Chapter 4

Differentiation Under Integral Sign (DUIS) and Error Function

4.1 Differentiation Under Integral Sign (DUIS)

The Differentiation Under Integral Sign, also known as Leibniz's Rule, is a fundamental technique in calculus that allows us to differentiate an integral with respect to a parameter. This method is particularly valuable when direct integration is challenging or when we need to establish relationships between different functions.

4.1.1 Basic Forms of DUIS

Rule I: Constant Limits of Integration

For an integral of the form $I(\alpha) = \int_a^b f(x, \alpha)\, dx$, where a and b are constants independent of α:

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial}{\partial\alpha}[f(x,\alpha)]\,dx \tag{4.1}$$

Rule II: Variable Limits of Integration (Leibniz's Rule)

For an integral of the form $I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)\, dx$, where $a(\alpha)$ and $b(\alpha)$ are functions of α:

$$\frac{d}{d\alpha}\left[\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx\right] = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha}[f(x,\alpha)]\,dx \tag{4.2}$$

$$+ f(b(\alpha),\alpha)\cdot\frac{db(\alpha)}{d\alpha} - f(a(\alpha),\alpha)\cdot\frac{da(\alpha)}{d\alpha} \tag{4.3}$$

4.1.2 Key Conditions for Validity

- The function $f(x, \alpha)$ must be continuous in both variables within the domain of integration.
- The partial derivative $\frac{\partial f}{\partial\alpha}$ must exist and be continuous within the domain.
- For variable limits, the functions $a(\alpha)$ and $b(\alpha)$ must be differentiable.

4.2 Error Function and Its Properties

The Error Function is a special function that appears frequently in probability, statistics, and solutions to differential equations, particularly those involving diffusion processes.

4.2.1 Definition of Error Function

Error Function

The Error Function, denoted by $\mathrm{erf}(x)$, is defined as:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du \tag{4.4}$$

Complementary Error Function

The Complementary Error Function, denoted by $\mathrm{erfc}(x)$, is defined as:

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du \tag{4.5}$$

4.3 Differentiation Under Integral Sign: Mathematical Foundation

DUIS with Constant Limits

Leibniz Rule I

Theorem 4.1. *Let $f(x,\alpha)$ be a function such that both f and $\frac{\partial f}{\partial \alpha}$ are continuous in the region $[a,b] \times [\alpha_0 - \delta, \alpha_0 + \delta]$ for some $\delta > 0$. Then for $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta)$:*

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\, dx\right] = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)\, dx \tag{4.6}$$

Proof. Define $I(\alpha) = \int_a^b f(x,\alpha)\, dx$.
For a small increment h, we have:

$$\begin{aligned} I(\alpha + h) - I(\alpha) &= \int_a^b [f(x,\alpha+h) - f(x,\alpha)]\, dx && (4.7) \\ & && (4.8) \end{aligned}$$

By the Mean Value Theorem, for each fixed $x \in [a,b]$, there exists $\theta_x \in (0,1)$ such that:

$$f(x,\alpha+h) - f(x,\alpha) = h \cdot \frac{\partial f}{\partial \alpha}(x, \alpha + \theta_x h) \tag{4.9}$$

Therefore:

$$\begin{aligned} \frac{I(\alpha+h) - I(\alpha)}{h} &= \int_a^b \frac{f(x,\alpha+h) - f(x,\alpha)}{h}\, dx && (4.10) \\ &= \int_a^b \frac{\partial f}{\partial \alpha}(x, \alpha + \theta_x h)\, dx && (4.11) \end{aligned}$$

As $h \to 0$, by the continuity of $\frac{\partial f}{\partial \alpha}$, we get:

$$\lim_{h\to 0} \frac{I(\alpha+h)-I(\alpha)}{h} = \int_a^b \lim_{h\to 0} \frac{\partial f}{\partial \alpha}(x, \alpha+\theta_x h)\, dx \tag{4.12}$$

$$= \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx \tag{4.13}$$

Hence:

$$\frac{d}{d\alpha} I(\alpha) = \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx \tag{4.14}$$

□

DUIS with Variable Limits

Leibniz Rule II

Theorem 4.2. *Let $f(x,\alpha)$ be a function such that both f and $\frac{\partial f}{\partial \alpha}$ are continuous in a region containing $\{(x,\alpha) : a(\alpha) \le x \le b(\alpha), \alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]\}$ for some $\delta > 0$. If $a(\alpha)$ and $b(\alpha)$ are differentiable functions, then:*

$$\frac{d}{d\alpha}\left[\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\, dx\right] = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x,\alpha)\, dx \tag{4.15}$$

$$+ f(b(\alpha),\alpha) \cdot \frac{db(\alpha)}{d\alpha} - f(a(\alpha),\alpha) \cdot \frac{da(\alpha)}{d\alpha} \tag{4.16}$$

Proof. Let $I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\, dx$.
We introduce an auxiliary function:

$$F(t,\alpha) = \int_{a(\alpha)}^{t} f(x,\alpha)\, dx \tag{4.17}$$

Note that $I(\alpha) = F(b(\alpha),\alpha)$. Using the chain rule:

$$\frac{dI}{d\alpha} = \frac{\partial F}{\partial t} \cdot \frac{db}{d\alpha} + \frac{\partial F}{\partial \alpha} \tag{4.18}$$

$$= f(b(\alpha),\alpha) \cdot \frac{db}{d\alpha} + \frac{\partial F}{\partial \alpha} \tag{4.19}$$

For the partial derivative $\frac{\partial F}{\partial \alpha}$, we have:

$$\frac{\partial F}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{a(\alpha)}^{t} f(x,\alpha)\, dx \tag{4.20}$$

$$= \int_{a(\alpha)}^{t} \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx - f(a(\alpha),\alpha) \cdot \frac{da}{d\alpha} \tag{4.21}$$

Substituting $t = b(\alpha)$ and combining:

$$\frac{dI}{d\alpha} = f(b(\alpha),\alpha) \cdot \frac{db}{d\alpha} + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx - f(a(\alpha),\alpha) \cdot \frac{da}{d\alpha} \tag{4.22}$$

$$= \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx + f(b(\alpha),\alpha) \cdot \frac{db}{d\alpha} - f(a(\alpha),\alpha) \cdot \frac{da}{d\alpha} \tag{4.23}$$

□

Special Case: When Only One Limit is Variable

Corollary 4.3. *If only the upper limit depends on α (i.e., a is constant), then:*

$$\frac{d}{d\alpha}\left[\int_a^{b(\alpha)} f(x,\alpha)\,dx\right] = \int_a^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx + f(b(\alpha),\alpha)\cdot\frac{db(\alpha)}{d\alpha} \tag{4.24}$$

Similarly, if only the lower limit depends on α (i.e., b is constant), then:

$$\frac{d}{d\alpha}\left[\int_{a(\alpha)}^{b} f(x,\alpha)\,dx\right] = \int_{a(\alpha)}^{b} \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx - f(a(\alpha),\alpha)\cdot\frac{da(\alpha)}{d\alpha} \tag{4.25}$$

4.4 Error Function: Mathematical Theory

Definition and Basic Properties

Error Function

The Error Function $\operatorname{erf}(x)$ is defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \tag{4.26}$$

Complementary Error Function

The Complementary Error Function $\operatorname{erfc}(x)$ is defined as:

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt \tag{4.27}$$

Basic Properties of Error Function

Theorem 4.4. *The error function satisfies the following properties:*

1. $\operatorname{erf}(0) = 0$
2. $\lim_{x\to\infty} \operatorname{erf}(x) = 1$
3. $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ *(odd function)*
4. $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$
5. $\operatorname{erfc}(-x) = 2 - \operatorname{erfc}(x)$

Proof. For property 1:

$$\operatorname{erf}(0) = \frac{2}{\sqrt{\pi}}\int_0^0 e^{-t^2}\,dt = 0 \tag{4.28}$$

For property 2:

$$\lim_{x\to\infty} \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt \tag{4.29}$$

The value of the improper integral $\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$, which is a well-known result derived by considering $\left(\int_0^\infty e^{-t^2}\,dt\right)^2$ and converting to polar coordinates. Hence:

$$\lim_{x\to\infty} \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = 1 \tag{4.30}$$

For property 3:

$$\begin{aligned}
\operatorname{erf}(-x) &= \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2}\, dt && (4.31)\\
&= \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-(-u)^2}\, d(-u) \quad \text{(substituting } t = -u\text{)} && (4.32)\\
&= \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2}\, (-du) && (4.33)\\
&= -\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2}\, du && (4.34)\\
&= -\operatorname{erf}(x) && (4.35)
\end{aligned}$$

Properties 4 and 5 follow directly from the definitions and property 3. □

4.4.1 Derivatives and Integrals

Theorem 4.5 (Derivative of Error Function)**.**

$$\frac{d}{dx}\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \tag{4.36}$$

Proof. Using the Fundamental Theorem of Calculus:

$$\begin{aligned}
\frac{d}{dx}\operatorname{erf}(x) &= \frac{d}{dx}\left[\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2}\, dt\right] && (4.37)\\
&= \frac{2}{\sqrt{\pi}} \cdot \frac{d}{dx}\left[\int_0^{x} e^{-t^2}\, dt\right] && (4.38)\\
&= \frac{2}{\sqrt{\pi}} \cdot e^{-x^2} && (4.39)
\end{aligned}$$

□

Corollary 4.6 (Higher Derivatives)**.**

$$\begin{aligned}
\frac{d^2}{dx^2}\operatorname{erf}(x) &= \frac{d}{dx}\left[\frac{2}{\sqrt{\pi}} e^{-x^2}\right] && (4.40)\\
&= \frac{2}{\sqrt{\pi}} \cdot (-2x) \cdot e^{-x^2} && (4.41)\\
&= -\frac{4x}{\sqrt{\pi}} e^{-x^2} && (4.42)
\end{aligned}$$

And similarly:

$$\frac{d^3}{dx^3}\operatorname{erf}(x) = \frac{4}{\sqrt{\pi}} e^{-x^2}(2x^2 - 1) \tag{4.43}$$

Theorem 4.7 (Related Integrals)**.** *The error function satisfies the following integral re-*

lationships:

$$\int e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + C \tag{4.44}$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \tag{4.45}$$

$$\int_0^\infty e^{-(x+a)^2}\,dx = \frac{\sqrt{\pi}}{2}[1-\operatorname{erf}(a)] = \frac{\sqrt{\pi}}{2}\operatorname{erfc}(a) \tag{4.46}$$

$$\int_0^\infty e^{-x^2}\cos(2ax)\,dx = \frac{\sqrt{\pi}}{2}e^{-a^2} \tag{4.47}$$

4.5 Connection Between DUIS and Error Function

The differentiation under integral sign technique is particularly powerful for establishing properties and evaluating integrals involving the error function. Here, we'll demonstrate how DUIS helps derive key results.

Theorem 4.8 (Gaussian Integral via DUIS)**.** *For $a > 0$:*

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \tag{4.48}$$

Proof. Let $I(a) = \int_0^\infty e^{-ax^2}\,dx$ for $a > 0$.
Differentiating with respect to a using DUIS:

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}[e^{-ax^2}]\,dx \tag{4.49}$$

$$= \int_0^\infty (-x^2)e^{-ax^2}\,dx \tag{4.50}$$

$$= -\int_0^\infty x^2 e^{-ax^2}\,dx \tag{4.51}$$

Using integration by parts with $u = x$ and $dv = xe^{-ax^2}\,dx$:

$$\int_0^\infty x^2 e^{-ax^2}\,dx = \left[-\frac{x}{2a}e^{-ax^2}\right]_0^\infty + \frac{1}{2a}\int_0^\infty e^{-ax^2}\,dx \tag{4.52}$$

$$= 0 + \frac{1}{2a}I(a) \tag{4.53}$$

Therefore:

$$\frac{dI}{da} = -\frac{1}{2a}I(a) \tag{4.54}$$

$$\frac{dI}{I} = -\frac{1}{2a}da \tag{4.55}$$

$$\ln I = -\frac{1}{2}\ln a + C \tag{4.56}$$

$$I(a) = \frac{K}{\sqrt{a}} \tag{4.57}$$

where K is a constant. To find K, we use a special case. For $a = 1$:

$$I(1) = \int_0^\infty e^{-x^2}\,dx = K \tag{4.58}$$

It is known that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$, so $K = \frac{\sqrt{\pi}}{2}$.
Therefore:

$$I(a) = \frac{1}{2}\sqrt{\frac{\pi}{a}} \tag{4.59}$$

□

This example illustrates how DUIS can be used to derive fundamental results related to the error function and Gaussian integrals by converting difficult direct integration problems into more manageable differential equations.

4.6 Advanced Properties and Series Expansions

4.6.1 Series Expansion of Error Function

Theorem 4.9 (Power Series for Error Function)**.** *The error function has the following power series expansion:*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \tag{4.60}$$

valid for all $x \in \mathbb{R}$.

Proof. Starting with the definition:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \tag{4.61}$$

We use the Taylor series for e^{-t^2}:

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \tag{4.62}$$

Substituting this into the integral:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\,dt \tag{4.63}$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n}\,dt \tag{4.64}$$

Evaluating the integral:

$$\int_0^x t^{2n}\,dt = \left[\frac{t^{2n+1}}{2n+1}\right]_0^x = \frac{x^{2n+1}}{2n+1} \tag{4.65}$$

Therefore:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1} \tag{4.66}$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \tag{4.67}$$

□

Theorem 4.10 (Asymptotic Expansion for erfc(x)). *For large positive x, the complementary error function has the asymptotic expansion:*

$$\operatorname{erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2x^2)^n} \tag{4.68}$$

where $(2n-1)!! = (2n-1) \cdot (2n-3) \cdot \ldots \cdot 3 \cdot 1$.

This asymptotic expansion is particularly useful for numerical approximations when x is large.

4.7 Examples with Step-by-Step Solutions

Example 1: DUIS with variable Limits

Verify the rule of differentiation under integral sign for the integral:

$$I(\alpha) = \int_0^{\alpha^2} \frac{1}{x+\alpha}\, dx \tag{4.69}$$

and show that $\frac{dI}{d\alpha} = \frac{1}{a+1}$.

