Chapter 2

Introduction to Ordinary Differential Equations

Differential equations are fundamental tools for modeling the behavior of dynamic systems in science, engineering, economics, and numerous other fields. They appear whenever we seek to understand how quantities change with respect to one another. This chapter introduces the key concepts and terminology of differential equations, preparing the foundation for the solution techniques presented in subsequent chapters.

Real-World Applications

Differential equations model countless phenomena:

- The trajectory of celestial bodies (planetary motion)
- The spread of diseases in a population (epidemiology)
- The flow of heat through materials (thermodynamics)
- The change in populations over time (ecology)
- The movement of electrical signals in circuits (electronics)
- The behavior of financial markets (economics)

2.1 Basic Concepts and Terminology

Definition 2.1 (Differential Equation)**.** *A* ***differential equation*** *is an equation that contains one or more derivatives of an unknown function. If the unknown function depends on a single variable, the equation is called an* ***ordinary differential equation*** *(ODE). If the unknown function depends on multiple variables, the equation is called a* ***partial differential equation*** *(PDE).*

Examples of Differential Equations

$$\frac{dy}{dx} + 2y = \sin x \quad \text{(First-order ODE)} \tag{2.1}$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^x \quad \text{(Second-order ODE)} \tag{2.2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{(Laplace's equation - a PDE)} \tag{2.3}$$

2.1.1 Order and Degree of Differential Equations

Definition 2.2 (Order of a Differential Equation). *The* ***order*** *of a differential equation is the highest derivative that appears in the equation.*

Definition 2.3 (Degree of a Differential Equation). *The* ***degree*** *of a differential equation is the power to which the highest-order derivative is raised, after the equation has been made free from radicals and fractions in the derivatives.*

Order and Degree Examples

$$\left(\frac{dy}{dx}\right)^2 + y = x^3 \quad \text{Order: 1, Degree: 2} \tag{2.4}$$

$$\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^4 = \sin x \quad \text{Order: 3, Degree: 1} \tag{2.5}$$

$$\sqrt{\frac{d^2y}{dx^2}} + y = e^x \quad \text{Order: 2, Degree: 1 (after squaring both sides)} \tag{2.6}$$

2.1.2 Classification of Differential Equations

Differential equations can be classified in several ways:

- **By type**: Ordinary (ODEs) or Partial (PDEs)
- **By order**: First-order, second-order, etc.
- **By linearity**: Linear or nonlinear
- **By homogeneity**: Homogeneous or non-homogeneous

Definition 2.4 (Linear Differential Equation). *A differential equation is* ***linear*** *if it can be written in the form:*

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{2.7}$$

where $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ *and* $g(x)$ *are functions of* x *only (not involving* y *or its derivatives).*

Definition 2.5 (Homogeneous Differential Equation). *A linear differential equation is* ***homogeneous*** *if* $g(x) = 0$. *Otherwise, it is* ***non-homogeneous****.*

Classification Examples

$$\frac{dy}{dx} + 2y = 0 \quad \text{(Linear, homogeneous, first-order ODE)} \tag{2.8}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin x \quad \text{(Linear, non-homogeneous, second-order ODE)} \tag{2.9}$$

$$\frac{dy}{dx} + y^2 = x \quad \text{(Nonlinear, first-order ODE due to } y^2 \text{ term)} \tag{2.10}$$

$$y\frac{d^2y}{dx^2} = x \quad \text{(Nonlinear, second-order ODE due to } y \text{ multiplying } \frac{d^2y}{dx^2}\text{)} \tag{2.11}$$

2.2 Solutions of Differential Equations

Definition 2.6 (Solution of a Differential Equation)**.** *A **solution** of a differential equation is a function that, when substituted into the equation, satisfies the equation identically for all values in its domain.*

Types of Solutions

For a differential equation of order n, we have:

- **General Solution**: Contains n arbitrary constants.
- **Particular Solution**: Obtained from the general solution by assigning specific values to the arbitrary constants, typically determined by initial or boundary conditions.
- **Singular Solution**: A solution that cannot be obtained from the general solution for any values of the constants.

