Chapter 4

Systems of Equations

4.1 Introduction to Systems of Linear Equations

Linear equations are among the most fundamental mathematical structures encountered across disciplines ranging from pure mathematics to engineering, economics, and the natural sciences. A *system of linear equations* consists of a collection of linear equations that we seek to solve simultaneously.

Definition 4.1. *A* ***linear equation*** *in n variables $x_1, x_2, \ldots, x_n$ is an equation of the form:*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{4.1}$$

where $a_1, a_2, \ldots, a_n, b$ are real (or complex) numbers, and the a_i are called the ***coefficients*** *of the equation.*

Definition 4.2. *A* ***system of linear equations*** *(or* ***linear system****) is a collection of m linear equations in n variables:*

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \tag{4.2}$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \tag{4.3}$$
$$\vdots \qquad\qquad\qquad \vdots \tag{4.4}$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \tag{4.5}$$

A **solution** to a system of linear equations is an ordered n-tuple $(s_1, s_2, \ldots, s_n)$ that satisfies all equations simultaneously when we substitute $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$. The set of all possible solutions is called the **solution set** of the system.

Mathematical Representation

There are several ways to represent a system of linear equations:

Representations of Linear Systems

1. **Standard Form:** Writing out each equation as shown above
2. **Vector Form:** $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$ for $i = 1, 2, \ldots, m$
3. **Matrix Form:** $Ax = b$ where A is the coefficient matrix, x is the variable vector, and b is the constant vector
4. **Augmented Matrix Form:** $[A|b]$ which combines the coefficient matrix with the constants

Each of these representations offers different advantages depending on the context. The standard form is explicit but can be cumbersome for large systems. The vector and matrix forms are more compact and reveal the underlying structure of the system.

Different Representations of a Linear System Consider the system:

$$2x_1 - 3x_2 + 5x_3 = 7 \quad (4.6)$$
$$4x_1 + 2x_2 - x_3 = -3 \quad (4.7)$$
$$x_1 - x_2 + x_3 = 1 \quad (4.8)$$

Vector Form:

$$(2, -3, 5) \cdot (x_1, x_2, x_3) = 7 \quad (4.9)$$
$$(4, 2, -1) \cdot (x_1, x_2, x_3) = -3 \quad (4.10)$$
$$(1, -1, 1) \cdot (x_1, x_2, x_3) = 1 \quad (4.11)$$

Matrix Form:

$$\begin{bmatrix} 2 & -3 & 5 \\ 4 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \quad (4.12)$$

Augmented Matrix Form:

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 7 \\ 4 & 2 & -1 & -3 \\ 1 & -1 & 1 & 1 \end{array}\right] \quad (4.13)$$

The matrix form of a linear system $Ax = b$ consists of three components:

Components of the Matrix Form

1. The **coefficient matrix** $A = [a_{ij}]_{m\times n}$ whose entries are the coefficients of the variables
2. The **variable vector** $x = [x_1, x_2, \dots, x_n]^T$
3. The **constant vector** $b = [b_1, b_2, \dots, b_m]^T$

4.2 Classification of Systems

Beyond the basic categorization based on the number of equations and variables, linear systems can be classified according to more fundamental properties that directly relate to their solution characteristics. These classifications provide deeper insights into the structure and behavior of linear systems.

Consistent vs. Inconsistent Systems

One of the most fundamental classifications of linear systems relates to whether they admit solutions at all.

Definition 4.3. *A system of linear equations is said to be* ***consistent*** *if it has at least one solution. If a system has no solution, it is called* ***inconsistent****.*

Geometrically, an inconsistent system represents a situation where the constraints imposed by the equations are mutually incompatible. For example, in a two-variable system, this could correspond to parallel lines that never intersect.

Determining Consistency from Augmented Matrix Form

A system $Ax = b$ is consistent if and only if the rank of the augmented matrix $[A|b]$ equals the rank of the coefficient matrix A, that is:

$$\text{rank}([A|b]) = \text{rank}(A) \tag{4.14}$$

If $\text{rank}([A|b]) \neq \text{rank}(A)$, then the system is inconsistent.

When we reduce an augmented matrix to Row Echelon Form, an inconsistent system will reveal itself through a row of the form $[0\ 0\ \cdots\ 0 \mid k]$ where $k \neq 0$. This represents an impossible equation $0 = k$.

Consistent and Inconsistent Systems **Consistent System:**

$$x_1 + 2x_2 = 5 \tag{4.15}$$
$$3x_1 - x_2 = 2 \tag{4.16}$$

The augmented matrix reduces to:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \tag{4.17}$$

This system has a unique solution: $x_1 = 1$, $x_2 = 2$.
Inconsistent System:

$$x_1 + 2x_2 = 5 \tag{4.18}$$
$$2x_1 + 4x_2 = 6 \tag{4.19}$$

The augmented matrix reduces to:

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & -4 \end{array}\right] \tag{4.20}$$

The row $[0\ 0 \mid -4]$ represents the equation $0 = -4$, which is impossible. Therefore, the system is inconsistent.

Homogeneous vs. Non-homogeneous Systems

Another important classification distinguishes between systems where all constants are zero and those where at least some constants are non-zero.

Definition 4.4. *A linear system $Ax = b$ is said to be **homogeneous** if $b = 0$ (the zero vector). If $b \neq 0$, the system is called **non-homogeneous**.*

Homogeneous systems always have at least one solution, namely the trivial solution where $x = 0$ (all variables equal to zero). The key question for homogeneous systems is whether they admit non-trivial solutions.

Properties of Homogeneous Systems

For a homogeneous system $Ax = 0$:

- It always has the trivial solution $x = 0$.
- It has non-trivial solutions if and only if $\text{rank}(A) < n$, where n is the number of variables.
- The set of all solutions forms a subspace of $\mathbb{R}^n$ called the null space or kernel of A.
- If x_1 and x_2 are solutions, then any linear combination $c_1x_1 + c_2x_2$ is also a solution (the principle of superposition).

For non-homogeneous systems, the solution structure is different. If a non-homogeneous system is consistent, its general solution can be expressed as the sum of a particular solution plus the general solution of the corresponding homogeneous system.

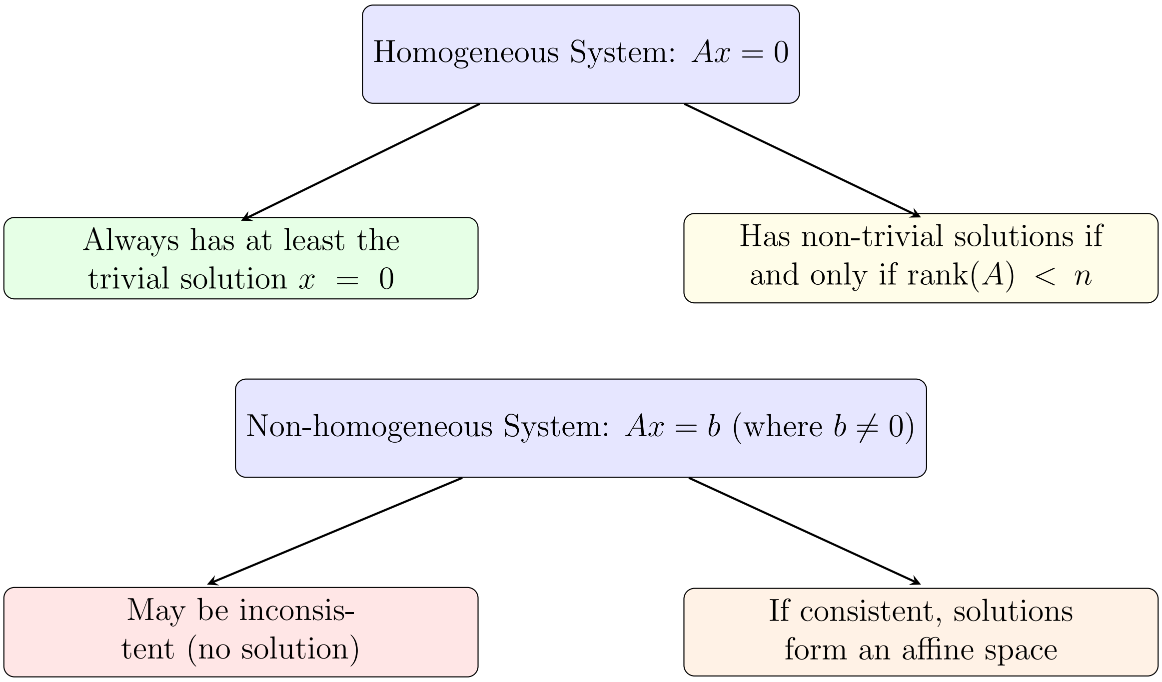

Figure 4.1: Comparison of homogeneous and non-homogeneous linear systems

Homogeneous and Non-homogeneous Systems **Homogeneous System:**

$$x_1 + 2x_2 - x_3 = 0 \tag{4.21}$$
$$2x_1 + 4x_2 - 2x_3 = 0 \tag{4.22}$$

The coefficient matrix has rank 1 (the second equation is a multiple of the first). Since the rank is less than the number of variables (3), this system has non-trivial solutions. The general solution can be expressed as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{4.23}$$

where s and t are free parameters.

Non-homogeneous System:

$$x_1 + 2x_2 - x_3 = 3 \tag{4.24}$$
$$2x_1 + 4x_2 - 2x_3 = 6 \tag{4.25}$$

This system is consistent (the second equation is a multiple of the first, and the constants are in the same proportion). A particular solution is $x_1 = 3$, $x_2 = 0$, $x_3 = 0$. The general solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{4.26}$$

Relationship Between Rank and Solution Existence

The rank of the coefficient matrix and the augmented matrix provides crucial information about the existence and uniqueness of solutions to a linear system.

Fundamental Theorem of Linear Systems

For a Non-homogeneous linear system $Ax = b$ with coefficient matrix A of size $m \times n$:

1. The system is consistent if and only if $\text{rank}(A) = \text{rank}([A|b])$.
2. If the system is consistent and $\text{rank}(A) = n$ (full column rank), then it has a unique solution.
3. If the system is consistent and $\text{rank}(A) < n$, then it has infinitely many solutions.

This theorem provides a complete classification of the solution behavior of linear systems based on the rank of the coefficient matrix and the consistency of the system.