Detailed Solution

We need to apply Leibniz's Rule for the case where one of the limits is variable. This corresponds to the formula:

$$\frac{d}{d\alpha}\left[\int_a^{b(\alpha)} f(x,\alpha)\, dx\right] = \int_a^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x,\alpha)\, dx + f(b(\alpha),\alpha) \cdot \frac{db(\alpha)}{d\alpha} \tag{4.70}$$

In our case:

$$f(x,\alpha) = \frac{1}{x+\alpha} \tag{4.71}$$

$$a = 0 \text{ (constant lower limit)} \tag{4.72}$$

$$b(\alpha) = \alpha^2 \text{ (variable upper limit)} \tag{4.73}$$

Step 1: Find $\frac{\partial f}{\partial \alpha}$.

$$\frac{\partial f}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(\frac{1}{x+\alpha}\right) \tag{4.74}$$

$$= \frac{\partial}{\partial \alpha}(x+\alpha)^{-1} \tag{4.75}$$

$$= -1 \cdot (x+\alpha)^{-2} \cdot \frac{\partial}{\partial \alpha}(x+\alpha) \tag{4.76}$$

$$= -1 \cdot (x+\alpha)^{-2} \cdot 1 \tag{4.77}$$

$$= -\frac{1}{(x+\alpha)^2} \tag{4.78}$$

Step 2: Find $\frac{db(\alpha)}{d\alpha}$.

$$\frac{db(\alpha)}{d\alpha} = \frac{d}{d\alpha}(\alpha^2) \tag{4.79}$$

$$= 2\alpha \tag{4.80}$$

Step 3: Find $f(b(\alpha),\alpha)$.

$$f(b(\alpha),\alpha) = f(\alpha^2,\alpha) \tag{4.81}$$
$$= \frac{1}{\alpha^2+\alpha} \tag{4.82}$$
$$= \frac{1}{\alpha(\alpha+1)} \tag{4.83}$$

Step 4: Apply Leibniz's Rule.

$$\frac{dI}{d\alpha} = \int_0^{\alpha^2} \frac{\partial f}{\partial \alpha}\,dx + f(b(\alpha),\alpha)\cdot\frac{db(\alpha)}{d\alpha} \tag{4.84}$$
$$= \int_0^{\alpha^2} -\frac{1}{(x+\alpha)^2}\,dx + \frac{1}{\alpha(\alpha+1)}\cdot 2\alpha \tag{4.85}$$
$$= -\int_0^{\alpha^2} \frac{1}{(x+\alpha)^2}\,dx + \frac{2}{\alpha+1} \tag{4.86}$$

Step 5: Evaluate the integral.

$$\int \frac{1}{(x+\alpha)^2}\,dx = \int (x+\alpha)^{-2}\,dx \tag{4.87}$$
$$= -(x+\alpha)^{-1} + C \tag{4.88}$$
$$= -\frac{1}{x+\alpha} + C \tag{4.89}$$

So:

$$\int_0^{\alpha^2} \frac{1}{(x+\alpha)^2}\,dx = \left[-\frac{1}{x+\alpha}\right]_0^{\alpha^2} \tag{4.90}$$
$$= -\frac{1}{\alpha^2+\alpha} + \frac{1}{0+\alpha} \tag{4.91}$$
$$= -\frac{1}{\alpha(\alpha+1)} + \frac{1}{\alpha} \tag{4.92}$$
$$= \frac{-1}{\alpha(\alpha+1)} + \frac{\alpha+1}{\alpha(\alpha+1)} \tag{4.93}$$
$$= \frac{\alpha}{\alpha(\alpha+1)} \tag{4.94}$$
$$= \frac{1}{\alpha+1} \tag{4.95}$$

Step 6: Calculate $\frac{dI}{d\alpha}$.

$$\frac{dI}{d\alpha} = -\int_0^{\alpha^2} \frac{1}{(x+\alpha)^2}\,dx + \frac{2}{\alpha+1} \tag{4.96}$$
$$= -\frac{1}{\alpha+1} + \frac{2}{\alpha+1} \tag{4.97}$$
$$= \frac{1}{\alpha+1} \tag{4.98}$$

Therefore, we have verified that $\frac{dI}{d\alpha} = \frac{1}{\alpha+1}$, as required.

Alternative Approach: We could also solve this by finding $I(\alpha)$ directly and then differentiating.
Step 1: Evaluate $I(\alpha)$.

$$I(\alpha) = \int_0^{\alpha^2} \frac{1}{x+\alpha}\, dx \tag{4.99}$$

(4.100)

Let $u = x + \alpha$, then $du = dx$ and when $x = 0$, $u = \alpha$; when $x = \alpha^2$, $u = \alpha^2 + \alpha$.

$$\begin{aligned} I(\alpha) &= \int_\alpha^{\alpha^2+\alpha} \frac{1}{u}\, du && (4.101)\\ &= [\ln|u|]_\alpha^{\alpha^2+\alpha} && (4.102)\\ &= \ln|\alpha^2+\alpha| - \ln|\alpha| && (4.103)\\ &= \ln\left|\frac{\alpha^2+\alpha}{\alpha}\right| && (4.104)\\ &= \ln|\alpha+1| && (4.105) \end{aligned}$$

Step 2: Differentiate $I(\alpha)$.

$$\begin{aligned} \frac{dI}{d\alpha} &= \frac{d}{d\alpha}\ln|\alpha+1| && (4.106)\\ &= \frac{1}{\alpha+1}\cdot\frac{d}{d\alpha}(\alpha+1) && (4.107)\\ &= \frac{1}{\alpha+1}\cdot 1 && (4.108)\\ &= \frac{1}{\alpha+1} && (4.109) \end{aligned}$$

This confirms our result from applying Leibniz's Rule directly.

Example 2: DUIS with Inverse Tangent Function

Verify the rule of differentiation under integral sign for the integral:

$$I(\alpha) = \int_0^{\alpha^2} \tan^{-1}(x/\alpha)\, dx \tag{4.110}$$

and show that $\frac{dI}{d\alpha} = -\frac{1}{2}\ln(\alpha^2+1) + 2\alpha\tan^{-1}(\alpha)$.

Detailed Solution

We need to apply Leibniz's Rule for the case where one of the limits is variable and the integrand also contains the parameter. The formula is:

$$\frac{d}{d\alpha}\left[\int_a^{b(\alpha)} f(x,\alpha)\, dx\right] = \int_a^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha)\, dx + f(b(\alpha),\alpha)\cdot\frac{db(\alpha)}{d\alpha} \tag{4.111}$$

In our case:

$$f(x,\alpha) = \tan^{-1}(x/\alpha) \tag{4.112}$$
$$a = 0 \text{ (constant lower limit)} \tag{4.113}$$
$$b(\alpha) = \alpha^2 \text{ (variable upper limit)} \tag{4.114}$$

Step 1: Find $\frac{\partial f}{\partial \alpha}$.

$$\frac{\partial f}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\tan^{-1}(x/\alpha)\right] \tag{4.115}$$

Using the chain rule, if $u = x/\alpha$, then:

$$\frac{\partial f}{\partial \alpha} = \frac{d}{du}\left[\tan^{-1}(u)\right]\cdot\frac{\partial u}{\partial \alpha} \tag{4.116}$$
$$= \frac{1}{1+u^2}\cdot\frac{\partial}{\partial \alpha}\left(\frac{x}{\alpha}\right) \tag{4.117}$$
$$= \frac{1}{1+(x/\alpha)^2}\cdot\left(-\frac{x}{\alpha^2}\right) \tag{4.118}$$
$$= -\frac{x}{\alpha^2}\cdot\frac{1}{1+(x/\alpha)^2} \tag{4.119}$$
$$= -\frac{x}{\alpha^2}\cdot\frac{\alpha^2}{\alpha^2+x^2} \tag{4.120}$$
$$= -\frac{x}{\alpha^2+x^2} \tag{4.121}$$

Step 2: Find $\frac{db(\alpha)}{d\alpha}$.

$$\frac{db(\alpha)}{d\alpha} = \frac{d}{d\alpha}(\alpha^2) \tag{4.122}$$
$$= 2\alpha \tag{4.123}$$

Step 3: Find $f(b(\alpha),\alpha)$.

$$f(b(\alpha),\alpha) = f(\alpha^2,\alpha) \tag{4.124}$$
$$= \tan^{-1}(\alpha^2/\alpha) \tag{4.125}$$
$$= \tan^{-1}(\alpha) \tag{4.126}$$

Step 4: Apply Leibniz's Rule.

$$\frac{dI}{d\alpha} = \int_0^{\alpha^2}\frac{\partial f}{\partial \alpha}\,dx + f(b(\alpha),\alpha)\cdot\frac{db(\alpha)}{d\alpha} \tag{4.127}$$
$$= \int_0^{\alpha^2} -\frac{x}{\alpha^2+x^2}\,dx + \tan^{-1}(\alpha)\cdot 2\alpha \tag{4.128}$$
$$= -\int_0^{\alpha^2}\frac{x}{\alpha^2+x^2}\,dx + 2\alpha\tan^{-1}(\alpha) \tag{4.129}$$

Step 5: Evaluate the integral.

$$\int\frac{x}{\alpha^2+x^2}\,dx = \frac{1}{2}\int\frac{2x}{\alpha^2+x^2}\,dx \tag{4.130}$$
$$= \frac{1}{2}\ln|\alpha^2+x^2| + C \tag{4.131}$$

So:

$$\int_0^{\alpha^2} \frac{x}{\alpha^2 + x^2}\, dx = \left[\frac{1}{2}\ln|\alpha^2 + x^2|\right]_0^{\alpha^2} \tag{4.132}$$
$$= \frac{1}{2}\ln|\alpha^2 + (\alpha^2)^2| - \frac{1}{2}\ln|\alpha^2 + 0^2| \tag{4.133}$$
$$= \frac{1}{2}\ln|\alpha^2 + \alpha^4| - \frac{1}{2}\ln|\alpha^2| \tag{4.134}$$
$$= \frac{1}{2}\ln|\alpha^2(1 + \alpha^2)| - \frac{1}{2}\ln|\alpha^2| \tag{4.135}$$
$$= \frac{1}{2}\ln|\alpha^2| + \frac{1}{2}\ln|1 + \alpha^2| - \frac{1}{2}\ln|\alpha^2| \tag{4.136}$$
$$= \frac{1}{2}\ln|1 + \alpha^2| \tag{4.137}$$

Step 6: Calculate $\frac{dI}{d\alpha}$.

$$\frac{dI}{d\alpha} = -\int_0^{\alpha^2} \frac{x}{\alpha^2 + x^2}\, dx + 2\alpha\tan^{-1}(\alpha) \tag{4.138}$$
$$= -\frac{1}{2}\ln|1 + \alpha^2| + 2\alpha\tan^{-1}(\alpha) \tag{4.139}$$
$$= -\frac{1}{2}\ln\big(\alpha^2 + 1\big) + 2\alpha\tan^{-1}(\alpha) \tag{4.140}$$

Therefore, we have verified that $\frac{dI}{d\alpha} = -\frac{1}{2}\ln(\alpha^2 + 1) + 2\alpha\tan^{-1}(\alpha)$, as required.

Example 3: DUIS with Sine and Cosine Functions

If $y = \int_0^x f(t)[\sin x\cos t - \cos x\sin t]\, dt$, then find the value of $\frac{d^2y}{dx^2} + y$.

Detailed Solution

This problem involves differentiating an integral where the parameter x appears both in the upper limit of integration and in the integrand.
Step 1: Let's identify the integral.

$$y = \int_0^x f(t)[\sin x\cos t - \cos x\sin t]\, dt \tag{4.141}$$

Step 2: Apply Leibniz's Rule to find $\frac{dy}{dx}$.
Using the formula:

$$\frac{d}{dx}\left[\int_a^{b(x)} g(t,x)\, dt\right] = \int_a^{b(x)} \frac{\partial g}{\partial x}(t,x)\, dt + g(b(x),x)\cdot\frac{db(x)}{dx} \tag{4.142}$$

In our case:

$$g(t,x) = f(t)[\sin x\cos t - \cos x\sin t] \tag{4.143}$$
$$a = 0 \text{ (constant lower limit)} \tag{4.144}$$
$$b(x) = x \text{ (variable upper limit)} \tag{4.145}$$

First, we calculate $\frac{\partial g}{\partial x}$:

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\{f(t)[\sin x \cos t - \cos x \sin t]\} \tag{4.146}$$
$$= f(t)\left[\frac{\partial}{\partial x}(\sin x \cos t) - \frac{\partial}{\partial x}(\cos x \sin t)\right] \tag{4.147}$$
$$= f(t)[(\cos x \cos t) - (-\sin x \sin t)] \tag{4.148}$$
$$= f(t)[\cos x \cos t + \sin x \sin t] \tag{4.149}$$
$$= f(t)\cos(x-t) \tag{4.150}$$

Now, we find $g(b(x), x)$:

$$g(b(x),x) = g(x,x) \tag{4.151}$$
$$= f(x)[\sin x \cos x - \cos x \sin x] \tag{4.152}$$
$$= f(x)[\sin x \cos x - \sin x \cos x] \tag{4.153}$$
$$= 0 \tag{4.154}$$

Since $\frac{db(x)}{dx} = 1$, we have:

$$\frac{dy}{dx} = \int_0^x \frac{\partial g}{\partial x}\,dt + g(b(x),x)\cdot\frac{db(x)}{dx} \tag{4.155}$$
$$= \int_0^x f(t)\cos(x-t)\,dt + 0\cdot 1 \tag{4.156}$$
$$= \int_0^x f(t)\cos(x-t)\,dt \tag{4.157}$$

Step 3: Find $\frac{d^2y}{dx^2}$ by differentiating $\frac{dy}{dx}$ with respect to x.
Using Leibniz's Rule again:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\int_0^x f(t)\cos(x-t)\,dt\right] \tag{4.158}$$

Let's denote $h(t,x) = f(t)\cos(x-t)$. We need to find $\frac{\partial h}{\partial x}$:

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}[f(t)\cos(x-t)] \tag{4.159}$$
$$= f(t)\cdot\frac{\partial}{\partial x}[\cos(x-t)] \tag{4.160}$$
$$= f(t)\cdot[-\sin(x-t)] \tag{4.161}$$
$$= -f(t)\sin(x-t) \tag{4.162}$$

Also, $h(b(x),x) = h(x,x) = f(x)\cos(x-x) = f(x)\cos(0) = f(x)$.
So:

$$\frac{d^2y}{dx^2} = \int_0^x \frac{\partial h}{\partial x}\,dt + h(b(x),x)\cdot\frac{db(x)}{dx} \tag{4.163}$$
$$= \int_0^x [-f(t)\sin(x-t)]\,dt + f(x)\cdot 1 \tag{4.164}$$
$$= -\int_0^x f(t)\sin(x-t)\,dt + f(x) \tag{4.165}$$

Step 4: Calculate $\frac{d^2y}{dx^2} + y$.

$$\frac{d^2y}{dx^2} + y = -\int_0^x f(t)\sin(x-t)\,dt + f(x) + \int_0^x f(t)[\sin x\cos t - \cos x\sin t]\,dt \tag{4.166}$$

Now, let's simplify:

$$\sin(x-t) = \sin x\cos t - \cos x\sin t \tag{4.167}$$

Substituting this:

$$\frac{d^2y}{dx^2} + y = -\int_0^x f(t)[\sin x\cos t - \cos x\sin t]\,dt + f(x) + \int_0^x f(t)[\sin x\cos t - \cos x\sin t]\,dt \tag{4.168}$$

$$= -\int_0^x f(t)[\sin x\cos t - \cos x\sin t]\,dt + \int_0^x f(t)[\sin x\cos t - \cos x\sin t]\,dt + f(x) \tag{4.169}$$

$$= f(x) \tag{4.170}$$

Therefore, $\frac{d^2y}{dx^2} + y = f(x)$.

