Chapter 3

Gamma Function and Beta Function

3.1 The Gamma Function

3.1.1 Definition and Basic Properties

Definition

For $n > 0$, the Gamma function is defined as:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \tag{3.1}$$

This definition gives an integral representation of the Gamma function valid for all positive real numbers. Let's explore some fundamental properties:

Factorial Property

For any positive integer n:

$$\Gamma(n) = (n-1)! \tag{3.2}$$

Proof

For $n = 1$:

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = [-e^{-x}]_0^\infty = 0 - (-1) = 1 = 0! \tag{3.3}$$

For $n > 1$, using integration by parts with $u = x^{n-1}$ and $dv = e^{-x}dx$:

$$\begin{aligned}
\Gamma(n) &= \int_0^\infty e^{-x} x^{n-1}\, dx && (3.4)\\
&= \left[-e^{-x} x^{n-1}\right]_0^\infty + \int_0^\infty e^{-x}(n-1)x^{n-2}\, dx && (3.5)\\
&= 0 + (n-1)\int_0^\infty e^{-x} x^{n-2}\, dx && (3.6)\\
&= (n-1)\Gamma(n-1) && (3.7)
\end{aligned}$$

By induction, we get:

$$\begin{aligned}
\Gamma(n) &= (n-1)(n-2)\cdots(2)(1)\Gamma(1) && (3.8)\\
&= (n-1)! && (3.9)
\end{aligned}$$

Recurrence Relation

For $n > 0$:

$$\Gamma(n+1) = n\Gamma(n) \tag{3.10}$$

This property follows directly from the integration by parts technique shown in the previous proof.

Special Values

Some important specific values of the Gamma function:

$$\Gamma(1) = 1 \tag{3.11}$$
$$\Gamma(2) = 1 \tag{3.12}$$
$$\Gamma(3) = 2 \tag{3.13}$$
$$\Gamma(4) = 6 \tag{3.14}$$
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{3.15}$$

Proof of $\Gamma\left(\frac{1}{2}\right)$

The special value $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ can be proven using a change of variables:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}x^{-1/2}\,dx \tag{3.16}$$

$$\tag{3.17}$$

Let $u = \sqrt{x}$, then $x = u^2$ and $dx = 2u\,du$:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-u^2}\frac{1}{u}2u\,du \tag{3.18}$$
$$= 2\int_0^\infty e^{-u^2}\,du \tag{3.19}$$
$$= \sqrt{\pi} \tag{3.20}$$

The last step follows from the well-known Gaussian integral: $\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$, and noting that our integral is half of this value since we integrate from 0 to ∞.

Duplication Formula

For all z where both sides are defined:

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z) \tag{3.21}$$

Reflection Formula

For all z not an integer:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{3.22}$$

Infinite Value

The Gamma function has poles at non-positive integers:

$$\Gamma(0) = \Gamma(-1) = \Gamma(-2) = \ldots = \infty \tag{3.23}$$

Scaling Property

For $k > 0$ and $n > 0$:

$$\int_0^\infty e^{-kx} x^{n-1}\, dx = \frac{\Gamma(n)}{k^n} \tag{3.24}$$

Proof

Using substitution $u = kx$, we get $dx = \frac{du}{k}$:

$$\int_0^\infty e^{-kx} x^{n-1}\, dx = \int_0^\infty e^{-u} \left(\frac{u}{k}\right)^{n-1} \frac{du}{k} \tag{3.25}$$

$$= \frac{1}{k^n} \int_0^\infty e^{-u} u^{n-1}\, du \tag{3.26}$$

$$= \frac{\Gamma(n)}{k^n} \tag{3.27}$$

Alternative Integral Form

For $n > 0$:

$$\Gamma(n) = 2 \int_0^\infty e^{-x^2} x^{2n-1}\, dx \tag{3.28}$$

Proof

Using substitution $u = x^2$, we get $dx = \frac{du}{2\sqrt{u}}$:

$$2 \int_0^\infty e^{-x^2} x^{2n-1}\, dx = 2 \int_0^\infty e^{-u} u^{n-1/2} \frac{du}{2\sqrt{u}} \tag{3.29}$$

$$= \int_0^\infty e^{-u} u^{n-1}\, du \tag{3.30}$$

$$= \Gamma(n) \tag{3.31}$$

Change of Variable Formula

For $n > 0$ and $a > 0$:

$$\int_0^\infty e^{-ky} y^{n-1}\, dy = \frac{\Gamma(n)}{k^n} \tag{3.32}$$

3.2 The Beta Function

3.2.1 Definition and Basic Properties

Definition

For $m, n > 0$, the Beta function is defined as:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \tag{3.33}$$

Symmetry

For all $m, n > 0$:

$$B(m,n) = B(n,m) \tag{3.34}$$

Proof

Using the substitution $u = 1 - x$, which gives $dx = -du$ and the limits change from $x = 0, 1$ to $u = 1, 0$:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \tag{3.35}$$

$$= \int_1^0 (1-u)^{m-1}u^{n-1}(-du) \tag{3.36}$$

$$= \int_0^1 (1-u)^{m-1}u^{n-1}\,du \tag{3.37}$$

$$= B(n,m) \tag{3.38}$$

Alternative Integral Representation

For $m, n > 0$:

$$B(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt \tag{3.39}$$

Proof

Using the substitution $x = \frac{t}{1+t}$ or equivalently $t = \frac{x}{1-x}$, we get $dx = \frac{dt}{(1+t)^2}$:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \tag{3.40}$$

$$= \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1}\left(1-\frac{t}{1+t}\right)^{n-1}\frac{dt}{(1+t)^2} \tag{3.41}$$

$$= \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1}\left(\frac{1}{1+t}\right)^{n-1}\frac{dt}{(1+t)^2} \tag{3.42}$$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m-1}}\frac{1}{(1+t)^{n-1}}\frac{dt}{(1+t)^2} \tag{3.43}$$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt \tag{3.44}$$

Relationship with Gamma Function

For $m, n > 0$:

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{3.45}$$

Proof

We start with the product of two Gamma functions:

$$\Gamma(m)\Gamma(n) = \int_0^\infty e^{-x} x^{m-1}\,dx \int_0^\infty e^{-y} y^{n-1}\,dy \tag{3.46}$$

$$= \int_0^\infty \int_0^\infty e^{-(x+y)} x^{m-1} y^{n-1}\,dx\,dy \tag{3.47}$$

Now, we use the transformation $x = ut$ and $y = u(1-t)$ with Jacobian u:

$$\Gamma(m)\Gamma(n) = \int_0^\infty \int_0^1 e^{-u}(ut)^{m-1}(u(1-t))^{n-1} u\,dt\,du \tag{3.48}$$

$$= \int_0^\infty e^{-u} u^{m+n-1}\,du \int_0^1 t^{m-1}(1-t)^{n-1}\,dt \tag{3.49}$$

$$= \Gamma(m+n)B(m,n) \tag{3.50}$$

Rearranging, we get:

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{3.51}$$

Special Values

For positive integers m and n:

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} \tag{3.52}$$

Reciprocal Formula

For p between 0 and 1:

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)} \tag{3.53}$$

3.2.2 Integrals Involving Gamma and Beta Functions

Many definite integrals can be evaluated in terms of Gamma and Beta functions:

Important Integral Formulas

- **Euler's Integral:**

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta = \frac{1}{2}B(m,n) = \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{3.54}$$

- **Wallis' Integral:**

$$\int_0^{\pi/2} \sin^n \theta \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \tag{3.55}$$

3.3 Reduction, Gamma and Beta Formula Sheet

3.3.1 Gamma Function

Property	**Formula**
Definition	$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx \quad \text{for } n > 0$
Factorial Property	$\Gamma(n) = (n-1)! \quad \text{for positive integers } n$
Recurrence Relation	$\Gamma(n+1) = n\Gamma(n) \quad \text{for } n > 0$
Special Values	$\Gamma(1) = 1, \Gamma(2) = 1, \Gamma(3) = 2, \Gamma(4) = 6$
Half-Integer Value	$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
Duplication Formula	$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$
Reflection Formula	$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad \text{for } z \neq \text{integer}$
Infinite Values	$\Gamma(0) = \Gamma(-1) = \Gamma(-2) = \ldots = \infty$
Scaling Property	$\int_0^\infty e^{-kx} x^{n-1} \, dx = \frac{\Gamma(n)}{k^n} \quad \text{for } k, n > 0$
Alternative Form	$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} \, dx \quad \text{for } n > 0$

Table 3.1: Key properties of the Gamma function

3.3.2 Beta Function

Property	**Formula**
Definition	$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, dx \quad \text{for } m, n > 0$
Symmetry	$B(m,n) = B(n,m) \quad \text{for } m, n > 0$
Alternative Form	$B(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} \, dt \quad \text{for } m, n > 0$
Relation to Gamma	$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{for } m, n > 0$
Integer Values	$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} \quad \text{for positive integers } m, n$
Reciprocal Formula	$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)} \quad \text{for } 0 < p < 1$

Table 3.2: Key properties of the Beta function

3.3.3 Important Integrals

Name	Formula
Euler's Integral	$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}B(m,n) = \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
Wallis' Integral	$\int_0^{\pi/2} \sin^n \theta \, d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$

Table 3.3: Important definite integrals related to Gamma and Beta functions

3.3.4 Powers of Sine and Cosine Integrals

Integral Domain	Formula
$\sin^n x$ on $[0, \frac{\pi}{2}]$ (even n)	$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \cdots \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$
$\sin^n x$ on $[0, \frac{\pi}{2}]$ (odd n)	$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \cdots \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1$
Symmetry	$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$
$\sin^m x \cos^n x$ on $[0, \frac{\pi}{2}]$	$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{\{(m-1)(m-3)\cdots\}\times\{(n-1)(n-3)\cdots\}}{(m+n)(m+n-2)(m+n-4)\cdots} \times p$
	where $p = \frac{\pi}{2}$ if m, n both even, otherwise $p = 1$
$\sin^n x$ on $[0, \pi]$	$\int_0^{\pi} \sin^n x \, dx = 2\int_0^{\frac{\pi}{2}} \sin^n x \, dx$
$\cos^n x$ on $[0, \pi]$ (even n)	$\int_0^{\pi} \cos^n x \, dx = 2\int_0^{\frac{\pi}{2}} \cos^n x \, dx$
$\cos^n x$ on $[0, \pi]$ (odd n)	$\int_0^{\pi} \cos^n x \, dx = 0$
$\sin^n x$ on $[0, 2\pi]$ (even n)	$\int_0^{2\pi} \sin^n x \, dx = 4\int_0^{\frac{\pi}{2}} \sin^n x \, dx$
$\sin^n x$ on $[0, 2\pi]$ (odd n)	$\int_0^{2\pi} \sin^n x \, dx = 0$
$\cos^n x$ on $[0, 2\pi]$ (even n)	$\int_0^{2\pi} \cos^n x \, dx = 4\int_0^{\frac{\pi}{2}} \cos^n x \, dx$
$\cos^n x$ on $[0, 2\pi]$ (odd n)	$\int_0^{2\pi} \cos^n x \, dx = 0$

Table 3.4: Formulas for integrals involving powers of sine and cosine

3.4 Solved Examples on Gamma Function

Example 1

Evaluate $\int_0^{\infty} x^4 e^{-x^4} \, dx$.

Detailed Solution

We'll evaluate this integral using substitution and the Gamma function.
Step 1: Recall the definition of the Gamma function:

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} \, dt \quad \text{for } n > 0 \tag{3.56}$$

Step 2: Looking at our integral $\int_0^\infty x^4 e^{-x^4}\,dx$, we need to transform it to match the form of the Gamma function.
Step 3: Let's make the substitution $t = x^4$. To find dx in terms of dt:

$$t = x^4 \tag{3.57}$$
$$\Rightarrow x = t^{1/4} \tag{3.58}$$

Differentiating both sides:

$$dx = \frac{d}{dt}(t^{1/4})\,dt \tag{3.59}$$
$$= \frac{1}{4}t^{-3/4}\,dt \tag{3.60}$$

Step 4: We also need to update the limits of integration:

$$\text{When } x = 0 \Rightarrow t = 0^4 = 0 \tag{3.61}$$
$$\text{When } x = \infty \Rightarrow t = \infty^4 = \infty \tag{3.62}$$

Step 5: Now, let's substitute everything into our original integral:

$$\int_0^\infty x^4 e^{-x^4}\,dx = \int_0^\infty (t^{1/4})^4 e^{-t} \cdot \frac{1}{4}t^{-3/4}\,dt \tag{3.63}$$
$$\tag{3.64}$$

Step 6: Let's simplify the integrand:

$$(t^{1/4})^4 = t^{4\cdot\frac{1}{4}} = t^1 = t \tag{3.65}$$
$$\tag{3.66}$$

Step 7: Continuing with the substitution:

$$\int_0^\infty x^4 e^{-x^4}\,dx = \int_0^\infty t \cdot e^{-t} \cdot \frac{1}{4}t^{-3/4}\,dt \tag{3.67}$$
$$= \frac{1}{4}\int_0^\infty t \cdot t^{-3/4} \cdot e^{-t}\,dt \tag{3.68}$$
$$= \frac{1}{4}\int_0^\infty t^{1-\frac{3}{4}} \cdot e^{-t}\,dt \tag{3.69}$$
$$= \frac{1}{4}\int_0^\infty t^{\frac{4-3}{4}} \cdot e^{-t}\,dt \tag{3.70}$$
$$= \frac{1}{4}\int_0^\infty t^{\frac{1}{4}} \cdot e^{-t}\,dt \tag{3.71}$$
$$\tag{3.72}$$

Step 8: Now, comparing with the Gamma function formula:

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt \tag{3.73}$$

We have $t^{\frac{1}{4}}$, so $n - 1 = \frac{1}{4}$ and $n = \frac{5}{4}$.

