Chapter 2

Reduction Formulae in Calculus

2.1 List of Reduction Formulae

2.1.1 Indefinite Integral Reduction Formulae

Key Indefinite Integral Reduction Formulae

1. **Powers of Sine:**

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx \tag{2.1}$$

2. **Powers of Cosine:**

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx \tag{2.2}$$

3. **Powers of Tangent:**

$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx \tag{2.3}$$

4. **Powers of Secant:**

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \tag{2.4}$$

5. **Product of Sine and Cosine Powers:**

$$\int \sin^m x \cos^n x\,dx = \frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^{n+2} x\,dx \tag{2.5}$$

6. **Product of x^n and e^x:**

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx \tag{2.6}$$

7. **Product of x^n and $\ln(x)$:**

$$\int x^n \ln(x)\,dx = \frac{x^{n+1}\ln(x)}{n+1} - \frac{x^{n+1}}{(n+1)^2} \tag{2.7}$$

2.1.2 Definite Integral Reduction Formulae

Key Definite Integral Reduction Formulae

1. **Powers of Sine from 0 to $\pi/2$:**

$$\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}, & n \text{ is even} \\ \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1, & n \text{ is odd} \end{cases} \tag{2.8}$$

2. **Symmetry of Sine and Cosine:**

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx \tag{2.9}$$

3. **Product of Sine and Cosine from 0 to $\pi/2$:**

$$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{\{(m-1)(m-3)\cdots 2 \text{ or } 1\} \times \{(n-1)(n-3)\cdots 2 \text{ or } 1\}}{(m+n)(m+n-2)(m+n-4)\cdots 2 \text{ or } 1} \times p \tag{2.10}$$

where

$$p = \begin{cases} \frac{\pi}{2}, & m \text{ and } n \text{ both are even} \\ 1, & \text{for other values of } m \text{ and } n \end{cases} \tag{2.11}$$

4. **Sine on $[0, \pi]$:**

$$\int_0^{\pi} \sin^n x\,dx = 2\int_0^{\pi/2} \sin^n x\,dx, \text{ for all positive integer of } n. \tag{2.12}$$

5. **Cosine on $[0, \pi]$:**

$$\int_0^{\pi} \cos^n x\,dx = \begin{cases} 2 \times \int_0^{\pi/2} \cos^n x\,dx; & n \text{ is even} \\ 0; & n \text{ is odd} \end{cases} \tag{2.13}$$

6. **Sine on $[0, 2\pi]$:**

$$\int_0^{2\pi} \sin^n x\,dx = \begin{cases} 4 \times \int_0^{\pi/2} \sin^n x\,dx; & n \text{ is even} \\ 0; & n \text{ is odd} \end{cases} \tag{2.14}$$

7. **Cosine on $[0, 2\pi]$:**

$$\int_0^{2\pi} \cos^n x\,dx = \begin{cases} 4 \times \int_0^{\pi/2} \cos^n x\,dx; & n \text{ is even} \\ 0; & n \text{ is odd} \end{cases} \tag{2.15}$$

2.1.3 Trigonometric Identities

Essential Trigonometric Identities

1. $\sin^2 x = \frac{1-\cos 2x}{2}$
2. $\cos^2 x = \frac{1+\cos 2x}{2}$
3. $\sin^2 x + \cos^2 x = 1$
4. $\sin x \cos x = \frac{\sin 2x}{2}$
5. $\sin^2 x \cos^2 x = \frac{1-\cos 4x}{8}$
6. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
7. $\cos(A+B) = \cos A \cos B - \sin A \sin B$
8. $\sin 2x = 2 \sin x \cos x$
9. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

2.1.4 Substitution Techniques

Key Substitution Techniques

1. For $\sqrt{a^2 - x^2}$, use $x = a \sin\theta$ or $x = a\cos\theta$
2. For $\sqrt{a^2 + x^2}$, use $x = a\tan\theta$
3. For $\sqrt{x^2 - a^2}$, use $x = a\sec\theta$

2.1.5 Common Types of Reduction Formulae

Categories of Reduction Formulae

Reduction formulae are commonly established for:

1. **Powers of Trigonometric Functions:** Such as $\int \sin^n(x)\,dx$, $\int \cos^n(x)\,dx$, $\int \tan^n(x)\,dx$
2. **Products of Trigonometric Functions:** Such as $\int \sin^m(x)\cos^n(x)\,dx$
3. **Powers of Algebraic Expressions:** Such as $\int (ax+b)^n\,dx$
4. **Products with Other Functions:** Such as $\int x^n e^x\,dx$, $\int x^n \ln(x)\,dx$

2.1.6 Mathematical Significance

Significance of Reduction Formulae

- Reduction formulae transform complex integrals into manageable forms through systematic reduction of powers.
- They reveal mathematical patterns and relationships between seemingly different integrals.
- They provide elegant solutions to integration problems that might otherwise require extensive substitutions or reference to integration tables.

2.2 Theory of Reduction Formulae

2.2.1 Mathematical Foundations

Reduction formulae are based on fundamental calculus techniques, primarily integration by parts and algebraic manipulation. The standard form of integration by parts is:

Integration by Parts

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx \tag{2.16}$$

This formula is strategically applied to establish recurrence relations between integrals with different powers.

2.2.2 General Approach to Deriving Reduction Formulae

General Method for Deriving Reduction Formulae

To derive a reduction formula for $\int f^n(x)\,dx$:

1. Identify a suitable decomposition of the integrand into factors $u(x)$ and $v'(x)$
2. Apply integration by parts
3. Algebraically manipulate the result to express the original integral in terms of a simpler integral
4. Solve for $\int f^n(x)\,dx$ to obtain the reduction formula

2.2.3 Standard Reduction Formulae

Powers of Sine

Powers of Sine

For $n > 1$:

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx \tag{2.17}$$

Derivation

$$\int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx \tag{2.18}$$

$$\tag{2.19}$$

Using integration by parts with $u = \sin^{n-1} x$ and $dv = \sin x\,dx$:

$$du = (n-1)\sin^{n-2} x \cos x\,dx \tag{2.20}$$

$$v = -\cos x \tag{2.21}$$

This gives:

$$\int \sin^n x\,dx = -\sin^{n-1} x \cos x - \int (n-1)\sin^{n-2} x \cos x \cdot (-\cos x)\,dx \tag{2.22}$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx \tag{2.23}$$

Substituting $\cos^2 x = 1 - \sin^2 x$:

$$\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx \tag{2.24}$$
$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx \tag{2.25}$$

Rearranging to solve for $\int \sin^n x\,dx$:

$$\int \sin^n x\,dx + (n-1)\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx \tag{2.26}$$
$$n\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx \tag{2.27}$$

Dividing by n:

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx \tag{2.28}$$

Powers of Cosine

Powers of Cosine

For $n > 1$:

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx \tag{2.29}$$

Complete Derivation

We want to find a reduction formula for $\int \cos^n x\,dx$ where $n > 1$.

$$\int \cos^n x\,dx = \int \cos^{n-1} x\cos x\,dx \tag{2.30}$$

We'll use integration by parts with the formula:

$$\int u\vartheta\,dx = u\int \vartheta\,dx - \int [\frac{du}{dx}\int \vartheta\,dx]\,dx \tag{2.31}$$