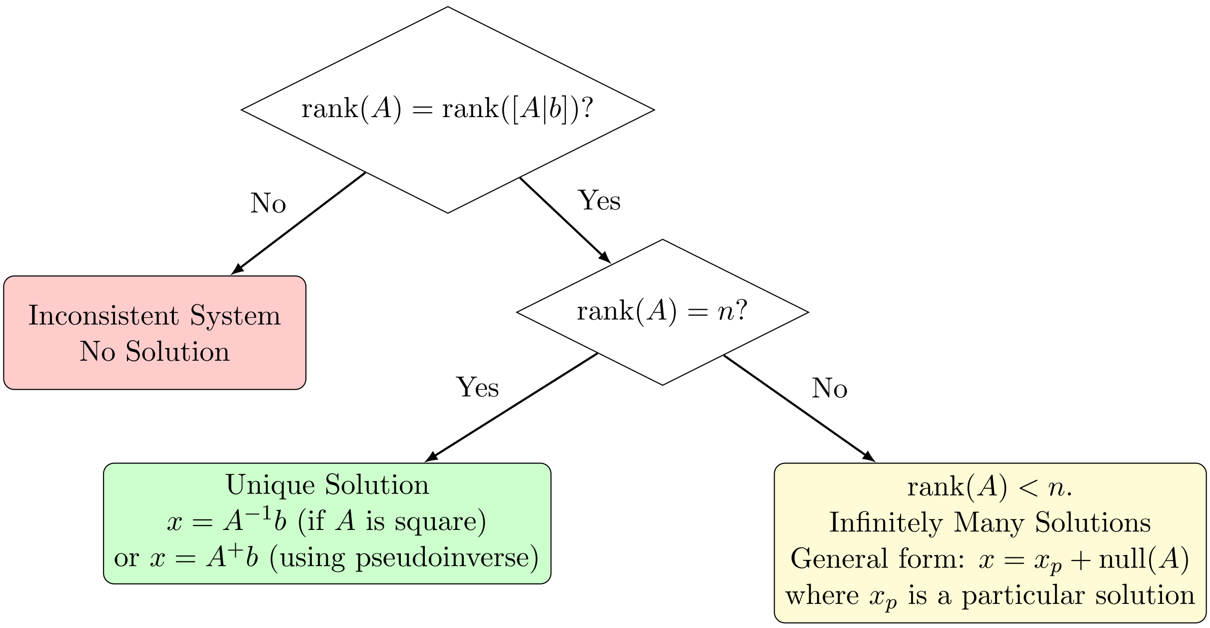

Figure 4.2: Decision tree for classifying Non-homogeneous linear systems based on rank

The concept of rank is particularly powerful because it provides a unified framework for analyzing all types of linear systems, regardless of their dimensions or specific coefficients.

4.3 Solved Examples

Que. Test for consistency and if consistent, solve the following system of equations:

Example 1:

$$3x + y + 2z = 3 \tag{4.27}$$
$$2x - 3y - z = -3 \tag{4.28}$$
$$x + 2y + z = 4 \tag{4.29}$$

Step-by-Step Solution to Example 1: We first write the augmented matrix for the system:

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{array}\right] \tag{4.30}$$

Now we apply row operations to reduce this matrix to Row Echelon Form:
Step 1: Swap rows 1 and 3 to have a 1 in the first position:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{array}\right] \tag{4.31}$$

Step 2: Eliminate entries below the first pivot:

$$R_2 \to R_2 - 2R_1 \tag{4.32}$$
$$R_3 \to R_3 - 3R_1 \tag{4.33}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{array}\right] \tag{4.34}$$

Step 3: Scale row 2 to get a leading 1:

$$R_2 \to -\frac{1}{7}R_2 \tag{4.35}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & \frac{3}{7} & \frac{11}{7} \\ 0 & -5 & -1 & -9 \end{array}\right] \tag{4.36}$$

Step 4: Eliminate entries below the second pivot:

$$R_3 \to R_3 + 5R_2 \tag{4.37}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & \frac{3}{7} & \frac{11}{7} \\ 0 & 0 & \frac{8}{7} & \frac{-8}{7} \end{array}\right] \tag{4.38}$$

Step 5: Scale row 3 to get a leading 1:

$$R_3 \to \frac{7}{8}R_3 \tag{4.39}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & \frac{3}{7} & \frac{11}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{4.40}$$

Rank Analysis: From the row echelon form, we can determine:

- rank(A) = 3 (we have 3 pivots in the coefficient matrix)
- rank($[A|b]$) = 3 (there are 3 pivots in the augmented matrix)

Since rank(A) = rank($[A|b]$), the system is consistent.
Also, rank(A) $= 3 = n$ (the number of variables), so the system has a unique solution.
Step 6: Now we can use back-substitution to find the solution:
From the third row: $z = -1$
From the second row: $y + \frac{3}{7}z = \frac{11}{7}$ Substituting $z = -1$: $y + \frac{3}{7}(-1) = \frac{11}{7}$ $y - \frac{3}{7} = \frac{11}{7}$ $y = \frac{11}{7} + \frac{3}{7} = \frac{14}{7} = 2$
From the first row: $x + 2y + z = 4$ Substituting $y = 2$ and $z = -1$: $x + 2(2) + (-1) = 4$ $x + 4 - 1 = 4$ $x = 1$
Therefore, the solution to the system is $x = 1$, $y = 2$, and $z = -1$.
Verification: We can verify our solution by substituting these values back into the original equations:
Equation 1: $3(1) + (2) + 2(-1) = 3 + 2 - 2 = 3$ ✓Equation 2: $2(1) - 3(2) - (-1) = 2 - 6 + 1 = -3$ ✓Equation 3: $(1) + 2(2) + (-1) = 1 + 4 - 1 = 4$ ✓
The solution $x = 1$, $y = 2$, $z = -1$ satisfies all three equations, confirming that our answer is correct.

Example 2:

$$2x_1 + x_2 + 2x_3 + x_4 = 6 \tag{4.41}$$
$$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36 \tag{4.42}$$
$$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1 \tag{4.43}$$
$$2x_1 + 2x_2 - x_3 + x_4 = 10 \tag{4.44}$$

Step-by-Step Solution to Example 2: We first write the augmented matrix for the system:

$$\left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right] \tag{4.45}$$

Now we apply row operations to reduce this matrix to Row Echelon Form:
Step 1: Scale the first row to have a leading 1:

$$R_1 \to \frac{1}{2}R_1 : \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 3 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right] \tag{4.46}$$

Step 2: Eliminate entries below the first pivot:

$$R_2 \to R_2 - 6R_1 \tag{4.47}$$
$$R_3 \to R_3 - 4R_1 \tag{4.48}$$
$$R_4 \to R_4 - 2R_1 \tag{4.49}$$

This gives:

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 3 \\ 0 & -9 & 0 & 9 & 18 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 1 & -3 & 0 & 4 \end{array}\right] \tag{4.50}$$

Step 3: Swap rows 2 and 3 to have a non-zero element in position (2,2):

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & -9 & 0 & 9 & 18 \\ 0 & 1 & -3 & 0 & 4 \end{array}\right] \tag{4.51}$$

Step 4: Eliminate entries below the second pivot:

$$R_3 \to R_3 + 9R_2 \tag{4.52}$$

$$R_4 \to R_4 - R_2 \tag{4.53}$$

This gives:

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 0 & -9 & -36 & -99 \\ 0 & 0 & -2 & 5 & 17 \end{array}\right] \tag{4.54}$$

Step 5: Scale row 3 to have a leading 1:

$$R_3 \to -\frac{1}{9}R_3 : \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 0 & 1 & 4 & 11 \\ 0 & 0 & -2 & 5 & 17 \end{array}\right] \tag{4.55}$$

Step 6: Eliminate entries below the third pivot:

$$R_4 \to R_4 + 2R_3 : \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 0 & 1 & 4 & 11 \\ 0 & 0 & 0 & 13 & 39 \end{array}\right] \tag{4.56}$$

Step 7: Scale row 4 to have a leading 1:

$$R_4 \to \frac{1}{13}R_4 : \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 0 & 1 & 4 & 11 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \tag{4.57}$$

Rank Analysis: From the row echelon form, we can determine:

- $\text{rank}(A) = 4$ (we have 4 pivots in the coefficient matrix)
- $\text{rank}([A|b]) = 4$ (there are 4 pivots in the augmented matrix)

Since $\text{rank}(A) = \text{rank}([A|b])$, the system is consistent.
Also, $\text{rank}(A) = 4 = n$ (the number of variables), so the system has a unique solution.
Step 8: Now we can use back-substitution to find the solution:
From the fourth row: $x_4 = 3$
From the third row: $x_3 + 4x_4 = 11$ Substituting $x_4 = 3$: $x_3 + 4(3) = 11$ $x_3 + 12 = 11$ $x_3 = -1$
From the second row: $x_2 - x_3 - 5x_4 = -13$ Substituting $x_3 = -1$ and $x_4 = 3$: $x_2 - (-1) - 5(3) = -13$ $x_2 + 1 - 15 = -13$ $x_2 - 14 = -13$ $x_2 = 1$
From the first row: $x_1 + \frac{1}{2}x_2 + x_3 + \frac{1}{2}x_4 = 3$ Substituting $x_2 = 1$, $x_3 = -1$, and $x_4 = 3$: $x_1 + \frac{1}{2}(1) + (-1) + \frac{1}{2}(3) = 3$ $x_1 + \frac{1}{2} - 1 + \frac{3}{2} = 3$ $x_1 + 1 = 3$ $x_1 = 2$
Therefore, the solution to the system is $x_1 = 2$, $x_2 = 1$, $x_3 = -1$, and $x_4 = 3$.

Verification: We can verify our solution by substituting these values back into the original equations:
Equation 1: $2(2) + (1) + 2(-1) + (3) = 4 + 1 - 2 + 3 = 6$ ✓
Equation 2: $6(2) - 6(1) + 6(-1) + 12(3) = 12 - 6 - 6 + 36 = 36$ ✓
Equation 3: $4(2) + 3(1) + 3(-1) - 3(3) = 8 + 3 - 3 - 9 = -1$ ✓
Equation 4: $2(2) + 2(1) - (-1) + (3) = 4 + 2 + 1 + 3 = 10$ ✓
The solution $x_1 = 2$, $x_2 = 1$, $x_3 = -1$, $x_4 = 3$ satisfies all four equations, confirming that our answer is correct.

Example 3: System with Different Constants Format

$$2x_1 + x_2 - x_3 + 3x_4 = 8 \tag{4.58}$$
$$x_1 + x_2 + x_3 - x_4 + 2 = 0 \tag{4.59}$$
$$3x_1 + 2x_2 - x_3 = 6 \tag{4.60}$$
$$4x_2 + 3x_3 + 2x_4 + 8 = 0 \tag{4.61}$$

Step-by-Step Solution to Example 3: First, we need to rearrange the equations to standard form with the constants on the right side:

$$2x_1 + x_2 - x_3 + 3x_4 = 8 \tag{4.62}$$
$$x_1 + x_2 + x_3 - x_4 = -2 \tag{4.63}$$
$$3x_1 + 2x_2 - x_3 = 6 \tag{4.64}$$
$$4x_2 + 3x_3 + 2x_4 = -8 \tag{4.65}$$

Now we write the augmented matrix for the system:

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 3 & 8 \\ 1 & 1 & 1 & -1 & -2 \\ 3 & 2 & -1 & 0 & 6 \\ 0 & 4 & 3 & 2 & -8 \end{array}\right] \tag{4.66}$$

We'll apply row operations to reduce this matrix to Row Echelon Form:
Step 1: Swap rows 1 and 2 to have a simpler leading coefficient:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 2 & 1 & -1 & 3 & 8 \\ 3 & 2 & -1 & 0 & 6 \\ 0 & 4 & 3 & 2 & -8 \end{array}\right] \tag{4.67}$$

Step 2: Eliminate entries below the first pivot:

$$R_2 \to R_2 - 2R_1 \tag{4.68}$$
$$R_3 \to R_3 - 3R_1 \tag{4.69}$$

This gives:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & -1 & -3 & 5 & 12 \\ 0 & -1 & -4 & 3 & 12 \\ 0 & 4 & 3 & 2 & -8 \end{array}\right] \tag{4.70}$$

Step 3: Scale row 2 to have a leading 1:

$$R_2 \to -R_2 : \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 3 & -5 & -12 \\ 0 & -1 & -4 & 3 & 12 \\ 0 & 4 & 3 & 2 & -8 \end{array}\right] \tag{4.71}$$

Step 4: Eliminate entries below the second pivot:

$$R_3 \to R_3 + R_2 \tag{4.72}$$
$$R_4 \to R_4 - 4R_2 \tag{4.73}$$

This gives:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 3 & -5 & -12 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & -9 & 22 & 40 \end{array}\right] \tag{4.74}$$

