each $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in V$, we need to show there exists a matrix $-A\in V$ such that $A+(-A)=O$.

Let $-A=\begin{bmatrix}-a&-b\\-c&-d\end{bmatrix}$. Then: $A+(-A)=\begin{bmatrix}a&b\\c&d\end{bmatrix}+\begin{bmatrix}-a&-b\\-c&-d\end{bmatrix}=\begin{bmatrix}a+(-a)&b+(-b)\\c+(-c)&d+(-d)\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}=O$

Therefore, for each $A\in V$, its additive inverse is $-A=\begin{bmatrix}-a&-b\\-c&-d\end{bmatrix}$.

6. Closure under scalar multiplication: For any scalar $k\in\mathbb{R}$ and any matrix $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in V$, the product $kA=\begin{bmatrix}ka&kb\\kc&kd\end{bmatrix}$ is in V.

This is true because the product of any real number with a real number is a real number, so all entries in kA are real numbers, making kA a 2×2 matrix with real entries.

For example, if $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ and $k=3$, then $kA=3\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}3&6\\9&12\end{bmatrix}\in V$.

7. Distributivity of scalar multiplication over vector addition: For any scalar $k\in\mathbb{R}$ and any matrices $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $B=\begin{bmatrix}p&q\\r&s\end{bmatrix}$ in V, we need to show that $k(A+B)=kA+kB$.

$k(A+B)=k\begin{bmatrix}a+p&b+q\\c+r&d+s\end{bmatrix}=\begin{bmatrix}k(a+p)&k(b+q)\\k(c+r)&k(d+s)\end{bmatrix}$

$kA+kB=\begin{bmatrix}ka&kb\\kc&kd\end{bmatrix}+\begin{bmatrix}kp&kq\\kr&ks\end{bmatrix}=\begin{bmatrix}ka+kp&kb+kq\\kc+kr&kd+ks\end{bmatrix}$

Since scalar multiplication distributes over addition of real numbers ($k(a+p)=ka+kp$, etc.), we have $k(A+B)=kA+kB$.

8. Distributivity of scalar multiplication over field addition: For any scalars $k,l\in\mathbb{R}$ and any matrix $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in V$, we need to show that $(k+l)A=kA+lA$.

$(k+l)A=(k+l)\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}(k+l)a&(k+l)b\\(k+l)c&(k+l)d\end{bmatrix}$

$kA + lA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} + \begin{bmatrix} la & lb \\ lc & ld \end{bmatrix} = \begin{bmatrix} ka+la & kb+lb \\ kc+lc & kd+ld \end{bmatrix}$

Since multiplication distributes over addition of real numbers ($(k+l)a = ka + la$, etc.), we have $(k+l)A = kA + lA$.

9. Compatibility of scalar multiplication with field multiplication: For any scalars $k, l \in \mathbb{R}$ and any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$, we need to show that $k(lA) = (kl)A$.

$k(lA) = k \begin{bmatrix} la & lb \\ lc & ld \end{bmatrix} = \begin{bmatrix} k(la) & k(lb) \\ k(lc) & k(ld) \end{bmatrix}$

$(kl)A = (kl) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (kl)a & (kl)b \\ (kl)c & (kl)d \end{bmatrix}$

Since multiplication of real numbers is associative ($k(la) = (kl)a$, etc.), we have $k(lA) = (kl)A$.

10. Scalar identity: For any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$, we need to show that $1A = A$.

$1A = 1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a & 1b \\ 1c & 1d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

Conclusion: Since all ten vector space axioms are satisfied, we have proven that $V = M_{2\times 2}(\mathbb{R})$ with the given operations of matrix addition and scalar multiplication forms a vector space over $\mathbb{R}$.

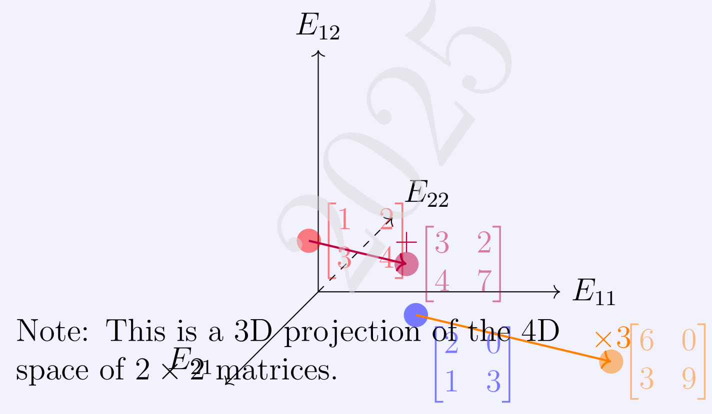
Additional observations: This vector space has dimension 4, as we can represent any 2×2 matrix as a linear combination of the following four basis matrices:

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Specifically, any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written as: $A = aE_{11} + bE_{12} + cE_{21} + dE_{22}$

Remark 7.13. *The standard operations on matrices demonstrate a particularly important example of a vector space. This space, $M_{2\times 2}(\mathbb{R})$, is not only fundamental in linear algebra but also serves as a bridge to more advanced topics such as linear transformations. Any linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ can be represented by a unique 2×2 matrix, establishing an isomorphism between the vector space of such transformations and $M_{2\times 2}(\mathbb{R})$.*

Geometric Visualization



Example 5: Positive Real Numbers with Modified Operations

Problem: Let V be the set of positive real numbers with addition and scalar multiplication defined as $x + y = xy$ and $c \cdot x = x^c$. Show that V is a vector space under these operations.

Solution: To prove that $V = \mathbb{R}^+$ (the set of positive real numbers) with the operations $x + y = xy$ and $c \cdot x = x^c$ forms a vector space, we need to verify all ten vector space axioms.

Note: It's important to recognize that we're redefining the operations here. The "addition" operation actually corresponds to multiplication of real numbers, and the "scalar multiplication" operation corresponds to exponentiation.

Let's verify each axiom systematically:

1. Closure under addition: For any $x, y \in V$, we need to show that $x + y \in V$.
Given that $x + y = xy$ in this context, and since the product of two positive real numbers is always a positive real number, we have $xy \in \mathbb{R}^+$. Therefore, $x + y \in V$.
For example, if $x = 2$ and $y = 3$, then $x + y = xy = 2 \cdot 3 = 6 \in \mathbb{R}^+$.

2. Commutativity of addition: For any $x, y \in V$, we need to show that $x + y = y + x$.
By definition, $x + y = xy$ and $y + x = yx$. Since multiplication of real numbers is commutative, $xy = yx$. Therefore, $x + y = y + x$.
For example, $2 + 3 = 2 \cdot 3 = 6$ and $3 + 2 = 3 \cdot 2 = 6$.

3. Associativity of addition: For any $x, y, z \in V$, we need to show that $(x + y) + z = x + (y + z)$.
By definition, $(x + y) + z = (xy) + z = (xy)z$ and $x + (y + z) = x + (yz) = x(yz)$. Since multiplication of real numbers is associative, $(xy)z = x(yz)$. Therefore, $(x + y) + z = x + (y + z)$.
For example, $(2 + 3) + 4 = (2 \cdot 3) + 4 = 6 + 4 = 6 \cdot 4 = 24$ and $2 + (3 + 4) = 2 + (3 \cdot 4) = 2 + 12 = 2 \cdot 12 = 24$.

4. Additive identity: We need to show there exists an element $e \in V$ such that $x+e = x$ for all $x \in V$.
By definition, $x + e = xe$. For this to equal x, we need $xe = x$, which means $e = 1$. Since $1 \in \mathbb{R}^+$, we have $e \in V$.
Therefore, 1 is the additive identity in V.
For example, $5 + 1 = 5 \cdot 1 = 5$.

5. Additive inverse: For each $x \in V$, we need to show there exists an element $-x \in V$ such that $x + (-x) = e$, where e is the additive identity (which we found to be 1).
By definition, $x + (-x) = x \cdot (-x)$. For this to equal 1, we need $x \cdot (-x) = 1$, which means $-x = \frac{1}{x}$.
Since $x > 0$, we know $\frac{1}{x} > 0$, so $\frac{1}{x} \in \mathbb{R}^+$, which means $-x \in V$.
Therefore, for any $x \in V$, its additive inverse is $-x = \frac{1}{x} \in V$.
For example, for $x = 5$, the additive inverse is $-x = \frac{1}{5} = 0.2$, and indeed $5+0.2 = 5\cdot0.2 = 1$.

6. Closure under scalar multiplication: For any scalar $c \in \mathbb{R}$ and any vector $x \in V$, we need to show that $c \cdot x \in V$.
By definition, $c\cdot x = x^c$. For $x > 0$ and any real number c, we know that $x^c > 0$. Therefore, $x^c \in \mathbb{R}^+$, which means $c \cdot x \in V$.
For example, if $c = 2$ and $x = 3$, then $c \cdot x = 3^2 = 9 \in \mathbb{R}^+$.

7. Distributivity of scalar multiplication over vector addition: For any scalar $c \in \mathbb{R}$ and any vectors $x, y \in V$, we need to show that $c \cdot (x + y) = c \cdot x + c \cdot y$.
By definition, $c \cdot (x + y) = c \cdot (xy) = (xy)^c$. Also, $c \cdot x + c \cdot y = x^c + y^c = x^c \cdot y^c = (xy)^c$, where the last equality follows from the laws of exponents: $(xy)^c = x^c \cdot y^c$ for positive real numbers.

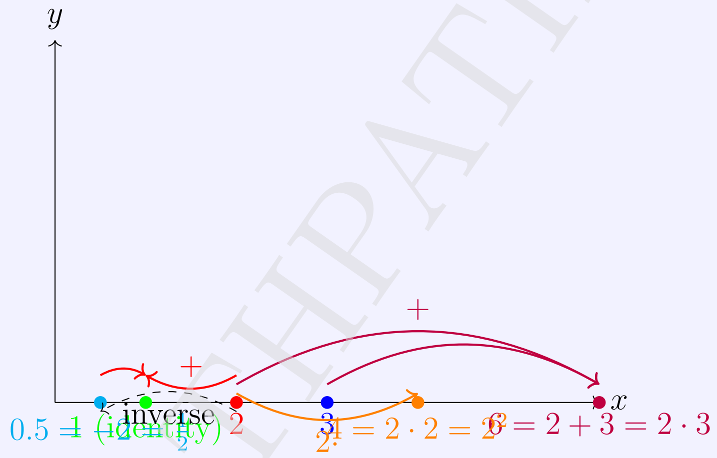
Therefore, $c \cdot (x + y) = c \cdot x + c \cdot y$.
For example, if $c = 2$, $x = 3$, and $y = 4$, then $c \cdot (x + y) = 2 \cdot (3 + 4) = 2 \cdot (3 \cdot 4) = 2 \cdot 12 = 12^2 = 144$ and $c \cdot x + c \cdot y = 2 \cdot 3 + 2 \cdot 4 = 3^2 + 4^2 = 9 + 16 = 9 \cdot 16 = 144$.
8. Distributivity of scalar multiplication over field addition: For any scalars $c, d \in \mathbb{R}$ and any vector $x \in V$, we need to show that $(c + d) \cdot x = c \cdot x + d \cdot x$.
By definition, $(c + d) \cdot x = x^{c+d}$. Also, $c \cdot x + d \cdot x = x^c + x^d = x^c \cdot x^d = x^{c+d}$, where the last equality follows from the laws of exponents: $x^c \cdot x^d = x^{c+d}$ for positive real numbers.
Therefore, $(c + d) \cdot x = c \cdot x + d \cdot x$.
For example, if $c = 2$, $d = 3$, and $x = 4$, then $(c + d) \cdot x = (2 + 3) \cdot 4 = 5 \cdot 4 = 4^5 = 1024$ and $c \cdot x + d \cdot x = 2 \cdot 4 + 3 \cdot 4 = 4^2 + 4^3 = 16 + 64 = 16 \cdot 64 = 1024$.
9. Compatibility of scalar multiplication with field multiplication: For any scalars $c, d \in \mathbb{R}$ and any vector $x \in V$, we need to show that $c \cdot (d \cdot x) = (cd) \cdot x$.
By definition, $c \cdot (d \cdot x) = c \cdot (x^d) = (x^d)^c$. Also, $(cd) \cdot x = x^{cd}$. By the laws of exponents, $(x^d)^c = x^{dc} = x^{cd}$.
Therefore, $c \cdot (d \cdot x) = (cd) \cdot x$.
For example, if $c = 2$, $d = 3$, and $x = 4$, then $c \cdot (d \cdot x) = 2 \cdot (3 \cdot 4) = 2 \cdot (4^3) = 2 \cdot 64 = 64^2 = 4096$ and $(cd) \cdot x = (2 \cdot 3) \cdot 4 = 6 \cdot 4 = 4^6 = 4096$.
10. Scalar identity: For any vector $x \in V$, we need to show that $1 \cdot x = x$.
By definition, $1 \cdot x = x^1 = x$.
Therefore, the scalar identity property holds.
For example, $1 \cdot 5 = 5^1 = 5$.
Conclusion: Since all ten vector space axioms are satisfied, we have proven that $V = \mathbb{R}^+$ with the operations $x + y = xy$ and $c \cdot x = x^c$ forms a vector space over $\mathbb{R}$.
Key insight: This example demonstrates that we can define vector space operations differently from the standard addition and multiplication, as long as these operations satisfy all the vector space axioms. Here, we've used multiplication of real numbers to represent vector addition and exponentiation to represent scalar multiplication. This is a one-dimensional vector space, with $\{e\}$ as a basis, where e is Euler's number (approximately 2.71828).

Remark 7.14. *This vector space is particularly interesting because it establishes an isomorphism between the additive structure of* $\mathbb{R}$ *and the multiplicative structure of* $\mathbb{R}^+$*. The natural logarithm function* $\ln : \mathbb{R}^+ \to \mathbb{R}$ *maps this vector space to the standard vector space* $\mathbb{R}$*, with the following properties:*

$$\ln(x + y) = \ln(xy) = \ln(x) + \ln(y)$$
$$\ln(c \cdot x) = \ln(x^c) = c \ln(x)$$

This connection illustrates why logarithms transform multiplication into addition, a property that has profound applications in various fields of mathematics and science.

Visualization of Vector Space Operations



Example 6: Determining Vector Spaces

Problem: Determine whether the following are vector spaces or not.
(i) $V = \mathbb{R}^2$ with operations

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$k(x_1, y_1) = (kx_1, ky_1)$$

(ii) $V = \mathbb{R}^2$ with operations

$$(x_1, y_1) + (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$$
$$k(x_1, y_1) = (kx_1, 0)$$

Solution:
Part (i): Let's examine whether $\mathbb{R}^2$ with the given operations forms a vector space. The operations are:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$k(x_1, y_1) = (kx_1, ky_1)$$

Let's verify all ten vector space axioms:
1. **Closure under addition:** For any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, their sum $(x_1 + x_2, y_1 + y_2)$ is in $\mathbb{R}^2$ since the sum of real numbers is a real number.
2. **Commutativity of addition:** For any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_2, y_2) + (x_1, y_1) = (x_2 + x_1, y_2 + y_1)$$

Since addition of real numbers is commutative, $x_1 + x_2 = x_2 + x_1$ and $y_1 + y_2 = y_2 + y_1$, so this axiom is satisfied.
3. **Associativity of addition:** For any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$:

$$\begin{aligned}((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (x_1 + x_2, y_1 + y_2) + (x_3, y_3) \\ &= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3)\end{aligned}$$

$$\begin{aligned}(x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) \\ &= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3))\end{aligned}$$

Since addition of real numbers is associative, $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$ and $(y_1 + y_2) + y_3 = y_1 + (y_2 + y_3)$, so this axiom is satisfied.
4. **Additive identity:** We need to find a vector (e_1, e_2) such that $(x, y) + (e_1, e_2) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$.

$$(x, y) + (e_1, e_2) = (x + e_1, y + e_2)$$

For this to equal (x, y), we need $x + e_1 = x$ and $y + e_2 = y$, which gives us $e_1 = 0$ and $e_2 = 0$.
So the additive identity is $(0, 0)$, and we can verify: $(x, y) + (0, 0) = (x + 0, y + 0) = (x, y)$.
5. **Additive inverse:** For each $(x, y) \in \mathbb{R}^2$, we need to find a vector (x', y') such that $(x, y) + (x', y') = (0, 0)$ (the additive identity).

$$(x, y) + (x', y') = (x + x', y + y') = (0, 0)$$

This gives us $x + x' = 0$ and $y + y' = 0$, so $x' = -x$ and $y' = -y$.
So the additive inverse of (x, y) is $(-x, -y)$, and we can verify: $(x, y) + (-x, -y) = (x + (-x), y + (-y)) = (0, 0)$.
6. **Closure under scalar multiplication:** For any scalar $k \in \mathbb{R}$ and any vector $(x, y) \in \mathbb{R}^2$, the product $k(x, y) = (kx, ky)$ is in $\mathbb{R}^2$ since the product of real numbers is a real number.
7. **Distributivity of scalar multiplication over vector addition:** For any scalar $k \in \mathbb{R}$ and vectors $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$:

$$\begin{aligned} k((x_1, y_1) + (x_2, y_2)) &= k(x_1 + x_2, y_1 + y_2) \\ &= (k(x_1 + x_2), k(y_1 + y_2)) \\ &= (kx_1 + kx_2, ky_1 + ky_2) \end{aligned}$$

$$\begin{aligned} k(x_1, y_1) + k(x_2, y_2) &= (kx_1, ky_1) + (kx_2, ky_2) \\ &= (kx_1 + kx_2, ky_1 + ky_2) \end{aligned}$$

Since these are equal, this axiom is satisfied.
8. **Distributivity of scalar multiplication over field addition:** For any scalars $k, l \in \mathbb{R}$ and vector $(x, y) \in \mathbb{R}^2$:

$$\begin{aligned} (k + l)(x, y) &= ((k + l)x, (k + l)y) \\ &= (kx + lx, ky + ly) \end{aligned}$$

$$\begin{aligned} k(x, y) + l(x, y) &= (kx, ky) + (lx, ly) \\ &= (kx + lx, ky + ly) \end{aligned}$$

Since these are equal, this axiom is satisfied.
9. **Compatibility of scalar multiplication with field multiplication:** For any scalars $k, l \in \mathbb{R}$ and vector $(x, y) \in \mathbb{R}^2$:

$$\begin{aligned} k(l(x, y)) &= k(lx, ly) \\ &= (k(lx), k(ly)) \\ &= ((kl)x, (kl)y) \end{aligned}$$

$$(kl)(x,y) = ((kl)x,(kl)y)$$

Since these are equal, this axiom is satisfied.
10. **Scalar identity:** For any vector $(x,y) \in \mathbb{R}^2$:

$$1(x,y) = (1x,1y) = (x,y)$$

So the scalar identity property holds.
Conclusion for Part (i): Since all ten vector space axioms are satisfied, $V = \mathbb{R}^2$ with the given operations is a vector space.
Part (ii): Let's examine whether $\mathbb{R}^2$ with the given operations forms a vector space. The operations are:

$$\begin{aligned} (x_1,y_1) + (x_2,y_2) &= (x_1 - x_2, y_1 - y_2) \\ k(x_1,y_1) &= (kx_1, 0) \end{aligned}$$

Let's check only the axioms that fail:
1. **Commutativity of addition:** For any $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$:

$$\begin{aligned} (x_1,y_1) + (x_2,y_2) &= (x_1 - x_2, y_1 - y_2) \\ (x_2,y_2) + (x_1,y_1) &= (x_2 - x_1, y_2 - y_1) \end{aligned}$$

These are not equal because $x_1 - x_2 \neq x_2 - x_1$ and $y_1 - y_2 \neq y_2 - y_1$ in general.
For example, if $(x_1,y_1) = (3,4)$ and $(x_2,y_2) = (1,2)$:

$$\begin{aligned} (3,4) + (1,2) &= (3-1, 4-2) = (2,2) \\ (1,2) + (3,4) &= (1-3, 2-4) = (-2,-2) \end{aligned}$$

Since $(2,2) \neq (-2,-2)$, commutativity fails.
2. **Associativity of addition:** For any $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in \mathbb{R}^2$:

$$\begin{aligned} ((x_1,y_1) + (x_2,y_2)) + (x_3,y_3) &= (x_1 - x_2, y_1 - y_2) + (x_3,y_3) \\ &= ((x_1 - x_2) - x_3, (y_1 - y_2) - y_3) \\ &= (x_1 - x_2 - x_3, y_1 - y_2 - y_3) \end{aligned}$$

$$\begin{aligned} (x_1,y_1) + ((x_2,y_2) + (x_3,y_3)) &= (x_1,y_1) + (x_2 - x_3, y_2 - y_3) \\ &= (x_1 - (x_2 - x_3), y_1 - (y_2 - y_3)) \\ &= (x_1 - x_2 + x_3, y_1 - y_2 + y_3) \end{aligned}$$

Since $x_1 - x_2 - x_3 \neq x_1 - x_2 + x_3$ and $y_1 - y_2 - y_3 \neq y_1 - y_2 + y_3$ in general, associativity fails.
3. **Distributivity of scalar multiplication over field addition:** For any scalars $k,l \in \mathbb{R}$ and vector $(x,y) \in \mathbb{R}^2$:

$$(k+l)(x,y) = ((k+l)x, 0)$$

$$\begin{aligned} k(x,y) + l(x,y) &= (kx,0) + (lx,0) \\ &= (kx - lx, 0 - 0) \\ &= (kx - lx, 0) \end{aligned}$$

Since $(k+l)x \neq kx - lx$ in general, this axiom fails.
4. **Scalar identity:** For any vector $(x, y) \in \mathbb{R}^2$, we need $1 \cdot (x, y) = (x, y)$.
By definition, $1 \cdot (x, y) = (1 \cdot x, 0) = (x, 0)$.
This does not equal (x, y) unless $y = 0$. For any vector with $y \neq 0$, the scalar identity property does not hold.
For example, $1 \cdot (3, 4) = (3, 0) \neq (3, 4)$.
Conclusion for Part (ii): $V = \mathbb{R}^2$ with the given operations is not a vector space because it fails several axioms, including commutativity of addition, associativity of addition, and distributivity of scalar multiplication over field addition.

Example 7: Identifying Non-Vector Spaces

Problem: Justify why the following sets are not vector spaces under the given operations.
(i) The set of all pairs of real numbers (x, y) with operations $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k(x, y) = (2kx, 2ky)$.
(ii) The set of all rational numbers w.r.t. usual addition and scalar multiplication.
(iii) The set of all 2×2 matrices with rational entries w.r.t. usual matrix addition and scalar multiplication.
Solution:
(i) Scalar identity axiom fails: For any vector v in a vector space, we must have $1 \cdot v = v$. Let's check:
For any vector (x, y):

$$\begin{aligned} 1 \cdot (x, y) &= (2 \cdot 1 \cdot x, 2 \cdot 1 \cdot y) \\ &= (2x, 2y) \end{aligned}$$

Since $(2x, 2y) \neq (x, y)$ for non-zero vectors, the scalar identity axiom fails.
For example, $1 \cdot (3, 4) = (2 \cdot 3, 2 \cdot 4) = (6, 8) \neq (3, 4)$.
Distributivity of scalar multiplication over field addition also fails: For any scalars a, b and vector v, we must have $(a + b) \cdot v = a \cdot v + b \cdot v$. Let's check:
For any scalars a, b and vector (x, y):

$$\begin{aligned} (a + b) \cdot (x, y) &= (2(a + b)x, 2(a + b)y) \\ &= (2ax + 2bx, 2ay + 2by) \end{aligned}$$

But:

$$\begin{aligned} a \cdot (x, y) + b \cdot (x, y) &= (2ax, 2ay) + (2bx, 2by) \\ &= (2ax + 2bx, 2ay + 2by) \end{aligned}$$

In this case, the results happen to be equal, so this axiom is actually satisfied.
Therefore, the set fails to be a vector space because the scalar identity axiom is not satisfied.
(ii) The set of all rational numbers w.r.t. usual addition and scalar multiplication.
For this to be a vector space, we need to specify what our scalar field is. If the scalar field is $\mathbb{Q}$ (rational numbers), then $\mathbb{Q}$ would indeed be a vector space over itself.
However, if the scalar field is $\mathbb{R}$ (real numbers), then **the closure under scalar multiplication axiom fails.** For any vector v in a vector space and any scalar k, $k \cdot v$ must be in the vector space.

For example, if we take the rational number $q = 1$ and multiply it by the irrational scalar $k = \sqrt{2}$, we get $k \cdot q = \sqrt{2} \cdot 1 = \sqrt{2}$, which is not a rational number.
Therefore, the set of all rational numbers is not a vector space over $\mathbb{R}$ because the closure under scalar multiplication axiom is not satisfied.
(iii) The set of all 2×2 matrices with rational entries w.r.t. usual matrix addition and scalar multiplication.
Similar to part (iii), we need to specify the scalar field. If the scalar field is $\mathbb{Q}$ (rational numbers), then the set of 2×2 matrices with rational entries would be a vector space over $\mathbb{Q}$.
However, if the scalar field is $\mathbb{R}$ (real numbers), then **the closure under scalar multiplication axiom fails.** For any matrix A with rational entries and any irrational scalar k, kA would have irrational entries, which would not be in our set.
For example, if we take a matrix with rational entries such as $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and multiply it by the irrational scalar $k = \pi$, we get:

$$kA = \pi \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix}$$

This matrix has irrational entries, so it's not in our set of matrices with rational entries.
Therefore, the set of all 2×2 matrices with rational entries is not a vector space over $\mathbb{R}$ because the closure under scalar multiplication axiom is not satisfied.