Step 9: Therefore:

$$\int_0^\infty x^4 e^{-x^4}\,dx = \frac{1}{4}\int_0^\infty t^{\frac{1}{4}} \cdot e^{-t}\,dt \tag{3.74}$$
$$= \frac{1}{4}\cdot\Gamma\left(\frac{5}{4}\right) \tag{3.75}$$

Step 10: We can compute $\Gamma\left(\frac{5}{4}\right)$ using the recurrence relation:

$$\Gamma(n+1) = n\cdot\Gamma(n) \tag{3.76}$$

With $n = \frac{1}{4}$:

$$\Gamma\left(\frac{5}{4}\right) = \Gamma\left(\frac{1}{4}+1\right) \tag{3.77}$$
$$= \frac{1}{4}\cdot\Gamma\left(\frac{1}{4}\right) \tag{3.78}$$

Step 11: Substituting back:

$$\int_0^\infty x^4 e^{-x^4}\,dx = \frac{1}{4}\cdot\Gamma\left(\frac{5}{4}\right) \tag{3.79}$$
$$= \frac{1}{4}\cdot\frac{1}{4}\cdot\Gamma\left(\frac{1}{4}\right) \tag{3.80}$$
$$= \frac{1}{16}\cdot\Gamma\left(\frac{1}{4}\right) \tag{3.81}$$

Therefore:

$$\boxed{\int_0^\infty x^4 e^{-x^4}\,dx = \frac{1}{16}\cdot\Gamma\left(\frac{1}{4}\right)} \tag{3.82}$$

Example 2

Evaluate $\int_0^\infty x^n e^{-\sqrt{ax}}\,dx$ where $a > 0$ and $n > -1$.

Detailed Solution

We'll evaluate this integral using substitution and the Gamma function.
Step 1: Recall the definition of the Gamma function:

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt \quad \text{for } n > 0 \tag{3.83}$$

Step 2: Looking at our integral $\int_0^\infty x^n e^{-\sqrt{ax}}\,dx$, we need to transform it to match the form of the Gamma function.
Step 3: Let's make the substitution $t = \sqrt{ax}$. To find dx in terms of dt:

$$t = \sqrt{ax} \tag{3.84}$$
$$\Rightarrow t^2 = ax \tag{3.85}$$
$$\Rightarrow x = \frac{t^2}{a} \tag{3.86}$$

Differentiating both sides:

$$dx = \frac{d}{dt}\left(\frac{t^2}{a}\right) dt \tag{3.87}$$

$$= \frac{2t}{a}\, dt \tag{3.88}$$

Step 4: We also need to update the limits of integration:

$$\text{When } x = 0 \Rightarrow t = \sqrt{a \cdot 0} = 0 \tag{3.89}$$

$$\text{When } x = \infty \Rightarrow t = \sqrt{a \cdot \infty} = \infty \tag{3.90}$$

Step 5: Now, let's substitute everything into our original integral:

$$\int_0^\infty x^n e^{-\sqrt{ax}}\, dx = \int_0^\infty \left(\frac{t^2}{a}\right)^n e^{-t} \cdot \frac{2t}{a}\, dt \tag{3.91}$$

$$\tag{3.92}$$

Step 6: Let's simplify the integrand:

$$\left(\frac{t^2}{a}\right)^n = \frac{t^{2n}}{a^n} \tag{3.93}$$

$$\tag{3.94}$$

Step 7: Continuing with the substitution:

$$\int_0^\infty x^n e^{-\sqrt{ax}}\, dx = \int_0^\infty \frac{t^{2n}}{a^n} \cdot e^{-t} \cdot \frac{2t}{a}\, dt \tag{3.95}$$

$$= \frac{2}{a^{n+1}} \int_0^\infty t^{2n} \cdot t \cdot e^{-t}\, dt \tag{3.96}$$

$$= \frac{2}{a^{n+1}} \int_0^\infty t^{2n+1} \cdot e^{-t}\, dt \tag{3.97}$$

$$\tag{3.98}$$

Step 8: Now, comparing with the Gamma function formula:

$$\Gamma(m) = \int_0^\infty t^{m-1} e^{-t}\, dt \tag{3.99}$$

We have t^{2n+1}, so $m - 1 = 2n + 1$ and $m = 2n + 2$.
Step 9: Therefore:

$$\int_0^\infty x^n e^{-\sqrt{ax}}\, dx = \frac{2}{a^{n+1}} \int_0^\infty t^{2n+1} \cdot e^{-t}\, dt \tag{3.100}$$

$$= \frac{2}{a^{n+1}} \cdot \Gamma(2n+2) \tag{3.101}$$

Step 10: We can express $\Gamma(2n+2)$ in terms of factorial for integer values, or leave it in terms of the Gamma function for non-integer values:

$$\Gamma(2n+2) = (2n+1)! \tag{3.102}$$

for integer values of n, and in general:

$$\Gamma(2n+2) = (2n+1)\cdot(2n)\cdot(2n-1)\cdots\Gamma(1) = (2n+1)\cdot(2n)\cdot(2n-1)\cdots 1 \quad (3.103)$$

for $2n+1$ a positive integer.
Therefore:

$$\boxed{\int_0^\infty x^n e^{-\sqrt{ax}}\,dx = \frac{2\Gamma(2n+2)}{a^{n+1}}} \quad (3.104)$$

If n is a non-negative integer, we can write this as:

$$\boxed{\int_0^\infty x^n e^{-\sqrt{ax}}\,dx = \frac{2(2n+1)!}{a^{n+1}}} \quad (3.105)$$

Example 3

Evaluate $\int_0^\infty a^{-4x^2}\,dx$ where $a > 1$.

Detailed Solution

Step 1: Let's put $a^{-4x^2} = e^{-t}$
Taking natural logarithm of both sides:

$$\log\left(a^{-4x^2}\right) = \log\left(e^{-t}\right) \quad (3.106)$$

$$-4x^2\log(a) = -t \quad (3.107)$$

$$4x^2\log(a) = t \quad (3.108)$$

Step 2: Solving for x:

$$x^2 = \frac{t}{4\log(a)} \quad (3.109)$$

$$x = \sqrt{\frac{t}{4\log(a)}} \quad (3.110)$$

Step 3: Find dx by differentiating the above equation:

$$dx = \frac{d}{dt}\left(\sqrt{\frac{t}{4\log(a)}}\right) dt \tag{3.111}$$

$$= \frac{1}{2}\left(\frac{t}{4\log(a)}\right)^{-1/2} \cdot \frac{1}{4\log(a)}\, dt \tag{3.112}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{t}{4\log(a)}}} \cdot \frac{1}{4\log(a)}\, dt \tag{3.113}$$

$$= \frac{1}{2} \cdot \frac{\sqrt{4\log(a)}}{\sqrt{t}} \cdot \frac{1}{4\log(a)}\, dt \tag{3.114}$$

$$= \frac{1}{2} \cdot \frac{2\sqrt{\log(a)}}{\sqrt{t}} \cdot \frac{1}{4\log(a)}\, dt \tag{3.115}$$

$$= \frac{\sqrt{\log(a)}}{2\sqrt{t} \cdot 2\log(a)}\, dt \tag{3.116}$$

$$= \frac{1}{4\sqrt{t}\sqrt{\log(a)}}\, dt \tag{3.117}$$

Step 4: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = 4 \cdot 0^2 \cdot \log(a) = 0 \tag{3.118}$$

$$\text{When } x = \infty \Rightarrow t = 4 \cdot \infty^2 \cdot \log(a) = \infty \tag{3.119}$$

Step 5: Substituting into our integral:

$$\int_0^\infty a^{-4x^2}\, dx = \int_0^\infty e^{-t} \cdot \frac{1}{4\sqrt{t}\sqrt{\log(a)}}\, dt \tag{3.120}$$

$$= \frac{1}{4\sqrt{\log(a)}} \int_0^\infty t^{-1/2} e^{-t}\, dt \tag{3.121}$$

Step 6: We can identify this integral as the Gamma function:

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt \tag{3.122}$$

With $t^{-1/2} = t^{n-1}$, we have $n - 1 = -1/2$, so $n = 1/2$. Therefore:

$$\int_0^\infty t^{-1/2} e^{-t}\, dt = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{3.123}$$

Step 7: Therefore:

$$\int_0^\infty a^{-4x^2}\, dx = \frac{1}{4\sqrt{\log(a)}} \cdot \sqrt{\pi} \tag{3.124}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{\log(a)}} \tag{3.125}$$

Therefore:

$$\boxed{\int_0^\infty a^{-4x^2}\, dx = \frac{\sqrt{\pi}}{4\sqrt{\log(a)}}} \tag{3.126}$$

Example 4

Evaluate $\int_0^\infty x^7 e^{-2x^2}\,dx$.

Detailed Solution

Step 1: Let's set $e^{-2x^2} = e^{-t}$
Taking natural logarithm of both sides:

$$\log\left(e^{-2x^2}\right) = \log(e^{-t}) \tag{3.127}$$
$$-2x^2 = -t \tag{3.128}$$
$$2x^2 = t \tag{3.129}$$

Step 2: Solving for x:

$$x^2 = \frac{t}{2} \tag{3.130}$$
$$x = \sqrt{\frac{t}{2}} \tag{3.131}$$

Step 3: Find dx by differentiating the above equation:

$$dx = \frac{d}{dt}\left(\sqrt{\frac{t}{2}}\right) dt \tag{3.132}$$
$$= \frac{1}{2}\left(\frac{t}{2}\right)^{-1/2} \cdot \frac{1}{2}\,dt \tag{3.133}$$
$$= \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{t}{2}}} \cdot \frac{1}{2}\,dt \tag{3.134}$$
$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{\sqrt{t}} \cdot \frac{1}{2}\,dt \tag{3.135}$$
$$= \frac{\sqrt{2}}{4\sqrt{t}}\,dt \tag{3.136}$$

Step 4: Now, we need to rewrite x^7 in terms of t:

$$x^7 = \left(\sqrt{\frac{t}{2}}\right)^7 \tag{3.137}$$
$$= \left(\frac{t}{2}\right)^{7/2} \tag{3.138}$$
$$= \frac{t^{7/2}}{2^{7/2}} \tag{3.139}$$

Step 5: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = 2 \cdot 0^2 = 0 \tag{3.140}$$
$$\text{When } x = \infty \Rightarrow t = 2 \cdot \infty^2 = \infty \tag{3.141}$$

Step 6: Substituting into our integral:

$$\int_0^\infty x^7 e^{-2x^2}\,dx = \int_0^\infty \frac{t^{7/2}}{2^{7/2}} \cdot e^{-t} \cdot \frac{\sqrt{2}}{4\sqrt{t}}\,dt \tag{3.142}$$

$$= \frac{\sqrt{2}}{4 \cdot 2^{7/2}} \int_0^\infty \frac{t^{7/2}}{\sqrt{t}} \cdot e^{-t}\,dt \tag{3.143}$$

$$= \frac{\sqrt{2}}{4 \cdot 2^{7/2}} \int_0^\infty t^{7/2-1/2} \cdot e^{-t}\,dt \tag{3.144}$$

$$= \frac{\sqrt{2}}{4 \cdot 2^{7/2}} \int_0^\infty t^3 \cdot e^{-t}\,dt \tag{3.145}$$

Step 7: Simplifying the coefficient:

$$\frac{\sqrt{2}}{4 \cdot 2^{7/2}} = \frac{\sqrt{2}}{4 \cdot 2^{3.5}} \tag{3.146}$$

$$= \frac{\sqrt{2}}{4 \cdot 2^3 \cdot 2^{0.5}} \tag{3.147}$$

$$= \frac{\sqrt{2}}{4 \cdot 8 \cdot \sqrt{2}} \tag{3.148}$$

$$= \frac{1}{32} \tag{3.149}$$

Step 8: We can identify the remaining integral as the Gamma function:

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\,dt \tag{3.150}$$

With $t^3 = t^{n-1}$, we have $n - 1 = 3$, so $n = 4$. Therefore:

$$\int_0^\infty t^3 e^{-t}\,dt = \Gamma(4) = 3! = 6 \tag{3.151}$$

Step 9: Therefore:

$$\int_0^\infty x^7 e^{-2x^2}\,dx = \frac{1}{32} \cdot 6 \tag{3.152}$$

$$= \frac{6}{32} \tag{3.153}$$

$$= \frac{3}{16} \tag{3.154}$$

Therefore:

$$\boxed{\int_0^\infty x^7 e^{-2x^2}\,dx = \frac{3}{16}} \tag{3.155}$$

Example 5

Evaluate $\int_0^\infty \sqrt[3]{x^2} e^{-3\sqrt{x}}\,dx$.

Detailed Solution

Step 1: Let's use the substitution $\sqrt{x} = t$, which gives $x = t^2$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(t^2)\,dt \tag{3.156}$$
$$= 2t\,dt \tag{3.157}$$

Step 3: Now, we need to rewrite $\sqrt[3]{x^2}$ in terms of t:

$$\sqrt[3]{x^2} = \sqrt[3]{(t^2)^2} \tag{3.158}$$
$$= \sqrt[3]{t^4} \tag{3.159}$$
$$= t^{4/3} \tag{3.160}$$

Step 4: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = \sqrt{0} = 0 \tag{3.161}$$
$$\text{When } x = \infty \Rightarrow t = \sqrt{\infty} = \infty \tag{3.162}$$

Step 5: Substituting into our integral:

$$\int_0^\infty \sqrt[3]{x^2}e^{-3\sqrt{x}}\,dx = \int_0^\infty t^{4/3} \cdot e^{-3t} \cdot 2t\,dt \tag{3.163}$$
$$= 2\int_0^\infty t^{4/3} \cdot t \cdot e^{-3t}\,dt \tag{3.164}$$
$$= 2\int_0^\infty t^{4/3+1} \cdot e^{-3t}\,dt \tag{3.165}$$
$$= 2\int_0^\infty t^{7/3} \cdot e^{-3t}\,dt \tag{3.166}$$

Step 6: We can now use the scaling property of the Gamma function:

$$\int_0^\infty e^{-kx}x^{n-1}\,dx = \frac{\Gamma(n)}{k^n} \tag{3.167}$$

In our case, we have $t^{7/3} \cdot e^{-3t}$, which corresponds to $n - 1 = 7/3$, so $n = 10/3$, and $k = 3$. Therefore:

$$\int_0^\infty t^{7/3} \cdot e^{-3t}\,dt = \frac{\Gamma\left(\frac{10}{3}\right)}{3^{10/3}} \tag{3.168}$$

Step 7: So our integral becomes:

$$\int_0^\infty \sqrt[3]{x^2}e^{-3\sqrt{x}}\,dx = 2 \cdot \frac{\Gamma\left(\frac{10}{3}\right)}{3^{10/3}} \tag{3.169}$$
$$= \frac{2\Gamma\left(\frac{10}{3}\right)}{3^{10/3}} \tag{3.170}$$

Example 6

Evaluate $\int_0^\infty \sqrt[4]{x}e^{-\sqrt{x}}\,dx$.