Example 4: DUIS with Sine Function

Find $\frac{dI}{d\alpha}$ if $I(\alpha) = \int_a^{\alpha^2} \frac{\sin\alpha x}{x}\,dx$.

Detailed Solution

We need to apply Leibniz's Rule for the case with a variable upper limit and a parameter α in the integrand.
Using the formula:

$$\frac{d}{d\alpha}\left[\int_a^{b(\alpha)} f(x,\alpha)\,dx\right] = \int_a^{b(\alpha)} \frac{\partial f}{\partial\alpha}(x,\alpha)\,dx + f(b(\alpha),\alpha)\cdot\frac{db(\alpha)}{d\alpha} \tag{4.171}$$

In our case:

$$f(x,\alpha) = \frac{\sin\alpha x}{x} \tag{4.172}$$

$$a = a \text{ (constant lower limit)} \tag{4.173}$$

$$b(\alpha) = \alpha^2 \text{ (variable upper limit)} \tag{4.174}$$

Step 1: Find $\frac{\partial f}{\partial\alpha}$.

$$\frac{\partial f}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(\frac{\sin\alpha x}{x}\right) \tag{4.175}$$

$$= \frac{1}{x}\cdot\frac{\partial}{\partial\alpha}(\sin\alpha x) \tag{4.176}$$

$$= \frac{1}{x}\cdot x\cos\alpha x \tag{4.177}$$

$$= \cos\alpha x \tag{4.178}$$

Step 2: Find $\frac{db(\alpha)}{d\alpha}$.

$$\frac{db(\alpha)}{d\alpha} = \frac{d}{d\alpha}(\alpha^2) \tag{4.179}$$
$$= 2\alpha \tag{4.180}$$

Step 3: Find $f(b(\alpha), \alpha)$.

$$f(b(\alpha), \alpha) = f(\alpha^2, \alpha) \tag{4.181}$$
$$= \frac{\sin \alpha \cdot \alpha^2}{\alpha^2} \tag{4.182}$$
$$= \sin \alpha^3 \tag{4.183}$$

Step 4: Apply Leibniz's Rule.

$$\frac{dI}{d\alpha} = \int_a^{\alpha^2} \frac{\partial f}{\partial \alpha}\, dx + f(b(\alpha), \alpha) \cdot \frac{db(\alpha)}{d\alpha} \tag{4.184}$$
$$= \int_a^{\alpha^2} \cos \alpha x\, dx + \sin \alpha^3 \cdot 2\alpha \tag{4.185}$$
$$= \int_a^{\alpha^2} \cos \alpha x\, dx + 2\alpha \sin \alpha^3 \tag{4.186}$$

Step 5: Evaluate the integral.

$$\int \cos \alpha x\, dx = \frac{1}{\alpha} \sin \alpha x + C \tag{4.187}$$

So:

$$\int_a^{\alpha^2} \cos \alpha x\, dx = \left[\frac{1}{\alpha} \sin \alpha x\right]_a^{\alpha^2} \tag{4.188}$$
$$= \frac{1}{\alpha} \sin \alpha \cdot \alpha^2 - \frac{1}{\alpha} \sin \alpha a \tag{4.189}$$
$$= \frac{1}{\alpha} \sin \alpha^3 - \frac{1}{\alpha} \sin \alpha a \tag{4.190}$$

Step 6: Calculate $\frac{dI}{d\alpha}$.

$$\frac{dI}{d\alpha} = \int_a^{\alpha^2} \cos \alpha x\, dx + 2\alpha \sin \alpha^3 \tag{4.191}$$
$$= \frac{1}{\alpha} \sin \alpha^3 - \frac{1}{\alpha} \sin \alpha a + 2\alpha \sin \alpha^3 \tag{4.192}$$
$$= \left(\frac{1}{\alpha} + 2\alpha\right) \sin \alpha^3 - \frac{1}{\alpha} \sin \alpha a \tag{4.193}$$
$$= \frac{1 + 2\alpha^2}{\alpha} \sin \alpha^3 - \frac{1}{\alpha} \sin \alpha a \tag{4.194}$$

Therefore, we have found that:

$$\frac{dI}{d\alpha} = \frac{1 + 2\alpha^2}{\alpha} \sin \alpha^3 - \frac{1}{\alpha} \sin \alpha a \tag{4.195}$$

Consolidation of Result:
Let's simplify our notation and write the final answer more cleanly:

$$\frac{dI}{d\alpha} = \frac{\sin\alpha^3}{\alpha}(1+2\alpha^2) - \frac{\sin\alpha a}{\alpha} \tag{4.196}$$

This form highlights the two main terms in our derivative: one arising from the parameter in the integrand and the other from the variable upper limit.

Example 5: Error Function Integration

Show that

$$\int_0^\infty e^{-(x+a)^2}\,dx = \frac{\sqrt{\pi}}{2}[1-\operatorname{erf}(a)] \tag{4.197}$$

Detailed Solution

We need to evaluate the integral $\int_0^\infty e^{-(x+a)^2}\,dx$ and express it in terms of the error function.
Step 1: Recall the definition of the error function.

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \tag{4.198}$$

Step 2: Make a substitution to transform our integral.
Let $u = x + a$, then $x = u - a$ and $dx = du$.
When $x = 0$, $u = a$. When $x = \infty$, $u = \infty$.
Using this substitution:

$$\int_0^\infty e^{-(x+a)^2}\,dx = \int_a^\infty e^{-u^2}\,du \tag{4.199}$$

Step 3: Split the integral into two parts.

$$\int_a^\infty e^{-u^2}\,du = \int_0^\infty e^{-u^2}\,du - \int_0^a e^{-u^2}\,du \tag{4.200}$$

Step 4: Use known values and the error function.
We know that $\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$.
Also, from the definition of the error function:

$$\int_0^a e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}\cdot\frac{2}{\sqrt{\pi}}\int_0^a e^{-u^2}\,du \tag{4.201}$$

$$= \frac{\sqrt{\pi}}{2}\cdot\operatorname{erf}(a) \tag{4.202}$$

Step 5: Combine the results.

$$\int_0^\infty e^{-(x+a)^2}\,dx = \int_a^\infty e^{-u^2}\,du \tag{4.203}$$

$$= \int_0^\infty e^{-u^2}\,du - \int_0^a e^{-u^2}\,du \tag{4.204}$$

$$= \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\cdot\operatorname{erf}(a) \tag{4.205}$$

$$= \frac{\sqrt{\pi}}{2}[1-\operatorname{erf}(a)] \tag{4.206}$$

Recall that $1 - \text{erf}(a) = \text{erfc}(a)$, the complementary error function. Therefore, we can also write:

$$\int_0^\infty e^{-(x+a)^2}\,dx = \frac{\sqrt{\pi}}{2}\cdot \text{erfc}(a) \tag{4.207}$$

Thus, we have shown that $\int_0^\infty e^{-(x+a)^2}\,dx = \frac{\sqrt{\pi}}{2}[1 - \text{erf}(a)]$.

Example 6: Error Function and Exponential Integral

Show that

$$\int_0^\infty e^{-x^2}\cos(2ax)\,dx = \frac{\sqrt{\pi}}{2}e^{-a^2} \tag{4.208}$$

Detailed Solution

Direct Method Using Differentiation Under Integral Sign:
Step 1: Define the function.

$$I(a) = \int_0^\infty e^{-x^2}\cos(2ax)\,dx \tag{4.209}$$

Step 2: Find $\frac{dI}{da}$.

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}[e^{-x^2}\cos(2ax)]\,dx \tag{4.210}$$

$$= \int_0^\infty e^{-x^2}\cdot\frac{\partial}{\partial a}[\cos(2ax)]\,dx \tag{4.211}$$

$$= \int_0^\infty e^{-x^2}\cdot[-2x\sin(2ax)]\,dx \tag{4.212}$$

$$= -2\int_0^\infty xe^{-x^2}\sin(2ax)\,dx \tag{4.213}$$

Step 3: Integrate by parts.
Let $u = \sin(2ax)$ and $dv = xe^{-x^2}\,dx$. Then:

$$du = 2a\cos(2ax)\,dx \tag{4.214}$$

$$v = -\frac{1}{2}e^{-x^2} \tag{4.215}$$

Therefore:

$$\int xe^{-x^2}\sin(2ax)\,dx = \sin(2ax)\cdot\left(-\frac{1}{2}e^{-x^2}\right) - \int\left(-\frac{1}{2}e^{-x^2}\right)\cdot 2a\cos(2ax)\,dx \tag{4.216}$$

$$= -\frac{1}{2}e^{-x^2}\sin(2ax) + a\int e^{-x^2}\cos(2ax)\,dx \tag{4.217}$$

$$= -\frac{1}{2}e^{-x^2}\sin(2ax) + aI(a) \tag{4.218}$$

Step 4: Substitute back to find the differential equation for $I(a)$.

$$\frac{dI}{da} = -2\int_0^\infty xe^{-x^2}\sin(2ax)\,dx \tag{4.219}$$

$$= -2\left[-\frac{1}{2}e^{-x^2}\sin(2ax)\right]_0^\infty - 2aI(a) \tag{4.220}$$

$$= 0 - 2aI(a) \tag{4.221}$$

$$= -2aI(a) \tag{4.222}$$

Step 5: Solve the differential equation.

$$\frac{dI}{da} = -2aI(a) \tag{4.223}$$

$$\frac{dI}{I} = -2a\,da \tag{4.224}$$

$$\ln|I| = -a^2 + C \tag{4.225}$$

$$I(a) = Ae^{-a^2} \tag{4.226}$$

To find the constant A, we use the fact that $I(0) = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

$$I(0) = Ae^0 \tag{4.227}$$

$$\frac{\sqrt{\pi}}{2} = A \tag{4.228}$$

$$\tag{4.229}$$

Therefore:

$$I(a) = \frac{\sqrt{\pi}}{2}e^{-a^2} \tag{4.230}$$

Which confirms our result.

Example 7: DUIS with Cosine Function in Limits

Show that

$$\frac{d}{d\alpha}\int_0^{1/a}\cos\alpha x^2\,dx = -\int_0^{1/a} x^2\sin\alpha x^2\,dx \tag{4.231}$$

Detailed Solution

We need to find the derivative with respect to α of the integral $\int_0^{1/a}\cos\alpha x^2\,dx$ using the differentiation under integral sign technique.
Step 1: Apply Leibniz's Rule.
Since the limits of integration (0 and $1/a$) are constants with respect to α, we use the simpler form of Leibniz's Rule:

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)\,dx \tag{4.232}$$

In our case:

$$f(x,\alpha) = \cos\alpha x^2 \tag{4.233}$$
$$a = 0 \text{ (constant lower limit)} \tag{4.234}$$
$$b = \frac{1}{a} \text{ (constant upper limit)} \tag{4.235}$$

Step 2: Find $\frac{\partial f}{\partial \alpha}$.

$$\frac{\partial f}{\partial \alpha} = \frac{\partial}{\partial \alpha}[\cos\alpha x^2] \tag{4.236}$$
$$= -\sin\alpha x^2 \cdot \frac{\partial}{\partial \alpha}[\alpha x^2] \tag{4.237}$$
$$= -\sin\alpha x^2 \cdot x^2 \tag{4.238}$$
$$= -x^2 \sin\alpha x^2 \tag{4.239}$$

Step 3: Apply Leibniz's Rule.

$$\frac{d}{d\alpha}\int_0^{1/a} \cos\alpha x^2\, dx = \int_0^{1/a} \frac{\partial f}{\partial \alpha}\, dx \tag{4.240}$$
$$= \int_0^{1/a} (-x^2 \sin\alpha x^2)\, dx \tag{4.241}$$
$$= -\int_0^{1/a} x^2 \sin\alpha x^2\, dx \tag{4.242}$$

Therefore, we have shown that:

$$\frac{d}{d\alpha}\int_0^{1/a} \cos\alpha x^2\, dx = -\int_0^{1/a} x^2 \sin\alpha x^2\, dx \tag{4.243}$$

Example 8: Evaluate $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})dx$

Evaluate the integral $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})dx$.

