Chapter 6

Eigen Values and Eigen Vectors

6.1 Introduction to Eigenvalues and Eigenvectors

In our exploration of linear algebra, we now arrive at one of the most powerful and elegant concepts: eigenvalues and eigenvectors. These concepts provide deep insights into the structure and behavior of linear transformations, with applications spanning from physics and engineering to computer science and data analysis.

Definition and Geometric Interpretation

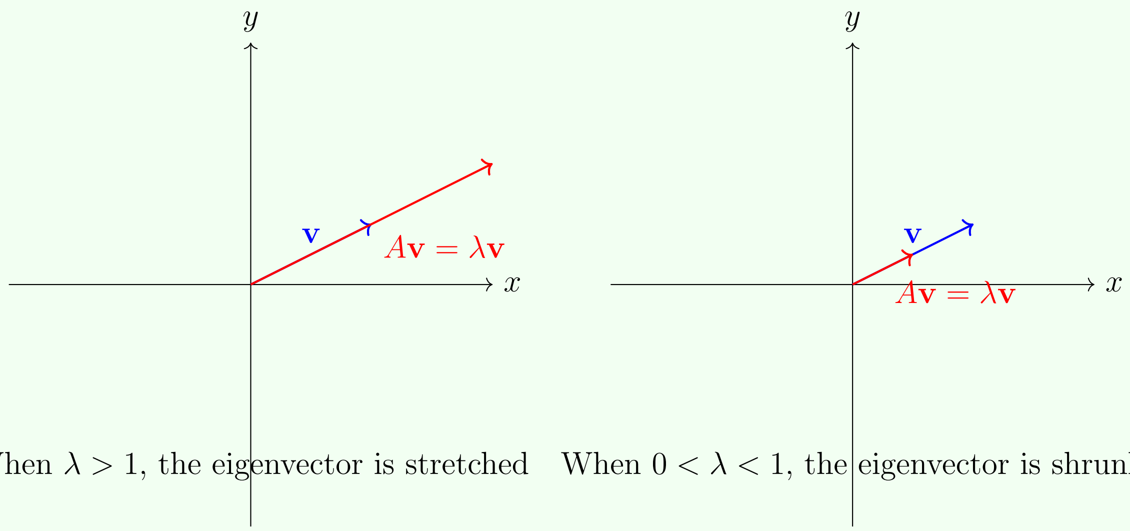
Definition 6.1. *Let A be an $n \times n$ matrix. A scalar λ is called an* ***eigenvalue*** *of A if there exists a non-zero vector $\mathbf{v}$ such that:*

$$A\mathbf{v} = \lambda\mathbf{v} \tag{6.1}$$

The non-zero vector $\mathbf{v}$ is called an ***eigenvector*** *corresponding to the eigenvalue λ.*

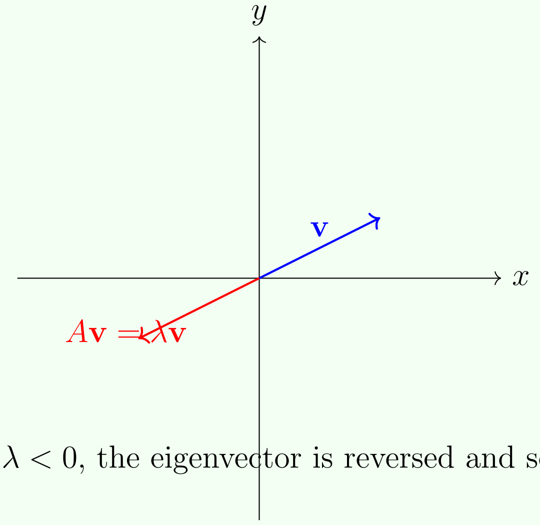
Geometric Interpretation

Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented by a matrix A. When A acts on most vectors, it changes both their direction and magnitude. However, eigenvectors are special vectors whose direction remains unchanged after the transformation—they are merely scaled by a factor of λ.



When $\lambda > 1$, the eigenvector is stretched

When $0 < \lambda < 1$, the eigenvector is shrunk



When $\lambda < 0$, the eigenvector is reversed and scaled

Remark 6.2. *The geometric interpretation reveals why eigenvalues and eigenvectors are so fundamental:*

- *They identify the directions that are invariant under a linear transformation (up to scaling).*
- *They decompose complex transformations into simpler stretching and shrinking operations along specific directions.*
- *The eigenvalues tell us precisely how much stretching or shrinking occurs in these special directions.*

The Eigenvalue Equation $A\mathbf{v} = \lambda\mathbf{v}$

The defining equation $A\mathbf{v} = \lambda\mathbf{v}$ can be rearranged to:

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \tag{6.2}$$

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \tag{6.3}$$

Where I is the $n \times n$ identity matrix. This equation states that $\mathbf{v}$ is in the null space of the matrix $(A - \lambda I)$. Since we require $\mathbf{v} \neq \mathbf{0}$, the matrix $(A - \lambda I)$ must be singular, which means:

$$\det(A - \lambda I) = 0 \tag{6.4}$$

This last equation is crucial for finding eigenvalues, as we'll see next.

Finding Eigenvalues and Eigenvectors

Consider the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Let's find its eigenvalues and eigenvectors.

First, we set up the equation:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \\
(3-\lambda)^2 - 1 &= 0 \\
(3-\lambda)^2 &= 1 \\
3-\lambda &= \pm 1 \\
\lambda &= 2 \text{ or } \lambda = 4
\end{aligned}$$

For $\lambda = 2$, we solve $(A - 2I)\mathbf{v} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives us $v_1 + v_2 = 0$, so an eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $\lambda = 4$, we solve $(A - 4I)\mathbf{v} = \mathbf{0}$:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives us $-v_1 + v_2 = 0$, so an eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Characteristic Polynomial and Characteristic Equation

Definition 6.3. *The* ***characteristic polynomial*** *of an $n \times n$ matrix A is:*

$$p_A(\lambda) = \det(A - \lambda I) \tag{6.5}$$

The equation $p_A(\lambda) = 0$ is called the ***characteristic equation****.*

The roots of the characteristic polynomial are precisely the eigenvalues of A. For an $n \times n$ matrix, the characteristic polynomial is of degree n, so there are at most n distinct eigenvalues.

Characteristic Polynomial

For the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, the characteristic polynomial is:

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda)(5-\lambda) \\ &= -\lambda^3 + 9\lambda^2 - 23\lambda + 15 \end{aligned}$$

The eigenvalues are $\lambda = 1, 3, 5$ (the diagonal elements, which is a special property of triangular matrices).

Property 6.4. *Important properties of the characteristic polynomial:*

1. *For an $n \times n$ matrix A, $p_A(\lambda) = (-1)^n \lambda^n + \ldots + \det(A)$*
2. *The coefficient of λ^{n-1} is $(-1)^{n-1}$ times the trace of A*
3. *The constant term is the determinant of A*
4. *Similar matrices have the same characteristic polynomial*

Relationship to Linear Transformations

Eigenvalues and eigenvectors provide profound insights into linear transformations. When a matrix A represents a linear transformation $T : V \to V$, eigenvectors indicate the directions in which T acts as a simple scaling.

Rotation Matrix

Consider the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ that rotates vectors in $\mathbb{R}^2$ by angle θ.
The characteristic polynomial is:

$$\begin{aligned} p_R(\lambda) &= \det(R - \lambda I) \\ &= \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} \\ &= (\cos\theta - \lambda)^2 + \sin^2\theta \\ &= \lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta \\ &= \lambda^2 - 2\lambda\cos\theta + 1 \end{aligned}$$

For $\theta \neq 0, \pi$, the eigenvalues are $\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$, which are complex numbers with magnitude 1. This reflects that rotations preserve lengths but change directions for all non-zero real vectors.

Dynamical Systems

Consider a linear dynamical system described by:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \tag{6.6}$$

If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then the solution along this direction is:

$$\mathbf{x}(t) = c\mathbf{v}e^{\lambda t} \tag{6.7}$$

This means:

- If $\lambda < 0$, the system is stable in the direction of $\mathbf{v}$
- If $\lambda > 0$, the system is unstable in the direction of $\mathbf{v}$
- If λ is complex with negative real part, the system exhibits damped oscillations

The eigenvalues completely determine the qualitative behavior of linear dynamical systems!

Remark 6.5. *The relationship between eigenvalues and linear transformations extends to:*

- *Powers of matrices:* $A^k\mathbf{v} = \lambda^k\mathbf{v}$ *for eigenvectors*
- *Matrix exponentials:* $e^{tA}\mathbf{v} = e^{t\lambda}\mathbf{v}$ *for eigenvectors*
- *Matrix functions in general:* $f(A)\mathbf{v} = f(\lambda)\mathbf{v}$ *for eigenvectors*

This property makes eigendecomposition extremely valuable for computing powers of matrices and solving linear differential equations.

6.2 Finding Eigenvalues and Eigenvectors

Having established the fundamental concepts of eigenvalues and eigenvectors, we now focus on systematic methods to find them. Our approach will utilize homogeneous systems, row echelon form, and rank theory—concepts we have thoroughly explored in previous chapters.