Detailed Solution

Step 1: Let's use the substitution $\sqrt{x} = t$, which gives $x = t^2$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(t^2)\,dt \tag{3.171}$$
$$= 2t\,dt \tag{3.172}$$

Step 3: Now, we need to rewrite $\sqrt[4]{x}$ in terms of t:

$$\sqrt[4]{x} = \sqrt[4]{t^2} \tag{3.173}$$
$$= (t^2)^{1/4} \tag{3.174}$$
$$= t^{2/4} \tag{3.175}$$
$$= t^{1/2} \tag{3.176}$$
$$= \sqrt{t} \tag{3.177}$$

Step 4: Also, we need to rewrite $e^{-\sqrt{x}}$ in terms of t:

$$e^{-\sqrt{x}} = e^{-\sqrt{t^2}} \tag{3.178}$$
$$= e^{-t} \tag{3.179}$$

Step 5: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = \sqrt{0} = 0 \tag{3.180}$$
$$\text{When } x = \infty \Rightarrow t = \sqrt{\infty} = \infty \tag{3.181}$$

Step 6: Substituting into our integral:

$$\int_0^\infty \sqrt[4]{x}e^{-\sqrt{x}}\,dx = \int_0^\infty \sqrt{t}\cdot e^{-t}\cdot 2t\,dt \tag{3.182}$$
$$= 2\int_0^\infty t^{1/2}\cdot t\cdot e^{-t}\,dt \tag{3.183}$$
$$= 2\int_0^\infty t^{1/2+1}\cdot e^{-t}\,dt \tag{3.184}$$
$$= 2\int_0^\infty t^{3/2}\cdot e^{-t}\,dt \tag{3.185}$$

Step 7: We can identify this integral as the Gamma function:

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt \tag{3.186}$$

With $t^{3/2} = t^{n-1}$, we have $n - 1 = 3/2$, so $n = 5/2$. Therefore:

$$\int_0^\infty t^{3/2}e^{-t}\,dt = \Gamma\left(\frac{5}{2}\right) \tag{3.187}$$

Step 8: We can simplify $\Gamma\left(\frac{5}{2}\right)$ using the recurrence relation $\Gamma(n+1) = n\Gamma(n)$:

$$\Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right) \tag{3.188}$$

$$= \frac{3}{2}\Gamma\left(\frac{3}{2}\right) \tag{3.189}$$

And:

$$\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right) \tag{3.190}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \tag{3.191}$$

$$= \frac{1}{2}\cdot\sqrt{\pi} \tag{3.192}$$

$$= \frac{\sqrt{\pi}}{2} \tag{3.193}$$

where we've used the known value $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
Combining these results:

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{\sqrt{\pi}}{2} \tag{3.194}$$

$$= \frac{3\sqrt{\pi}}{4} \tag{3.195}$$

Step 9: Therefore:

$$\int_0^\infty \sqrt[4]{x}e^{-\sqrt{x}}\,dx = 2\cdot\Gamma\left(\frac{5}{2}\right) \tag{3.196}$$

$$= 2\cdot\frac{3\sqrt{\pi}}{4} \tag{3.197}$$

$$= \frac{3\sqrt{\pi}}{2} \tag{3.198}$$

Therefore:

$$\boxed{\int_0^\infty \sqrt[4]{x}e^{-\sqrt{x}}\,dx = \frac{3\sqrt{\pi}}{2}} \tag{3.199}$$

Example 7

Evaluate $\int_0^\infty \frac{x^2}{3^{x^2}}\,dx$.

Detailed Solution

Step 1: Let's set $3^{-x^2} = e^{-t}$

Taking natural logarithm of both sides:

$$\log\left(3^{-x^2}\right) = \log\left(e^{-t}\right) \tag{3.200}$$
$$-x^2\log(3) = -t \tag{3.201}$$
$$x^2\log(3) = t \tag{3.202}$$

Step 2: Solving for x:

$$x^2 = \frac{t}{\log(3)} \tag{3.203}$$
$$x = \sqrt{\frac{t}{\log(3)}} \tag{3.204}$$

Step 3: Find dx by differentiating:

$$dx = \frac{d}{dt}\left(\sqrt{\frac{t}{\log(3)}}\right)dt \tag{3.205}$$
$$= \frac{1}{2}\left(\frac{t}{\log(3)}\right)^{-1/2}\cdot\frac{1}{\log(3)}\,dt \tag{3.206}$$
$$= \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{t}{\log(3)}}}\cdot\frac{1}{\log(3)}\,dt \tag{3.207}$$
$$= \frac{1}{2}\cdot\frac{\sqrt{\log(3)}}{\sqrt{t}}\cdot\frac{1}{\log(3)}\,dt \tag{3.208}$$
$$= \frac{1}{2\sqrt{t}\cdot\sqrt{\log(3)}}\,dt \tag{3.209}$$

Step 4: Now, we need to rewrite x^2 in terms of t:

$$x^2 = \frac{t}{\log(3)} \tag{3.210}$$

Step 5: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = 0^2\cdot\log(3) = 0 \tag{3.211}$$
$$\text{When } x = \infty \Rightarrow t = \infty^2\cdot\log(3) = \infty \tag{3.212}$$

Step 6: Substituting into our integral:

$$\int_0^\infty \frac{x^2}{3^{x^2}}\,dx = \int_0^\infty \frac{t}{\log(3)}\cdot e^{-t}\cdot\frac{1}{2\sqrt{t}\cdot\sqrt{\log(3)}}\,dt \tag{3.213}$$
$$= \frac{1}{2\log(3)\cdot\sqrt{\log(3)}}\int_0^\infty \frac{t}{\sqrt{t}}\cdot e^{-t}\,dt \tag{3.214}$$
$$= \frac{1}{2\log(3)\cdot\sqrt{\log(3)}}\int_0^\infty t^{1/2}\cdot e^{-t}\,dt \tag{3.215}$$
$$= \frac{1}{2(\log(3))^{3/2}}\int_0^\infty t^{1/2}\cdot e^{-t}\,dt \tag{3.216}$$

Step 7: We can identify this integral as the Gamma function:

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt \tag{3.217}$$

With $t^{1/2} = t^{n-1}$, we have $n-1 = 1/2$, so $n = 3/2$. Therefore:

$$\int_0^\infty t^{1/2} e^{-t}\, dt = \Gamma\left(\frac{3}{2}\right) \tag{3.218}$$

Step 8: We know that $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \sqrt{\pi} = \frac{\sqrt{\pi}}{2}$. Therefore:

$$\int_0^\infty \frac{x^2}{3^{x^2}}\, dx = \frac{1}{2(\log(3))^{3/2}} \cdot \frac{\sqrt{\pi}}{2} \tag{3.219}$$

$$= \frac{\sqrt{\pi}}{4(\log(3))^{3/2}} \tag{3.220}$$

Therefore:

$$\boxed{\int_0^\infty \frac{x^2}{3^{x^2}}\, dx = \frac{\sqrt{\pi}}{4(\log(3))^{3/2}}} \tag{3.221}$$

Example 8

Evaluate $\int_0^\infty \frac{x^4}{4^x}\, dx$.

Detailed Solution

Step 1: Let's set $4^{-x} = e^{-t}$
Taking natural logarithm of both sides:

$$\log\left(4^{-x}\right) = \log\left(e^{-t}\right) \tag{3.222}$$

$$-x\log(4) = -t \tag{3.223}$$

$$x\log(4) = t \tag{3.224}$$

Step 2: Solving for x:

$$x = \frac{t}{\log(4)} \tag{3.225}$$

Step 3: Find dx by differentiating:

$$dx = \frac{d}{dt}\left(\frac{t}{\log(4)}\right) dt \tag{3.226}$$

$$= \frac{1}{\log(4)}\, dt \tag{3.227}$$

Step 4: Now, we need to rewrite x^4 in terms of t:

$$x^4 = \left(\frac{t}{\log(4)}\right)^4 \tag{3.228}$$

$$= \frac{t^4}{(\log(4))^4} \tag{3.229}$$

Step 5: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = 0 \cdot \log(4) = 0 \tag{3.230}$$
$$\text{When } x = \infty \Rightarrow t = \infty \cdot \log(4) = \infty \tag{3.231}$$

Step 6: Substituting into our integral:

$$\int_0^\infty \frac{x^4}{4^x}\,dx = \int_0^\infty \frac{t^4}{(\log(4))^4} \cdot e^{-t} \cdot \frac{1}{\log(4)}\,dt \tag{3.232}$$
$$= \frac{1}{(\log(4))^4 \cdot \log(4)} \int_0^\infty t^4 \cdot e^{-t}\,dt \tag{3.233}$$
$$= \frac{1}{(\log(4))^5} \int_0^\infty t^4 \cdot e^{-t}\,dt \tag{3.234}$$

Step 7: We can identify this integral as the Gamma function:

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\,dt \tag{3.235}$$

With $t^4 = t^{n-1}$, we have $n - 1 = 4$, so $n = 5$. Therefore:

$$\int_0^\infty t^4 e^{-t}\,dt = \Gamma(5) \tag{3.236}$$

Step 8: We know that $\Gamma(5) = 4! = 24$. Therefore:

$$\int_0^\infty \frac{x^4}{4^x}\,dx = \frac{1}{(\log(4))^5} \cdot 24 \tag{3.237}$$
$$= \frac{24}{(\log(4))^5} \tag{3.238}$$

Step 9: Since $\log(4) = \log(2^2) = 2\log(2)$, we can write:

$$\int_0^\infty \frac{x^4}{4^x}\,dx = \frac{24}{(2\log(2))^5} \tag{3.239}$$
$$= \frac{24}{32 \cdot (\log(2))^5} \tag{3.240}$$
$$= \frac{3}{4 \cdot (\log(2))^5} \tag{3.241}$$

Therefore:

$$\boxed{\int_0^\infty \frac{x^4}{4^x}\,dx = \frac{24}{(\log(4))^5} = \frac{3}{4 \cdot (\log(2))^5}} \tag{3.242}$$

Example 9

Evaluate $\int_0^\infty x^m e^{-ax^n}\,dx$ where $a > 0$.

Detailed Solution

Step 1: Let's use the substitution $x^n = t$, which gives $x = t^{1/n}$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(t^{1/n})\,dt \tag{3.243}$$
$$= \frac{1}{n}t^{1/n-1}\,dt \tag{3.244}$$
$$= \frac{1}{n}t^{(1-n)/n}\,dt \tag{3.245}$$

Step 3: Now, we need to rewrite x^m in terms of t:

$$x^m = (t^{1/n})^m \tag{3.246}$$
$$= t^{m/n} \tag{3.247}$$

Step 4: Also, we need to rewrite e^{-ax^n} in terms of t:

$$e^{-ax^n} = e^{-at} \tag{3.248}$$

Step 5: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = 0^n = 0 \tag{3.249}$$
$$\text{When } x = \infty \Rightarrow t = \infty^n = \infty \tag{3.250}$$

Step 6: Substituting into our integral:

$$\int_0^\infty x^m e^{-ax^n}\,dx = \int_0^\infty t^{m/n} \cdot e^{-at} \cdot \frac{1}{n}t^{(1-n)/n}\,dt \tag{3.251}$$
$$= \frac{1}{n}\int_0^\infty t^{m/n} \cdot t^{(1-n)/n} \cdot e^{-at}\,dt \tag{3.252}$$
$$= \frac{1}{n}\int_0^\infty t^{(m+1-n)/n} \cdot e^{-at}\,dt \tag{3.253}$$

Step 7: Let's set $\frac{m+1-n}{n} = p - 1$ for some p, to match the gamma function form. Then:

$$p - 1 = \frac{m+1-n}{n} \tag{3.254}$$
$$p = \frac{m+1-n}{n} + 1 \tag{3.255}$$
$$= \frac{m+1-n+n}{n} \tag{3.256}$$
$$= \frac{m+1}{n} \tag{3.257}$$

So our integral becomes:

$$\int_0^\infty x^m e^{-ax^n}\,dx = \frac{1}{n}\int_0^\infty t^{p-1} \cdot e^{-at}\,dt \tag{3.258}$$

Step 8: We can now use the scaling property of the Gamma function:

$$\int_0^\infty t^{p-1}e^{-at}\,dt = \frac{\Gamma(p)}{a^p} \tag{3.259}$$

Therefore:

$$\int_0^\infty x^m e^{-ax^n}\,dx = \frac{1}{n}\cdot\frac{\Gamma(p)}{a^p} \tag{3.260}$$

$$= \frac{1}{n}\cdot\frac{\Gamma\left(\frac{m+1}{n}\right)}{a^{(m+1)/n}} \tag{3.261}$$

Therefore:

$$\boxed{\int_0^\infty x^m e^{-ax^n}\,dx = \frac{1}{n}\cdot\frac{\Gamma\left(\frac{m+1}{n}\right)}{a^{(m+1)/n}}} \tag{3.262}$$

This formula is valid for $a > 0$ and assuming $\frac{m+1}{n} > 0$ for the Gamma function to be defined.

Example 10

Evaluate $\int_0^1 x^m(\log x)^n\,dx$ where m, n are real numbers with $m > -1$ and $n \geq 0$.