Step 5: Scale row 3 to have a leading 1:

$$R_3 \to -R_3 : \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 3 & -5 & -12 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -9 & 22 & 40 \end{array}\right] \tag{4.75}$$

Step 6: Eliminate entries below the third pivot:

$$R_4 \to R_4 + 9R_3 : \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 3 & -5 & -12 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 40 & 40 \end{array}\right] \tag{4.76}$$

Step 7: Scale row 4 to have a leading 1:

$$R_4 \to \frac{1}{40}R_4 : \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 3 & -5 & -12 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \tag{4.77}$$

Rank Analysis: From the row echelon form, we can determine:

- $\text{rank}(A) = 4$ (we have 4 pivots in the coefficient matrix)
- $\text{rank}([A|b]) = 4$ (there are 4 pivots in the augmented matrix)

Since $\text{rank}(A) = \text{rank}([A|b])$, the system is consistent.
Also, $\text{rank}(A) = 4 = n$ (the number of variables), so the system has a unique solution.
Step 8: Now we use back-substitution to find the solution:
From the fourth row: $x_4 = 1$
From the third row: $x_3 + 2x_4 = 0$ Substituting $x_4 = 1$: $x_3 + 2(1) = 0$ $x_3 + 2 = 0$ $x_3 = -2$
From the second row: $x_2 + 3x_3 - 5x_4 = -12$ Substituting $x_3 = -2$ and $x_4 = 1$: $x_2 + 3(-2) - 5(1) = -12$ $x_2 - 6 - 5 = -12$ $x_2 - 11 = -12$ $x_2 = -1$
From the first row: $x_1 + x_2 + x_3 - x_4 = -2$ Substituting $x_2 = -1$, $x_3 = -2$, and $x_4 = 1$: $x_1 + (-1) + (-2) - (1) = -2$ $x_1 - 4 = -2$ $x_1 = 2$
Therefore, the solution to the system is $x_1 = 2$, $x_2 = -1$, $x_3 = -2$, and $x_4 = 1$.

Verification: We can verify our solution by substituting these values back into the original equations:
Equation 1: $2(2) + (-1) - (-2) + 3(1) = 4 - 1 + 2 + 3 = 8$ ✓
Equation 2: $2 + (-1) + (-2) - (1) + 2 = 2 - 1 - 2 - 1 + 2 = 0$ ✓
Equation 3: $3(2) + 2(-1) - (-2) = 6 - 2 + 2 = 6$ ✓
Equation 4: $4(-1) + 3(-2) + 2(1) + 8 = -4 - 6 + 2 + 8 = 0$ ✓
The solution $x_1 = 2$, $x_2 = -1$, $x_3 = -2$, $x_4 = 1$ satisfies all four equations, confirming that our answer is correct.

Example 4: System with More Variables Than Equations

$$x_1 + x_2 + 2x_3 + x_4 = 5 \tag{4.78}$$
$$2x_1 + 3x_2 - x_3 - 2x_4 = 2 \tag{4.79}$$
$$4x_1 + 5x_2 + 3x_3 = 7 \tag{4.80}$$

Step-by-Step Solution to Example 4: We first write the augmented matrix for the system:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{array}\right] \tag{4.81}$$

Now we apply row operations to reduce this matrix to Row Echelon Form:
Step 1: The first pivot is already 1, so we eliminate entries below it:

$$R_2 \to R_2 - 2R_1 \tag{4.82}$$
$$R_3 \to R_3 - 4R_1 \tag{4.83}$$

This gives:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 1 & -5 & -4 & -13 \end{array}\right] \tag{4.84}$$

Step 2: The second pivot is already 1, so we eliminate entries below it:

$$R_3 \to R_3 - R_2 \tag{4.85}$$

This gives:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right] \tag{4.86}$$

Rank Analysis: From the row echelon form, we see that the last row is of the form $[0\ 0\ 0\ 0 \mid -5]$, which represents the equation $0 = -5$. This is a contradiction.
Therefore:

- $\text{rank}(A) = 2$ (we have 2 pivots in the coefficient matrix)
- $\text{rank}([A|b]) = 3$ (the last row adds another pivot in the augmented matrix)

Since $\text{rank}(A) < \text{rank}([A|b])$, the system is inconsistent and has no solution.
Explanation: The last row of our reduced augmented matrix gives us the equation $0 = -5$, which is clearly impossible. This indicates that the original system of equations contains contradictory constraints, and therefore no values of x_1, x_2, x_3, and x_4 can simultaneously satisfy all three equations.

This is an example of an overdetermined system (more independent equations than unknowns) that has no solution. If we were to visualize this geometrically, the three equations would represent planes in 4D space that have no common intersection point.
Note that although we have 4 variables and 3 equations (which might suggest the system is underdetermined), the rank analysis reveals that the 3 equations are actually inconsistent with each other, making the system impossible to solve.

Example 5: System with Infinitely Many Solutions

$$2x_1 - 3x_2 + 5x_3 = 1 \tag{4.87}$$
$$3x_1 + x_2 - 3x_3 = 2 \tag{4.88}$$
$$x_1 + 4x_2 - 6x_3 = 1 \tag{4.89}$$

Step-by-Step Solution to Example 5: We first write the augmented matrix for the system:

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 3 & 1 & -3 & 2 \\ 1 & 4 & -6 & 1 \end{array}\right] \tag{4.90}$$

Now we apply row operations to reduce this matrix to Row Echelon Form:
Step 1: Swap rows 1 and 3 to have a 1 in the first position:

$$\left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 3 & 1 & -3 & 2 \\ 2 & -3 & 5 & 1 \end{array}\right] \tag{4.91}$$

Step 2: Eliminate entries below the first pivot:

$$R_2 \to R_2 - 3R_1 \tag{4.92}$$
$$R_3 \to R_3 - 2R_1 \tag{4.93}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & -11 & 15 & -1 \\ 0 & -11 & 17 & -1 \end{array}\right] \tag{4.94}$$

Step 3: Scale row 2 to get a leading 1:

$$R_2 \to -\frac{1}{11}R_2 \tag{4.95}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & 1 & -\frac{15}{11} & \frac{1}{11} \\ 0 & -11 & 17 & -1 \end{array}\right] \tag{4.96}$$

Step 4: Eliminate entries below the second pivot:

$$R_3 \to R_3 + 11R_2 \tag{4.97}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & 1 & -\frac{15}{11} & \frac{1}{11} \\ 0 & 0 & 2 & 0 \end{array}\right] \tag{4.98}$$

Step 5: Scale row 3 to get a leading 1:

$$R_3 \to \frac{1}{2}R_3 \tag{4.99}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & 1 & -\frac{15}{11} & \frac{1}{11} \\ 0 & 0 & 1 & 0 \end{array}\right] \tag{4.100}$$

Rank Analysis: From the row echelon form, we can determine:

- $\text{rank}(A) = 3$ (we have 3 pivots in the coefficient matrix)
- $\text{rank}([A|b]) = 3$ (there are 3 pivots in the augmented matrix)

Since $\text{rank}(A) = \text{rank}([A|b])$, the system is consistent.
Also, $\text{rank}(A) = 3 = n$ (the number of variables), so the system has a unique solution.
Step 6: Now we can use back-substitution to find the solution:
From the third row: $x_3 = 0$
From the second row: $x_2 - \frac{15}{11}x_3 = \frac{1}{11}$ Substituting $x_3 = 0$: $x_2 = \frac{1}{11}$
From the first row: $x_1 + 4x_2 - 6x_3 = 1$ Substituting $x_2 = \frac{1}{11}$ and $x_3 = 0$: $x_1 + 4(\frac{1}{11}) = 1$ $x_1 + \frac{4}{11} = 1$ $x_1 = 1 - \frac{4}{11} = \frac{11-4}{11} = \frac{7}{11}$
Therefore, the solution to the system is $x_1 = \frac{7}{11}$, $x_2 = \frac{1}{11}$, and $x_3 = 0$.
Verification: We can verify our solution by substituting these values back into the original equations:
Equation 1: $2(\frac{7}{11}) - 3(\frac{1}{11}) + 5(0) = \frac{14}{11} - \frac{3}{11} = \frac{11}{11} = 1$ ✓
Equation 2: $3(\frac{7}{11}) + (\frac{1}{11}) - 3(0) = \frac{21}{11} + \frac{1}{11} = \frac{22}{11} = 2$ ✓
Equation 3: $(\frac{7}{11}) + 4(\frac{1}{11}) - 6(0) = \frac{7}{11} + \frac{4}{11} = \frac{11}{11} = 1$ ✓
The solution $x_1 = \frac{7}{11}$, $x_2 = \frac{1}{11}$, $x_3 = 0$ satisfies all three equations, confirming that our answer is correct.

Example 6: System with Infinitely Many Solutions

$$5x + 3y + 7z = 4 \tag{4.101}$$
$$3x + 26y + 2z = 9 \tag{4.102}$$
$$7x + 2y + 10z = 5 \tag{4.103}$$

Step-by-Step Solution to Example 6:
First, we will divide row 1 by 5 to get a leading 1:

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right] \tag{4.104}$$

Next, eliminate the first column in rows 2 and 3:

$$R_2 \to R_2 - 3R_1 \tag{4.105}$$
$$R_3 \to R_3 - 7R_1 \tag{4.106}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 0 & 26 - \frac{9}{5} & 2 - \frac{21}{5} & 9 - \frac{12}{5} \\ 0 & 2 - \frac{21}{5} & 10 - \frac{49}{5} & 5 - \frac{28}{5} \end{array}\right] \tag{4.107}$$

Simplifying:

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 0 & \frac{121}{5} & -\frac{11}{5} & \frac{33}{5} \\ 0 & -\frac{11}{5} & \frac{1}{5} & -\frac{3}{5} \end{array}\right] \tag{4.108}$$

I notice that row 3 $= -\frac{1}{11} \times$ row 2. So we have:

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 0 & \frac{121}{5} & -\frac{11}{5} & \frac{33}{5} \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4.109}$$

Now I'll simplify row 2 by dividing by $\frac{121}{5}$:

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 0 & 1 & -\frac{11}{121} & \frac{33}{121} \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4.110}$$

Now eliminate the second column in row 1:

$$R_1 \to R_1 - \frac{3}{5}R_2 \tag{4.111}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{5} + \frac{3}{5} \cdot \frac{11}{121} & \frac{4}{5} - \frac{3}{5} \cdot \frac{33}{121} \\ 0 & 1 & -\frac{11}{121} & \frac{33}{121} \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4.112}$$

Simplifying the first row:

$$\frac{7}{5} + \frac{3}{5} \cdot \frac{11}{121} = \frac{7}{5} + \frac{33}{5 \cdot 121} = \frac{847}{5 \cdot 121} + \frac{33}{5 \cdot 121} = \frac{880}{5 \cdot 121} = \frac{176}{121} \tag{4.113}$$

$$\frac{4}{5} - \frac{3}{5} \cdot \frac{33}{121} = \frac{4}{5} - \frac{99}{5 \cdot 121} = \frac{484}{5 \cdot 121} - \frac{99}{5 \cdot 121} = \frac{385}{5 \cdot 121} = \frac{77}{121} \tag{4.114}$$

So our final row echelon form is:

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{176}{121} & \frac{77}{121} \\ 0 & 1 & -\frac{11}{121} & \frac{33}{121} \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4.115}$$