The Homogeneous System Approach

Recall that for an $n \times n$ matrix A, a scalar λ is an eigenvalue if and only if there exists a non-zero vector $\mathbf{v}$ such that:

$$A\mathbf{v} = \lambda\mathbf{v} \tag{6.8}$$

This can be rewritten as:

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \tag{6.9}$$

This is a homogeneous system of linear equations. For a non-trivial solution $\mathbf{v} \neq \mathbf{0}$ to exist, the system must be singular, which means:

$$\det(A - \lambda I) = 0 \tag{6.10}$$

Property 6.6. *The eigenvalues of matrix A are precisely the values of λ that cause the homogeneous system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to have non-trivial solutions.*

Using Rank to Determine Eigenvalues

An alternative perspective uses the concept of rank:

Theorem 6.7. *A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if:*

$$\operatorname{rank}(A - \lambda I) < n \tag{6.11}$$

Proof. For $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to have a non-trivial solution, the null space of $(A - \lambda I)$ must have dimension at least 1. By the Rank-Nullity Theorem:

$$\dim(\operatorname{null}(A - \lambda I)) + \operatorname{rank}(A - \lambda I) = n$$

Thus, $\dim(\operatorname{null}(A - \lambda I)) \geq 1$ if and only if $\operatorname{rank}(A - \lambda I) < n$. □

Using Rank to Find Eigenvalues

Consider the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

For a value λ to be an eigenvalue, we need $\operatorname{rank}(A - \lambda I) < 3$. Let's examine:

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 3-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{bmatrix}$$

The characteristic equation is $\det(A - \lambda I) = 0$, which we can compute as:

$$\begin{aligned}
\det(A - \lambda I) &= (2-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & -1 \\ 1 & 1-\lambda \end{vmatrix} + 0 \\
&= (2-\lambda)((3-\lambda)(1-\lambda) - (-1)(-1)) - (0 \cdot (1-\lambda) - (-1) \cdot 1) \\
&= (2-\lambda)((3-\lambda)(1-\lambda) - 1) + 1 \\
&= (2-\lambda)(3 - 4\lambda + \lambda^2 - 1) + 1 \\
&= (2-\lambda)(2 - 4\lambda + \lambda^2) + 1 \\
&= 4 - 8\lambda + 2\lambda^2 - 2\lambda + 4\lambda^2 - \lambda^3 + 1 \\
&= -\lambda^3 + 6\lambda^2 - 10\lambda + 5
\end{aligned}$$

Setting this equal to zero gives us the characteristic equation:

$$-\lambda^3 + 6\lambda^2 - 10\lambda + 5 = 0 \tag{6.12}$$

Solving this equation (which may require numerical methods for more complex matrices) gives us the eigenvalues.

Finding Eigenvectors using Row Echelon Form

Once we have found the eigenvalues, we can find the corresponding eigenvectors using row echelon form:

Algorithm 1 Finding Eigenvectors using Row Echelon Form

```
for each eigenvalue λ do
    Form the matrix A − λI
    Reduce A − λI to row echelon form using Gaussian elimination
    Solve the homogeneous system (A − λI)v = 0
    The non-trivial solutions form the eigenvectors for λ
end for
```

Finding Eigenvectors using Row Echelon Form

Let's find the eigenvectors for the matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$.
First, we find the eigenvalues by solving $\det(A - \lambda I) = 0$:

$$\begin{aligned} \det \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} &= (4-\lambda)(3-\lambda) - 2 \cdot 1 \\ &= 12 - 4\lambda - 3\lambda + \lambda^2 - 2 \\ &= \lambda^2 - 7\lambda + 10 \end{aligned}$$

Setting this equal to zero and solving:

$$\begin{aligned} \lambda^2 - 7\lambda + 10 &= 0 \\ (\lambda - 5)(\lambda - 2) &= 0 \end{aligned}$$

So the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 2$.
For $\lambda_1 = 5$, we form the matrix:

$$A - 5I = \begin{bmatrix} 4-5 & 2 \\ 1 & 3-5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

Reducing to row echelon form:

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

This gives us the equation $v_1 - 2v_2 = 0$ or $v_1 = 2v_2$.
If we set $v_2 = 1$ (any non-zero value will work), then $v_1 = 2$, giving us an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
For $\lambda_2 = 2$, we form the matrix:

$$A - 2I = \begin{bmatrix} 4-2 & 2 \\ 1 & 3-2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

Reducing to row echelon form:

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

This gives us the equation $v_1 + v_2 = 0$ or $v_1 = -v_2$.
If we set $v_2 = 1$, then $v_1 = -1$, giving us an eigenvector $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Existence of Non-trivial Solutions

The existence of non-trivial solutions to the homogeneous system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is directly related to the singularity of the matrix $A - \lambda I$.

Theorem 6.8. *For an $n \times n$ matrix A and a scalar λ, the following are equivalent:*

1. *λ is an eigenvalue of A*
2. $\det(A - \lambda I) = 0$
3. $\operatorname{rank}(A - \lambda I) < n$
4. *The homogeneous system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has non-trivial solutions*

Property 6.9. *If λ is an eigenvalue of A with algebraic multiplicity k (i.e., λ is a root of multiplicity k in the characteristic polynomial), then:*

$$\dim(\ker(A - \lambda I)) \leq k \tag{6.13}$$

The dimension of $\ker(A - \lambda I)$ is called the geometric multiplicity of λ.

Multiple Eigenvalues

Consider the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

The characteristic polynomial is:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{bmatrix} \\ &= (3-\lambda)^2 \cdot (4-\lambda) \\ &= (3-\lambda)^2(4-\lambda) \end{aligned}$$

Setting this equal to zero, we get:

$$(3-\lambda)^2(4-\lambda) = 0$$

So $\lambda_1 = 3$ with algebraic multiplicity 2, and $\lambda_2 = 4$ with algebraic multiplicity 1.
For $\lambda_1 = 3$, we have:

$$A - 3I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In row echelon form:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives us $v_2 = 0$ and $v_3 = 0$, with v_1 free. So an eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Note that even though the algebraic multiplicity of $\lambda_1 = 3$ is 2, the geometric multiplicity (dimension of the eigenspace) is only 1. This is because the matrix $A - 3I$ has rank 2.
For $\lambda_2 = 4$, we have:

$$A - 4I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In row echelon form:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives us $v_1 = v_2$ and v_3 is free. If we set $v_3 = 1$ and $v_2 = 0$, then $v_1 = 0$, giving us an eigenvector $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Computational Considerations

Numerical Stability

Finding eigenvalues by solving $\det(A - \lambda I) = 0$ directly can be numerically unstable for large matrices. More sophisticated algorithms like the QR algorithm are preferred in practice.
However, once the eigenvalues are determined, using row echelon form to find eigenvectors is still an effective approach.

Summary of the Process

1. Form the characteristic equation $\det(A - \lambda I) = 0$
2. Solve for the eigenvalues λ
3. For each eigenvalue λ:
   (a) Form the matrix $A - \lambda I$
   (b) Reduce to row echelon form
   (c) Solve the homogeneous system $(A - \lambda I)\mathbf{v} = \mathbf{0}$
   (d) The solutions form the eigenspace E_λ corresponding to λ

Solving Systems of Differential Equations

Consider a system of linear first-order differential equations:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \tag{6.14}$$

If we can find the eigenvalues λ_i and corresponding eigenvectors $\mathbf{v}_i$ of matrix A, then the general solution is:

$$\mathbf{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}\mathbf{v}_i \tag{6.15}$$

Where c_i are constants determined by initial conditions. This approach dramatically simplifies solving coupled differential equations.

6.3 Solved Examples for Eigenvalues and Eigenvectors

In this section, we will work through detailed examples to illustrate the process of finding eigenvalues and eigenvectors using the concepts of homogeneous systems, row echelon form, and rank.

6.3.1 Type I: Asymmetric/Symmetric matrix with Non-repeated eigenvalues.

Example 1: Finding Eigenvalues and Eigenvectors

Consider the matrix $\mathbf{A} = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors.

Solution:

First, we notice that, it is a non-symmetric 3×3 matrix.

Step 1: The characteristic equation for a 3×3 matrix is given by:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \tag{6.16}$$

where:

$$\begin{aligned} S_1 &= \text{Trace of } A && (6.17)\\ S_2 &= \text{Sum of minors of diagonal elements} && (6.18)\\ |A| &= \text{Determinant of matrix } A && (6.19) \end{aligned}$$

Calculate S_1, the trace of A:

$$S_1 = 4 + 3 + (-3) = 4 \tag{6.20}$$

Calculate S_2, the sum of minors of diagonal elements:

$$\begin{aligned} S_2 &= \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix} && (6.21)\\ &= (3 \cdot (-3) - 2 \cdot (-4)) + (4 \cdot (-3) - 6 \cdot (-1)) + (4 \cdot 3 - 6 \cdot 1) && (6.22)\\ &= (-9 - (-8)) + (-12 - (-6)) + (12 - 6) && (6.23)\\ &= -1 + (-6) + 6 && (6.24)\\ &= -1 && (6.25) \end{aligned}$$

Calculate $|A|$, the determinant of matrix A:

$$\begin{aligned} |A| &= 4\begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix} + 6\begin{vmatrix} 1 & 3 \\ -1 & -4 \end{vmatrix} && (6.26)\\ &= 4(3 \cdot (-3) - 2 \cdot (-4)) - 6(1 \cdot (-3) - 2 \cdot (-1)) + 6(1 \cdot (-4) - 3 \cdot (-1)) && (6.27)\\ &= 4(-9 + 8) - 6(-3 + 2) + 6(-4 + 3) && (6.28)\\ &= 4(-1) - 6(-1) + 6(-1) && (6.29)\\ &= -4 + 6 - 6 && (6.30)\\ &= -4 && (6.31) \end{aligned}$$

Substituting these values into the characteristic equation:

$$\begin{aligned} \lambda^3 - 4\lambda^2 + (-1)\lambda - (-4) &= 0 && (6.32)\\ \lambda^3 - 4\lambda^2 - \lambda + 4 &= 0 && (6.33) \end{aligned}$$