Let:

$$u = \cos^{n-1} x \tag{2.32}$$
$$\vartheta = \cos x \tag{2.33}$$

This gives:

$$\frac{du}{dx} = (n-1)\cos^{n-2} x\cdot(-\sin x) = -(n-1)\cos^{n-2} x\sin x \tag{2.34}$$
$$\int \vartheta\,dx = \int \cos x\,dx = \sin x \tag{2.35}$$

Applying the integration by parts formula:

$$\int \cos^n x\,dx = \cos^{n-1} x\sin x - \int [-(n-1)\cos^{n-2} x\sin x\cdot\sin x]\,dx \tag{2.36}$$
$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\sin^2 x\,dx \tag{2.37}$$

Using the identity $\sin^2 x = 1 - \cos^2 x$:

$$\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx \tag{2.38}$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx \tag{2.39}$$

Rearranging to isolate $\int \cos^n x\,dx$:

$$\int \cos^n x\,dx + (n-1)\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx \tag{2.40}$$

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx \tag{2.41}$$

Dividing by n:

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx \tag{2.42}$$

Therefore, the reduction formula for powers of cosine is:

$$\boxed{\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx} \tag{2.43}$$

Powers of Tangent

Powers of Tangent

For $n \neq 1$:

$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx \tag{2.44}$$

Complete Derivation

We want to find a reduction formula for $\int \tan^n x\,dx$ where $n \neq 1$.
We'll use the identity $\tan^2 x = \sec^2 x - 1$ to begin the derivation:

$$\int \tan^n x\,dx = \int \tan^{n-2} x \cdot \tan^2 x\,dx \tag{2.45}$$

$$= \int \tan^{n-2} x \cdot (\sec^2 x - 1)\,dx \tag{2.46}$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx \tag{2.47}$$

For the first integral, we can make the substitution $u = \tan x$, which gives $\frac{du}{dx} = \sec^2 x$:

$$\int \tan^{n-2} x \sec^2 x \, dx = \int \tan^{n-2} x \frac{du}{dx} \, dx \tag{2.48}$$
$$= \int \tan^{n-2} x \, du \tag{2.49}$$
$$= \int u^{n-2} \, du \tag{2.50}$$
$$= \frac{u^{n-1}}{n-1} + C \quad (\text{for } n \neq 1) \tag{2.51}$$
$$= \frac{\tan^{n-1} x}{n-1} + C \tag{2.52}$$

Substituting back into our original expression:

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx \tag{2.53}$$

Therefore, the reduction formula for powers of tangent is:

$$\boxed{\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx} \tag{2.54}$$

Note that this formula is valid for $n \neq 1$. For $n = 1$, we have:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \tag{2.55}$$
$$= -\int \frac{-\sin x}{\cos x} \, dx \tag{2.56}$$
$$= -\int \frac{d(\cos x)}{\cos x} \tag{2.57}$$
$$= -\ln|\cos x| + C \tag{2.58}$$
$$= \ln|\sec x| + C \tag{2.59}$$

Powers of Secant

Powers of Secant

For $n > 2$:

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \tag{2.60}$$

Complete Derivation

We want to find a reduction formula for $\int \sec^n x \, dx$ where $n > 2$.
We can rewrite the integral as:

$$\int \sec^n x \, dx = \int \sec^{n-2} x \cdot \sec^2 x \, dx \tag{2.61}$$

Using integration by parts with the formula:

$$\int u\vartheta\,dx = u\int \vartheta\,dx - \int [\frac{du}{dx}\int \vartheta\,dx]\,dx \tag{2.62}$$

Let:

$$u = \sec^{n-2} x \tag{2.63}$$
$$\vartheta = \sec^2 x \tag{2.64}$$

This gives:

$$\frac{du}{dx} = (n-2)\sec^{n-2} x \tan x \tag{2.65}$$
$$\int \vartheta\,dx = \int \sec^2 x\,dx = \tan x \tag{2.66}$$

Applying the integration by parts formula:

$$\int \sec^{n-2} x \cdot \sec^2 x\,dx = \sec^{n-2} x \tan x - \int [(n-2)\sec^{n-2} x \tan x \cdot \tan x]\,dx \tag{2.67}$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx \tag{2.68}$$

Using the identity $\tan^2 x = \sec^2 x - 1$:

$$\int \sec^{n-2} x \cdot \sec^2 x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx \tag{2.69}$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx \tag{2.70}$$

Rearranging to isolate $\int \sec^n x\,dx$:

$$\int \sec^n x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx \tag{2.71}$$
$$(n-1)\int \sec^n x\,dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx \tag{2.72}$$

Dividing by $(n-1)$:

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \tag{2.73}$$

Therefore, the reduction formula for powers of secant is:

$$\boxed{\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx} \tag{2.74}$$

Product of Sine and Cosine Powers

Product of Sine and Cosine Powers

For $m > 0$:

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^{n+2} x \, dx \tag{2.75}$$

Complete Derivation

We want to find a reduction formula for $\int \sin^m x \cos^n x \, dx$ where $m > 0$.
We begin by rewriting the integral:

$$\int \sin^m x \cos^n x \, dx = \int \sin^{m-1} x \cos^n x \sin x \, dx \tag{2.76}$$

Using integration by parts with the formula:

$$\int u\vartheta \, dx = u \int \vartheta \, dx - \int \left[\frac{du}{dx} \int \vartheta \, dx \right] dx \tag{2.77}$$

Let:

$$u = \sin^{m-1} x \cos^n x \tag{2.78}$$

$$\vartheta = \sin x \tag{2.79}$$

This gives:

$$\frac{du}{dx} = (m-1) \sin^{m-2} x \cos^{n+1} x \tag{2.80}$$

$$+ \sin^{m-1} x \cdot n \cos^{n-1} x \cdot (-\sin x) \tag{2.81}$$

$$= (m-1) \sin^{m-2} x \cos^{n+1} x - n \sin^m x \cos^{n-1} x \tag{2.82}$$

And:

$$\int \vartheta \, dx = \int \sin x \, dx = -\cos x \tag{2.83}$$

Applying the integration by parts formula:

$$\int \sin^{m-1} x \cos^n x \sin x \, dx = \sin^{m-1} x \cos^n x \cdot (-\cos x) \tag{2.84}$$

$$- \int \left[(m-1) \sin^{m-2} x \cos^{n+1} x - n \sin^m x \cos^{n-1} x \right] \cdot (-\cos x) \, dx \tag{2.85}$$