Step 2: Now, following your approach, since the rank is 2 and we have 3 unknowns, we set $z = t$.
From row 2, we have:

$$y - \frac{11}{121}t = \frac{33}{121} \tag{4.116}$$

$$y = \frac{33}{121} + \frac{11}{121}t = \frac{33 + 11t}{121} \tag{4.117}$$

From row 1, we have:

$$x + \frac{176}{121}t = \frac{77}{121} \tag{4.118}$$

$$x = \frac{77}{121} - \frac{176}{121}t = \frac{77 - 176t}{121} \tag{4.119}$$

Let's simplify these fractions:

$$y = \frac{33 + 11t}{121} = \frac{3 + t}{11} \tag{4.120}$$
$$x = \frac{77 - 176t}{121} = \frac{7 - 16t}{11} \tag{4.121}$$

Therefore, the general solution to the system is:

$$x = \frac{7 - 16t}{11} \tag{4.122}$$
$$y = \frac{3 + t}{11} \tag{4.123}$$
$$z = t \tag{4.124}$$

where t is any real number.
Verification: Let's verify our solution by substituting it back into the original equations:
Equation 1:

$$5x + 3y + 7z = 5 \cdot \frac{7 - 16t}{11} + 3 \cdot \frac{3 + t}{11} + 7t \tag{4.125}$$
$$= \frac{5(7 - 16t) + 3(3 + t) + 77t}{11} \tag{4.126}$$
$$= \frac{35 - 80t + 9 + 3t + 77t}{11} \tag{4.127}$$
$$= \frac{44 + 0t}{11} = 4\checkmark \tag{4.128}$$

Equation 2:

$$3x + 26y + 2z = 3 \cdot \frac{7 - 16t}{11} + 26 \cdot \frac{3 + t}{11} + 2t \tag{4.129}$$
$$= \frac{3(7 - 16t) + 26(3 + t) + 22t}{11} \tag{4.130}$$
$$= \frac{21 - 48t + 78 + 26t + 22t}{11} \tag{4.131}$$
$$= \frac{99 + 0t}{11} = 9\checkmark \tag{4.132}$$

Equation 3:

$$7x + 2y + 10z = 7 \cdot \frac{7 - 16t}{11} + 2 \cdot \frac{3 + t}{11} + 10t \tag{4.133}$$
$$= \frac{7(7 - 16t) + 2(3 + t) + 110t}{11} \tag{4.134}$$
$$= \frac{49 - 112t + 6 + 2t + 110t}{11} \tag{4.135}$$
$$= \frac{55 + 0t}{11} = 5\checkmark \tag{4.136}$$

The solution $x = \frac{7-16t}{11}$, $y = \frac{3+t}{11}$, and $z = t$ satisfies all three original equations, verifying that our answer is correct.

Example 7: System with Infinitely Many Solutions

$$2x - y - z = 2 \tag{4.137}$$
$$x + 2y + z = 2 \tag{4.138}$$
$$4x - 7y - 5z = 2 \tag{4.139}$$

Step-by-Step Solution to Example 7: We first write the augmented matrix for the system:

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right] \tag{4.140}$$

Now we apply row operations to reduce this matrix to Row Echelon Form:
Step 1: Start with row operations to eliminate entries in the first column. Swap rows 1 and 2 to get a leading 1:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right] \tag{4.141}$$

Step 2: Eliminate entries below the first pivot:

$$R_2 \to R_2 - 2R_1 \tag{4.142}$$
$$R_3 \to R_3 - 4R_1 \tag{4.143}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{array}\right] \tag{4.144}$$

Step 3: Scale row 2 to get a leading 1:

$$R_2 \to -\frac{1}{5}R_2 \tag{4.145}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \\ 0 & -15 & -9 & -6 \end{array}\right] \tag{4.146}$$

Step 4: Eliminate entries below the second pivot:

$$R_3 \to R_3 + 15R_2 \tag{4.147}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & -9 + 15 \cdot \frac{3}{5} & -6 + 15 \cdot \frac{2}{5} \end{array}\right] \tag{4.148}$$

Computing the third row:

$$-9 + 15 \cdot \frac{3}{5} = -9 + 9 = 0 \tag{4.149}$$
$$-6 + 15 \cdot \frac{2}{5} = -6 + 6 = 0 \tag{4.150}$$

So we have:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4.151}$$

Rank Analysis: From the row echelon form, we can determine:

- rank$(A) = 2$ (we have 2 pivots in the coefficient matrix)
- rank$([A|b]) = 2$ (there are 2 pivots in the augmented matrix)

Since rank$(A) =$ rank$([A|b]) = 2 < n = 3$ (number of variables), the system is consistent and has infinitely many solutions.

Step 5: since the rank is 2 and we have 3 unknowns, we need to parameterize one variable. Let's set $z = t$ (where t is a free parameter).

From the second row, we have:

$$y + \frac{3}{5}t = \frac{2}{5} \tag{4.152}$$
$$\Rightarrow y = \frac{2}{5} - \frac{3}{5}t \tag{4.153}$$
$$= \frac{2-3t}{5} \tag{4.154}$$

From the first row, we have:

$$x + 2y + t = 2 \tag{4.155}$$
$$\Rightarrow x = 2 - 2y - t \tag{4.156}$$
$$= 2 - 2 \cdot \frac{2-3t}{5} - t \tag{4.157}$$
$$= 2 - \frac{4-6t}{5} - t \tag{4.158}$$
$$= 2 - \frac{4-6t}{5} - \frac{5t}{5} \tag{4.159}$$
$$= 2 - \frac{4-6t+5t}{5} \tag{4.160}$$
$$= 2 - \frac{4-t}{5} \tag{4.161}$$
$$= 2 - \frac{4}{5} + \frac{t}{5} \tag{4.162}$$
$$= \frac{10-4+t}{5} \tag{4.163}$$
$$= \frac{6+t}{5} \tag{4.164}$$

Therefore, the general solution to the system is:

$$x = \frac{6+t}{5} \tag{4.165}$$
$$y = \frac{2-3t}{5} \tag{4.166}$$
$$z = t \tag{4.167}$$

where t is any real number.

Verification: Let's verify our solution by substituting it back into the original equations:

Equation 1:

$$\begin{aligned} 2x - y - z &= 2 \cdot \frac{6+t}{5} - \frac{2-3t}{5} - t && (4.168)\\ &= \frac{12+2t}{5} - \frac{2-3t}{5} - t && (4.169)\\ &= \frac{12+2t-2+3t-5t}{5} && (4.170)\\ &= \frac{10+0t}{5} = 2\checkmark && (4.171) \end{aligned}$$

Equation 2:

$$\begin{aligned} x + 2y + z &= \frac{6+t}{5} + 2 \cdot \frac{2-3t}{5} + t && (4.172)\\ &= \frac{6+t}{5} + \frac{4-6t}{5} + t && (4.173)\\ &= \frac{6+t+4-6t+5t}{5} && (4.174)\\ &= \frac{10+0t}{5} = 2\checkmark && (4.175) \end{aligned}$$

Equation 3:

$$\begin{aligned} 4x - 7y - 5z &= 4 \cdot \frac{6+t}{5} - 7 \cdot \frac{2-3t}{5} - 5t && (4.176)\\ &= \frac{24+4t}{5} - \frac{14-21t}{5} - 5t && (4.177)\\ &= \frac{24+4t-14+21t-25t}{5} && (4.178)\\ &= \frac{10+0t}{5} = 2\checkmark && (4.179) \end{aligned}$$

The solution $x = \frac{6+t}{5}$, $y = \frac{2-3t}{5}$, and $z = t$ satisfies all three equations, confirming that our answer is correct.

Example 8: Determine the value of (λ) for consistency. Also find corrosponding solution

$$\begin{aligned} x + 2y + z &= 3 && (4.180)\\ x + y + z &= \lambda && (4.181)\\ 3x + y + 3z &= \lambda^2 && (4.182) \end{aligned}$$

Step-by-Step Solution to Example 8: We first write the augmented matrix for the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 1 & 1 & 1 & \lambda\\ 3 & 1 & 3 & \lambda^2 \end{array}\right] \tag{4.183}$$

Now we apply row operations to reduce this matrix to Row Echelon Form:

Step 1: Eliminate entries below the first pivot:

$$R_2 \to R_2 - R_1 \tag{4.184}$$
$$R_3 \to R_3 - 3R_1 \tag{4.185}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & -5 & 0 & \lambda^2 - 9 \end{array}\right] \tag{4.186}$$

Step 2: Scale row 2 to get a leading 1:

$$R_2 \to -R_2 \tag{4.187}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 3 - \lambda \\ 0 & -5 & 0 & \lambda^2 - 9 \end{array}\right] \tag{4.188}$$

Step 3: Eliminate entries below the second pivot:

$$R_3 \to R_3 + 5R_2 \tag{4.189}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 3 - \lambda \\ 0 & 0 & 0 & \lambda^2 - 9 + 5(3 - \lambda) \end{array}\right] \tag{4.190}$$

Computing the third row:

$$\lambda^2 - 9 + 5(3 - \lambda) = \lambda^2 - 9 + 15 - 5\lambda \tag{4.191}$$
$$= \lambda^2 - 5\lambda + 6 \tag{4.192}$$

So we have:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 3 - \lambda \\ 0 & 0 & 0 & \lambda^2 - 5\lambda + 6 \end{array}\right] \tag{4.193}$$

Consistency Analysis: For the system to be consistent, we need the last row to be all zeros, which means:

$$\lambda^2 - 5\lambda + 6 = 0 \tag{4.194}$$
$$(\lambda - 2)(\lambda - 3) = 0 \tag{4.195}$$

So the system is consistent when $\lambda = 2$ or $\lambda = 3$.
Case 1: When $\lambda = 2$
Substituting back into our row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 3 - 2 = 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4.196}$$

From the second row, we have:

$$y = 1 \tag{4.197}$$

From the first row, we have:

$$x + 2y + z = 3 \tag{4.198}$$
$$x + 2(1) + z = 3 \tag{4.199}$$
$$x + z = 1 \tag{4.200}$$

Since we have 2 equations and 3 unknowns, we need to parameterize one variable.
Let's set $z = t$ (where t is a free parameter).
Then:

$$x + t = 1 \tag{4.201}$$
$$\Rightarrow x = 1 - t \tag{4.202}$$

Therefore, when $\lambda = 2$, the general solution is:

$$x = 1 - t \tag{4.203}$$
$$y = 1 \tag{4.204}$$
$$z = t \tag{4.205}$$

where t is any real number.
Case 2: When $\lambda = 3$
Substituting back into our row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 3 - 3 = 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4.206}$$

From the second row, we have:

$$y = 0 \tag{4.207}$$

From the first row, we have:

$$x + 2y + z = 3 \tag{4.208}$$
$$x + 2(0) + z = 3 \tag{4.209}$$
$$x + z = 3 \tag{4.210}$$

Let's set $z = t$ (where t is a free parameter).
Then:

$$x + t = 3 \tag{4.211}$$
$$\Rightarrow x = 3 - t \tag{4.212}$$

Therefore, when $\lambda = 3$, the general solution is:

$$x = 3 - t \tag{4.213}$$
$$y = 0 \tag{4.214}$$
$$z = t \tag{4.215}$$

where t is any real number.