Let's check if $\lambda = 1$ is a root:

$$1^3 - 4(1)^2 - 1 + 4 = 1 - 4 - 1 + 4 = 0 \tag{6.34}$$

Since $\lambda = 1$ is a root, we can factor out $(\lambda - 1)$. Let's use synthetic division to find the other factors:

$$\begin{array}{c|cccc} 1 & 1 & -4 & -1 & 4 \\ & & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$

So we have:

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = (\lambda - 1)(\lambda^2 - 3\lambda - 4) \tag{6.35}$$
$$= (\lambda - 1)(\lambda - 4)(\lambda + 1) \tag{6.36}$$
$$\tag{6.37}$$

Therefore, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 4$.
Step 2: Find the eigenvector corresponding to $\lambda_1 = 1$.
For $\lambda_1 = 1$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 4-1 & 6 & 6 \\ 1 & 3-1 & 2 \\ -1 & -4 & -3-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.38}$$
$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.39}$$

As $\lambda_1 = 1$ is an eigenvalue with multiplicity 1, we consider any two linearly independent rows. Let's use rows 1 and 3 to form:

$$3x + 6y + 6z = 0 \tag{6.40}$$
$$-x - 4y - 4z = 0 \tag{6.41}$$

Using Cramer's Rule to find the proportional relationships between variables:

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ -4 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 6 \\ -1 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 6 \\ -1 & -4 \end{vmatrix}} \tag{6.42}$$
$$\frac{x}{6(-4) - 6(-4)} = \frac{-y}{3(-4) - 6(-1)} = \frac{z}{3(-4) - (-1)6} \tag{6.43}$$
$$\frac{x}{0} = \frac{-y}{-12 + 6} = \frac{z}{-12 + 6} \tag{6.44}$$
$$\frac{x}{0} = \frac{-y}{-6} = \frac{z}{-6} \tag{6.45}$$
$$\tag{6.46}$$

Therefore, an eigenvector corresponding to $\lambda_1 = 1$ is:

$$\vec{X}_1 = \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{6.47}$$

Step 3: Find the eigenvector corresponding to $\lambda_2 = -1$.

For $\lambda_2 = -1$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 4-(-1) & 6 & 6 \\ 1 & 3-(-1) & 2 \\ -1 & -4 & -3-(-1) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.48}$$

$$\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.49}$$

Let's consider the first and second rows:

$$5x + 6y + 6z = 0 \tag{6.50}$$

$$x + 4y + 2z = 0 \tag{6.51}$$

Using Cramer's Rule:

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ 4 & 2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & 6 \\ 1 & 4 \end{vmatrix}} \tag{6.52}$$

$$\frac{x}{6 \cdot 2 - 6 \cdot 4} = \frac{-y}{5 \cdot 2 - 6 \cdot 1} = \frac{z}{5 \cdot 4 - 6 \cdot 1} \tag{6.53}$$

$$\frac{x}{12 - 24} = \frac{-y}{10 - 6} = \frac{z}{20 - 6} \tag{6.54}$$

$$\frac{x}{-12} = \frac{-y}{4} = \frac{z}{14} \tag{6.55}$$

$$\frac{x}{-6} = \frac{-y}{-2} = \frac{z}{7} \tag{6.56}$$

Therefore, an eigenvector corresponding to $\lambda_2 = -1$ is:

$$\vec{X}_2 = \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix} \tag{6.57}$$

Step 4: Find the eigenvector corresponding to $\lambda_3 = 4$.
For $\lambda_3 = 4$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 4-4 & 6 & 6 \\ 1 & 3-4 & 2 \\ -1 & -4 & -3-4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.58}$$

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.59}$$

Let's consider the first and second rows:

$$0x + 6y + 6z = 0 \tag{6.60}$$

$$x - y + 2z = 0 \tag{6.61}$$

Using Cramer's Rule:

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ -1 & 2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 6 \\ 1 & -1 \end{vmatrix}} \tag{6.62}$$

$$\frac{x}{18} = \frac{-y}{-6} = \frac{z}{-6} \tag{6.63}$$

Therefore, an eigenvector corresponding to $\lambda_3 = 4$ is:

$$\vec{X}_3 = \begin{bmatrix} 18 \\ 6 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \tag{6.64}$$

In summary, the eigenvalues and corresponding eigenvectors of matrix A are:

$$\lambda_1 = 1, \qquad \vec{X}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{6.65}$$

$$\lambda_2 = -1, \qquad \vec{X}_2 = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix} \tag{6.66}$$

$$\lambda_3 = 4, \qquad \vec{X}_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \tag{6.67}$$

Example 2: Finding Eigenvalues and Eigenvectors

Consider the matrix $\mathbf{A} = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors.

Solution:

First, we notice that it is a non-symmetric 3×3 matrix.

Step 1: The characteristic equation for a 3×3 matrix is given by:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \tag{6.68}$$

where:

$$S_1 = \text{Trace of } A \tag{6.69}$$

$$S_2 = \text{Sum of minors of diagonal elements} \tag{6.70}$$

$$|A| = \text{Determinant of matrix } A \tag{6.71}$$

Calculate S_1, the trace of A:

$$S_1 = 8 + (-3) + 1 = 6 \tag{6.72}$$

Calculate S_2, the sum of minors of diagonal elements:

$$S_2 = \begin{vmatrix} -3 & -2 \\ -4 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -8 \\ 4 & -3 \end{vmatrix} \tag{6.73}$$

$$= ((-3)\cdot 1 - (-2)\cdot(-4)) + (8\cdot 1 - (-2)\cdot 3) + (8\cdot(-3) - (-8)\cdot 4) \tag{6.74}$$

$$= (-3 - 8) + (8 + 6) + (-24 + 32) \tag{6.75}$$

$$= -11 + 14 + 8 \tag{6.76}$$

$$= 11 \tag{6.77}$$

Calculate $|A|$, the determinant of matrix A:

$$|A| = 8\begin{vmatrix}-3 & -2\\ -4 & 1\end{vmatrix} - (-8)\begin{vmatrix}4 & -2\\ 3 & 1\end{vmatrix} + (-2)\begin{vmatrix}4 & -3\\ 3 & -4\end{vmatrix} \tag{6.78}$$
$$= 8((-3)\cdot 1 - (-2)\cdot(-4)) + 8(4\cdot 1 - (-2)\cdot 3) + (-2)(4\cdot(-4) - (-3)\cdot 3) \tag{6.79}$$
$$= 8(-3-8) + 8(4+6) + (-2)(-16+9) \tag{6.80}$$
$$= 8(-11) + 8(10) + (-2)(-7) \tag{6.81}$$
$$= -88 + 80 + 14 \tag{6.82}$$
$$= 6 \tag{6.83}$$

so our characteristic equation is:

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \tag{6.84}$$

Let's try some values to find a root:
For $\lambda = 1$:

$$1^3 - 6(1)^2 + 11(1) - 6 = 1 - 6 + 11 - 6 = 0 \tag{6.85}$$

$\lambda = 1$ is a root.
let's use synthetic division with $\lambda = 1$:

1	1	-6	11	-6
		1	-5	6
	1	-5	6	0

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda^2 - 5\lambda + 6) \quad = 0 \tag{6.86}$$
$$= (\lambda-1)(\lambda-2)(\lambda-3) \quad = 0 \tag{6.87}$$
$$\tag{6.88}$$

And its roots are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.
Step 2: Find the eigenvector corresponding to $\lambda_1 = 1$. For $\lambda_1 = 1$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix}7 & -8 & -2\\ 4 & -4 & -2\\ 3 & -4 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \tag{6.89}$$

As $\lambda_1 = 1$ is eigenvalue with multiplicity 1, we will consider any two linearly independent rows. Let's consider the first and third rows:

$$7x - 8y - 2z = 0 \tag{6.90}$$
$$3x - 4y + 0z = 0 \tag{6.91}$$

Using Cramer's Rule to find the proportional relationships:

$$\frac{x}{\begin{vmatrix}-8 & -2\\ -4 & 0\end{vmatrix}} = \frac{y}{\begin{vmatrix}7 & -2\\ 3 & 0\end{vmatrix}} = \frac{z}{\begin{vmatrix}7 & -8\\ 3 & -4\end{vmatrix}} \tag{6.92}$$

Computing these determinants:

$$\begin{vmatrix}-8 & -2\\ -4 & 0\end{vmatrix} = (-8)(0) - (-2)(-4) \quad = 0 - 8 \ = -8 \tag{6.93}$$

$$\begin{vmatrix} 7 & -2 \\ 3 & 0 \end{vmatrix} = (7)(0) - (-2)(3) \qquad = 0 + 6 \ = 6 \tag{6.94}$$

$$\begin{vmatrix} 7 & -8 \\ 3 & -4 \end{vmatrix} = (7)(-4) - (-8)(3) \qquad = -28 + 24 \ = -4 \tag{6.95}$$

Therefore:

$$\frac{x}{-8} = \frac{y}{6} = \frac{z}{-4} \tag{6.96}$$

These ratios can be simplified:

$$\frac{x}{2} = \frac{y}{-\frac{3}{2}} = \frac{z}{1} \tag{6.97}$$

Setting $z = 1$ for convenience, we get $y = \frac{3}{2}$ and $x = 2$. Therefore, an eigenvector corresponding to $\lambda_1 = 1$ is:

$$\vec{X}_1 = \begin{bmatrix} 2 & \frac{3}{2} & 1 \end{bmatrix} \tag{6.98}$$

Step 3: Find the eigenvector corresponding to $\lambda_2 = 2$.
For $\lambda_2 = 2$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 8-2 & -8 & -2 \\ 4 & -3-2 & -2 \\ 3 & -4 & 1-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.99}$$

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.100}$$

As $\lambda_2 = 2$ is eigenvalue with multiplicity 1, we will consider any two linearly independent rows. Let's consider the first and second rows:

$$6x - 8y - 2z = 0 \tag{6.101}$$

$$4x - 5y - 2z = 0 \tag{6.102}$$

Using Cramer's Rule:

$$\frac{x}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}} \tag{6.103}$$

$$\frac{x}{(-8)(-2) - (-2)(-5)} = \frac{-y}{6(-2) - (-2)(4)} = \frac{z}{6(-5) - (-8)(4)} \tag{6.104}$$

$$\frac{x}{16 - 10} = \frac{y}{-12 + 8} = \frac{z}{-30 + 32} \tag{6.105}$$

$$\frac{x}{6} = \frac{-y}{-4} = \frac{z}{2} \tag{6.106}$$

$$\frac{x}{3} = \frac{-y}{-2} = \frac{z}{1} \tag{6.107}$$

Setting $z = 1$, we get $y = 2$ and $x = 3$.