Example 8: Matrix Sets as Vector Spaces

Problem: Determine whether the following are vector spaces or not with usual matrix addition and scalar multiplication:
(i) $V = \left\{ \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} | a, b \in \mathbb{R} \right\}$
(ii) $V = \left\{ \begin{bmatrix} a+b & a \\ b & a+b \end{bmatrix} | a, b \in \mathbb{R} \right\}$
(iii) $V = \left\{ \begin{bmatrix} 0 & a \\ b & c \end{bmatrix} | a, b, c \in \mathbb{R} \right\}$
Solution:
(i) $V = \left\{ \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} | a, b \in \mathbb{R} \right\}$
Let's check if this set forms a vector space under the usual matrix operations.
Closure under addition: Let's take two matrices from the set and add them:

$$\begin{bmatrix} 1 & a_1 \\ b_1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & a_2 \\ b_2 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & a_1 + a_2 \\ b_1 + b_2 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & a_1 + a_2 \\ b_1 + b_2 & 2 \end{bmatrix}$$

The result does not have 1's on the diagonal, so it's not in our set V. Therefore, the set is not closed under addition, and the closure under addition axiom fails.
Conclusion for (i): The set is not a vector space because it fails the closure under addition axiom.
(ii) $V = \left\{ \begin{bmatrix} a+b & a \\ b & a+b \end{bmatrix} | a, b \in \mathbb{R} \right\}$
Let's verify if this set forms a vector space.
Closure under addition: Take two matrices from the set and add them:

$$\begin{bmatrix} a_1 + b_1 & a_1 \\ b_1 & a_1 + b_1 \end{bmatrix} + \begin{bmatrix} a_2 + b_2 & a_2 \\ b_2 & a_2 + b_2 \end{bmatrix} = \begin{bmatrix} (a_1 + b_1) + (a_2 + b_2) & a_1 + a_2 \\ b_1 + b_2 & (a_1 + b_1) + (a_2 + b_2) \end{bmatrix}$$

Let $a_3 = a_1 + a_2$ and $b_3 = b_1 + b_2$. Then:

$$\begin{bmatrix} (a_1+b_1)+(a_2+b_2) & a_1+a_2 \\ b_1+b_2 & (a_1+b_1)+(a_2+b_2) \end{bmatrix} = \begin{bmatrix} a_3+b_3 & a_3 \\ b_3 & a_3+b_3 \end{bmatrix}$$

This is in our set V, so closure under addition is satisfied.
Closure under scalar multiplication: For any scalar k and any matrix in V:

$$k\begin{bmatrix} a+b & a \\ b & a+b \end{bmatrix} = \begin{bmatrix} k(a+b) & ka \\ kb & k(a+b) \end{bmatrix}$$

Let $a' = ka$ and $b' = kb$. Then:

$$\begin{bmatrix} k(a+b) & ka \\ kb & k(a+b) \end{bmatrix} = \begin{bmatrix} a'+b' & a' \\ b' & a'+b' \end{bmatrix}$$

This is in our set V, so closure under scalar multiplication is satisfied.
Additive identity: We need to check if the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in our set V.
The zero matrix corresponds to $a = 0$ and $b = 0$, giving $\begin{bmatrix} 0+0 & 0 \\ 0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is in V.
Additive inverse: For any matrix $\begin{bmatrix} a+b & a \\ b & a+b \end{bmatrix}$ in V, its additive inverse is $\begin{bmatrix} -(a+b) & -a \\ -b & -(a+b) \end{bmatrix}$.
Let $a' = -a$ and $b' = -b$. Then:

$$\begin{bmatrix} -(a+b) & -a \\ -b & -(a+b) \end{bmatrix} = \begin{bmatrix} a'+b' & a' \\ b' & a'+b' \end{bmatrix}$$

This is in our set V, so the additive inverse axiom is satisfied.
Since V satisfies closure under addition, closure under scalar multiplication, contains the additive identity, and contains additive inverses, and the standard matrix operations satisfy all the other vector space axioms, we conclude that V is a vector space.
Conclusion for (ii): The set is a vector space.
(iii) $V = \left\{ \begin{bmatrix} 0 & a \\ b & c \end{bmatrix} | a, b, c \in \mathbb{R} \right\}$
Let's check if this set forms a vector space.
Closure under addition: Take two matrices from the set and add them:

$$\begin{bmatrix} 0 & a_1 \\ b_1 & c_1 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ b_2 & c_2 \end{bmatrix} = \begin{bmatrix} 0+0 & a_1+a_2 \\ b_1+b_2 & c_1+c_2 \end{bmatrix} = \begin{bmatrix} 0 & a_1+a_2 \\ b_1+b_2 & c_1+c_2 \end{bmatrix}$$

This is in our set V, so closure under addition is satisfied.
Closure under scalar multiplication: For any scalar k and any matrix in V:

$$k\begin{bmatrix} 0 & a \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & ka \\ kb & kc \end{bmatrix}$$

This is in our set V, so closure under scalar multiplication is satisfied.
Additive identity: The zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ corresponds to $a = 0$, $b = 0$, and $c = 0$, which is in V.

Additive inverse: For any matrix $\begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$ in V, its additive inverse is $\begin{bmatrix} 0 & -a \\ -b & -c \end{bmatrix}$, which is also in V.
Since V satisfies closure under addition, closure under scalar multiplication, contains the additive identity, and contains additive inverses, and the standard matrix operations satisfy all the other vector space axioms, we conclude that V is a vector space.
Conclusion for (iii): The set is a vector space.
Summary: (i) Not a vector space (fails closure under addition) (ii) Is a vector space (iii) Is a vector space

Remark 7.15. *For case (ii), we can interpret this set geometrically as the set of all matrices of the form* $\begin{bmatrix} a+b & a \\ b & a+b \end{bmatrix}$*, which can be written as* $a\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$*. Therefore, this set is a 2-dimensional subspace of* $M_{2\times 2}(\mathbb{R})$ *with basis* $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$.
For case (iii), the set is the subspace of $M_{2\times 2}(\mathbb{R})$ *consisting of matrices with a zero in the top-left corner. This is a 3-dimensional subspace with basis* $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

7.2 Subspaces

In our exploration of vector spaces, we now turn to one of the most fundamental concepts: subspaces. Just as in set theory where we study subsets, in linear algebra we are interested in identifying and analyzing certain subsets of a vector space that inherit the vector space structure. These special subsets, called subspaces, provide powerful tools for understanding the internal structure of vector spaces and their transformations.

7.2.1 Definition and Properties

Definition 7.16. *Let V be a vector space over a field $\mathbb{F}$. A subset $W \subseteq V$ is called a* ***subspace*** *of V if W is itself a vector space over $\mathbb{F}$ with the same vector addition and scalar multiplication operations inherited from V.*

This definition requires that W satisfy all the vector space axioms. However, we can simplify the verification process with the following theorem:

Theorem 7.17 (Subspace Test)**.** *Let V be a vector space over a field $\mathbb{F}$ and W be a subset of V. Then W is a subspace of V if and only if the following three conditions hold:*

1. *W is non-empty, i.e., $W \neq \emptyset$.*
2. *W is closed under vector addition: For all $\vec{u}, \vec{v} \in W$, we have $\vec{u} + \vec{v} \in W$.*
3. *W is closed under scalar multiplication: For all $\vec{v} \in W$ and $c \in \mathbb{F}$, we have $c\vec{v} \in W$.*

Proof. ($\Rightarrow$) If W is a subspace of V, then W is a vector space in its own right. So, by the vector space axioms, W is non-empty (as it contains at least the zero vector) and is closed under vector addition and scalar multiplication.

($\Leftarrow$) Now, assume that W is non-empty and closed under vector addition and scalar multiplication. We need to verify all the vector space axioms:

1. Since W is non-empty, let $\vec{v} \in W$. Then, by closure under scalar multiplication, $0\vec{v} = \vec{0} \in W$.

2. The operations of vector addition and scalar multiplication are the same as in V, so all the properties such as commutativity, associativity, distributivity, etc., are inherited from V.

3. The zero vector $\vec{0}$ is in W as shown above.
4. For any $\vec{v} \in W$, by closure under scalar multiplication, $(-1)\vec{v} = -\vec{v} \in W$, so additive inverses exist in W.

Therefore, W is a vector space over $\mathbb{F}$, and hence a subspace of V. □

An immediate consequence of this theorem is the following useful property:

Property 7.18. *Every subspace W of a vector space V contains the zero vector $\vec{0}$ of V.*

Note

Since every subspace must contain the zero vector, we can simplify the subspace test to check only two conditions:
1. Closure under vector addition: $\vec{u}, \vec{v} \in W \implies \vec{u} + \vec{v} \in W$.
2. Closure under scalar multiplication: $\vec{v} \in W, c \in \mathbb{F} \implies c\vec{v} \in W$.

The non-emptiness condition becomes redundant as long as we verify that $\vec{0} \in W$.

Another important property of subspaces follows directly from the closure properties:

Property 7.19 (Closure under Linear Combinations)**.** *Let W be a subspace of a vector space V. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in W$ and $c_1, c_2, \ldots, c_m \in \mathbb{F}$, then the linear combination $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m \in W$.*

7.2.2 Testing for Subspaces

Let's develop a systematic approach to determine whether a given subset of a vector space is a subspace.

Testing for Subspaces

Determine whether the following subsets are subspaces of the given vector spaces:
1. $W = \{\vec{v} \in \mathbb{R}^3 : \vec{v} \cdot \vec{a} = 0\}$ for some fixed non-zero vector $\vec{a} \in \mathbb{R}^3$.
2. $U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$.
3. $P_n^{\text{even}} = \{p \in P_n : p(-x) = p(x) \text{ for all } x \in \mathbb{R}\}$, the set of all even polynomials of degree at most n.

Solution

1. Let's test $W = \{\vec{v} \in \mathbb{R}^3 : \vec{v} \cdot \vec{a} = 0\}$:
(a) $\vec{0} \cdot \vec{a} = 0$, so $\vec{0} \in W$. Thus, W is non-empty.
(b) Let $\vec{u}, \vec{v} \in W$. Then $\vec{u} \cdot \vec{a} = 0$ and $\vec{v} \cdot \vec{a} = 0$.
For vector addition: $(\vec{u} + \vec{v}) \cdot \vec{a} = \vec{u} \cdot \vec{a} + \vec{v} \cdot \vec{a} = 0 + 0 = 0$.
So $\vec{u} + \vec{v} \in W$.
(c) Let $\vec{v} \in W$ and $c \in \mathbb{R}$. Then $\vec{v} \cdot \vec{a} = 0$.
For scalar multiplication: $(c\vec{v}) \cdot \vec{a} = c(\vec{v} \cdot \vec{a}) = c \cdot 0 = 0$.
So $c\vec{v} \in W$.
Since all three conditions are satisfied, W is a subspace of $\mathbb{R}^3$.
2. Let's test $U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$:
First, let's check if $\vec{0} = (0, 0, 0) \in U$:
$0 + 0 + 0 = 0 \neq 1$. Thus, $\vec{0} \notin U$.
Since a subspace must contain the zero vector, U is not a subspace of $\mathbb{R}^3$.
3. Let's test $P_n^{\text{even}} = \{p \in P_n : p(-x) = p(x) \text{ for all } x \in \mathbb{R}\}$:
(a) The zero polynomial $p(x) = 0$ satisfies $p(-x) = 0 = p(x)$, so P_n^{even} is non-empty.
(b) Let $p, q \in P_n^{\text{even}}$. So $p(-x) = p(x)$ and $q(-x) = q(x)$ for all $x \in \mathbb{R}$.
For vector addition: $(p + q)(-x) = p(-x) + q(-x) = p(x) + q(x) = (p + q)(x)$.

So $p + q \in P_n^{\text{even}}$.
(c) Let $p \in P_n^{\text{even}}$ and $c \in \mathbb{R}$. So $p(-x) = p(x)$ for all $x \in \mathbb{R}$.
For scalar multiplication: $(cp)(-x) = c \cdot p(-x) = c \cdot p(x) = (cp)(x)$.
So $cp \in P_n^{\text{even}}$.
Since all three conditions are satisfied, P_n^{even} is a subspace of P_n.

These examples illustrate a common pattern: sets defined by homogeneous linear equations (like $\vec{v} \cdot \vec{a} = 0$) typically form subspaces, while those defined by non-homogeneous equations (like $x + y + z = 1$) typically do not.

Practice Problem

Determine whether the following sets are subspaces of the indicated vector spaces:
1. $W = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 0\}$ in $\mathbb{R}^3$.
2. $U = \{A \in M_{n\times n} : A^T = -A\}$, the set of all $n \times n$ skew-symmetric matrices.
3. $V = \{f \in C[0, 1] : f(0) = f(1)\}$, the set of continuous functions on $[0, 1]$ that have the same value at 0 and 1.

7.2.3 Common Subspaces

Linear algebra frequently deals with several important subspaces associated with matrices. Let's explore these fundamental subspaces.

Null Space

Definition 7.20. *Let A be an $m \times n$ matrix. The **null space** (or **kernel**) of A, denoted by* null(A) *or* ker(A)*, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{0}$. That is:*

$$\text{null}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$$

Theorem 7.21. *The null space of any matrix A is a subspace of $\mathbb{R}^n$, where n is the number of columns in A.*

Proof. Let's verify the three conditions for null(A):
1. $A\vec{0} = \vec{0}$, so $\vec{0} \in \text{null}(A)$. Thus, null$(A)$ is non-empty.
2. Let $\vec{u}, \vec{v} \in \text{null}(A)$. Then $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$.
For vector addition: $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$.
So $\vec{u} + \vec{v} \in \text{null}(A)$.
3. Let $\vec{v} \in \text{null}(A)$ and $c \in \mathbb{R}$. Then $A\vec{v} = \vec{0}$.
For scalar multiplication: $A(c\vec{v}) = cA\vec{v} = c\vec{0} = \vec{0}$.
So $c\vec{v} \in \text{null}(A)$.
Therefore, null(A) is a subspace of $\mathbb{R}^n$. □

The null space represents the solution set of the homogeneous system $A\vec{x} = \vec{0}$. Its dimension, called the **nullity** of A, provides important information about the matrix:

$$\text{nullity}(A) = \dim(\text{null}(A)) = n - \text{rank}(A)$$

This relation is part of the Rank-Nullity Theorem, which we will explore in detail later.

Column Space

Definition 7.22. *Let A be an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$. The* ***column space*** *of A, denoted by $\text{col}(A)$, is the span of the columns of A. That is:*

$$\text{col}(A) = \text{span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} = \{\vec{v} \in \mathbb{R}^m : \vec{v} = A\vec{x} \textit{ for some } \vec{x} \in \mathbb{R}^n\}$$

Theorem 7.23. *The column space of any matrix A is a subspace of $\mathbb{R}^m$, where m is the number of rows in A.*

Proof. Since $\text{col}(A)$ is defined as the span of a set of vectors, and the span of any set of vectors forms a subspace (as we'll prove in a separate theorem), $\text{col}(A)$ is a subspace of $\mathbb{R}^m$. □

The column space represents the set of all possible outputs of the matrix transformation $T(\vec{x}) = A\vec{x}$. Its dimension, called the **rank** of A, equals the number of linearly independent columns in A.

Row Space

Definition 7.24. *Let A be an $m \times n$ matrix. The* ***row space*** *of A, denoted by $\text{row}(A)$, is the span of the rows of A. Equivalently, $\text{row}(A) = \text{col}(A^T)$, where A^T is the transpose of A.*

Theorem 7.25. *The row space of any matrix A is a subspace of $\mathbb{R}^n$, where n is the number of columns in A.*

Proof. Since $\text{row}(A) = \text{col}(A^T)$, and we've already established that the column space of any matrix is a subspace, $\text{row}(A)$ is a subspace of $\mathbb{R}^n$. □

An important theorem in linear algebra establishes that the row space and column space of a matrix have the same dimension:

Theorem 7.26. *For any matrix A, $\dim(\text{row}(A)) = \dim(\text{col}(A)) = \text{rank}(A)$.*

Computing Common Subspaces Using Row Echelon Form

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$. Find bases for $\text{null}(A)$, $\text{col}(A)$, and $\text{row}(A)$, and verify the Rank-Nullity theorem.

Solution

We'll use the row echelon form (REF) of A to analyze all three subspaces systematically:
Step 1: Find the row echelon form of A using Gaussian elimination:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 2: Determine the rank of A.
From the REF, we can see that there is only one non-zero row, so $\text{rank}(A) = 1$.
Step 3: Find a basis for the row space of A.
The non-zero rows of the REF form a basis for the row space. Thus:

$$\text{row}(A) = \text{span}\{(1, 2, 3)\}$$

Step 4: Find a basis for the null space of A.
To find $\text{null}(A)$, we solve the homogeneous system $A\vec{x} = \vec{0}$ using the REF:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives us the equation $x_1 + 2x_2 + 3x_3 = 0$.
Solving for the basic variable x_1 in terms of the free variables x_2 and x_3:

$$x_1 = -2x_2 - 3x_3$$

Now we find the basis vectors by setting each free variable to 1 (one at a time) and the others to 0:
When $x_2 = 1, x_3 = 0$: $\vec{v}_1 = (-2, 1, 0)^T$ When $x_2 = 0, x_3 = 1$: $\vec{v}_2 = (-3, 0, 1)^T$
Thus, a basis for null(A) is $\{\vec{v}_1, \vec{v}_2\} = \{(-2, 1, 0)^T, (-3, 0, 1)^T\}$.
The nullity of A is the dimension of the null space:

$$\text{nullity}(A) = \dim(\text{null}(A)) = 2$$

Step 5: Find a basis for the column space of A.
The pivot columns of the original matrix A form a basis for the column space. From the REF, we can see that only the first column is a pivot column. Therefore:

$$\text{col}(A) = \text{span}\{(1, 2)^T\}$$

This confirms that $\text{rank}(A) = \dim(\text{col}(A)) = 1$.
Step 6: Verify the Rank-Nullity Theorem.
The Rank-Nullity Theorem states that:

$$\text{rank}(A) + \text{nullity}(A) = n$$

where n is the number of columns in A.
In our case:

$$1 + 2 = 3$$

Since A is a 2×3 matrix, $n = 3$, and the theorem is verified.
We can also observe the fundamental relationships between these subspaces:
- $\dim(\text{row}(A)) = \text{rank}(A) = 1$
- $\dim(\text{col}(A)) = \text{rank}(A) = 1$
- $\dim(\text{null}(A)) = \text{nullity}(A) = n - \text{rank}(A) = 3 - 1 = 2$

7.2.4 Sum and Intersection of Subspaces

When working with multiple subspaces, two natural operations arise: taking their sum and their intersection.

Definition 7.27. *Let U and W be subspaces of a vector space V. The* ***sum*** *of U and W, denoted by $U + W$, is defined as:*

$$U + W = \{\vec{u} + \vec{w} : \vec{u} \in U, \vec{w} \in W\}$$

Theorem 7.28. *If U and W are subspaces of a vector space V, then $U + W$ is also a subspace of V.*

Proof. Let's verify the three conditions for $U + W$:
1. Since U and W are subspaces, both contain $\vec{0}$. Thus, $\vec{0} = \vec{0} + \vec{0} \in U + W$, so $U + W$ is non-empty.
2. Let $\vec{u}_1 + \vec{w}_1, \vec{u}_2 + \vec{w}_2 \in U + W$, where $\vec{u}_1, \vec{u}_2 \in U$ and $\vec{w}_1, \vec{w}_2 \in W$.
For vector addition: $(\vec{u}_1 + \vec{w}_1) + (\vec{u}_2 + \vec{w}_2) = (\vec{u}_1 + \vec{u}_2) + (\vec{w}_1 + \vec{w}_2)$.

Since U and W are subspaces, $\vec{u}_1 + \vec{u}_2 \in U$ and $\vec{w}_1 + \vec{w}_2 \in W$.
So $(\vec{u}_1 + \vec{w}_1) + (\vec{u}_2 + \vec{w}_2) \in U + W$.
3. Let $\vec{u} + \vec{w} \in U + W$, where $\vec{u} \in U$ and $\vec{w} \in W$, and let $c \in \mathbb{F}$.
For scalar multiplication: $c(\vec{u} + \vec{w}) = c\vec{u} + c\vec{w}$.
Since U and W are subspaces, $c\vec{u} \in U$ and $c\vec{w} \in W$.
So $c(\vec{u} + \vec{w}) \in U + W$.
Therefore, $U + W$ is a subspace of V. □

Definition 7.29. *Let U and W be subspaces of a vector space V. The **intersection** of U and W, denoted by $U \cap W$, is the set of all vectors that belong to both U and W.*

Theorem 7.30. *If U and W are subspaces of a vector space V, then $U \cap W$ is also a subspace of V.*

Proof. Let's verify the three conditions for $U \cap W$:
1. Since U and W are subspaces, both contain $\vec{0}$. Thus, $\vec{0} \in U \cap W$, so $U \cap W$ is non-empty.
2. Let $\vec{v}_1, \vec{v}_2 \in U \cap W$. Then $\vec{v}_1, \vec{v}_2 \in U$ and $\vec{v}_1, \vec{v}_2 \in W$.
Since U and W are subspaces, $\vec{v}_1 + \vec{v}_2 \in U$ and $\vec{v}_1 + \vec{v}_2 \in W$.
So $\vec{v}_1 + \vec{v}_2 \in U \cap W$.
3. Let $\vec{v} \in U \cap W$ and $c \in \mathbb{F}$. Then $\vec{v} \in U$ and $\vec{v} \in W$.
Since U and W are subspaces, $c\vec{v} \in U$ and $c\vec{v} \in W$.
So $c\vec{v} \in U \cap W$.
Therefore, $U \cap W$ is a subspace of V. □

An important relationship between the dimensions of the sum and intersection of subspaces is given by the following theorem:

Theorem 7.31 (Dimension Formula)**.** *If U and W are finite-dimensional subspaces of a vector space V, then:*

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Sum and Intersection of Subspaces

Let $U = \text{span}\{(1, 1, 0), (0, 1, 1)\}$ and $W = \text{span}\{(1, 0, 1), (0, 1, 0)\}$ be subspaces of $\mathbb{R}^3$. Find bases for $U + W$ and $U \cap W$, and verify the dimension formula.

Solution

First, let's find a basis for $U + W$:
The vectors that span U and W are $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, and $(0, 1, 0)$. To find a basis for $U + W$, we need to identify a linearly independent subset that spans the same space. Using Gaussian elimination on the matrix formed by these vectors:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

After row reduction, we find that the rank is 3, meaning that a basis for $U + W$ consists of three linearly independent vectors, such as $\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$.
So $\dim(U + W) = 3$.
Now, let's find $U \cap W$:
A vector $\vec{v} = (v_1, v_2, v_3)$ is in $U \cap W$ if it can be written as:

$$\vec{v} = c_1(1, 1, 0) + c_2(0, 1, 1) = d_1(1, 0, 1) + d_2(0, 1, 0)$$

This gives us the system of equations:

$$\begin{aligned} c_1 &= d_1 \\ c_1 + c_2 &= d_2 \\ c_2 &= d_1 \end{aligned}$$

Solving this system, we get $c_1 = c_2 = d_1 = d_2$, which means $\vec{v} = c_1((1,1,0)+(0,1,1)) = c_1(1,2,1)$.
So $U \cap W = \text{span}\{(1,2,1)\}$ and $\dim(U \cap W) = 1$.
Verifying the dimension formula:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

$$3 = 2 + 2 - 1$$

Indeed, the formula holds!

Applications of Subspaces

Subspaces find numerous applications in various fields:
1. **Signal Processing**: In signal processing, the null space of a filter matrix represents the signals that get eliminated by the filter.
2. **Image Compression**: In image compression, we can use subspaces to approximate images with fewer dimensions, reducing storage requirements while preserving essential features.
3. **Differential Equations**: The solution space of a homogeneous linear differential equation forms a subspace, which helps in understanding the behavior of the system.
4. **Control Systems**: In control theory, the controllable subspace indicates which states of a system can be reached through proper control inputs.
5. **Quantum Mechanics**: In quantum mechanics, subspaces of Hilbert spaces represent the possible states of physical systems.

7.2.5 Additional Solved Examples

Example 1

Let $V = M_{2\times 2}(\mathbb{R})$ be the vector space with respect to usual matrix addition and scalar multiplication. If W is the set of all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a, b \in \mathbb{R}$, then show that W is a vector subspace of V.

Detailed Solution

To prove that W is a subspace of V, we need to verify the three conditions of the subspace test:
1. W is non-empty 2. W is closed under addition 3. W is closed under scalar multiplication
Let's verify each of these conditions:

Step 1: Show that W is non-empty.
We need to show that W contains at least one element. The simplest approach is to check if the zero matrix belongs to W.

The zero matrix in $M_{2\times 2}(\mathbb{R})$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

According to the definition of W, matrices in W have the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b \in \mathbb{R}$.

If we set $a = 0$ and $b = 0$, we get $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is indeed in W.

Therefore, W is non-empty.

Step 2: Show that W is closed under addition.
Let's take two arbitrary matrices from W and check if their sum is also in W.
Let $A = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$ be elements of W, where $a_1, b_1, a_2, b_2 \in \mathbb{R}$.
Now, let's compute their sum:

$$A + B = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \tag{7.1}$$

$$= \begin{bmatrix} a_1 + a_2 & 0 + 0 \\ 0 + 0 & b_1 + b_2 \end{bmatrix} \tag{7.2}$$

$$= \begin{bmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{bmatrix} \tag{7.3}$$

Since $a_1 + a_2 \in \mathbb{R}$ and $b_1 + b_2 \in \mathbb{R}$ (because $\mathbb{R}$ is closed under addition), the matrix $A + B$ is of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b \in \mathbb{R}$.
Therefore, $A + B \in W$, which means W is closed under addition.

Step 3: Show that W is closed under scalar multiplication.
Let's take an arbitrary matrix from W and multiply it by any scalar, then check if the result is also in W.
Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be an element of W, where $a, b \in \mathbb{R}$, and let $c \in \mathbb{R}$ be any scalar.
Now, let's compute the scalar multiplication:

$$c \cdot A = c \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \tag{7.4}$$

$$= \begin{bmatrix} c \cdot a & c \cdot 0 \\ c \cdot 0 & c \cdot b \end{bmatrix} \tag{7.5}$$

$$= \begin{bmatrix} c \cdot a & 0 \\ 0 & c \cdot b \end{bmatrix} \tag{7.6}$$

Since $c \cdot a \in \mathbb{R}$ and $c \cdot b \in \mathbb{R}$ (because $\mathbb{R}$ is closed under multiplication), the matrix $c \cdot A$ is of the form $\begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}$ where $a', b' \in \mathbb{R}$.
Therefore, $c \cdot A \in W$, which means W is closed under scalar multiplication.

Conclusion:
Since W satisfies all three conditions of the subspace test:

- W is non-empty (it contains the zero matrix)
- W is closed under addition
- W is closed under scalar multiplication

We have proven that W is indeed a subspace of $V = M_{2\times 2}(\mathbb{R})$.

Note: The set W actually consists of all 2×2 diagonal matrices and is commonly denoted as $D_2(\mathbb{R})$. This is an important subspace of $M_{2\times 2}(\mathbb{R})$ with dimension 2, as it can be parametrized by the two values a and b on the diagonal.

Example 2

Let V be the set of real valued functions defined on closed interval $[0, 1]$ which is a vector space for usual addition and scalar multiplication. Let $W = \{f \in V | f(1/2) = 0\}$. Then show that W is a vector subspace of V.

Detailed Solution

To prove that W is a subspace of V, we need to verify the three conditions of the subspace test:
1. W is non-empty 2. W is closed under addition 3. W is closed under scalar multiplication
Let's verify each of these conditions:

Step 1: Show that W is non-empty.
We need to show that W contains at least one element. The simplest approach is to check if the zero function belongs to W.
The zero function in V is defined as $f(x) = 0$ for all $x \in [0, 1]$.
According to the definition of W, functions in W satisfy $f(1/2) = 0$.
For the zero function, we have $f(1/2) = 0$, which satisfies this condition.
Therefore, the zero function belongs to W, so W is non-empty.

Step 2: Show that W is closed under addition.
Let's take two arbitrary functions from W and check if their sum is also in W.
Let $f, g \in W$. By definition, this means that $f(1/2) = 0$ and $g(1/2) = 0$.
Now, let's consider the sum function $f + g$, defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in [0, 1]$.
At $x = 1/2$, we have:

$$\begin{aligned}(f + g)(1/2) &= f(1/2) + g(1/2) &&(7.7)\\ &= 0 + 0 &&(7.8)\\ &= 0 &&(7.9)\end{aligned}$$

Therefore, $(f + g)(1/2) = 0$, which means $f + g \in W$.
This proves that W is closed under addition.

