Chapter 2

Normal Form

2.1 Normal Form of Matrix

When studying matrices, one powerful way to understand their structure is to reduce them to simpler, canonical representations. The Normal Form of a matrix is one such representation that reveals fundamental properties of the matrix, particularly its rank.

Definition 2.1. *The* ***Normal Form*** *of a matrix is a standardized representation achieved through elementary row and column operations that highlights the matrix's rank structure.*

Theorem 2.2 (Normal Form Theorem)**.** *Any matrix A of size $m \times n$ with rank r can be reduced, by applying a sequence of elementary row and column operations, to the unique block matrix form:*

$$N = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m \times n}$$

Where:

- *I_r is the identity matrix of size $r \times r$.*
- *r is the* ***rank*** *of the matrix A.*
- *The 0 symbols represent zero matrices of appropriate sizes to fill the remaining $m \times n$ dimensions.*

Two $m \times n$ matrices are equivalent (one can be transformed into the other via elementary row and column operations) if and only if they have the same Normal Form (i.e., they have the same rank).

2.2 Matrix Rank

Before delving deeper into Normal Form, let's establish a solid understanding of matrix rank, which plays a central role in this theory.

Definition 2.3. *The* ***rank*** *of a matrix A, denoted* $\operatorname{rank}(A)$*, is the maximum number of linearly independent rows or columns in the matrix.*

Equivalent Definitions of Rank

For a matrix A, the following are equivalent:

1. The maximum number of linearly independent rows
2. The maximum number of linearly independent columns
3. The order of the largest non-singular square submatrix
4. The dimension of the row space or column space
5. The order of maximum non-vanishing(non-zero) minor
6. The number of pivots in the normal form or row echelon form
7. The number of non-zero rows after reducing the matrix to normal form or row echleon form

Rank Properties

For matrices A and B of compatible dimensions:

1. $0 \leq \text{rank}(A) \leq \min(m, n)$ for an $m \times n$ matrix A
2. $\text{rank}(A^T) = \text{rank}(A)$
3. $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$
4. $\text{rank}(A \cdot B) \leq \min(\text{rank}(A), \text{rank}(B))$
5. If A is square, A is invertible if and only if $\text{rank}(A) = n$
6. Elementary row or column operations do not change the rank of a matrix

2.3 Matrix Reduction to Normal Form: Visual Guide

STEP 1:

$$\begin{matrix} \boxed{a_{11}} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{matrix}$$

Find your first pivot element at position (1,1).
If $a_{11} = 0$, locate a non-zero element elsewhere in the matrix and use row/column swaps to move it to position (1,1).

STEP 2:

$$\begin{matrix} 1 & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{matrix}$$

Transform the pivot element to 1 by dividing the entire first row by a_{11} or by swapping rows or columns. This gives us our first proper pivot at position (1,1).

STEP 3:

$$\begin{matrix} 1 & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a'_{m2} & a'_{m3} & \cdots & a'_{mn} \end{matrix}$$

Zero out all elements below the pivot by subtracting multiples of the first row from each row below: $R_i \leftarrow R_i - a_{i1} \cdot R_1$ for each row $i = 2, 3, \ldots, m$

STEP 4:

$$\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a''_{22} & a''_{23} & \cdots & a''_{2n} \\ 0 & a''_{32} & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a''_{m2} & a''_{m3} & \cdots & a''_{mn} \end{matrix}$$

Zero out all elements to the right of the pivot by using column operations: $C_j \leftarrow C_j - a_{1j} \cdot C_1$ for each column $j = 2, 3, \ldots, n$

STEP 5:

$$\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \boxed{a''_{22}} & a''_{23} & \cdots & a''_{2n} \\ 0 & a''_{32} & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a''_{m2} & a''_{m3} & \cdots & a''_{mn} \end{matrix}$$

Move to the next pivot position (2,2). Repeat the entire process on the remaining submatrix (excluding the first row and column).

STEP 6:

$$\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{matrix}$$

After repeating the process for all possible pivots, the matrix is in normal form with r leading 1's along the diagonal (where r is the rank of the matrix).

Color Legend

Current pivot | Pivot = 1 | Zeroed element

What distinguishes the Normal Form from the Row Echelon Form is this critical step: we must also create zeros to the right of the pivot in the first row. This requires column operations, which are not used in the standard row reduction.