Therefore, an eigenvector corresponding to $\lambda_2 = 2$ is:

$$\vec{X}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \tag{6.108}$$

Step 4: Find the eigenvector corresponding to $\lambda_3 = 3$. For $\lambda_3 = 3$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.109}$$

As $\lambda_3 = 3$ is eigenvalue with multiplicity 1, we will consider any two linearly independent rows. Let's consider the first and second rows:

$$5x - 8y - 2z = 0 \tag{6.110}$$

$$4x - 6y - 2z = 0 \tag{6.111}$$

Using Cramer's Rule to find the proportional relationships:

$$\frac{x}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}} \tag{6.112}$$

Computing these determinants:

$$\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix} = (-8)(-2) - (-2)(-6) \qquad = 16 - 12 \ = 4 \tag{6.113}$$

$$\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix} = (5)(-2) - (-2)(4) \qquad = -10 + 8 \ = -2 \tag{6.114}$$

$$\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix} = (5)(-6) - (-8)(4) \qquad = -30 + 32 \ = 2 \tag{6.115}$$

Therefore:

$$\frac{x}{4} = \frac{y}{-2} = \frac{z}{2} \tag{6.116}$$

Simplifying these ratios:

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{1} \tag{6.117}$$

Setting $z = 1$, we get $y = -1$ and $x = 2$. Let's verify these values in the original equations:

$$5(2) - 8(-1) - 2(1) = 10 + 8 - 2 = 16 \tag{6.118}$$

This isn't zero, indicating an error in our calculation. Let me solve this directly. Subtracting the second equation from the first:

$$x - 2y = 0 \tag{6.119}$$

So $x = 2y$. Substituting into the first equation:

$$5(2y) - 8y - 2z = 0 \; 10y - 8y - 2z = 0 \; 2y - 2z = 0 \tag{6.120}$$

So $y = z$. And since $x = 2y$, we have $x = 2z$ as well. Setting $z = 1$, we get $y = 1$ and $x = 2$. Therefore, the correct eigenvector corresponding to $\lambda_3 = 3$ is:

$$\vec{X}_3 = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \tag{6.121}$$

In summary, the eigenvalues and corresponding eigenvectors of matrix A are:

$$\lambda_1 = 1, \qquad \vec{X}_1 = \begin{bmatrix} 2 \\ \frac{3}{2} \\ 1 \end{bmatrix} \tag{6.122}$$

$$\lambda_2 = 2, \qquad \vec{X}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \tag{6.123}$$

$$\lambda_3 = 3, \qquad \vec{X}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \tag{6.124}$$

Example 3: Finding Eigenvalues and Eigenvectors

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors.

Solution:

First, we notice that it is a non-symmetric 3×3 matrix.

Step 1: The characteristic equation for a 3×3 matrix is given by:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \tag{6.125}$$

where:

$$S_1 = \text{Trace of } A \tag{6.126}$$

$$S_2 = \text{Sum of minors of diagonal elements} \tag{6.127}$$

$$|A| = \text{Determinant of matrix } A \tag{6.128}$$

Calculate S_1, the trace of A:

$$S_1 = 1 + 2 + 3 = 6 \tag{6.129}$$

Calculate S_2, the sum of minors of diagonal elements:

$$S_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \tag{6.130}$$

$$= (2 \cdot 3 - 1 \cdot 2) + (1 \cdot 3 - (-1) \cdot 2) + (1 \cdot 2 - 0 \cdot 1) \tag{6.131}$$

$$= (6 - 2) + (3 + 2) + (2 - 0) \tag{6.132}$$

$$= 4 + 5 + 2 \tag{6.133}$$

$$= 11 \tag{6.134}$$

Calculate $|A|$, the determinant of matrix A:

$$\begin{aligned}
|A| &= 1\begin{vmatrix}2 & 1\\ 2 & 3\end{vmatrix} - 0\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix} + (-1)\begin{vmatrix}1 & 2\\ 2 & 2\end{vmatrix} && (6.135)\\
&= 1(2\cdot 3 - 1\cdot 2) - 0(1\cdot 3 - 1\cdot 2) + (-1)(1\cdot 2 - 2\cdot 2) && (6.136)\\
&= 1(6-2) - 0(3-2) + (-1)(2-4) && (6.137)\\
&= 1(4) - 0(1) + (-1)(-2) && (6.138)\\
&= 4 + 0 + 2 && (6.139)\\
&= 6 && (6.140)
\end{aligned}$$

So our characteristic equation is:

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \qquad (6.141)$$

Let's try some values to find a root:
For $\lambda = 1$:

$$1^3 - 6(1)^2 + 11(1) - 6 = 1 - 6 + 11 - 6 = 0 \qquad (6.142)$$

$\lambda = 1$ is a root.
Let's use synthetic division with $\lambda = 1$:

1	1	-6	11	-6
		1	-5	6
	1	-5	6	0

So we have:

$$\begin{aligned}
\lambda^3 - 6\lambda^2 + 11\lambda - 6 &= (\lambda - 1)(\lambda^2 - 5\lambda + 6) && (6.143)\\
&= (\lambda - 1)(\lambda - 2)(\lambda - 3) && (6.144)\\
& && (6.145)
\end{aligned}$$

Therefore, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.
Step 2: Find the eigenvector corresponding to $\lambda_1 = 1$.
For $\lambda_1 = 1$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix}0 & 0 & -1\\ 1 & 1 & 1\\ 2 & 2 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \qquad (6.146)$$

Let's consider the first and second rows:

$$\begin{aligned}
0x + 0y - z &= 0 && (6.147)\\
x + y + z &= 0 && (6.148)
\end{aligned}$$

From the first equation, we get $z = 0$. Substituting this into the second equation:

$$\begin{aligned}
x + y &= 0 && (6.149)\\
x &= -y && (6.150)
\end{aligned}$$

Setting $y = 1$ for simplicity, we get $x = -1$ and $z = 0$.

Therefore, an eigenvector corresponding to $\lambda_1 = 1$ is:

$$\vec{X}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{6.151}$$

Step 3: Find the eigenvector corresponding to $\lambda_2 = 2$.
For $\lambda_2 = 2$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.152}$$

Let's consider the first and third rows:

$$-x - z = 0 \tag{6.153}$$
$$2x + 2y + z = 0 \tag{6.154}$$

Using the correct form of Cramer's Rule for homogeneous systems:

$$\frac{x}{\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}} \tag{6.155}$$

Substituting our coefficients:

$$\frac{x}{\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 0 \\ 2 & 2 \end{vmatrix}} \tag{6.156}$$

Computing these determinants:

$$\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} = 0 \cdot 1 - (-1) \cdot 2 = 0 + 2 = 2 \tag{6.157}$$
$$\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = (-1) \cdot 1 - (-1) \cdot 2 = -1 + 2 = 1 \tag{6.158}$$
$$\begin{vmatrix} -1 & 0 \\ 2 & 2 \end{vmatrix} = (-1) \cdot 2 - 0 \cdot 2 = -2 - 0 = -2 \tag{6.159}$$

Therefore:

$$\frac{x}{2} = \frac{-y}{1} = \frac{z}{-2} \tag{6.160}$$

Simplifying:

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{-2} \tag{6.161}$$

Setting $z = 2$ for simplicity, we get $y = 1$ and $x = -2$.
Therefore, an eigenvector corresponding to $\lambda_2 = 2$ is:

$$\vec{X}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \tag{6.162}$$

Step 4: Find the eigenvector corresponding to $\lambda_3 = 3$.
For $\lambda_3 = 3$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.163}$$

Let's consider the first and second rows:

$$-2x - z = 0 \tag{6.164}$$

$$x - y + z = 0 \tag{6.165}$$

Using the correct form of Cramer's Rule for homogeneous systems:

$$\frac{x}{\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \tag{6.166}$$

Substituting our coefficients:

$$\frac{x}{\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix}} \tag{6.167}$$

Computing these determinants:

$$\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = 0 \cdot 1 - (-1) \cdot (-1) = 0 - 1 = -1 \tag{6.168}$$

$$\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = (-2) \cdot 1 - (-1) \cdot 1 = -2 + 1 = -1 \tag{6.169}$$

$$\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} = (-2) \cdot (-1) - 0 \cdot 1 = 2 - 0 = 2 \tag{6.170}$$

Therefore:

$$\frac{x}{-1} = \frac{-y}{-1} = \frac{z}{2} \tag{6.171}$$

Simplifying:

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{2} \tag{6.172}$$

Therefore, the eigenvector corresponding to $\lambda_3 = 3$ is:

$$\vec{X}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \tag{6.173}$$

In summary, the eigenvalues and corresponding eigenvectors of matrix A are:

$$\lambda_1 = 1, \qquad \vec{X}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{6.174}$$

$$\lambda_2 = 2, \qquad \vec{X}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \tag{6.175}$$

$$\lambda_3 = 3, \qquad \vec{X}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \tag{6.176}$$

Geometrical Interpretation of Example 3

For the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$, and corresponding eigenvectors $\vec{X}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{X}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, and $\vec{X}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, let's examine the geometrical interpretation.