Detailed Solution

Let $I(a) = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})dx$ (1)
Now using DUIS Rule-I:

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{e^{-x}}{x}(1-e^{-ax})\right] dx \tag{4.244}$$
$$= \int_0^\infty \frac{e^{-x}}{x}\frac{\partial}{\partial a}(1-e^{-ax})dx \tag{4.245}$$
$$= \int_0^\infty \frac{e^{-x}}{x}\{0-(e^{-ax})(-x)\}dx \tag{4.246}$$
$$= \int_0^\infty e^{-x}e^{-ax}dx \tag{4.247}$$
$$= \int_0^\infty e^{-(1+a)x}dx \tag{4.248}$$

Evaluating this integral:

$$\frac{dI}{da} = \left[\frac{e^{-(1+a)x}}{-(1+a)}\right]_0^\infty \tag{4.249}$$
$$= \frac{1}{-(1+a)}[e^{-\infty} - e^0] \tag{4.250}$$
$$= \frac{1}{-(1+a)}[0 - 1] \tag{4.251}$$
$$= \frac{1}{1+a} \tag{4.252}$$

We have

$$\frac{dI}{da} = \frac{1}{a+1} \tag{4.253}$$

This is a variable separable differential equation.
Separating variables and integrating, we get

$$\int dI = \int \frac{da}{a+1} + c \tag{4.254}$$
$$\Rightarrow I(a) = \log(a+1) + c \qquad (2) \tag{4.255}$$

where c is the integrating constant.
Now we need to calculate the value of c from equations (1) and (2).
Putting $a = 0$ in (1) and (2), we have
From (1):

$$I(0) = \int_0^\infty \frac{e^{-x}}{x}(1 - e^0)dx \tag{4.256}$$
$$= \int_0^\infty \frac{e^{-x}}{x}(1 - 1)dx = 0 \tag{4.257}$$

From (2):

$$I(0) = \log(0+1) + c \tag{4.258}$$
$$\Rightarrow I(0) = c = 0 \tag{4.259}$$

Hence, (2) becomes

$$I(a) = \log(a+1) \tag{4.260}$$
$$\Rightarrow \int_0^\infty \frac{e^{-x}}{x}(1 - e^{-ax})dx = \log(a+1) \tag{4.261}$$

This is the desired result.

Example 9: Prove that $\int_0^1 \frac{x^a-1}{\log x}dx$

Prove that $\int_0^1 \frac{x^a-1}{\log x}dx = \log(a+1)$.

Detailed Solution

Let $I(a) = \int_0^1 \frac{x^a-1}{\log x} dx$ (1)
Now using DUIS Rule-I:

$$\frac{dI}{da} = \int_0^1 \frac{\partial}{\partial a}\left[\frac{x^a-1}{\log x}\right] dx \tag{4.262}$$
$$= \int_0^1 \frac{\partial}{\partial a}\left[\frac{x^a-1}{\log x}\right] dx \tag{4.263}$$
$$= \int_0^1 \left[\frac{x^a \log x}{\log x}\right] dx \tag{4.264}$$
$$= \int_0^1 x^a dx \tag{4.265}$$
$$= \left[\frac{x^{a+1}}{a+1}\right]_0^1 \tag{4.266}$$
$$= \frac{1}{a+1} \tag{4.267}$$

We have

$$\frac{dI}{da} = \frac{1}{a+1} \tag{4.268}$$

This is a variable separable differential equation.
Separating variables and integrating, we get

$$\int dI = \int \frac{da}{a+1} + c \tag{4.269}$$
$$\Rightarrow I(a) = \log(a+1) + c \qquad (2) \tag{4.270}$$

where c is the integrating constant.
Now we need to calculate the value of c from equations (1) and (2).
Putting $a = 0$ in (1) and (2), we have
From (1):

$$I(0) = \int_0^1 \frac{x^0-1}{\log x} dx \tag{4.271}$$
$$= \int_0^1 \frac{1-1}{\log x} dx = 0 \tag{4.272}$$

From (2):

$$I(0) = \log(0+1) + c \tag{4.273}$$
$$\Rightarrow I(0) = c = 0 \tag{4.274}$$

Hence, (2) becomes

$$I(a) = \log(a+1) \tag{4.275}$$
$$\Rightarrow \int_0^1 \frac{x^a-1}{\log x} dx = \log(a+1) \tag{4.276}$$

This is the desired result.

Example 10: Evaluate $\int_0^\infty \frac{\log\left(1+ax^2\right)}{x^2}dx$

Evaluate the integral $\int_0^\infty \frac{\log\left(1+ax^2\right)}{x^2}dx$.

Detailed Solution

Let $I(a) = \int_0^\infty \frac{\log\left(1+ax^2\right)}{x^2}dx$ (1)
Now using DUIS Rule-I:

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{\log(1+ax^2)}{x^2}\right]dx \tag{4.277}$$
$$= \int_0^\infty \frac{1}{x^2}\frac{\partial}{\partial a}[\log\left(1+ax^2\right)]dx \tag{4.278}$$
$$= \int_0^\infty \frac{1}{x^2}\left[\frac{1}{1+ax^2}(x^2)\right]dx \tag{4.279}$$
$$= \int_0^\infty \frac{1}{1+ax^2}dx \tag{4.280}$$

Let's evaluate this integral with a substitution. Let $u = \sqrt{a}x$, then $dx = \frac{du}{\sqrt{a}}$.
Also, when $x \to 0 \Rightarrow u \to 0$ and when $x \to \infty \Rightarrow u \to \infty$.

$$\frac{dI}{da} = \int_0^\infty \frac{1}{1+ax^2}dx \tag{4.281}$$
$$= \int_0^\infty \frac{1}{1+\frac{u^2}{a}a}\cdot\frac{du}{\sqrt{a}} \tag{4.282}$$
$$= \int_0^\infty \frac{1}{1+u^2}\cdot\frac{du}{\sqrt{a}} \tag{4.283}$$
$$= \frac{1}{\sqrt{a}}\int_0^\infty \frac{1}{1+u^2}du \tag{4.284}$$
$$= \frac{1}{\sqrt{a}}\left[\tan^{-1}(u)\right]_0^\infty \tag{4.285}$$
$$= \frac{1}{\sqrt{a}}[\tan^{-1}(\infty) - \tan^{-1}(0)] \tag{4.286}$$
$$= \frac{1}{\sqrt{a}}\left[\frac{\pi}{2} - 0\right] \tag{4.287}$$
$$= \frac{\pi}{2\sqrt{a}} \tag{4.288}$$

We have

$$\frac{dI}{da} = \frac{\pi}{2\sqrt{a}} \tag{4.289}$$

This is a variable separable differential equation.

Separating variables and integrating, we get

$$\int dI = \frac{\pi}{2}\int \frac{da}{\sqrt{a}} + c \tag{4.290}$$
$$= \frac{\pi}{2}(a^{-\frac{1}{2}+1})_{-\frac{1}{2}+1} + c \tag{4.291}$$
$$= \frac{\pi}{2}\cdot\frac{a^{\frac{1}{2}}}{\frac{1}{2}} + c \tag{4.292}$$
$$= \pi\sqrt{a} + c \tag{4.293}$$

where c is the integrating constant.
Now we need to calculate the value of c from equations (1) and (2).
Putting $a = 0$ in (1):

$$I(0) = \int_0^\infty \frac{\log(1+0)}{x^2}dx \tag{4.294}$$
$$= \int_0^\infty \frac{\log(1)}{x^2}dx = 0 \tag{4.295}$$

From our integrated equation:

$$I(0) = \pi\sqrt{0} + c \tag{4.296}$$
$$\Rightarrow I(0) = c = 0 \tag{4.297}$$

Hence, our solution becomes

$$I(a) = \pi\sqrt{a} \tag{4.298}$$
$$\Rightarrow \int_0^\infty \frac{\log(1+ax^2)}{x^2}dx = \pi\sqrt{a} \tag{4.299}$$

This is the desired result.

Example 11: Verify the rule of differentiation under integral sign for $\int_a^{a^2}\log(ax)dx$

Verify the rule of differentiation under integral sign for the integral

$$\int_a^{a^2}\log(ax)dx \tag{4.300}$$

Detailed Solution

Let $I(a) = \int_a^{a^2}\log(ax)dx$ (1)
Now using DUIS Rule-II (Leibnitz's Rule), we have:

$$\frac{dI}{da} = \int_a^{a^2}\frac{\partial}{\partial a}[\log(ax)]dx + \log\left(a\cdot a^2\right)\frac{\partial}{\partial a}(a^2) - \log(a\cdot a)\frac{\partial}{\partial a}(a) \tag{4.301}$$
$$= \int_a^{a^2}\frac{1}{ax}\cdot x\,dx + \log\left(a^3\right)\cdot 2a - \log\left(a^2\right)\cdot 1 \tag{4.302}$$
$$= \int_a^{a^2}\frac{1}{a}\,dx + 2a\log\left(a^3\right) - \log\left(a^2\right) \tag{4.303}$$

Evaluating the integral:

$$\frac{dI}{da} = \frac{1}{a}[x]_a^{a^2} + 2a\log\left(a^3\right) - \log\left(a^2\right) \tag{4.304}$$
$$= \frac{1}{a}(a^2 - a) + 2a\log\left(a^3\right) - \log\left(a^2\right) \tag{4.305}$$
$$= \frac{a^2 - a}{a} + 2a\log\left(a^3\right) - \log\left(a^2\right) \tag{4.306}$$
$$= a - 1 + 2a\log\left(a^3\right) - \log\left(a^2\right) \tag{4.307}$$
$$= a - 1 + 2a \cdot 3\log(a) - 2\log(a) \tag{4.308}$$
$$= a - 1 + 6a\log(a) - 2\log(a) \qquad (1) \tag{4.309}$$

Now, let's compute $I(a)$ directly and then differentiate it to verify our result.

$$I(a) = \int_a^{a^2} \log(ax) \cdot 1\,dx \tag{4.310}$$
$$= [x\log(ax)]_a^{a^2} - \int_a^{a^2} x \cdot \frac{1}{ax} \cdot a\,dx \tag{4.311}$$
$$= [x\log(ax)]_a^{a^2} - \int_a^{a^2} 1\,dx \tag{4.312}$$
$$= [x\log(ax)]_a^{a^2} - [x]_a^{a^2} \tag{4.313}$$
$$= a^2\log\left(a \cdot a^2\right) - a\log(a \cdot a) - (a^2 - a) \tag{4.314}$$
$$= a^2\log\left(a^3\right) - a\log\left(a^2\right) - a^2 + a \tag{4.315}$$
$$= 3a^2\log(a) - 2a\log(a) - a^2 + a \tag{4.316}$$

Differentiating this with respect to a:

$$\frac{dI}{da} = \frac{d}{da}[3a^2\log(a) - 2a\log(a) - a^2 + a] \tag{4.317}$$
$$= 3\frac{d}{da}[a^2\log(a)] - 2\frac{d}{da}[a\log(a)] - \frac{d}{da}[a^2] + \frac{d}{da}[a] \tag{4.318}$$

Using the product rule:

$$\frac{d}{da}[a^2\log(a)] = 2a\log(a) + a^2 \cdot \frac{1}{a} \tag{4.319}$$
$$= 2a\log(a) + a \tag{4.320}$$

And similarly:

$$\frac{d}{da}[a\log(a)] = \log(a) + a \cdot \frac{1}{a} \tag{4.321}$$
$$= \log(a) + 1 \tag{4.322}$$

Therefore:

$$\frac{dI}{da} = 3[2a\log(a) + a] - 2[\log(a) + 1] - 2a + 1 \tag{4.323}$$
$$= 6a\log(a) + 3a - 2\log(a) - 2 - 2a + 1 \tag{4.324}$$
$$= 6a\log(a) - 2\log(a) + 3a - 2a - 2 + 1 \tag{4.325}$$
$$= 6a\log(a) - 2\log(a) + a - 1 \tag{4.326}$$
$$= a - 1 + 6a\log(a) - 2\log(a) \qquad (2) \tag{4.327}$$

Comparing (1) and (2), we can see that both results are identical. Hence, the rule of differentiation under integral sign is verified for this case.

Example 12: Prove that erf(0)

Prove that $\operatorname{erf}(0) = 0$

Detailed Solution

The error function of x is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{4.328}$$

To find $\operatorname{erf}(0)$, we substitute $x = 0$ in the definition:

$$\operatorname{erf}(0) = \frac{2}{\sqrt{\pi}} \int_0^0 e^{-u^2} du \tag{4.329}$$

$$= \frac{2}{\sqrt{\pi}} \cdot 0 \tag{4.330}$$

$$= 0 \tag{4.331}$$

Hence, $\operatorname{erf}(0) = 0$.

Example 13: Prove that erf(∞)

Prove that $\operatorname{erf}(\infty) = 1$

Detailed Solution

The error function of x is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{4.332}$$

Put $x = \infty$, we have

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du \qquad (1) \tag{4.333}$$

To evaluate this integral, we'll use a substitution method.
Let $u^2 = t$, then $2u\,du = dt \Rightarrow du = \frac{dt}{2\sqrt{t}}$
When $u \to 0 \Rightarrow t \to 0$ and when $u \to \infty \Rightarrow t \to \infty$.
Substituting this into equation (1):

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}} \tag{4.334}$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \int_0^\infty t^{-1/2} e^{-t} dt \tag{4.335}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty t^{1/2-1} e^{-t} dt \tag{4.336}$$

This integral takes the form of the Gamma function $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$.
For $n = 1/2$, we have $\Gamma(1/2) = \sqrt{\pi}$.

Therefore:

$$\operatorname{erf}(\infty) = \frac{1}{\sqrt{\pi}} \cdot \Gamma(1/2) \tag{4.337}$$
$$= \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} \tag{4.338}$$
$$= 1 \tag{4.339}$$

Hence, $\operatorname{erf}(\infty) = 1$.

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Example 14: Prove that the error function is an odd function

Prove that the error function is an odd function, i.e., $\operatorname{erf}(-x) = -\operatorname{erf}(x)$.

Detailed Solution

The error function of x is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{4.340}$$

Replace x by $-x$, we have

$$\operatorname{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-u^2} du \qquad (1) \tag{4.341}$$

To evaluate this integral with a negative upper limit, we'll use substitution.
Let $u = -t$, then $du = -dt$. When $u \to 0 \Rightarrow t \to 0$ and when $u \to -x \Rightarrow t \to x$.
Substituting this into equation (1):

$$\operatorname{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-(-t)^2} (-dt) \tag{4.342}$$
$$= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} (-dt) \tag{4.343}$$
$$= -\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{4.344}$$
$$= -\operatorname{erf}(x) \tag{4.345}$$

Therefore, $\operatorname{erf}(-x) = -\operatorname{erf}(x)$, which proves that the error function is an odd function.

Example 15: Prove that $\operatorname{erf}(x) + \operatorname{erfc}(x)$

Prove that $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$.

Detailed Solution

The error function and complementary error function of x are given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{4.346}$$
$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du \tag{4.347}$$

Taking the sum of these two functions:

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du + \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du \tag{4.348}$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_0^x e^{-u^2} du + \int_x^\infty e^{-u^2} du \right] \tag{4.349}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du \tag{4.350}$$

We know from the previous example that

$$\int_0^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{2} \tag{4.351}$$

Therefore:

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} \tag{4.352}$$

$$= 1 \tag{4.353}$$

Hence, we have proven that $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$.