For each column j (where $j = 2, 3, \ldots, n$), we perform:

$$C_j \leftarrow C_j - a'_{1j} \cdot C_1 \tag{2.1}$$

These column operations eliminate all elements in the first row to the right of the pivot, replacing them with zeros. These operations also affect other elements in the matrix, resulting in new values that we denote with double primes.

After completing steps 3 and 4, we have isolated our first pivot with zeros above, below, to the left, and to the right. This "cross pattern" of zeros surrounding the pivot is characteristic of the Normal

Example Reduction to Normal Form

Let's visualize the process with a concrete example, reducing a 3×4 matrix to normal form:

Initial 3×4 Matrix

$$\begin{matrix} 2 & 1 & 3 & 4 \\ 1 & 0 & 2 & 1 \\ 3 & 2 & 1 & 5 \end{matrix}$$

Step 1:

$$\begin{matrix} 2 & 1 & 3 & 4 \\ 1 & 0 & 2 & 1 \\ 3 & 2 & 1 & 5 \end{matrix}$$

The element at (1,1) is non-zero, so it can serve as our pivot

Step 2:

$$\begin{matrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 \\ 1 & 0 & 2 & 1 \\ 3 & 2 & 1 & 5 \end{matrix}$$

Make the pivot 1 by dividing the first row by 2.

Step 3:

$$\begin{matrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -1 \\ 0 & \frac{1}{2} & -\frac{7}{2} & -1 \end{matrix}$$

Zero out the elements below the pivot by subtracting appropriate multiples of the first row.

$R_2 \leftarrow R_2 - 1 \cdot R_1$

$R_3 \leftarrow R_3 - 3 \cdot R_1$

Step 4:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -1 \\ 0 & \frac{1}{2} & -\frac{7}{2} & -1 \end{bmatrix}$$

Zero out the elements to the right of the pivot using column operations.

$C_2 \leftarrow C_2 - \frac{1}{2} \cdot C_1$

$C_3 \leftarrow C_3 - \frac{3}{2} \cdot C_1$

$C_4 \leftarrow C_4 - 2 \cdot C_1$

Step 5:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -1 \\ 0 & \frac{1}{2} & -\frac{7}{2} & -1 \end{bmatrix}$$

Move to the next pivot position (2,2). The element is non-zero, so we can use it as a pivot.

Step 6:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & \frac{1}{2} & -\frac{7}{2} & -1 \end{bmatrix}$$

Make the second pivot 1 by dividing the second row by $-\frac{1}{2}$.

Step 7:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

Zero out the elements below the second pivot by subtracting multiples of the second row.

$R_3 \leftarrow R_3 - \frac{1}{2} \cdot R_2$

Step 8:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

Zero out the elements to the right of the second pivot using column operations.

$C_3 \leftarrow C_3 - (-1) \cdot C_2$

$C_4 \leftarrow C_4 - 2 \cdot C_2$

Step 9:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

Move to the next pivot position (3,3). The element is non-zero, so we can use it as a pivot.

Step 10:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix}$$

Make the third pivot 1 by dividing the third row by -3.

Step 11:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Zero out the elements to the right of the third pivot using column operations.
$C_4 \leftarrow C_4 - \frac{2}{3} \cdot C_3$

Final Result

The final matrix is now in normal form with pivots in positions (1,1), (2,2), and (3,3).
The rank of the original matrix is 3.

2.4 Additional Solved Examples

Matrix with Full Rank $r = 2$ Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 9 \end{bmatrix}$.

- **k=1:** Prepare (1,1). $a_{11} = 2 \neq 0$. OK.
- Make Pivot '1': $R_1 \to \frac{1}{2}R_1$

$$\implies \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$$

- Zeros Below: $R_2 \to R_2 - 3R_1$

$$\implies \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

- Zeros Right: $C_2 \to C_2 - 2C_1$

$$\implies \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

- **k=2:** Prepare (2,2). $a_{22} = 3 \neq 0$. OK.
- Make Pivot '1': $R_2 \to \frac{1}{3}R_2$

$$\implies \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Zeros Below: None.
- Zeros Right: None.

Result for A: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$. (Standard Normal Form). Rank $r = 2$.

Rectangular Matrix with Rank $r = 2$ Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 9 \end{bmatrix}$.