Geometrical Meaning of Eigenvalues and Eigenvectors:

When a matrix $\mathbf{A}$ operates on a vector $\vec{v}$, it generally changes both its direction and magnitude. However, eigenvectors are special vectors that maintain their direction when multiplied by the matrix, only being stretched or compressed by a factor equal to the corresponding eigenvalue.

For our matrix $\mathbf{A}$, we can visualize the action of $\mathbf{A}$ on each eigenvector:

1. **For eigenvector $\vec{X}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ with eigenvalue $\lambda_1 = 1$:**

When $\mathbf{A}$ operates on $\vec{X}_1$, the result is: $\mathbf{A}\vec{X}_1 = 1 \cdot \vec{X}_1 = \vec{X}_1$

Geometrically, this means that $\vec{X}_1$ is unchanged by the transformation. The vector $\vec{X}_1$ represents a direction in 3D space where the linear transformation represented by $\mathbf{A}$ causes neither stretching nor compression. This is a *fixed direction* of the transformation.

2. **For eigenvector $\vec{X}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ with eigenvalue $\lambda_2 = 2$:**

When $\mathbf{A}$ operates on $\vec{X}_2$, the result is: $\mathbf{A}\vec{X}_2 = 2 \cdot \vec{X}_2$

Geometrically, this means that $\vec{X}_2$ is stretched by a factor of 2 in its own direction. Any vector along the line defined by $\vec{X}_2$ will be doubled in length but maintain its direction when transformed by $\mathbf{A}$.

3. **For eigenvector $\vec{X}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ with eigenvalue $\lambda_3 = 3$:**

When $\mathbf{A}$ operates on $\vec{X}_3$, the result is: $\mathbf{A}\vec{X}_3 = 3 \cdot \vec{X}_3$

Geometrically, this means that $\vec{X}_3$ is stretched by a factor of 3 in its own direction. Any vector along the line defined by $\vec{X}_3$ will be tripled in length while maintaining its direction.

The Overall Geometric Transformation:

Since we have three distinct eigenvalues with three linearly independent eigenvectors, the matrix $\mathbf{A}$ is diagonalizable. This means the action of $\mathbf{A}$ can be geometrically understood as:

1. A pure scaling operation when viewed in the eigenbasis (the coordinate system formed by the eigenvectors).

2. In this eigenbasis, the transformation scales by factors of 1, 2, and 3 along the three principal axes defined by $\vec{X}_1$, $\vec{X}_2$, and $\vec{X}_3$, respectively.

Geometrically, the transformation represented by $\mathbf{A}$ takes a unit sphere and transforms it into an ellipsoid, where: - Along the direction of $\vec{X}_1$, there is no scaling (scale factor = 1) - Along the direction of $\vec{X}_2$, there is a doubling of length (scale factor = 2) - Along the direction of $\vec{X}_3$, there is a tripling of length (scale factor = 3)

Specific Geometric Features:

1. **Principal Axes:** The eigenvectors $\vec{X}_1$, $\vec{X}_2$, and $\vec{X}_3$ form the principal axes of the ellipsoid resulting from the transformation.

2. **Invariant Line:** The line passing through the origin in the direction of $\vec{X}_1$ remains unchanged by the transformation, as points on this line are only multiplied by 1.

3. **Maximum Stretching:** The direction of maximum stretching is along $\vec{X}_3$, where the scaling factor is 3.

4. **Volume Change:** The determinant of $\mathbf{A}$ (which equals 6) represents the factor by which volumes are scaled by the transformation. This equals the product of all eigenvalues: $1\times2\times3 = 6$.

In summary, the transformation represented by matrix $\mathbf{A}$ is a non-uniform scaling in 3D space, with different scaling factors along three principal directions defined by the eigenvectors. Understanding these eigenvectors and eigenvalues gives us a complete geometric picture of how the transformation affects any vector in 3D space.

Example 4: Finding Eigenvalues and Eigenvectors

Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors.

Solution:

First, we notice that this is a symmetric 3×3 matrix, which is a special case where all eigenvalues are real.

Step 1: The characteristic equation for a 3×3 matrix is given by:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \tag{6.177}$$

where:

$$S_1 = \text{Trace of } A \tag{6.178}$$
$$S_2 = \text{Sum of minors of diagonal elements} \tag{6.179}$$
$$|A| = \text{Determinant of matrix } A \tag{6.180}$$

Calculate S_1, the trace of A:

$$S_1 = 3 + 5 + 3 = 11 \tag{6.181}$$

Calculate S_2, the sum of minors of diagonal elements:

$$S_2 = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} \tag{6.182}$$
$$= (5 \cdot 3 - (-1) \cdot (-1)) + (3 \cdot 3 - 1 \cdot 1) + (3 \cdot 5 - (-1) \cdot (-1)) \tag{6.183}$$
$$= (15 - 1) + (9 - 1) + (15 - 1) \tag{6.184}$$
$$= 14 + 8 + 14 \tag{6.185}$$
$$= 36 \tag{6.186}$$

Calculate $|A|$, the determinant of matrix A:

$$|A| = 3\begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} -1 & 5 \\ 1 & -1 \end{vmatrix} \tag{6.187}$$
$$= 3(5 \cdot 3 - (-1) \cdot (-1)) - (-1)((-1) \cdot 3 - (-1) \cdot 1) + 1((-1) \cdot (-1) - 5 \cdot 1) \tag{6.188}$$
$$= 3(15 - 1) - (-1)(-3 - (-1)) + 1(1 - 5) \tag{6.189}$$
$$= 3(14) - (-1)(-2) + 1(-4) \tag{6.190}$$
$$= 42 - 2 - 4 \tag{6.191}$$
$$= 36 \tag{6.192}$$

So our characteristic equation is:

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0 \tag{6.193}$$

To find the roots, let's use synthetic division. Let's try $\lambda = 1$:

1	1	-11	36	-36
		1	-10	26
	1	-10	26	-10

Since the remainder is not zero, $\lambda = 1$ is not a root.
Let's try $\lambda = 2$:

2	1	-11	36	-36
		2	-18	36
	1	-9	18	0

Since the remainder is zero, $\lambda = 2$ is a root of the characteristic equation.
Now, we can factor out $(\lambda - 2)$:

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = (\lambda - 2)(\lambda^2 - 9\lambda + 18) \tag{6.194}$$

$$\tag{6.195}$$

We can factor the quadratic term:

$$\lambda^2 - 9\lambda + 18 = (\lambda - 3)(\lambda - 6) \tag{6.196}$$

Therefore, the characteristic equation can be written as:

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = (\lambda - 2)(\lambda - 3)(\lambda - 6) \tag{6.197}$$

$$\tag{6.198}$$

The eigenvalues of matrix A are $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = 6$.
Step 2: Find the eigenvector corresponding to $\lambda_1 = 2$.
For $\lambda_1 = 2$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 3-2 & -1 & 1 \\ -1 & 5-2 & -1 \\ 1 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.199}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.200}$$

We notice that the first and third rows are identical, indicating linear dependency. Let's consider the first and second rows:

$$x - y + z = 0 \tag{6.201}$$

$$-x + 3y - z = 0 \tag{6.202}$$

Using Cramer's Rule to find the proportional relationships:

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}} \tag{6.203}$$

Computing these determinants:

$$\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = (-1)(-1) - 1 \cdot 3 = 1 - 3 = -2 \tag{6.204}$$

$$\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 1 \cdot (-1) - 1 \cdot (-1) = -1 + 1 = 0 \tag{6.205}$$

$$\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = 1 \cdot 3 - (-1) \cdot (-1) = 3 - 1 = 2 \tag{6.206}$$

Since the middle determinant is zero, we cannot directly apply Cramer's Rule. Let's solve the system directly:
From the first equation:

$$x - y + z = 0 \tag{6.207}$$

$$x + z = y \tag{6.208}$$

From the second equation:

$$-x + 3y - z = 0 \tag{6.209}$$

$$3y = x + z \tag{6.210}$$

Substituting the expression for y from the first equation:

$$3(x + z) = x + z \tag{6.211}$$

$$3x + 3z = x + z \tag{6.212}$$

$$2x + 2z = 0 \tag{6.213}$$

$$x + z = 0 \tag{6.214}$$

$$x = -z \tag{6.215}$$

So we have $x = -z$ and $y = x + z = -z + z = 0$.
Let's set $z = 1$ for simplicity, which gives $x = -1$ and $y = 0$.
Therefore, an eigenvector corresponding to $\lambda_1 = 2$ is:

$$\vec{X}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{6.216}$$

Step 3: Find the eigenvector corresponding to $\lambda_2 = 3$.
For $\lambda_2 = 3$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 3-3 & -1 & 1 \\ -1 & 5-3 & -1 \\ 1 & -1 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.217}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.218}$$