Simplifying:

$$\int \sin^{m-1} x \cos^n x \sin x \, dx = -\sin^{m-1} x \cos^{n+1} x \tag{2.86}$$

$$+ \int (m-1) \sin^{m-2} x \cos^{n+2} x \, dx \tag{2.87}$$

$$- \int n \sin^m x \cos^n x \, dx \tag{2.88}$$

Rearranging to isolate $\int \sin^m x \cos^n x\, dx$:

$$\int \sin^m x \cos^n x\, dx + n \int \sin^m x \cos^n x\, dx = -\sin^{m-1} x \cos^{n+1} x \tag{2.89}$$

$$+ (m-1) \int \sin^{m-2} x \cos^{n+2} x\, dx \tag{2.90}$$

Therefore:

$$(m+n) \int \sin^m x \cos^n x\, dx = -\sin^{m-1} x \cos^{n+1} x \tag{2.91}$$

$$+ (m-1) \int \sin^{m-2} x \cos^{n+2} x\, dx \tag{2.92}$$

Dividing by $(m+n)$:

$$\int \sin^m x \cos^n x\, dx = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} \tag{2.93}$$

$$+ \frac{m-1}{m+n} \int \sin^{m-2} x \cos^{n+2} x\, dx \tag{2.94}$$

Adjusting the sign:

$$\int \sin^m x \cos^n x\, dx = \frac{\sin^{m-1} x \cos^{n+1} x}{m+n} \tag{2.95}$$

$$+ \frac{m-1}{m+n} \int \sin^{m-2} x \cos^{n+2} x\, dx \tag{2.96}$$

Therefore, the reduction formula for products of sine and cosine powers is:

$$\boxed{\int \sin^m x \cos^n x\, dx = \frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^{n+2} x\, dx} \tag{2.97}$$

Product of x^n and e^x

Product of x^n and e^x

$$\int x^n e^x\, dx = x^n e^x - n \int x^{n-1} e^x\, dx \tag{2.98}$$

Complete Derivation

We want to find a reduction formula for $\int x^n e^x\, dx$.
Using integration by parts with the formula:

$$\int u\vartheta\, dx = u \int \vartheta\, dx - \int [\frac{du}{dx} \int \vartheta\, dx]\, dx \tag{2.99}$$

Let:

$$u = x^n \tag{2.100}$$

$$\vartheta = e^x \tag{2.101}$$

This gives:

$$\frac{du}{dx} = nx^{n-1} \tag{2.102}$$

$$\int \vartheta\,dx = \int e^x\,dx = e^x \tag{2.103}$$

Applying the integration by parts formula:

$$\int x^n e^x\,dx = x^n \cdot e^x - \int [nx^{n-1} \cdot e^x]\,dx \tag{2.104}$$

$$= x^n e^x - n \int x^{n-1} e^x\,dx \tag{2.105}$$

Therefore, the reduction formula for the product of x^n and e^x is:

$$\boxed{\int x^n e^x\,dx = x^n e^x - n \int x^{n-1} e^x\,dx} \tag{2.106}$$

Note that the base case for this reduction is:

$$\int x^0 e^x\,dx = \int e^x\,dx = e^x + C \tag{2.107}$$

Therefore, we can use the reduction formula to express $\int x^n e^x\,dx$ in terms of elementary functions:

$$\int x^n e^x\,dx = x^n e^x - n \int x^{n-1} e^x\,dx \tag{2.108}$$

$$= x^n e^x - n(x^{n-1} e^x - (n-1) \int x^{n-2} e^x\,dx) \tag{2.109}$$

$$= x^n e^x - nx^{n-1} e^x + n(n-1) \int x^{n-2} e^x\,dx \tag{2.110}$$

$$\vdots \tag{2.111}$$

After continuing this process, we get:

$$\int x^n e^x\,dx = e^x \left(x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^n n!\right) + C \tag{2.112}$$

Product of x^n and $\ln(x)$

Product of x^n and $\ln(x)$

$$\int x^n \ln(x)\,dx = \frac{x^{n+1}\ln(x)}{n+1} - \frac{x^{n+1}}{(n+1)^2} \tag{2.113}$$

Complete Derivation

We want to find a reduction formula for $\int x^n \ln(x)\,dx$.

Using integration by parts with the formula:

$$\int u\vartheta\,dx = u\int \vartheta\,dx - \int [\frac{du}{dx}\int \vartheta\,dx]\,dx \tag{2.114}$$

Let:

$$u = \ln(x) \tag{2.115}$$
$$\vartheta = x^n \tag{2.116}$$

This gives:

$$\frac{du}{dx} = \frac{1}{x} \tag{2.117}$$
$$\int \vartheta\,dx = \int x^n\,dx = \frac{x^{n+1}}{n+1} \tag{2.118}$$

Applying the integration by parts formula:

$$\int x^n \ln(x)\,dx = \ln(x)\cdot\frac{x^{n+1}}{n+1} - \int [\frac{1}{x}\cdot\frac{x^{n+1}}{n+1}]\,dx \tag{2.119}$$
$$= \frac{x^{n+1}\ln(x)}{n+1} - \frac{1}{n+1}\int x^n\,dx \tag{2.120}$$
$$= \frac{x^{n+1}\ln(x)}{n+1} - \frac{1}{n+1}\cdot\frac{x^{n+1}}{n+1} \tag{2.121}$$
$$= \frac{x^{n+1}\ln(x)}{n+1} - \frac{x^{n+1}}{(n+1)^2} \tag{2.122}$$

Therefore, the formula for the product of x^n and $\ln(x)$ is:

$$\boxed{\int x^n \ln(x)\,dx = \frac{x^{n+1}\ln(x)}{n+1} - \frac{x^{n+1}}{(n+1)^2}} \tag{2.123}$$

2.2.4 Terminating the Recursion

Base Cases for Recursion

Every reduction formula eventually terminates at a base case:

- For $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$, the base cases are:

$$\int \sin^0 x\,dx = \int 1\,dx = x + C \tag{2.124}$$
$$\int \sin^1 x\,dx = -\cos x + C \tag{2.125}$$
$$\int \cos^0 x\,dx = \int 1\,dx = x + C \tag{2.126}$$
$$\int \cos^1 x\,dx = \sin x + C \tag{2.127}$$

- For $\int \tan^n x\,dx$, the base cases are:

$$\int \tan^0 x\,dx = \int 1\,dx = x + C \tag{2.128}$$

$$\int \tan^1 x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C \tag{2.129}$$

2.3 Solved Examples

Example 1

If $I_n = \int_0^{\frac{\pi}{4}} \sin^{2n} x\,dx$, prove that $I_n = \left(1 - \frac{1}{2n}\right) I_{n-1} - \frac{1}{n2^{n+1}}$.

Detailed Solution

We need to establish a relationship between I_n and I_{n-1} where:

$$I_n = \int_0^{\frac{\pi}{4}} \sin^{2n} x\,dx \tag{2.130}$$

$$I_{n-1} = \int_0^{\frac{\pi}{4}} \sin^{2n-2} x\,dx \tag{2.131}$$

Step 1: First, we'll rewrite the integral to apply integration by parts:

$$I_n = \int_0^{\frac{\pi}{4}} \sin^{2n} x\,dx = \int_0^{\frac{\pi}{4}} \sin^{2n-1} x \cdot \sin x\,dx \tag{2.132}$$

Step 2: We'll use the integration by parts formula:

$$\int u\vartheta\,dx = u\int \vartheta\,dx - \int \left[\frac{du}{dx}\int \vartheta\,dx\right] dx \tag{2.133}$$

Let:

$$u = \sin^{2n-1} x \tag{2.134}$$

$$\vartheta = \sin x \tag{2.135}$$

Then:

$$\frac{du}{dx} = (2n-1)\sin^{2n-2} x\cos x \tag{2.136}$$

$$\int \vartheta\,dx = \int \sin x\,dx = -\cos x \tag{2.137}$$

Step 3: Applying the integration by parts formula:

$$\int_0^{\frac{\pi}{4}} \sin^{2n} x\,dx = \left[\sin^{2n-1} x \cdot (-\cos x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \left[(2n-1)\sin^{2n-2} x\cos x \cdot (-\cos x)\right] dx \tag{2.138}$$

$$= \left[-\sin^{2n-1} x\cos x\right]_0^{\frac{\pi}{4}} + (2n-1)\int_0^{\frac{\pi}{4}} \sin^{2n-2} x\cos^2 x\,dx \tag{2.139}$$