- **k=1:** Prepare (1,1). $a_{11} = 1 \neq 0$. OK.
- Make Pivot '1': Already 1.
- Zeros Below: $R_2 \to R_2 - 2R_1$

$$\implies \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

- Zeros Right: $C_2 \to C_2 - 2C_1$; $C_3 \to C_3 - 3C_1$

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- **k=2:** Prepare (2,2). $a_{22} = 0$. Column 2 below is also 0. Look right: $a_{23} = 3 \neq 0$ in column $j = 3$. Swap columns $C_2 \leftrightarrow C_3$.

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

  Now $a_{22} = 3 \neq 0$. OK.
- Make Pivot '1': $R_2 \to \frac{1}{3}R_2$

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Zeros Below: None.
- Zeros Right: None.

Result for B: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [I_2|0]$. (Standard Normal Form). Rank $r = 2$.

Matrix Requiring Row Swaps Let $C = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$.

- **k=1:** Prepare (1,1). $a_{11} = 0$. Look below: $a_{21} = 1 \neq 0$. Swap rows $R_1 \leftrightarrow R_2$.

$$\implies \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

  Now $a_{11} = 1 \neq 0$. OK.
- Make Pivot '1': Already 1.
- Zeros Below: $a_{21} = 0$. OK.
- Zeros Right: Eliminate $a_{12} = 3$. $C_2 \to C_2 - 3C_1$.

$$\implies \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

- **k=2:** Prepare (2,2). $a_{22} = 2 \neq 0$. OK.
- Make Pivot '1': $R_2 \to \frac{1}{2}R_2$

$$\implies \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Zeros Below: None.
- Zeros Right: None.

Result for C: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$. (Standard Normal Form). Rank $r = 2$.

3×3 Matrix with Full Rank $r = 3$ Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 11 \end{bmatrix}$.

k=1:

- Prepare (1,1): $a_{11} = 1 \neq 0$. OK.
- Make Pivot '1': Already 1.
- Zeros Below: $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$.

$$\implies \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

- Zeros Right: $C_2 \to C_2 - 2C_1$; $C_3 \to C_3 - 3C_1$.

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

k=2:

- Prepare (2,2): $a_{22} = 1 \neq 0$. OK.
- Make Pivot '1': Already 1.
- Zeros Below: $R_3 \to R_3 - R_2$.

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- Zeros Right: $C_3 \to C_3 - C_2$.

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

k=3:

- Prepare (3,3): $a_{33} = 1 \neq 0$. OK.
- Make Pivot '1': Already 1.
- Zeros Below: None.
- Zeros Right: None.

Result for A: The Normal Form is $N = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The rank is $r = 3$.

3×4 Matrix with Rank $r = 2$ Let $B = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 5 & 4 \\ 1 & 1 & 0 & 7 \end{bmatrix}$.

k=1:

- Prepare (1,1): $a_{11} = 1 \neq 0$. OK.
- Make Pivot '1': Already 1.
- Zeros Below: $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$.

$$\implies \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

- Zeros Right: $C_2 \to C_2 - C_1$; $C_3 \to C_3 - 2C_1$; $C_4 \to C_4 - 3C_1$.

$$\implies \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

k=2:

- Prepare (2,2): $a_{22} = 0$. Column 2 below is 0. Look right: $a_{23} = 1 \neq 0$ in column $j = 3$. Swap columns $C_2 \leftrightarrow C_3$.

$$\implies \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & -2 & 0 & 4 \end{bmatrix}$$

  Now $a_{22} = 1 \neq 0$. OK.
- Make Pivot '1': Already 1.
- Zeros Below: $R_3 \to R_3 + 2R_2$.

$$\implies \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Zeros Right: $C_4 \to C_4 - (-2)C_2 = C_4 + 2C_2$.

$$\implies \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

k=3:

- Prepare (3,3): $a_{33} = 0$. The entire remaining submatrix is zero. Stop.

Result for B: The Normal Form is $N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$. The rank is $r = 2$.

4×3 Matrix with Rank $r = 2$ Let $C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 0 & 0 & 1 \\ 3 & 6 & 1 \end{bmatrix}$.

k=1:

- Prepare (1,1): $a_{11} = 1 \neq 0$. OK.
- Make Pivot '1': Already 1.
- Zeros Below: $R_2 \to R_2 - 2R_1$; $R_4 \to R_4 - 3R_1$. (R3 already 0).

$$\implies \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

- Zeros Right: $C_2 \to C_2 - 2C_1$; $C_3 \to C_3 - C_1$.