Let's consider the first and third rows:

$$-y + z = 0 \tag{6.219}$$

$$x - y = 0 \tag{6.220}$$

From these equations, we get $y = z$ and $x = y$, which means $x = y = z$.
Setting $z = 1$ for simplicity, we get $x = 1$ and $y = 1$.
Let's verify this solution in the second equation:

$$-x + 2y - z = -1 + 2 \cdot 1 - 1 = -1 + 2 - 1 = 0 \tag{6.221}$$

Therefore, an eigenvector corresponding to $\lambda_2 = 3$ is:

$$\vec{X}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{6.222}$$

Step 4: Find the eigenvector corresponding to $\lambda_3 = 6$.
For $\lambda_3 = 6$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 3-6 & -1 & 1 \\ -1 & 5-6 & -1 \\ 1 & -1 & 3-6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.223}$$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.224}$$

Let's consider the first and second rows:

$$-3x - y + z = 0 \tag{6.225}$$

$$-x - y - z = 0 \tag{6.226}$$

Using Cramer's Rule to find the proportional relationships:

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}} \tag{6.227}$$

Computing these determinants:

$$\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = (-1)(-1) - 1 \cdot (-1) = 1 + 1 = 2 \tag{6.228}$$

$$\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix} = (-3)(-1) - 1 \cdot (-1) = 3 + 1 = 4 \tag{6.229}$$

$$\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix} = (-3)(-1) - (-1) \cdot (-1) = 3 - 1 = 2 \tag{6.230}$$

Therefore:

$$\frac{x}{2} = \frac{-y}{4} = \frac{z}{2} \tag{6.231}$$

Simplifying:

$$\frac{x}{1} = \frac{-y}{2} = \frac{z}{1} \tag{6.232}$$

So $x = z$ and $y = -2x$.
Setting $x = 1$ for simplicity, we get $y = -2$ and $z = 1$.

Therefore, an eigenvector corresponding to $\lambda_3 = 6$ is:

$$\vec{X}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \tag{6.233}$$

In summary, the eigenvalues and corresponding eigenvectors of matrix A are:

$$\lambda_1 = 2, \qquad \vec{X}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{6.234}$$

$$\lambda_2 = 3, \qquad \vec{X}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{6.235}$$

$$\lambda_3 = 6, \qquad \vec{X}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \tag{6.236}$$

Example 5: Finding Eigenvalues and Eigenvectors

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors.

Solution:

First, we notice that this is a symmetric 3×3 matrix, which means all eigenvalues are real and eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

Step 1: The characteristic equation for a 3×3 matrix is given by:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \tag{6.237}$$

where:

$$S_1 = \text{Trace of } A \tag{6.238}$$
$$S_2 = \text{Sum of minors of diagonal elements} \tag{6.239}$$
$$|A| = \text{Determinant of matrix } A \tag{6.240}$$

Calculate S_1, the trace of A:

$$S_1 = 1 + 5 + 3 = 9 \tag{6.241}$$

Calculate S_2, the sum of minors of diagonal elements:

$$S_2 = \begin{vmatrix} 5 & 4 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix} \tag{6.242}$$
$$= (5 \cdot 3 - 4 \cdot 4) + (1 \cdot 3 - (-4) \cdot (-4)) + (1 \cdot 5 - 0 \cdot 0) \tag{6.243}$$
$$= (15 - 16) + (3 - 16) + (5 - 0) \tag{6.244}$$
$$= -1 - 13 + 5 \tag{6.245}$$
$$= -9 \tag{6.246}$$

Calculate $|A|$, the determinant of matrix A:

$$\begin{aligned}
|A| &= 1\begin{vmatrix}5 & 4\\4 & 3\end{vmatrix} - 0\begin{vmatrix}0 & 4\\-4 & 3\end{vmatrix} + (-4)\begin{vmatrix}0 & 5\\-4 & 4\end{vmatrix} && (6.247)\\
&= 1(5\cdot 3 - 4\cdot 4) - 0(0\cdot 3 - 4\cdot(-4)) + (-4)(0\cdot 4 - 5\cdot(-4)) && (6.248)\\
&= 1(15-16) - 0(0+16) + (-4)(0+20) && (6.249)\\
&= 1(-1) - 0(16) + (-4)(20) && (6.250)\\
&= -1 - 0 - 80 && (6.251)\\
&= -81 && (6.252)
\end{aligned}$$

So our characteristic equation is:

$$\lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0 \tag{6.253}$$

To find the roots, let's factor this polynomial. Let's try $\lambda = -3$:

-3	1	-9	-9	81
		-3	36	-81
	1	-12	27	0

Since the remainder is zero, $\lambda = -3$ is a root of the characteristic equation.
Now, we can factor out $(\lambda + 3)$:

$$\lambda^3 - 9\lambda^2 - 9\lambda + 81 = (\lambda+3)(\lambda^2 - 12\lambda + 27) \tag{6.254}$$
$$\tag{6.255}$$

We can factor the quadratic term using the quadratic formula:

$$\begin{aligned}
\lambda &= \frac{12 \pm \sqrt{144-108}}{2} && (6.256)\\
&= \frac{12 \pm \sqrt{36}}{2} && (6.257)\\
&= \frac{12 \pm 6}{2} && (6.258)\\
&= 6 \pm 3 && (6.259)
\end{aligned}$$

This gives us $\lambda = 9$ and $\lambda = 3$.
Therefore, the characteristic equation can be written as:

$$\lambda^3 - 9\lambda^2 - 9\lambda + 81 = (\lambda+3)(\lambda-9)(\lambda-3) \tag{6.260}$$
$$\tag{6.261}$$

The eigenvalues of matrix A are $\lambda_1 = -3$, $\lambda_2 = 3$, and $\lambda_3 = 9$.
Step 2: Find the eigenvector corresponding to $\lambda_1 = -3$.
For $\lambda_1 = -3$, we form the system $(A - \lambda I)X = 0$:

$$\begin{aligned}
\begin{bmatrix}1-(-3) & 0 & -4\\ 0 & 5-(-3) & 4\\ -4 & 4 & 3-(-3)\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} &= \begin{bmatrix}0\\0\\0\end{bmatrix} && (6.262)\\
\begin{bmatrix}4 & 0 & -4\\ 0 & 8 & 4\\ -4 & 4 & 6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} &= \begin{bmatrix}0\\0\\0\end{bmatrix} && (6.263)
\end{aligned}$$

Let's consider the first and third rows:

$$4x - 4z = 0 \tag{6.264}$$
$$-4x + 4y + 6z = 0 \tag{6.265}$$

From the first equation, we get $x = z$. Substituting into the second equation:

$$-4z + 4y + 6z = 0 \tag{6.266}$$
$$4y + 2z = 0 \tag{6.267}$$
$$y = -\frac{z}{2} \tag{6.268}$$

So we have $x = z$ and $y = -\frac{z}{2}$.
Setting $z = 2$ for simplicity (to get integer values), we get $x = 2$ and $y = -1$.
Therefore, an eigenvector corresponding to $\lambda_1 = -3$ is:

$$\vec{X}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \tag{6.269}$$

Step 3: Find the eigenvector corresponding to $\lambda_2 = 3$.
For $\lambda_2 = 3$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 1-3 & 0 & -4 \\ 0 & 5-3 & 4 \\ -4 & 4 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.270}$$
$$\begin{bmatrix} -2 & 0 & -4 \\ 0 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.271}$$

Let's consider the first and third rows:

$$-2x - 4z = 0 \tag{6.272}$$
$$-4x + 4y = 0 \tag{6.273}$$

Using Cramer's Rule to find the proportional relationships:

$$\frac{x}{\begin{vmatrix} 0 & -4 \\ 4 & 0 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & -4 \\ -4 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 0 \\ -4 & 4 \end{vmatrix}} \tag{6.274}$$

Computing these determinants:

$$\begin{vmatrix} 0 & -4 \\ 4 & 0 \end{vmatrix} = 0 \cdot 0 - (-4) \cdot 4 = 0 + 16 = 16 \tag{6.275}$$
$$\begin{vmatrix} -2 & -4 \\ -4 & 0 \end{vmatrix} = (-2) \cdot 0 - (-4) \cdot (-4) = 0 - 16 = -16 \tag{6.276}$$
$$\begin{vmatrix} -2 & 0 \\ -4 & 4 \end{vmatrix} = (-2) \cdot 4 - 0 \cdot (-4) = -8 - 0 = -8 \tag{6.277}$$

Therefore:

$$\frac{x}{16} = \frac{-y}{-16} = \frac{z}{-8} \tag{6.278}$$

Simplifying:

$$\frac{x}{16} = \frac{y}{16} = \frac{z}{-8} = \frac{-z}{8} \tag{6.279}$$

Further simplifying:

$$\frac{x}{2} = \frac{y}{2} = \frac{-z}{1} \tag{6.280}$$

So $x = 2k$, $y = 2k$, and $z = -k$ for some constant k.
Setting $k = 1$ for simplicity, we get $x = 2$, $y = 2$, and $z = -1$.
Therefore, an eigenvector corresponding to $\lambda_2 = 3$ is:

$$\vec{X}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \tag{6.281}$$

Step 4: Find the eigenvector corresponding to $\lambda_3 = 9$.
For $\lambda_3 = 9$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 1-9 & 0 & -4 \\ 0 & 5-9 & 4 \\ -4 & 4 & 3-9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.282}$$

$$\begin{bmatrix} -8 & 0 & -4 \\ 0 & -4 & 4 \\ -4 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.283}$$

Let's consider the first and second rows:

$$-8x - 4z = 0 \tag{6.284}$$

$$-4y + 4z = 0 \tag{6.285}$$

From these equations, we get $x = -\frac{z}{2}$ and $y = z$.
Setting $z = 1$ for simplicity, we get $x = -\frac{1}{2}$ and $y = 1$.
Therefore, an eigenvector corresponding to $\lambda_3 = 9$ is:

$$\vec{X}_3 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} \tag{6.286}$$

For simplicity and to work with integer values, we can multiply this vector by 2:

$$\vec{X}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \tag{6.287}$$

Verification of Orthogonality:
Since A is a symmetric matrix, eigenvectors corresponding to distinct eigenvalues should be orthogonal to each other. Let's verify:
$\vec{X}_1 \cdot \vec{X}_2 = 2 \cdot 2 + (-1) \cdot 2 + 2 \cdot (-1) = 4 - 2 - 2 = 0$
$\vec{X}_1 \cdot \vec{X}_3 = 2 \cdot (-1) + (-1) \cdot 2 + 2 \cdot 2 = -2 - 2 + 4 = 0$
$\vec{X}_2 \cdot \vec{X}_3 = 2 \cdot (-1) + 2 \cdot 2 + (-1) \cdot 2 = -2 + 4 - 2 = 0$

This confirms our calculations and illustrates an important property of symmetric matrices.
In summary, the eigenvalues and corresponding eigenvectors of matrix A are:

$$\lambda_1 = -3, \qquad \vec{X}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \tag{6.288}$$

$$\lambda_2 = 3, \qquad \vec{X}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \tag{6.289}$$

$$\lambda_3 = 9, \qquad \vec{X}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \tag{6.290}$$

Example 6: Finding Eigenvalues and Eigenvectors

Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors.