Step 4: Evaluating the first term at the limits:

$$\left[-\sin^{2n-1}x\cos x\right]_0^{\frac{\pi}{4}} = -\sin^{2n-1}\frac{\pi}{4}\cos\frac{\pi}{4} + \sin^{2n-1}0\cos 0 \tag{2.140}$$

$$= -\sin^{2n-1}\frac{\pi}{4}\cos\frac{\pi}{4} + 0\cdot 1 \tag{2.141}$$

$$= -\sin^{2n-1}\frac{\pi}{4}\cos\frac{\pi}{4} \tag{2.142}$$

At $x = \frac{\pi}{4}$, we have $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, so:

$$-\sin^{2n-1}\frac{\pi}{4}\cos\frac{\pi}{4} = -\left(\frac{1}{\sqrt{2}}\right)^{2n-1}\cdot\frac{1}{\sqrt{2}} \tag{2.143}$$

$$= -\frac{1}{2^{\frac{2n-1}{2}}}\cdot\frac{1}{2^{\frac{1}{2}}} \tag{2.144}$$

$$= -\frac{1}{2^n} \tag{2.145}$$

Step 5: For the second term, we use the identity $\cos^2 x = 1 - \sin^2 x$:

$$(2n-1)\int_0^{\frac{\pi}{4}}\sin^{2n-2}x\cos^2 x\,dx = (2n-1)\int_0^{\frac{\pi}{4}}\sin^{2n-2}x(1-\sin^2 x)\,dx \tag{2.146}$$

$$= (2n-1)\int_0^{\frac{\pi}{4}}\sin^{2n-2}x\,dx - (2n-1)\int_0^{\frac{\pi}{4}}\sin^{2n}x\,dx \tag{2.147}$$

$$= (2n-1)I_{n-1} - (2n-1)I_n \tag{2.148}$$

Step 6: Combining all terms:

$$I_n = -\frac{1}{2^n} + (2n-1)I_{n-1} - (2n-1)I_n \tag{2.149}$$

$$2nI_n = -\frac{1}{2^n} + (2n-1)I_{n-1} \tag{2.150}$$

$$I_n = -\frac{1}{2n\cdot 2^n} + \frac{2n-1}{2n}I_{n-1} \tag{2.151}$$

$$I_n = \left(1 - \frac{1}{2n}\right)I_{n-1} - \frac{1}{n\cdot 2^{n+1}} \tag{2.152}$$

Therefore:

$$\boxed{I_n = \left(1 - \frac{1}{2n}\right)I_{n-1} - \frac{1}{n\cdot 2^{n+1}}} \tag{2.153}$$

This proves the desired formula.

Example 2

If $I_n = \int_0^{\frac{\pi}{4}}\cos^{2n}x\,dx$, prove that $I_n = \left(1 - \frac{1}{2n}\right)I_{n-1} + \frac{1}{n2^{n+1}}$.

Detailed Solution

We need to establish a relationship between I_n and I_{n-1} where:

$$I_n = \int_0^{\frac{\pi}{4}} \cos^{2n} x \, dx \tag{2.154}$$

$$I_{n-1} = \int_0^{\frac{\pi}{4}} \cos^{2n-2} x \, dx \tag{2.155}$$

Step 1: First, we'll rewrite the integral to apply integration by parts:

$$I_n = \int_0^{\frac{\pi}{4}} \cos^{2n} x \, dx = \int_0^{\frac{\pi}{4}} \cos^{2n-1} x \cdot \cos x \, dx \tag{2.156}$$

Step 2: We'll use the integration by parts formula:

$$\int u\vartheta \, dx = u \int \vartheta \, dx - \int \left[\frac{du}{dx} \int \vartheta \, dx \right] dx \tag{2.157}$$

Let:

$$u = \cos^{2n-1} x \tag{2.158}$$

$$\vartheta = \cos x \tag{2.159}$$

Then:

$$\frac{du}{dx} = (2n-1)\cos^{2n-2} x \cdot (-\sin x) = -(2n-1)\cos^{2n-2} x \sin x \tag{2.160}$$

$$\int \vartheta \, dx = \int \cos x \, dx = \sin x \tag{2.161}$$

Step 3: Applying the integration by parts formula:

$$\int_0^{\frac{\pi}{4}} \cos^{2n} x \, dx = \left[\cos^{2n-1} x \cdot \sin x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \left[-(2n-1)\cos^{2n-2} x \sin x \cdot \sin x\right] dx \tag{2.162}$$

$$= \left[\cos^{2n-1} x \sin x\right]_0^{\frac{\pi}{4}} + (2n-1)\int_0^{\frac{\pi}{4}} \cos^{2n-2} x \sin^2 x \, dx \tag{2.163}$$

Step 4: Evaluating the first term at the limits:

$$\left[\cos^{2n-1} x \sin x\right]_0^{\frac{\pi}{4}} = \cos^{2n-1} \frac{\pi}{4} \sin \frac{\pi}{4} - \cos^{2n-1} 0 \sin 0 \tag{2.164}$$

$$= \cos^{2n-1} \frac{\pi}{4} \sin \frac{\pi}{4} - 1 \cdot 0 \tag{2.165}$$

$$= \cos^{2n-1} \frac{\pi}{4} \sin \frac{\pi}{4} \tag{2.166}$$

At $x = \frac{\pi}{4}$, we have $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, so:

$$\cos^{2n-1} \frac{\pi}{4} \sin \frac{\pi}{4} = \left(\frac{1}{\sqrt{2}}\right)^{2n-1} \cdot \frac{1}{\sqrt{2}} \tag{2.167}$$

$$= \frac{1}{2^{\frac{2n-1}{2}}} \cdot \frac{1}{2^{\frac{1}{2}}} \tag{2.168}$$

$$= \frac{1}{2^n} \tag{2.169}$$

Step 5: For the second term, we use the identity $\sin^2 x = 1 - \cos^2 x$:

$$(2n-1)\int_0^{\frac{\pi}{4}} \cos^{2n-2} x \sin^2 x \, dx = (2n-1)\int_0^{\frac{\pi}{4}} \cos^{2n-2} x(1-\cos^2 x)\, dx \tag{2.170}$$

$$= (2n-1)\int_0^{\frac{\pi}{4}} \cos^{2n-2} x \, dx - (2n-1)\int_0^{\frac{\pi}{4}} \cos^{2n} x \, dx \tag{2.171}$$

$$= (2n-1)I_{n-1} - (2n-1)I_n \tag{2.172}$$

Step 6: Combining all terms:

$$I_n = \frac{1}{2^n} + (2n-1)I_{n-1} - (2n-1)I_n \tag{2.173}$$

$$2nI_n = \frac{1}{2^n} + (2n-1)I_{n-1} \tag{2.174}$$

$$I_n = \frac{1}{2n \cdot 2^n} + \frac{2n-1}{2n} I_{n-1} \tag{2.175}$$

$$I_n = \left(1 - \frac{1}{2n}\right) I_{n-1} + \frac{1}{n \cdot 2^{n+1}} \tag{2.176}$$

Therefore:

$$\boxed{I_n = \left(1 - \frac{1}{2n}\right) I_{n-1} + \frac{1}{n \cdot 2^{n+1}}} \tag{2.177}$$

This proves the desired formula.