Step 3: Show that W is closed under scalar multiplication.
Let's take an arbitrary function from W and multiply it by any scalar, then check if the result is also in W.
Let $f \in W$ and let $c \in \mathbb{R}$ be any scalar.
By definition, $f(1/2) = 0$.
Now, let's consider the scalar multiple function cf, defined by $(cf)(x) = c \cdot f(x)$ for all $x \in [0, 1]$.
At $x = 1/2$, we have:

$$\begin{aligned}(cf)(1/2) &= c \cdot f(1/2) &&(7.10)\\ &= c \cdot 0 &&(7.11)\\ &= 0 &&(7.12)\end{aligned}$$

Therefore, $(cf)(1/2) = 0$, which means $cf \in W$.

This proves that W is closed under scalar multiplication.

Conclusion:
Since W satisfies all three conditions of the subspace test:

- W is non-empty (it contains the zero function)
- W is closed under addition
- W is closed under scalar multiplication

We have proven that W is indeed a subspace of V.

Note: The set W represents all functions in V that have a root at $x = 1/2$. This is an important example of how constraints on function values can define subspaces. Similar constructions arise in many applications, such as boundary value problems in differential equations, where functions are required to vanish at specific points.

Example 3

Show that the set W of all solutions of m linear homogeneous equations in n unknowns forms a subspace of $\mathbb{R}^n$.

Detailed Solution

Let's begin by formalizing the problem. A system of m linear homogeneous equations in n unknowns can be written as:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \tag{7.13}$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \tag{7.14}$$
$$\vdots \tag{7.15}$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \tag{7.16}$$

This system can be represented in matrix form as $A\vec{x} = \vec{0}$, where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7.17}$$

The set W is the collection of all vectors $\vec{x} \in \mathbb{R}^n$ that satisfy $A\vec{x} = \vec{0}$. In other words, $W = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$, which is precisely the null space (or kernel) of the matrix A.
To prove that W is a subspace of $\mathbb{R}^n$, we need to verify the three conditions of the subspace test:
1. W is non-empty 2. W is closed under addition 3. W is closed under scalar multiplication
Let's verify each of these conditions:

Step 1: Show that W is non-empty.
We need to show that W contains at least one element. The zero vector $\vec{0} = (0, 0, \ldots, 0) \in$

$\mathbb{R}^n$ satisfies:

$$A\vec{0} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7.18}$$

$$= \begin{bmatrix} a_{11} \cdot 0 + a_{12} \cdot 0 + \cdots + a_{1n} \cdot 0 \\ a_{21} \cdot 0 + a_{22} \cdot 0 + \cdots + a_{2n} \cdot 0 \\ \vdots \\ a_{m1} \cdot 0 + a_{m2} \cdot 0 + \cdots + a_{mn} \cdot 0 \end{bmatrix} \tag{7.19}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0} \tag{7.20}$$

Thus, $\vec{0} \in W$, and W is non-empty.

Step 2: Show that W is closed under addition.
Let's take two arbitrary vectors from W and check if their sum is also in W.
Let $\vec{u}, \vec{v} \in W$. By definition, this means that $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$.
Now, let's consider their sum $\vec{u} + \vec{v}$. We need to show that $A(\vec{u} + \vec{v}) = \vec{0}$:

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} \quad \text{(by linearity of matrix multiplication)} \tag{7.21}$$

$$= \vec{0} + \vec{0} \quad (\text{since } \vec{u}, \vec{v} \in W) \tag{7.22}$$

$$= \vec{0} \tag{7.23}$$

Therefore, $\vec{u} + \vec{v} \in W$, which means W is closed under addition.

Step 3: Show that W is closed under scalar multiplication.
Let's take an arbitrary vector from W and multiply it by any scalar, then check if the result is also in W.
Let $\vec{u} \in W$ and let $c \in \mathbb{R}$ be any scalar. By definition, $A\vec{u} = \vec{0}$.
Now, let's consider the scalar multiple $c\vec{u}$. We need to show that $A(c\vec{u}) = \vec{0}$:

$$A(c\vec{u}) = cA\vec{u} \quad \text{(by linearity of matrix multiplication)} \tag{7.24}$$

$$= c\vec{0} \quad (\text{since } \vec{u} \in W) \tag{7.25}$$

$$= \vec{0} \tag{7.26}$$

Therefore, $c\vec{u} \in W$, which means W is closed under scalar multiplication.

Conclusion:
Since W satisfies all three conditions of the subspace test:

- W is non-empty (it contains the zero vector)
- W is closed under addition
- W is closed under scalar multiplication

We have proven that W is indeed a subspace of $\mathbb{R}^n$.

Note: This result is fundamental in linear algebra and has many applications. It shows that the solution set of any homogeneous linear system forms a subspace. Furthermore, if the coefficient matrix A has rank r, then the dimension of this subspace is $n - r$, as given by the Rank-Nullity Theorem. This means we can always parametrize the solution space using $n - r$ free variables, which is a key insight for solving linear systems.

Example 4

Check whether the following are subspaces. Justify your answer.
(i) $W = \{(a, 0, 0) \mid a \in \mathbb{R}\}$ of $\mathbb{R}^3$.
(ii) $W = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ of $\mathbb{R}^3$.
(iii) $W = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$ of $\mathbb{R}^2$.
(iv) $W = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0, a, b, c \in \mathbb{R}\}$ of $\mathbb{R}^3$.
(v) $W = \{(x, y) \in \mathbb{R}^2 \mid x = 3y\}$ of $\mathbb{R}^2$.
(vi) $W = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 = 0, a_i \in \mathbb{R}, i = 0, 1, 2, 3\}$ of P_3.
(vii) $W = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } y = 0\}$ of $\mathbb{R}^3$.
(viii) The set of all pairs of real numbers (x, y) with the operation $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k(x, y) = (2kx, 2ky)$.

Solution

To determine whether each set is a subspace, we will verify the three conditions of the subspace test: 1. W is non-empty (contains the zero vector) 2. W is closed under addition 3. W is closed under scalar multiplication

(i) $W = \{(a, 0, 0) \mid a \in \mathbb{R}\}$ of $\mathbb{R}^3$
Step 1: Is W non-empty? The zero vector in $\mathbb{R}^3$ is $(0, 0, 0)$. If we set $a = 0$, we get $(0, 0, 0) \in W$. So W is non-empty.
Step 2: Is W closed under addition? Let $(a_1, 0, 0), (a_2, 0, 0) \in W$. Then:

$$(a_1, 0, 0) + (a_2, 0, 0) = (a_1 + a_2, 0, 0) \tag{7.27}$$

Since $a_1 + a_2 \in \mathbb{R}$, we have $(a_1 + a_2, 0, 0) \in W$. So W is closed under addition.
Step 3: Is W closed under scalar multiplication? Let $(a, 0, 0) \in W$ and $c \in \mathbb{R}$. Then:

$$c \cdot (a, 0, 0) = (ca, 0, 0) \tag{7.28}$$

Since $ca \in \mathbb{R}$, we have $(ca, 0, 0) \in W$. So W is closed under scalar multiplication.
Conclusion: Since all three conditions are satisfied, W is a subspace of $\mathbb{R}^3$.

(ii) $W = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ of $\mathbb{R}^3$
Step 1: Is W non-empty? The zero vector $(0, 0, 0)$ satisfies $0^2 + 0^2 + 0^2 = 0 \leq 1$. So $(0, 0, 0) \in W$, and W is non-empty.
Step 2: Is W closed under addition? Let's consider two vectors in W: $\vec{u} = (x_1, y_1, z_1)$ and $\vec{v} = (x_2, y_2, z_2)$, where $x_1^2 + y_1^2 + z_1^2 \leq 1$ and $x_2^2 + y_2^2 + z_2^2 \leq 1$.
Their sum is $\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$.
For a counterexample, consider $\vec{u} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\vec{v} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
We have:

$$x_1^2 + y_1^2 + z_1^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \tag{7.29}$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \tag{7.30}$$
$$= \frac{3}{4} \leq 1 \tag{7.31}$$

So $\vec{u} \in W$. Similarly, $\vec{v} \in W$.
Now, for their sum:

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^2 = (1)^2 + (1)^2 + (1)^2 \tag{7.32}$$
$$= 1 + 1 + 1 \tag{7.33}$$
$$= 3 > 1 \tag{7.34}$$

Therefore, $\vec{u} + \vec{v} \notin W$. This shows that W is not closed under addition.
Conclusion: Since W fails the second condition, W is not a subspace of $\mathbb{R}^3$.

(iii) $W = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$ **of** $\mathbb{R}^2$
Step 1: Is W non-empty? The zero vector $(0, 0)$ satisfies $0 \geq 0$ and $0 \geq 0$. So $(0, 0) \in W$, and W is non-empty.
Step 2: Is W closed under addition? Let $(x_1, y_1), (x_2, y_2) \in W$. Then $x_1 \geq 0$, $y_1 \geq 0$, $x_2 \geq 0$, and $y_2 \geq 0$.
Their sum is $(x_1 + x_2, y_1 + y_2)$.
Since $x_1 \geq 0$ and $x_2 \geq 0$, we have $x_1 + x_2 \geq 0$. Similarly, $y_1 + y_2 \geq 0$.
Therefore, $(x_1 + x_2, y_1 + y_2) \in W$. So W is closed under addition.
Step 3: Is W closed under scalar multiplication? Let $(x, y) \in W$ and $c \in \mathbb{R}$. Then $x \geq 0$ and $y \geq 0$.
For $c \cdot (x, y) = (cx, cy)$, we need to check if $(cx, cy) \in W$, i.e., if $cx \geq 0$ and $cy \geq 0$.
If $c < 0$, then $cx < 0$ (assuming $x > 0$). This means $(cx, cy) \notin W$.
For a specific counterexample, let $(1, 1) \in W$ and $c = -1$. Then:

$$c \cdot (1, 1) = (-1) \cdot (1, 1) = (-1, -1) \tag{7.35}$$

Since $-1 < 0$, we have $(-1, -1) \notin W$. This shows that W is not closed under scalar multiplication.
Conclusion: Since W fails the third condition, W is not a subspace of $\mathbb{R}^2$.

(iv) $W = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0, a, b, c \in \mathbb{R}\}$ **of** $\mathbb{R}^3$
This statement is ambiguous. There are two possible interpretations:
Interpretation 1: For any fixed $a, b, c \in \mathbb{R}$ (not all zero), W is the set of all (x, y, z) such that $ax + by + cz = 0$.
Interpretation 2: W is the set of all (x, y, z) such that there exist $a, b, c \in \mathbb{R}$ with $ax + by + cz = 0$.
Let's analyze Interpretation 1, which is the more standard one:
For fixed $a, b, c \in \mathbb{R}$ (not all zero), $W = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0\}$.
Step 1: Is W non-empty? The zero vector $(0, 0, 0)$ satisfies $a \cdot 0 + b \cdot 0 + c \cdot 0 = 0$. So $(0, 0, 0) \in W$, and W is non-empty.
Step 2: Is W closed under addition? Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in W$. Then:

$$ax_1 + by_1 + cz_1 = 0 \tag{7.36}$$
$$ax_2 + by_2 + cz_2 = 0 \tag{7.37}$$

Their sum is $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$. We need to check if this satisfies the equation:

$$\begin{aligned} a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) &= ax_1 + ax_2 + by_1 + by_2 + cz_1 + cz_2 && (7.38)\\ &= (ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2) && (7.39)\\ &= 0 + 0 && (7.40)\\ &= 0 && (7.41) \end{aligned}$$

Therefore, $(x_1 + x_2, y_1 + y_2, z_1 + z_2) \in W$. So W is closed under addition.
Step 3: Is W closed under scalar multiplication? Let $(x, y, z) \in W$ and $k \in \mathbb{R}$. Then $ax + by + cz = 0$.
For $k \cdot (x, y, z) = (kx, ky, kz)$, we need to check if $(kx, ky, kz) \in W$, i.e., if $a(kx) + b(ky) + c(kz) = 0$.

$$\begin{aligned} a(kx) + b(ky) + c(kz) &= k(ax + by + cz) && (7.42)\\ &= k \cdot 0 && (7.43)\\ &= 0 && (7.44) \end{aligned}$$

Therefore, $(kx, ky, kz) \in W$. So W is closed under scalar multiplication.
Conclusion: Since all three conditions are satisfied, W is a subspace of $\mathbb{R}^3$.
Note: This is the equation of a plane through the origin, which is indeed a subspace of $\mathbb{R}^3$.

(v) $W = \{(x, y) \in \mathbb{R}^2 \mid x = 3y\}$ **of** $\mathbb{R}^2$
Step 1: Is W non-empty? The zero vector $(0, 0)$ satisfies $0 = 3 \cdot 0$. So $(0, 0) \in W$, and W is non-empty.
Step 2: Is W closed under addition? Let $(x_1, y_1), (x_2, y_2) \in W$. Then $x_1 = 3y_1$ and $x_2 = 3y_2$.
Their sum is $(x_1 + x_2, y_1 + y_2)$. We need to check if $x_1 + x_2 = 3(y_1 + y_2)$:

$$x_1 + x_2 = 3y_1 + 3y_2 \tag{7.45}$$
$$= 3(y_1 + y_2) \tag{7.46}$$

Therefore, $(x_1 + x_2, y_1 + y_2) \in W$. So W is closed under addition.
Step 3: Is W closed under scalar multiplication? Let $(x, y) \in W$ and $c \in \mathbb{R}$. Then $x = 3y$.
For $c \cdot (x, y) = (cx, cy)$, we need to check if $(cx, cy) \in W$, i.e., if $cx = 3(cy)$:

$$cx = c \cdot 3y \tag{7.47}$$
$$= 3(cy) \tag{7.48}$$

Therefore, $(cx, cy) \in W$. So W is closed under scalar multiplication.
Conclusion: Since all three conditions are satisfied, W is a subspace of $\mathbb{R}^2$.
Note: This is the equation of a line through the origin, which is indeed a subspace of $\mathbb{R}^2$.

(vi) $W = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 = 0, a_i \in \mathbb{R}, i = 0, 1, 2, 3\}$ **of** P_3
Step 1: Is W non-empty? The zero polynomial $0 + 0x + 0x^2 + 0x^3$ satisfies $a_0 = 0$. So $0 + 0x + 0x^2 + 0x^3 \in W$, and W is non-empty.
Step 2: Is W closed under addition? Let $p(x) = 0 + a_1x + a_2x^2 + a_3x^3$ and $q(x) = 0 + b_1x + b_2x^2 + b_3x^3$ be in W.
Their sum is:

$$p(x) + q(x) = (0 + a_1x + a_2x^2 + a_3x^3) + (0 + b_1x + b_2x^2 + b_3x^3) \tag{7.49}$$
$$= 0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \tag{7.50}$$

The constant term is 0, so $p(x) + q(x) \in W$. Therefore, W is closed under addition.
Step 3: Is W closed under scalar multiplication? Let $p(x) = 0 + a_1x + a_2x^2 + a_3x^3 \in W$ and $c \in \mathbb{R}$.
Then:

$$c \cdot p(x) = c \cdot (0 + a_1x + a_2x^2 + a_3x^3) \tag{7.51}$$
$$= 0 + ca_1x + ca_2x^2 + ca_3x^3 \tag{7.52}$$

The constant term is 0, so $c \cdot p(x) \in W$. Therefore, W is closed under scalar multiplication.
Conclusion: Since all three conditions are satisfied, W is a subspace of P_3.
Note: This subspace consists of all polynomials in P_3 with no constant term, i.e., polynomials of the form $a_1x + a_2x^2 + a_3x^3$.

(vii) $W = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0$ **or** $y = 0\}$ **of** $\mathbb{R}^3$
Step 1: Is W non-empty? The zero vector $(0, 0, 0)$ satisfies both $x = 0$ and $y = 0$. So $(0, 0, 0) \in W$, and W is non-empty.

Step 2: Is W closed under addition? Let's consider two vectors in W: $\vec{u} = (0,1,1)$ (where $x=0$) and $\vec{v} = (1,0,1)$ (where $y=0$).
Both $\vec{u}$ and $\vec{v}$ are in W.
Their sum is $\vec{u}+\vec{v} = (0,1,1)+(1,0,1) = (1,1,2)$.
In $\vec{u}+\vec{v}$, neither x nor y is zero. Therefore, $\vec{u}+\vec{v} \notin W$. This shows that W is not closed under addition.
Conclusion: Since W fails the second condition, W is not a subspace of $\mathbb{R}^3$.

(viii) The set of all pairs of real numbers (x,y) with the operation $(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$ and $k(x,y) = (2kx, 2ky)$
For this problem, we need to check if $\mathbb{R}^2$ with the given operations forms a vector space. If it does, then the entire set would be a subspace of itself. However, the scalar multiplication operation is non-standard.
Step 1: Is $\mathbb{R}^2$ non-empty? Yes, it contains $(0,0)$.
Step 2: Is $\mathbb{R}^2$ closed under the given addition? Yes, for any $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, we have $(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2) \in \mathbb{R}^2$.
Step 3: Is $\mathbb{R}^2$ closed under the given scalar multiplication? Yes, for any $(x,y) \in \mathbb{R}^2$ and $k \in \mathbb{R}$, we have $k(x,y) = (2kx, 2ky) \in \mathbb{R}^2$.
However, to be a vector space, the operations must satisfy certain axioms. Let's check if the scalar multiplication satisfies the necessary properties:
(a) $1\cdot(x,y) = (x,y)$ for all $(x,y) \in \mathbb{R}^2$:

$$1\cdot(x,y) = (2\cdot 1\cdot x, 2\cdot 1\cdot y) \tag{7.53}$$
$$= (2x, 2y) \tag{7.54}$$

But this is not equal to (x,y) unless $x=y=0$. So this property is not satisfied.
(b) $(k_1\cdot k_2)\cdot(x,y) = k_1\cdot(k_2\cdot(x,y))$ for all $k_1,k_2 \in \mathbb{R}$ and $(x,y) \in \mathbb{R}^2$:

$$(k_1\cdot k_2)\cdot(x,y) = (2(k_1\cdot k_2)x, 2(k_1\cdot k_2)y) \tag{7.55}$$
$$= (2k_1k_2x, 2k_1k_2y) \tag{7.56}$$

And:

$$k_1\cdot(k_2\cdot(x,y)) = k_1\cdot(2k_2x, 2k_2y) \tag{7.57}$$
$$= (2k_1\cdot 2k_2x, 2k_1\cdot 2k_2y) \tag{7.58}$$
$$= (4k_1k_2x, 4k_1k_2y) \tag{7.59}$$

These are not equal, so this property is not satisfied.
Conclusion: Since the scalar multiplication operation doesn't satisfy all the required properties for a vector space, $\mathbb{R}^2$ with the given operations is not a vector space. Therefore, it's not meaningful to ask whether it's a subspace.

Example 5

State only one axiom that fails to hold for each of the following sets W to be subspaces of the respective real vector spaces V with the standard operations.
(i) $W = \{(x,y) \mid x^2 = y^2\}, V = \mathbb{R}^2$
(ii) $W = \{(x,y) \mid xy \geq 0\}, V = \mathbb{R}^2$
(iii) $W = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}, V = \mathbb{R}^3$
(iv) $W = \{A_{n\times n} \mid AX = 0 \Rightarrow X = 0\}, V = M_{n\times n}$
(v) $W = \{f \mid f(x) \leq 0; \forall x\}, V = F(-\infty,\infty)$

(vi) $W = \{f \mid f(0) = 0; \forall x\}, V = F(-\infty, \infty)$

Solution

For each set W, we will identify one axiom of a subspace that fails to hold. The three axioms that need to be satisfied for a subset to be a subspace are:

1. Non-emptiness (contains the zero element)
2. Closure under addition
3. Closure under scalar multiplication

(i) $W = \{(x, y) \mid x^2 = y^2\}, V = \mathbb{R}^2$
The set W consists of points where $x^2 = y^2$, which implies $y = \pm x$. This means W is the union of two lines through the origin: $y = x$ and $y = -x$.
Let's check for closure under addition: Take $(1, 1) \in W$ and $(1, -1) \in W$. $(1, 1)+(1, -1) = (2, 0)$
For $(2, 0)$ to be in W, we would need $(2)^2 = (0)^2$, which gives $4 = 0$, which is false.
Therefore, W is not closed under addition.

(ii) $W = \{(x, y) \mid xy \geq 0\}, V = \mathbb{R}^2$
The set W consists of points where $xy \geq 0$, which corresponds to the first and third quadrants of the plane (including the axes).
Let's check for closure under scalar multiplication: Take $(1, 1) \in W$ since $1 \cdot 1 = 1 > 0$.
For scalar $c = -1$: $(-1) \cdot (1, 1) = (-1, -1)$
For $(-1, -1)$ to be in W, we would need $(-1)(-1) \geq 0$, which gives $1 \geq 0$, which is true.
Let's check another vector: $(1, 0) \in W$ since $1 \cdot 0 = 0 \geq 0$. For scalar $c = -1$: $(-1) \cdot (1, 0) = (-1, 0)$
For $(-1, 0)$ to be in W, we would need $(-1)(0) \geq 0$, which gives $0 \geq 0$, which is true.
It seems the set is closed under scalar multiplication. Let's check for closure under addition:
Take $(1, 0) \in W$ and $(0, -1) \in W$. $(1, 0) + (0, -1) = (1, -1)$
For $(1, -1)$ to be in W, we would need $(1)(-1) \geq 0$, which gives $-1 \geq 0$, which is false.
Therefore, W is not closed under addition.

(iii) $W = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}, V = \mathbb{R}^3$
The set W is the unit sphere centered at the origin.
First, let's check if the zero vector is in W: For $(0, 0, 0)$, we have $0^2 + 0^2 + 0^2 = 0 \neq 1$
Therefore, W does not contain the zero vector, so the non-emptiness condition (in the sense of containing the zero vector) fails.

(iv) $W = \{A_{n\times n} \mid AX = 0 \Rightarrow X = 0\}, V = M_{n\times n}$
The set W consists of $n \times n$ matrices A for which the equation $AX = 0$ implies $X = 0$.
This means A has trivial null space, which implies A is invertible.
Let's check if the zero matrix is in W: Let $A = 0$ (the zero matrix). For any non-zero vector X, we have $AX = 0X = 0$.
This means that for $A = 0$, the equation $AX = 0$ is satisfied by any vector X, not just by $X = 0$. Therefore, the zero matrix does not belong to W.
Thus, W does not contain the zero element of V, so the non-emptiness condition (in the sense of containing the zero element) fails.

(v) $W = \{f \mid f(x) \leq 0; \forall x\}, V = F(-\infty, \infty)$
The set W consists of all functions f such that $f(x) \leq 0$ for all $x \in \mathbb{R}$.
Let's check if the zero function is in W: For the zero function $f(x) = 0$, we have $f(x) = 0 \leq 0$ for all $x \in \mathbb{R}$. So the zero function is in W.
Let's check for closure under addition: Take $f(x) = -1$ and $g(x) = -2$. Both $f, g \in W$

since $f(x) = -1 < 0$ and $g(x) = -2 < 0$ for all x. $(f+g)(x) = f(x)+g(x) = -1+(-2) = -3$
Since $(f+g)(x) = -3 < 0$ for all $x \in \mathbb{R}$, we have $f+g \in W$.
Let's check for closure under scalar multiplication: Take $f(x) = -1 \in W$ and scalar $c = -1$. $(c \cdot f)(x) = c \cdot f(x) = (-1) \cdot (-1) = 1$
Since $(c \cdot f)(x) = 1 > 0$, we have $c \cdot f \notin W$.
Therefore, W is not closed under scalar multiplication.

(vi) $W = \{f \mid f(0) = 0; \forall x\}, V = F(-\infty, \infty)$
The set W consists of all functions f such that $f(0) = 0$.
Let's check if the zero function is in W: For the zero function $f(x) = 0$, we have $f(0) = 0$, so the zero function is in W.
Let's check for closure under addition: Take $f, g \in W$, so $f(0) = 0$ and $g(0) = 0$. $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$
So $f+g \in W$, and W is closed under addition.
Let's check for closure under scalar multiplication: Take $f \in W$ and scalar $c \in \mathbb{R}$. $(c \cdot f)(0) = c \cdot f(0) = c \cdot 0 = 0$
So $c \cdot f \in W$, and W is closed under scalar multiplication.
It appears that W satisfies all three axioms of a subspace. Therefore, W is actually a subspace of $V = F(-\infty, \infty)$. The question statement might be incorrect for this part, as no axiom fails.

Example 6

Determine whether the set of all vectors $(x, y, z) \in \mathbb{R}^3$ that satisfy each of the following equations forms a subspace of $\mathbb{R}^3$. Justify your answer.
(a) $2x - 3y + z = 0$ (homogeneous equation)
(b) $2x - 3y + z = 5$ (non-homogeneous equation)

Solution

To determine whether a set forms a subspace, we need to verify three conditions: 1. The set contains the zero vector 2. The set is closed under addition 3. The set is closed under scalar multiplication

(a) For the homogeneous equation $2x - 3y + z = 0$:
Step 1: Does the set contain the zero vector? Let's check if $(0, 0, 0)$ satisfies the equation: $2(0) - 3(0) + 0 = 0$ $0 = 0$ The zero vector is in the set.
Step 2: Is the set closed under addition? Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be any two vectors in the set. Then: $2x_1 - 3y_1 + z_1 = 0$ and $2x_2 - 3y_2 + z_2 = 0$
Now consider their sum $(x_1+x_2, y_1+y_2, z_1+z_2)$. Does it satisfy the equation? $2(x_1+x_2) - 3(y_1+y_2) + (z_1+z_2) = 2x_1 + 2x_2 - 3y_1 - 3y_2 + z_1 + z_2 = (2x_1 - 3y_1 + z_1) + (2x_2 - 3y_2 + z_2) = 0 + 0 = 0$
The set is closed under addition.
Step 3: Is the set closed under scalar multiplication? Let (x, y, z) be any vector in the set and c be any scalar. Then: $2x - 3y + z = 0$
Now consider the scalar multiple $c(x, y, z) = (cx, cy, cz)$. Does it satisfy the equation? $2(cx) - 3(cy) + (cz) = c(2x - 3y + z) = c \cdot 0 = 0$
The set is closed under scalar multiplication.
Conclusion: Since all three conditions are satisfied, the set of vectors satisfying the homogeneous equation $2x - 3y + z = 0$ forms a subspace of $\mathbb{R}^3$.
Geometrically, this set represents a plane through the origin in $\mathbb{R}^3$.

(b) For the non-homogeneous equation $2x - 3y + z = 5$:
Step 1: Does the set contain the zero vector? Let's check if $(0, 0, 0)$ satisfies the equation: $2(0) - 3(0) + 0 = 5$ $0 = 5$ The zero vector is not in the set.
Since the first condition fails, the set cannot be a subspace of $\mathbb{R}^3$.
For completeness, let's also check the other conditions:
Step 2: Is the set closed under addition? Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be any two vectors in the set. Then: $2x_1 - 3y_1 + z_1 = 5$ and $2x_2 - 3y_2 + z_2 = 5$
Now consider their sum $(x_1+x_2, y_1+y_2, z_1+z_2)$. Does it satisfy the equation? $2(x_1+x_2) - 3(y_1+y_2) + (z_1+z_2) = 2x_1 + 2x_2 - 3y_1 - 3y_2 + z_1 + z_2 = (2x_1 - 3y_1 + z_1) + (2x_2 - 3y_2 + z_2) = 5 + 5 = 10 \neq 5$
The set is not closed under addition.
Step 3: Is the set closed under scalar multiplication? Let (x, y, z) be any vector in the set and c be any scalar. Then: $2x - 3y + z = 5$
Now consider the scalar multiple $c(x, y, z) = (cx, cy, cz)$. Does it satisfy the equation? $2(cx) - 3(cy) + (cz) = c(2x - 3y + z) = c \cdot 5 = 5c$
This equals 5 only when $c = 1$. For all other values of c, the equation is not satisfied.
The set is not closed under scalar multiplication.
Conclusion: The set of vectors satisfying the non-homogeneous equation $2x - 3y + z = 5$ does not form a subspace of $\mathbb{R}^3$ because it fails all three conditions of the subspace test.
Geometrically, this set represents a plane that does not pass through the origin in $\mathbb{R}^3$.