Detailed Solution

Step 1: Let's use the substitution $\log x = -t$, which gives $x = e^{-t}$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(e^{-t})\,dt \tag{3.263}$$

$$= -e^{-t}\,dt \tag{3.264}$$

Step 3: Now, we need to rewrite x^m in terms of t:

$$x^m = (e^{-t})^m \tag{3.265}$$

$$= e^{-mt} \tag{3.266}$$

Step 4: Also, we need to rewrite $(\log x)^n$ in terms of t:

$$(\log x)^n = (-t)^n \tag{3.267}$$

$$= (-1)^n \cdot t^n \tag{3.268}$$

Step 5: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = -\log(0) = \infty \tag{3.269}$$

$$\text{When } x = 1 \Rightarrow t = -\log(1) = 0 \tag{3.270}$$

Step 6: Substituting into our integral:

$$\int_0^1 x^m(\log x)^n\,dx = \int_\infty^0 e^{-mt}\cdot(-1)^n\cdot t^n\cdot(-e^{-t})\,dt \tag{3.271}$$

$$= (-1)^n\cdot(-1)\int_\infty^0 e^{-mt}\cdot t^n\cdot e^{-t}\,dt \tag{3.272}$$

$$= (-1)^{n+1}\int_\infty^0 t^n\cdot e^{-(m+1)t}\,dt \tag{3.273}$$

Step 7: Change the limits of integration:

$$\int_0^1 x^m (\log x)^n \, dx = (-1)^{n+1} \cdot (-1) \int_0^\infty t^n \cdot e^{-(m+1)t} \, dt \tag{3.274}$$

$$= (-1)^n \int_0^\infty t^n \cdot e^{-(m+1)t} \, dt \tag{3.275}$$

Step 8: The integral $\int_0^\infty t^n e^{-at} \, dt$ is the gamma function $\Gamma(n+1)/a^{n+1}$ when $a > 0$ and $n > -1$. In our case, $a = m + 1$ and we have assumed $m > -1$, so $a > 0$.

$$\int_0^1 x^m (\log x)^n \, dx = (-1)^n \cdot \frac{\Gamma(n+1)}{(m+1)^{n+1}} \tag{3.276}$$

Step 9: Recall that $\Gamma(n+1) = n!$ when n is a non-negative integer. So for integer values of n:

$$\int_0^1 x^m (\log x)^n \, dx = (-1)^n \cdot \frac{n!}{(m+1)^{n+1}} \tag{3.277}$$

Therefore:

$$\boxed{\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n \cdot \Gamma(n+1)}{(m+1)^{n+1}}} \tag{3.278}$$

For integer values of n, this simplifies to:

$$\boxed{\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n \cdot n!}{(m+1)^{n+1}}} \tag{3.279}$$

This formula is valid for $m > -1$ and $n \geq 0$.

Example 11

Evaluate $\int_0^1 (x \log x)^4 \, dx$.

Detailed Solution

Step 1: Let's use the substitution $\log x = -t$, which gives $x = e^{-t}$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(e^{-t}) \, dt \tag{3.280}$$

$$= -e^{-t} \, dt \tag{3.281}$$

Step 3: Now, we need to rewrite $(x \log x)^4$ in terms of t:

$$(x \log x)^4 = (e^{-t} \cdot (-t))^4 \tag{3.282}$$

$$= (e^{-t} \cdot (-t))^4 \tag{3.283}$$

$$= e^{-4t} \cdot (-t)^4 \tag{3.284}$$

$$= e^{-4t} \cdot t^4 \tag{3.285}$$

Step 4: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = -\log(0) = \infty \tag{3.286}$$

$$\text{When } x = 1 \Rightarrow t = -\log(1) = 0 \tag{3.287}$$

Step 5: Substituting into our integral:

$$\int_0^1 (x\log x)^4\,dx = \int_\infty^0 e^{-4t}\cdot t^4\cdot(-e^{-t})\,dt \tag{3.288}$$
$$= (-1)\int_\infty^0 e^{-4t}\cdot t^4\cdot e^{-t}\,dt \tag{3.289}$$
$$= (-1)\int_\infty^0 e^{-(4+1)t}\cdot t^4\,dt \tag{3.290}$$
$$= (-1)\int_\infty^0 e^{-5t}\cdot t^4\,dt \tag{3.291}$$

Step 6: Change the limits of integration:

$$\int_0^1 (x\log x)^4\,dx = (-1)\cdot(-1)\int_0^\infty e^{-5t}\cdot t^4\,dt \tag{3.292}$$
$$= \int_0^\infty e^{-5t}\cdot t^4\,dt \tag{3.293}$$

Step 7: The integral $\int_0^\infty t^n e^{-at}\,dt$ is the gamma function $\Gamma(n+1)/a^{n+1}$ when $a>0$ and $n>-1$. In our case, $a=5$ and $n=4$.

$$\int_0^1 (x\log x)^4\,dx = \frac{\Gamma(4+1)}{5^{4+1}} \tag{3.294}$$
$$= \frac{\Gamma(5)}{5^5} \tag{3.295}$$

Step 8: Since $\Gamma(5)=4!=24$, we have:

$$\int_0^1 (x\log x)^4\,dx = \frac{24}{5^5} \tag{3.296}$$
$$= \frac{24}{3125} \tag{3.297}$$

Therefore:

$$\boxed{\int_0^1 (x\log x)^4\,dx = \frac{24}{3125} = \frac{24}{5^5}} \tag{3.298}$$

Example 12

Evaluate $\int_0^1 \frac{x\,dx}{\sqrt{\log\left(\frac{1}{x}\right)}}$.

Detailed Solution

Step 1: Let's use the substitution $\log\left(\frac{1}{x}\right)=t$, which gives $\log(x)=-t$ and $x=e^{-t}$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(e^{-t})\,dt \tag{3.299}$$
$$= -e^{-t}\,dt \tag{3.300}$$

Step 3: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = \log\left(\frac{1}{0}\right) = \infty \tag{3.301}$$

$$\text{When } x = 1 \Rightarrow t = \log\left(\frac{1}{1}\right) = 0 \tag{3.302}$$

Step 4: Substituting into our integral:

$$\int_0^1 \frac{x\,dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \int_\infty^0 \frac{e^{-t}\cdot(-e^{-t})}{\sqrt{t}}\,dt \tag{3.303}$$

$$= \int_\infty^0 \frac{-e^{-2t}}{\sqrt{t}}\,dt \tag{3.304}$$

$$= (-1)\int_\infty^0 \frac{e^{-2t}}{\sqrt{t}}\,dt \tag{3.305}$$

Step 5: Change the limits of integration:

$$\int_0^1 \frac{x\,dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = (-1)\cdot(-1)\int_0^\infty \frac{e^{-2t}}{\sqrt{t}}\,dt \tag{3.306}$$

$$= \int_0^\infty \frac{e^{-2t}}{\sqrt{t}}\,dt \tag{3.307}$$

$$= \int_0^\infty t^{-1/2}\cdot e^{-2t}\,dt \tag{3.308}$$

Step 6: The integral $\int_0^\infty t^{n-1}e^{-kt}\,dt = \frac{\Gamma(n)}{k^n}$ for $n > 0$ and $k > 0$.
In our case, $n = 1/2$ and $k = 2$:

$$\int_0^1 \frac{x\,dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \frac{\Gamma(1/2)}{2^{1/2}} \tag{3.309}$$

Step 7: We know that $\Gamma(1/2) = \sqrt{\pi}$, so:

$$\int_0^1 \frac{x\,dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \frac{\sqrt{\pi}}{\sqrt{2}} \tag{3.310}$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}} \tag{3.311}$$

Therefore:

$$\boxed{\int_0^1 \frac{x\,dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \frac{\sqrt{\pi}}{\sqrt{2}}} \tag{3.312}$$

Example 13

Evaluate $\int_0^1 \left[\log\left(\frac{1}{y}\right)\right]^{n-1} dy$ where $n > 0$.

Detailed Solution

Step 1: Let's use the substitution $\log\left(\frac{1}{y}\right) = t$, which gives $\log(y) = -t$ and $y = e^{-t}$.
Step 2: Find dy by differentiating:

$$dy = \frac{d}{dt}(e^{-t})\,dt \tag{3.313}$$
$$= -e^{-t}\,dt \tag{3.314}$$

Step 3: The limits of integration transform as:

$$\text{When } y = 0 \Rightarrow t = \log\left(\frac{1}{0}\right) = \infty \tag{3.315}$$
$$\text{When } y = 1 \Rightarrow t = \log\left(\frac{1}{1}\right) = 0 \tag{3.316}$$

Step 4: Substituting into our integral:

$$\int_0^1 \left[\log\left(\frac{1}{y}\right)\right]^{n-1} dy = \int_\infty^0 t^{n-1} \cdot (-e^{-t})\,dt \tag{3.317}$$
$$= -\int_\infty^0 t^{n-1} \cdot e^{-t}\,dt \tag{3.318}$$

Step 5: Change the limits of integration:

$$\int_0^1 \left[\log\left(\frac{1}{y}\right)\right]^{n-1} dy = -(-1)\int_0^\infty t^{n-1} \cdot e^{-t}\,dt \tag{3.319}$$
$$= \int_0^\infty t^{n-1} \cdot e^{-t}\,dt \tag{3.320}$$

Step 6: The integral $\int_0^\infty t^{n-1}e^{-t}\,dt = \Gamma(n)$ for $n > 0$, which is the definition of the gamma function.
Therefore:

$$\boxed{\int_0^1 \left[\log\left(\frac{1}{y}\right)\right]^{n-1} dy = \Gamma(n)} \tag{3.321}$$

For integer values of n, we have $\Gamma(n) = (n-1)!$, so the integral equals $(n-1)!$ when n is a positive integer.

Example 14

Evaluate $\int_0^1 x^{n-1}\left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx$ where $n > 0$.

Detailed Solution

Step 1: Let's use the substitution $\log\left(\frac{1}{x}\right) = t$, which gives $\log(x) = -t$ and $x = e^{-t}$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(e^{-t})\,dt \tag{3.322}$$
$$= -e^{-t}\,dt \tag{3.323}$$

Step 3: Now, we need to rewrite x^{n-1} in terms of t:

$$x^{n-1} = (e^{-t})^{n-1} \tag{3.324}$$
$$= e^{-(n-1)t} \tag{3.325}$$

Step 4: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = \log\left(\frac{1}{0}\right) = \infty \tag{3.326}$$
$$\text{When } x = 1 \Rightarrow t = \log\left(\frac{1}{1}\right) = 0 \tag{3.327}$$

Step 5: Substituting into our integral:

$$\int_0^1 x^{n-1}\left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx = \int_\infty^0 e^{-(n-1)t} \cdot t^{n-1} \cdot (-e^{-t})\, dt \tag{3.328}$$
$$= -\int_\infty^0 e^{-(n-1)t} \cdot t^{n-1} \cdot e^{-t}\, dt \tag{3.329}$$
$$= -\int_\infty^0 e^{-(n-1+1)t} \cdot t^{n-1}\, dt \tag{3.330}$$
$$= -\int_\infty^0 e^{-nt} \cdot t^{n-1}\, dt \tag{3.331}$$

Step 6: Change the limits of integration:

$$\int_0^1 x^{n-1}\left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx = -(-1)\int_0^\infty e^{-nt} \cdot t^{n-1}\, dt \tag{3.332}$$
$$= \int_0^\infty e^{-nt} \cdot t^{n-1}\, dt \tag{3.333}$$

Step 7: Using the formula $\int_0^\infty t^{n-1}e^{-kt}\, dt = \frac{\Gamma(n)}{k^n}$ for $n > 0$ and $k > 0$. In our case, $k = n$:

$$\int_0^1 x^{n-1}\left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx = \frac{\Gamma(n)}{n^n} \tag{3.334}$$

Therefore:

$$\boxed{\int_0^1 x^{n-1}\left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx = \frac{\Gamma(n)}{n^n}} \tag{3.335}$$

For integer values of n, we have $\Gamma(n) = (n-1)!$, so the integral equals $\frac{(n-1)!}{n^n}$ when n is a positive integer.

Example 15

Evaluate $\int_0^\infty e^{-h^2x^n} dx$ where $h > 0$ and $n > 0$.

Detailed Solution

Step 1: Let's use the substitution $x^n = t$, which gives $x = t^{1/n}$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(t^{1/n})\,dt \tag{3.336}$$
$$= \frac{1}{n}t^{1/n-1}\,dt \tag{3.337}$$
$$= \frac{1}{n}t^{(1-n)/n}\,dt \tag{3.338}$$

Step 3: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = 0^n = 0 \tag{3.339}$$
$$\text{When } x = \infty \Rightarrow t = \infty^n = \infty \tag{3.340}$$

Step 4: Substituting into our integral:

$$\int_0^\infty e^{-h^2x^n}\,dx = \int_0^\infty e^{-h^2t}\cdot\frac{1}{n}t^{(1-n)/n}\,dt \tag{3.341}$$
$$= \frac{1}{n}\int_0^\infty t^{(1-n)/n}\cdot e^{-h^2t}\,dt \tag{3.342}$$

Step 5: Using the formula $\int_0^\infty t^{p-1}e^{-qt}\,dt = \frac{\Gamma(p)}{q^p}$ for $p > 0$ and $q > 0$.
In our case, $p = \frac{1-n}{n} + 1 = \frac{1}{n}$ and $q = h^2$:

$$\int_0^\infty e^{-h^2x^n}\,dx = \frac{1}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)}{(h^2)^{1/n}} \tag{3.343}$$
$$= \frac{1}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)}{h^{2/n}} \tag{3.344}$$

Therefore:

$$\boxed{\int_0^\infty e^{-h^2x^n}\,dx = \frac{1}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)}{h^{2/n}}} \tag{3.345}$$

Special cases:
For $n = 1$, the integral becomes:

$$\int_0^\infty e^{-h^2x}dx = \frac{\Gamma(1)}{h^2} = \frac{1}{h^2} \tag{3.346}$$

For $n = 2$, the integral becomes:

$$\int_0^\infty e^{-h^2x^2}dx = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{h} = \frac{1}{2}\cdot\frac{\sqrt{\pi}}{h} = \frac{\sqrt{\pi}}{2h} \tag{3.347}$$

Example 16

Evaluate $\int_0^\infty x^{n-1}e^{-h^2x^2}\,dx$ where $h > 0$ and $n > 0$.