Verification: Let's verify our solutions by substituting them back into the original equations:
For $\lambda = 2$:
Equation 1:

$$x + 2y + z = (1 - t) + 2(1) + t \tag{4.216}$$
$$= 1 - t + 2 + t \tag{4.217}$$
$$= 3\checkmark \tag{4.218}$$

Equation 2:

$$x + y + z = (1 - t) + 1 + t \tag{4.219}$$
$$= 1 - t + 1 + t \tag{4.220}$$
$$= 2 = \lambda\checkmark \tag{4.221}$$

Equation 3:

$$3x + y + 3z = 3(1 - t) + 1 + 3t \tag{4.222}$$
$$= 3 - 3t + 1 + 3t \tag{4.223}$$
$$= 4 = \lambda^2 = 2^2 = 4\checkmark \tag{4.224}$$

For $\lambda = 3$:
Equation 1:

$$x + 2y + z = (3 - t) + 2(0) + t \tag{4.225}$$
$$= 3 - t + 0 + t \tag{4.226}$$
$$= 3\checkmark \tag{4.227}$$

Equation 2:

$$x + y + z = (3 - t) + 0 + t \tag{4.228}$$
$$= 3 - t + t \tag{4.229}$$
$$= 3 = \lambda\checkmark \tag{4.230}$$

Equation 3:

$$3x + y + 3z = 3(3 - t) + 0 + 3t \tag{4.231}$$
$$= 9 - 3t + 3t \tag{4.232}$$
$$= 9 = \lambda^2 = 3^2 = 9\checkmark \tag{4.233}$$

Therefore, the system is consistent exactly when $\lambda = 2$ or $\lambda = 3$, with the following solutions:
For $\lambda = 2$:

$$x = 1 - t \tag{4.234}$$
$$y = 1 \tag{4.235}$$
$$z = t \tag{4.236}$$

For $\lambda = 3$:

$$x = 3 - t \tag{4.237}$$
$$y = 0 \tag{4.238}$$
$$z = t \tag{4.239}$$

where t is any real number in both cases.

Example 9: Investigating Values of (λ) and (μ)

$$2x + 3y + 5z = 9 \tag{4.240}$$
$$7x + 3y - 2z = 8 \tag{4.241}$$
$$2x + 3y + \lambda z = \mu \tag{4.242}$$

have (i) no solution (ii) an infinite Solution (iii) a unique solution.

Step-by-Step Solution to Example 9: We first write the augmented matrix for the system:

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right] \tag{4.243}$$

Now we apply row operations to reduce this matrix to Row Echelon Form:
Step 1: We'll use the first row to eliminate entries in the first column below it:

$$R_2 \to R_2 - \frac{7}{2}R_1 \tag{4.244}$$
$$R_3 \to R_3 - R_1 \tag{4.245}$$

This gives:

$$R_2 : \left(7 - \frac{7}{2} \cdot 2\right) x + \left(3 - \frac{7}{2} \cdot 3\right) y + \left(-2 - \frac{7}{2} \cdot 5\right) z = 8 - \frac{7}{2} \cdot 9 \tag{4.246}$$
$$= 7 - 7x + 3 - \frac{21}{2}y - 2 - \frac{35}{2}z = 8 - \frac{63}{2} \tag{4.247}$$
$$= 0x - \frac{15}{2}y - \frac{39}{2}z = -\frac{47}{2} \tag{4.248}$$

$$R_3 : (2-2)x + (3-3)y + (\lambda - 5)z = \mu - 9 \tag{4.249}$$
$$= 0x + 0y + (\lambda - 5)z = \mu - 9 \tag{4.250}$$

So the matrix becomes:

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -\frac{15}{2} & -\frac{39}{2} & -\frac{47}{2} \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right] \tag{4.251}$$

Step 2: Now we scale the second row to simplify:

$$R_2 \to -\frac{2}{15}R_2 \tag{4.252}$$

This gives:

$$R_2 : -\frac{2}{15} \cdot \left(0x - \frac{15}{2}y - \frac{39}{2}z = -\frac{47}{2}\right) \tag{4.253}$$
$$= 0x + y + \frac{26}{15}z = \frac{47}{15} \tag{4.254}$$

So the matrix becomes:

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & 1 & \frac{26}{15} & \frac{47}{15} \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right] \tag{4.255}$$

Step 3: Use the second row to eliminate entries in the second column:

$$R_1 \to R_1 - 3R_2 \tag{4.256}$$

This gives:

$$R_1 :(2)x + (3-3\cdot 1)y + (5 - 3\cdot\frac{26}{15})z = 9 - 3\cdot\frac{47}{15} \tag{4.257}$$

$$= 2x + 0y + \left(5 - \frac{26}{5}\right)z = 9 - \frac{47}{5} \tag{4.258}$$

$$= 2x + 0y + \frac{25-26}{5}z = \frac{45-47}{5} \tag{4.259}$$

$$= 2x - \frac{1}{5}z = -\frac{2}{5} \tag{4.260}$$

So the matrix becomes:

$$\left[\begin{array}{ccc|c} 2 & 0 & -\frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{26}{15} & \frac{47}{15} \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right] \tag{4.261}$$

Step 4: Scale the first row to get a leading 1:

$$R_1 \to \frac{1}{2}R_1 \tag{4.262}$$

This gives:

$$R_1 :\frac{1}{2}\cdot\left(2x - \frac{1}{5}z = -\frac{2}{5}\right) \tag{4.263}$$

$$= x - \frac{1}{10}z = -\frac{1}{5} \tag{4.264}$$

So the matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{10} & -\frac{1}{5} \\ 0 & 1 & \frac{26}{15} & \frac{47}{15} \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right] \tag{4.265}$$

Analysis of Cases: Now we need to analyze the different cases based on the values of λ and μ.
The last row of our row echelon form is crucial:

$$[0 \quad 0 \quad \lambda-5 \quad \mu-9] \tag{4.266}$$

Case (i): No Solution For the system to have no solution, we need a contradiction in the form of a row with all zeros in the coefficient matrix but a non-zero constant. This occurs when:

$$\lambda - 5 = 0 \quad \text{(i.e., } \lambda = 5) \tag{4.267}$$

$$\mu - 9 \neq 0 \quad \text{(i.e., } \mu \neq 9) \tag{4.268}$$

Therefore, the system has no solution when $\lambda = 5$ and $\mu \neq 9$.

Case (ii): Infinite Solutions For the system to have infinitely many solutions, we need the rank of the coefficient matrix to be less than the number of variables (3). This happens when:

$$\lambda - 5 = 0 \quad \text{(i.e., } \lambda = 5) \tag{4.269}$$

$$\mu - 9 = 0 \quad \text{(i.e., } \mu = 9) \tag{4.270}$$

In this case, the row echelon form becomes:

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{10} & -\frac{1}{5} \\ 0 & 1 & \frac{26}{15} & \frac{47}{15} \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4.271}$$

We have 2 pivot columns but 3 variables, so we need to parameterize one variable.
Let's set $z = t$ (where t is a free parameter).
From the second row:

$$y + \frac{26}{15}t = \frac{47}{15} \tag{4.272}$$

$$\Rightarrow y = \frac{47}{15} - \frac{26}{15}t \tag{4.273}$$

$$= \frac{47 - 26t}{15} \tag{4.274}$$

From the first row:

$$x - \frac{1}{10}t = -\frac{1}{5} \tag{4.275}$$

$$\Rightarrow x = -\frac{1}{5} + \frac{1}{10}t \tag{4.276}$$

$$= \frac{-2 + t}{10} \tag{4.277}$$

Therefore, when $\lambda = 5$ and $\mu = 9$, the general solution is:

$$x = \frac{-2 + t}{10} \tag{4.278}$$

$$y = \frac{47 - 26t}{15} \tag{4.279}$$

$$z = t \tag{4.280}$$

where t is any real number.
Case (iii): Unique Solution For the system to have a unique solution, we need the rank of the coefficient matrix to equal the number of variables (3). This happens when:

$$\lambda - 5 \neq 0 \quad \text{(i.e., } \lambda \neq 5) \tag{4.281}$$

In this case, μ can be any value since it doesn't affect the rank of the coefficient matrix. Let's solve for this case. First, we scale the third row to get a leading 1:

$$R_3 \to \frac{1}{\lambda - 5} R_3 \tag{4.282}$$

This gives:

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{10} & -\frac{1}{5} \\ 0 & 1 & \frac{26}{15} & \frac{47}{15} \\ 0 & 0 & 1 & \frac{\mu-9}{\lambda-5} \end{array}\right] \tag{4.283}$$

Now we can back-substitute to find the unique solution.
From the third row:

$$z = \frac{\mu - 9}{\lambda - 5} \tag{4.284}$$

From the second row:

$$y + \frac{26}{15}z = \frac{47}{15} \tag{4.285}$$
$$\Rightarrow y = \frac{47}{15} - \frac{26}{15} \cdot \frac{\mu - 9}{\lambda - 5} \tag{4.286}$$
$$= \frac{47}{15} - \frac{26(\mu - 9)}{15(\lambda - 5)} \tag{4.287}$$

From the first row:

$$x - \frac{1}{10}z = -\frac{1}{5} \tag{4.288}$$
$$\Rightarrow x = -\frac{1}{5} + \frac{1}{10} \cdot \frac{\mu - 9}{\lambda - 5} \tag{4.289}$$
$$= -\frac{1}{5} + \frac{\mu - 9}{10(\lambda - 5)} \tag{4.290}$$

Therefore, when $\lambda \neq 5$, the system has the unique solution:

$$x = -\frac{1}{5} + \frac{\mu - 9}{10(\lambda - 5)} \tag{4.291}$$
$$y = \frac{47}{15} - \frac{26(\mu - 9)}{15(\lambda - 5)} \tag{4.292}$$
$$z = \frac{\mu - 9}{\lambda - 5} \tag{4.293}$$

Summary:
- (i) **No Solution:** $\lambda = 5$, $\mu \neq 9$
- (ii) **Infinite Solutions:** $\lambda = 5$, $\mu = 9$
- (iii) **Unique Solution:** $\lambda \neq 5$, μ can be any value

Verification: Let's verify our analysis:
For case (ii), when $\lambda = 5$ and $\mu = 9$, let's substitute our parametric solution into the original equations:
Equation 1:

$$2x + 3y + 5z = 2 \cdot \frac{-2 + t}{10} + 3 \cdot \frac{47 - 26t}{15} + 5t \tag{4.294}$$
$$= \frac{-4 + 2t}{10} + \frac{141 - 78t}{15} + 5t \tag{4.295}$$
$$= \frac{-6 + 3t}{15} + \frac{141 - 78t}{15} + 5t \tag{4.296}$$
$$= \frac{-6 + 3t + 141 - 78t}{15} + 5t \tag{4.297}$$
$$= \frac{135 - 75t}{15} + 5t \tag{4.298}$$
$$= 9 - 5t + 5t \tag{4.299}$$
$$= 9\checkmark \tag{4.300}$$

Equation 2:

$$7x + 3y - 2z = 7 \cdot \frac{-2+t}{10} + 3 \cdot \frac{47-26t}{15} - 2t \tag{4.301}$$
$$= \frac{-14+7t}{10} + \frac{141-78t}{15} - 2t \tag{4.302}$$
$$= \frac{-21+10.5t}{15} + \frac{141-78t}{15} - 2t \tag{4.303}$$
$$= \frac{-21+10.5t+141-78t}{15} - 2t \tag{4.304}$$
$$= \frac{120-67.5t}{15} - 2t \tag{4.305}$$
$$= 8 - 4.5t - 2t \tag{4.306}$$
$$= 8 - 6.5t + 6.5t \tag{4.307}$$
$$= 8\checkmark \tag{4.308}$$

Equation 3 (with $\lambda = 5$ and $\mu = 9$):

$$2x + 3y + 5z = 2 \cdot \frac{-2+t}{10} + 3 \cdot \frac{47-26t}{15} + 5t \tag{4.309}$$
$$= 9 \text{ (as shown above)} \tag{4.310}$$
$$= \mu\checkmark \tag{4.311}$$