Solution:
First, we notice that this is a symmetric 3×3 matrix, which means all eigenvalues are real and eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.
Step 1: The characteristic equation for a 3×3 matrix is given by:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \tag{6.291}$$

where:

$$S_1 = \text{Trace of } A \tag{6.292}$$
$$S_2 = \text{Sum of minors of diagonal elements} \tag{6.293}$$
$$|A| = \text{Determinant of matrix } A \tag{6.294}$$

Calculate S_1, the trace of A:

$$S_1 = 3 + 3 + 3 = 9 \tag{6.295}$$

Calculate S_2, the sum of minors of diagonal elements:

$$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \tag{6.296}$$
$$= (3 \cdot 3 - (-1) \cdot (-1)) + (3 \cdot 3 - 1 \cdot 1) + (3 \cdot 3 - 1 \cdot 1) \tag{6.297}$$
$$= (9 - 1) + (9 - 1) + (9 - 1) \tag{6.298}$$
$$= 8 + 8 + 8 \tag{6.299}$$
$$= 24 \tag{6.300}$$

Calculate $|A|$, the determinant of matrix A:

$$|A| = 3\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} \tag{6.301}$$
$$= 3(3 \cdot 3 - (-1) \cdot (-1)) - 1(1 \cdot 3 - (-1) \cdot 1) + 1(1 \cdot (-1) - 3 \cdot 1) \tag{6.302}$$
$$= 3(9 - 1) - 1(3 + 1) + 1(-1 - 3) \tag{6.303}$$
$$= 3(8) - 1(4) + 1(-4) \tag{6.304}$$
$$= 24 - 4 - 4 \tag{6.305}$$
$$= 16 \tag{6.306}$$

So our characteristic equation is:

$$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0 \tag{6.307}$$

To find the roots, let's try some values. Let's try $\lambda = 1$:

1	1	-9	24	-16
		1	-8	16
	1	-8	16	0

Since the remainder is zero, $\lambda = 1$ is a root. Now, we can factor out $(\lambda - 1)$:

$$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = (\lambda - 1)(\lambda^2 - 8\lambda + 16) \tag{6.308}$$
$$= (\lambda - 1)(\lambda - 4)^2 \tag{6.309}$$

Therefore, the eigenvalues of matrix A are $\lambda_1 = 1$ (with algebraic multiplicity 1) and $\lambda_2 = 4$ (with algebraic multiplicity 2).
Step 2: Find the eigenvector corresponding to $\lambda_1 = 1$.
For $\lambda_1 = 1$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 3-1 & 1 & 1 \\ 1 & 3-1 & -1 \\ 1 & -1 & 3-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.310}$$
$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.311}$$

Let's consider the first and second rows:

$$2x + y + z = 0 \tag{6.312}$$
$$x + 2y - z = 0 \tag{6.313}$$

Using Cramer's Rule to find the proportional relationships:

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}} \tag{6.314}$$

Computing these determinants:

$$\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 1 \cdot (-1) - 1 \cdot 2 = -1 - 2 = -3 \tag{6.315}$$
$$\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 2 \cdot (-1) - 1 \cdot 1 = -2 - 1 = -3 \tag{6.316}$$
$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot 1 = 4 - 1 = 3 \tag{6.317}$$

Therefore:

$$\frac{x}{-3} = \frac{-y}{-3} = \frac{z}{3} \tag{6.318}$$

Simplifying:

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{1} \tag{6.319}$$

So $x = -z$ and $y = -z$.
Setting $z = 1$ for simplicity, we get $x = -1$ and $y = -1$.
Therefore, an eigenvector corresponding to $\lambda_1 = 1$ is:

$$\vec{X}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \tag{6.320}$$

Step 3: Find the eigenvectors corresponding to $\lambda_2 = 4$.
For $\lambda_2 = 4$, we form the system $(A - \lambda I)X = 0$:

$$\begin{bmatrix} 3-4 & 1 & 1 \\ 1 & 3-4 & -1 \\ 1 & -1 & 3-4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.321}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.322}$$

We notice that the second and third rows are identical, indicating linear dependency. Let's consider the first and second rows:

$$-x + y + z = 0 \tag{6.323}$$

$$x - y - z = 0 \tag{6.324}$$

Adding these equations:

$$0 = 0 \tag{6.325}$$

This means the equations are linearly dependent as well. In fact, the second equation is just the negative of the first. So we have only one independent equation:

$$-x + y + z = 0 \tag{6.326}$$

$$y + z = x \tag{6.327}$$

Since $\lambda_2 = 4$ has algebraic multiplicity 2, we expect to find two linearly independent eigenvectors corresponding to this eigenvalue. For a symmetric matrix, we can find orthogonal eigenvectors for this eigenspace.
Let's find a basis for the eigenspace. We have one constraint: $y + z = x$. This means we have two free variables, so the eigenspace has dimension 2, as expected.
We can choose two linearly independent vectors that satisfy this constraint. Let's set $y = 1$ and $z = 0$, which gives $x = 1$. This gives us our first eigenvector:

$$\vec{X}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{6.328}$$

For the second eigenvector, let's set $y = 0$ and $z = 1$, which gives $x = 1$. This gives us:

$$\vec{X}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{6.329}$$

Let's verify that these vectors are eigenvectors and are orthogonal to $\vec{X}_1$:
$\vec{X}_1 \cdot \vec{X}_2 = (-1) \cdot 1 + (-1) \cdot 1 + 1 \cdot 0 = -1 - 1 + 0 = -2$
This is not zero, which means $\vec{X}_2$ is not orthogonal to $\vec{X}_1$. Let's modify our approach.
Since we have the constraint $y + z = x$, let's set $y = 1$ and $z = 0$, which gives $x = 1$. This gives us:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{6.330}$$

And for $y = 0$ and $z = 1$, we get $x = 1$, which gives us:

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{6.331}$$

These two vectors form a basis for the eigenspace corresponding to $\lambda_2 = 4$. However, they are not orthogonal to $\vec{X}_1$.
Let's use the Gram-Schmidt process to construct an orthogonal basis for the eigenspace that is also orthogonal to $\vec{X}_1$.
We know $\vec{X}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.
Let's verify if $\vec{v}_1$ and $\vec{v}_2$ are orthogonal to each other:
$\vec{v}_1 \cdot \vec{v}_2 = 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 1$
They are not orthogonal. Let's apply the Gram-Schmidt process to create an orthogonal basis.
First, we'll use $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as our first basis vector.
For the second basis vector, we project $\vec{v}_2$ onto $\vec{v}_1$ and subtract:

$$\vec{v}_2' = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 \tag{6.332}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1 \cdot 1 + 0 \cdot 1 + 1 \cdot 0}{1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{6.333}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{6.334}$$

$$= \begin{bmatrix} 1 - \frac{1}{2} \\ 0 - \frac{1}{2} \\ 1 - 0 \end{bmatrix} \tag{6.335}$$

$$= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \tag{6.336}$$

Now we have two orthogonal vectors in the eigenspace of $\lambda_2 = 4$:

$$\vec{X}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{6.337}$$

$$\vec{X}_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \tag{6.338}$$

Let's verify that they are eigenvectors and orthogonal to each other:
$\vec{X}_2 \cdot \vec{X}_3 = 1 \cdot \frac{1}{2} + 1 \cdot (-\frac{1}{2}) + 0 \cdot 1 = \frac{1}{2} - \frac{1}{2} + 0 = 0$
For clarity, let's multiply $\vec{X}_3$ by 2 to get integer values:

$$\vec{X}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \tag{6.339}$$

Let's verify that $\vec{X}_2$ and $\vec{X}_3$ are eigenvectors corresponding to $\lambda_2 = 4$:
For $\vec{X}_2$:

$$\mathbf{A}\vec{X}_2 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{6.340}$$

$$= \begin{bmatrix} 3 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 + 3 \cdot 1 + (-1) \cdot 0 \\ 1 \cdot 1 + (-1) \cdot 1 + 3 \cdot 0 \end{bmatrix} \tag{6.341}$$

$$= \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} \tag{6.342}$$

$$= 4 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{6.343}$$

$$= \lambda_2 \vec{X}_2 \tag{6.344}$$

For $\vec{X}_3$:

$$\mathbf{A}\vec{X}_3 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \tag{6.345}$$