Detailed Solution

Step 1: Let's use the substitution $h^2x^2 = t$, which gives $x = \frac{\sqrt{t}}{h}$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}\left(\frac{\sqrt{t}}{h}\right) dt \tag{3.348}$$
$$= \frac{1}{h} \cdot \frac{1}{2} t^{-1/2}\, dt \tag{3.349}$$
$$= \frac{1}{2h} t^{-1/2}\, dt \tag{3.350}$$

Step 3: Now, we need to rewrite x^{n-1} in terms of t:

$$x^{n-1} = \left(\frac{\sqrt{t}}{h}\right)^{n-1} \tag{3.351}$$
$$= \frac{t^{(n-1)/2}}{h^{n-1}} \tag{3.352}$$

Step 4: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = h^2 \cdot 0^2 = 0 \tag{3.353}$$
$$\text{When } x = \infty \Rightarrow t = h^2 \cdot \infty^2 = \infty \tag{3.354}$$

Step 5: Substituting into our integral:

$$\int_0^\infty x^{n-1} e^{-h^2x^2} dx = \int_0^\infty \frac{t^{(n-1)/2}}{h^{n-1}} \cdot e^{-t} \cdot \frac{1}{2h} t^{-1/2}\, dt \tag{3.355}$$
$$= \frac{1}{2h \cdot h^{n-1}} \int_0^\infty t^{(n-1)/2} \cdot t^{-1/2} \cdot e^{-t}\, dt \tag{3.356}$$
$$= \frac{1}{2h^n} \int_0^\infty t^{(n-1)/2-1/2} \cdot e^{-t}\, dt \tag{3.357}$$
$$= \frac{1}{2h^n} \int_0^\infty t^{(n-2)/2} \cdot e^{-t}\, dt \tag{3.358}$$
$$= \frac{1}{2h^n} \int_0^\infty t^{n/2-1} \cdot e^{-t}\, dt \tag{3.359}$$

Step 6: Using the formula $\int_0^\infty t^{p-1}e^{-t}\, dt = \Gamma(p)$ for $p > 0$.
In our case, $p = \frac{n}{2}$:

$$\int_0^\infty x^{n-1} e^{-h^2x^2} dx = \frac{1}{2h^n} \cdot \Gamma\left(\frac{n}{2}\right) \tag{3.360}$$

Therefore:

$$\boxed{\int_0^\infty x^{n-1} e^{-h^2x^2} dx = \frac{1}{2h^n} \cdot \Gamma\left(\frac{n}{2}\right)} \tag{3.361}$$

Special cases:
For $n = 1$, the integral becomes:

$$\int_0^\infty x^0 e^{-h^2x^2} dx = \frac{1}{2h} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{1}{2h} \cdot \sqrt{\pi} = \frac{\sqrt{\pi}}{2h} \tag{3.362}$$

For $n = 2$, the integral becomes:

$$\int_0^\infty x e^{-h^2x^2} dx = \frac{1}{2h^2} \cdot \Gamma(1) = \frac{1}{2h^2} \tag{3.363}$$

Example 17

Evaluate $\int_0^\infty \sqrt{y}e^{-\sqrt{y}}dy$.

Detailed Solution

Step 1: Let's use the substitution $\sqrt{y} = t$, which gives $y = t^2$.
Step 2: Find dy by differentiating:

$$dy = \frac{d}{dt}(t^2)\, dt \tag{3.364}$$
$$= 2t\, dt \tag{3.365}$$

Step 3: Now, we need to rewrite $\sqrt{y}$ in terms of t:

$$\sqrt{y} = \sqrt{t^2} \tag{3.366}$$
$$= t \tag{3.367}$$

Step 4: The limits of integration transform as:

$$\text{When } y = 0 \Rightarrow t = \sqrt{0} = 0 \tag{3.368}$$
$$\text{When } y = \infty \Rightarrow t = \sqrt{\infty} = \infty \tag{3.369}$$

Step 5: Substituting into our integral:

$$\int_0^\infty \sqrt{y}e^{-\sqrt{y}}dy = \int_0^\infty t \cdot e^{-t} \cdot 2t\, dt \tag{3.370}$$
$$= 2\int_0^\infty t^2 \cdot e^{-t}\, dt \tag{3.371}$$

Step 6: Using the formula $\int_0^\infty t^n e^{-t}\, dt = \Gamma(n+1)$ for $n > -1$.
In our case, $n = 2$:

$$\int_0^\infty \sqrt{y}e^{-\sqrt{y}}dy = 2 \cdot \Gamma(3) \tag{3.372}$$
$$= 2 \cdot 2! \tag{3.373}$$
$$= 2 \cdot 2 \tag{3.374}$$
$$= 4 \tag{3.375}$$

Therefore:

$$\boxed{\int_0^\infty \sqrt{y}e^{-\sqrt{y}}dy = 4} \tag{3.376}$$

Example 18

Evaluate $\int_0^1 \frac{dx}{\sqrt{-\log x}}$.

Detailed Solution

Step 1: Let's use the substitution $-\log x = t$, which gives $\log x = -t$ and $x = e^{-t}$.

Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(e^{-t})\,dt \tag{3.377}$$
$$= -e^{-t}\,dt \tag{3.378}$$

Step 3: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = -\log(0) = \infty \tag{3.379}$$
$$\text{When } x = 1 \Rightarrow t = -\log(1) = 0 \tag{3.380}$$

Step 4: Substituting into our integral:

$$\int_0^1 \frac{dx}{\sqrt{-\log x}} = \int_\infty^0 \frac{-e^{-t}}{\sqrt{t}}\,dt \tag{3.381}$$
$$= -\int_\infty^0 \frac{e^{-t}}{\sqrt{t}}\,dt \tag{3.382}$$

Step 5: Change the limits of integration:

$$\int_0^1 \frac{dx}{\sqrt{-\log x}} = -(-1)\int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt \tag{3.383}$$
$$= \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt \tag{3.384}$$
$$= \int_0^\infty t^{-1/2} \cdot e^{-t}\,dt \tag{3.385}$$

Step 6: Using the formula $\int_0^\infty t^{n-1}e^{-t}\,dt = \Gamma(n)$ for $n > 0$.
In our case, $n = 1/2$:

$$\int_0^1 \frac{dx}{\sqrt{-\log x}} = \Gamma\left(\frac{1}{2}\right) \tag{3.386}$$
$$= \sqrt{\pi} \tag{3.387}$$

Therefore:

$$\boxed{\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}} \tag{3.388}$$

Example 19

Evaluate $\int_0^\infty e^{-x^4}\,dx$.

Detailed Solution

Step 1: Let's use the substitution $x^4 = t$, which gives $x = t^{1/4}$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(t^{1/4})\,dt \tag{3.389}$$
$$= \frac{1}{4}t^{-3/4}\,dt \tag{3.390}$$

Step 3: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = 0^4 = 0 \tag{3.391}$$
$$\text{When } x = \infty \Rightarrow t = \infty^4 = \infty \tag{3.392}$$

Step 4: Substituting into our integral:

$$\int_0^\infty e^{-x^4} dx = \int_0^\infty e^{-t} \cdot \frac{1}{4} t^{-3/4} \, dt \tag{3.393}$$
$$= \frac{1}{4} \int_0^\infty t^{-3/4} \cdot e^{-t} \, dt \tag{3.394}$$

Step 5: Using the formula $\int_0^\infty t^{n-1} e^{-t} \, dt = \Gamma(n)$ for $n > 0$.
In our case, $n = 1/4$:

$$\int_0^\infty e^{-x^4} dx = \frac{1}{4} \cdot \Gamma\left(\frac{1}{4}\right) \tag{3.395}$$

Therefore:

$$\boxed{\int_0^\infty e^{-x^4} dx = \frac{1}{4} \cdot \Gamma\left(\frac{1}{4}\right)} \tag{3.396}$$

Using properties of the gamma function, we can write this in terms of $\Gamma(1/4)$. Note that $\Gamma(1/4) \approx 3.6256$.

Example 20

Evaluate $\int_0^\infty \sqrt{y} e^{-y^3} dy$.

Detailed Solution

Step 1: Let's use the substitution $y^3 = t$, which gives $y = t^{1/3}$.
Step 2: Find dy by differentiating:

$$dy = \frac{d}{dt}(t^{1/3}) \, dt \tag{3.397}$$
$$= \frac{1}{3} t^{-2/3} \, dt \tag{3.398}$$

Step 3: Now, we need to rewrite $\sqrt{y}$ in terms of t:

$$\sqrt{y} = \sqrt{t^{1/3}} \tag{3.399}$$
$$= t^{1/6} \tag{3.400}$$

Step 4: The limits of integration transform as:

$$\text{When } y = 0 \Rightarrow t = 0^3 = 0 \tag{3.401}$$
$$\text{When } y = \infty \Rightarrow t = \infty^3 = \infty \tag{3.402}$$

Step 5: Substituting into our integral:

$$\int_0^\infty \sqrt{y} e^{-y^3} dy = \int_0^\infty t^{1/6} \cdot e^{-t} \cdot \frac{1}{3} t^{-2/3} \, dt \tag{3.403}$$

$$= \frac{1}{3} \int_0^\infty t^{1/6} \cdot t^{-2/3} \cdot e^{-t} \, dt \tag{3.404}$$

$$= \frac{1}{3} \int_0^\infty t^{1/6-2/3} \cdot e^{-t} \, dt \tag{3.405}$$

$$= \frac{1}{3} \int_0^\infty t^{-1/2} \cdot e^{-t} \, dt \tag{3.406}$$

Step 6: Using the formula $\int_0^\infty t^{n-1} e^{-t} \, dt = \Gamma(n)$ for $n > 0$.
In our case, $n = 1/2$:

$$\int_0^\infty \sqrt{y} e^{-y^3} dy = \frac{1}{3} \cdot \Gamma\left(\frac{1}{2}\right) \tag{3.407}$$

$$= \frac{1}{3} \cdot \sqrt{\pi} \tag{3.408}$$

$$= \frac{\sqrt{\pi}}{3} \tag{3.409}$$

Therefore:

$$\boxed{\int_0^\infty \sqrt{y} e^{-y^3} dy = \frac{\sqrt{\pi}}{3}} \tag{3.410}$$

3.5 Solved Examples on Beta Function

Example 1

Evaluate $\int_3^7 \sqrt{(7-x)(x-3)} dx$ using the Beta function.

Detailed Solution

Step 1: First, let's rearrange the integrand to identify the form required for the Beta function. We notice that the limits of integration are from 3 to 7, which correspond to the roots of the expressions inside the square root.
Step 2: Let's make the substitution $x = 3 + 4t$, which transforms the interval $[3, 7]$ to $[0, 1]$. This gives:

$$dx = 4 \, dt \tag{3.411}$$

$$x - 3 = 4t \tag{3.412}$$

$$7 - x = 7 - (3 + 4t) = 4 - 4t = 4(1 - t) \tag{3.413}$$

Step 3: Substituting into our integral:

$$\int_3^7 \sqrt{(7-x)(x-3)}dx = \int_0^1 \sqrt{4(1-t)\cdot 4t}\cdot 4\,dt \tag{3.414}$$

$$= \int_0^1 \sqrt{16t(1-t)}\cdot 4\,dt \tag{3.415}$$

$$= \int_0^1 4\sqrt{t(1-t)}\cdot 4\,dt \tag{3.416}$$

$$= 16\int_0^1 \sqrt{t(1-t)}\,dt \tag{3.417}$$

$$= 16\int_0^1 \sqrt{t}\cdot\sqrt{1-t}\,dt \tag{3.418}$$

$$= 16\int_0^1 t^{1/2}\cdot(1-t)^{1/2}\,dt \tag{3.419}$$

Step 4: The integral $\int_0^1 t^{m-1}(1-t)^{n-1}dt$ is the Beta function $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. In our case, $m = 3/2$ and $n = 3/2$:

$$\int_3^7 \sqrt{(7-x)(x-3)}dx = 16\int_0^1 t^{3/2-1}\cdot(1-t)^{3/2-1}\,dt \tag{3.420}$$

$$= 16\cdot B\left(\frac{3}{2},\frac{3}{2}\right) \tag{3.421}$$

$$= 16\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} \tag{3.422}$$

Step 5: Recall that $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\cdot\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$ and $\Gamma(3) = 2! = 2$:

$$\int_3^7 \sqrt{(7-x)(x-3)}dx = 16\cdot\frac{\frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{\pi}}{2}}{2} \tag{3.423}$$

$$= 16\cdot\frac{\pi/4}{2} \tag{3.424}$$

$$= 16\cdot\frac{\pi}{8} \tag{3.425}$$

$$= 2\pi \tag{3.426}$$

Therefore:

$$\boxed{\int_3^7 \sqrt{(7-x)(x-3)}dx = 2\pi} \tag{3.427}$$

Example 2

Evaluate $\int_a^b \sqrt{(b-x)(x-a)}dx$ using the Beta function.

Detailed Solution

Step 1: First, let's use the substitution: $(x-a) = (b-a)t$, which gives $x = a + (b-a)t$.
Step 2: Calculate dx by differentiating:

$$dx = (b-a)\,dt \tag{3.428}$$

Step 3: Determine how the expressions in the integrand transform:

$$x - a = (b - a)t \tag{3.429}$$

$$b - x = b - [a + (b - a)t] = b - a - (b - a)t = (b - a)(1 - t) \tag{3.430}$$

Step 4: The limits of integration transform as:

$$\text{When } x = a \Rightarrow t = \frac{a-a}{b-a} = 0 \tag{3.431}$$

$$\text{When } x = b \Rightarrow t = \frac{b-a}{b-a} = 1 \tag{3.432}$$

Step 5: Substituting into our integral:

$$\int_a^b \sqrt{(b-x)(x-a)}dx = \int_0^1 \sqrt{(b-a)(1-t)\cdot(b-a)t}\cdot(b-a)\,dt \tag{3.433}$$

$$= \int_0^1 \sqrt{(b-a)^2\cdot t(1-t)}\cdot(b-a)\,dt \tag{3.434}$$

$$= \int_0^1 (b-a)\cdot\sqrt{t(1-t)}\cdot(b-a)\,dt \tag{3.435}$$

$$= (b-a)^2\int_0^1 \sqrt{t(1-t)}\,dt \tag{3.436}$$

$$= (b-a)^2\int_0^1 \sqrt{t}\cdot\sqrt{1-t}\,dt \tag{3.437}$$

$$= (b-a)^2\int_0^1 t^{1/2}\cdot(1-t)^{1/2}\,dt \tag{3.438}$$

Step 6: The integral $\int_0^1 t^{m-1}(1-t)^{n-1}dt$ is the Beta function $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. In our case, $m = 3/2$ and $n = 3/2$:

$$\int_a^b \sqrt{(b-x)(x-a)}dx = (b-a)^2\int_0^1 t^{3/2-1}\cdot(1-t)^{3/2-1}\,dt \tag{3.439}$$

$$= (b-a)^2\cdot B\left(\frac{3}{2},\frac{3}{2}\right) \tag{3.440}$$

$$= (b-a)^2\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} \tag{3.441}$$

Step 7: Recall that $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\cdot\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$ and $\Gamma(3) = 2! = 2$:

$$\int_a^b \sqrt{(b-x)(x-a)}dx = (b-a)^2\cdot\frac{\frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{\pi}}{2}}{2} \tag{3.442}$$

$$= (b-a)^2\cdot\frac{\pi/4}{2} \tag{3.443}$$

$$= (b-a)^2\cdot\frac{\pi}{8} \tag{3.444}$$

$$= \frac{\pi(b-a)^2}{8} \tag{3.445}$$

Therefore:

$$\boxed{\int_a^b \sqrt{(b-x)(x-a)}dx = \frac{\pi(b-a)^2}{8}} \tag{3.446}$$

Example 3

Evaluate $\int_2^5 (x-2)^{\frac{1}{4}}(5-x)^{\frac{1}{4}}dx$ using the Beta function.