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

k=2:

- Prepare (2,2): $a_{22} = 0$. Column 2 below is 0. Look right: $a_{23} = 1 \neq 0$ in column $j = 3$. Swap columns $C_2 \leftrightarrow C_3$.

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

  Now $a_{22} = 1 \neq 0$. OK.
- Make Pivot '1': Already 1.
- Zeros Below: $R_3 \to R_3 - R_2$; $R_4 \to R_4 + 2R_2$.

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Zeros Right: None (no columns $j > 2$).

k=3:

- Prepare (3,3): $a_{33} = 0$. The remaining submatrix is all zero. Stop.

Result for C: The Normal Form is $N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}$. The rank is $r = 2$.

3×3 Matrix with Rank $r = 2$ (Requires Swaps) Let $D = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 5 & 4 \end{bmatrix}$.

k=1:

- Prepare (1,1): $a_{11} = 0$. Look below: $a_{21} = 1 \neq 0$. Swap rows $R_1 \leftrightarrow R_2$.

$$\implies \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 5 & 4 \end{bmatrix}$$

  Now $a_{11} = 1 \neq 0$. OK.
- Make Pivot '1': Already 1.
- Zeros Below: $R_3 \to R_3 - 2R_1$. (R2 already 0).

$$\implies \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

- Zeros Right: $C_2 \to C_2 - 2C_1$; $C_3 \to C_3 - C_1$.

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

k=2:

- Prepare (2,2): $a_{22} = 1 \neq 0$. OK.
- Make Pivot '1': Already 1.
- Zeros Below: $R_3 \to R_3 - R_2$.

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- Zeros Right: $C_3 \to C_3 - 2C_2$.

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

k=3:

- Prepare (3,3): $a_{33} = 0$. The remaining submatrix is zero. Stop.

Result for D: The Normal Form is $N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$. The rank is $r = 2$.

2.5 Connection to Normal Form Theory

The visual guide presented in this chapter illustrates the algorithmic procedure that transforms any matrix to its Normal Form. This transformation process directly demonstrates several key concepts from matrix theory:

Key Observations

1. **Rank Visualization**: The number of pivot elements (green boxes) in the final form equals the rank of the matrix.
2. **Submatrix Focus**: Each step operates on progressively smaller submatrices, working diagonally through the matrix.
3. **Systematic Elimination**: The process demonstrates how systematic elimination using both row and column operations leads to the cleanest possible form of a matrix.
4. **Role of Operations**: Row operations create zeros below pivots, while column operations create zeros to the right of pivots—both are essential for reaching Normal Form.

Contrast with Row Echelon Form

It's instructive to compare this process with the computation of Row Echelon Form:

Normal Form vs. Row Echelon Form Procedures

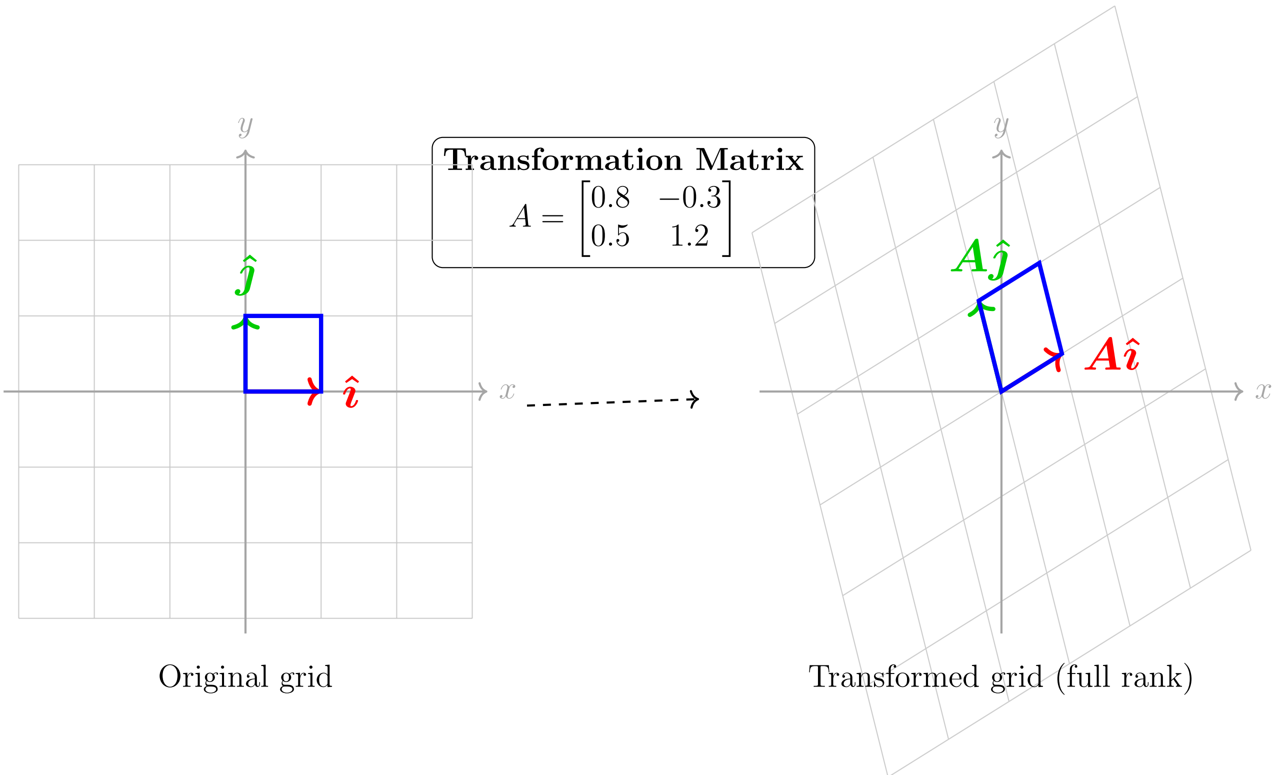
- **Normal Form**: Uses both row and column operations to create a symmetric pattern of zeros around each pivot, resulting in a block diagonal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.
- **Row Echelon Form**: Uses only row operations, creating zeros below pivots but maintaining coefficient values to the right of pivots, resulting in the form $\begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix}$ where F represents values relating to free variables.

2.6 Rank Visualization

The rank of a matrix can be visualized as the dimension of the image space after transformation.

Full Rank Transformation

A 2×2 matrix with rank 2 transforms the plane to a plane:



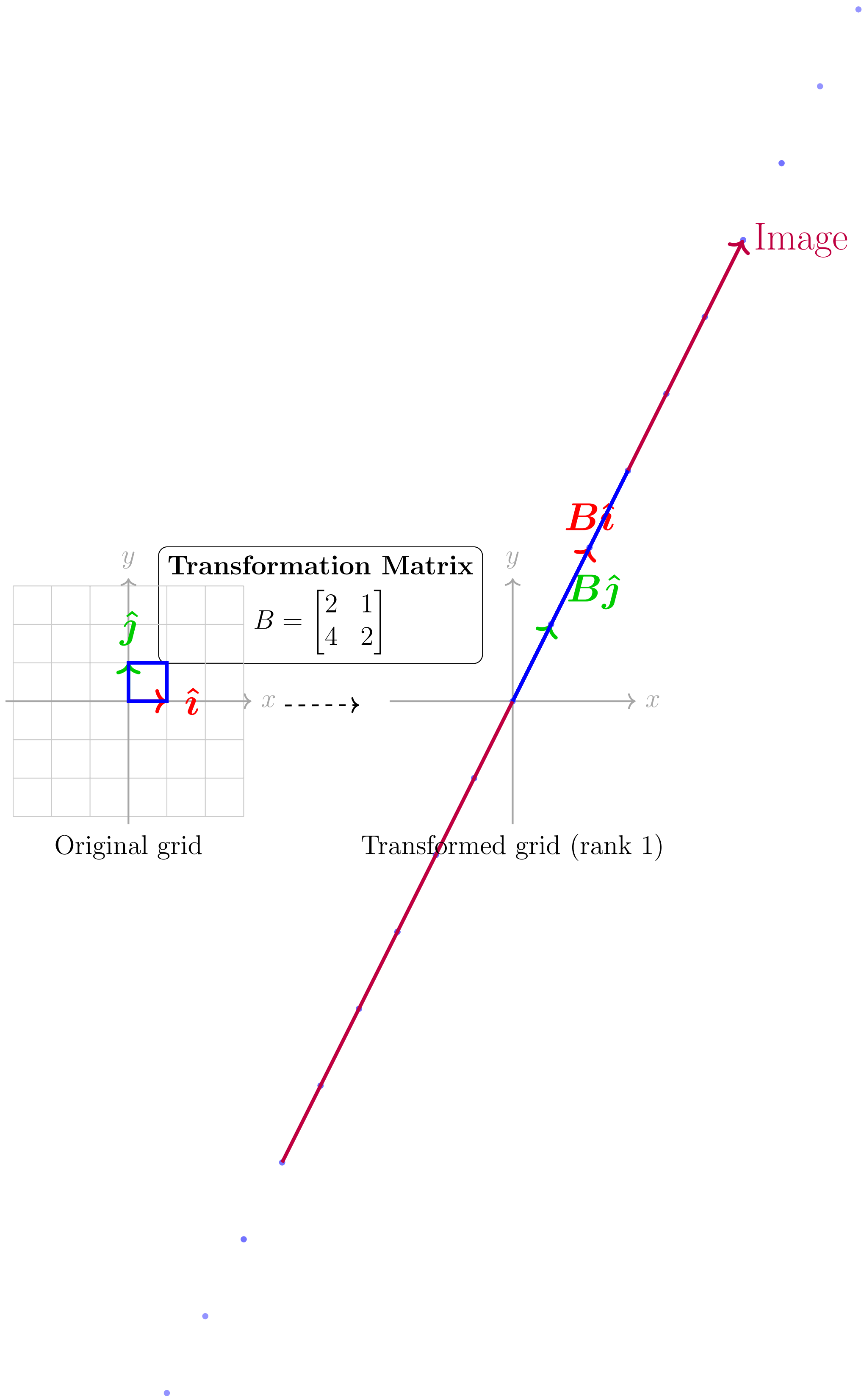
Original grid

Transformed grid (full rank)

The transformed space remains two-dimensional.

Rank 1 Transformation

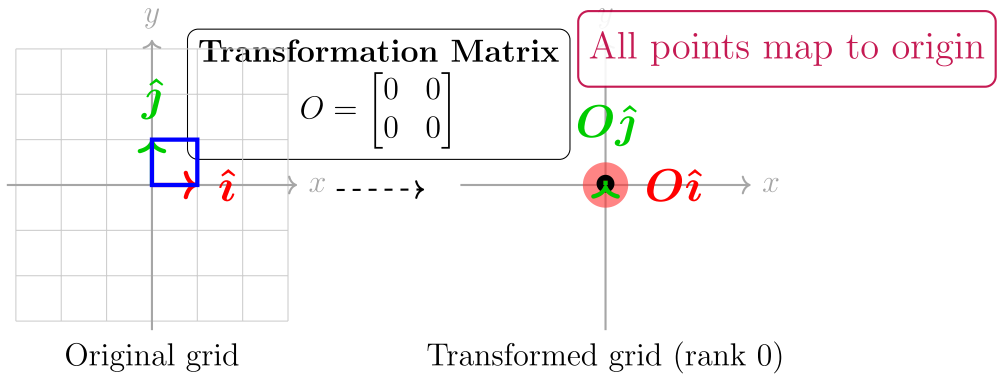
A 2×2 matrix with rank 1 collapses the plane to a line:



The transformed vectors all lie on a single line through the origin.

Rank 0 Transformation

A zero matrix collapses everything to a single point (the origin):



All vectors are mapped to the origin.

2.7 Significance of Normal Form

Matrix Equivalence

Definition 2.4. *Two matrices A and B are called **equivalent** if one can be obtained from the other by a sequence of elementary row and column operations.*

Theorem 2.5 (Equivalence Theorem)**.** *Two matrices are equivalent if and only if they have the same rank.*

This is a powerful result because it means that the Normal Form provides a complete invariant for matrix equivalence under elementary row and column operations.

Invariants Under Equivalence

The only invariant under elementary row and column operations is the rank of the matrix. Other properties like determinant, eigenvalues, or trace are not preserved.