Example of Solutions

Consider the first-order differential equation $\frac{dy}{dx} = 2xy$:

$$\text{General Solution} : y = Ce^{x^2} \quad (\text{where } C \text{ is an arbitrary constant}) \tag{2.12}$$

$$\text{Particular Solution} : y = 3e^{x^2} \quad (\text{when } C = 3) \tag{2.13}$$

$$\text{Trivial Solution} : y = 0 \quad (\text{when } C = 0) \tag{2.14}$$

To verify that $y = Ce^{x^2}$ is indeed a solution, we substitute it into the original equation:

$$\frac{dy}{dx} = \frac{d}{dx}(Ce^{x^2}) \tag{2.15}$$

$$= C \cdot \frac{d}{dx}(e^{x^2}) \tag{2.16}$$

$$= C \cdot e^{x^2} \cdot \frac{d}{dx}(x^2) \tag{2.17}$$

$$= C \cdot e^{x^2} \cdot 2x \tag{2.18}$$

$$= 2x \cdot Ce^{x^2} \tag{2.19}$$

$$= 2xy \tag{2.20}$$

Thus, $y = Ce^{x^2}$ satisfies the differential equation $\frac{dy}{dx} = 2xy$ and is its general solution.

Let's visualize some solutions to the above differential equation:

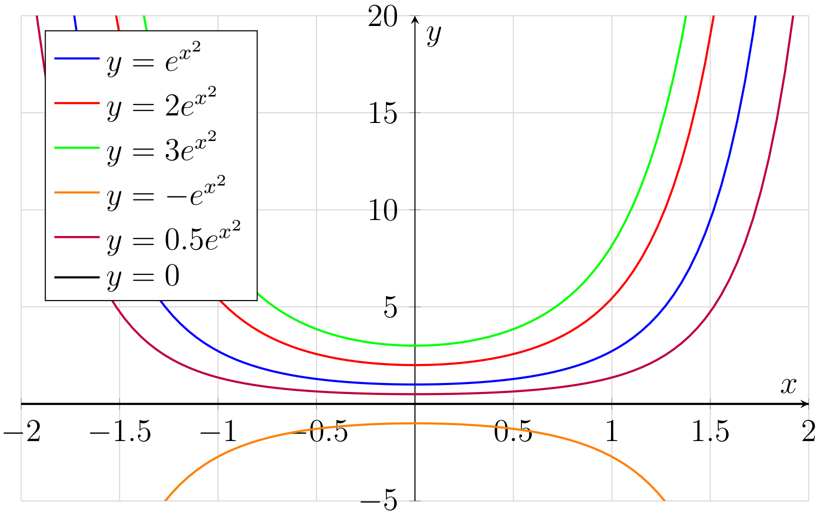

Figure 2.1: Family of solutions to the differential equation $\frac{dy}{dx} = 2xy$

2.2.1 Initial Value Problems

Definition 2.7 (Initial Value Problem)**.** *An **initial value problem** (IVP) consists of a differential equation together with an initial condition that specifies the value of the unknown function (and possibly its derivatives) at a particular point.*

Initial Value Problem Example

Consider the differential equation $\frac{dy}{dx} = 2xy$ with initial condition $y(0) = 3$.
The general solution is $y = Ce^{x^2}$. To find the particular solution that satisfies the initial condition:

$$y(0) = Ce^{0^2} = Ce^0 = C \cdot 1 = C = 3 \tag{2.21}$$

Therefore, the solution to the initial value problem is $y = 3e^{x^2}$.

Existence and Uniqueness

For a first-order initial value problem $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$, if $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous in a rectangle containing the point (x_0, y_0), then:

- There exists a solution to the IVP.
- The solution is unique in some interval containing x_0.

This is known as the Picard-Lindelöf theorem or the existence and uniqueness theorem.

2.3 Formation of Ordinary Differential Equations

One common way to form differential equations is by eliminating arbitrary constants from a given relation between the dependent and independent variables.