7.2.6 Challenging Examples

Example 7

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$. Determine the general solution to both the homogeneous system $A\vec{x} = \vec{0}$ and the non-homogeneous system $A\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. Then explain the relationship between these solution sets.

Solution

We'll solve both systems by using Gaussian elimination to find the row echelon form of the augmented matrices.

Step 1: Solve the homogeneous system $A\vec{x} = \vec{0}$.
First, let's set up the augmented matrix $[A|\vec{0}]$:

$$[A|\vec{0}] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 6 & 1 & 0 \end{bmatrix}$$

Now, we perform row operations to get the row echelon form:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 6 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 3 & 6 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3-3R_1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we can solve this system. From the second row, we get:

$$-2z = 0 \implies z = 0$$

From the first row:

$$x + 2y + z = 0 \implies x + 2y + 0 = 0 \implies x = -2y$$

So, the general solution to the homogeneous system is:

$$\vec{x}_h = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Where y is a free parameter. The solution set forms a one-dimensional subspace (a line through the origin) in $\mathbb{R}^3$.

Step 2: Solve the non-homogeneous system $A\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$.

Let's set up the augmented matrix $[A|\vec{b}]$:

$$[A|\vec{b}] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 1 & 5 \end{bmatrix}$$

We perform the same row operations:

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 1 & 5 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -2 & -4 \\ 3 & 6 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3-3R_1} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -2 & -4 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the second row:

$$-2z = -4 \implies z = 2$$

From the first row:

$$x + 2y + z = 3 \implies x + 2y + 2 = 3 \implies x + 2y = 1 \implies x = 1 - 2y$$

So, the general solution to the non-homogeneous system is:

$$\vec{x}_p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-2y \\ y \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Step 3: Relationship between the solution sets.

Notice that the general solution to the non-homogeneous system can be written as:

$$\vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

This is the sum of a particular solution $\vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and the general solution to the homogeneous system $\vec{x}_h = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

This exemplifies the general relationship between solutions to homogeneous and non-homogeneous systems:

- The general solution to a non-homogeneous system $A\vec{x} = \vec{b}$ is the sum of:
  - Any particular solution to $A\vec{x} = \vec{b}$ (e.g., $\vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$)
  - The general solution to the corresponding homogeneous system $A\vec{x} = \vec{0}$ (e.g., $\vec{x}_h = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$)
- Geometrically, the solution set to a non-homogeneous system is an affine subspace (a translated subspace), not a true subspace because it doesn't contain the zero vector.
- The solution set to $A\vec{x} = \vec{b}$ is parallel to the solution set of $A\vec{x} = \vec{0}$ but shifted by a particular solution $\vec{x}_p$.

Example 8

Consider the system of equations:

$$x_1 + 2x_2 - x_3 = 0 \tag{7.60}$$
$$2x_1 + 4x_2 - 2x_3 = 0 \tag{7.61}$$
$$3x_1 + 6x_2 - 3x_3 = 0 \tag{7.62}$$

Determine if this system is homogeneous, find its solution space, and explain why the solution space is a subspace of $\mathbb{R}^3$.

Solution

Step 1: Determine if the system is homogeneous.
A system of linear equations is homogeneous if the right-hand side of all equations is zero. Since all equations have zero on the right-hand side, this system is homogeneous.

Step 2: Find the solution space.
We can express this system in matrix form as $A\vec{x} = \vec{0}$, where:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's solve this system using row reduction on the augmented matrix $[A|\vec{0}]$:

$$[A|\vec{0}] = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 0 \\ 3 & 6 & -3 & 0 \end{bmatrix}$$

Applying row operations:

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 0 \\ 3 & 6 & -3 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 6 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have only one linearly independent equation: $x_1 + 2x_2 - x_3 = 0$, which can be rewritten as $x_1 = -2x_2 + x_3$.
This means we have two free variables, x_2 and x_3. The general solution is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the solution space is spanned by the vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, forming a two-dimensional subspace (a plane through the origin) in $\mathbb{R}^3$.

Step 3: Explain why the solution space is a subspace of $\mathbb{R}^3$.
The solution space of a homogeneous system of linear equations always forms a subspace. We can verify this by checking the three conditions:

1. **Non-empty**: The zero vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is always a solution to a homogeneous system, so the solution space is non-empty.
2. **Closure under addition**: If $\vec{u}$ and $\vec{v}$ are solutions to $A\vec{x} = \vec{0}$, then $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$. By linearity of matrix multiplication:

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$$

Thus, $\vec{u} + \vec{v}$ is also a solution, demonstrating closure under addition.
3. **Closure under scalar multiplication**: If $\vec{u}$ is a solution to $A\vec{x} = \vec{0}$ and c is a scalar, then $A\vec{u} = \vec{0}$. By linearity of matrix multiplication:

$$A(c\vec{u}) = cA\vec{u} = c\vec{0} = \vec{0}$$

Thus, $c\vec{u}$ is also a solution, demonstrating closure under scalar multiplication.
Since all three conditions are satisfied, the solution space is a subspace of $\mathbb{R}^3$.
In this specific example, we determined that the solution space is a 2-dimensional subspace (a plane through the origin) spanned by the vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Example 9

Consider the non-homogeneous system:

$$2x - y + 3z = 4 \tag{7.63}$$
$$4x - 2y + 6z = 8 \tag{7.64}$$
$$6x - 3y + 9z = 12 \tag{7.65}$$

(a) Determine whether this system is consistent. If so, find the general solution.
(b) Does the solution set form a subspace of $\mathbb{R}^3$? Justify your answer.
(c) Identify the corresponding homogeneous system and find its solution space.
(d) Explain the geometric relationship between the solution sets of the non-homogeneous and homogeneous systems.

Solution

(a) Determine whether the system is consistent and find the general solution.
Let's express this system in matrix form as $A\vec{x} = \vec{b}$, where:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ 6 & -3 & 9 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$

We'll use row reduction on the augmented matrix $[A|\vec{b}]$:

$$[A|\vec{b}] = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 4 & -2 & 6 & 8 \\ 6 & -3 & 9 & 12 \end{bmatrix}$$

Applying row operations:

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 4 & -2 & 6 & 8 \\ 6 & -3 & 9 & 12 \end{bmatrix} \xrightarrow{R_1 \div 2} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 2 \\ 4 & -2 & 6 & 8 \\ 6 & -3 & 9 & 12 \end{bmatrix}$$

$$\xrightarrow{R_2 - 4R_1, R_3 - 6R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the second and third rows are zeros, we've verified that the system is consistent. We have one equation with three variables, meaning two variables are free.
From the first row: $x - \frac{1}{2}y + \frac{3}{2}z = 2$, which can be rearranged as $x = 2 + \frac{1}{2}y - \frac{3}{2}z$.
Taking y and z as free parameters, the general solution is:

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + \frac{1}{2}y - \frac{3}{2}z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

(b) Does the solution set form a subspace of $\mathbb{R}^3$?
No, the solution set does not form a subspace of $\mathbb{R}^3$. To be a subspace, it must contain the zero vector, be closed under addition, and be closed under scalar multiplication.

The zero vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not a solution to the non-homogeneous system. We can verify this by substituting into the original equation:

$$A\vec{0} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ 6 & -3 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix} = \vec{b}$$

Since the solution set doesn't contain the zero vector, it cannot be a subspace.

(c) Identify the corresponding homogeneous system and find its solution space.
The corresponding homogeneous system is obtained by replacing the right-hand side with zeros:

$$2x - y + 3z = 0 \tag{7.66}$$
$$4x - 2y + 6z = 0 \tag{7.67}$$
$$6x - 3y + 9z = 0 \tag{7.68}$$

Using row reduction on the augmented matrix:

$$\begin{bmatrix} 2 & -1 & 3 & 0 \\ 4 & -2 & 6 & 0 \\ 6 & -3 & 9 & 0 \end{bmatrix} \xrightarrow{R_1 \div 2} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 0 \\ 4 & -2 & 6 & 0 \\ 6 & -3 & 9 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 - 4R_1, R_3 - 6R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the first row: $x - \frac{1}{2}y + \frac{3}{2}z = 0$, which can be rearranged as $x = \frac{1}{2}y - \frac{3}{2}z$.
The general solution to the homogeneous system is:

$$\vec{x}_h = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}y - \frac{3}{2}z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

This solution space is a 2-dimensional subspace (a plane through the origin) spanned by the vectors $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$.

(d) Explain the geometric relationship between the solution sets.
Comparing the solutions:

$$\text{Non-homogeneous solution:} \quad \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \tag{7.69}$$

$$\text{Homogeneous solution:} \quad \vec{x}_h = y \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \tag{7.70}$$

Geometrically:

- The solution to the homogeneous system forms a 2-dimensional subspace (a plane through the origin) in $\mathbb{R}^3$.
- The solution to the non-homogeneous system is obtained by translating this plane away from the origin by the vector $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, resulting in an affine subspace (a plane not through the origin).
- Any particular solution to the non-homogeneous system (e.g., $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$) can serve as the translation vector.
- The general principle is: If S_h is the solution space of $A\vec{x} = \vec{0}$ and $\vec{p}$ is any particular solution to $A\vec{x} = \vec{b}$, then the solution set to $A\vec{x} = \vec{b}$ is $S = \vec{p} + S_h = \{\vec{p} + \vec{v} \mid \vec{v} \in S_h\}$.

This is why the solution set to a non-homogeneous system is an affine subspace (a shifted subspace) rather than a true subspace.

Example 10

Consider the matrix $A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 6 & 1 & 5 \\ 1 & 3 & 1 & 4 \end{bmatrix}$.

(a) Find the rank, nullity, and a basis for the null space of A.

(b) Determine whether the system $A\vec{x} = \begin{bmatrix} 4 \\ 9 \\ 6 \end{bmatrix}$ is consistent. If it is, find the general solution.

(c) For what values of k is the system $A\vec{x} = \begin{bmatrix} 2 \\ k \\ 3 \end{bmatrix}$ consistent?

(d) Explain how your answers to parts (a), (b), and (c) are connected to the concepts of homogeneous and non-homogeneous systems.

Solution

(a) Find the rank, nullity, and a basis for the null space of A.

First, let's find the row echelon form (REF) of A through Gaussian elimination:

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 6 & 1 & 5 \\ 1 & 3 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 3 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now that we have the REF, we can determine:

- **Rank**: The number of non-zero rows in the REF is 3, so $\text{rank}(A) = 3$.
- **Nullity**: By the Rank-Nullity Theorem, for an $m \times n$ matrix, $\text{rank}(A) + \text{nullity}(A) = n$. Since A is a 3×4 matrix with $\text{rank}(A) = 3$, we have $\text{nullity}(A) = 4 - 3 = 1$.

To find a basis for the null space, we need to solve the homogeneous system $A\vec{x} = \vec{0}$. From the row echelon form, we can write the system as:

$$x_1 + 3x_2 + 0x_3 + 2x_4 = 0 \tag{7.71}$$
$$0x_1 + 0x_2 + x_3 + x_4 = 0 \tag{7.72}$$
$$0x_1 + 0x_2 + 0x_3 + x_4 = 0 \tag{7.73}$$

From the third equation, we have $x_4 = 0$. From the second equation, $x_3 + x_4 = 0 \implies x_3 = 0$. From the first equation, $x_1 + 3x_2 + 2x_4 = 0 \implies x_1 + 3x_2 = 0 \implies x_1 = -3x_2$. Taking x_2 as a free parameter, the general solution is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, a basis for the null space of A is $\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \}$.

(b) Determine whether the system $A\vec{x} = \begin{bmatrix} 4 \\ 9 \\ 6 \end{bmatrix}$ is consistent.

To check consistency, we'll use row reduction on the augmented matrix $[A|\vec{b}]$:

$$[A|\vec{b}] = \begin{bmatrix} 1 & 3 & 0 & 2 & 4 \\ 2 & 6 & 1 & 5 & 9 \\ 1 & 3 & 1 & 4 & 6 \end{bmatrix}$$

$$\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 3 & 1 & 4 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 3 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 3 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Now, we can back-substitute to find the solution:
From the third row: $x_4 = 1$ From the second row: $x_3 + x_4 = 1 \implies x_3 + 1 = 1 \implies x_3 = 0$
From the first row: $x_1 + 3x_2 + 2x_4 = 4 \implies x_1 + 3x_2 + 2(1) = 4 \implies x_1 + 3x_2 = 2 \implies x_1 = 2 - 3x_2$
Taking x_2 as a free parameter, the general solution is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - 3x_2 \\ x_2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the system is consistent, and its solution is the sum of a particular solution $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and the general solution to the homogeneous system $x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(c) For what values of k is the system $A\vec{x} = \begin{bmatrix} 2 \\ k \\ 3 \end{bmatrix}$ consistent?

We'll use row reduction on the augmented matrix $[A|\vec{b}]$ with the parametric right-hand side:

$$[A|\vec{b}] = \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 2 & 6 & 1 & 5 & k \\ 1 & 3 & 1 & 4 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & k-4 \\ 1 & 3 & 1 & 4 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & k-4 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & k-4 \\ 0 & 0 & 0 & 1 & 5-k \end{bmatrix}$$

For the system to be consistent, we need all rows of the augmented matrix to be consistent. Looking at the third row, we have $x_4 = 5 - k$. This means:

- If $k \neq 5$, then $x_4 = 5 - k$ is a specific value, and the system has a unique solution for x_4.
- If $k = 5$, then the third row becomes $0x_1 + 0x_2 + 0x_3 + 1x_4 = 0$, which is inconsistent since x_4 cannot equal 0 and satisfy this equation.

Therefore, the system $A\vec{x} = \begin{bmatrix} 2 \\ k \\ 3 \end{bmatrix}$ is consistent if and only if $k \neq 5$.

(d) Explain how your answers are connected to the concepts of homogeneous and non-homogeneous systems.

The solutions to parts (a), (b), and (c) demonstrate several important connections between homogeneous and non-homogeneous systems:

1. **Structure of solution spaces**:
   - In part (a), we found that the null space of A (the solution to the homogeneous system $A\vec{x} = \vec{0}$) is a 1-dimensional subspace spanned by $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

- In part (b), we found that the solution to the non-homogeneous system $A\vec{x} = \begin{bmatrix}4\\9\\6\end{bmatrix}$ is $\vec{x} = \begin{bmatrix}2\\0\\0\\1\end{bmatrix} + x_2 \begin{bmatrix}-3\\1\\0\\0\end{bmatrix}$, which is an affine subspace (a line not through the origin) obtained by translating the null space.

2. **Rank-Nullity Theorem and consistency**:
   - In part (a), we determined that $\text{rank}(A) = 3$ and $\text{nullity}(A) = 1$, which satisfies the Rank-Nullity Theorem: $\text{rank}(A) + \text{nullity}(A) = n = 4$.
   - The fact that $\text{rank}(A) = m = 3$ (the number of rows of A) means A has full row rank, which guarantees that the system $A\vec{x} = \vec{b}$ is consistent for any $\vec{b} \in \mathbb{R}^3$, with one exception revealed in part (c).
3. **Consistency conditions**:
   - In part (c), we discovered that the system $A\vec{x} = \begin{bmatrix}2\\k\\3\end{bmatrix}$ is consistent if and only if $k \neq 5$.
   - This demonstrates the general result that a non-homogeneous system $A\vec{x} = \vec{b}$ is consistent if and only if $\text{rank}([A|\vec{b}]) = \text{rank}(A)$.
   - When $k = 5$, the augmented matrix $[A|\vec{b}]$ has rank 4, which is greater than $\text{rank}(A) = 3$, making the system inconsistent.
4. **Dimension of solution spaces**:
   - The dimension of the solution space for a homogeneous system equals the nullity of the coefficient matrix, which in this case is 1.
   - The solution set of a consistent non-homogeneous system has the same "shape" as the null space but is translated away from the origin by a particular solution.

These examples illustrate the fundamental relationship between homogeneous and non-homogeneous systems: if $\vec{p}$ is a particular solution to the non-homogeneous system $A\vec{x} = \vec{b}$, then the general solution is $\vec{x} = \vec{p} + \vec{x}_h$, where $\vec{x}_h$ is the general solution to the corresponding homogeneous system $A\vec{x} = \vec{0}$.

7.3 Linear Independence and Dependence

One of the most fundamental concepts in vector spaces is the notion of linear independence. It allows us to identify the essential vectors that generate a vector space and eliminate redundancy in vector representations.

7.3.1 Definitions and Geometric Interpretation

Definition 7.32. *A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is* ***linearly dependent*** *if there exist scalars $c_1, c_2, \ldots, c_k$, not all zero, such that:*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \tag{7.74}$$

A set of vectors that is not linearly dependent is called ***linearly independent****.*

In other words, a set of vectors is linearly independent if the only solution to the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ is the trivial solution where all $c_i = 0$.

Example: Linear Dependence in $\mathbb{R}^2$

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ in $\mathbb{R}^2$.

We can see that $\mathbf{v}_2 = 2\mathbf{v}_1$, which means we can write $2\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$. Since we have found a non-trivial linear combination that equals the zero vector, these vectors are linearly dependent.

Example: Linear Independence in $\mathbb{R}^2$

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$.

To check if these are linearly dependent, we need to determine if there exist scalars c_1 and c_2, not both zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$.
This gives us:

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7.75}$$

Which yields the system:

$$c_1 = 0 \tag{7.76}$$
$$c_2 = 0 \tag{7.77}$$

Since the only solution is $c_1 = c_2 = 0$, these vectors are linearly independent.

Geometric Interpretation:

In $\mathbb{R}^2$ and $\mathbb{R}^3$, linear independence has clear geometric interpretations:

- In $\mathbb{R}^2$:
  - Two vectors are linearly independent if and only if neither is a scalar multiple of the other (they don't lie on the same line through the origin).
- In $\mathbb{R}^3$:
  - Two vectors are linearly independent if and only if they don't lie on the same line through the origin.
  - Three vectors are linearly independent if and only if they don't lie in the same plane through the origin.

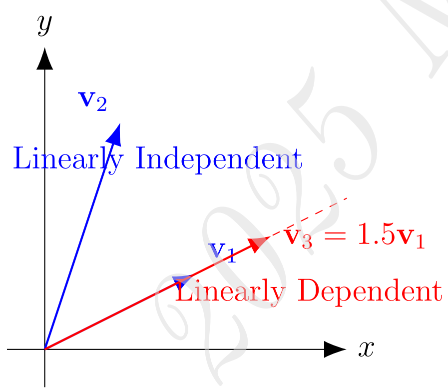


Figure 7.1: Geometric illustration of linear independence and dependence in $\mathbb{R}^2$

Property 7.33. *Important properties of linear independence:*

1. *The empty set* $\{\}$ *is linearly independent by convention.*

2. *A set containing the zero vector* $\{\mathbf{0}, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is always linearly dependent.*
3. *If a set of vectors is linearly dependent, then at least one vector in the set can be expressed as a linear combination of the others.*
4. *Adding a vector to a linearly independent set may result in either a linearly independent or dependent set.*
5. *Removing a vector from a linearly dependent set may result in either a linearly independent or dependent set.*

7.3.2 Testing for Linear Independence

To test whether a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, we need to determine if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \tag{7.78}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$.

Matrix Method: We can formulate this as a matrix equation. Let A be the matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Then we need to solve the homogeneous system:

$$A\mathbf{c} = \mathbf{0} \tag{7.79}$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_k)^T$.

Theorem 7.34. *A set of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is linearly independent if and only if the matrix* $A = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_k]$ *has full column rank, i.e.,* $\text{rank}(A) = k$.

Example: Testing Linear Independence Using Matrices

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ in $\mathbb{R}^3$.

We form the matrix:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 3 & 7 & 2 \end{pmatrix}$$

We can row reduce this matrix to determine its rank:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 3 & 7 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the rank of A is 2, which is less than the number of vectors (3), the set of vectors is linearly dependent.

Problem: Determine Linear Independence

Determine whether the following set of vectors is linearly independent or dependent in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 7 \\ 1 \end{pmatrix}$$

Solution

We form the matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & 7 \\ 1 & 0 & 1 \end{pmatrix}$$

Row reducing this matrix:

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & 7 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the rank of A is 3, which equals the number of vectors, the set is linearly independent.

Determinant Method for Square Matrices: When the number of vectors equals the dimension of the space (i.e., we have n vectors in $\mathbb{R}^n$), we can use the determinant:

Theorem 7.35. *A set of n vectors in $\mathbb{R}^n$ is linearly independent if and only if the determinant of the matrix formed by these vectors as columns is non-zero.*

Example: Using Determinant to Test Independence

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$ in $\mathbb{R}^3$.

The determinant is:

$$\det \begin{pmatrix} 2 & 1 & 4 \\ 1 & -1 & 0 \\ 3 & 2 & 5 \end{pmatrix} = 2 \cdot (-1) \cdot 5 + 1 \cdot 0 \cdot 3 + 4 \cdot 1 \cdot 2 - 4 \cdot (-1) \cdot 3 - 2 \cdot 0 \cdot 2 - 5 \cdot 1 \cdot 1$$

$$= -10 + 0 + 8 + 12 - 0 - 5 = 5$$

Since the determinant is non-zero, the set of vectors is linearly independent.

7.3.3 Relation to Systems of Equations

Linear independence is closely related to systems of linear equations in several important ways.

Theorem 7.36. *Consider a homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix. The columns of A are linearly dependent if and only if the system has non-trivial solutions.*

This connection provides another method for testing linear independence:

Algorithm: Testing Linear Independence via System of Equations

1. Form the matrix A with the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ as its columns.
2. Solve the homogeneous system $A\mathbf{x} = \mathbf{0}$.
3. If the system has only the trivial solution $\mathbf{x} = \mathbf{0}$, then the vectors are linearly independent.
4. If the system has non-trivial solutions, then the vectors are linearly dependent, and the non-zero entries in a non-trivial solution indicate which vectors participate in a dependence relation.

Example: Finding Dependence Relations

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 7 \\ -2 \\ 5 \end{pmatrix}$ in $\mathbb{R}^3$.
To determine if they are linearly dependent and find potential dependency relations, we solve:

$$c_1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 7 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives us the system:

$$c_1 + 3c_2 + 7c_3 = 0 \tag{7.80}$$
$$-c_1 + 0c_2 - 2c_3 = 0 \tag{7.81}$$
$$2c_1 + c_2 + 5c_3 = 0 \tag{7.82}$$

Converting this to a matrix and row reducing:

$$\begin{pmatrix} 1 & 3 & 7 \\ -1 & 0 & -2 \\ 2 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 7 \\ 0 & 3 & 5 \\ 0 & -5 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 7 \\ 0 & 3 & 5 \\ 0 & 0 & -2/3 \end{pmatrix}$$

This system has only the trivial solution $c_1 = c_2 = c_3 = 0$, so the vectors are linearly independent.

Application: Consistency of Non-homogeneous Systems

Linear independence plays a crucial role in determining when a non-homogeneous system $A\mathbf{x} = \mathbf{b}$ has a solution.
Let A be an $m \times n$ matrix and consider the augmented matrix $[A|\mathbf{b}]$. The system $A\mathbf{x} = \mathbf{b}$ is consistent (has at least one solution) if and only if $\text{rank}(A) = \text{rank}([A|\mathbf{b}])$.
When $\mathbf{b}$ is not in the column space of A, the system is inconsistent because $\mathbf{b}$ cannot be expressed as a linear combination of the columns of A.

Theorem 7.37 (Fundamental Theorem on Linear Systems)**.** *For a system of linear equations* $A\mathbf{x} = \mathbf{b}$*:*

1. *The system has a unique solution if and only if the columns of A are linearly independent and their number equals the number of unknowns.*
2. *The system has infinitely many solutions if and only if either the columns of A are linearly dependent or their number is greater than the number of unknowns, and* $\mathbf{b}$ *is in the column space of A.*
3. *The system has no solution if and only if* $\mathbf{b}$ *is not in the column space of A.*

The Connection to Basis: Linear independence is a key requirement for a set of vectors to form a basis of a vector space, which we will explore in the next section.

7.3.4 Additional Solved Examples

Example 1: Linear Combination Using Row Echelon Form

Show that the vector $\mathbf{w} = (9, 2, 7)$ is a linear combination of the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in $\mathbb{R}^3$.

Detailed Solution

To determine if $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, we need to find scalars a and b such that:

$$\mathbf{w} = a\mathbf{u} + b\mathbf{v} \tag{7.83}$$

This is equivalent to solving the system:

$$a\mathbf{u} + b\mathbf{v} = \mathbf{w} \tag{7.84}$$

We can represent this as a matrix equation. Let's form a matrix A with columns $\mathbf{u}$ and $\mathbf{v}$, and we want to find a vector $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ such that $A\mathbf{x} = \mathbf{w}$.

Step 1: Form the augmented matrix $[A|\mathbf{w}]$:

$$\begin{bmatrix} 1 & 6 & | & 9 \\ 2 & 4 & | & 2 \\ -1 & 2 & | & 7 \end{bmatrix} \tag{7.85}$$

Step 2: Use Gaussian elimination to convert this to row echelon form:
First row operation - no change needed as first element is already 1:

$$\begin{bmatrix} 1 & 6 & | & 9 \\ 2 & 4 & | & 2 \\ -1 & 2 & | & 7 \end{bmatrix} \tag{7.86}$$

Eliminate the first column below the first row: $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 + R_1$

$$\begin{bmatrix} 1 & 6 & | & 9 \\ 0 & -8 & | & -16 \\ 0 & 8 & | & 16 \end{bmatrix} \tag{7.87}$$

Make the second pivot: $R_2 \leftarrow -\frac{1}{8}R_2$

$$\begin{bmatrix} 1 & 6 & | & 9 \\ 0 & 1 & | & 2 \\ 0 & 8 & | & 16 \end{bmatrix} \tag{7.88}$$

Eliminate the second column below the second row: $R_3 \leftarrow R_3 - 8R_2$

$$\begin{bmatrix} 1 & 6 & | & 9 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{7.89}$$

Step 3: Convert to reduced row echelon form: $R_1 \leftarrow R_1 - 6R_2$

$$\begin{bmatrix} 1 & 0 & | & -3 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{7.90}$$

Step 4: Analyze the result:
From the row echelon form, we can see that:

$$a = -3 \tag{7.91}$$
$$b = 2 \tag{7.92}$$

The system is consistent because the rank of the coefficient matrix equals the rank of the augmented matrix:

$$\text{rank}(A) = \text{rank}([A|\mathbf{w}]) = 2 \tag{7.93}$$

This means $\mathbf{w}$ is in the column space of A, or equivalently, $\mathbf{w}$ is a linear combination of the columns of A.
Step 5: Verify our solution:

$$a\mathbf{u} + b\mathbf{v} = (-3)(1, 2, -1) + (2)(6, 4, 2) \tag{7.94}$$
$$= (-3, -6, 3) + (12, 8, 4) \tag{7.95}$$
$$= (9, 2, 7) \tag{7.96}$$
$$= \mathbf{w} \quad \checkmark \tag{7.97}$$

Therefore, $\mathbf{w} = (9, 2, 7)$ can be expressed as the linear combination $\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$.
Conclusion: This example demonstrates that a vector is in the span of a set of vectors if and only if the augmented matrix $[A|\mathbf{w}]$ has the same rank as the coefficient matrix A.