For case (iii), with $\lambda \neq 5$, substituting back into the original equations is straightforward but tedious. The key is that we have a system of full rank (3), which guarantees a unique solution.
Therefore, our analysis confirms the three cases:

- (i) **No Solution:** $\lambda = 5$, $\mu \neq 9$
- (ii) **Infinite Solutions:** $\lambda = 5$, $\mu = 9$
- (iii) **Unique Solution:** $\lambda \neq 5$, μ can be any value

Example 10: Show that the system of equations

$$3x + 4y + 5z = a \tag{4.312}$$
$$4x + 5y + 6z = b \tag{4.313}$$
$$5x + 6y + 7z = c \tag{4.314}$$

will be consistent only if a + c = 2 b

Step-by-Step Solution to Example 10: We first write the augmented matrix for the system:

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{array}\right] \tag{4.315}$$

Now we apply row operations to reduce this matrix to Row Echelon Form:

Step 1: We'll use the first row to eliminate entries in the first column below it:

$$R_2 \to R_2 - \frac{4}{3}R_1 \tag{4.316}$$

$$R_3 \to R_3 - \frac{5}{3}R_1 \tag{4.317}$$

For the second row:

$$R_2 = \left(4 - \frac{4}{3} \cdot 3\right)x + \left(5 - \frac{4}{3} \cdot 4\right)y + \left(6 - \frac{4}{3} \cdot 5\right)z = b - \frac{4}{3}a \tag{4.318}$$

$$= 0x + \left(5 - \frac{16}{3}\right)y + \left(6 - \frac{20}{3}\right)z = b - \frac{4}{3}a \tag{4.319}$$

$$= 0x + \frac{15 - 16}{3}y + \frac{18 - 20}{3}z = b - \frac{4}{3}a \tag{4.320}$$

$$= 0x - \frac{1}{3}y - \frac{2}{3}z = b - \frac{4}{3}a \tag{4.321}$$

For the third row:

$$R_3 = \left(5 - \frac{5}{3} \cdot 3\right)x + \left(6 - \frac{5}{3} \cdot 4\right)y + \left(7 - \frac{5}{3} \cdot 5\right)z = c - \frac{5}{3}a \tag{4.322}$$

$$= 0x + \left(6 - \frac{20}{3}\right)y + \left(7 - \frac{25}{3}\right)z = c - \frac{5}{3}a \tag{4.323}$$

$$= 0x + \frac{18 - 20}{3}y + \frac{21 - 25}{3}z = c - \frac{5}{3}a \tag{4.324}$$

$$= 0x - \frac{2}{3}y - \frac{4}{3}z = c - \frac{5}{3}a \tag{4.325}$$

So the matrix becomes:

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 0 & -\frac{1}{3} & -\frac{2}{3} & b - \frac{4}{3}a \\ 0 & -\frac{2}{3} & -\frac{4}{3} & c - \frac{5}{3}a \end{array}\right] \tag{4.326}$$

Step 2: Scale the second row to simplify:

$$R_2 \to -3R_2 \tag{4.327}$$

This gives:

$$R_2 = 0x + 1y + 2z = -3 \cdot \left(b - \frac{4}{3}a\right) \tag{4.328}$$

$$= 0x + 1y + 2z = -3b + 4a \tag{4.329}$$

So the matrix becomes:

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 0 & 1 & 2 & -3b + 4a \\ 0 & -\frac{2}{3} & -\frac{4}{3} & c - \frac{5}{3}a \end{array}\right] \tag{4.330}$$

Step 3: Eliminate entries in the second column below the second pivot:

$$R_3 \to R_3 + \frac{2}{3}R_2 \tag{4.331}$$

This gives:

$$R_3 = 0x + \left(-\frac{2}{3} + \frac{2}{3} \cdot 1\right) y + \left(-\frac{4}{3} + \frac{2}{3} \cdot 2\right) z = \left(c - \frac{5}{3}a\right) + \frac{2}{3} \cdot (-3b + 4a) \tag{4.332}$$

$$= 0x + 0y + \left(-\frac{4}{3} + \frac{4}{3}\right) z = c - \frac{5}{3}a - 2b + \frac{8}{3}a \tag{4.333}$$

$$= 0x + 0y + 0z = c - \frac{5}{3}a - 2b + \frac{8}{3}a \tag{4.334}$$

$$= 0x + 0y + 0z = c + \frac{3}{3}a - 2b \tag{4.335}$$

$$= 0x + 0y + 0z = c + a - 2b \tag{4.336}$$

So the matrix becomes:

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 0 & 1 & 2 & -3b + 4a \\ 0 & 0 & 0 & c + a - 2b \end{array}\right] \tag{4.337}$$

Consistency Analysis: For the system to be consistent, the last row must be equivalent to $0 = 0$. This requires:

$$c + a - 2b = 0 \tag{4.338}$$

$$\Rightarrow a + c = 2b \tag{4.339}$$

Therefore, the system is consistent if and only if $a + c = 2b$.
When this condition is satisfied, the third equation becomes redundant, and we effectively have a system of two equations with three unknowns, which has infinitely many solutions.
Solution When Consistent: When $a + c = 2b$, the system reduces to:

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 0 & 1 & 2 & -3b + 4a \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4.340}$$

Let's parameterize by setting $z = t$ (where t is a free parameter).
From the second row:

$$y + 2t = -3b + 4a \tag{4.341}$$

$$\Rightarrow y = -3b + 4a - 2t \tag{4.342}$$

From the first row:

$$3x + 4y + 5t = a \tag{4.343}$$

$$\Rightarrow 3x + 4(-3b + 4a - 2t) + 5t = a \tag{4.344}$$

$$\Rightarrow 3x - 12b + 16a - 8t + 5t = a \tag{4.345}$$

$$\Rightarrow 3x = a - 16a + 12b + 3t \tag{4.346}$$

$$\Rightarrow 3x = -15a + 12b + 3t \tag{4.347}$$

$$\Rightarrow x = -5a + 4b + t \tag{4.348}$$

Therefore, when $a + c = 2b$, the general solution is:

$$x = -5a + 4b + t \tag{4.349}$$

$$y = -3b + 4a - 2t \tag{4.350}$$

$$z = t \tag{4.351}$$

where t is any real number.
Verification: Let's verify that $a + c = 2b$ is indeed necessary for consistency.
If we substitute the general solution into the original equations:
Equation 1:

$$\begin{aligned}
3x + 4y + 5z &= 3(-5a + 4b + t) + 4(-3b + 4a - 2t) + 5t && (4.352)\\
&= -15a + 12b + 3t - 12b + 16a - 8t + 5t && (4.353)\\
&= -15a + 16a + 0b + 0t && (4.354)\\
&= a\checkmark && (4.355)
\end{aligned}$$

Equation 2:

$$\begin{aligned}
4x + 5y + 6z &= 4(-5a + 4b + t) + 5(-3b + 4a - 2t) + 6t && (4.356)\\
&= -20a + 16b + 4t - 15b + 20a - 10t + 6t && (4.357)\\
&= 0a + b + 0t && (4.358)\\
&= b\checkmark && (4.359)
\end{aligned}$$

Equation 3:

$$\begin{aligned}
5x + 6y + 7z &= 5(-5a + 4b + t) + 6(-3b + 4a - 2t) + 7t && (4.360)\\
&= -25a + 20b + 5t - 18b + 24a - 12t + 7t && (4.361)\\
&= -25a + 24a + 20b - 18b + 0t && (4.362)\\
&= -a + 2b && (4.363)\\
&= c \text{ (if and only if } a + c = 2b)\checkmark && (4.364)
\end{aligned}$$

This confirms that the system is consistent if and only if $a + c = 2b$.

Example 11: Examining Non-trivial Solutions

$$\begin{aligned}
4x_1 - x_2 + x_3 &= 0 && (4.365)\\
x_1 + 2x_2 - x_3 &= 0 && (4.366)\\
3x_1 + x_2 + 5x_3 &= 0 && (4.367)
\end{aligned}$$

Step-by-Step Solution to Example 11: First, note that this is a homogeneous system of linear equations, i.e., all the constant terms are zero. Such a system always has the trivial solution $x_1 = x_2 = x_3 = 0$.
Method 1: Determinant Approach
One way to determine if the system has non-trivial solutions is to check the determinant of the coefficient matrix. A homogeneous system has non-trivial solutions if and only if the determinant of the coefficient matrix is zero.
Let's compute the determinant of the coefficient matrix:

$$A = \begin{bmatrix} 4 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 5 \end{bmatrix} \tag{4.368}$$

To compute the determinant, we'll use the cofactor expansion along the first row:

$$|A| = 4 \cdot \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \tag{4.369}$$

$$= 4 \cdot (2 \cdot 5 - (-1) \cdot 1) + 1 \cdot (1 \cdot 5 - (-1) \cdot 3) + 1 \cdot (1 \cdot 1 - 2 \cdot 3) \tag{4.370}$$

$$= 4 \cdot (10 + 1) + 1 \cdot (5 + 3) + 1 \cdot (1 - 6) \tag{4.371}$$

$$= 4 \cdot 11 + 1 \cdot 8 + 1 \cdot (-5) \tag{4.372}$$

$$= 44 + 8 - 5 \tag{4.373}$$

$$= 47 \tag{4.374}$$

Since $|A| = 47 \neq 0$, the coefficient matrix is invertible, and the system has only the trivial solution $x_1 = x_2 = x_3 = 0$.

Method 2: Rank Approach

To determine if the system has non-trivial solutions, we find the rank of the coefficient matrix by reducing it to row echelon form. For a homogeneous system:

- If rank equals the number of unknowns, then only the trivial solution exists.
- If rank is less than the number of unknowns, then non-trivial solutions exist.

Let's find the rank by reducing the coefficient matrix to row echelon form:

$$A = \begin{bmatrix} 4 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 5 \end{bmatrix} \tag{4.375}$$

Step 1: Swap rows 1 and 2 to get a leading 1:

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \tag{4.376}$$

Step 2: Eliminate entries below the first pivot:

$$R_2 \to R_2 - 4R_1 \tag{4.377}$$

$$R_3 \to R_3 - 3R_1 \tag{4.378}$$

This gives:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -9 & 5 \\ 0 & -5 & 8 \end{bmatrix} \tag{4.379}$$

Step 3: Scale the second row to get a leading 1:

$$R_2 \to -\frac{1}{9} R_2 \tag{4.380}$$

This gives:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{5}{9} \\ 0 & -5 & 8 \end{bmatrix} \tag{4.381}$$

Step 4: Eliminate entries below the second pivot:

$$R_3 \to R_3 + 5R_2 \tag{4.382}$$

This gives:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{5}{9} \\ 0 & 0 & \frac{47}{9} \end{bmatrix} \tag{4.383}$$

Step 5: Scale the third row to get a leading 1:

$$R_3 \to \frac{9}{47} R_3 \tag{4.384}$$

This gives:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{5}{9} \\ 0 & 0 & 1 \end{bmatrix} \tag{4.385}$$

From the row echelon form, we can see that the rank of the coefficient matrix is 3, which equals the number of unknowns.
Since $\text{rank}(A) = 3 =$ number of unknowns, the system has only the trivial solution $x_1 = x_2 = x_3 = 0$.
This confirms our earlier conclusion based on the determinant calculation.
Summary: The given system of equations has only the trivial solution $x_1 = x_2 = x_3 = 0$. This is because the determinant of the coefficient matrix is $|A| = 47 \neq 0$, which means the matrix is invertible and the system has a unique solution. Since all constant terms in the original equations are zero, this unique solution must be the trivial solution.