Example 16: Show that $\frac{d}{dx} \operatorname{erf}(ax)$

Show that $\frac{d}{dx}[\operatorname{erf}(ax)] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$ and evaluate $\int_0^t \operatorname{erf}(ax)dx$.

Detailed Solution

The error function of x is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{4.354}$$

Therefore:

$$\operatorname{erf}(ax) = \frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-u^2} du \tag{4.355}$$

To find the derivative with respect to x, we'll use the Leibnitz rule (DUIS Rule):

$$\frac{d}{dx}[\operatorname{erf}(ax)] = \frac{2}{\sqrt{\pi}} \frac{d}{dx} \left[\int_0^{ax} e^{-u^2} du \right] \tag{4.356}$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_0^{ax} \frac{\partial}{\partial x}(e^{-u^2}) du + \frac{d(ax)}{dx} e^{-(ax)^2} - \frac{d(0)}{dx} e^{-0^2} \right] \tag{4.357}$$

$$= \frac{2}{\sqrt{\pi}} [0 + a \cdot e^{-a^2x^2} - 0] \tag{4.358}$$

$$= \frac{2a}{\sqrt{\pi}} e^{-a^2x^2} \tag{4.359}$$

Therefore, $\frac{d}{dx}[\operatorname{erf}(ax)] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$.
Now, let's evaluate $\int_0^t \operatorname{erf}(ax)dx$.
Consider:

$$\int_0^t \operatorname{erf}(ax) \cdot 1\, dx = [x \cdot \operatorname{erf}(ax)]_0^t - \int_0^t x \cdot \frac{d}{dx}[\operatorname{erf}(ax)] dx \tag{4.360}$$

Using the derivative we just calculated:

$$\int_0^t \operatorname{erf}(ax)dx = [x \cdot \operatorname{erf}(ax)]_0^t - \int_0^t x \cdot \frac{2a}{\sqrt{\pi}} e^{-a^2x^2} dx \tag{4.361}$$

$$= t \cdot \operatorname{erf}(at) - 0 \cdot \operatorname{erf}(0) - \frac{2a}{\sqrt{\pi}} \int_0^t x \cdot e^{-a^2x^2} dx \tag{4.362}$$

To evaluate the remaining integral, let $a^2x^2 = u \Rightarrow 2a^2x\,dx = du \Rightarrow x\,dx = \frac{du}{2a^2}$. When $x \to 0 \Rightarrow u \to 0$ and when $x \to t \Rightarrow u \to a^2t^2$.

$$\int_0^t \operatorname{erf}(ax)dx = t \cdot \operatorname{erf}(at) - \frac{2a}{\sqrt{\pi}} \int_0^{a^2t^2} \frac{du}{2a^2} \cdot e^{-u} \tag{4.363}$$

$$= t \cdot \operatorname{erf}(at) - \frac{1}{a\sqrt{\pi}} \int_0^{a^2t^2} e^{-u} du \tag{4.364}$$

$$= t \cdot \operatorname{erf}(at) - \frac{1}{a\sqrt{\pi}} [-e^{-u}]_0^{a^2t^2} \tag{4.365}$$

$$= t \cdot \operatorname{erf}(at) - \frac{1}{a\sqrt{\pi}} [-e^{-a^2t^2} + e^0] \tag{4.366}$$

$$= t \cdot \operatorname{erf}(at) - \frac{1}{a\sqrt{\pi}} [-e^{-a^2t^2} + 1] \tag{4.367}$$

$$= t \cdot \operatorname{erf}(at) + \frac{1}{a\sqrt{\pi}} [e^{-a^2t^2} - 1] \tag{4.368}$$

$$= t \cdot \operatorname{erf}(at) + \frac{e^{-a^2t^2} - 1}{a\sqrt{\pi}} \tag{4.369}$$

Therefore, $\int_0^t \operatorname{erf}(ax)dx = t \cdot \operatorname{erf}(at) + \frac{e^{-a^2t^2}-1}{a\sqrt{\pi}}$.

4.8 Additional solved Examples on DUIS

Example 1

Prove that $\int_0^\infty \left(\frac{e^{-ax}-e^{-bx}}{x}\right) dx = \log\left(\frac{b}{a}\right)$, $a > 0$, $b > 0$.

Detailed Solution

Step 1: Let's approach this problem using differentiation under the integral sign (DUIS). We'll define a function $F(t)$ where t is a parameter:

$$F(t) = \int_0^\infty \frac{e^{-tx}}{x} dx, \quad t > 0 \tag{4.370}$$

Step 2: Our original integral can be expressed as:

$$\int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right) dx = F(a) - F(b) \tag{4.371}$$

Step 3: Now we'll differentiate $F(t)$ with respect to t:

$$\frac{dF(t)}{dt} = \frac{d}{dt}\int_0^\infty \frac{e^{-tx}}{x}\,dx \tag{4.372}$$
$$= \int_0^\infty \frac{\partial}{\partial t}\left(\frac{e^{-tx}}{x}\right)dx \tag{4.373}$$
$$= \int_0^\infty \frac{-x\cdot e^{-tx}}{x}\,dx \tag{4.374}$$
$$= -\int_0^\infty e^{-tx}\,dx \tag{4.375}$$

Step 4: We know that $\int_0^\infty e^{-tx}\,dx = \frac{1}{t}$ for $t > 0$. Therefore:

$$\frac{dF(t)}{dt} = -\frac{1}{t} \tag{4.376}$$

Step 5: Integrating both sides with respect to t:

$$F(t) = \int\left(-\frac{1}{t}\right)dt \tag{4.377}$$
$$= -\log|t| + C \tag{4.378}$$
$$= -\log(t) + C \quad (\text{since } t > 0) \tag{4.379}$$

where C is a constant of integration.
Step 6: Using this result, we can evaluate the difference:

$$F(a) - F(b) = [-\log(a) + C] - [-\log(b) + C] \tag{4.380}$$
$$= -\log(a) + \log(b) \tag{4.381}$$
$$= \log\left(\frac{b}{a}\right) \tag{4.382}$$

Step 7: Therefore:

$$\int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right)dx = \log\left(\frac{b}{a}\right) \tag{4.383}$$

Note: We did not need to determine the constant of integration C because it cancels out when we take the difference $F(a) - F(b)$.
Therefore:

$$\boxed{\int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right)dx = \log\left(\frac{b}{a}\right)} \tag{4.384}$$

Example 2

Evaluate $\int_0^\infty \frac{e^{-x}-e^{-ax}}{x\sec x}dx$, where $a > 0$.

Detailed Solution

Step 1: Let's define a function $I(a)$ for our integral:

$$I(a) = \int_0^\infty \frac{e^{-x} - e^{-ax}}{x \sec x} dx \tag{4.385}$$

Step 2: We'll use differentiation under the integral sign (DUIS) with respect to parameter a:

$$\frac{dI}{da} = I'(a) = \int_0^\infty \frac{\partial}{\partial a} \left(\frac{e^{-x} - e^{-ax}}{x \sec x} \right) dx \tag{4.386}$$

$$= \int_0^\infty \frac{0 + e^{-ax}(-x)}{x \sec x} dx \tag{4.387}$$

$$= \int_0^\infty \frac{-xe^{-ax}}{x \sec x} dx \tag{4.388}$$

$$= -\int_0^\infty \frac{e^{-ax}}{\sec x} dx \tag{4.389}$$

$$= -\int_0^\infty e^{-ax} \cos x \, dx \tag{4.390}$$

Step 3: Using the standard result that $\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$ for $a > 0$, we have:

$$I'(a) = -\int_0^\infty e^{-ax} \cos x \, dx \tag{4.391}$$

$$= -\frac{a}{a^2+1} \tag{4.392}$$

Step 4: Now we integrate with respect to a to find $I(a)$:

$$I(a) = \int \left(-\frac{a}{a^2+1} \right) da \tag{4.393}$$

$$= -\frac{1}{2} \int \frac{2a}{a^2+1} da \tag{4.394}$$

$$= -\frac{1}{2} \ln\left(a^2+1\right) + C \tag{4.395}$$

where C is a constant of integration.
Step 5: To determine the value of C, we note from the original integral that when $a = 1$:

$$I(1) = \int_0^\infty \frac{e^{-x} - e^{-x}}{x \sec x} dx \tag{4.396}$$

$$= \int_0^\infty \frac{0}{x \sec x} dx \tag{4.397}$$

$$= 0 \tag{4.398}$$

Step 6: Using this condition, we have:

$$I(1) = -\frac{1}{2} \ln\left(1^2+1\right) + C = 0 \tag{4.399}$$

$$-\frac{1}{2} \ln(2) + C = 0 \tag{4.400}$$

$$C = \frac{1}{2} \ln(2) \tag{4.401}$$

Step 7: Substituting this value of C back into our expression for $I(a)$:

$$I(a) = -\frac{1}{2}\ln\big(a^2+1\big) + \frac{1}{2}\ln(2) \tag{4.402}$$

$$= \frac{1}{2}\ln(2) - \frac{1}{2}\ln\big(a^2+1\big) \tag{4.403}$$

$$= \frac{1}{2}\ln\left(\frac{2}{a^2+1}\right) \tag{4.404}$$

Therefore:

$$\boxed{\int_0^\infty \frac{e^{-x}-e^{-ax}}{x\sec x}dx = \frac{1}{2}\ln\left(\frac{2}{a^2+1}\right)} \tag{4.405}$$

Example 3

Evaluate $\int_0^1 \frac{x^a-x^b}{\log x}dx = \log\left(\frac{a+1}{b+1}\right)$.

Detailed Solution

Step 1: Let's define the function:

$$I(a) = \int_0^1 \frac{x^a-x^b}{\log x}dx \tag{4.406}$$

where b is treated as a fixed parameter, and we'll differentiate with respect to a.
Step 2: Applying differentiation under the integral sign (DUIS):

$$\frac{dI(a)}{da} = \frac{d}{da}\int_0^1 \frac{x^a-x^b}{\log x}dx \tag{4.407}$$

$$= \int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a-x^b}{\log x}\right)dx \tag{4.408}$$

$$= \int_0^1 \frac{x^a\log x}{\log x}dx \tag{4.409}$$

$$= \int_0^1 x^a dx \tag{4.410}$$

$$= \left[\frac{x^{a+1}}{a+1}\right]_0^1 \tag{4.411}$$

$$= \frac{1}{a+1} \tag{4.412}$$

Step 3: Integrating this result with respect to a:

$$I(a) = \int \frac{1}{a+1}da \tag{4.413}$$

$$= \log(a+1) + C \tag{4.414}$$

where C is a constant of integration.
Step 4: To determine C, we need a boundary condition.

Let's consider what happens when $a = b$:

$$I(b) = \int_0^1 \frac{x^b - x^b}{\log x} dx \tag{4.415}$$
$$= \int_0^1 \frac{0}{\log x} dx \tag{4.416}$$
$$= 0 \tag{4.417}$$

Step 5: Using this boundary condition:

$$I(b) = \log(b+1) + C = 0 \tag{4.418}$$
$$\Rightarrow C = -\log(b+1) \tag{4.419}$$

Step 6: Therefore:

$$I(a) = \log(a+1) - \log(b+1) \tag{4.420}$$
$$= \log\left(\frac{a+1}{b+1}\right) \tag{4.421}$$

Therefore:

$$\boxed{\int_0^1 \frac{x^a - x^b}{\log x} dx = \log\left(\frac{a+1}{b+1}\right)} \tag{4.422}$$

Example 4

Prove that $\int_0^\infty \frac{e^{-ax}}{x} \sin x \, dx = \cot^{-1} a$ and deduce that $\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}$.

Detailed Solution

Step 1: Let's define the function:

$$I(a) = \int_0^\infty \frac{e^{-ax}}{x} \sin x \, dx \tag{4.423}$$

where $a > 0$.

Step 2: We'll differentiate $I(a)$ with respect to a using differentiation under the integral sign (DUIS):

$$\frac{dI(a)}{da} = \frac{d}{da} \int_0^\infty \frac{e^{-ax}}{x} \sin x \, dx \tag{4.424}$$
$$= \int_0^\infty \frac{\partial}{\partial a} \left(\frac{e^{-ax}}{x} \sin x \right) dx \tag{4.425}$$
$$= \int_0^\infty \frac{-x \cdot e^{-ax}}{x} \sin x \, dx \tag{4.426}$$
$$= -\int_0^\infty e^{-ax} \sin x \, dx \tag{4.427}$$

Step 3: We know the standard result for the Laplace transform of $\sin x$:

$$\int_0^\infty e^{-ax} \sin x \, dx = \frac{1}{a^2 + 1} \tag{4.428}$$

for $a > 0$.
Step 4: Therefore:

$$\frac{dI(a)}{da} = -\int_0^\infty e^{-ax} \sin x \, dx \tag{4.429}$$

$$= -\frac{1}{a^2+1} \tag{4.430}$$

Step 5: Now we need to integrate this result to find $I(a)$:

$$I(a) = \int \left(-\frac{1}{a^2+1}\right) da \tag{4.431}$$

$$= -\int \frac{1}{a^2+1} \, da \tag{4.432}$$

Step 6: Using the standard integral $\int \frac{1}{a^2+1} \, da = \tan^{-1} a + C$:

$$I(a) = -\tan^{-1} a + C \tag{4.433}$$

where C is a constant of integration.
Step 7: To find the value of C, we need a boundary condition. Let's find the limit of $I(a)$ as $a \to \infty$:
As $a \to \infty$, $e^{-ax} \to 0$ for any fixed $x > 0$. Thus:

$$\lim_{a\to\infty} I(a) = \lim_{a\to\infty} \int_0^\infty \frac{e^{-ax}}{x} \sin x \, dx = 0 \tag{4.434}$$

Step 8: Using this boundary condition:

$$\lim_{a\to\infty} I(a) = \lim_{a\to\infty} [-\tan^{-1} a + C] = 0 \tag{4.435}$$

$$\Rightarrow -\frac{\pi}{2} + C = 0 \tag{4.436}$$

$$\Rightarrow C = \frac{\pi}{2} \tag{4.437}$$

Step 9: Therefore:

$$I(a) = -\tan^{-1} a + \frac{\pi}{2} \tag{4.438}$$

Step 10: Using the identity $\cot^{-1} a = \frac{\pi}{2} - \tan^{-1} a$, we have:

$$I(a) = -\tan^{-1} a + \frac{\pi}{2} \tag{4.439}$$

$$= \frac{\pi}{2} - \tan^{-1} a \tag{4.440}$$

$$= \cot^{-1} a \tag{4.441}$$

Step 11: Thus, we have proven:

$$\int_0^\infty \frac{e^{-ax}}{x} \sin x \, dx = \cot^{-1} a \tag{4.442}$$

Step 12: To deduce the value of $\int_0^\infty \frac{\sin x}{x}\,dx$, we take the limit as $a \to 0^+$ in our main result:

$$\lim_{a\to 0^+} I(a) = \lim_{a\to 0^+} \cot^{-1} a \tag{4.443}$$

$$= \cot^{-1} 0 \tag{4.444}$$

$$= \frac{\pi}{2} \tag{4.445}$$

Step 13: Also:

$$\lim_{a\to 0^+} I(a) = \lim_{a\to 0^+} \int_0^\infty \frac{e^{-ax}}{x} \sin x\,dx \tag{4.446}$$

$$= \int_0^\infty \lim_{a\to 0^+} \frac{e^{-ax}}{x} \sin x\,dx \tag{4.447}$$

$$= \int_0^\infty \frac{\sin x}{x}\,dx \tag{4.448}$$

Step 14: Therefore:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} \tag{4.449}$$

Therefore:

$$\boxed{\int_0^\infty \frac{e^{-ax}}{x} \sin x\,dx = \cot^{-1} a} \tag{4.450}$$

and

$$\boxed{\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}} \tag{4.451}$$

Example 5

Evaluate $\int_0^\infty \frac{(1-\cos mx)}{x} e^{-x} dx = \log\sqrt{1+m^2}$.