Detailed Solution

Step 1: Let's use the suggested substitution: $(x-2) = (5-2)t$, which gives $x = 2+3t$.
Step 2: Calculate dx by differentiating:

$$dx = 3\,dt \tag{3.447}$$

Step 3: Determine how the expressions in the integrand transform:

$$x - 2 = 3t \tag{3.448}$$
$$5 - x = 5 - (2+3t) = 3 - 3t = 3(1-t) \tag{3.449}$$

Step 4: The limits of integration transform as:

$$\text{When } x = 2 \Rightarrow t = \frac{2-2}{3} = 0 \tag{3.450}$$
$$\text{When } x = 5 \Rightarrow t = \frac{5-2}{3} = 1 \tag{3.451}$$

Step 5: Substituting into our integral:

$$\int_2^5 (x-2)^{\frac{1}{4}}(5-x)^{\frac{1}{4}}dx = \int_0^1 (3t)^{\frac{1}{4}} \cdot (3(1-t))^{\frac{1}{4}} \cdot 3\,dt \tag{3.452}$$
$$= \int_0^1 3^{\frac{1}{4}} \cdot t^{\frac{1}{4}} \cdot 3^{\frac{1}{4}} \cdot (1-t)^{\frac{1}{4}} \cdot 3\,dt \tag{3.453}$$
$$= \int_0^1 3^{\frac{1}{4}+\frac{1}{4}} \cdot t^{\frac{1}{4}} \cdot (1-t)^{\frac{1}{4}} \cdot 3\,dt \tag{3.454}$$
$$= \int_0^1 3^{\frac{1}{2}} \cdot t^{\frac{1}{4}} \cdot (1-t)^{\frac{1}{4}} \cdot 3\,dt \tag{3.455}$$
$$= 3 \cdot 3^{\frac{1}{2}} \int_0^1 t^{\frac{1}{4}} \cdot (1-t)^{\frac{1}{4}}\,dt \tag{3.456}$$
$$= 3 \cdot \sqrt{3} \int_0^1 t^{\frac{1}{4}} \cdot (1-t)^{\frac{1}{4}}\,dt \tag{3.457}$$

Step 6: The integral $\int_0^1 t^{m-1}(1-t)^{n-1}dt$ is the Beta function $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. In our case, $m = 5/4$ and $n = 5/4$:

$$\int_2^5 (x-2)^{\frac{1}{4}}(5-x)^{\frac{1}{4}}dx = 3 \cdot \sqrt{3} \int_0^1 t^{\frac{5}{4}-1} \cdot (1-t)^{\frac{5}{4}-1}\,dt \tag{3.458}$$
$$= 3 \cdot \sqrt{3} \cdot B\left(\frac{5}{4}, \frac{5}{4}\right) \tag{3.459}$$
$$= 3 \cdot \sqrt{3} \cdot \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{2}\right)} \tag{3.460}$$

Example 4

Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta$ using the Beta function.

Detailed Solution

Step 1: We know that the Beta function has an important property:

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,d\theta \tag{3.461}$$

Step 2: Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we have:

$$\begin{aligned}
\sqrt{\tan\theta} &= \sqrt{\frac{\sin\theta}{\cos\theta}} && (3.462)\\
&= \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}} && (3.463)\\
&= \frac{(\sin\theta)^{1/2}}{(\cos\theta)^{1/2}} && (3.464)
\end{aligned}$$

Step 3: Therefore, our integral becomes:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta &= \int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{1/2}}{(\cos\theta)^{1/2}}\,d\theta && (3.465)\\
&= \int_0^{\frac{\pi}{2}} (\sin\theta)^{1/2}(\cos\theta)^{-1/2}\,d\theta && (3.466)
\end{aligned}$$

Step 4: Comparing with the Beta function formula, we have $2p-1 = 1/2$ and $2q-1 = -1/2$, which gives $p = 3/4$ and $q = 1/4$. Thus:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta &= \int_0^{\frac{\pi}{2}} (\sin\theta)^{2(3/4)-1}(\cos\theta)^{2(1/4)-1}\,d\theta && (3.467)\\
&= \frac{1}{2}B\left(\frac{3}{4},\frac{1}{4}\right) && (3.468)\\
&= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} && (3.469)
\end{aligned}$$

Step 5: Since $\Gamma(1) = 1$ and using the property $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}$, we have:

$$\begin{aligned}
\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) &= \Gamma\left(\frac{3}{4}\right)\Gamma\left(1-\frac{3}{4}\right) && (3.470)\\
&= \frac{\pi}{\sin\left(\frac{3\pi}{4}\right)} && (3.471)\\
&= \frac{\pi}{\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right)} && (3.472)\\
&= \frac{\pi}{\cos\left(\frac{\pi}{4}\right)} && (3.473)\\
&= \frac{\pi}{\frac{1}{\sqrt{2}}} && (3.474)\\
&= \pi\cdot\sqrt{2} && (3.475)\\
&= \sqrt{2}\pi && (3.476)
\end{aligned}$$

Step 6: Therefore:

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \frac{1}{2} \cdot \sqrt{2}\pi \tag{3.477}$$

$$= \frac{\sqrt{2}\pi}{2} \tag{3.478}$$

$$= \frac{\pi}{\sqrt{2}} \tag{3.479}$$

$$= \frac{\pi}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \tag{3.480}$$

$$= \frac{\pi \cdot \sqrt{2}}{2} \tag{3.481}$$

Therefore:

$$\boxed{\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \frac{\pi \cdot \sqrt{2}}{2}} \tag{3.482}$$

Example 5

Prove that $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \pi$

Detailed Solution

Step 1: Let's first evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}}$ using the Beta function.
Step 2: We know that the Beta function has the representation:

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta \tag{3.483}$$

Step 3: For the first integral, $\frac{1}{\sqrt{\sin\theta}} = (\sin\theta)^{-1/2}$. Comparing with the Beta function formula, we have $2p-1 = -1/2$, which gives $p = 1/4$. Also, there's no $\cos\theta$ term, which means $2q-1 = 0$, giving $q = 1/2$.
Step 4: Therefore:

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\frac{\pi}{2}} (\sin\theta)^{-1/2}(\cos\theta)^0\, d\theta \tag{3.484}$$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{2(1/4)-1}(\cos\theta)^{2(1/2)-1}\, d\theta \tag{3.485}$$

$$= \frac{1}{2}B\left(\frac{1}{4}, \frac{1}{2}\right) \tag{3.486}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \tag{3.487}$$

Step 5: We know that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. We can also use the reflection formula $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}$:

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) \tag{3.488}$$

$$= \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} \tag{3.489}$$

$$= \frac{\pi}{\frac{1}{\sqrt{2}}} \tag{3.490}$$

$$= \sqrt{2}\pi \tag{3.491}$$

Step 6: So we have:

$$\Gamma\left(\frac{1}{4}\right) = \frac{\sqrt{2}\pi}{\Gamma\left(\frac{3}{4}\right)} \tag{3.492}$$

Step 7: Substituting back:

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} = \frac{1}{2}\cdot\frac{\frac{\sqrt{2}\pi}{\Gamma\left(\frac{3}{4}\right)}\cdot\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)} \tag{3.493}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}\pi\cdot\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)^2} \tag{3.494}$$

$$= \frac{\sqrt{2}\pi^{3/2}}{2\Gamma\left(\frac{3}{4}\right)^2} \tag{3.495}$$

Step 8: Now let's evaluate the second integral $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta$.
Step 9: For this integral, $\sqrt{\sin\theta} = (\sin\theta)^{1/2}$. Comparing with the Beta function formula, we have $2p-1 = 1/2$, which gives $p = 3/4$. Also, there's no $\cos\theta$ term, which means $2q-1 = 0$, giving $q = 1/2$.
Step 10: Therefore:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \int_0^{\frac{\pi}{2}} (\sin\theta)^{1/2}(\cos\theta)^0\, d\theta \tag{3.496}$$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{2(3/4)-1}(\cos\theta)^{2(1/2)-1}\, d\theta \tag{3.497}$$

$$= \frac{1}{2}B\left(\frac{3}{4},\frac{1}{2}\right) \tag{3.498}$$

$$= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} \tag{3.499}$$

Step 11: Using $\Gamma\left(\frac{5}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\cdot\sqrt{\pi}}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} \tag{3.500}$$

$$= \frac{1}{2}\cdot\frac{4\Gamma\left(\frac{3}{4}\right)\cdot\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)} \tag{3.501}$$

Step 12: Using the relation from Step 6:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \frac{1}{2} \cdot \frac{4\Gamma\left(\frac{3}{4}\right) \cdot \sqrt{\pi}}{\frac{\sqrt{2\pi}}{\Gamma\left(\frac{3}{4}\right)}} \tag{3.502}$$

$$= \frac{1}{2} \cdot \frac{4\Gamma\left(\frac{3}{4}\right)^2 \cdot \sqrt{\pi}}{\sqrt{2\pi}} \tag{3.503}$$

$$= \frac{2\Gamma\left(\frac{3}{4}\right)^2 \cdot \sqrt{\pi}}{\sqrt{2\pi}} \tag{3.504}$$

$$= \frac{2\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi}} \tag{3.505}$$

Step 13: Now, let's compute the product of the two integrals:

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \frac{\sqrt{2}\pi^{3/2}}{2\Gamma\left(\frac{3}{4}\right)^2} \cdot \frac{2\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi}} \tag{3.506}$$

$$= \frac{\sqrt{2}\pi^{3/2} \cdot 2\Gamma\left(\frac{3}{4}\right)^2}{2\Gamma\left(\frac{3}{4}\right)^2 \cdot \sqrt{2\pi}} \tag{3.507}$$

$$= \frac{\pi^{3/2}}{\sqrt{\pi}} \tag{3.508}$$

$$= \pi \tag{3.509}$$

Therefore:

$$\boxed{\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \pi} \tag{3.510}$$

Example 6

Prove that $\int_0^1 x^{m-1}(1-x^2)^{n-1}dx = \frac{1}{2}\beta\left(\frac{m}{2}, n\right)$, where $m > 0$ and $n > 0$.

Detailed Solution

Step 1: Let's use the substitution $x^2 = t$, which gives $x = \sqrt{t}$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(\sqrt{t})\, dt \tag{3.511}$$

$$= \frac{1}{2}t^{-1/2}\, dt \tag{3.512}$$

Step 3: Now, we need to rewrite x^{m-1} in terms of t:

$$x^{m-1} = (\sqrt{t})^{m-1} \tag{3.513}$$

$$= t^{(m-1)/2} \tag{3.514}$$

Step 4: Also, rewrite $(1-x^2)^{n-1}$ in terms of t:

$$(1-x^2)^{n-1} = (1-t)^{n-1} \tag{3.515}$$

Step 5: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = 0^2 = 0 \tag{3.516}$$
$$\text{When } x = 1 \Rightarrow t = 1^2 = 1 \tag{3.517}$$

Step 6: Substituting into our integral:

$$\int_0^1 x^{m-1}(1-x^2)^{n-1}dx = \int_0^1 t^{(m-1)/2} \cdot (1-t)^{n-1} \cdot \frac{1}{2}t^{-1/2}\,dt \tag{3.518}$$
$$= \frac{1}{2}\int_0^1 t^{(m-1)/2} \cdot t^{-1/2} \cdot (1-t)^{n-1}\,dt \tag{3.519}$$
$$= \frac{1}{2}\int_0^1 t^{(m-1)/2-1/2} \cdot (1-t)^{n-1}\,dt \tag{3.520}$$
$$= \frac{1}{2}\int_0^1 t^{(m-2)/2} \cdot (1-t)^{n-1}\,dt \tag{3.521}$$
$$= \frac{1}{2}\int_0^1 t^{m/2-1} \cdot (1-t)^{n-1}\,dt \tag{3.522}$$

Step 7: The integral $\int_0^1 t^{p-1}(1-t)^{q-1}dt$ is the Beta function $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. In our case, $p = m/2$ and $q = n$:

$$\int_0^1 x^{m-1}(1-x^2)^{n-1}dx = \frac{1}{2}\int_0^1 t^{m/2-1} \cdot (1-t)^{n-1}\,dt \tag{3.523}$$
$$= \frac{1}{2} \cdot B\left(\frac{m}{2}, n\right) \tag{3.524}$$
$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{m}{2}\right)\Gamma(n)}{\Gamma\left(\frac{m}{2}+n\right)} \tag{3.525}$$

Therefore:

$$\boxed{\int_0^1 x^{m-1}(1-x^2)^{n-1}dx = \frac{1}{2}B\left(\frac{m}{2}, n\right)} \tag{3.526}$$

Example 7

Prove that $\beta(m,n) = \beta(m,n+1) + \beta(m+1,n)$.

Detailed Solution

Step 1: Let's start by recalling the definition of the Beta function in terms of the Gamma function:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{3.527}$$

Step 2: Another useful definition of the Beta function is the integral representation:

$$\beta(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt \tag{3.528}$$

We'll use this representation to prove the given identity.