Formation by Eliminating Constants

Consider the family of circles centered on the y-axis:

$$(x - 0)^2 + (y - b)^2 = a^2 \tag{2.22}$$

where a and b are arbitrary constants. To find the differential equation satisfied by all circles in this family:
Step 1: Expand the equation.

$$x^2 + (y-b)^2 = a^2 \tag{2.23}$$
$$x^2 + y^2 - 2by + b^2 = a^2 \tag{2.24}$$

Step 2: Differentiate both sides with respect to x.

$$2x + 2y\frac{dy}{dx} - 2b\frac{dy}{dx} = 0 \tag{2.25}$$
$$2x + 2(y-b)\frac{dy}{dx} = 0 \tag{2.26}$$

Step 3: Solve for b from the original equation. From $x^2 + y^2 - 2by + b^2 = a^2$, we can rearrange to get:

$$b^2 - 2by + (y^2 + x^2 - a^2) = 0 \tag{2.27}$$

This is a quadratic in b. For this to have a solution, the discriminant must be zero:

$$(2y)^2 - 4(1)(y^2 + x^2 - a^2) = 0 \tag{2.28}$$
$$4y^2 - 4y^2 - 4x^2 + 4a^2 = 0 \tag{2.29}$$
$$-4x^2 + 4a^2 = 0 \tag{2.30}$$
$$a^2 = x^2 \tag{2.31}$$

Step 4: From the differential equation in Step 2, we can substitute $b = y - x\frac{dy}{dx}$ (which we get by solving for b) and replace a^2 with x^2:

$$2x + 2(y-b)\frac{dy}{dx} = 0 \tag{2.32}$$
$$2x + 2\left(y - \left(y - x\frac{dy}{dx}\right)\right)\frac{dy}{dx} = 0 \tag{2.33}$$
$$2x + 2\left(x\frac{dy}{dx}\right)\frac{dy}{dx} = 0 \tag{2.34}$$
$$2x + 2x\left(\frac{dy}{dx}\right)^2 = 0 \tag{2.35}$$
$$x\left(1 + \left(\frac{dy}{dx}\right)^2\right) = 0 \tag{2.36}$$

Since x can be zero (which corresponds to circles touching the y-axis), the differential equation of the family is:

$$1 + \left(\frac{dy}{dx}\right)^2 = 0 \tag{2.37}$$

Wait, this equation doesn't have real solutions! Let's double-check our work...
The error occurred when we solved for b. Let's try again using a different approach.
From the original equation:

$$x^2 + (y-b)^2 = a^2 \tag{2.38}$$
$$\tag{2.39}$$

Differentiating with respect to x:

$$2x + 2(y - b)\frac{dy}{dx} = 0 \tag{2.40}$$

$$\tag{2.41}$$

This gives us:

$$y - b = -\frac{x}{\frac{dy}{dx}} \tag{2.42}$$

$$\tag{2.43}$$

Squaring both sides and substituting into the original equation:

$$x^2 + \left(-\frac{x}{\frac{dy}{dx}}\right)^2 = a^2 \tag{2.44}$$

$$x^2 + \frac{x^2}{\left(\frac{dy}{dx}\right)^2} = a^2 \tag{2.45}$$

$$\tag{2.46}$$

Since we found earlier that $a^2 = x^2$, we substitute:

$$x^2 + \frac{x^2}{\left(\frac{dy}{dx}\right)^2} = x^2 \tag{2.47}$$

$$\frac{x^2}{\left(\frac{dy}{dx}\right)^2} = 0 \tag{2.48}$$

$$\tag{2.49}$$

This would imply that either $x = 0$ or $\frac{dy}{dx} = \infty$, which doesn't seem right.
Let's start over with a simpler example: consider the family of straight lines $y = mx + c$ where m and c are constants. To find the differential equation:
Differentiate with respect to x:

$$\frac{dy}{dx} = m \tag{2.50}$$

Differentiating again:

$$\frac{d^2y}{dx^2} = 0 \tag{2.51}$$

So the differential equation for the family of straight lines is $\frac{d^2y}{dx^2} = 0$.

2.3.1 Physical Examples Leading to Differential Equations

Many physical phenomena naturally lead to differential equations through the mathematical modeling process.