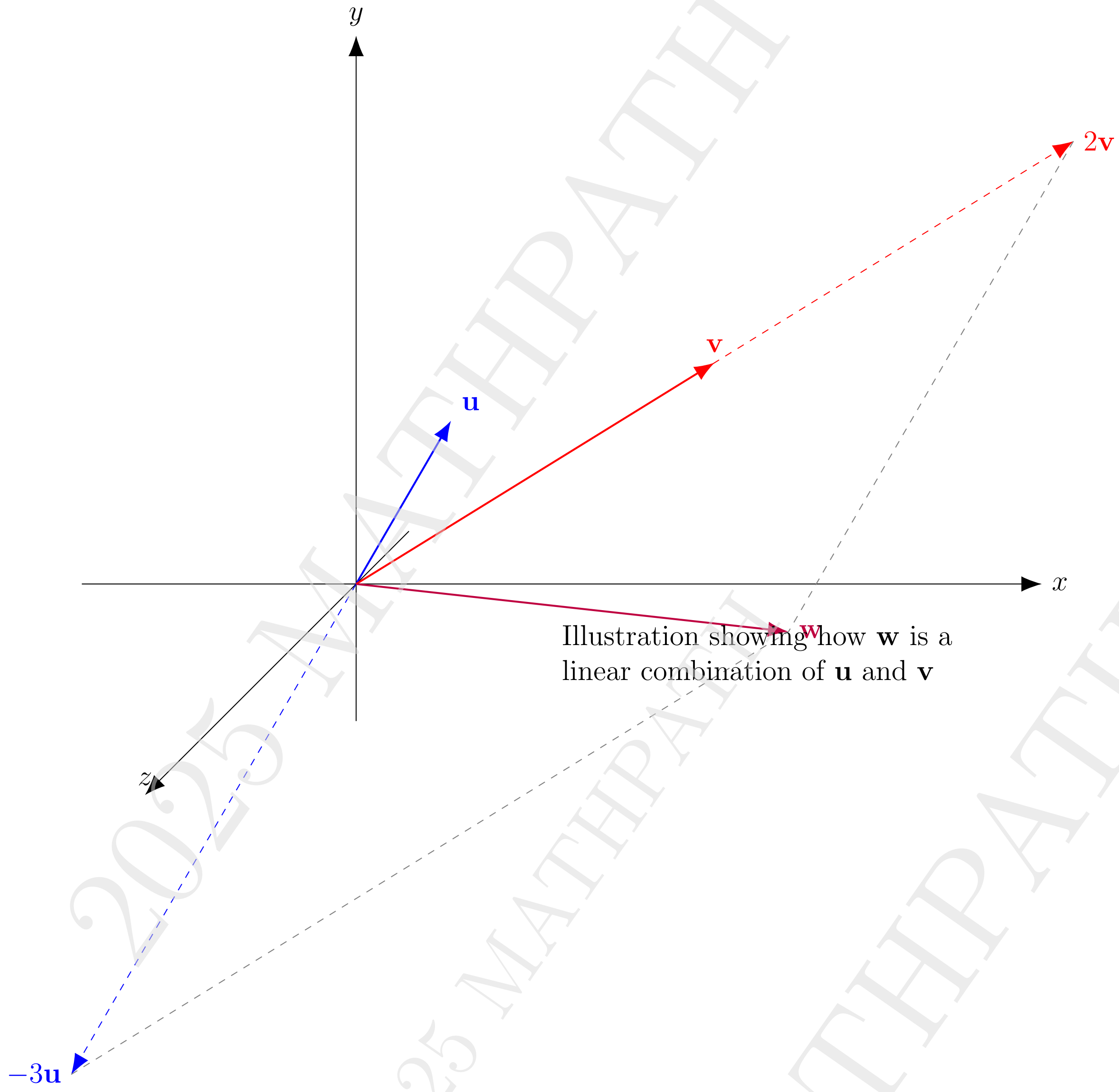


Figure 7.2: Geometric interpretation of $\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$

Example 2: Linear Combination Using Matrix Rank Analysis

Express the vector $\mathbf{v} = (7, 4, -3)$ as a linear combination of the vectors $\mathbf{v}_1 = (1, -2, -5)$ and $\mathbf{v}_2 = (2, 5, 6)$ in $\mathbb{R}^3$.

Detailed Solution

We need to determine if there exist scalars c_1 and c_2 such that:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \tag{7.98}$$

Step 1: Formulate this as a matrix equation $A\mathbf{x} = \mathbf{v}$ where $A = [\mathbf{v}_1\ \mathbf{v}_2]$ and $\mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

Create the augmented matrix $[A|\mathbf{v}]$:

$$\begin{bmatrix} 1 & 2 & | & 7 \\ -2 & 5 & | & 4 \\ -5 & 6 & | & -3 \end{bmatrix} \tag{7.99}$$

Step 2: Apply row operations to transform the matrix into row echelon form.
First row operation - no change needed as the first element is already 1:

$$\begin{bmatrix} 1 & 2 & | & 7 \\ -2 & 5 & | & 4 \\ -5 & 6 & | & -3 \end{bmatrix} \tag{7.100}$$

Eliminate the first column below the first row: $R_2 \leftarrow R_2 + 2R_1$ and $R_3 \leftarrow R_3 + 5R_1$

$$\begin{bmatrix} 1 & 2 & | & 7 \\ 0 & 9 & | & 18 \\ 0 & 16 & | & 32 \end{bmatrix} \tag{7.101}$$

Make the second pivot: $R_2 \leftarrow \frac{1}{9}R_2$

$$\begin{bmatrix} 1 & 2 & | & 7 \\ 0 & 1 & | & 2 \\ 0 & 16 & | & 32 \end{bmatrix} \tag{7.102}$$

Eliminate the second column below the second row: $R_3 \leftarrow R_3 - 16R_2$

$$\begin{bmatrix} 1 & 2 & | & 7 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{7.103}$$

Step 3: Convert to reduced row echelon form: $R_1 \leftarrow R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{7.104}$$

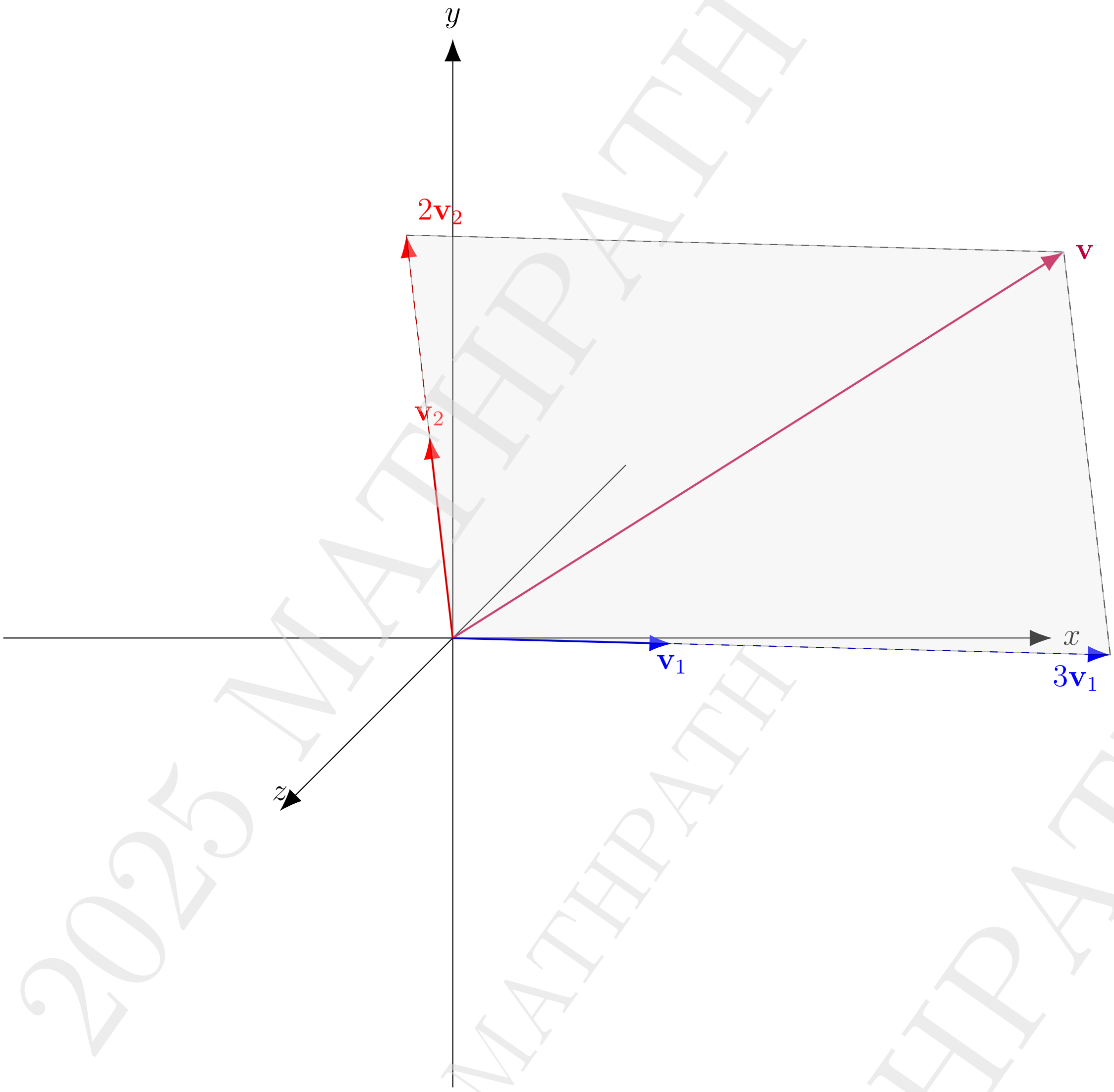
Step 4: Analyze the result and determine the parameters:
From the reduced row echelon form, we read:

$$c_1 = 3 \tag{7.105}$$
$$c_2 = 2 \tag{7.106}$$

Observe that $\text{rank}(A) = \text{rank}([A|\mathbf{v}]) = 2$, which confirms that $\mathbf{v}$ lies in the column space of A, meaning it can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
Step 5: Verify our solution:

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= 3(1,-2,-5) + 2(2,5,6) && (7.107) \\ &= (3,-6,-15) + (4,10,12) && (7.108) \\ &= (7,4,-3) && (7.109) \\ &= \mathbf{v} \quad \checkmark && (7.110) \end{aligned}$$

Therefore, $\mathbf{v} = 3\mathbf{v}_1 + 2\mathbf{v}_2$.
Geometric Interpretation: The vector $\mathbf{v}$ lies in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. If we consider the parallelogram formed by $3\mathbf{v}_1$ and $2\mathbf{v}_2$, the vector $\mathbf{v}$ is precisely the diagonal of this parallelogram.



Geometric illustration of $\mathbf{v} = 3\mathbf{v}_1 + 2\mathbf{v}_2$,
showing how $\mathbf{v}$ lies in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$

Figure 7.3: Geometric representation of the linear combination $\mathbf{v} = 3\mathbf{v}_1 + 2\mathbf{v}_2$

Remark 7.38. *This example demonstrates an important concept: When vectors in $\mathbb{R}^3$ are linearly independent but don't form a basis (since we only have two vectors), they span a plane through the origin. Any vector that lies in this plane can be expressed as a linear combination of these vectors. The fact that the rank of the augmented matrix equals the rank of the coefficient matrix (both equal to 2) confirms that* $\mathbf{v}$ *lies in this plane.*

Example 3: Testing Linear Combinations

Let $\mathbf{v}_1 = (1, 2, 3)$ and $\mathbf{v}_2 = (0, -1, -2)$ be vectors in $\mathbb{R}^3$. Show that:

1. $\mathbf{u} = (-1, -4, -7)$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$
2. $\mathbf{v} = (4, 7, 11)$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$

Detailed Solution

We'll use the matrix row reduction approach to determine whether each vector can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Part (1): Show that $\mathbf{u} = (-1, -4, -7)$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Step 1: Set up the augmented matrix with columns $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{u}$:

$$\begin{bmatrix} 1 & 0 & | & -1 \\ 2 & -1 & | & -4 \\ 3 & -2 & | & -7 \end{bmatrix} \tag{7.111}$$

Step 2: Apply row operations to transform the matrix into row echelon form.
First row operation - no change needed as the first element is already 1:

$$\begin{bmatrix} 1 & 0 & | & -1 \\ 2 & -1 & | & -4 \\ 3 & -2 & | & -7 \end{bmatrix} \tag{7.112}$$

Eliminate the first column below the first row: $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & | & -1 \\ 0 & -1 & | & -2 \\ 0 & -2 & | & -4 \end{bmatrix} \tag{7.113}$$

Make the second pivot (multiply by -1 to get a leading 1): $R_2 \leftarrow -R_2$

$$\begin{bmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 2 \\ 0 & -2 & | & -4 \end{bmatrix} \tag{7.114}$$

Eliminate the second column below the second row: $R_3 \leftarrow R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{7.115}$$

Step 3: Convert to reduced row echelon form (already done in this case).
Step 4: Analyze the result and determine the parameters:
From the reduced row echelon form, we read:

$$c_1 = -1 \tag{7.116}$$
$$c_2 = 2 \tag{7.117}$$

The system is consistent because the last row is all zeros, indicating that $\text{rank}(A) = \text{rank}([A|\mathbf{u}]) = 2$. This confirms that $\mathbf{u}$ lies in the column space of A, meaning it can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
Step 5: Verify our solution:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = (-1)(1,2,3) + 2(0,-1,-2) \tag{7.118}$$
$$= (-1,-2,-3) + (0,-2,-4) \tag{7.119}$$
$$= (-1,-4,-7) \tag{7.120}$$
$$= \mathbf{u} \quad \checkmark \tag{7.121}$$

Therefore, $\mathbf{u} = -1\mathbf{v}_1 + 2\mathbf{v}_2$.
Part (2): Show that $\mathbf{v} = (4, 7, 11)$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
Step 1: Set up the augmented matrix with columns $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}$:

$$\begin{bmatrix} 1 & 0 & | & 4 \\ 2 & -1 & | & 7 \\ 3 & -2 & | & 11 \end{bmatrix} \tag{7.122}$$

Step 2: Apply row operations to transform the matrix into row echelon form.
Eliminate the first column below the first row: $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & | & 4 \\ 0 & -1 & | & -1 \\ 0 & -2 & | & -1 \end{bmatrix} \tag{7.123}$$

Make the second pivot: $R_2 \leftarrow -R_2$

$$\begin{bmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & 1 \\ 0 & -2 & | & -1 \end{bmatrix} \tag{7.124}$$

Eliminate the second column below the second row: $R_3 \leftarrow R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & 1 \\ 0 & 0 & | & 1 \end{bmatrix} \tag{7.125}$$

Step 3: Analyze the result.
The last row of the augmented matrix reads $0 \cdot c_1 + 0 \cdot c_2 = 1$, which is a contradiction. This means the system is inconsistent and has no solution.
The rank of the coefficient matrix is $\text{rank}(A) = 2$, while the rank of the augmented matrix is $\text{rank}([A|\mathbf{v}]) = 3$. Since $\text{rank}(A) \neq \text{rank}([A|\mathbf{v}])$, the system is inconsistent, confirming that $\mathbf{v}$ is not in the column space of A.
Therefore, $\mathbf{v} = (4, 7, 11)$ cannot be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
Geometric Interpretation: The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ span a 2-dimensional subspace (a plane) in $\mathbb{R}^3$. The vector $\mathbf{v} = (4, 7, 11)$ lies outside this plane, which is why it cannot be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
To further illustrate, let's try to find what values of c_1 and c_2 would come closest to representing $\mathbf{v}$. From the first two rows of our reduced system, we get:

$$c_1 = 4 \tag{7.126}$$
$$c_2 = 1 \tag{7.127}$$

Let's see what vector these coefficients produce:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 4(1, 2, 3) + 1(0, -1, -2) \tag{7.128}$$
$$= (4, 8, 12) + (0, -1, -2) \tag{7.129}$$
$$= (4, 7, 10) \tag{7.130}$$

This vector $(4, 7, 10)$ is the projection of $\mathbf{v} = (4, 7, 11)$ onto the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. The difference $(0, 0, 1)$ represents how far $\mathbf{v}$ is from the plane.
Conclusion:

1. $\mathbf{u} = (-1, -4, -7)$ is a linear combination: $\mathbf{u} = -1\mathbf{v}_1 + 2\mathbf{v}_2$
2. $\mathbf{v} = (4, 7, 11)$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, as proven by the inconsistent system of equations.

Example 4: Linear Combinations in Vector Space of Polynomials

Express $P(x) = 7 + 8x + 9x^2$ as a linear combination of $P_1 = 2 + x + 4x^2$, $P_2 = 1 - x + 3x^2$, and $P_3 = 2 + x + 5x^2$.

Detailed Solution

To express $P(x)$ as a linear combination of P_1, P_2, and P_3, we need to find scalars c_1, c_2, and c_3 such that:

$$P(x) = c_1P_1(x) + c_2P_2(x) + c_3P_3(x) \tag{7.131}$$

Let's first recognize that polynomials form a vector space, where the coefficients of each power of x can be treated separately. In this case, we can set up a system of linear equations by comparing the coefficients of each power of x on both sides.

Step 1: Write out the linear combination in terms of coefficients:

$$7 + 8x + 9x^2 = c_1(2 + x + 4x^2) + c_2(1 - x + 3x^2) + c_3(2 + x + 5x^2) \tag{7.132}$$

$$= (2c_1 + c_2 + 2c_3) + (c_1 - c_2 + c_3)x + (4c_1 + 3c_2 + 5c_3)x^2 \tag{7.133}$$

Step 2: Compare coefficients to obtain a system of linear equations:

$$2c_1 + c_2 + 2c_3 = 7 \quad \text{(constant term)} \tag{7.134}$$

$$c_1 - c_2 + c_3 = 8 \quad \text{(coefficient of } x\text{)} \tag{7.135}$$

$$4c_1 + 3c_2 + 5c_3 = 9 \quad \text{(coefficient of } x^2\text{)} \tag{7.136}$$

Step 3: Set up the augmented matrix for this system:

$$\begin{bmatrix} 2 & 1 & 2 & | & 7 \\ 1 & -1 & 1 & | & 8 \\ 4 & 3 & 5 & | & 9 \end{bmatrix} \tag{7.137}$$

Step 4: Apply row operations to transform the matrix into row echelon form.
First, let's make the first entry a 1 (it's already done):

$$\begin{bmatrix} 2 & 1 & 2 & | & 7 \\ 1 & -1 & 1 & | & 8 \\ 4 & 3 & 5 & | & 9 \end{bmatrix} \tag{7.138}$$

Let's swap the first and second rows to have 1 as the first pivot:

$$\begin{bmatrix} 1 & -1 & 1 & | & 8 \\ 2 & 1 & 2 & | & 7 \\ 4 & 3 & 5 & | & 9 \end{bmatrix} \tag{7.139}$$

Eliminate the first column below the first row: $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - 4R_1$

$$\begin{bmatrix} 1 & -1 & 1 & | & 8 \\ 0 & 3 & 0 & | & -9 \\ 0 & 7 & 1 & | & -23 \end{bmatrix} \tag{7.140}$$

Make the second pivot a 1: $R_2 \leftarrow \frac{1}{3}R_2$

$$\begin{bmatrix} 1 & -1 & 1 & | & 8 \\ 0 & 1 & 0 & | & -3 \\ 0 & 7 & 1 & | & -23 \end{bmatrix} \tag{7.141}$$

Eliminate the second column below the second row: $R_3 \leftarrow R_3 - 7R_2$

$$\begin{bmatrix} 1 & -1 & 1 & | & 8 \\ 0 & 1 & 0 & | & -3 \\ 0 & 0 & 1 & | & -2 \end{bmatrix} \tag{7.142}$$

The matrix is now in row echelon form, and we can see that the system has a unique solution.

Step 5: Convert to reduced row echelon form to find the values of c_1, c_2, and c_3.
Eliminate the second column above the second row: $R_1 \leftarrow R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 1 & | & 5 \\ 0 & 1 & 0 & | & -3 \\ 0 & 0 & 1 & | & -2 \end{bmatrix} \tag{7.143}$$

Eliminate the third column above the third row: $R_1 \leftarrow R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & | & 7 \\ 0 & 1 & 0 & | & -3 \\ 0 & 0 & 1 & | & -2 \end{bmatrix} \tag{7.144}$$

Step 6: Read off the solution:

$$c_1 = 7 \tag{7.145}$$
$$c_2 = -3 \tag{7.146}$$
$$c_3 = -2 \tag{7.147}$$

Step 7: Verify our solution:

$$c_1P_1(x) + c_2P_2(x) + c_3P_3(x) = 7(2 + x + 4x^2) + (-3)(1 - x + 3x^2) + (-2)(2 + x + 5x^2) \tag{7.148}$$
$$= (14 + 7x + 28x^2) + (-3 + 3x - 9x^2) + (-4 - 2x - 10x^2) \tag{7.149}$$
$$= 14 - 3 - 4 + 7x + 3x - 2x + 28x^2 - 9x^2 - 10x^2 \tag{7.150}$$
$$= 7 + 8x + 9x^2 \tag{7.151}$$
$$= P(x) \quad \checkmark \tag{7.152}$$

Therefore, $P(x) = 7P_1(x) - 3P_2(x) - 2P_3(x)$.

Remark 7.39. *This example illustrates a key concept: the vector space of polynomials. Just as we can express vectors in $\mathbb{R}^n$ as linear combinations of other vectors, we can express polynomials as linear combinations of other polynomials. The coefficients of the powers of x serve as the "components" of the polynomial vector.*

The approach used here generalizes to any vector space. We:

1. *Identify the "components" of the vectors (in this case, the coefficients of different powers of x)*
2. *Set up a system of linear equations by equating corresponding components*
3. *Solve the system using row reduction to find the coefficients of the linear combination*

This method works because polynomials form a vector space with operations of addition and scalar multiplication, satisfying all the vector space axioms.

Example 5: Linear Combinations in the Space of Matrices

Which of the following are linear combinations of $A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ -2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$?

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Detailed Solution

In this problem, we are working in the vector space of 2×2 matrices. To determine if a matrix is a linear combination of the given matrices, we need to check if there exist scalars c_1, c_2, and c_3 such that:

$$c_1A + c_2B + c_3C = D \tag{7.153}$$

where D is the matrix we are testing.

Part (a): The zero matrix is trivially a linear combination of any set of matrices (with all coefficients being zero), so the answer is yes, the zero matrix is a linear combination of A, B, and C.

Part (b): Is the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ a linear combination of A, B, and C?

We need to find values of c_1, c_2, and c_3 such that:

$$c_1 \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 0 \\ -2 & 4 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \tag{7.154}$$

$$\tag{7.155}$$

This expands to:

$$\begin{bmatrix} -c_1 + 2c_2 + c_3 & c_1 + c_3 \\ -2c_2 + c_3 & 2c_1 + 4c_2 - c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \tag{7.156}$$

Comparing corresponding entries gives us the system:

$$-c_1 + 2c_2 + c_3 = 1 \quad \text{(from top-left entry)} \tag{7.157}$$
$$c_1 + c_3 = 1 \quad \text{(from top-right entry)} \tag{7.158}$$
$$-2c_2 + c_3 = 2 \quad \text{(from bottom-left entry)} \tag{7.159}$$
$$2c_1 + 4c_2 - c_3 = 2 \quad \text{(from bottom-right entry)} \tag{7.160}$$

Let's set up the augmented matrix for this system:

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & -2 & 1 & 2 \\ 2 & 4 & -1 & 2 \end{array}\right] \tag{7.161}$$

Let's apply row operations to solve this system:

$R_1 \leftrightarrow R_2$ (swap rows 1 and 2)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -1 & 2 & 1 & 1 \\ 0 & -2 & 1 & 2 \\ 2 & 4 & -1 & 2 \end{array}\right] \tag{7.162}$$

Eliminate the first column: $R_2 \leftarrow R_2 + R_1$ and $R_4 \leftarrow R_4 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 2 & 2 & | & 2 \\ 0 & -2 & 1 & | & 2 \\ 0 & 4 & -3 & | & 0 \end{bmatrix} \tag{7.163}$$

Make the second pivot a 1: $R_2 \leftarrow \frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & -2 & 1 & | & 2 \\ 0 & 4 & -3 & | & 0 \end{bmatrix} \tag{7.164}$$

Eliminate the second column: $R_3 \leftarrow R_3 + 2R_2$ and $R_4 \leftarrow R_4 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 3 & | & 4 \\ 0 & 0 & -7 & | & -4 \end{bmatrix} \tag{7.165}$$

Make the third pivot a 1: $R_3 \leftarrow \frac{1}{3}R_3$

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & \frac{4}{3} \\ 0 & 0 & -7 & | & -4 \end{bmatrix} \tag{7.166}$$

Eliminate the third column: $R_1 \leftarrow R_1 - R_3$, $R_2 \leftarrow R_2 - R_3$, and $R_4 \leftarrow R_4 + 7R_3$

$$\begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{3} \\ 0 & 1 & 0 & | & -\frac{1}{3} \\ 0 & 0 & 1 & | & \frac{4}{3} \\ 0 & 0 & 0 & | & \frac{28}{3} - 4 \end{bmatrix} \tag{7.167}$$

Simplifying the last row:

$$\begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{3} \\ 0 & 1 & 0 & | & -\frac{1}{3} \\ 0 & 0 & 1 & | & \frac{4}{3} \\ 0 & 0 & 0 & | & \frac{28}{3} - \frac{12}{3} = \frac{16}{3} \end{bmatrix} \tag{7.168}$$

The last row indicates that $0 = \frac{16}{3}$, which is a contradiction. Therefore, the system is inconsistent and has no solution.

This means that $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ cannot be expressed as a linear combination of A, B, and C.

Conclusion:

1. The zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a linear combination of A, B, and C (trivially, with all coefficients being zero).
2. The matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not a linear combination of A, B, and C.

Remark 7.40. *The key insight is that to determine if a matrix is a linear combination of other* *matrices, we need to set up and solve a system of linear equations. The number of equations will be equal to the number of entries in each matrix.*

Also note that the zero matrix is always in the span of any set of matrices, as it can be represented by taking all coefficients to be zero. However, this doesn't tell us anything about the linear independence of the original set of matrices.