Step 3: Let's consider the right side and express it using the integral representation:

$$\beta(m, n+1) + \beta(m+1, n) = \int_0^1 t^{m-1}(1-t)^{(n+1)-1}\,dt + \int_0^1 t^{(m+1)-1}(1-t)^{n-1}\,dt \tag{3.529}$$

$$= \int_0^1 t^{m-1}(1-t)^{n}\,dt + \int_0^1 t^{m}(1-t)^{n-1}\,dt \tag{3.530}$$

Step 4: We can combine these integrals:

$$\beta(m, n+1) + \beta(m+1, n) = \int_0^1 \left[t^{m-1}(1-t)^{n} + t^{m}(1-t)^{n-1}\right]\,dt \tag{3.531}$$

Step 5: Let's focus on the integrand and see if we can simplify it:

$$t^{m-1}(1-t)^{n} + t^{m}(1-t)^{n-1} = t^{m-1}(1-t)^{n-1}(1-t) + t^{m-1}t(1-t)^{n-1} \tag{3.532}$$

$$= t^{m-1}(1-t)^{n-1}[(1-t)+t] \tag{3.533}$$

$$= t^{m-1}(1-t)^{n-1} \tag{3.534}$$

Step 6: Substituting this back into our integral:

$$\beta(m, n+1) + \beta(m+1, n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt \tag{3.535}$$

$$= \beta(m, n) \tag{3.536}$$

Therefore:

$$\boxed{\beta(m, n) = \beta(m, n+1) + \beta(m+1, n)} \tag{3.537}$$

Alternative Proof Using Gamma Functions:

Step 1: Let's express the right side using the Gamma function definition of the Beta function:

$$\beta(m, n+1) + \beta(m+1, n) = \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)} + \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} \tag{3.538}$$

Step 2: Using the recursive property of the Gamma function, $\Gamma(x+1) = x\Gamma(x)$, we have:

$$\Gamma(n+1) = n\Gamma(n) \tag{3.539}$$

$$\Gamma(m+1) = m\Gamma(m) \tag{3.540}$$

$$\Gamma(m+n+1) = (m+n)\Gamma(m+n) \tag{3.541}$$

Step 3: Substituting these relations:

$$\beta(m, n+1) + \beta(m+1, n) = \frac{\Gamma(m)\cdot n\Gamma(n)}{(m+n)\Gamma(m+n)} + \frac{m\Gamma(m)\cdot\Gamma(n)}{(m+n)\Gamma(m+n)} \tag{3.542}$$

$$= \frac{\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)}[n+m] \tag{3.543}$$

$$= \frac{\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)}\cdot(m+n) \tag{3.544}$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{3.545}$$

$$= \beta(m, n) \tag{3.546}$$

Therefore:

$$\boxed{\beta(m, n) = \beta(m, n+1) + \beta(m+1, n)} \tag{3.547}$$

Example 8

Evaluate $\int_0^1 x^3(1-\sqrt{x})^5\,dx$ using the Beta function.

Detailed Solution

Step 1: Let's use the substitution $\sqrt{x}=t$, which gives $x=t^2$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(t^2)\,dt \tag{3.548}$$
$$= 2t\,dt \tag{3.549}$$

Step 3: Now, we need to rewrite x^3 in terms of t:

$$x^3 = (t^2)^3 \tag{3.550}$$
$$= t^6 \tag{3.551}$$

Step 4: Also, rewrite $(1-\sqrt{x})^5$ in terms of t:

$$(1-\sqrt{x})^5 = (1-t)^5 \tag{3.552}$$

Step 5: The limits of integration transform as:

$$\text{When } x=0 \Rightarrow t=\sqrt{0}=0 \tag{3.553}$$
$$\text{When } x=1 \Rightarrow t=\sqrt{1}=1 \tag{3.554}$$

Step 6: Substituting into our integral:

$$\int_0^1 x^3(1-\sqrt{x})^5\,dx = \int_0^1 t^6\cdot(1-t)^5\cdot 2t\,dt \tag{3.555}$$
$$= 2\int_0^1 t^7\cdot(1-t)^5\,dt \tag{3.556}$$

Step 7: The integral $\int_0^1 t^{p-1}(1-t)^{q-1}\,dt$ is the Beta function $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. In our case, $p=8$ and $q=6$:

$$\int_0^1 x^3(1-\sqrt{x})^5\,dx = 2\int_0^1 t^{8-1}\cdot(1-t)^{6-1}\,dt \tag{3.557}$$
$$= 2\cdot B(8,6) \tag{3.558}$$
$$= 2\cdot\frac{\Gamma(8)\Gamma(6)}{\Gamma(14)} \tag{3.559}$$

Step 8: Using the property $\Gamma(n)=(n-1)!$ for positive integers:

$$\int_0^1 x^3(1-\sqrt{x})^5\,dx = 2\cdot\frac{7!\cdot 5!}{13!} \tag{3.560}$$
$$= 2\cdot\frac{5040\cdot 120}{6227020800} \tag{3.561}$$
$$= 2\cdot\frac{604800}{6227020800} \tag{3.562}$$
$$= \frac{1209600}{6227020800} \tag{3.563}$$

Step 9: Simplifying the fraction:

$$\frac{1209600}{6227020800} = \frac{1}{5148} \tag{3.564}$$

Therefore:

$$\boxed{\int_0^1 x^3(1-\sqrt{x})^5\,dx = \frac{1}{5148}} \tag{3.565}$$

Example 9

Prove that $\int_1^\infty \frac{x^{\frac{n}{2}-1}}{(1+x)^n}dx = \frac{1}{2}\beta\left(\frac{n}{2},\frac{n}{2}\right)$.

Detailed Solution

Step 1: Let's use a simpler substitution. If we set $t = \frac{x}{1+x}$, then $x = \frac{t}{1-t}$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}\left(\frac{t}{1-t}\right)dt \tag{3.566}$$
$$= \frac{1\cdot(1-t) - t\cdot(-1)}{(1-t)^2}\,dt \tag{3.567}$$
$$= \frac{1-t+t}{(1-t)^2}\,dt \tag{3.568}$$
$$= \frac{1}{(1-t)^2}\,dt \tag{3.569}$$

Step 3: The limits of integration transform as:

$$\text{When } x = 1 \Rightarrow t = \frac{1}{1+1} = \frac{1}{2} \tag{3.570}$$
$$\text{When } x = \infty \Rightarrow t = \frac{\infty}{1+\infty} = 1 \tag{3.571}$$

Step 4: Let's rewrite the integrand in terms of t:

$$\frac{x^{\frac{n}{2}-1}}{(1+x)^n} = \frac{\left(\frac{t}{1-t}\right)^{\frac{n}{2}-1}}{\left(1+\frac{t}{1-t}\right)^n} \tag{3.572}$$
$$= \frac{\left(\frac{t}{1-t}\right)^{\frac{n}{2}-1}}{\left(\frac{1-t+t}{1-t}\right)^n} \tag{3.573}$$
$$= \frac{\left(\frac{t}{1-t}\right)^{\frac{n}{2}-1}}{\left(\frac{1}{1-t}\right)^n} \tag{3.574}$$
$$= \frac{t^{\frac{n}{2}-1}}{(1-t)^{\frac{n}{2}-1}}\cdot(1-t)^n \tag{3.575}$$
$$= t^{\frac{n}{2}-1}\cdot(1-t)^{n-\frac{n}{2}+1} \tag{3.576}$$
$$= t^{\frac{n}{2}-1}\cdot(1-t)^{\frac{n}{2}+1} \tag{3.577}$$

Step 5: Substituting into our integral:

$$\int_1^\infty \frac{x^{\frac{n}{2}-1}}{(1+x)^n} dx = \int_{\frac{1}{2}}^1 t^{\frac{n}{2}-1} \cdot (1-t)^{\frac{n}{2}+1} \cdot \frac{1}{(1-t)^2}\, dt \tag{3.578}$$

$$= \int_{\frac{1}{2}}^1 t^{\frac{n}{2}-1} \cdot (1-t)^{\frac{n}{2}-1}\, dt \tag{3.579}$$

Step 6: Now recall the definition of the Beta function:

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt \tag{3.580}$$

Step 7: Due to the symmetry property of the Beta function when $p = q$, and using the substitution $t = 1-u$, we can show:

$$\int_0^{\frac{1}{2}} t^{p-1}(1-t)^{p-1}\, dt = \int_{\frac{1}{2}}^1 t^{p-1}(1-t)^{p-1}\, dt = \frac{1}{2}\beta(p,p) \tag{3.581}$$

Step 8: In our case, $p = \frac{n}{2}$, so:

$$\int_{\frac{1}{2}}^1 t^{\frac{n}{2}-1} \cdot (1-t)^{\frac{n}{2}-1}\, dt = \frac{1}{2}\beta\left(\frac{n}{2}, \frac{n}{2}\right) \tag{3.582}$$

Therefore:

$$\boxed{\int_1^\infty \frac{x^{\frac{n}{2}-1}}{(1+x)^n} dx = \frac{1}{2}\beta\left(\frac{n}{2}, \frac{n}{2}\right)} \tag{3.583}$$

This proves the given relationship using a direct substitution approach.

Example 10

Evaluate $\int_0^2 x(8-x^3)^{\frac{1}{3}} dx$.

Detailed Solution

Step 1: Let's use the substitution $x^3 = 8t$, which gives $x = 2 \cdot t^{1/3}$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(2 \cdot t^{1/3})\, dt \tag{3.584}$$

$$= 2 \cdot \frac{1}{3} t^{-2/3}\, dt \tag{3.585}$$

$$= \frac{2}{3} t^{-2/3}\, dt \tag{3.586}$$

Step 3: Now, we need to rewrite x in terms of t:

$$x = 2 \cdot t^{1/3} \tag{3.587}$$

Step 4: Also, rewrite $(8-x^3)^{1/3}$ in terms of t:

$$(8-x^3)^{1/3} = (8-8t)^{1/3} \tag{3.588}$$
$$= (8(1-t))^{1/3} \tag{3.589}$$
$$= 2\cdot(1-t)^{1/3} \tag{3.590}$$

Step 5: The limits of integration transform as:

$$\text{When } x=0 \Rightarrow t = \frac{0^3}{8} = 0 \tag{3.591}$$
$$\text{When } x=2 \Rightarrow t = \frac{2^3}{8} = \frac{8}{8} = 1 \tag{3.592}$$

Step 6: Substituting into our integral:

$$\int_0^2 x(8-x^3)^{1/3}\,dx = \int_0^1 (2\cdot t^{1/3})\cdot(2\cdot(1-t)^{1/3})\cdot\frac{2}{3}t^{-2/3}\,dt \tag{3.593}$$
$$= \int_0^1 2\cdot t^{1/3}\cdot 2\cdot(1-t)^{1/3}\cdot\frac{2}{3}t^{-2/3}\,dt \tag{3.594}$$
$$= \frac{8}{3}\int_0^1 t^{1/3}\cdot t^{-2/3}\cdot(1-t)^{1/3}\,dt \tag{3.595}$$
$$= \frac{8}{3}\int_0^1 t^{1/3-2/3}\cdot(1-t)^{1/3}\,dt \tag{3.596}$$
$$= \frac{8}{3}\int_0^1 t^{-1/3}\cdot(1-t)^{1/3}\,dt \tag{3.597}$$
$$= \frac{8}{3}\int_0^1 t^{2/3-1}\cdot(1-t)^{4/3-1}\,dt \tag{3.598}$$

Step 7: The integral $\int_0^1 t^{p-1}(1-t)^{q-1}\,dt$ is the Beta function $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. In our case, $p = 2/3$ and $q = 4/3$:

$$\int_0^2 x(8-x^3)^{1/3}\,dx = \frac{8}{3}\cdot B\left(\frac{2}{3},\frac{4}{3}\right) \tag{3.599}$$
$$= \frac{8}{3}\cdot\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma(2)} \tag{3.600}$$

Example 11

Evaluate $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx$.

Detailed Solution

Step 1: Let's first expand the numerator:

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx = \int_0^\infty \frac{x^8-x^{14}}{(1+x)^{24}}dx \tag{3.601}$$
$$= \int_0^\infty \frac{x^8}{(1+x)^{24}}dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}dx \tag{3.602}$$

Step 2: We can use an alternative form of the Beta function for integrals of the form:

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \beta(m,n) \tag{3.603}$$

where $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.
Step 3: For the first integral, $\int_0^\infty \frac{x^8}{(1+x)^{24}}dx$, we have $m-1=8$, so $m=9$. Also, $m+n=24$, which means $9+n=24$, giving $n=15$.
Step 4: Similarly, for the second integral, $\int_0^\infty \frac{x^{14}}{(1+x)^{24}}dx$, we have $m-1=14$, so $m=15$. Also, $m+n=24$, which means $15+n=24$, giving $n=9$.
Step 5: Therefore:

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx = \int_0^\infty \frac{x^8}{(1+x)^{24}}dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}dx \tag{3.604}$$

$$= \beta(9,15) - \beta(15,9) \tag{3.605}$$

Step 6: Due to the symmetry property of the Beta function, $\beta(m,n) = \beta(n,m)$, we have $\beta(15,9) = \beta(9,15)$. Thus:

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx = \beta(9,15) - \beta(15,9) \tag{3.606}$$

$$= \beta(9,15) - \beta(9,15) \tag{3.607}$$

$$= 0 \tag{3.608}$$

Therefore:

$$\boxed{\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx = 0} \tag{3.609}$$

This result can also be understood intuitively. When we substitute $u = \frac{1}{x}$ in the integral, the symmetry of the expression leads to cancellation.

Example 12

Evaluate $\int_0^\infty \frac{x^8-x^5}{(1+x^3)^5}dx$.