Detailed Solution

Step 1: Let's define the function:

$$I(m) = \int_0^\infty \frac{(1-\cos mx)}{x} e^{-x} dx \tag{4.452}$$

Step 2: We'll differentiate $I(m)$ with respect to m using differentiation under the integral sign (DUIS):

$$\frac{dI(m)}{dm} = \frac{d}{dm}\int_0^\infty \frac{(1-\cos mx)}{x}e^{-x}dx \tag{4.453}$$
$$= \int_0^\infty \frac{\partial}{\partial m}\left(\frac{1-\cos mx}{x}\right)e^{-x}dx \tag{4.454}$$
$$= \int_0^\infty \frac{x\sin mx}{x}e^{-x}dx \tag{4.455}$$
$$= \int_0^\infty \sin mx \cdot e^{-x}dx \tag{4.456}$$

Step 3: We know the Laplace transform of $\sin mx$:

$$\int_0^\infty \sin mx \cdot e^{-sx}dx = \frac{m}{s^2+m^2} \tag{4.457}$$

for $s > 0$. In our case, $s = 1$, so:

$$\frac{dI(m)}{dm} = \int_0^\infty \sin mx \cdot e^{-x}dx \tag{4.458}$$
$$= \frac{m}{1+m^2} \tag{4.459}$$

Step 4: Now we integrate with respect to m to find $I(m)$:

$$I(m) = \int \frac{m}{1+m^2}dm \tag{4.460}$$
$$= \frac{1}{2}\int \frac{2m}{1+m^2}dm \tag{4.461}$$
$$= \frac{1}{2}\ln\left(1+m^2\right) + C \tag{4.462}$$

where C is a constant of integration.
Step 5: To determine the value of C, we evaluate $I(0)$:

$$I(0) = \int_0^\infty \frac{(1-\cos 0)}{x}e^{-x}dx \tag{4.463}$$
$$= \int_0^\infty \frac{1-1}{x}e^{-x}dx \tag{4.464}$$
$$= \int_0^\infty 0 \cdot e^{-x}dx \tag{4.465}$$
$$= 0 \tag{4.466}$$

Step 6: Using this boundary condition:

$$I(0) = \frac{1}{2}\ln\left(1+0^2\right) + C = 0 \tag{4.467}$$
$$\Rightarrow \frac{1}{2}\ln(1) + C = 0 \tag{4.468}$$
$$\Rightarrow 0 + C = 0 \tag{4.469}$$
$$\Rightarrow C = 0 \tag{4.470}$$

Step 7: Therefore:

$$I(m) = \frac{1}{2}\ln\left(1+m^2\right) \tag{4.471}$$
$$= \ln\left(1+m^2\right)^{1/2} \tag{4.472}$$
$$= \ln\sqrt{1+m^2} \tag{4.473}$$

Therefore:

$$\boxed{\int_0^\infty \frac{(1-\cos mx)}{x} e^{-x}\,dx = \ln\sqrt{1+m^2}} \tag{4.474}$$

Example 6

Verify the differentiation under the integral sign (DUIS) rule for $\int_0^\infty e^{-ax}\cos bx\,dx$, where a is the parameter.

Detailed Solution

Step 1: Let's define the function:

$$I(a) = \int_0^\infty e^{-ax}\cos bx\,dx \tag{4.475}$$

where $a > 0$ and b is a constant.
Step 2: We know the standard result for this integral:

$$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2} \tag{4.476}$$

for $a > 0$.
Step 3: To verify the DUIS rule, we need to check if:

$$\frac{dI(a)}{da} = \int_0^\infty \frac{\partial}{\partial a}\left(e^{-ax}\cos bx\right)dx \tag{4.477}$$

Step 4: Let's calculate both sides separately and verify they are equal.
First, differentiating $I(a)$ with respect to a:

$$\frac{dI(a)}{da} = \frac{d}{da}\left(\frac{a}{a^2+b^2}\right) \tag{4.478}$$
$$= \frac{(a^2+b^2)\cdot 1 - a\cdot 2a}{(a^2+b^2)^2} \tag{4.479}$$
$$= \frac{a^2+b^2-2a^2}{(a^2+b^2)^2} \tag{4.480}$$
$$= \frac{b^2-a^2}{(a^2+b^2)^2} \tag{4.481}$$

Step 5: Now, let's calculate the right-hand side:

$$\int_0^\infty \frac{\partial}{\partial a}\left(e^{-ax}\cos bx\right)dx = \int_0^\infty \frac{\partial}{\partial a}\left(e^{-ax}\right)\cos bx\,dx \tag{4.482}$$
$$= \int_0^\infty (-x)e^{-ax}\cos bx\,dx \tag{4.483}$$
$$= -\int_0^\infty xe^{-ax}\cos bx\,dx \tag{4.484}$$

Step 6: To calculate this integral, we can use integration by parts or utilize known Laplace transforms. Let's proceed with the latter approach.
The Laplace transform of $x \cos bx$ with parameter a is:

$$\int_0^\infty xe^{-ax} \cos bx \, dx = \frac{a^2 - b^2}{(a^2 + b^2)^2} \tag{4.485}$$

for $a > 0$.
Step 7: Therefore:

$$-\int_0^\infty xe^{-ax} \cos bx \, dx = -\frac{a^2 - b^2}{(a^2 + b^2)^2} \tag{4.486}$$

$$= \frac{b^2 - a^2}{(a^2 + b^2)^2} \tag{4.487}$$

Step 8: Comparing the results from steps 4 and 7:

$$\frac{dI(a)}{da} = \frac{b^2 - a^2}{(a^2 + b^2)^2} \tag{4.488}$$

$$\int_0^\infty \frac{\partial}{\partial a}(e^{-ax} \cos bx) \, dx = \frac{b^2 - a^2}{(a^2 + b^2)^2} \tag{4.489}$$

Since both expressions are identical, the DUIS rule is verified for this integral.
Therefore:

$$\boxed{\frac{d}{da}\int_0^\infty e^{-ax} \cos bx \, dx = \int_0^\infty \frac{\partial}{\partial a}(e^{-ax} \cos bx) \, dx} \tag{4.490}$$

This confirms that the differentiation under the integral sign rule is valid for this specific case.

Example 7

Verify the differentiation under the integral sign (DUIS) rule for $\int_a^{a^2} \frac{1}{x+a} dx$, where a is the parameter.

Detailed Solution

Step 1: Let's define the function:

$$I(a) = \int_a^{a^2} \frac{1}{x + a} dx \tag{4.491}$$

where $a > 0$ is a parameter.
Step 2: When we have an integral with variable limits of integration that depend on the parameter, the general formula for differentiation under the integral sign is:

$$\frac{d}{da}\int_{g(a)}^{h(a)} f(x, a)dx = \int_{g(a)}^{h(a)} \frac{\partial f(x, a)}{\partial a} dx + f(h(a), a) \cdot \frac{dh(a)}{da} - f(g(a), a) \cdot \frac{dg(a)}{da} \tag{4.492}$$

Step 3: For our integral, we have:

$$g(a) = a \tag{4.493}$$
$$h(a) = a^2 \tag{4.494}$$
$$f(x, a) = \frac{1}{x+a} \tag{4.495}$$

Step 4: Let's first evaluate the integral directly to have a reference for verification.

$$I(a) = \int_a^{a^2} \frac{1}{x+a} dx \tag{4.496}$$
$$= [\ln|x+a|]_a^{a^2} \tag{4.497}$$
$$= \ln|a^2+a| - \ln|a+a| \tag{4.498}$$
$$= \ln|a(a+1)| - \ln|2a| \tag{4.499}$$
$$= \ln a + \ln(a+1) - \ln 2 - \ln a \tag{4.500}$$
$$= \ln(a+1) - \ln 2 \tag{4.501}$$
$$= \ln\left(\frac{a+1}{2}\right) \tag{4.502}$$

Step 5: Now, let's compute the derivative of $I(a)$ directly:

$$\frac{dI(a)}{da} = \frac{d}{da} \ln\left(\frac{a+1}{2}\right) \tag{4.503}$$
$$= \frac{1}{\frac{a+1}{2}} \cdot \frac{1}{2} \tag{4.504}$$
$$= \frac{1}{a+1} \tag{4.505}$$

Step 6: Next, let's calculate the derivative using the DUIS formula.
First, we compute the partial derivative of $f(x, a)$ with respect to a:

$$\frac{\partial f(x,a)}{\partial a} = \frac{\partial}{\partial a}\left(\frac{1}{x+a}\right) \tag{4.506}$$
$$= \frac{\partial}{\partial a}(x+a)^{-1} \tag{4.507}$$
$$= -1 \cdot (x+a)^{-2} \cdot 1 \tag{4.508}$$
$$= -\frac{1}{(x+a)^2} \tag{4.509}$$

Step 7: Next, we need:

$$f(h(a), a) = f(a^2, a) = \frac{1}{a^2+a} \tag{4.510}$$
$$\frac{dh(a)}{da} = \frac{d(a^2)}{da} = 2a \tag{4.511}$$
$$f(g(a), a) = f(a, a) = \frac{1}{a+a} = \frac{1}{2a} \tag{4.512}$$
$$\frac{dg(a)}{da} = \frac{d(a)}{da} = 1 \tag{4.513}$$

Step 8: Now, substituting into the DUIS formula:

$$\frac{dI(a)}{da} = \int_a^{a^2} \frac{\partial f(x,a)}{\partial a} dx + f(h(a),a)\cdot\frac{dh(a)}{da} - f(g(a),a)\cdot\frac{dg(a)}{da} \tag{4.514}$$

$$= \int_a^{a^2} -\frac{1}{(x+a)^2} dx + \frac{1}{a^2+a}\cdot 2a - \frac{1}{2a}\cdot 1 \tag{4.515}$$

$$= -\int_a^{a^2} \frac{1}{(x+a)^2} dx + \frac{2a}{a^2+a} - \frac{1}{2a} \tag{4.516}$$

Step 9: Let's evaluate the integral:

$$\int_a^{a^2} \frac{1}{(x+a)^2} dx = \left[-\frac{1}{x+a}\right]_a^{a^2} \tag{4.517}$$

$$= -\frac{1}{a^2+a} + \frac{1}{a+a} \tag{4.518}$$

$$= -\frac{1}{a^2+a} + \frac{1}{2a} \tag{4.519}$$

Step 10: Substituting this back:

$$\frac{dI(a)}{da} = -\left(-\frac{1}{a^2+a} + \frac{1}{2a}\right) + \frac{2a}{a^2+a} - \frac{1}{2a} \tag{4.520}$$

$$= \frac{1}{a^2+a} - \frac{1}{2a} + \frac{2a}{a^2+a} - \frac{1}{2a} \tag{4.521}$$

$$= \frac{1+2a}{a^2+a} - \frac{2}{2a} \tag{4.522}$$

$$= \frac{1+2a}{a(a+1)} - \frac{1}{a} \tag{4.523}$$

$$= \frac{1+2a}{a(a+1)} - \frac{a+1}{a(a+1)} \tag{4.524}$$

$$= \frac{1+2a-(a+1)}{a(a+1)} \tag{4.525}$$

$$= \frac{1+2a-a-1}{a(a+1)} \tag{4.526}$$

$$= \frac{a}{a(a+1)} \tag{4.527}$$

$$= \frac{1}{a+1} \tag{4.528}$$

Step 11: Comparing the results from steps 5 and 10:

$$\frac{dI(a)}{da} = \frac{1}{a+1} \tag{4.529}$$

$$\frac{dI(a)}{da} \text{ (using DUIS)} = \frac{1}{a+1} \tag{4.530}$$

Since both expressions are identical, the DUIS rule is verified for this integral with variable limits.

Therefore:

$$\boxed{\frac{d}{da}\int_a^{a^2} \frac{1}{x+a} dx = \int_a^{a^2} \frac{\partial}{\partial a}\left(\frac{1}{x+a}\right) dx + \frac{1}{a^2+a}\cdot 2a - \frac{1}{2a}\cdot 1} \tag{4.531}$$

This confirms that the differentiation under the integral sign rule, including the additional terms for variable limits, is valid for this specific case.

Example 8

Verify the differentiation under the integral sign (DUIS) rule for $\int_0^{a^2} \tan^{-1}(x/a)dx$, where a is the parameter.