Example 3

If $I_n = \int_0^{\frac{\pi}{4}} \frac{\sin(2n-1)x}{\sin x}\, dx$, then prove that $n(I_{n+1} - I_n) = \sin \frac{n\pi}{2}$ and hence find I_3.

Detailed Solution

Let's use a more direct method with integration by substitution.
Step 1: First, we need to express $I_{n+1} - I_n$:

$$I_{n+1} - I_n = \int_0^{\frac{\pi}{4}} \frac{\sin(2n+1)x}{\sin x}\, dx - \int_0^{\frac{\pi}{4}} \frac{\sin(2n-1)x}{\sin x}\, dx \tag{2.178}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}\, dx \tag{2.179}$$

Step 2: Using the identity $\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$:

$$\sin(2n+1)x - \sin(2n-1)x = 2\sin x \cos(2nx) \tag{2.180}$$

Step 3: Substituting this:

$$I_{n+1} - I_n = \int_0^{\frac{\pi}{4}} \frac{2 \sin x \cos(2nx)}{\sin x}\, dx \tag{2.181}$$

$$= 2 \int_0^{\frac{\pi}{4}} \cos(2nx)\, dx \tag{2.182}$$

$$= 2 \cdot \frac{\sin(2nx)}{2n}\Big|_0^{\frac{\pi}{4}} \tag{2.183}$$

$$= \frac{\sin\left(2n \cdot \frac{\pi}{4}\right) - \sin(0)}{n} \tag{2.184}$$

$$= \frac{\sin\left(\frac{n\pi}{2}\right)}{n} \tag{2.185}$$

Therefore:

$$n(I_{n+1} - I_n) = \sin \frac{n\pi}{2} \tag{2.186}$$

Step 4: To find I_3, we need to establish a recursive formula and an initial value. From our result:

$$I_{n+1} - I_n = \frac{\sin \frac{n\pi}{2}}{n} \tag{2.187}$$

Since we need I_3, let's compute:

$$I_2 - I_1 = \frac{\sin \frac{\pi}{2}}{1} = \frac{1}{1} = 1 \tag{2.188}$$

$$I_3 - I_2 = \frac{\sin \pi}{2} = \frac{0}{2} = 0 \tag{2.189}$$

Therefore:

$$I_3 = I_2 = I_1 + 1 \tag{2.190}$$

Step 5: We need to find I_1:

$$I_1 = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x}\, dx \tag{2.191}$$

$$= \int_0^{\frac{\pi}{4}} 1\, dx \tag{2.192}$$

$$= x\Big|_0^{\frac{\pi}{4}} \tag{2.193}$$

$$= \frac{\pi}{4} \tag{2.194}$$

Step 6: Now we can compute I_3:

$$I_3 = I_1 + 1 \tag{2.195}$$

$$= \frac{\pi}{4} + 1 \tag{2.196}$$

Therefore:

$$\boxed{I_3 = \frac{\pi}{4} + 1} \tag{2.197}$$

Example 4

If $I_n = \int_0^{\frac{\pi}{4}} \tan^n\theta\, d\theta$, prove that $I_n = \frac{1}{n-1} - I_{n-2}$. Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^6\theta\, d\theta$.

Detailed Solution

We need to establish a relationship between I_n and I_{n-2} where:

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n\theta\, d\theta \tag{2.198}$$

Step 1: First, we'll split $\tan^n\theta$ into $\tan^{n-2}\theta\cdot\tan^2\theta$ and use the identity $\tan^2\theta = \sec^2\theta - 1$:

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{4}} \tan^n\theta\, d\theta && (2.199)\\
&= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\cdot\tan^2\theta\, d\theta && (2.200)\\
&= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\cdot(\sec^2\theta - 1)\, d\theta && (2.201)\\
&= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\cdot\sec^2\theta\, d\theta - \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\, d\theta && (2.202)\\
&= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\cdot\sec^2\theta\, d\theta - I_{n-2} && (2.203)
\end{aligned}$$

Step 2: For the first integral, we'll use the formula $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$.
We know that $\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$, so $\sec^2\theta$ is the derivative of $\tan\theta$.
If we set $f(\theta) = \tan\theta$, then $f'(\theta) = \sec^2\theta$, and $[f(\theta)]^{n-2} = \tan^{n-2}\theta$.
Using the formula with $n-2$ in place of n:

$$\begin{aligned}
\int \tan^{n-2}\theta\cdot\sec^2\theta\, d\theta &= \int [f(\theta)]^{n-2} f'(\theta)\, d\theta && (2.204)\\
&= \frac{[f(\theta)]^{n-2+1}}{n-2+1} && (2.205)\\
&= \frac{\tan^{n-1}\theta}{n-1} && (2.206)
\end{aligned}$$

Step 3: Evaluating this at the limits:

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\cdot\sec^2\theta\, d\theta &= \left.\frac{\tan^{n-1}\theta}{n-1}\right|_0^{\frac{\pi}{4}} && (2.207)\\
&= \frac{\tan^{n-1}\frac{\pi}{4}}{n-1} - \frac{\tan^{n-1}0}{n-1} && (2.208)\\
&= \frac{1^{n-1}}{n-1} - \frac{0^{n-1}}{n-1} && (2.209)\\
&= \frac{1}{n-1} && (2.210)
\end{aligned}$$

since $\tan\frac{\pi}{4} = 1$ and $\tan 0 = 0$.
Step 4: Therefore:

$$I_n = \frac{1}{n-1} - I_{n-2} \tag{2.211}$$

Step 5: To evaluate $\int_0^{\frac{\pi}{4}} \tan^6\theta\,d\theta = I_6$, we need to find I_4, I_2, and I_0 first.
Using our reduction formula:

$$I_6 = \frac{1}{6-1} - I_{6-2} \quad (2.212)$$
$$= \frac{1}{5} - I_4 \quad (2.213)$$
$$I_4 = \frac{1}{4-1} - I_{4-2} \quad (2.214)$$
$$= \frac{1}{3} - I_2 \quad (2.215)$$
$$I_2 = \frac{1}{2-1} - I_{2-2} \quad (2.216)$$
$$= 1 - I_0 \quad (2.217)$$

Step 6: We need to compute I_0:

$$I_0 = \int_0^{\frac{\pi}{4}} \tan^0\theta\,d\theta \quad (2.218)$$
$$= \int_0^{\frac{\pi}{4}} 1\,d\theta \quad (2.219)$$
$$= \theta\Big|_0^{\frac{\pi}{4}} \quad (2.220)$$
$$= \frac{\pi}{4} - 0 \quad (2.221)$$
$$= \frac{\pi}{4} \quad (2.222)$$

Step 7: Now we can calculate I_2:

$$I_2 = 1 - I_0 \quad (2.223)$$
$$= 1 - \frac{\pi}{4} \quad (2.224)$$
$$= \frac{4}{4} - \frac{\pi}{4} \quad (2.225)$$
$$= \frac{4-\pi}{4} \quad (2.226)$$