Detailed Solution

Step 1: Let's split this integral into two parts:

$$\int_0^\infty \frac{x^8-x^5}{(1+x^3)^5}dx = \int_0^\infty \frac{x^8}{(1+x^3)^5}dx - \int_0^\infty \frac{x^5}{(1+x^3)^5}dx \tag{3.610}$$

Step 2: For the first integral, let's use the substitution $t = x^3$, which gives $x = t^{1/3}$.
Step 3: Find dx by differentiating:

$$dx = \frac{d}{dt}(t^{1/3})\,dt \tag{3.611}$$

$$= \frac{1}{3}t^{-2/3}\,dt \tag{3.612}$$

Step 4: Rewrite the first integral in terms of t:

$$\int_0^\infty \frac{x^8}{(1+x^3)^5}dx = \int_0^\infty \frac{(t^{1/3})^8}{(1+t)^5} \cdot \frac{1}{3}t^{-2/3}\,dt \tag{3.613}$$

$$= \int_0^\infty \frac{t^{8/3} \cdot \frac{1}{3}t^{-2/3}}{(1+t)^5}\,dt \tag{3.614}$$

$$= \frac{1}{3}\int_0^\infty \frac{t^{8/3-2/3}}{(1+t)^5}\,dt \tag{3.615}$$

$$= \frac{1}{3}\int_0^\infty \frac{t^2}{(1+t)^5}\,dt \tag{3.616}$$

Step 5: Similarly, for the second integral:

$$\int_0^\infty \frac{x^5}{(1+x^3)^5}dx = \int_0^\infty \frac{(t^{1/3})^5}{(1+t)^5} \cdot \frac{1}{3}t^{-2/3}\,dt \tag{3.617}$$

$$= \int_0^\infty \frac{t^{5/3} \cdot \frac{1}{3}t^{-2/3}}{(1+t)^5}\,dt \tag{3.618}$$

$$= \frac{1}{3}\int_0^\infty \frac{t^{5/3-2/3}}{(1+t)^5}\,dt \tag{3.619}$$

$$= \frac{1}{3}\int_0^\infty \frac{t^1}{(1+t)^5}\,dt \tag{3.620}$$

Step 6: The integral $\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt = \beta(m,n)$, where $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.
For the first integral, we have $t^2/(1+t)^5$, so $m-1=2$, which gives $m=3$. Also, $m+n=5$, which means $3+n=5$, giving $n=2$. Thus:

$$\frac{1}{3}\int_0^\infty \frac{t^2}{(1+t)^5}\,dt = \frac{1}{3} \cdot \beta(3,2) \tag{3.621}$$

Step 7: Similarly, for the second integral, we have $t^1/(1+t)^5$, so $m-1=1$, which gives $m=2$. Also, $m+n=5$, which means $2+n=5$, giving $n=3$. Thus:

$$\frac{1}{3}\int_0^\infty \frac{t^1}{(1+t)^5}\,dt = \frac{1}{3} \cdot \beta(2,3) \tag{3.622}$$

Step 8: Using the symmetry property of the Beta function, $\beta(m,n) = \beta(n,m)$, we have $\beta(2,3) = \beta(3,2)$. Thus:

$$\int_0^\infty \frac{x^8 - x^5}{(1+x^3)^5}dx = \frac{1}{3} \cdot \beta(3,2) - \frac{1}{3} \cdot \beta(2,3) \tag{3.623}$$

$$= \frac{1}{3} \cdot \beta(3,2) - \frac{1}{3} \cdot \beta(3,2) \tag{3.624}$$

$$= 0 \tag{3.625}$$

Therefore:

$$\boxed{\int_0^\infty \frac{x^8 - x^5}{(1+x^3)^5}dx = 0} \tag{3.626}$$

Example 13

Evaluate $\int_0^1 x^{-\frac{1}{3}}(1-x)^{-\frac{2}{3}}(1+2x)^{-1}dx = \frac{1}{3^{\frac{2}{3}}}\beta\left(\frac{2}{3},\frac{1}{3}\right)$.

Detailed Solution

For simplicity, let's try a direct approach using the Beta function. Let's substitute $y = 1-x$:

Step 1 (new approach): With $y = 1-x$, we have $x = 1-y$ and $dx = -dy$. The limits transform:

$$\text{When } x = 0 \Rightarrow y = 1 \tag{3.627}$$
$$\text{When } x = 1 \Rightarrow y = 0 \tag{3.628}$$

Step 2 (new approach): Our integral becomes:

$$\int_0^1 x^{-\frac{1}{3}}(1-x)^{-\frac{2}{3}}(1+2x)^{-1}dx = \int_1^0 (1-y)^{-\frac{1}{3}}y^{-\frac{2}{3}}(1+2(1-y))^{-1}(-dy) \tag{3.629}$$
$$= \int_0^1 (1-y)^{-\frac{1}{3}}y^{-\frac{2}{3}}(3-2y)^{-1}dy \tag{3.630}$$

Step 3 (new approach): Let's now use the substitution $z = \frac{y}{3-2y}$. This gives $y = \frac{3z}{1+2z}$. Finding dy:

$$dy = \frac{d}{dz}\left(\frac{3z}{1+2z}\right)dz \tag{3.631}$$
$$= \frac{3(1+2z)-3z\cdot 2}{(1+2z)^2}dz \tag{3.632}$$
$$= \frac{3+6z-6z}{(1+2z)^2}dz \tag{3.633}$$
$$= \frac{3}{(1+2z)^2}dz \tag{3.634}$$

The limits transform:

$$\text{When } y = 0 \Rightarrow z = \frac{0}{3-2\cdot 0} = 0 \tag{3.635}$$
$$\text{When } y = 1 \Rightarrow z = \frac{1}{3-2\cdot 1} = 1 \tag{3.636}$$

Step 4 (new approach): Rewriting the integrand:

$$(1-y)^{-\frac{1}{3}} = \left(1-\frac{3z}{1+2z}\right)^{-\frac{1}{3}} \tag{3.637}$$
$$= \left(\frac{1+2z-3z}{1+2z}\right)^{-\frac{1}{3}} \tag{3.638}$$
$$= \left(\frac{1-z}{1+2z}\right)^{-\frac{1}{3}} \tag{3.639}$$
$$= \left(\frac{1+2z}{1-z}\right)^{\frac{1}{3}} \tag{3.640}$$

$$y^{-\frac{2}{3}} = \left(\frac{3z}{1+2z}\right)^{-\frac{2}{3}} \tag{3.641}$$

$$= \left(\frac{1+2z}{3z}\right)^{\frac{2}{3}} \tag{3.642}$$

$$= \frac{(1+2z)^{\frac{2}{3}}}{(3z)^{\frac{2}{3}}} \tag{3.643}$$

$$= \frac{(1+2z)^{\frac{2}{3}}}{3^{\frac{2}{3}}z^{\frac{2}{3}}} \tag{3.644}$$

$$(3-2y)^{-1} = \left(3 - 2\cdot\frac{3z}{1+2z}\right)^{-1} \tag{3.645}$$

$$= \left(3 - \frac{6z}{1+2z}\right)^{-1} \tag{3.646}$$

$$= \left(\frac{3(1+2z)-6z}{1+2z}\right)^{-1} \tag{3.647}$$

$$= \left(\frac{3+6z-6z}{1+2z}\right)^{-1} \tag{3.648}$$

$$= \left(\frac{3}{1+2z}\right)^{-1} \tag{3.649}$$

$$= \frac{1+2z}{3} \tag{3.650}$$

Step 5 (new approach): Substituting all these into our integral:

$$\int_0^1 (1-y)^{-\frac{1}{3}} y^{-\frac{2}{3}} (3-2y)^{-1} dy = \int_0^1 \left(\frac{1+2z}{1-z}\right)^{\frac{1}{3}} \cdot \frac{(1+2z)^{\frac{2}{3}}}{3^{\frac{2}{3}}z^{\frac{2}{3}}} \cdot \frac{1+2z}{3} \cdot \frac{3}{(1+2z)^2}\, dz \tag{3.651}$$

$$= \int_0^1 \frac{(1+2z)^{\frac{1}{3}} \cdot (1+2z)^{\frac{2}{3}} \cdot (1+2z)\cdot 3}{(1-z)^{\frac{1}{3}} \cdot 3^{\frac{2}{3}}z^{\frac{2}{3}} \cdot 3 \cdot (1+2z)^2}\, dz \tag{3.652}$$

$$= \frac{1}{3^{\frac{2}{3}}} \int_0^1 \frac{(1+2z)^{\frac{1}{3}+\frac{2}{3}+1-2}}{(1-z)^{\frac{1}{3}} \cdot z^{\frac{2}{3}}}\, dz \tag{3.653}$$

$$= \frac{1}{3^{\frac{2}{3}}} \int_0^1 \frac{(1+2z)^0}{(1-z)^{\frac{1}{3}} \cdot z^{\frac{2}{3}}}\, dz \tag{3.654}$$

$$= \frac{1}{3^{\frac{2}{3}}} \int_0^1 \frac{1}{(1-z)^{\frac{1}{3}} \cdot z^{\frac{2}{3}}}\, dz \tag{3.655}$$

$$= \frac{1}{3^{\frac{2}{3}}} \int_0^1 (1-z)^{-\frac{1}{3}} \cdot z^{-\frac{2}{3}}\, dz \tag{3.656}$$

Step 6 (new approach): This matches the form of the Beta function $\int_0^1 t^{p-1}(1-t)^{q-1}\, dt = \beta(p,q)$ with $p = 1 - \frac{2}{3} = \frac{1}{3}$ and $q = 1 - \frac{1}{3} = \frac{2}{3}$. Using $\beta(p,q) = \beta(q,p)$, we have:

$$\frac{1}{3^{\frac{2}{3}}}\int_0^1 (1-z)^{-\frac{1}{3}} \cdot z^{-\frac{2}{3}}\,dz = \frac{1}{3^{\frac{2}{3}}} \cdot \beta\left(\frac{1}{3}, \frac{2}{3}\right) \tag{3.657}$$

$$= \frac{1}{3^{\frac{2}{3}}} \cdot \beta\left(\frac{2}{3}, \frac{1}{3}\right) \tag{3.658}$$

Example 14

Evaluate $\int_0^1 \left(1 - x^{\frac{1}{n}}\right)^m dx$.

Detailed Solution

Step 1: Let's use the substitution $x^{\frac{1}{n}} = t$, which gives $x = t^n$.
Step 2: Find dx by differentiating:

$$dx = \frac{d}{dt}(t^n)\,dt \tag{3.659}$$

$$= nt^{n-1}\,dt \tag{3.660}$$

Step 3: The limits of integration transform as:

$$\text{When } x = 0 \Rightarrow t = 0^{\frac{1}{n}} = 0 \tag{3.661}$$

$$\text{When } x = 1 \Rightarrow t = 1^{\frac{1}{n}} = 1 \tag{3.662}$$

Step 4: Rewrite the integrand in terms of t:

$$\left(1 - x^{\frac{1}{n}}\right)^m = (1-t)^m \tag{3.663}$$

Step 5: Substituting into our integral:

$$\int_0^1 \left(1 - x^{\frac{1}{n}}\right)^m dx = \int_0^1 (1-t)^m \cdot nt^{n-1}\,dt \tag{3.664}$$

$$= n\int_0^1 (1-t)^m \cdot t^{n-1}\,dt \tag{3.665}$$

Step 6: The integral $\int_0^1 t^{p-1}(1-t)^{q-1}\,dt$ is the Beta function $\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. In our case, $p = n$ and $q = m+1$:

$$\int_0^1 \left(1 - x^{\frac{1}{n}}\right)^m dx = n\int_0^1 t^{n-1} \cdot (1-t)^m\,dt \tag{3.666}$$

$$= n \cdot \beta(n, m+1) \tag{3.667}$$

$$= n \cdot \frac{\Gamma(n)\Gamma(m+1)}{\Gamma(n+m+1)} \tag{3.668}$$

Step 7: For positive integers, $\Gamma(k) = (k-1)!$, so:

$$\Gamma(n) = (n-1)! \tag{3.669}$$

$$\Gamma(m+1) = m! \tag{3.670}$$

$$\Gamma(n+m+1) = (n+m)! \tag{3.671}$$

Step 8: Therefore:

$$\int_0^1 \left(1 - x^{\frac{1}{n}}\right)^m dx = n \cdot \frac{(n-1)! \cdot m!}{(n+m)!} \tag{3.672}$$
$$= \frac{n \cdot (n-1)! \cdot m!}{(n+m)!} \tag{3.673}$$
$$= \frac{n! \cdot m!}{(n+m)!} \tag{3.674}$$

Therefore:

$$\boxed{\int_0^1 \left(1 - x^{\frac{1}{n}}\right)^m dx = \frac{m!n!}{(m+n)!}} \tag{3.675}$$

This formula is valid for positive integers m and n. For non-integer values, the formula would involve gamma functions instead of factorials.

Example 15

Evaluate $\int_{-1}^{1}(1+x)^m(1-x)^n dx = 2^{m+n+1} \cdot \frac{m! \cdot n!}{(m+n+1)!}$, where $m > 0, n > 0$.

Detailed Solution

Step 1: Let's use the substitution $x = 2t - 1$, which maps the interval $[-1, 1]$ to $[0, 1]$.
Step 2: Find dx:

$$dx = 2\,dt \tag{3.676}$$

Step 3: The limits of integration transform as:

$$\text{When } x = -1 \Rightarrow t = \frac{-1+1}{2} = 0 \tag{3.677}$$
$$\text{When } x = 1 \Rightarrow t = \frac{1+1}{2} = 1 \tag{3.678}$$

Step 4: Now, let's rewrite the expressions in the integrand:

$$(1+x)^m = (1+(2t-1))^m \tag{3.679}$$
$$= (2t)^m \tag{3.680}$$
$$= 2^m \cdot t^m \tag{3.681}$$

$$(1-x)^n = (1-(2t-1))^n \tag{3.682}$$
$$= (2-2t)^n \tag{3.683}$$
$$= 2^n(1-t)^n \tag{3.684}$$

Step 5: Substituting into our integral:

$$\int_{-1}^{1}(1+x)^m(1-x)^n dx = \int_0^1 2^m \cdot t^m \cdot 2^n(1-t)^n \cdot 2\,dt \tag{3.685}$$
$$= 2^{m+n+1}\int_0^1 t^m(1-t)^n\,dt \tag{3.686}$$

Step 6: The integral $\int_0^1 t^{p-1}(1-t)^{q-1}\,dt$ is the Beta function $\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. In our case, $p = m+1$ and $q = n+1$:

$$\int_{-1}^{1} (1+x)^m (1-x)^n dx = 2^{m+n+1} \int_0^1 t^{(m+1)-1}(1-t)^{(n+1)-1}\,dt \tag{3.687}$$

$$= 2^{m+n+1} \cdot \beta(m+1, n+1) \tag{3.688}$$

$$= 2^{m+n+1} \cdot \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} \tag{3.689}$$

Step 7: For positive integers, $\Gamma(k) = (k-1)!$, so:

$$\Gamma(m+1) = m! \tag{3.690}$$

$$\Gamma(n+1) = n! \tag{3.691}$$

$$\Gamma(m+n+2) = (m+n+1)! \tag{3.692}$$

Step 8: Therefore:

$$\int_{-1}^{1} (1+x)^m (1-x)^n dx = 2^{m+n+1} \cdot \frac{m! \cdot n!}{(m+n+1)!} \tag{3.693}$$