Example 12: Examining Non-trivial Solutions

$$x_1 + 2x_2 - 3x_4 = 0 \tag{4.386}$$
$$2x_1 - x_2 + x_3 + 7x_4 = 0 \tag{4.387}$$
$$4x_1 + 3x_2 + 2x_3 + 2x_4 = 0 \tag{4.388}$$

Step-by-Step Solution to Example 12: First, note that this is a homogeneous system of linear equations with 3 equations and 4 unknowns. To determine whether non-trivial solutions exist, we'll use the rank approach.
Rank Approach: For a homogeneous system:

- If rank equals the number of unknowns, then only the trivial solution exists.
- If rank is less than the number of unknowns, then non-trivial solutions exist.

We have 3 equations and 4 unknowns. If the rank of the coefficient matrix is less than 4, then non-trivial solutions exist.
Let's find the rank by reducing the coefficient matrix to row echelon form:

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \tag{4.389}$$

Step 1: Use the first row to eliminate entries in the first column below it:

$$R_2 \to R_2 - 2R_1 \tag{4.390}$$
$$R_3 \to R_3 - 4R_1 \tag{4.391}$$

This gives:

$$R_2 :(2-2\cdot 1)x_1 + (-1-2\cdot 2)x_2 + (1-2\cdot 0)x_3 + (7-2\cdot(-3))x_4 = 0-2\cdot 0 \quad (4.392)$$
$$= 0x_1 + (-1-4)x_2 + 1x_3 + (7+6)x_4 = 0 \quad (4.393)$$
$$= 0x_1 - 5x_2 + x_3 + 13x_4 = 0 \quad (4.394)$$

$$R_3 :(4-4\cdot 1)x_1 + (3-4\cdot 2)x_2 + (2-4\cdot 0)x_3 + (2-4\cdot(-3))x_4 = 0-4\cdot 0 \quad (4.395)$$
$$= 0x_1 + (3-8)x_2 + 2x_3 + (2+12)x_4 = 0 \quad (4.396)$$
$$= 0x_1 - 5x_2 + 2x_3 + 14x_4 = 0 \quad (4.397)$$

So the matrix becomes:

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \quad (4.398)$$

Step 2: Scale the second row to get a leading coefficient of 1:

$$R_2 \to -\frac{1}{5}R_2 \quad (4.399)$$

This gives:

$$R_2 :0x_1 + 1x_2 - \frac{1}{5}x_3 - \frac{13}{5}x_4 = 0 \quad (4.400)$$

So the matrix becomes:

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & -\frac{1}{5} & -\frac{13}{5} \\ 0 & -5 & 2 & 14 \end{bmatrix} \quad (4.401)$$

Step 3: Eliminate entries below the second pivot:

$$R_3 \to R_3 + 5R_2 \quad (4.402)$$

This gives:

$$R_3 :0x_1 + (-5+5\cdot 1)x_2 + (2+5\cdot(-\frac{1}{5}))x_3 + (14+5\cdot(-\frac{13}{5}))x_4 = 0+5\cdot 0 \quad (4.403)$$
$$= 0x_1 + 0x_2 + (2-1)x_3 + (14-13)x_4 = 0 \quad (4.404)$$
$$= 0x_1 + 0x_2 + 1x_3 + 1x_4 = 0 \quad (4.405)$$

So the matrix becomes:

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & -\frac{1}{5} & -\frac{13}{5} \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad (4.406)$$

Step 4: Now eliminate entries above the third pivot:

$$R_2 \to R_2 + \frac{1}{5}R_3 \quad (4.407)$$

This gives:

$$R_2 :0x_1 + 1x_2 + (-\frac{1}{5} + \frac{1}{5}\cdot 1)x_3 + (-\frac{13}{5} + \frac{1}{5}\cdot 1)x_4 = 0 + \frac{1}{5}\cdot 0 \quad (4.408)$$
$$= 0x_1 + 1x_2 + 0x_3 + (-\frac{13}{5} + \frac{1}{5})x_4 = 0 \quad (4.409)$$
$$= 0x_1 + 1x_2 + 0x_3 - \frac{12}{5}x_4 = 0 \quad (4.410)$$

So the matrix becomes:

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -\frac{12}{5} \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{4.411}$$

Step 5: Finally, eliminate entries above the third pivot in the first row:

$$R_1 \to R_1 - 0 \cdot R_3 = R_1 \tag{4.412}$$

This step doesn't change anything because the entry is already zero.
So the final row echelon form is:

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -\frac{12}{5} \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{4.413}$$

From the row echelon form, we can see that the rank of the coefficient matrix is 3, which is less than the number of unknowns (4).
Since $\text{rank}(A) = 3 < 4 =$ number of unknowns, the system has non-trivial solutions.
Finding the Non-trivial Solutions: To find the non-trivial solutions, we parameterize one of the variables. Let's set $x_4 = t$ (where t is a free parameter).
From the third row:

$$x_3 + x_4 = 0 \tag{4.414}$$

$$x_3 + t = 0 \tag{4.415}$$

$$\Rightarrow x_3 = -t \tag{4.416}$$

From the second row:

$$x_2 - \frac{12}{5}x_4 = 0 \tag{4.417}$$

$$x_2 - \frac{12}{5} \cdot t = 0 \tag{4.418}$$

$$\Rightarrow x_2 = \frac{12}{5}t \tag{4.419}$$

From the first row:

$$x_1 + 2x_2 - 3x_4 = 0 \tag{4.420}$$

$$x_1 + 2 \cdot \frac{12}{5}t - 3t = 0 \tag{4.421}$$

$$x_1 + \frac{24}{5}t - 3t = 0 \tag{4.422}$$

$$x_1 + \frac{24 - 15}{5}t = 0 \tag{4.423}$$

$$x_1 + \frac{9}{5}t = 0 \tag{4.424}$$

$$\Rightarrow x_1 = -\frac{9}{5}t \tag{4.425}$$

Therefore, the general solution to the system is:

$$x_1 = -\frac{9}{5}t \tag{4.426}$$

$$x_2 = \frac{12}{5}t \tag{4.427}$$

$$x_3 = -t \tag{4.428}$$

$$x_4 = t \tag{4.429}$$

where t is any real number except zero (for a non-trivial solution).
Verification: Let's verify our solution by substituting it back into the original equations:
Equation 1:

$$\begin{aligned} x_1 + 2x_2 - 3x_4 &= -\frac{9}{5}t + 2 \cdot \frac{12}{5}t - 3t && (4.430)\\ &= -\frac{9}{5}t + \frac{24}{5}t - 3t && (4.431)\\ &= -\frac{9}{5}t + \frac{24}{5}t - \frac{15}{5}t && (4.432)\\ &= \frac{-9+24-15}{5}t && (4.433)\\ &= \frac{0}{5}t = 0✓ && (4.434) \end{aligned}$$

Equation 2:

$$\begin{aligned} 2x_1 - x_2 + x_3 + 7x_4 &= 2 \cdot (-\frac{9}{5}t) - \frac{12}{5}t + (-t) + 7t && (4.435)\\ &= -\frac{18}{5}t - \frac{12}{5}t - t + 7t && (4.436)\\ &= -\frac{18+12}{5}t - t + 7t && (4.437)\\ &= -\frac{30}{5}t - t + 7t && (4.438)\\ &= -6t - t + 7t && (4.439)\\ &= 0t = 0✓ && (4.440) \end{aligned}$$

Equation 3:

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 + 2x_4 &= 4 \cdot (-\frac{9}{5}t) + 3 \cdot \frac{12}{5}t + 2 \cdot (-t) + 2t && (4.441)\\ &= -\frac{36}{5}t + \frac{36}{5}t - 2t + 2t && (4.442)\\ &= -\frac{36}{5}t + \frac{36}{5}t + 0t && (4.443)\\ &= 0t = 0✓ && (4.444) \end{aligned}$$

Our solution satisfies all three original equations, confirming that the general solution is:

$$\begin{aligned} x_1 &= -\frac{9}{5}t && (4.445)\\ x_2 &= \frac{12}{5}t && (4.446)\\ x_3 &= -t && (4.447)\\ x_4 &= t && (4.448) \end{aligned}$$

where t is any real number.
Summary: The given system of equations has infinitely many non-trivial solutions because the rank of the coefficient matrix (3) is less than the number of unknowns (4). The

general solution is:

$$x_1 = -\frac{9}{5}t \tag{4.449}$$
$$x_2 = \frac{12}{5}t \tag{4.450}$$
$$x_3 = -t \tag{4.451}$$
$$x_4 = t \tag{4.452}$$

where t is any non-zero real number.

Example 12: Examining Non-trivial Solutions

$$x + y + 3z = 0 \tag{4.453}$$
$$x - y + z = 0 \tag{4.454}$$
$$-x + 2y = 0 \tag{4.455}$$
$$x - y + z = 0 \tag{4.456}$$

Step-by-Step Solution to Example 12: First, note that we have 4 equations but only 3 unknowns. Additionally, we can see that the second and fourth equations are identical. So effectively, we have 3 distinct equations with 3 unknowns:

$$x + y + 3z = 0 \quad \text{(Equation 1)} \tag{4.457}$$
$$x - y + z = 0 \quad \text{(Equation 2)} \tag{4.458}$$
$$-x + 2y = 0 \quad \text{(Equation 3)} \tag{4.459}$$

Method 1: Determinant Approach
One way to determine if the system has non-trivial solutions is to check the determinant of the coefficient matrix. A homogeneous system has non-trivial solutions if and only if the determinant of the coefficient matrix is zero.
Let's compute the determinant of the coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \end{bmatrix} \tag{4.460}$$

To compute the determinant, we'll use the cofactor expansion along the first row:

$$|A| = 1 \cdot \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} \tag{4.461}$$
$$= 1 \cdot ((-1) \cdot 0 - 1 \cdot 2) - 1 \cdot (1 \cdot 0 - 1 \cdot (-1)) + 3 \cdot (1 \cdot 2 - (-1) \cdot (-1)) \tag{4.462}$$
$$= 1 \cdot (-2) - 1 \cdot 1 + 3 \cdot (2 - 1) \tag{4.463}$$
$$= -2 - 1 + 3 \cdot 1 \tag{4.464}$$
$$= -3 + 3 \tag{4.465}$$
$$= 0 \tag{4.466}$$

Since $|A| = 0$, the coefficient matrix is singular, and the system has non-trivial solutions.

Method 2: Rank Approach
To confirm this result and find the non-trivial solutions, we'll determine the rank of the coefficient matrix by reducing it to row echelon form. For a homogeneous system:

- If rank equals the number of unknowns, then only the trivial solution exists.
- If rank is less than the number of unknowns, then non-trivial solutions exist.