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change of temperature of an object is proportional to the difference between its temperature and the ambient temperature.
If we denote the temperature of the object at time t as $T(t)$ and the ambient temperature as T_a, then:

$$\frac{dT}{dt} = -k(T - T_a) \tag{2.52}$$

where $k > 0$ is the cooling constant.
This is a first-order linear differential equation that models heat transfer in many real-world situations.
Application: Suppose a cup of coffee at 90°C is placed in a room at 20°C. If after 5 minutes, the coffee's temperature is 70°C, when will the coffee be at a drinkable temperature of 50°C?
Solution: We need to solve the initial value problem:

$$\frac{dT}{dt} = -k(T - 20) \tag{2.53}$$
$$T(0) = 90 \tag{2.54}$$
$$T(5) = 70 \tag{2.55}$$

The general solution to this equation is:

$$T(t) = 20 + Ce^{-kt} \tag{2.56}$$

Using the initial condition:

$$90 = 20 + Ce^{0} \tag{2.57}$$
$$C = 70 \tag{2.58}$$

So $T(t) = 20 + 70e^{-kt}$. Using the second condition:

$$70 = 20 + 70e^{-5k} \tag{2.59}$$
$$50 = 70e^{-5k} \tag{2.60}$$
$$\frac{50}{70} = e^{-5k} \tag{2.61}$$
$$\ln\left(\frac{5}{7}\right) = -5k \tag{2.62}$$
$$k = \frac{-\ln\left(\frac{5}{7}\right)}{5} \approx 0.0682 \tag{2.63}$$

Now we can find when $T(t) = 50C$:

$$50 = 20 + 70e^{-0.0682t} \tag{2.64}$$
$$30 = 70e^{-0.0682t} \tag{2.65}$$
$$\frac{30}{70} = e^{-0.0682t} \tag{2.66}$$
$$\ln\left(\frac{3}{7}\right) = -0.0682t \tag{2.67}$$
$$t = \frac{-\ln\left(\frac{3}{7}\right)}{0.0682} \approx 12.7 \text{ minutes} \tag{2.68}$$

Therefore, the coffee will be at a drinkable temperature after approximately 12.7 minutes.

Simple Harmonic Motion

A mass attached to a spring follows the equation:

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{2.69}$$

where m is the mass, k is the spring constant, and x is the displacement from equilibrium. This is a second-order linear differential equation with constant coefficients, and it describes the oscillatory motion of the spring-mass system.
The general solution is:

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \tag{2.70}$$

where $\omega = \sqrt{\frac{k}{m}}$ is the natural frequency of the system, and A and B are constants determined by initial conditions.
Application: If a 0.5 kg mass is attached to a spring with spring constant $k = 2$ N/m and released from a position 10 cm from equilibrium with zero initial velocity, find the equation of motion.
Solution: We have $m = 0.5$ kg, $k = 2$ N/m, $x(0) = 0.1$ m, and $\frac{dx}{dt}(0) = 0$.
The natural frequency is:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{0.5}} = 2 \text{ rad/s} \tag{2.71}$$

The general solution is:

$$x(t) = A\cos(2t) + B\sin(2t) \tag{2.72}$$

Using the initial conditions:

$$x(0) = 0.1 \tag{2.73}$$
$$A\cos(0) + B\sin(0) = 0.1 \tag{2.74}$$
$$A = 0.1 \tag{2.75}$$

And:

$$\frac{dx}{dt}(0) = 0 \tag{2.76}$$
$$-2A\sin(0) + 2B\cos(0) = 0 \tag{2.77}$$
$$2B = 0 \tag{2.78}$$
$$B = 0 \tag{2.79}$$

Therefore, the equation of motion is:

$$x(t) = 0.1\cos(2t) \tag{2.80}$$

This represents simple harmonic motion with amplitude 10 cm and period $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$ seconds.

Population Growth

The simplest model for population growth is the Malthusian model:

$$\frac{dP}{dt} = rP \tag{2.81}$$

where $P(t)$ is the population at time t and r is the growth rate constant.
This is a first-order linear differential equation whose general solution is:

$$P(t) = P_0 e^{rt} \tag{2.82}$$

where P_0 is the initial population.
Application: If a bacterial colony initially contains 1000 cells and grows at a rate of 5% per hour, how many cells will be present after 12 hours?
Solution: We have $P_0 = 1000$, $r = 0.05$ per hour, and $t = 12$ hours.
Using the solution:

$$P(12) = 1000e^{0.05\cdot 12} = 1000e^{0.6} = 1000 \cdot 1.822 = 1822 \tag{2.83}$$

After 12 hours, there will be approximately 1822 cells in the colony.
A more realistic model for population growth is the logistic model:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{2.84}$$

where K is the carrying capacity of the environment. This model accounts for limited resources and leads to bounded population growth.