Example 6: Testing for Linear Dependence

Test the following vectors for linear dependence and find a relation between them if dependent:
$\mathbf{v}_1 = (2, -1, 3, 2), \mathbf{v}_2 = (1, 3, 4, 2), \mathbf{v}_3 = (3, -5, 2, 2)$

Detailed Solution

To test whether the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, we need to determine if there exist scalars c_1, c_2, c_3, not all zero, such that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \tag{7.169}$$

This gives us the homogeneous system:

$$c_1 \begin{pmatrix} 2 \\ -1 \\ 3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ -5 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.170}$$

Which expands to:

$$2c_1 + c_2 + 3c_3 = 0 \tag{7.171}$$
$$-c_1 + 3c_2 - 5c_3 = 0 \tag{7.172}$$
$$3c_1 + 4c_2 + 2c_3 = 0 \tag{7.173}$$
$$2c_1 + 2c_2 + 2c_3 = 0 \tag{7.174}$$

We can express this as a matrix equation $A\mathbf{c} = \mathbf{0}$, where:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \tag{7.175}$$

Step 1: We'll apply row operations to find the rank of matrix A.
Starting with:

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \tag{7.176}$$

$R_1 \leftrightarrow \frac{1}{2}R_1$ (Scale the first row to make the leading entry 1):

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \tag{7.177}$$

$R_2 \leftarrow R_2 + R_1$, $R_3 \leftarrow R_3 - 3R_1$, $R_4 \leftarrow R_4 - 2R_1$ (Eliminate first column):

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{7}{2} & -\frac{7}{2} \\ 0 & \frac{5}{2} & -\frac{5}{2} \\ 0 & 1 & -1 \end{bmatrix} \tag{7.178}$$

$R_2 \leftrightarrow R_4$ (Swap rows to get a leading 1 in the second position):

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -1 \\ 0 & \frac{5}{2} & -\frac{5}{2} \\ 0 & \frac{7}{2} & -\frac{7}{2} \end{bmatrix} \tag{7.179}$$

$R_3 \leftarrow R_3 - \frac{5}{2}R_2$, $R_4 \leftarrow R_4 - \frac{7}{2}R_2$ (Eliminate second column):

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.180}$$

Step 2: Analyze the rank and dependence.
The matrix A has rank 2, which is less than the number of vectors (3). Therefore, the vectors are linearly dependent.
Step 3: Find the linear dependence relation.
From the row echelon form, we can express the system as:

$$c_1 + \frac{1}{2}c_2 + \frac{3}{2}c_3 = 0 \quad (1) \tag{7.181}$$

$$c_2 - c_3 = 0 \quad (2) \tag{7.182}$$

From equation (2), we get $c_2 = c_3$.
Substituting this into equation (1):

$$c_1 + \frac{1}{2}c_3 + \frac{3}{2}c_3 = 0 \tag{7.183}$$

$$c_1 + 2c_3 = 0 \tag{7.184}$$

$$c_1 = -2c_3 \tag{7.185}$$

Setting $c_3 = 1$ gives us $c_1 = -2$ and $c_2 = 1$.
Therefore, the linear dependence relation is:

$$-2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0} \tag{7.186}$$

Rearranging to solve for $\mathbf{v}_1$:

$$-2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0} \tag{7.187}$$

$$-2\mathbf{v}_1 = -\mathbf{v}_2 - \mathbf{v}_3 \tag{7.188}$$

$$\mathbf{v}_1 = \frac{1}{2}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 \tag{7.189}$$

$$2\mathbf{v}_1 = \mathbf{v}_2 + \mathbf{v}_3 \tag{7.190}$$

Step 4: Verify our result.
Let's substitute the actual vectors:

$$2(2, -1, 3, 2) = (1, 3, 4, 2) + (3, -5, 2, 2) \tag{7.191}$$

$$(4, -2, 6, 4) = (4, -2, 6, 4) \quad \checkmark \tag{7.192}$$

Indeed, $2\mathbf{v}_1 = \mathbf{v}_2 + \mathbf{v}_3$ is the correct linear dependence relation.

Example 7: Testing for Linear Dependence

Test the following vectors for linear dependence and find a relation between them if dependent:
$\mathbf{v}_1 = (1, 2, 4), \mathbf{v}_2 = (2, -1, 3), \mathbf{v}_3 = (0, 1, 2), \mathbf{v}_4 = (-3, 7, 2)$

Detailed Solution

To test whether the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent, we need to determine if there exist scalars c_1, c_2, c_3, c_4, not all zero, such that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0} \tag{7.193}$$

This gives us the homogeneous system:

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_4 \begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.194}$$

Which expands to:

$$c_1 + 2c_2 + 0c_3 - 3c_4 = 0 \tag{7.195}$$
$$2c_1 - c_2 + c_3 + 7c_4 = 0 \tag{7.196}$$
$$4c_1 + 3c_2 + 2c_3 + 2c_4 = 0 \tag{7.197}$$

We can express this as a matrix equation $A\mathbf{c} = \mathbf{0}$, where:

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \tag{7.198}$$

$R_2 \leftarrow R_2 - 2R_1$:

$$R_2 = (2, -1, 1, 7) - 2(1, 2, 0, -3) \tag{7.199}$$
$$= (2, -1, 1, 7) - (2, 4, 0, -6) \tag{7.200}$$
$$= (0, -5, 1, 13) \tag{7.201}$$

$R_3 \leftarrow R_3 - 4R_1$:

$$R_3 = (4, 3, 2, 2) - 4(1, 2, 0, -3) \tag{7.202}$$
$$= (4, 3, 2, 2) - (4, 8, 0, -12) \tag{7.203}$$
$$= (0, -5, 2, 14) \tag{7.204}$$

So we have:

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \tag{7.205}$$

$R_2 \leftarrow -\frac{1}{5}R_2$:

$$R_2 = -\frac{1}{5}(0, -5, 1, 13) \tag{7.206}$$
$$= (0, 1, -\frac{1}{5}, -\frac{13}{5}) \tag{7.207}$$

So we have:

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & -\frac{1}{5} & -\frac{13}{5} \\ 0 & -5 & 2 & 14 \end{bmatrix} \tag{7.208}$$

$R_1 \leftarrow R_1 - 2R_2$:

$$R_1 = (1, 2, 0, -3) - 2(0, 1, -\frac{1}{5}, -\frac{13}{5}) \tag{7.209}$$
$$= (1, 2, 0, -3) - (0, 2, -\frac{2}{5}, -\frac{26}{5}) \tag{7.210}$$
$$= (1, 0, \frac{2}{5}, -3 + \frac{26}{5}) \tag{7.211}$$
$$= (1, 0, \frac{2}{5}, \frac{-15 + 26}{5}) \tag{7.212}$$
$$= (1, 0, \frac{2}{5}, \frac{11}{5}) \tag{7.213}$$

$R_3 \leftarrow R_3 + 5R_2$:

$$R_3 = (0, -5, 2, 14) + 5(0, 1, -\frac{1}{5}, -\frac{13}{5}) \tag{7.214}$$
$$= (0, -5, 2, 14) + (0, 5, -1, -13) \tag{7.215}$$
$$= (0, 0, 1, 1) \tag{7.216}$$

So we have:

$$\begin{bmatrix} 1 & 0 & \frac{2}{5} & \frac{11}{5} \\ 0 & 1 & -\frac{1}{5} & -\frac{13}{5} \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{7.217}$$

$R_1 \leftarrow R_1 - \frac{2}{5}R_3$:

$$R_1 = (1, 0, \frac{2}{5}, \frac{11}{5}) - \frac{2}{5}(0, 0, 1, 1) \tag{7.218}$$
$$= (1, 0, \frac{2}{5}, \frac{11}{5}) - (0, 0, \frac{2}{5}, \frac{2}{5}) \tag{7.219}$$
$$= (1, 0, 0, \frac{11 - 2}{5}) \tag{7.220}$$
$$= (1, 0, 0, \frac{9}{5}) \tag{7.221}$$

$R_2 \leftarrow R_2 + \frac{1}{5}R_3$:

$$R_2 = (0, 1, -\frac{1}{5}, -\frac{13}{5}) + \frac{1}{5}(0, 0, 1, 1) \tag{7.222}$$
$$= (0, 1, -\frac{1}{5}, -\frac{13}{5}) + (0, 0, \frac{1}{5}, \frac{1}{5}) \tag{7.223}$$
$$= (0, 1, 0, -\frac{13 - 1}{5}) \tag{7.224}$$
$$= (0, 1, 0, -\frac{12}{5}) \tag{7.225}$$

So we have:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{9}{5} \\ 0 & 1 & 0 & -\frac{12}{5} \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{7.226}$$

From this row echelon form, we can express the system as:

$$c_1 + \frac{9}{5}c_4 = 0 \quad (1) \tag{7.227}$$
$$c_2 - \frac{12}{5}c_4 = 0 \quad (2) \tag{7.228}$$
$$c_3 + c_4 = 0 \quad (3) \tag{7.229}$$

From these equations:

$$c_1 = -\frac{9}{5}c_4 \tag{7.230}$$
$$c_2 = \frac{12}{5}c_4 \tag{7.231}$$
$$c_3 = -c_4 \tag{7.232}$$

Setting $c_4 = 5$ to clear fractions:

$$c_1 = -\frac{9}{5} \cdot 5 = -9 \tag{7.233}$$
$$c_2 = \frac{12}{5} \cdot 5 = 12 \tag{7.234}$$
$$c_3 = -1 \cdot 5 = -5 \tag{7.235}$$
$$c_4 = 5 \tag{7.236}$$

Therefore, the linear dependence relation is:

$$-9\mathbf{v}_1 + 12\mathbf{v}_2 - 5\mathbf{v}_3 + 5\mathbf{v}_4 = \mathbf{0} \tag{7.237}$$

Step 5: Verify our result by direct calculation.
Let's verify by substituting the actual vectors:

$$-9\mathbf{v}_1 + 12\mathbf{v}_2 - 5\mathbf{v}_3 + 5\mathbf{v}_4 \tag{7.238}$$
$$= -9(1, 2, 4) + 12(2, -1, 3) - 5(0, 1, 2) + 5(-3, 7, 2) \tag{7.239}$$
$$= (-9, -18, -36) + (24, -12, 36) - (0, 5, 10) + (-15, 35, 10) \tag{7.240}$$
$$= (-9 + 24 + 0 - 15, -18 - 12 - 5 + 35, -36 + 36 - 10 + 10) \tag{7.241}$$
$$= (0, 0, 0) \quad \checkmark \tag{7.242}$$

The relation is verified. Therefore, the vectors are linearly dependent with the relation:

$$-9\mathbf{v}_1 + 12\mathbf{v}_2 - 5\mathbf{v}_3 + 5\mathbf{v}_4 = \mathbf{0} \tag{7.243}$$

Example 8: Testing for Linear Dependence

Test the following vectors for linear dependence and find a relation between them if dependent:
$\mathbf{v}_1 = (1, 1, -1, 1), \mathbf{v}_2 = (1, -1, 2, -1), \mathbf{v}_3 = (3, 1, 0, 1)$

Detailed Solution

To test whether the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, we need to determine if there exist scalars c_1, c_2, c_3, not all zero, such that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \tag{7.244}$$

This gives us the homogeneous system:

$$c_1 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 2 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.245}$$

Which expands to:

$$c_1 + c_2 + 3c_3 = 0 \tag{7.246}$$
$$c_1 - c_2 + c_3 = 0 \tag{7.247}$$
$$-c_1 + 2c_2 + 0c_3 = 0 \tag{7.248}$$
$$c_1 - c_2 + c_3 = 0 \tag{7.249}$$

We can express this as a matrix equation $A\mathbf{c} = \mathbf{0}$, where:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \tag{7.250}$$

Step 1: Apply row operations to transform the matrix into row echelon form. Starting with the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 3 & | & 0 \\ 1 & -1 & 1 & | & 0 \\ -1 & 2 & 0 & | & 0 \\ 1 & -1 & 1 & | & 0 \end{bmatrix} \tag{7.251}$$

$R_2 \leftarrow R_2 - R_1$, $R_3 \leftarrow R_3 + R_1$, $R_4 \leftarrow R_4 - R_1$ (Eliminate first column except for the pivot):

$$\begin{bmatrix} 1 & 1 & 3 & | & 0 \\ 0 & -2 & -2 & | & 0 \\ 0 & 3 & 3 & | & 0 \\ 0 & -2 & -2 & | & 0 \end{bmatrix} \tag{7.252}$$

$R_2 \leftarrow -\frac{1}{2}R_2$ (Scale second row to make the leading entry 1):

$$\begin{bmatrix} 1 & 1 & 3 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 3 & 3 & | & 0 \\ 0 & -2 & -2 & | & 0 \end{bmatrix} \tag{7.253}$$

$R_1 \leftarrow R_1 - R_2$, $R_3 \leftarrow R_3 - 3R_2$, $R_4 \leftarrow R_4 + 2R_2$ (Eliminate second column except for the pivot):

$$\begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \tag{7.254}$$

Step 2: Analyze the rank and dependence.
The matrix A has rank 2, which is less than the number of vectors (3). Therefore, the vectors are linearly dependent.
Step 3: Find the linear dependence relation.
From the row echelon form, we can express the system as:

$$c_1 + 2c_3 = 0 \quad (1) \tag{7.255}$$
$$c_2 + c_3 = 0 \quad (2) \tag{7.256}$$

From equation (2), we get $c_2 = -c_3$.
Substituting into equation (1):

$$c_1 + 2c_3 = 0 \tag{7.257}$$
$$c_1 = -2c_3 \tag{7.258}$$

Let $c_3 = 1$, then $c_1 = -2$ and $c_2 = -1$.
Therefore, the linear dependence relation is:

$$-2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0} \tag{7.259}$$

Rearranging to match the expected form:

$$2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0} \tag{7.260}$$

Step 4: Verify our result by direct calculation.
Let's verify by substituting the actual vectors:

$$2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 \tag{7.261}$$
$$= 2(1, 1, -1, 1) + (1, -1, 2, -1) - (3, 1, 0, 1) \tag{7.262}$$
$$= (2, 2, -2, 2) + (1, -1, 2, -1) - (3, 1, 0, 1) \tag{7.263}$$
$$= (2 + 1 - 3, 2 - 1 - 1, -2 + 2 - 0, 2 - 1 - 1) \tag{7.264}$$
$$= (0, 0, 0, 0) \quad \checkmark \tag{7.265}$$

The relation is verified. Therefore, the vectors are linearly dependent with the relation:

$$2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0} \tag{7.266}$$

Example 9: Testing for Linear Dependence

Test the following vectors for linear dependence and find a relation between them if dependent:
$\mathbf{v}_1 = (2, 2, 7, -1), \mathbf{v}_2 = (3, -1, 2, 4), \mathbf{v}_3 = (1, 1, 3, 1)$

Detailed Solution

To test whether the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, we need to determine if there exist scalars c_1, c_2, c_3, not all zero, such that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \tag{7.267}$$

This gives us the homogeneous system:

$$c_1 \begin{pmatrix} 2 \\ 2 \\ 7 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ -1 \\ 2 \\ 4 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.268}$$

Which expands to:

$$2c_1 + 3c_2 + c_3 = 0 \tag{7.269}$$
$$2c_1 - c_2 + c_3 = 0 \tag{7.270}$$
$$7c_1 + 2c_2 + 3c_3 = 0 \tag{7.271}$$
$$-c_1 + 4c_2 + c_3 = 0 \tag{7.272}$$

We can express this as a matrix equation $A\mathbf{c} = \mathbf{0}$, where:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 1 \\ 7 & 2 & 3 \\ -1 & 4 & 1 \end{bmatrix} \tag{7.273}$$

Step 1: Apply row operations to transform the matrix into row echelon form. Starting with the augmented matrix:

$$\begin{bmatrix} 2 & 3 & 1 & | & 0 \\ 2 & -1 & 1 & | & 0 \\ 7 & 2 & 3 & | & 0 \\ -1 & 4 & 1 & | & 0 \end{bmatrix} \tag{7.274}$$

$R_1 \leftrightarrow \frac{1}{2}R_1$ (Scale first row to make the leading entry 1):

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & | & 0 \\ 2 & -1 & 1 & | & 0 \\ 7 & 2 & 3 & | & 0 \\ -1 & 4 & 1 & | & 0 \end{bmatrix} \tag{7.275}$$

$R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - 7R_1$, $R_4 \leftarrow R_4 + R_1$ (Eliminate first column):

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & | & 0 \\ 0 & -4 & 0 & | & 0 \\ 0 & -\frac{19}{2} & -\frac{1}{2} & | & 0 \\ 0 & \frac{11}{2} & \frac{3}{2} & | & 0 \end{bmatrix} \tag{7.276}$$

$R_2 \leftarrow -\frac{1}{4}R_2$ (Scale second row to make the leading entry 1):

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & -\frac{19}{2} & -\frac{1}{2} & | & 0 \\ 0 & \frac{11}{2} & \frac{3}{2} & | & 0 \end{bmatrix} \tag{7.277}$$

$R_1 \leftarrow R_1 - \frac{3}{2}R_2$, $R_3 \leftarrow R_3 + \frac{19}{2}R_2$, $R_4 \leftarrow R_4 - \frac{11}{2}R_2$ (Eliminate second column):

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & -\frac{1}{2} & | & 0 \\ 0 & 0 & \frac{3}{2} & | & 0 \end{bmatrix} \tag{7.278}$$

$R_3 \leftarrow -2R_3$ (Scale third row to make the leading entry 1):

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & \frac{3}{2} & | & 0 \end{bmatrix} \tag{7.279}$$

$R_1 \leftarrow R_1 - \frac{1}{2}R_3$, $R_4 \leftarrow R_4 - \frac{3}{2}R_3$ (Eliminate third column):

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \tag{7.280}$$

Step 2: Analyze the rank and dependence.
The matrix A has rank 3, which equals the number of vectors. Therefore, the vectors are linearly independent.
Step 3: Interpretation and conclusion.
Since the vectors are linearly independent, there is no non-trivial linear combination of these vectors that equals the zero vector. The only solution to the system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ is the trivial solution $c_1 = c_2 = c_3 = 0$.
Therefore, there is no linear dependence relation among these vectors.

Example 10: Testing for Linear Dependence

Test the following vectors for linear dependence and find a relation between them if dependent:
$\mathbf{v}_1 = (1, 2, -1, 0), \mathbf{v}_2 = (1, 3, 1, 2), \mathbf{v}_3 = (4, 2, 1, 0), \mathbf{v}_4 = (6, 1, 0, 1)$

Detailed Solution

To test whether the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent, we need to determine if there exist scalars c_1, c_2, c_3, c_4, not all zero, such that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0} \tag{7.281}$$

This gives us the homogeneous system:

$$c_1 \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ 2 \\ 1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 6 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.282}$$

Which expands to:

$$c_1 + c_2 + 4c_3 + 6c_4 = 0 \tag{7.283}$$
$$2c_1 + 3c_2 + 2c_3 + c_4 = 0 \tag{7.284}$$
$$-c_1 + c_2 + c_3 + 0c_4 = 0 \tag{7.285}$$
$$0c_1 + 2c_2 + 0c_3 + c_4 = 0 \tag{7.286}$$

We can express this as a matrix equation $A\mathbf{c} = \mathbf{0}$, where:

$$A = \begin{pmatrix} 1 & 1 & 4 & 6 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \tag{7.287}$$

Step 1: Apply row operations to transform the matrix into row echelon form. Starting with the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 4 & 6 & | & 0 \\ 2 & 3 & 2 & 1 & | & 0 \\ -1 & 1 & 1 & 0 & | & 0 \\ 0 & 2 & 0 & 1 & | & 0 \end{pmatrix} \tag{7.288}$$

$R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + R_1$ (Eliminate first column):

$$\begin{pmatrix} 1 & 1 & 4 & 6 & | & 0 \\ 0 & 1 & -6 & -11 & | & 0 \\ 0 & 2 & 5 & 6 & | & 0 \\ 0 & 2 & 0 & 1 & | & 0 \end{pmatrix} \tag{7.289}$$

$R_3 \leftarrow R_3 - 2R_2$, $R_4 \leftarrow R_4 - 2R_2$ (Eliminate second column):

$$\begin{pmatrix} 1 & 1 & 4 & 6 & | & 0 \\ 0 & 1 & -6 & -11 & | & 0 \\ 0 & 0 & 17 & 28 & | & 0 \\ 0 & 0 & 12 & 23 & | & 0 \end{pmatrix} \tag{7.290}$$

$R_3 \leftarrow \frac{1}{17}R_3$ (Scale third row):

$$\begin{pmatrix} 1 & 1 & 4 & 6 & | & 0 \\ 0 & 1 & -6 & -11 & | & 0 \\ 0 & 0 & 1 & \frac{28}{17} & | & 0 \\ 0 & 0 & 12 & 23 & | & 0 \end{pmatrix} \tag{7.291}$$

$R_1 \leftarrow R_1 - 4R_3$, $R_2 \leftarrow R_2 + 6R_3$, $R_4 \leftarrow R_4 - 12R_3$ (Eliminate third column):

$$\begin{pmatrix} 1 & 1 & 0 & 6 - \frac{112}{17} & | & 0 \\ 0 & 1 & 0 & -11 + \frac{168}{17} & | & 0 \\ 0 & 0 & 1 & \frac{28}{17} & | & 0 \\ 0 & 0 & 0 & 23 - \frac{336}{17} & | & 0 \end{pmatrix} \tag{7.292}$$

Simplifying:

$$\begin{pmatrix} 1 & 1 & 0 & -\frac{10}{17} & | & 0 \\ 0 & 1 & 0 & -\frac{19}{17} & | & 0 \\ 0 & 0 & 1 & \frac{28}{17} & | & 0 \\ 0 & 0 & 0 & \frac{55}{17} & | & 0 \end{pmatrix} \tag{7.293}$$

$R_4 \leftarrow \frac{17}{55}R_4$ (Scale fourth row):

$$\begin{pmatrix} 1 & 1 & 0 & -\frac{10}{17} & | & 0 \\ 0 & 1 & 0 & -\frac{19}{17} & | & 0 \\ 0 & 0 & 1 & \frac{28}{17} & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{pmatrix} \tag{7.294}$$

$R_1 \leftarrow R_1 + \frac{10}{17}R_4$, $R_2 \leftarrow R_2 + \frac{19}{17}R_4$, $R_3 \leftarrow R_3 - \frac{28}{17}R_4$ (Eliminate fourth column):

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \tag{7.295}$$

Step 2: Analyze the rank and dependence.
The row echelon form shows that the matrix A has rank 4, which equals the number of vectors. Therefore, the vectors are linearly independent.
Step 3: Interpretation and conclusion.
Since the vectors are linearly independent, there is no non-trivial linear combination of these vectors that equals the zero vector. The only solution to the system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$ is the trivial solution $c_1 = c_2 = c_3 = c_4 = 0$.
Therefore, there is no linear dependence relation among these vectors.

7.4 Span, Basis and Dimension

In our exploration of vector spaces, we now turn to the fundamental concepts of span, basis, and dimension. These concepts allow us to characterize the structure and size of vector spaces in precise mathematical terms.

7.4.1 Span of a Set of Vectors

Definition 7.41. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in a vector space V. The **span** of these vectors, denoted by $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, is the set of all linear combinations of these vectors:*

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k : c_1, c_2, \ldots, c_k \in \mathbb{R}\} \tag{7.296}$$

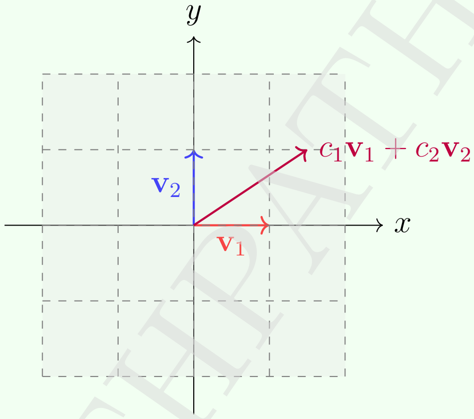
The span of a set of vectors forms a subspace of the vector space V. In fact, it is the smallest subspace that contains all the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Example: Span in $\mathbb{R}^2$

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$.
The span $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ consists of all vectors of the form:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{7.297}$$

This means $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbb{R}^2$, i.e., these two vectors span the entire $\mathbb{R}^2$ plane.



Example: Span in $\mathbb{R}^3$

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$.

The span $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set of all vectors of the form:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} \tag{7.298}$$

This represents the xy-plane in $\mathbb{R}^3$, which is a two-dimensional subspace of $\mathbb{R}^3$.

Linear Independence and Linear Dependence

Before defining a basis, we need to revise the concept of linear independence from previous section , which is crucial for characterizing the minimal set of vectors needed to span a space.

Definition 7.42. *A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is **linearly independent** if the equation*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0} \tag{7.299}$$

has only the trivial solution $c_1 = c_2 = \ldots = c_k = 0$.

*If there exists at least one set of coefficients, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$, then the vectors are said to be **linearly dependent**.*

Example: Testing Linear Independence

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix}$.

To determine if these vectors are linearly independent, we examine the equation:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \tag{7.300}$$

This expands to:

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.301}$$

Which gives us the system of equations:

$$c_1 + 2c_2 + 5c_3 = 0 \quad (7.302)$$
$$2c_1 + c_2 + 5c_3 = 0 \quad (7.303)$$
$$c_1 + 0c_2 + 2c_3 = 0 \quad (7.304)$$

Solving this system (you can use row reduction), we get $c_1 = -2c_3$, $c_2 = -c_3$, with c_3 as a free variable.
Since we can have non-zero values for the coefficients (e.g., $c_1 = -2$, $c_2 = -1$, $c_3 = 1$), the vectors are linearly dependent. In fact, $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$.

Property 7.43. *Some important properties of linear independence (from previous section):*

1. *A set containing the zero vector is always linearly dependent.*
2. *A set with a single non-zero vector is always linearly independent.*
3. *If a set of vectors is linearly dependent, then at least one vector in the set can be expressed as a linear combination of the others.*
4. *Adding a vector to a linearly independent set may result in either a linearly independent or dependent set.*
5. *Removing a vector from a linearly dependent set may result in either a linearly independent or dependent set.*

7.4.2 Basis of a Vector Space

Definition 7.44. *A **basis** for a vector space V is a set of vectors $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ that satisfies the following two conditions:*

1. *The vectors in $\mathcal{B}$ are linearly independent.*
2. *The vectors in $\mathcal{B}$ span the vector space V, i.e.,* $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = V$.

A basis is essentially a minimal set of vectors needed to generate or span the entire vector space. The basis vectors serve as "coordinate axes" for the space.

Theorem 7.45. *For any vector $\mathbf{v}$ in a vector space V with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, there exists a unique set of scalars $c_1, c_2, \ldots, c_n$ such that:*

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n \quad (7.305)$$

*These coefficients $c_1, c_2, \ldots, c_n$ are called the **coordinates** of vector $\mathbf{v}$ with respect to the basis $\mathcal{B}$.*

Proof. Since $\mathcal{B}$ spans V, we know that $\mathbf{v}$ can be expressed as a linear combination of the basis vectors, so existence of such coefficients is guaranteed.
For uniqueness, suppose there are two different sets of coefficients:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n \quad (7.306)$$
$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \ldots + d_n\mathbf{v}_n \quad (7.307)$$

Subtracting the second equation from the first:

$$\mathbf{0} = (c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \ldots + (c_n - d_n)\mathbf{v}_n \quad (7.308)$$

Since the basis vectors are linearly independent, the only solution is $c_i - d_i = 0$ for all i, which means $c_i = d_i$. This proves uniqueness. □

7.4.3 Standard Basis in $\mathbb{R}^n$

Definition 7.46. *The **standard basis** for $\mathbb{R}^n$ consists of the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, where $\mathbf{e}_i$ has a 1 in the ith position and 0s elsewhere. That is:*

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{7.309}$$

The standard basis is particularly useful because the coordinates of a vector with respect to the standard basis are simply the components of the vector itself.