Step 8: Calculating I_4:

$$I_4 = \frac{1}{3} - I_2 \quad (2.227)$$
$$= \frac{1}{3} - \left(\frac{4-\pi}{4}\right) \quad (2.228)$$
$$= \frac{1}{3} - \frac{4-\pi}{4} \quad (2.229)$$
$$= \frac{4}{12} - \frac{3(4-\pi)}{12} \quad (2.230)$$
$$= \frac{4-12+3\pi}{12} \quad (2.231)$$
$$= \frac{-8+3\pi}{12} \quad (2.232)$$
$$= \frac{3\pi-8}{12} \quad (2.233)$$

Step 9: Finally, calculating I_6:

$$I_6 = \frac{1}{5} - I_4 \tag{2.234}$$
$$= \frac{1}{5} - \frac{3\pi - 8}{12} \tag{2.235}$$
$$= \frac{12}{60} - \frac{5(3\pi - 8)}{60} \tag{2.236}$$
$$= \frac{12 - 15\pi + 40}{60} \tag{2.237}$$
$$= \frac{52 - 15\pi}{60} \tag{2.238}$$

Therefore:

$$\boxed{\int_0^{\frac{\pi}{4}} \tan^6\theta \, d\theta = \frac{52 - 15\pi}{60}} \tag{2.239}$$

Example 5

If $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n\theta \, d\theta$, prove that $I_n = \frac{1}{n-1} - I_{n-2}$. Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^6\theta \, d\theta$.

Detailed Solution

We need to establish the reduction formula for:

$$I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n\theta \, d\theta \tag{2.240}$$

Step 1: First, we rewrite $\cot^n\theta$ by splitting off two powers:

$$I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n\theta \, d\theta \tag{2.241}$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2}\theta \cdot \cot^2\theta \, d\theta \tag{2.242}$$

Step 2: Using the identity $\cot^2\theta = \csc^2\theta - 1$:

$$I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2}\theta \cdot (\csc^2\theta - 1) \, d\theta \tag{2.243}$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2}\theta \cdot \csc^2\theta \, d\theta - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2}\theta \, d\theta \tag{2.244}$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2}\theta \cdot \csc^2\theta \, d\theta - I_{n-2} \tag{2.245}$$

Step 3: We know that $\frac{d}{d\theta}(\cot\theta) = -\csc^2\theta$, so:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2}\theta \cdot \csc^2\theta \, d\theta = -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2}\theta \cdot (-\csc^2\theta) \, d\theta \tag{2.246}$$
$$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2}\theta \cdot \frac{d}{d\theta}(\cot\theta) \, d\theta \tag{2.247}$$

Step 4: Using the formula $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$ with $f(\theta) = \cot\theta$ and $n = n-2$:

$$-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2}\theta \cdot \frac{d}{d\theta}(\cot\theta)\,d\theta = -\left[\frac{(\cot\theta)^{n-2+1}}{n-2+1}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \tag{2.248}$$

$$= -\left[\frac{(\cot\theta)^{n-1}}{n-1}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \tag{2.249}$$

$$= -\frac{1}{n-1}\left[\cot^{n-1}\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \tag{2.250}$$

$$= -\frac{1}{n-1}\left[\cot^{n-1}\frac{\pi}{2} - \cot^{n-1}\frac{\pi}{4}\right] \tag{2.251}$$

$$= -\frac{1}{n-1}[0-1] \tag{2.252}$$

$$= -\frac{1}{n-1}(-1) \tag{2.253}$$

$$= \frac{1}{n-1} \tag{2.254}$$

Since $\cot\frac{\pi}{2} = 0$ and $\cot\frac{\pi}{4} = 1$.
Step 5: Combining our results:

$$I_n = \frac{1}{n-1} - I_{n-2} \tag{2.255}$$

Step 6: To evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^6\theta\,d\theta = I_6$, we need to use the recurrence relation repeatedly. Using the reduction formula:

$$I_6 = \frac{1}{5} - I_4 \tag{2.256}$$

$$I_4 = \frac{1}{3} - I_2 \tag{2.257}$$

$$I_2 = \frac{1}{1} - I_0 \tag{2.258}$$

Step 7: For I_0:

$$I_0 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1\,d\theta \tag{2.259}$$

$$= \theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \tag{2.260}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} \tag{2.261}$$

$$= \frac{\pi}{4} \tag{2.262}$$

Step 8: Now we calculate I_2:

$$I_2 = \frac{1}{1} - I_0 \tag{2.263}$$

$$= 1 - \frac{\pi}{4} \tag{2.264}$$

Step 9: Calculating I_4:

$$I_4 = \frac{1}{3} - I_2 \tag{2.265}$$
$$= \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) \tag{2.266}$$
$$= \frac{1}{3} - 1 + \frac{\pi}{4} \tag{2.267}$$
$$= -\frac{2}{3} + \frac{\pi}{4} \tag{2.268}$$

Step 10: Finally, calculating I_6:

$$I_6 = \frac{1}{5} - I_4 \tag{2.269}$$
$$= \frac{1}{5} - \left(-\frac{2}{3} + \frac{\pi}{4}\right) \tag{2.270}$$
$$= \frac{1}{5} + \frac{2}{3} - \frac{\pi}{4} \tag{2.271}$$
$$= \frac{3}{15} + \frac{10}{15} - \frac{\pi}{4} \tag{2.272}$$
$$= \frac{13}{15} - \frac{\pi}{4} \tag{2.273}$$

Therefore:

$$\boxed{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^6 \theta \, d\theta = \frac{13}{15} - \frac{\pi}{4}} \tag{2.274}$$

Example 6

Evaluate $\int_0^\pi x \sin^7 x \cos^4 x \, dx$.

Detailed Solution

We need to evaluate:

$$I = \int_0^\pi x \sin^7 x \cos^4 x \, dx \tag{1}$$
(2.275)

Step 1: Using the substitution property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$:

$$I = \int_0^\pi (\pi - x) \sin^7(\pi - x) \cos^4(\pi - x) \, dx \tag{2.276}$$

Step 2: Using the identities $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$:

$$I = \int_0^\pi (\pi - x) \sin^7 x \, (-\cos x)^4 \, dx \tag{2.277}$$
$$= \int_0^\pi (\pi - x) \sin^7 x \, \cos^4 x \, dx \quad (\text{since } (-\cos x)^4 = \cos^4 x) \tag{2.278}$$

Step 3: Expanding the integrand:

$$I = \int_0^\pi \pi \sin^7 x \cos^4 x \, dx - \int_0^\pi x \sin^7 x \cos^4 x \, dx \quad (2.279)$$

$$= \pi \int_0^\pi \sin^7 x \cos^4 x \, dx - I \quad (2.280)$$

Step 4: Solving for I:

$$2I = \pi \int_0^\pi \sin^7 x \cos^4 x \, dx \quad (2.281)$$

$$(2.282)$$

Step 5: For the integral $\int_0^\pi \sin^7 x \cos^4 x \, dx$, we can use symmetry properties. For m, n positive integers, we have:

$$\int_0^\pi \sin^m x \cos^n x \, dx = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2\int_0^{\pi/2} \sin^m x \cos^n x \, dx, & \text{if } n \text{ is even} \end{cases} \quad (2.283)$$

Since $n = 4$ is even:

$$\int_0^\pi \sin^7 x \cos^4 x \, dx = 2\int_0^{\pi/2} \sin^7 x \cos^4 x \, dx \quad (2.284)$$