Detailed Solution

Step 1: Let's define the function:

$$I(a) = \int_0^{a^2} \tan^{-1}(x/a)dx \tag{4.532}$$

where $a > 0$ is a parameter.
Step 2: When we have an integral with variable limits of integration that depend on the parameter, the general formula for differentiation under the integral sign is:

$$\frac{d}{da}\int_{g(a)}^{h(a)} f(x,a)dx = \int_{g(a)}^{h(a)} \frac{\partial f(x,a)}{\partial a}dx + f(h(a),a)\cdot\frac{dh(a)}{da} - f(g(a),a)\cdot\frac{dg(a)}{da} \tag{4.533}$$

Step 3: For our integral, we have:

$$g(a) = 0 \tag{4.534}$$
$$h(a) = a^2 \tag{4.535}$$
$$f(x,a) = \tan^{-1}(x/a) \tag{4.536}$$

Step 4: First, let's evaluate the integral directly to have a reference for verification. We can use integration by parts:

$$\int \tan^{-1}(x/a)dx = x\tan^{-1}(x/a) - \int \frac{x}{1+(x/a)^2}\cdot\frac{1}{a}dx \tag{4.537}$$
$$= x\tan^{-1}(x/a) - \int \frac{x}{a^2+x^2}\cdot a dx \tag{4.538}$$
$$= x\tan^{-1}(x/a) - \frac{a}{2}\int \frac{2x}{a^2+x^2}dx \tag{4.539}$$
$$= x\tan^{-1}(x/a) - \frac{a}{2}\ln\left(a^2+x^2\right) + C \tag{4.540}$$

Step 5: Evaluating at the limits:

$$I(a) = \int_0^{a^2} \tan^{-1}(x/a)dx \tag{4.541}$$
$$= \left[x\tan^{-1}(x/a) - \frac{a}{2}\ln\left(a^2+x^2\right)\right]_0^{a^2} \tag{4.542}$$
$$= a^2\tan^{-1}(a) - \frac{a}{2}\ln\left(a^2+a^4\right) - \left[0 - \frac{a}{2}\ln\left(a^2\right)\right] \tag{4.543}$$
$$= a^2\tan^{-1}(a) - \frac{a}{2}\ln\left(a^2(1+a^2)\right) + \frac{a}{2}\ln\left(a^2\right) \tag{4.544}$$
$$= a^2\tan^{-1}(a) - \frac{a}{2}\ln\left(a^2\right) - \frac{a}{2}\ln\left(1+a^2\right) + \frac{a}{2}\ln\left(a^2\right) \tag{4.545}$$
$$= a^2\tan^{-1}(a) - \frac{a}{2}\ln\left(1+a^2\right) \tag{4.546}$$

Step 6: Now, let's compute the derivative of $I(a)$ directly:

$$\frac{dI(a)}{da} = \frac{d}{da}\left[a^2\tan^{-1}(a) - \frac{a}{2}\ln\left(1+a^2\right)\right] \tag{4.547}$$
$$= 2a\tan^{-1}(a) + a^2\cdot\frac{1}{1+a^2} - \frac{1}{2}\ln\left(1+a^2\right) - \frac{a}{2}\cdot\frac{2a}{1+a^2} \tag{4.548}$$
$$= 2a\tan^{-1}(a) + \frac{a^2}{1+a^2} - \frac{1}{2}\ln\left(1+a^2\right) - \frac{a^2}{1+a^2} \tag{4.549}$$
$$= 2a\tan^{-1}(a) - \frac{1}{2}\ln\left(1+a^2\right) \tag{4.550}$$

Step 7: Next, let's calculate the derivative using the DUIS formula. First, we compute the partial derivative of $f(x,a)$ with respect to a:

$$\frac{\partial f(x,a)}{\partial a} = \frac{\partial}{\partial a}\left(\tan^{-1}(x/a)\right) \tag{4.551}$$
$$= \frac{\partial}{\partial a}(\tan^{-1}(x/a)) \tag{4.552}$$
$$= \frac{1}{1+(x/a)^2}\cdot\frac{\partial}{\partial a}(x/a) \tag{4.553}$$
$$= \frac{1}{1+(x/a)^2}\cdot\left(-\frac{x}{a^2}\right) \tag{4.554}$$
$$= -\frac{x}{a^2}\cdot\frac{1}{1+(x/a)^2} \tag{4.555}$$
$$= -\frac{x}{a^2}\cdot\frac{a^2}{a^2+x^2} \tag{4.556}$$
$$= -\frac{x}{a^2+x^2} \tag{4.557}$$

Step 8: Next, we need:

$$f(h(a),a) = f(a^2,a) = \tan^{-1}(a^2/a) = \tan^{-1}(a) \tag{4.558}$$
$$\frac{dh(a)}{da} = \frac{d(a^2)}{da} = 2a \tag{4.559}$$
$$f(g(a),a) = f(0,a) = \tan^{-1}(0/a) = \tan^{-1}(0) = 0 \tag{4.560}$$
$$\frac{dg(a)}{da} = \frac{d(0)}{da} = 0 \tag{4.561}$$

Step 9: Now, substituting into the DUIS formula:

$$\frac{dI(a)}{da} = \int_0^{a^2}\frac{\partial f(x,a)}{\partial a}dx + f(h(a),a)\cdot\frac{dh(a)}{da} - f(g(a),a)\cdot\frac{dg(a)}{da} \tag{4.562}$$
$$= \int_0^{a^2} -\frac{x}{a^2+x^2}dx + \tan^{-1}(a)\cdot 2a - 0\cdot 0 \tag{4.563}$$
$$= -\int_0^{a^2}\frac{x}{a^2+x^2}dx + 2a\tan^{-1}(a) \tag{4.564}$$

Step 10: Let's evaluate the integral:

$$\int_0^{a^2} \frac{x}{a^2+x^2}dx = \frac{1}{2}\int_0^{a^2} \frac{2x}{a^2+x^2}dx \tag{4.565}$$

$$= \frac{1}{2}\left[\ln(a^2+x^2)\right]_0^{a^2} \tag{4.566}$$

$$= \frac{1}{2}\ln(a^2+a^4) - \frac{1}{2}\ln(a^2) \tag{4.567}$$

$$= \frac{1}{2}\ln\left(\frac{a^2(1+a^2)}{a^2}\right) \tag{4.568}$$

$$= \frac{1}{2}\ln(1+a^2) \tag{4.569}$$

Step 11: Substituting this back:

$$\frac{dI(a)}{da} = -\frac{1}{2}\ln(1+a^2) + 2a\tan^{-1}(a) \tag{4.570}$$

$$= 2a\tan^{-1}(a) - \frac{1}{2}\ln(1+a^2) \tag{4.571}$$

Step 12: Comparing the results from steps 6 and 11:

$$\frac{dI(a)}{da} = 2a\tan^{-1}(a) - \frac{1}{2}\ln(1+a^2) \tag{4.572}$$

$$\frac{dI(a)}{da}\text{ (using DUIS)} = 2a\tan^{-1}(a) - \frac{1}{2}\ln(1+a^2) \tag{4.573}$$

Since both expressions are identical, the DUIS rule is verified for this integral with variable limits.
Therefore:

$$\boxed{\frac{d}{da}\int_0^{a^2} \tan^{-1}(x/a)dx = \int_0^{a^2} \frac{\partial}{\partial a}\left(\tan^{-1}(x/a)\right)dx + \tan^{-1}(a)\cdot 2a} \tag{4.574}$$

This confirms that the differentiation under the integral sign rule, including the additional term for the variable upper limit, is valid for this specific case.

Example 9

Find $\frac{dI}{da}$, if $I(a) = \int_a^{a^2} \frac{\sin ax}{x}dx$.

Detailed Solution

Step 1: We need to find the derivative of:

$$I(a) = \int_a^{a^2} \frac{\sin ax}{x}dx \tag{4.575}$$

where both the limits of integration and the integrand depend on the parameter a.
Step 2: When differentiating an integral with variable limits, we use the formula:

$$\frac{d}{da}\int_{g(a)}^{h(a)} f(x,a)dx = \int_{g(a)}^{h(a)} \frac{\partial f(x,a)}{\partial a}dx + f(h(a),a)\cdot\frac{dh(a)}{da} - f(g(a),a)\cdot\frac{dg(a)}{da} \tag{4.576}$$

In our case:

$$g(a) = a \tag{4.577}$$
$$h(a) = a^2 \tag{4.578}$$
$$f(x, a) = \frac{\sin ax}{x} \tag{4.579}$$

Step 3: First, let's find $\frac{\partial f(x,a)}{\partial a}$:

$$\frac{\partial f(x, a)}{\partial a} = \frac{\partial}{\partial a}\left(\frac{\sin ax}{x}\right) \tag{4.580}$$
$$= \frac{1}{x} \cdot \frac{\partial}{\partial a}(\sin ax) \tag{4.581}$$
$$= \frac{1}{x} \cdot x \cos ax \tag{4.582}$$
$$= \cos ax \tag{4.583}$$

Step 4: Next, we need to evaluate:

$$f(h(a), a) = f(a^2, a) = \frac{\sin a \cdot a^2}{a^2} = \sin a \tag{4.584}$$
$$\frac{dh(a)}{da} = \frac{d}{da}(a^2) = 2a \tag{4.585}$$
$$f(g(a), a) = f(a, a) = \frac{\sin a \cdot a}{a} = \sin a \tag{4.586}$$
$$\frac{dg(a)}{da} = \frac{d}{da}(a) = 1 \tag{4.587}$$

Step 5: Now, let's substitute all these values into our differentiation formula:

$$\frac{dI}{da} = \int_a^{a^2} \frac{\partial f(x, a)}{\partial a} dx + f(h(a), a) \cdot \frac{dh(a)}{da} - f(g(a), a) \cdot \frac{dg(a)}{da} \tag{4.588}$$
$$= \int_a^{a^2} \cos ax \, dx + \sin a \cdot 2a - \sin a \cdot 1 \tag{4.589}$$
$$= \int_a^{a^2} \cos ax \, dx + \sin a \cdot (2a - 1) \tag{4.590}$$

Step 6: Let's evaluate the integral:

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C \tag{4.591}$$

Evaluating at the limits:

$$\int_a^{a^2} \cos ax \, dx = \left[\frac{1}{a} \sin ax\right]_a^{a^2} \tag{4.592}$$
$$= \frac{1}{a} \sin a \cdot a^2 - \frac{1}{a} \sin a \cdot a \tag{4.593}$$
$$= \frac{1}{a} \sin a \cdot (a^2 - a) \tag{4.594}$$
$$= \frac{1}{a} \sin a \cdot a(a - 1) \tag{4.595}$$
$$= \sin a \cdot (a - 1) \tag{4.596}$$

Step 7: Substituting this result back:

$$\frac{dI}{da} = \sin a \cdot (a-1) + \sin a \cdot (2a-1) \tag{4.597}$$
$$= \sin a \cdot [(a-1) + (2a-1)] \tag{4.598}$$
$$= \sin a \cdot (3a-2) \tag{4.599}$$

Therefore:

$$\boxed{\frac{dI}{da} = \sin a \cdot (3a-2)} \tag{4.600}$$

Example 10

Show that $\frac{d}{da}\int_{\sqrt{a}}^{1/a} \cos ax^2 dx = -\int_{\sqrt{a}}^{1/a} x^2 \sin ax^2 dx - \frac{1}{a^2}\cos\left(\frac{1}{a}\right) - \frac{1}{2\sqrt{a}}\cos a$.

Detailed Solution

Step 1: Let's define the function:

$$I(a) = \int_{\sqrt{a}}^{1/a} \cos ax^2 dx \tag{4.601}$$

where both the limits of integration and the integrand depend on the parameter a.
Step 2: When differentiating an integral with variable limits, we use the formula:

$$\frac{d}{da}\int_{g(a)}^{h(a)} f(x,a)dx = \int_{g(a)}^{h(a)} \frac{\partial f(x,a)}{\partial a}dx + f(h(a),a)\cdot\frac{dh(a)}{da} - f(g(a),a)\cdot\frac{dg(a)}{da} \tag{4.602}$$

In our case:

$$g(a) = \sqrt{a} \tag{4.603}$$
$$h(a) = \frac{1}{a} \tag{4.604}$$
$$f(x,a) = \cos ax^2 \tag{4.605}$$

Step 3: First, let's find $\frac{\partial f(x,a)}{\partial a}$:

$$\frac{\partial f(x,a)}{\partial a} = \frac{\partial}{\partial a}(\cos ax^2) \tag{4.606}$$
$$= -\sin ax^2 \cdot \frac{\partial}{\partial a}(ax^2) \tag{4.607}$$
$$= -\sin ax^2 \cdot x^2 \tag{4.608}$$
$$= -x^2 \sin ax^2 \tag{4.609}$$

Step 4: Next, we need to evaluate:

$$f(h(a),a) = f\left(\frac{1}{a},a\right) = \cos a\left(\frac{1}{a}\right)^2 = \cos\left(\frac{1}{a}\right) \tag{4.610}$$
$$\frac{dh(a)}{da} = \frac{d}{da}\left(\frac{1}{a}\right) = -\frac{1}{a^2} \tag{4.611}$$
$$f(g(a),a) = f(\sqrt{a},a) = \cos a(\sqrt{a})^2 = \cos a \cdot a = \cos a \tag{4.612}$$
$$\frac{dg(a)}{da} = \frac{d}{da}(\sqrt{a}) = \frac{1}{2\sqrt{a}} \tag{4.613}$$

Step 5: Now, let's substitute all these values into our differentiation formula:

$$\frac{dI}{da} = \int_{g(a)}^{h(a)} \frac{\partial f(x,a)}{\partial a} dx + f(h(a),a) \cdot \frac{dh(a)}{da} - f(g(a),a) \cdot \frac{dg(a)}{da} \tag{4.614}$$

$$= \int_{\sqrt{a}}^{1/a} (-x^2 \sin ax^2) dx + \cos\left(\frac{1}{a}\right) \cdot \left(-\frac{1}{a^2}\right) - \cos a \cdot \frac{1}{2\sqrt{a}} \tag{4.615}$$

$$= -\int_{\sqrt{a}}^{1/a} x^2 \sin ax^2 dx - \frac{1}{a^2} \cos\left(\frac{1}{a}\right) - \frac{1}{2\sqrt{a}} \cos a \tag{4.616}$$

Therefore:

$$\boxed{\frac{d}{da} \int_{\sqrt{a}}^{1/a} \cos ax^2 dx = -\int_{\sqrt{a}}^{1/a} x^2 \sin ax^2 dx - \frac{1}{a^2} \cos\left(\frac{1}{a}\right) - \frac{1}{2\sqrt{a}} \cos a} \tag{4.617}$$

Thus, we have verified the given identity.

Example 11

Prove that $\int_0^x \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3} \tan^{-1}(x/a) + \frac{x}{2a^2(x^2+a^2)}$.

Detailed Solution

Step 1: First, let's consider this integral as a function of the upper limit:

$$I(x) = \int_0^x \frac{dt}{(t^2+a^2)^2} \tag{4.618}$$

Note that I've changed the variable of integration to t to avoid confusion with the upper limit x.