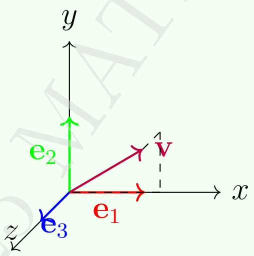
Example: Standard Basis in $\mathbb{R}^3$

The standard basis in $\mathbb{R}^3$ consists of:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{7.310}$$

Any vector $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbb{R}^3$ can be uniquely expressed as:

$$\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 \tag{7.311}$$



7.4.4 Finding a Basis for a Subspace

Finding a basis for a given subspace is a fundamental problem in linear algebra. Here, we outline a general approach for finding a basis for a subspace.

Problem: Finding a Basis for $\text{span}\{S\}$

Given a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, find a basis for the subspace $W = \text{span}\{S\}$.

Solution

To find a basis for the subspace spanned by a set of vectors, we follow these steps:

1. Form a matrix A whose rows (or columns, depending on your convention) are the given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$.
2. Row reduce the matrix A to its row echelon form (REF) or reduced row echelon form (RREF).
3. The non-zero rows of the row echelon form constitute a basis for the row space of A, which is precisely the span of the original vectors.

Let's illustrate this with an example.

Example: Finding a Basis

Consider the set of vectors in $\mathbb{R}^4$:

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 7 \\ 9 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\} \tag{7.312}$$

We want to find a basis for $W = \text{span}\{S\}$.
Step 1: Form a matrix A with these vectors as rows:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 5 & 7 & 9 \\ 0 & 1 & 2 & 3 \end{pmatrix} \tag{7.313}$$

Step 2: Row reduce this matrix to RREF:

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.314}$$

Step 3: The non-zero rows of the RREF form a basis for W:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\} \tag{7.315}$$

This basis tells us that W is a two-dimensional subspace of $\mathbb{R}^4$.

Another common problem is finding a basis that includes specific vectors:

Problem: Extending to a Basis

Given a linearly independent set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V, find additional vectors to form a basis for V.

Solution

To extend a linearly independent set to a basis for a finite-dimensional vector space, we can use the following algorithm:

1. Start with the given linearly independent set S.
2. Add a vector $\mathbf{w}$ not in $\text{span}\{S\}$ to S.
3. Check if the new set is linearly independent. If not, discard $\mathbf{w}$ and try another vector.
4. Repeat steps 2-3 until you have a basis (i.e., a linearly independent set that spans the space).

In $\mathbb{R}^n$, a practical approach is to include standard basis vectors $\mathbf{e}_i$ that are not in the span of the current set.

7.4.5 Dimension of a Vector Space

Definition 7.47. *The **dimension** of a vector space V, denoted by $\dim(V)$, is the number of vectors in any basis for V.*

Theorem 7.48. *All bases for a vector space V have the same number of elements.*

Proof. The proof relies on the fact that if we have two bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for a vector space V, then each vector in $\mathcal{B}_1$ can be expressed as a linear combination of vectors in $\mathcal{B}_2$, and vice versa. Through algebraic manipulation and properties of linear independence, it can be shown that $|\mathcal{B}_1| = |\mathcal{B}_2|$.

(For a complete proof, one typically uses the Replacement Theorem or similar techniques to establish this fundamental result.) □

Example: Dimension of Common Vector Spaces

- $\dim(\mathbb{R}^n) = n$, as seen from the standard basis.
- $\dim(\mathbb{P}_n)$ (the space of polynomials of degree at most n) is $n+1$, with basis $\{1, x, x^2, \ldots, x^n\}$.
- $\dim(M_{m\times n})$ (the space of $m \times n$ matrices) is mn.
- $\dim(\{\mathbf{0}\}) = 0$, as the zero vector space has no basis (empty set).

7.4.6 The Dimension Theorem and Rank Nullity Theorem

The dimension theorem relates the dimension of a linear transformation's domain, kernel, and image.

Theorem 7.49 (Dimension Theorem)**.** *Let $T : V \to W$ be a linear transformation where V and W are finite-dimensional vector spaces. Then:*

$$\dim(V) = \dim(\ker(T)) + \dim(\operatorname{im}(T)) \tag{7.316}$$

where $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$ is the kernel (or null space) of T, and $\operatorname{im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$ is the image (or range) of T.

Proof. Let $\dim(\ker(T)) = k$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for $\ker(T)$.
This basis can be extended to a basis for V: $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ where $n = \dim(V)$.
Consider the set $\{T(\mathbf{v}_{k+1}), T(\mathbf{v}_{k+2}), \ldots, T(\mathbf{v}_n)\}$. We can show that:

1. This set spans $\operatorname{im}(T)$.
2. This set is linearly independent.

Therefore, $\{T(\mathbf{v}_{k+1}), T(\mathbf{v}_{k+2}), \ldots, T(\mathbf{v}_n)\}$ forms a basis for $\operatorname{im}(T)$, with $\dim(\operatorname{im}(T)) = n - k = \dim(V) - \dim(\ker(T))$.
Rearranging gives us $\dim(V) = \dim(\ker(T)) + \dim(\operatorname{im}(T))$. □

Application: Matrix Rank and Nullity

For a matrix $\mathbf{A}$, the dimension theorem relates the rank and nullity:

$$\operatorname{rank}(\mathbf{A}) + \operatorname{nullity}(\mathbf{A}) = n \tag{7.317}$$

where n is the number of columns in $\mathbf{A}$, $\operatorname{rank}(\mathbf{A}) = \dim(\operatorname{col}(\mathbf{A}))$, and $\operatorname{nullity}(\mathbf{A}) = \dim(\operatorname{null}(\mathbf{A}))$.
This relationship has many practical applications:

- Determining if a system of linear equations has a unique solution
- Finding the dimension of solution spaces

- Analyzing the invertibility of matrices
- Understanding linear transformations between vector spaces

Summary

In this section, we have explored the fundamental concepts of span, basis, and dimension in vector spaces:

- The span of a set of vectors is the collection of all possible linear combinations of those vectors.
- A basis is a linearly independent set of vectors that spans a vector space.
- Every vector in a vector space can be uniquely represented as a linear combination of basis vectors.
- The dimension of a vector space is the number of vectors in any basis for that space.
- The dimension theorem relates the dimensions of a linear transformation's domain, kernel, and image.

These concepts form the backbone of vector space theory and have wide-ranging applications in mathematics, physics, engineering, and computer science.

7.4.7 Exercises

1. Determine whether the following set of vectors is linearly independent in $\mathbb{R}^3$: $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}3\\1\\1\end{pmatrix}\right\}$
2. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors: $\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\\1\end{pmatrix}\right\}$
3. If $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation with $\dim(\ker(T)) = 2$, what is $\dim(\mathrm{im}(T))$?
4. Show that the polynomials $1, 1 + x, 1 + x + x^2$ form a basis for $\mathbb{P}_2$.
5. Find a basis for the solution space of the homogeneous system:

$$\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 2x_1 + 4x_2 - 2x_3 = 0 \end{cases} \tag{7.318}$$

6. Let W be the subspace of $\mathbb{R}^3$ consisting of all vectors of the form $(a, b, a + b)$. Find a basis for W and determine $\dim(W)$.

7.4.8 Solutions to Exercises

Solution to Exercise 1

Determine whether the following set of vectors is linearly independent in $\mathbb{R}^3$: $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}3\\1\\1\end{pmatrix}\right\}$

To determine linear independence, we need to check if the equation

$$c_1 \begin{pmatrix}1\\1\\0\end{pmatrix} + c_2 \begin{pmatrix}2\\0\\1\end{pmatrix} + c_3 \begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{7.319}$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$.
Writing this as a system of equations:

$$c_1 + 2c_2 + 3c_3 = 0 \tag{7.320}$$
$$c_1 + 0c_2 + c_3 = 0 \tag{7.321}$$
$$0c_1 + c_2 + c_3 = 0 \tag{7.322}$$

We can express this as an augmented matrix and apply row reduction:

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{7.323}$$

Step 1: Subtract row 1 from row 2.

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{7.324}$$

Step 2: Add 2 times row 3 to row 2.

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{7.325}$$

Step 3: Subtract 2 times row 3 from row 1.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{7.326}$$

The reduced row echelon form has only two non-zero rows, which means the system has a free variable. We can set $c_3 = t$ (where t is a parameter) and solve:
From row 3: $c_2 + c_3 = 0 \implies c_2 = -c_3 = -t$ From row 1: $c_1 + c_3 = 0 \implies c_1 = -c_3 = -t$
Thus, the general solution is $c_1 = -t$, $c_2 = -t$, $c_3 = t$ for any value of t.
Since there are non-trivial solutions (solutions where not all $c_i = 0$), the vectors are linearly dependent.
This is further confirmed by noting that the third vector can be expressed as a linear combination of the first two:

$$\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \tag{7.327}$$

Solution to Exercise 2

Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}$

To find a basis, we'll form a matrix with these vectors as columns and row reduce to find the linearly independent columns.

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \tag{7.328}$$

Now we row reduce this matrix to RREF:
Step 1: The first entry is already a leading 1.

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \tag{7.329}$$

Step 2: Subtract row 1 from row 3.

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \tag{7.330}$$

Step 3: Subtract row 2 from row 4.

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.331}$$

The RREF has two non-zero rows, indicating that the rank of the matrix is 2. This means the dimension of the subspace is 2, and we need two linearly independent vectors for our basis.
The leading columns (columns with leading 1's) in the RREF are columns 1 and 2. These correspond to the original vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{7.332}$$

Therefore, a basis for the subspace is:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{7.333}$$

We can verify that the other vectors can be expressed as linear combinations of these basis vectors:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{7.334}$$

$$\begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{7.335}$$

Solution to Exercise 3

If $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation with $\dim(\ker(T)) = 2$, what is $\dim(\mathrm{im}(T))$?
We can apply the dimension theorem, which states:

$$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T)) \tag{7.336}$$

Given information: - $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation - $\dim(\ker(T)) = 2$ - $\dim(\mathbb{R}^4) = 4$ (domain dimension)
Substituting into the dimension theorem:

$$\dim(\mathbb{R}^4) = \dim(\ker(T)) + \dim(\mathrm{im}(T)) \tag{7.337}$$
$$4 = 2 + \dim(\mathrm{im}(T)) \tag{7.338}$$
$$\dim(\mathrm{im}(T)) = 4 - 2 = 2 \tag{7.339}$$

Therefore, $\dim(\mathrm{im}(T)) = 2$.
This means the image of T is a 2-dimensional subspace of $\mathbb{R}^3$. We can also interpret this result in terms of the rank-nullity theorem for matrices: if T can be represented by a matrix A with 4 columns, then $\mathrm{rank}(A) = 2$ and $\mathrm{nullity}(A) = 2$.

Solution to Exercise 4

Show that the polynomials $1, 1 + x, 1 + x + x^2$ form a basis for $\mathbb{P}_2$.
Let's denote these polynomials as:

$$p_1(x) = 1 \tag{7.340}$$
$$p_2(x) = 1 + x \tag{7.341}$$
$$p_3(x) = 1 + x + x^2 \tag{7.342}$$

To show that these polynomials form a basis for $\mathbb{P}_2$, we need to verify two conditions: 1. They are linearly independent. 2. They span $\mathbb{P}_2$.
First, let's check linear independence. We need to determine if the equation

$$c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = 0 \tag{7.343}$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$.
Substituting the polynomials:

$$c_1 \cdot 1 + c_2 \cdot (1 + x) + c_3 \cdot (1 + x + x^2) = 0 \tag{7.344}$$
$$c_1 + c_2 + c_3 + c_2x + c_3x + c_3x^2 = 0 \tag{7.345}$$

For two polynomials to be equal, their coefficients must be equal. This gives us the system:

$$c_1 + c_2 + c_3 = 0 \quad (\text{coefficient of } x^0) \tag{7.346}$$
$$c_2 + c_3 = 0 \quad (\text{coefficient of } x^1) \tag{7.347}$$
$$c_3 = 0 \quad (\text{coefficient of } x^2) \tag{7.348}$$

From the third equation, we have $c_3 = 0$. Substituting this into the second equation: $c_2 + 0 = 0 \implies c_2 = 0$.
Substituting these values into the first equation: $c_1 + 0 + 0 = 0 \implies c_1 = 0$.
Since the only solution is $c_1 = c_2 = c_3 = 0$, the polynomials are linearly independent.
Next, we need to check if they span $\mathbb{P}_2$. The space $\mathbb{P}_2$ consists of all polynomials of the form $a + bx + cx^2$ where $a, b, c \in \mathbb{R}$. We need to show that any polynomial in $\mathbb{P}_2$ can be expressed as a linear combination of $p_1(x)$, $p_2(x)$, and $p_3(x)$.

Let $p(x) = a+bx+cx^2$ be an arbitrary polynomial in $\mathbb{P}_2$. We want to find scalars d_1, d_2, d_3 such that:

$$d_1p_1(x) + d_2p_2(x) + d_3p_3(x) = p(x) \tag{7.349}$$
$$d_1 \cdot 1 + d_2 \cdot (1+x) + d_3 \cdot (1+x+x^2) = a+bx+cx^2 \tag{7.350}$$
$$d_1 + d_2 + d_3 + d_2x + d_3x + d_3x^2 = a+bx+cx^2 \tag{7.351}$$

Comparing coefficients:

$$d_1 + d_2 + d_3 = a \quad \text{(coefficient of } x^0) \tag{7.352}$$
$$d_2 + d_3 = b \quad \text{(coefficient of } x^1) \tag{7.353}$$
$$d_3 = c \quad \text{(coefficient of } x^2) \tag{7.354}$$

From the third equation, $d_3 = c$. Substituting into the second equation: $d_2 + c = b \implies d_2 = b - c$. Substituting into the first equation: $d_1 + (b-c) + c = a \implies d_1 = a - b$. Therefore, for any polynomial $p(x) = a + bx + cx^2$, we can express it as:

$$p(x) = (a-b) \cdot 1 + (b-c) \cdot (1+x) + c \cdot (1+x+x^2) \tag{7.355}$$
$$= (a-b)p_1(x) + (b-c)p_2(x) + cp_3(x) \tag{7.356}$$

Since we can represent any polynomial in $\mathbb{P}_2$ as a linear combination of $p_1(x)$, $p_2(x)$, and $p_3(x)$, they span $\mathbb{P}_2$.
Having verified both linear independence and spanning property, we conclude that $\{1, 1+x, 1+x+x^2\}$ forms a basis for $\mathbb{P}_2$.
Note that $\dim(\mathbb{P}_2) = 3$, which matches the number of polynomials in our basis.

Solution to Exercise 5

Find a basis for the solution space of the homogeneous system:

$$\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 2x_1 + 4x_2 - 2x_3 = 0 \end{cases} \tag{7.357}$$

First, we'll express this as an augmented matrix and reduce it to row echelon form:

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 0 \end{pmatrix} \tag{7.358}$$

Step 1: Subtract 2 times row 1 from row 2.

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.359}$$

We can see that the second equation is a multiple of the first equation, so our system is equivalent to:

$$x_1 + 2x_2 - x_3 = 0 \tag{7.360}$$

Solving for x_1:

$$x_1 = x_3 - 2x_2 \tag{7.361}$$

We have two free variables: x_2 and x_3. We can parametrize the solution as:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 - 2x_2 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{7.362}$$

Therefore, a basis for the solution space consists of the vectors:

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{7.363}$$

The solution space is a 2-dimensional subspace of $\mathbb{R}^3$, which aligns with the fact that we have one independent constraint equation in three variables.
To verify this is indeed a basis: 1. These vectors are linearly independent (cannot express one as a multiple of the other). 2. Any solution to the system can be written as a linear combination of these two vectors.

Solution to Exercise 6

Let W be the subspace of $\mathbb{R}^3$ consisting of all vectors of the form $(a, b, a+b)$. Find a basis for W and determine $\dim(W)$.
First, we'll identify the structure of vectors in W. A vector $(x, y, z) \in W$ if and only if $z = x + y$. This gives us a constraint equation:

$$z - x - y = 0 \tag{7.364}$$

This is equivalent to saying W is the solution space of the homogeneous equation:

$$-x - y + z = 0 \tag{7.365}$$

We can parametrize the solution with two free variables. If we choose $x = a$ and $y = b$ as parameters, then $z = a + b$. This gives:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ a+b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{7.366}$$

Therefore, a basis for W is:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \tag{7.367}$$

To verify this is a basis: 1. Linear independence: We need to check if $c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ implies $c_1 = c_2 = 0$.
Expanding:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.368}$$

This gives us the system:

$$c_1 = 0 \tag{7.369}$$
$$c_2 = 0 \tag{7.370}$$
$$c_1 + c_2 = 0 \tag{7.371}$$

The only solution is $c_1 = c_2 = 0$, confirming linear independence.
2. Spanning property: Any vector in W can be written in the form $(a, b, a+b)$, which is precisely $a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.
Therefore, $\mathcal{B}$ is a basis for W, and $\dim(W) = 2$.
We can interpret this geometrically: W is a 2-dimensional plane in $\mathbb{R}^3$ passing through the origin with normal vector $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

7.4.9 Additional Solved Examples

Example 1: Determining Bases for $\mathbb{R}^3$

Determine which of the following sets of vectors are bases for $\mathbb{R}^3$.
(a) $\{(3, 3, 3), (4, 4, 0), (5, 0, 0)\}$
(b) $\{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$
Note: For sets with exactly n vectors in $\mathbb{R}^n$ (like these examples with 3 vectors in $\mathbb{R}^3$), we can use the determinant method as an efficient approach. The vectors form a basis if and only if the determinant of the matrix formed by these vectors is non-zero.

Solution to Example 1(a)

To determine if a set of vectors forms a basis for $\mathbb{R}^3$, we need to check two conditions:
1. The vectors are linearly independent.
2. The vectors span $\mathbb{R}^3$.

For a set of three vectors in $\mathbb{R}^3$, these conditions are satisfied if and only if the vectors are linearly independent, which is equivalent to having rank $= 3$.
Let's denote the vectors as:

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \tag{7.372}$$

Method 1: Row Reduction Approach
To determine linear independence, we check if the equation:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \tag{7.373}$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$.
This is equivalent to solving the homogeneous system:

$$3c_1 + 4c_2 + 5c_3 = 0 \tag{7.374}$$
$$3c_1 + 4c_2 + 0c_3 = 0 \tag{7.375}$$
$$3c_1 + 0c_2 + 0c_3 = 0 \tag{7.376}$$

We form the coefficient matrix and row reduce it to determine its rank:

$$A = \begin{pmatrix} 3 & 4 & 5 \\ 3 & 4 & 0 \\ 3 & 0 & 0 \end{pmatrix} \tag{7.377}$$

Row reduction to row echelon form:
Step 1: Divide the first row by 3 to get a leading 1.

$$\begin{pmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 3 & 4 & 0 \\ 3 & 0 & 0 \end{pmatrix} \tag{7.378}$$

Step 2: Eliminate the entries below the first pivot.

$$R_2 \leftarrow R_2 - 3R_1 \tag{7.379}$$

$$R_3 \leftarrow R_3 - 3R_1 \tag{7.380}$$

This gives:

$$\begin{pmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & 0 & -5 \\ 0 & -4 & -5 \end{pmatrix} \tag{7.381}$$

Step 3: Divide the second row by -5 to get a leading 1.

$$\begin{pmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & 0 & 1 \\ 0 & -4 & -5 \end{pmatrix} \tag{7.382}$$

Step 4: Eliminate the entry below the second pivot and the entry in the first row.

$$R_3 \leftarrow R_3 + 5R_2 \tag{7.383}$$

$$R_1 \leftarrow R_1 - \frac{5}{3}R_2 \tag{7.384}$$

This gives:

$$\begin{pmatrix} 1 & \frac{4}{3} & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{pmatrix} \tag{7.385}$$

Step 5: Divide the third row by -4 to get a leading 1.

$$\begin{pmatrix} 1 & \frac{4}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{7.386}$$

Step 6: Eliminate the remaining entry in the first row.

$$R_1 \leftarrow R_1 - \frac{4}{3}R_3 \tag{7.387}$$

This gives the row echelon form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{7.388}$$

Since we have 3 pivots (leading 1s), the rank of matrix A is 3, which equals the number of vectors and the dimension of $\mathbb{R}^3$.
Therefore, the vectors are linearly independent and form a basis for $\mathbb{R}^3$.
Method 2: Determinant Approach
For 3×3 matrices, we can use the determinant as a more efficient approach. Let's arrange the given vectors as columns of a matrix and compute its determinant:

$$\det(A) = \begin{vmatrix} 3 & 4 & 5 \\ 3 & 4 & 0 \\ 3 & 0 & 0 \end{vmatrix} \tag{7.389}$$

$$= 3\begin{vmatrix} 4 & 0 \\ 0 & 0 \end{vmatrix} - 4\begin{vmatrix} 3 & 0 \\ 3 & 0 \end{vmatrix} + 5\begin{vmatrix} 3 & 4 \\ 3 & 0 \end{vmatrix} \tag{7.390}$$

$$= 3(0) - 4(0) + 5(3 \cdot 0 - 4 \cdot 3) \tag{7.391}$$

$$= 5(-12) = -60 \neq 0 \tag{7.392}$$

Since the determinant is non-zero, the matrix is invertible, which confirms that the vectors are linearly independent and form a basis for $\mathbb{R}^3$.
This determinant method is particularly efficient for small matrices where we need to check if vectors form a basis.

Solution to Example 1(b)

Let's examine the set $\{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$.
We denote the vectors as:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \tag{7.393}$$

Method 1: Row Reduction Approach
To determine if they form a basis for $\mathbb{R}^3$, we check if they are linearly independent by solving the homogeneous system:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \tag{7.394}$$

$$c_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + c_3\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.395}$$

This gives us the system of equations:

$$c_1 + 4c_2 + 7c_3 = 0 \tag{7.396}$$

$$2c_1 + 5c_2 + 8c_3 = 0 \tag{7.397}$$

$$3c_1 + 6c_2 + 9c_3 = 0 \tag{7.398}$$

We form the coefficient matrix and row reduce it:

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \tag{7.399}$$

Row reduction to row echelon form:

Step 1: Eliminate the entries below the first pivot.

$$R_2 \leftarrow R_2 - 2R_1 \tag{7.400}$$
$$R_3 \leftarrow R_3 - 3R_1 \tag{7.401}$$

This gives:

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \tag{7.402}$$

Step 2: Divide the second row by -3 to get a leading 1.

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{pmatrix} \tag{7.403}$$

Step 3: Eliminate the entries in the first and third rows.

$$R_1 \leftarrow R_1 - 4R_2 \tag{7.404}$$
$$R_3 \leftarrow R_3 + 6R_2 \tag{7.405}$$

This gives:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.406}$$

We have only 2 pivots (leading 1s), so the rank of matrix A is 2, which is less than the dimension of $\mathbb{R}^3$.
This means the vectors are linearly dependent. From our row echelon form, we have:

$$c_1 = c_3 \tag{7.407}$$
$$c_2 = -2c_3 \tag{7.408}$$

Taking $c_3 = 1$, we get $c_1 = 1$ and $c_2 = -2$. So we have:

$$\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0} \tag{7.409}$$
$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.410}$$

Let's verify this:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 1-8+7 \\ 2-10+8 \\ 3-12+9 \end{pmatrix} \tag{7.411}$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.412}$$

This confirms that the vectors are linearly dependent, and therefore they do not form a basis for $\mathbb{R}^3$.

Method 2: Determinant Approach
As with part (a), we can directly compute the determinant of the matrix formed by these vectors:

$$\det(A) = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} \tag{7.413}$$
$$= 1(5 \cdot 9 - 8 \cdot 6) - 4(2 \cdot 9 - 8 \cdot 3) + 7(2 \cdot 6 - 5 \cdot 3) \tag{7.414}$$
$$= 1(45 - 48) - 4(18 - 24) + 7(12 - 15) \tag{7.415}$$
$$= 1(-3) - 4(-6) + 7(-3) \tag{7.416}$$
$$= -3 + 24 - 21 = 0 \tag{7.417}$$

Since the determinant is zero, the matrix is not invertible, confirming that the vectors are linearly dependent and do not form a basis for $\mathbb{R}^3$.
This determinant approach provides a faster alternative to row reduction for 3×3 matrices, allowing us to quickly determine if vectors form a basis.

Example 2: Basis for Polynomial Space

Show that the set $B = \{1, x, x^2, x^3, \ldots, x^n\}$ is a basis for the vector space $\mathbb{P}_n$ (polynomials of degree $\leq n$).

Solution

To show that $B = \{1, x, x^2, x^3, \ldots, x^n\}$ is a basis for $\mathbb{P}_n$, we need to verify two conditions:
1. B is linearly independent.
2. B spans $\mathbb{P}_n$.

Let's denote the elements of B as:

$$p_0(x) = 1 \tag{7.418}$$
$$p_1(x) = x \tag{7.419}$$
$$p_2(x) = x^2 \tag{7.420}$$
$$\vdots \tag{7.421}$$
$$p_n(x) = x^n \tag{7.422}$$

Step 1: Proving Linear Independence
We need to show that if

$$c_0 p_0(x) + c_1 p_1(x) + c_2 p_2(x) + \cdots + c_n p_n(x) = 0 \tag{7.423}$$

for all $x \in \mathbb{R}$, then $c_0 = c_1 = c_2 = \cdots = c_n = 0$.
Substituting the expressions for each $p_i(x)$:

$$c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 + \cdots + c_n \cdot x^n = 0 \tag{7.424}$$

This gives us the polynomial:

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0 \tag{7.425}$$

For this polynomial to be equal to 0 for all values of x, all coefficients must be zero (by the fundamental theorem of algebra, specifically the identity theorem for polynomials).

Therefore:

$$c_0 = c_1 = c_2 = \cdots = c_n = 0 \tag{7.426}$$

This proves that the set B is linearly independent.
Step 2: Proving B Spans $\mathbb{P}_n$
We need to show that any polynomial $p(x) \in \mathbb{P}_n$ can be expressed as a linear combination of the elements in B.
By definition, any polynomial $p(x) \in \mathbb{P}_n$ can be written in the form:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \tag{7.427}$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants.
We can rewrite this as:

$$p(x) = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n \tag{7.428}$$
$$= a_0 p_0(x) + a_1 p_1(x) + a_2 p_2(x) + \cdots + a_n p_n(x) \tag{7.429}$$

This shows that $p(x)$ can be expressed as a linear combination of the elements in B, proving that B spans $\mathbb{P}_n$.
Conclusion
Since $B = \{1, x, x^2, x^3, \ldots, x^n\}$ is both linearly independent and spans $\mathbb{P}_n$, it is a basis for $\mathbb{P}_n$.
Dimension of $\mathbb{P}_n$
From this result, we can also determine that the dimension of $\mathbb{P}_n$ is $n + 1$, as this is the number of vectors in the basis B.