Let's find the rank by reducing the augmented matrix to row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{array}\right] \tag{4.467}$$

Step 1: Use the first row to eliminate entries in the first column below it:

$$R_2 \to R_2 - R_1 \tag{4.468}$$
$$R_3 \to R_3 + R_1 \tag{4.469}$$

This gives:

$$R_2 :(1-1)x + (-1-1)y + (1-3)z = 0 - 0 \tag{4.470}$$
$$= 0x - 2y - 2z = 0 \tag{4.471}$$

$$R_3 :(-1+1)x + (2+1)y + (0+3)z = 0 + 0 \tag{4.472}$$
$$= 0x + 3y + 3z = 0 \tag{4.473}$$

So the matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \tag{4.474}$$

Step 2: Scale the second row to simplify:

$$R_2 \to -\frac{1}{2}R_2 \tag{4.475}$$

This gives:

$$R_2 :0x + 1y + 1z = 0 \tag{4.476}$$

So the matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \tag{4.477}$$

Step 3: Eliminate entries below the second pivot:

$$R_3 \to R_3 - 3R_2 \tag{4.478}$$

This gives:

$$R_3 :0x + (3 - 3\cdot 1)y + (3 - 3\cdot 1)z = 0 - 3\cdot 0 \tag{4.479}$$
$$= 0x + 0y + 0z = 0 \tag{4.480}$$

So the matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4.481}$$

Step 4: Eliminate entries above the second pivot:

$$R_1 \to R_1 - R_2 \tag{4.482}$$

This gives:

$$R_1 : (1-0)x + (1-1)y + (3-1)z = 0 - 0 \tag{4.483}$$
$$= 1x + 0y + 2z = 0 \tag{4.484}$$

So the matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4.485}$$

From the row echelon form, we can see that the rank of the coefficient matrix is 2, which is less than the number of unknowns (3).
Since $\text{rank}(A) = 2 < 3 =$ number of unknowns, the system has non-trivial solutions, confirming our determinant calculation.
Finding the Non-trivial Solutions: To find the non-trivial solutions, we'll parameterize one of the variables. Let's set $z = t$ (where t is a free parameter).
From the second row of our row echelon form:

$$y + z = 0 \tag{4.486}$$
$$y + t = 0 \tag{4.487}$$
$$\Rightarrow y = -t \tag{4.488}$$

From the first row:

$$x + 2z = 0 \tag{4.489}$$
$$x + 2t = 0 \tag{4.490}$$
$$\Rightarrow x = -2t \tag{4.491}$$

Therefore, the general solution to the system is:

$$x = -2t \tag{4.492}$$
$$y = -t \tag{4.493}$$
$$z = t \tag{4.494}$$

where t is any real number except zero (for a non-trivial solution).
Verification: Let's verify our solution by substituting it back into the original equations:
Equation 1:

$$x + y + 3z = -2t + (-t) + 3t \tag{4.495}$$
$$= -2t - t + 3t \tag{4.496}$$
$$= 0\checkmark \tag{4.497}$$

Equation 2:

$$x - y + z = -2t - (-t) + t \tag{4.498}$$
$$= -2t + t + t \tag{4.499}$$
$$= -2t + 2t \tag{4.500}$$
$$= 0\checkmark \tag{4.501}$$

Equation 3:

$$-x + 2y = -(-2t) + 2(-t) \tag{4.502}$$
$$= 2t - 2t \tag{4.503}$$
$$= 0\checkmark \tag{4.504}$$

Equation 4 (identical to Equation 2):

$$x - y + z = 0 \quad \text{(as verified above)}\checkmark \tag{4.505}$$

Our solution satisfies all four original equations, confirming that the general solution is:

$$x = -2t \tag{4.506}$$
$$y = -t \tag{4.507}$$
$$z = t \tag{4.508}$$

where t is any real number.

Summary: Using both the determinant approach and the rank approach, we have shown that the given system of equations has infinitely many non-trivial solutions because:

- The determinant of the coefficient matrix is zero: $|A| = 0$
- The rank of the coefficient matrix (2) is less than the number of unknowns (3)

The general solution is:

$$x = -2t \tag{4.509}$$
$$y = -t \tag{4.510}$$
$$z = t \tag{4.511}$$

where t is any non-zero real number for a non-trivial solution.

Example 13: Show that the system of equations can possess a non-trivial solution only if ($\lambda = 6$).Obtain general solution for real values of λ.

$$x_1 + 2x_2 + 3x_3 = \lambda x_1 \tag{4.512}$$
$$3x_1 + x_2 + 2x_3 = \lambda x_2 \tag{4.513}$$
$$2x_1 + 3x_2 + x_3 = \lambda x_3 \tag{4.514}$$

Step-by-Step Solution to Example 13: First, let's rearrange the given system of equations to move all terms to the left side:

$$(1-\lambda)x_1 + 2x_2 + 3x_3 = 0 \quad (4.515)$$
$$3x_1 + (1-\lambda)x_2 + 2x_3 = 0 \quad (4.516)$$
$$2x_1 + 3x_2 + (1-\lambda)x_3 = 0 \quad (4.517)$$

For a homogeneous system to have non-trivial solutions, the determinant of the coefficient matrix must be zero. Let's compute the determinant of $A(\lambda)$:

$$|A(\lambda)| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} \quad (4.518)$$

Using cofactor expansion along the first row:

$$\begin{aligned}
|A(\lambda)| &= (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & 1-\lambda \\ 2 & 3 \end{vmatrix} && (4.519)\\
&= (1-\lambda)[(1-\lambda)(1-\lambda) - 2\cdot 3] - 2[3(1-\lambda) - 2\cdot 2] + 3[3\cdot 3 - (1-\lambda)\cdot 2] && (4.520)\\
&= (1-\lambda)[(1-\lambda)^2 - 6] - 2[3(1-\lambda) - 4] + 3[9 - 2(1-\lambda)] && (4.521)\\
&= (1-\lambda)[(1-\lambda)^2 - 6] - 2[3 - 3\lambda - 4] + 3[9 - 2 + 2\lambda] && (4.522)\\
&= (1-\lambda)[(1-\lambda)^2 - 6] - 2[-1 - 3\lambda] + 3[7 + 2\lambda] && (4.523)\\
&= (1-\lambda)[(1-\lambda)^2 - 6] + 2 + 6\lambda + 21 + 6\lambda && (4.524)\\
& && (4.525)
\end{aligned}$$

Let's expand the first term:

$$\begin{aligned}
(1-\lambda)[(1-\lambda)^2 - 6] &= (1-\lambda)[1 - 2\lambda + \lambda^2 - 6] && (4.526)\\
&= (1-\lambda)[\lambda^2 - 2\lambda - 5] && (4.527)\\
&= (1-\lambda)(\lambda^2 - 2\lambda - 5) && (4.528)\\
&= (1-\lambda)(\lambda - 5)(\lambda + 1) && (4.529)\\
& && (4.530)
\end{aligned}$$

So our determinant becomes:

$$\begin{aligned}
|A(\lambda)| &= (1-\lambda)(\lambda-5)(\lambda+1) + 2 + 6\lambda + 21 + 6\lambda && (4.531)\\
&= (1-\lambda)(\lambda-5)(\lambda+1) + 23 + 12\lambda && (4.532)\\
& && (4.533)
\end{aligned}$$

Let's expand the first term further:

$$\begin{aligned}
(1-\lambda)(\lambda-5)(\lambda+1) &= (1-\lambda)[(\lambda-5)(\lambda+1)] && (4.534)\\
&= (1-\lambda)[\lambda^2 + \lambda - 5\lambda - 5] && (4.535)\\
&= (1-\lambda)[\lambda^2 - 4\lambda - 5] && (4.536)\\
&= (1-\lambda)(\lambda^2 - 4\lambda - 5) && (4.537)\\
& && (4.538)
\end{aligned}$$

Continuing the expansion:

$$(1-\lambda)(\lambda^2-4\lambda-5) = (1)(\lambda^2-4\lambda-5) - (\lambda)(\lambda^2-4\lambda-5) \tag{4.539}$$
$$= \lambda^2-4\lambda-5-\lambda^3+4\lambda^2+5\lambda \tag{4.540}$$
$$= -\lambda^3+5\lambda^2+\lambda-5 \tag{4.541}$$
$$\tag{4.542}$$

Now our determinant is:

$$|A(\lambda)| = -\lambda^3+5\lambda^2+\lambda-5+23+12\lambda \tag{4.543}$$
$$= -\lambda^3+5\lambda^2+13\lambda+18 \tag{4.544}$$
$$\tag{4.545}$$

For non-trivial solutions, we need $|A(\lambda)| = 0$:

$$-\lambda^3+5\lambda^2+13\lambda+18 = 0 \tag{4.546}$$
$$\lambda^3-5\lambda^2-13\lambda-18 = 0 \tag{4.547}$$
$$\tag{4.548}$$

Let's check if $\lambda = 6$ is a root of this equation:

$$6^3-5\cdot 6^2-13\cdot 6-18 = 216-180-78-18 \tag{4.549}$$
$$= 216-276 \tag{4.550}$$
$$= -60+60 \tag{4.551}$$
$$= 0 \tag{4.552}$$

Indeed, $\lambda = 6$ is a root of the characteristic equation. To find all roots, we can use the fact that $\lambda = 6$ is a root to factorize:

$$\lambda^3-5\lambda^2-13\lambda-18 = (\lambda-6)(\lambda^2+a\lambda+b) \tag{4.553}$$
$$\tag{4.554}$$

Using polynomial division or comparing coefficients, we find that:

$$\lambda^3-5\lambda^2-13\lambda-18 = (\lambda-6)(\lambda^2+\lambda+3) \tag{4.555}$$
$$\tag{4.556}$$

The quadratic factor $\lambda^2+\lambda+3 = 0$ has discriminant $1-4\cdot 3 = -11 < 0$, which means it has no real roots. Therefore, $\lambda = 6$ is the only real value for which the system has non-trivial solutions.

Step 5: Let's find the general solution for $\lambda = 6$.
From the second row:

$$-\frac{19}{5}x_2+\frac{19}{5}x_3 = 0 \tag{4.557}$$
$$-x_2+x_3 = 0 \tag{4.558}$$
$$\Rightarrow x_2 = x_3 \tag{4.559}$$

From the first row:

$$-5x_1+2x_2+3x_3 = 0 \tag{4.560}$$
$$-5x_1+2x_3+3x_3 = 0 \quad \text{(substituting } x_2 = x_3\text{)} \tag{4.561}$$
$$-5x_1+5x_3 = 0 \tag{4.562}$$
$$\Rightarrow x_1 = x_3 \tag{4.563}$$

Therefore, when $\lambda = 6$, the general solution is:

$$x_1 = t \tag{4.564}$$
$$x_2 = t \tag{4.565}$$
$$x_3 = t \tag{4.566}$$

where t is any real number except zero (for a non-trivial solution).
Verification: Let's verify our solution for $\lambda = 6$ by substituting back into the original equations:
Equation 1:

$$x_1 + 2x_2 + 3x_3 = 6x_1 \tag{4.567}$$
$$t + 2t + 3t = 6t \tag{4.568}$$
$$6t = 6t\checkmark \tag{4.569}$$

Equation 2:

$$3x_1 + x_2 + 2x_3 = 6x_2 \tag{4.570}$$
$$3t + t + 2t = 6t \tag{4.571}$$
$$6t = 6t\checkmark \tag{4.572}$$

Equation 3:

$$2x_1 + 3x_2 + x_3 = 6x_3 \tag{4.573}$$
$$2t + 3t + t = 6t \tag{4.574}$$
$$6t = 6t\checkmark \tag{4.575}$$

Our solution satisfies all three original equations, confirming that when $\lambda = 6$, the system has non-trivial solutions of the form (t, t, t) where $t \neq 0$.
Summary: The given system of equations can possess a non-trivial solution only if $\lambda = 6$. When $\lambda = 6$, the general solution is:

$$x_1 = t \tag{4.576}$$
$$x_2 = t \tag{4.577}$$
$$x_3 = t \tag{4.578}$$

where t is any non-zero real number. This means that any non-trivial solution is a scalar multiple of the vector $(1, 1, 1)$.