Step 2: We'll use differentiation under the integral sign (DUIS), but in an indirect way. Instead of directly calculating the integral, let's define a more general function:

$$J(a) = \int_0^x \frac{dt}{t^2+a^2} \tag{4.619}$$

Step 3: It's well-known that:

$$\int \frac{dt}{t^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{t}{a}\right) + C \tag{4.620}$$

Therefore:

$$J(a) = \int_0^x \frac{dt}{t^2+a^2} \tag{4.621}$$

$$= \left[\frac{1}{a} \tan^{-1}\left(\frac{t}{a}\right)\right]_0^x \tag{4.622}$$

$$= \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{a} \tan^{-1}(0) \tag{4.623}$$

$$= \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \tag{4.624}$$

Step 4: Notice that our original integral $I(x)$ is related to $J(a)$ by differentiation with respect to a:

$$\frac{\partial J(a)}{\partial a} = \frac{\partial}{\partial a}\int_0^x \frac{dt}{t^2+a^2} \tag{4.625}$$

$$= \int_0^x \frac{\partial}{\partial a}\left(\frac{1}{t^2+a^2}\right) dt \tag{4.626}$$

$$= \int_0^x \frac{\partial}{\partial a}(t^2+a^2)^{-1} dt \tag{4.627}$$

$$= \int_0^x (-1)(t^2+a^2)^{-2}\cdot 2a\, dt \tag{4.628}$$

$$= -2a\int_0^x \frac{dt}{(t^2+a^2)^2} \tag{4.629}$$

$$= -2a\cdot I(x) \tag{4.630}$$

Step 5: Therefore:

$$I(x) = -\frac{1}{2a}\frac{\partial J(a)}{\partial a} \tag{4.631}$$

Step 6: Now, let's compute $\frac{\partial J(a)}{\partial a}$:

$$\frac{\partial J(a)}{\partial a} = \frac{\partial}{\partial a}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right] \tag{4.632}$$

$$= \frac{\partial}{\partial a}\left(\frac{1}{a}\right)\cdot\tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{a}\cdot\frac{\partial}{\partial a}\left[\tan^{-1}\left(\frac{x}{a}\right)\right] \tag{4.633}$$

$$= -\frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{a}\cdot\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{\partial}{\partial a}\left(\frac{x}{a}\right) \tag{4.634}$$

$$= -\frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{a}\cdot\frac{1}{1+\frac{x^2}{a^2}}\cdot\left(-\frac{x}{a^2}\right) \tag{4.635}$$

$$= -\frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{a}\cdot\frac{x}{a^2}\cdot\frac{a^2}{a^2+x^2} \tag{4.636}$$

$$= -\frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right) - \frac{x}{a^2}\cdot\frac{1}{a^2+x^2} \tag{4.637}$$

$$= -\frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right) - \frac{x}{a^2(a^2+x^2)} \tag{4.638}$$

$$\tag{4.639}$$

Step 7: Substituting this result into our expression for $I(x)$:

$$I(x) = -\frac{1}{2a}\frac{\partial J(a)}{\partial a} \tag{4.640}$$

$$= -\frac{1}{2a}\left[-\frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right) - \frac{x}{a^2(a^2+x^2)}\right] \tag{4.641}$$

$$= \frac{1}{2a}\cdot\frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{2a}\cdot\frac{x}{a^2(a^2+x^2)} \tag{4.642}$$

$$= \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^3(a^2+x^2)} \tag{4.643}$$

$$= \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} \tag{4.644}$$

Therefore:

$$\boxed{\int_0^x \frac{dt}{(t^2+a^2)^2} = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)}} \tag{4.645}$$

which matches the result we were asked to prove.

4.9 Additional Solved Examples on Error Function

Example 1

Show that $\int_a^b e^{-x^2}dx = \frac{\sqrt{\pi}}{2}[erf(b) - erf(a)]$.

Detailed Solution

Given: Error function and complementary error function of x are given by:

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du \tag{4.646}$$

$$erf_c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}du \tag{4.647}$$

We know that:

$$erf(\infty) = 1 \tag{4.648}$$

This implies:

$$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}du = 1 \tag{4.649}$$

Let's split the integral from 0 to ∞ into three parts:

$$\frac{2}{\sqrt{\pi}}\left[\int_0^a e^{-u^2}du + \int_a^b e^{-u^2}du + \int_b^\infty e^{-u^2}du\right] = 1 \tag{4.650}$$

Recognizing the definition of error functions:

$$erf(a) + \frac{2}{\sqrt{\pi}}\int_a^b e^{-u^2}du + erf_c(b) = 1 \tag{4.651}$$

Step 1: We also know that $erf(x) + erf_c(x) = 1$, which gives us:

$$erf_c(b) = 1 - erf(b) \tag{4.652}$$

Step 2: Substituting this into our equation:

$$erf(a) + \frac{2}{\sqrt{\pi}}\int_a^b e^{-u^2}du + 1 - erf(b) = 1 \tag{4.653}$$

Step 3: Simplifying:

$$erf(a) + \frac{2}{\sqrt{\pi}}\int_a^b e^{-u^2}du - erf(b) = 0 \tag{4.654}$$

$$\frac{2}{\sqrt{\pi}}\int_a^b e^{-u^2}du = erf(b) - erf(a) \tag{4.655}$$

Step 4: Rearranging to solve for the integral:

$$\int_a^b e^{-u^2} du = \frac{\sqrt{\pi}}{2}[erf(b) - erf(a)] \tag{4.656}$$

Step 5: Changing the variable back to x:

$$\int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2}[erf(b) - erf(a)] \tag{4.657}$$

Therefore:

$$\boxed{\int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2}[erf(b) - erf(a)]} \tag{4.658}$$

Example 2

Show that $\int_0^\infty e^{-x^2-2ax} dx = \frac{\sqrt{\pi}}{2} e^{a^2}[1 - erf(a)]$.

Detailed Solution

Given: Error function and complementary error function of x are given by:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{4.659}$$

$$erf_c(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du \tag{4.660}$$

Step 1: Let's start by manipulating the integrand to complete the square in the exponent:

$$\text{L.H.S.} = \int_0^\infty e^{-x^2-2ax} dx \tag{4.661}$$

$$= \int_0^\infty e^{-x^2-2ax-a^2+a^2} dx \tag{4.662}$$

$$= \int_0^\infty e^{-(x+a)^2} \cdot e^{a^2} dx \tag{4.663}$$

$$= e^{a^2} \int_0^\infty e^{-(x+a)^2} dx \tag{4.664}$$

Step 2: Let's make the substitution $u = x + a$, which gives $dx = du$. The limits transform as:

$$\text{When } x = 0 \Rightarrow u = a \tag{4.665}$$

$$\text{When } x = \infty \Rightarrow u = \infty \tag{4.666}$$

Step 3: Applying this substitution:

$$\text{L.H.S.} = e^{a^2} \int_a^\infty e^{-u^2} du \tag{4.667}$$

$$= e^{a^2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} \int_a^\infty e^{-u^2} du \tag{4.668}$$

$$= \frac{\sqrt{\pi}}{2} e^{a^2} \cdot erf_c(a) \tag{4.669}$$

Step 4: Using the property $erf_c(x) = 1 - erf(x)$:

$$\text{L.H.S.} = \frac{\sqrt{\pi}}{2} e^{a^2} \cdot [1 - erf(a)] \tag{4.670}$$

$$= \frac{\sqrt{\pi}}{2} e^{a^2} [1 - erf(a)] \tag{4.671}$$

Therefore:

$$\boxed{\int_0^\infty e^{-x^2-2ax} dx = \frac{\sqrt{\pi}}{2} e^{a^2} [1 - erf(a)]} \tag{4.672}$$

Example 3

Find $\frac{1}{x}\frac{d}{da}[erf_c(ax)] = -\frac{1}{a}\frac{d}{dx}[erf(ax)]$.

Detailed Solution

Given: Error function and complementary error function of x are given by:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{4.673}$$

$$erf_c(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du \tag{4.674}$$

Therefore:

$$erf(ax) = \frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-u^2} du \tag{4.675}$$

$$erf_c(ax) = \frac{2}{\sqrt{\pi}} \int_{ax}^\infty e^{-u^2} du \tag{4.676}$$

Step 1: Let's compute the left-hand side using differentiation under the integral sign (DUIS):

$$\text{L.H.S.} = \frac{1}{x}\frac{d}{da}[erf_c(ax)] \tag{4.677}$$

$$= \frac{1}{x}\frac{d}{da}\left[\frac{2}{\sqrt{\pi}} \int_{ax}^\infty e^{-u^2} du\right] \tag{4.678}$$

$$= \frac{1}{x}\frac{2}{\sqrt{\pi}}\left[\int_{ax}^\infty \frac{\partial}{\partial a}(e^{-u^2}) du + \frac{d(\infty)}{da} e^{-\infty} - \frac{d(ax)}{da} e^{-(ax)^2}\right] \tag{4.679}$$

Step 2: Analyzing each term:

$$\frac{\partial}{\partial a}(e^{-u^2}) = 0 \quad \text{(since } u \text{ is the integration variable, independent of } a\text{)} \tag{4.680}$$

$$\frac{d(\infty)}{da} = 0 \tag{4.681}$$

$$\frac{d(ax)}{da} = x \tag{4.682}$$

Step 3: Substituting these values:

$$\text{L.H.S.} = \frac{1}{x}\frac{2}{\sqrt{\pi}}\left[0 + 0 - x \cdot e^{-(ax)^2}\right] \tag{4.683}$$

$$= \frac{1}{x}\frac{2}{\sqrt{\pi}}[-x \cdot e^{-a^2x^2}] \tag{4.684}$$

$$= -\frac{2}{\sqrt{\pi}}e^{-a^2x^2} \tag{4.685}$$

Step 4: Now let's compute the right-hand side:

$$\text{R.H.S.} = -\frac{1}{a}\frac{d}{dx}[erf(ax)] \tag{4.686}$$

$$= -\frac{1}{a}\frac{d}{dx}\left[\frac{2}{\sqrt{\pi}}\int_0^{ax} e^{-u^2}du\right] \tag{4.687}$$

Step 5: Using Leibniz's rule:

$$\frac{d}{dx}\left[\frac{2}{\sqrt{\pi}}\int_0^{ax} e^{-u^2}du\right] = \frac{2}{\sqrt{\pi}}\left[\frac{d(ax)}{dx}e^{-(ax)^2} - \frac{d(0)}{dx}e^{-0^2}\right] \tag{4.688}$$

$$= \frac{2}{\sqrt{\pi}}[a \cdot e^{-a^2x^2} - 0] \tag{4.689}$$

$$= \frac{2a}{\sqrt{\pi}}e^{-a^2x^2} \tag{4.690}$$

Step 6: Therefore:

$$\text{R.H.S.} = -\frac{1}{a} \cdot \frac{2a}{\sqrt{\pi}}e^{-a^2x^2} \tag{4.691}$$

$$= -\frac{2}{\sqrt{\pi}}e^{-a^2x^2} \tag{4.692}$$

Step 7: Comparing the results:

$$\text{L.H.S.} = -\frac{2}{\sqrt{\pi}}e^{-a^2x^2} \tag{4.693}$$

$$\text{R.H.S.} = -\frac{2}{\sqrt{\pi}}e^{-a^2x^2} \tag{4.694}$$

Since L.H.S. = R.H.S., we have proven that:

$$\boxed{\frac{1}{x}\frac{d}{da}[erf_c(ax)] = -\frac{1}{a}\frac{d}{dx}[erf(ax)]} \tag{4.695}$$

Example 4

Prove that $\frac{d}{dt}erf\sqrt{t} = \frac{e^{-t}}{\sqrt{\pi t}}$ and $\frac{d}{dt}erf_c\sqrt{t} = \frac{-e^{-t}}{\sqrt{\pi t}}$.

Detailed Solution

Given: Error function and complementary error function are defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{4.696}$$

$$erf_c(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du \tag{4.697}$$

Part 1: Let's prove $\frac{d}{dt} erf\sqrt{t} = \frac{e^{-t}}{\sqrt{\pi t}}$
Step 1: Using the chain rule:

$$\frac{d}{dt} erf\sqrt{t} = \frac{d(erf\sqrt{t})}{d\sqrt{t}} \cdot \frac{d\sqrt{t}}{dt} \tag{4.698}$$

Step 2: Computing each derivative:

$$\frac{d(erf\sqrt{t})}{d\sqrt{t}} = \left.\frac{d(erf(x))}{dx}\right|_{x=\sqrt{t}} \tag{4.699}$$

$$= \frac{2}{\sqrt{\pi}} e^{-(\sqrt{t})^2} \tag{4.700}$$

$$= \frac{2}{\sqrt{\pi}} e^{-t} \tag{4.701}$$

And:

$$\frac{d\sqrt{t}}{dt} = \frac{1}{2\sqrt{t}} \tag{4.702}$$

Step 3: Multiplying these results:

$$\frac{d}{dt} erf\sqrt{t} = \frac{2}{\sqrt{\pi}} e^{-t} \cdot \frac{1}{2\sqrt{t}} \tag{4.703}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{e^{-t}}{\sqrt{t}} \tag{4.704}$$

$$= \frac{e^{-t}}{\sqrt{\pi t}} \tag{4.705}$$

Part 2: Let's prove $\frac{d}{dt} erf_c\sqrt{t} = \frac{-e^{-t}}{\sqrt{\pi t}}$
Step 4: We know that $erf_c(x) = 1 - erf(x)$, therefore:

$$erf_c\sqrt{t} = 1 - erf\sqrt{t} \tag{4.706}$$

Step 5: Taking the derivative:

$$\frac{d}{dt} erf_c\sqrt{t} = \frac{d}{dt}(1 - erf\sqrt{t}) \tag{4.707}$$

$$= -\frac{d}{dt} erf\sqrt{t} \tag{4.708}$$

$$= -\frac{e^{-t}}{\sqrt{\pi t}} \tag{4.709}$$

Therefore:

$$\boxed{\frac{d}{dt} erf\sqrt{t} = \frac{e^{-t}}{\sqrt{\pi t}}} \tag{4.710}$$

and

$$\boxed{\frac{d}{dt}erf_c\sqrt{t} = \frac{-e^{-t}}{\sqrt{\pi t}}} \tag{4.711}$$