Example 3: Finding Coordinate Vectors

Find the coordinate vector for $\mathbf{u}$ relative to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
(a) $\mathbf{u} = (3, -1, 2)$, $\mathbf{v}_1 = (3, 3, 3)$, $\mathbf{v}_2 = (4, 4, 0)$, $\mathbf{v}_3 = (5, 0, 0)$
(b) $\mathbf{u} = (5, 5, 5)$, $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-4, 5, 6)$, $\mathbf{v}_3 = (7, -8, 9)$

Solution to Example 3(a)

To find the coordinate vector of $\mathbf{u}$ relative to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we need to determine the scalars c_1, c_2, c_3 such that:

$$\mathbf{u} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \tag{7.430}$$

Given:

$$\mathbf{u} = (3, -1, 2) \tag{7.431}$$
$$\mathbf{v}_1 = (3, 3, 3) \tag{7.432}$$
$$\mathbf{v}_2 = (4, 4, 0) \tag{7.433}$$
$$\mathbf{v}_3 = (5, 0, 0) \tag{7.434}$$

Substituting these vectors, we get:

$$(3, -1, 2) = c_1(3, 3, 3) + c_2(4, 4, 0) + c_3(5, 0, 0) \tag{7.435}$$
$$= (3c_1 + 4c_2 + 5c_3, 3c_1 + 4c_2, 3c_1) \tag{7.436}$$

This gives us the system of equations:

$$3c_1 + 4c_2 + 5c_3 = 3 \quad \text{(from first component)} \tag{7.437}$$
$$3c_1 + 4c_2 = -1 \quad \text{(from second component)} \tag{7.438}$$
$$3c_1 = 2 \quad \text{(from third component)} \tag{7.439}$$

We can solve this system using the method of elimination:
From the third equation, we get:

$$c_1 = \frac{2}{3} \tag{7.440}$$

Substituting this value into the second equation:

$$3 \cdot \frac{2}{3} + 4c_2 = -1 \tag{7.441}$$
$$2 + 4c_2 = -1 \tag{7.442}$$
$$4c_2 = -3 \tag{7.443}$$
$$c_2 = -\frac{3}{4} \tag{7.444}$$

Substituting the values of c_1 and c_2 into the first equation:

$$3 \cdot \frac{2}{3} + 4 \cdot \left(-\frac{3}{4}\right) + 5c_3 = 3 \tag{7.445}$$
$$2 - 3 + 5c_3 = 3 \tag{7.446}$$
$$-1 + 5c_3 = 3 \tag{7.447}$$
$$5c_3 = 4 \tag{7.448}$$
$$c_3 = \frac{4}{5} \tag{7.449}$$

Therefore, the coordinate vector of $\mathbf{u}$ relative to the basis B is:

$$[\mathbf{u}]_B = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{3}{4} \\ \frac{4}{5} \end{pmatrix} \tag{7.450}$$

Verification: Let's confirm that this coordinate vector gives us the original vector $\mathbf{u}$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \frac{2}{3}(3,3,3) + \left(-\frac{3}{4}\right)(4,4,0) + \frac{4}{5}(5,0,0) \tag{7.451}$$
$$= (2,2,2) + (-3,-3,0) + (4,0,0) \tag{7.452}$$
$$= (2-3+4, 2-3+0, 2+0+0) \tag{7.453}$$
$$= (3,-1,2) = \mathbf{u} \tag{7.454}$$

This confirms our answer is correct.

Solution to Example 3(b)

We need to find the coordinate vector of $\mathbf{u}$ relative to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Given:

$$\mathbf{u} = (5,5,5) \tag{7.455}$$
$$\mathbf{v}_1 = (1,2,3) \tag{7.456}$$
$$\mathbf{v}_2 = (-4,5,6) \tag{7.457}$$
$$\mathbf{v}_3 = (7,-8,9) \tag{7.458}$$

Method: System of Equations with Row Reduction
We need to find scalars c_1, c_2, c_3 such that:

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \tag{7.459}$$

Substituting the vectors:

$$(5,5,5) = c_1(1,2,3) + c_2(-4,5,6) + c_3(7,-8,9) \tag{7.460}$$
$$= (c_1 - 4c_2 + 7c_3, 2c_1 + 5c_2 - 8c_3, 3c_1 + 6c_2 + 9c_3) \tag{7.461}$$

This gives us the system of equations:

$$c_1 - 4c_2 + 7c_3 = 5 \quad \text{(from first component)} \tag{7.462}$$
$$2c_1 + 5c_2 - 8c_3 = 5 \quad \text{(from second component)} \tag{7.463}$$
$$3c_1 + 6c_2 + 9c_3 = 5 \quad \text{(from third component)} \tag{7.464}$$

We'll solve this system using row reduction on the augmented matrix:

$$\begin{pmatrix} 1 & -4 & 7 & 5 \\ 2 & 5 & -8 & 5 \\ 3 & 6 & 9 & 5 \end{pmatrix} \tag{7.465}$$

Now, we'll perform row operations to find the solution. Let's use arrow notation to clearly show the operations:
Step 1: Eliminate the entries below the first pivot.

$$R_2 \to R_2 - 2R_1 \tag{7.466}$$
$$R_3 \to R_3 - 3R_1 \tag{7.467}$$

This gives:

$$\begin{pmatrix} 1 & -4 & 7 & 5 \\ 0 & 13 & -22 & -5 \\ 0 & 18 & -12 & -10 \end{pmatrix} \tag{7.468}$$

Step 2: Make the second pivot 1.

$$R_2 \to \frac{1}{13}R_2 \tag{7.469}$$

This gives:

$$\begin{pmatrix} 1 & -4 & 7 & 5 \\ 0 & 1 & -\frac{22}{13} & -\frac{5}{13} \\ 0 & 18 & -12 & -10 \end{pmatrix} \tag{7.470}$$

Step 3: Eliminate the entry below the second pivot.

$$R_3 \to R_3 - 18R_2 \tag{7.471}$$

This gives:

$$\begin{pmatrix} 1 & -4 & 7 & 5 \\ 0 & 1 & -\frac{22}{13} & -\frac{5}{13} \\ 0 & 0 & -12 + 18 \cdot \frac{22}{13} & -10 + 18 \cdot \frac{5}{13} \end{pmatrix} \tag{7.472}$$

Let's compute the third row precisely:

$$-12 + 18 \cdot \frac{22}{13} = -12 + \frac{18 \cdot 22}{13} = -12 + \frac{396}{13} = \frac{-156 + 396}{13} = \frac{240}{13} \tag{7.473}$$

$$-10 + 18 \cdot \frac{5}{13} = -10 + \frac{18 \cdot 5}{13} = -10 + \frac{90}{13} = \frac{-130 + 90}{13} = -\frac{40}{13} \tag{7.474}$$

So the matrix becomes:

$$\begin{pmatrix} 1 & -4 & 7 & 5 \\ 0 & 1 & -\frac{22}{13} & -\frac{5}{13} \\ 0 & 0 & \frac{240}{13} & -\frac{40}{13} \end{pmatrix} \tag{7.475}$$

Step 4: Make the third pivot 1.

$$R_3 \to \frac{13}{240} R_3 \tag{7.476}$$

This gives:

$$\begin{pmatrix} 1 & -4 & 7 & 5 \\ 0 & 1 & -\frac{22}{13} & -\frac{5}{13} \\ 0 & 0 & 1 & -\frac{40/13}{240/13} = -\frac{40}{240} = -\frac{1}{6} \end{pmatrix} \tag{7.477}$$

Now the matrix is in row echelon form:

$$\begin{pmatrix} 1 & -4 & 7 & 5 \\ 0 & 1 & -\frac{22}{13} & -\frac{5}{13} \\ 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix} \tag{7.478}$$

To get the reduced row echelon form (RREF), we continue with elimination:
Step 5: Eliminate the entries above the third pivot.

$$R_1 \to R_1 - 7R_3 \tag{7.479}$$

$$R_2 \to R_2 + \frac{22}{13} R_3 \tag{7.480}$$

Let's calculate the new values:

$$R_1 : 7 - 7 \cdot 1 = 0 \tag{7.481}$$

$$R_1 : 5 - 7 \cdot \left(-\frac{1}{6}\right) = 5 + \frac{7}{6} = \frac{30 + 7}{6} = \frac{37}{6} \tag{7.482}$$

$$R_2 : -\frac{22}{13} + \frac{22}{13} \cdot 1 = 0 \tag{7.483}$$

$$R_2 : -\frac{5}{13} + \frac{22}{13} \cdot \left(-\frac{1}{6}\right) = -\frac{5}{13} - \frac{22}{13} \cdot \frac{1}{6} = -\frac{5}{13} - \frac{22}{78} = -\frac{30}{78} - \frac{22}{78} = -\frac{52}{78} = -\frac{2}{3} \tag{7.484}$$

This gives:

$$\begin{pmatrix} 1 & -4 & 0 & \frac{37}{6} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix} \tag{7.485}$$

Step 6: Eliminate the entry above the second pivot.

$$R_1 \to R_1 + 4R_2 \tag{7.486}$$

This gives:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{37}{6} + 4 \cdot \left(-\frac{2}{3}\right) \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix} \tag{7.487}$$

Let's calculate the last step:

$$\frac{37}{6} + 4 \cdot \left(-\frac{2}{3}\right) = \frac{37}{6} - \frac{8}{3} = \frac{37}{6} - \frac{16}{6} = \frac{37-16}{6} = \frac{21}{6} = \frac{7}{2} \tag{7.488}$$

The final RREF is:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{7}{2} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix} \tag{7.489}$$

From the RREF, we can read off the values of c_1, c_2, c_3:

$$c_1 = \frac{7}{2} \tag{7.490}$$

$$c_2 = -\frac{2}{3} \tag{7.491}$$

$$c_3 = -\frac{1}{6} \tag{7.492}$$

Therefore, the coordinate vector of $\mathbf{u}$ relative to the basis B is:

$$[\mathbf{u}]_B = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ -\frac{2}{3} \\ -\frac{1}{6} \end{pmatrix} \tag{7.493}$$

Verification: Let's confirm that this coordinate vector gives us the original vector $\mathbf{u}$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \frac{7}{2}(1,2,3) + \left(-\frac{2}{3}\right)(-4,5,6) + \left(-\frac{1}{6}\right)(7,-8,9) \tag{7.494}$$

$$= \left(\frac{7}{2}, 7, \frac{21}{2}\right) + \left(\frac{8}{3}, -\frac{10}{3}, -4\right) + \left(-\frac{7}{6}, \frac{4}{3}, -\frac{3}{2}\right) \tag{7.495}$$

Computing each component:

$$\text{First component}: \frac{7}{2} + \frac{8}{3} - \frac{7}{6} = \frac{21}{6} + \frac{16}{6} - \frac{7}{6} = \frac{21+16-7}{6} = \frac{30}{6} = 5 \tag{7.496}$$

$$\text{Second component}: 7 - \frac{10}{3} + \frac{4}{3} = 7 - \frac{10-4}{3} = 7 - \frac{6}{3} = 7 - 2 = 5 \tag{7.497}$$

$$\text{Third component}: \frac{21}{2} - 4 - \frac{3}{2} = \frac{21}{2} - \frac{8}{2} - \frac{3}{2} = \frac{21-8-3}{2} = \frac{10}{2} = 5 \tag{7.498}$$

This gives us $(5,5,5)$, which is indeed our original vector $\mathbf{u}$, confirming our answer is correct.

Example 4: Basis and Dimension of Solution Spaces

Determine the basis and dimension for the solution space of the following:
(a) $x_1 + 2x_2 + 3x_3 = 0$, $x_1 + 5x_2 - x_3 = 0$, $3x_1 - 5x_2 + 8x_3 = 0$
(b) $x_1 + 2x_2 + 4x_3 + 3x_4 = 0$, $x_1 + 2x_2 - 4x_3 + 3x_4 = 0$, $2x_1 + 5x_2 - 2x_3 + x_4 = 0$
Note: Finding a basis for the solution space of a homogeneous system $A\mathbf{x} = \mathbf{0}$ is equivalent to finding a basis for the null space (or kernel) of matrix A. The row echelon form of A allows us to identify the free variables, which we can then use to construct the basis vectors for the null space.

Solution to Example 4(a)

The solution space of a homogeneous system of linear equations is the null space of the coefficient matrix. To find a basis for this solution space, we need to row reduce the coefficient matrix and find the general solution.
Step 1: Form the coefficient matrix for the system.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & -1 \\ 3 & -5 & 8 \end{pmatrix} \tag{7.499}$$

Step 2: Apply row reduction to get the matrix in row echelon form (REF).
$R_2 \to R_2 - R_1$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & -4 \\ 3 & -5 & 8 \end{pmatrix} \tag{7.500}$$

$R_3 \to R_3 - 3R_1$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & -4 \\ 0 & -11 & -1 \end{pmatrix} \tag{7.501}$$

$R_3 \to R_3 + \frac{11}{3}R_2$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & -4 \\ 0 & 0 & -1 + \frac{11}{3} \cdot (-\frac{4}{3}) \end{pmatrix} \tag{7.502}$$

Let's calculate the entry in position $(3, 3)$:

$$-1 + \frac{11}{3} \cdot (-\frac{4}{3}) = -1 - \frac{11 \cdot 4}{3 \cdot 3} \tag{7.503}$$

$$= -1 - \frac{44}{9} \tag{7.504}$$

$$= -\frac{9}{9} - \frac{44}{9} \tag{7.505}$$

$$= -\frac{53}{9} \tag{7.506}$$

So the matrix becomes:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & -4 \\ 0 & 0 & -\frac{53}{9} \end{pmatrix} \tag{7.507}$$

Step 3: Continue the row reduction to obtain the reduced row echelon form (RREF).
$R_3 \to -\frac{9}{53}R_3$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & -4 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.508}$$

$R_2 \to R_2 + \frac{4}{3}R_3$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.509}$$

$R_1 \to R_1 - 3R_3$:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.510}$$

$R_2 \to \frac{1}{3}R_2$:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.511}$$

$R_1 \to R_1 - 2R_2$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.512}$$

Step 4: Analyze the RREF to determine the basis and dimension of the solution space.
The RREF shows that the coefficient matrix has full rank (rank = 3) with 3 pivot positions. Since the system has 3 variables and the rank of the coefficient matrix is 3, by the rank-nullity theorem:

$$\dim(\ker(A)) = n - \text{rank}(A) = 3 - 3 = 0 \tag{7.513}$$

Therefore, the solution space has dimension 0, which means it contains only the zero vector $(0, 0, 0)$. The basis for this solution space is the empty set $\{\}$, and the only solution to the system is the trivial solution.

Solution to Example 4(b)

We follow the same approach for the second system.
Step 1: Form the coefficient matrix for the system.

$$A = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 1 & 2 & -4 & 3 \\ 2 & 5 & -2 & 1 \end{pmatrix} \tag{7.514}$$

Step 2: Apply row reduction to get the matrix in row echelon form (REF).

$R_2 \to R_2 - R_1$:

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & -8 & 0 \\ 2 & 5 & -2 & 1 \end{pmatrix} \tag{7.515}$$

$R_3 \to R_3 - 2R_1$:

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & -8 & 0 \\ 0 & 1 & -10 & -5 \end{pmatrix} \tag{7.516}$$

$R_2 \to -\frac{1}{8}R_2$:

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -10 & -5 \end{pmatrix} \tag{7.517}$$

$R_3 \to R_3 + 10R_2$:

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -5 \end{pmatrix} \tag{7.518}$$

Step 3: Continue the row reduction to obtain the reduced row echelon form (RREF).
$R_1 \to R_1 - 4R_2$:

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -5 \end{pmatrix} \tag{7.519}$$

$R_1 \to R_1 - 2R_3$:

$$\begin{pmatrix} 1 & 0 & 0 & 13 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -5 \end{pmatrix} \tag{7.520}$$

Step 4: Analyze the RREF to determine the basis and dimension of the solution space.
The RREF shows that the coefficient matrix has rank 3 (there are 3 pivot columns: columns 1, 2, and 3). The pivot variables are x_1, x_3, and x_2. The non-pivot variable is x_4, which is a free variable.
We can express the pivot variables in terms of the free variable:

$$x_1 = -13x_4 \tag{7.521}$$
$$x_2 = 5x_4 \tag{7.522}$$
$$x_3 = 0 \tag{7.523}$$

Therefore, the general solution to the system is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -13x_4 \\ 5x_4 \\ 0 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -13 \\ 5 \\ 0 \\ 1 \end{pmatrix} \tag{7.524}$$

By the rank-nullity theorem:

$$\dim(\ker(A)) = n - \text{rank}(A) = 4 - 3 = 1 \tag{7.525}$$

Therefore, the solution space has dimension 1. A basis for this solution space is:

$$\left\{ \begin{pmatrix} -13 \\ 5 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{7.526}$$

To verify that this is indeed a solution, let's substitute it back into the original system:
For the first equation:

$$1 \cdot (-13) + 2 \cdot 5 + 4 \cdot 0 + 3 \cdot 1 = -13 + 10 + 0 + 3 \tag{7.527}$$
$$= 0 \tag{7.528}$$

For the second equation:

$$1 \cdot (-13) + 2 \cdot 5 + (-4) \cdot 0 + 3 \cdot 1 = -13 + 10 + 0 + 3 \tag{7.529}$$
$$= 0 \tag{7.530}$$

For the third equation:

$$2 \cdot (-13) + 5 \cdot 5 + (-2) \cdot 0 + 1 \cdot 1 = -26 + 25 + 0 + 1 \tag{7.531}$$
$$= 0 \tag{7.532}$$

Therefore, $\mathbf{v} = (-13, 5, 0, 1)$ is indeed a solution, and since the dimension of the solution space is 1, this vector forms a basis for the solution space.

Example 5: Finding Bases for Row and Column Spaces

Find the basis for the row space and column space of A if

$$A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 0 \end{pmatrix} \tag{7.533}$$

Solution to Example 5

To find bases for the row and column spaces of matrix A, we will use the row echelon form (REF) approach. The row echelon form preserves the row space, and the pivot columns of the REF identify the basis vectors for the column space.
Step 1: Find the row echelon form of A.
Let's apply elementary row operations to reduce A to row echelon form.
$R_2 \to R_2 - 2R_1$:

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ -1 & 3 & 2 & 0 \end{pmatrix} \tag{7.534}$$

$R_3 \to R_3 + R_1$:

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 2 \end{pmatrix} \tag{7.535}$$

$R_3 \to R_3 + R_2$:

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{7.536}$$

$R_2 \to -\frac{1}{7}R_2$:

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{7.537}$$

$R_3 \to -\frac{1}{2}R_3$:

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.538}$$

The matrix is now in row echelon form. The pivot positions are in columns 1, 2, and 4.

Step 2: Find the rank of the matrix.
From the row echelon form, we can see that there are 3 non-zero rows, so the rank of A is 3.

Step 3: Find a basis for the row space.
The non-zero rows of the row echelon form span the row space. However, to find a basis for the original row space, we need to use the corresponding rows from the original matrix A. Since row operations preserve the row space, and all three rows are linearly independent (as evident from the REF having three non-zero rows), a basis for the row space is given by the three rows of the original matrix A:

$$\text{Basis for row space of } A = \{(1, 4, 5, 2), (2, 1, 3, 0), (-1, 3, 2, 0)\} \tag{7.539}$$

Step 4: Find a basis for the column space.
To find a basis for the column space, we identify the columns of A that correspond to the pivot positions in the REF. The pivot positions are in columns 1, 2, and 4. Therefore, a basis for the column space consists of columns 1, 2, and 4 of the original matrix A:

$$\text{Basis for column space of } A = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\} \tag{7.540}$$

Step 5: Verification of the dimensions.
We have found that:

- The row space has dimension 3 (with 3 basis vectors)
- The column space has dimension 3 (with 3 basis vectors)

This is consistent with the rank-nullity theorem, as the rank of A is 3.
Alternative approach for the row space basis:
Since row operations preserve the row space, we can also use the non-zero rows of the REF as a basis for the row space:

$$\text{Alternative basis for row space of } A = \left\{(1,4,5,2),(0,1,1,\frac{4}{7}),(0,0,0,1)\right\} \tag{7.541}$$

This basis is more convenient in some applications as it clearly shows the pivot positions and displays the reduced structure of the row space.
Note: The choice between using the original rows or the rows of the REF as a basis for the row space depends on the specific context and application.

Example 6: Finding Bases for Row and Column Spaces

Find the basis for the row space and column space of A if

$$A = \begin{pmatrix} 2 & -3 & 6 \\ 1 & -1 & 5 \\ -1 & 2 & 0 \\ 4 & 1 & 1 \end{pmatrix} \tag{7.542}$$

Solution to Example 6

To find bases for the row and column spaces of matrix A, we will apply row reduction to find the row echelon form (REF). This will allow us to determine the rank of the matrix and identify the appropriate basis vectors.
Step 1: Find the row echelon form of A.
Let's apply elementary row operations to reduce A to row echelon form.
$R_2 \to R_2 - \frac{1}{2}R_1$:

$$\begin{pmatrix} 2 & -3 & 6 \\ 0 & \frac{1}{2} & 2 \\ -1 & 2 & 0 \\ 4 & 1 & 1 \end{pmatrix} \tag{7.543}$$

$R_3 \to R_3 + \frac{1}{2}R_1$:

$$\begin{pmatrix} 2 & -3 & 6 \\ 0 & \frac{1}{2} & 2 \\ 0 & \frac{1}{2} & 3 \\ 4 & 1 & 1 \end{pmatrix} \tag{7.544}$$

$R_4 \to R_4 - 2R_1$:

$$\begin{pmatrix} 2 & -3 & 6 \\ 0 & \frac{1}{2} & 2 \\ 0 & \frac{1}{2} & 3 \\ 0 & 7 & -11 \end{pmatrix} \tag{7.545}$$

$R_3 \to R_3 - R_2$:

$$\begin{pmatrix} 2 & -3 & 6 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & 1 \\ 0 & 7 & -11 \end{pmatrix} \tag{7.546}$$

$R_4 \to R_4 - 14R_2$:

$$\begin{pmatrix} 2 & -3 & 6 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -39 \end{pmatrix} \tag{7.547}$$

$R_4 \to R_4 + 39R_3$:

$$\begin{pmatrix} 2 & -3 & 6 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.548}$$

The matrix is now in row echelon form with three non-zero rows. Therefore, the rank of A is 3.

Step 2: Find a basis for the row space.

The non-zero rows of the row echelon form span the row space. Since row operations preserve the row space, the original rows corresponding to the non-zero rows in the REF form a basis for the row space. Since we have 3 non-zero rows in the REF, we take the first 3 rows of the original matrix A as a basis for the row space:

$$\text{Basis for row space of } A = \{(2, -3, 6), (1, -1, 5), (-1, 2, 0)\} \tag{7.549}$$

Step 3: Find a basis for the column space.

To find a basis for the column space, we identify the columns of A that correspond to the pivot positions in the REF. The pivot positions are in columns 1, 2, and 3. Therefore, a basis for the column space consists of all three columns of the original matrix A:

$$\text{Basis for column space of } A = \left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 5 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{7.550}$$

Step 4: Verify that these are indeed bases.

To be a basis, a set of vectors must be linearly independent and span the respective space. For the row space basis: - The row echelon form already confirms that these rows are linearly independent (since they correspond to the non-zero rows in the REF). - The row space has dimension equal to the rank of the matrix, which is 3. Since we have exactly 3 vectors in our basis, they span the row space.

For the column space basis: - Since the pivot columns of the original matrix are linearly independent (as determined by the REF), our column space basis consists of linearly independent vectors. - The dimension of the column space equals the rank of the matrix, which is 3. Since we have 3 vectors in our column space basis, they span the column space.

Alternative approach for the row space basis:
We can also use the non-zero rows of the REF as an alternative basis for the row space:

$$\text{Alternative basis for row space of } A = \left\{(2, -3, 6), (0, \frac{1}{2}, 2), (0, 0, 1)\right\} \tag{7.551}$$

This alternative basis is in echelon form, which can be useful for certain applications.

Example 7: Rank

Find the rank and nullity of the matrix and hence verify the dimension theorem.

$$A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 0 \end{pmatrix} \tag{7.552}$$

Solution to Example 7

To find the rank and nullity of matrix A, we'll reduce the matrix to row echelon form (REF) and apply the dimension theorem, which states:

$$\text{rank}(A) + \text{nullity}(A) = n \tag{7.553}$$

where n is the number of columns in A.
Step 1: Find the row echelon form of A.
Let's apply elementary row operations to reduce A to row echelon form.
$R_2 \to R_2 - 2R_1$:

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ -1 & 3 & 2 & 0 \end{pmatrix} \tag{7.554}$$

$R_3 \to R_3 + R_1$:

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 2 \end{pmatrix} \tag{7.555}$$

$R_3 \to R_3 + R_2$:

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{7.556}$$

$R_2 \to -\frac{1}{7}R_2$:

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{7.557}$$

$R_3 \to -\frac{1}{2}R_3$:

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.558}$$

The matrix is now in row echelon form. We can identify the pivot positions (leading 1's) in columns 1, 2, and 4.

Step 2: Determine the rank of the matrix.

The rank of a matrix equals the number of non-zero rows in its row echelon form, or equivalently, the number of pivot positions. From the REF, we can see that:

$$\text{rank}(A) = 3 \tag{7.559}$$

Step 3: Determine the nullity of the matrix.

The nullity of a matrix is the dimension of its null space, which equals the number of free variables in the solution to $A\vec{x} = \vec{0}$. The number of free variables equals the number of columns minus the rank.

Given that matrix A has 4 columns and rank 3:

$$\text{nullity}(A) = n - \text{rank}(A) = 4 - 3 = 1 \tag{7.560}$$

Step 4: Verify the dimension theorem.

The dimension theorem states that $\text{rank}(A) + \text{nullity}(A) = n$, where n is the number of columns in A.

Let's verify:

$$\text{rank}(A) + \text{nullity}(A) = 3 + 1 \tag{7.561}$$
$$= 4 = n \tag{7.562}$$

Therefore, the dimension theorem is verified for matrix A.

Step 5: Find a basis for the null space.

Since $\text{nullity}(A) = 1$, the null space has dimension 1. To find a basis for the null space, we solve the homogeneous system $A\vec{x} = \vec{0}$.

From the row echelon form, we have:

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.563}$$

This gives us the system of equations:

$$x_1 + 4x_2 + 5x_3 + 2x_4 = 0 \tag{7.564}$$
$$x_2 + x_3 + \frac{4}{7}x_4 = 0 \tag{7.565}$$
$$x_4 = 0 \tag{7.566}$$

From the third equation, we have $x_4 = 0$. Substituting into the second equation: $x_2 + x_3 = 0$, which gives $x_2 = -x_3$. Substituting into the first equation: $x_1 + 4(-x_3) + 5x_3 + 2(0) = 0$, which gives $x_1 - 4x_3 + 5x_3 = 0$, thus $x_1 = -x_3$.

If we take $x_3 = t$ (where t is a parameter), then:

$$x_1 = -t \tag{7.567}$$
$$x_2 = -t \tag{7.568}$$
$$x_3 = t \tag{7.569}$$
$$x_4 = 0 \tag{7.570}$$

Therefore, the general solution to $A\vec{x} = \vec{0}$ is:

$$\vec{x} = t \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \tag{7.571}$$

A basis for the null space of A is:

$$\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \} \tag{7.572}$$

Conclusion:
- Rank of $A = 3$
- Nullity of $A = 1$
- The dimension theorem is verified: $\text{rank}(A) + \text{nullity}(A) = 4$
- A basis for the null space consists of the vector $(-1, -1, 1, 0)$