Step 6: Thus:

$$2I = \pi \cdot 2 \int_0^{\pi/2} \sin^7 x \cos^4 x \, dx \quad (2.285)$$

$$I = \pi \int_0^{\pi/2} \sin^7 x \cos^4 x \, dx \quad (2.286)$$

Step 7: We need to evaluate $\int_0^{\pi/2} \sin^7 x \cos^4 x \, dx$. We can use the formula:

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\{(m-1)(m-3)\cdots 2 \text{ or } 1\} \times \{(n-1)(n-3)\cdots 2 \text{ or } 1\}}{(m+n)(m+n-2)(m+n-4)\cdots 2 \text{ or } 1} \times p \quad (2.287)$$

where

$$p = \begin{cases} \frac{\pi}{2}, & m \text{ and } n \text{ both are even} \\ 1, & \text{for other values of } m \text{ and } n \end{cases} \quad (2.288)$$

Step 8: In our case, $m = 7$ and $n = 4$. Since m is odd and n is even, $p = 1$. Computing the numerator:

$$(m-1)(m-3)(m-5) \times (n-1)(n-3) \quad (2.289)$$

$$= (7-1)(7-3)(7-5) \times (4-1)(4-3) \quad (2.290)$$

$$= 6 \times 4 \times 2 \times 3 \times 1 \quad (2.291)$$

$$= 144 \quad (2.292)$$

Computing the denominator:

$$(m+n)(m+n-2)(m+n-4)(m+n-6)(m+n-8)(m+n-10) \tag{2.293}$$
$$= (7+4)(11-2)(11-4)(11-6)(11-8)(11-10) \tag{2.294}$$
$$= 11 \times 9 \times 7 \times 5 \times 3 \times 1 \tag{2.295}$$
$$= 10395 \tag{2.296}$$

Step 9: Therefore:

$$\int_0^{\pi/2} \sin^7 x \cos^4 x \, dx = \frac{6 \times 4 \times 2 \times 3 \times 1}{11 \times 9 \times 7 \times 5 \times 3 \times 1} \times 1 \tag{2.297}$$
$$= \frac{144}{10395} \tag{2.298}$$
$$= \frac{16}{1155} \tag{2.299}$$

Step 10: Finally:

$$I = \pi \int_0^{\pi/2} \sin^7 x \cos^4 x \, dx \tag{2.300}$$
$$= \pi \cdot \frac{16}{1155} \tag{2.301}$$
$$= \frac{16\pi}{1155} \tag{2.302}$$

Therefore:

$$\boxed{\int_0^{\pi} x \sin^7 x \cos^4 x \, dx = \frac{16\pi}{1155}} \tag{2.303}$$

Example 7

Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx$.

We can directly apply the reduction formula for powers of sine from 0 to $\frac{\pi}{2}$:

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \begin{cases} \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}, & n \text{ is even} \\ \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1, & n \text{ is odd} \end{cases} \tag{2.304}$$

For $n = 6$ (even case):

$$\int_0^{\frac{\pi}{2}} \sin^6 x \, dx = \frac{6-1}{6} \times \frac{6-3}{6-2} \times \frac{6-5}{6-4} \times \frac{\pi}{2} \tag{2.305}$$
$$= \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \tag{2.306}$$
$$= \frac{5 \times 3 \times 1}{6 \times 4 \times 2} \times \frac{\pi}{2} \tag{2.307}$$
$$= \frac{15}{48} \times \frac{\pi}{2} \tag{2.308}$$
$$= \frac{5\pi}{32} \tag{2.309}$$

Therefore:

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^6 x\,dx = \frac{5\pi}{32}} \tag{2.310}$$

Example 8

Evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x\,dx$.

We'll use the symmetry property of sine and cosine:

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx \tag{2.311}$$

Therefore:

$$\int_0^{\frac{\pi}{2}} \cos^5 x\,dx = \int_0^{\frac{\pi}{2}} \sin^5 x\,dx \tag{2.312}$$

Now we can apply the reduction formula for powers of sine with $n = 5$ (odd case):

$$\int_0^{\frac{\pi}{2}} \sin^5 x\,dx = \frac{5-1}{5} \times \frac{5-3}{5-2} \times \frac{5-5}{5-4} \times 1 \tag{2.313}$$

$$= \frac{4}{5} \times \frac{2}{3} \times 1 \tag{2.314}$$

$$= \frac{8}{15} \tag{2.315}$$

Therefore:

$$\boxed{\int_0^{\frac{\pi}{2}} \cos^5 x\,dx = \frac{8}{15}} \tag{2.316}$$

Example 9

Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x\,dx$.

We'll use the product formula for sine and cosine:

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx = \frac{\{(m-1)(m-3)\cdots 2 \text{ or } 1\} \times \{(n-1)(n-3)\cdots 2 \text{ or } 1\}}{(m+n)(m+n-2)(m+n-4)\cdots 2 \text{ or } 1} \times p \tag{2.317}$$

where

$$p = \begin{cases} \frac{\pi}{2}, & m \text{ and } n \text{ both are even} \\ 1, & \text{for other values of } m \text{ and } n \end{cases} \tag{2.318}$$

For $m = 3$ (odd) and $n = 2$ (even), $p = 1$.

Computing the numerator:

$$\{(m-1)(m-3)\cdots 2 \text{ or } 1\} \times \{(n-1)(n-3)\cdots 2 \text{ or } 1\} = \{(3-1)\} \times \{(2-1)\} \tag{2.319}$$
$$= 2 \times 1 \tag{2.320}$$
$$= 2 \tag{2.321}$$

Computing the denominator:

$$(m+n)(m+n-2)(m+n-4)\cdots 2 \text{ or } 1 = (3+2)(5-2)(5-4) \tag{2.322}$$
$$= 5 \times 3 \times 1 \tag{2.323}$$
$$= 15 \tag{2.324}$$

Therefore:

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx = \frac{2}{15} \times 1 \tag{2.325}$$
$$= \frac{2}{15} \tag{2.326}$$

Therefore:

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx = \frac{2}{15}} \tag{2.327}$$

Example 10

Evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x \, dx$.

We'll use the product formula for sine and cosine:

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{\{(m-1)(m-3)\cdots 2 \text{ or } 1\} \times \{(n-1)(n-3)\cdots 2 \text{ or } 1\}}{(m+n)(m+n-2)(m+n-4)\cdots 2 \text{ or } 1} \times p \tag{2.328}$$

For $m = 4$ (even) and $n = 6$ (even), $p = \frac{\pi}{2}$.
Computing the numerator:

$$\{(m-1)(m-3)\cdots 2 \text{ or } 1\} \times \{(n-1)(n-3)\cdots 2 \text{ or } 1\} \tag{2.329}$$
$$= \{(4-1)(4-3)\} \times \{(6-1)(6-3)(6-5)\} \tag{2.330}$$
$$= \{3 \times 1\} \times \{5 \times 3 \times 1\} \tag{2.331}$$
$$= 3 \times 15 \tag{2.332}$$
$$= 45 \tag{2.333}$$

Computing the denominator:

$$(m+n)(m+n-2)(m+n-4)\cdots 2 \text{ or } 1 \tag{2.334}$$
$$= (4+6)(10-2)(10-4)(10-6)(10-8) \tag{2.335}$$
$$= 10 \times 8 \times 6 \times 4 \times 2 \tag{2.336}$$
$$= 3840 \tag{2.337}$$

Therefore:

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x \, dx = \frac{45}{3840} \times \frac{\pi}{2} \tag{2.338}$$

$$= \frac{45\pi}{7680} \tag{2.339}$$

$$= \frac{3\pi}{512} \tag{2.340}$$

Therefore:

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x \, dx = \frac{3\pi}{512}} \tag{2.341}$$

Example 11

Evaluate $\int_0^{\pi} \sin^5 x \, dx$.

We'll use the formula for sine on $[0, \pi]$:

$$\int_0^{\pi} \sin^n x \, dx = 2 \int_0^{\frac{\pi}{2}} \sin^n x \, dx, \text{ for all positive integers } n. \tag{2.342}$$

First, we calculate $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$ using the reduction formula for powers of sine with $n = 5$ (odd case):

$$\int_0^{\frac{\pi}{2}} \sin^5 x \, dx = \frac{5-1}{5} \times \frac{5-3}{5-2} \times 1 \tag{2.343}$$

$$= \frac{4}{5} \times \frac{2}{3} \tag{2.344}$$

$$= \frac{8}{15} \tag{2.345}$$

Therefore:

$$\int_0^{\pi} \sin^5 x \, dx = 2 \times \int_0^{\frac{\pi}{2}} \sin^5 x \, dx \tag{2.346}$$

$$= 2 \times \frac{8}{15} \tag{2.347}$$

$$= \frac{16}{15} \tag{2.348}$$

Therefore:

$$\boxed{\int_0^{\pi} \sin^5 x \, dx = \frac{16}{15}} \tag{2.349}$$

Example 12

Evaluate $\int_0^{\pi} \cos^4 x \, dx$.

We'll use the formula for cosine on $[0, \pi]$:

$$\int_0^{\pi} \cos^n x\, dx = \begin{cases} 2 \times \int_0^{\frac{\pi}{2}} \cos^n x\, dx; & n \text{ is even} \\ 0; & n \text{ is odd} \end{cases} \tag{2.350}$$

Since $n = 4$ is even, we have:

$$\int_0^{\pi} \cos^4 x\, dx = 2 \times \int_0^{\frac{\pi}{2}} \cos^4 x\, dx \tag{2.351}$$

Using the symmetry property $\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$:

$$\int_0^{\frac{\pi}{2}} \cos^4 x\, dx = \int_0^{\frac{\pi}{2}} \sin^4 x\, dx \tag{2.352}$$

Applying the reduction formula for powers of sine with $n = 4$ (even case):

$$\int_0^{\frac{\pi}{2}} \sin^4 x\, dx = \frac{4-1}{4} \times \frac{4-3}{4-2} \times \frac{\pi}{2} \tag{2.353}$$

$$= \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \tag{2.354}$$

$$= \frac{3\pi}{16} \tag{2.355}$$

Therefore:

$$\int_0^{\pi} \cos^4 x\, dx = 2 \times \frac{3\pi}{16} \tag{2.356}$$

$$= \frac{3\pi}{8} \tag{2.357}$$

Therefore:

$$\boxed{\int_0^{\pi} \cos^4 x\, dx = \frac{3\pi}{8}} \tag{2.358}$$

Example 13

Evaluate $\int_0^{\pi} \cos^3 x\, dx$.

We'll use the formula for cosine on $[0, \pi]$:

$$\int_0^{\pi} \cos^n x\, dx = \begin{cases} 2 \times \int_0^{\frac{\pi}{2}} \cos^n x\, dx; & n \text{ is even} \\ 0; & n \text{ is odd} \end{cases} \tag{2.359}$$

Since $n = 3$ is odd, we immediately have:

$$\int_0^{\pi} \cos^3 x\, dx = 0 \tag{2.360}$$

Therefore:

$$\boxed{\int_0^{\pi} \cos^3 x\, dx = 0} \tag{2.361}$$

Example 14

Evaluate $\int_0^{2\pi} \sin^6 x\, dx$.

We'll use the formula for sine on $[0, 2\pi]$:

$$\int_0^{2\pi} \sin^n x\, dx = \begin{cases} 4 \times \int_0^{\frac{\pi}{2}} \sin^n x\, dx; & n \text{ is even} \\ 0; & n \text{ is odd} \end{cases} \tag{2.362}$$

Since $n = 6$ is even, we have:

$$\int_0^{2\pi} \sin^6 x\, dx = 4 \times \int_0^{\frac{\pi}{2}} \sin^6 x\, dx \tag{2.363}$$

We already computed this in Example 9:

$$\int_0^{\frac{\pi}{2}} \sin^6 x\, dx = \frac{5\pi}{32} \tag{2.364}$$

Therefore:

$$\int_0^{2\pi} \sin^6 x\, dx = 4 \times \frac{5\pi}{32} \tag{2.365}$$

$$= \frac{5\pi}{8} \tag{2.366}$$

Therefore:

$$\boxed{\int_0^{2\pi} \sin^6 x\, dx = \frac{5\pi}{8}} \tag{2.367}$$

Example 15

Evaluate $\int_0^{2\pi} \sin^5 x\, dx$.

We'll use the formula for sine on $[0, 2\pi]$:

$$\int_0^{2\pi} \sin^n x\, dx = \begin{cases} 4 \times \int_0^{\frac{\pi}{2}} \sin^n x\, dx; & n \text{ is even} \\ 0; & n \text{ is odd} \end{cases} \tag{2.368}$$

Since $n = 5$ is odd, we immediately have:

$$\int_0^{2\pi} \sin^5 x\, dx = 0 \tag{2.369}$$

Therefore:

$$\boxed{\int_0^{2\pi} \sin^5 x\, dx = 0} \tag{2.370}$$

Example 16

Evaluate $\int_0^{2\pi} \cos^4 x\, dx$.

We'll use the formula for cosine on $[0, 2\pi]$:

$$\int_0^{2\pi} \cos^n x\,dx = \begin{cases} 4 \times \int_0^{\frac{\pi}{2}} \cos^n x\,dx; & n \text{ is even} \\ 0; & n \text{ is odd} \end{cases} \tag{2.371}$$

Since $n = 4$ is even, we have:

$$\int_0^{2\pi} \cos^4 x\,dx = 4 \times \int_0^{\frac{\pi}{2}} \cos^4 x\,dx \tag{2.372}$$

Using the symmetry property $\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$:

$$\int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \int_0^{\frac{\pi}{2}} \sin^4 x\,dx \tag{2.373}$$

Applying the reduction formula for powers of sine with $n = 4$ (even case):

$$\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \frac{4-1}{4} \times \frac{4-3}{4-2} \times \frac{\pi}{2} \tag{2.374}$$

$$= \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \tag{2.375}$$

$$= \frac{3\pi}{16} \tag{2.376}$$

Therefore:

$$\int_0^{2\pi} \cos^4 x\,dx = 4 \times \frac{3\pi}{16} \tag{2.377}$$

$$= \frac{3\pi}{4} \tag{2.378}$$

Therefore:

$$\boxed{\int_0^{2\pi} \cos^4 x\,dx = \frac{3\pi}{4}} \tag{2.379}$$