$$= \begin{bmatrix} 3 \cdot 1 + 1 \cdot (-1) + 1 \cdot 2 \\ 1 \cdot 1 + 3 \cdot (-1) + (-1) \cdot 2 \\ 1 \cdot 1 + (-1) \cdot (-1) + 3 \cdot 2 \end{bmatrix} \tag{6.346}$$

$$= \begin{bmatrix} 3 - 1 + 2 \\ 1 - 3 - 2 \\ 1 + 1 + 6 \end{bmatrix} \tag{6.347}$$

$$= \begin{bmatrix} 4 \\ -4 \\ 8 \end{bmatrix} \tag{6.348}$$

$$= 4 \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \tag{6.349}$$

$$= \lambda_2 \vec{X}_3 \tag{6.350}$$

Finally, let's check if $\vec{X}_1$ is orthogonal to both $\vec{X}_2$ and $\vec{X}_3$:
$\vec{X}_1 \cdot \vec{X}_2 = (-1) \cdot 1 + (-1) \cdot 1 + 1 \cdot 0 = -1 - 1 + 0 = -2$
This is not zero, indicating that $\vec{X}_1$ and $\vec{X}_2$ are not orthogonal. Let's recalculate $\vec{X}_1$.
For $\lambda_1 = 1$, we have the equations:

$$2x + y + z = 0 \tag{6.351}$$

$$x + 2y - z = 0 \tag{6.352}$$

$$x - y + 2z = 0 \tag{6.353}$$

Adding the first two equations:

$$3x + 3y = 0 \tag{6.354}$$

$$x + y = 0 \tag{6.355}$$

$$x = -y \tag{6.356}$$

Substituting into the third equation:

$$(-y) - y + 2z = 0 \tag{6.357}$$

$$-2y + 2z = 0 \tag{6.358}$$

$$y = z \tag{6.359}$$

So we have $x = -y$ and $y = z$, which means $x = -z$.
Setting $z = 1$ for simplicity, we get $y = 1$ and $x = -1$.
Therefore, the correct eigenvector corresponding to $\lambda_1 = 1$ is:

$$\vec{X}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \tag{6.360}$$

Let's check if this is orthogonal to $\vec{X}_2$ and $\vec{X}_3$:

$\vec{X}_1 \cdot \vec{X}_2 = (-1) \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = -1 + 1 + 0 = 0$
$\vec{X}_1 \cdot \vec{X}_3 = (-1) \cdot 1 + 1 \cdot (-1) + 1 \cdot 2 = -1 - 1 + 2 = 0$
In summary, the eigenvalues and corresponding eigenvectors of matrix A are:

$$\lambda_1 = 1, \qquad \vec{X}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \tag{6.361}$$

$$\lambda_2 = 4 \text{ (with multiplicity 2)}, \qquad \vec{X}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{X}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \tag{6.362}$$

6.3.2 Type II: Asymmetric Matrix with Repeated Eigenvalues

Remark 6.10. *For an asymmetric matrix with a repeated eigenvalue $\lambda = m$ with algebraic multiplicity k, the number of linearly independent eigenvectors corresponding to this eigenvalue may be less than or equal to k. Specifically, there exist $(n - r)$ linearly independent eigenvectors, where r is the rank of the matrix $[\mathbf{A} - m\mathbf{I}]$.*

Example 1: Finding Eigenvalues and Eigenvectors for an Asymmetric Matrix with Repeated Eigenvalues

Consider the matrix $\mathbf{A} = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors.

Solution:
We first observe that the given matrix is asymmetric (not symmetric).
Step 1: Finding the Eigenvalues
The characteristic equation for a 3×3 matrix is given by:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \tag{6.363}$$

where:

$$S_1 = \text{Trace of } A \tag{6.364}$$
$$S_2 = \text{Sum of minors of diagonal elements} \tag{6.365}$$
$$|A| = \text{Determinant of matrix } A \tag{6.366}$$

Calculate S_1, the trace of A:

$$S_1 = -9 + 3 + 7 = 1 \tag{6.367}$$

Calculate S_2, the sum of minors of diagonal elements:

$$S_2 = \begin{vmatrix} 3 & 4 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -16 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -8 & 3 \end{vmatrix} \tag{6.368}$$
$$= (3 \cdot 7 - 4 \cdot 8) + (-9 \cdot 7 - 4 \cdot (-16)) + (-9 \cdot 3 - 4 \cdot (-8)) \tag{6.369}$$
$$= (21 - 32) + (-63 + 64) + (-27 + 32) \tag{6.370}$$
$$= -11 + 1 + 5 \tag{6.371}$$
$$= -5 \tag{6.372}$$

Calculate $|A|$, the determinant of matrix A:

$$\begin{aligned}
|A| &= -9\begin{vmatrix}3 & 4\\ 8 & 7\end{vmatrix} - 4\begin{vmatrix}-8 & 4\\ -16 & 7\end{vmatrix} + 4\begin{vmatrix}-8 & 3\\ -16 & 8\end{vmatrix} && (6.373)\\
&= -9(21-32) - 4(-56+64) + 4(-64+48) && (6.374)\\
&= -9(-11) - 4(8) + 4(-16) && (6.375)\\
&= 99 - 32 - 64 && (6.376)\\
&= 3 && (6.377)
\end{aligned}$$

So our characteristic equation is:

$$\lambda^3 - \lambda^2 - 5\lambda - 3 = 0 \qquad (6.378)$$

Let's check if $\lambda = 1$ is a root:

$$1^3 - 1^2 - 5(1) - 3 = 1 - 1 - 5 - 3 = -8 \neq 0 \qquad (6.379)$$

So $\lambda = 1$ is not a root.
Let's check if $\lambda = -1$ is a root:

$$(-1)^3 - (-1)^2 - 5(-1) - 3 = -1 - 1 + 5 - 3 = 0 \qquad (6.380)$$

So $\lambda = -1$ is a root.
Using synthetic division with $\lambda = -1$:

-1	1	-1	-5	-3
		-1	2	3
	1	-2	-3	0

The quotient is $\lambda^2 - 2\lambda - 3$, which can be factored as $(\lambda - 3)(\lambda + 1)$.
So we have:

$$\begin{aligned}
\lambda^3 - \lambda^2 - 5\lambda - 3 &= (\lambda+1)(\lambda^2 - 2\lambda - 3) && (6.381)\\
&= (\lambda+1)(\lambda-3)(\lambda+1) && (6.382)\\
&= (\lambda+1)^2(\lambda-3) && (6.383)
\end{aligned}$$

Therefore, the eigenvalues of matrix A are:

$$\begin{aligned}
\lambda_1 &= 3 \text{ (with algebraic multiplicity 1)} && (6.384)\\
\lambda_2 &= -1 \text{ (with algebraic multiplicity 2)} && (6.385)
\end{aligned}$$

Step 2: Finding the Eigenvectors
(i) For eigenvalue $\lambda_1 = 3$:
For $\lambda_1 = 3$, we form the system $(\mathbf{A} - \lambda_1\mathbf{I})\vec{x} = \vec{0}$:

$$\begin{bmatrix}-9-3 & 4 & 4\\ -8 & 3-3 & 4\\ -16 & 8 & 7-3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \qquad (6.386)$$

$$\begin{bmatrix}-12 & 4 & 4\\ -8 & 0 & 4\\ -16 & 8 & 4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \qquad (6.387)$$

The eigenvalue $\lambda_1 = 3$ is not repeated (algebraic multiplicity 1), so we need to consider any two linearly independent rows. Let's consider the second and third rows:

$$-8x + 0y + 4z = 0 \tag{6.388}$$
$$-16x + 8y + 4z = 0 \tag{6.389}$$

Using Cramer's Rule to find the proportional relationships:

$$\frac{x}{\begin{vmatrix} 0 & 4 \\ 8 & 4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -8 & 4 \\ -16 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -8 & 0 \\ -16 & 8 \end{vmatrix}} \tag{6.390}$$

Computing these determinants:

$$\begin{vmatrix} 0 & 4 \\ 8 & 4 \end{vmatrix} = 0 \cdot 4 - 4 \cdot 8 = -32 \tag{6.391}$$
$$\begin{vmatrix} -8 & 4 \\ -16 & 4 \end{vmatrix} = (-8) \cdot 4 - 4 \cdot (-16) = -32 + 64 = 32 \tag{6.392}$$
$$\begin{vmatrix} -8 & 0 \\ -16 & 8 \end{vmatrix} = (-8) \cdot 8 - 0 \cdot (-16) = -64 \tag{6.393}$$

Therefore:

$$\frac{x}{-32} = \frac{-y}{32} = \frac{z}{-64} \tag{6.394}$$

Simplifying:

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{2} \tag{6.395}$$

Setting $z = 2$ for simplicity, we get $x = 1$ and $y = 1$.
Therefore, an eigenvector corresponding to $\lambda_1 = 3$ is:

$$\vec{X}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \tag{6.396}$$

(ii) For eigenvalue $\lambda_2 = -1$ (with algebraic multiplicity 2):
For $\lambda_2 = -1$, we form the system $(\mathbf{A} - \lambda_2 \mathbf{I})\vec{x} = \vec{0}$:

$$\begin{bmatrix} -9-(-1) & 4 & 4 \\ -8 & 3-(-1) & 4 \\ -16 & 8 & 7-(-1) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.397}$$
$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.398}$$

We observe that the first and second rows are identical, and the third row is a scalar multiple ($2\times$) of the first row. This means the rank of $[\mathbf{A} - (-1)\mathbf{I}] = [\mathbf{A} + \mathbf{I}]$ is 1.
Since $n = 3$ and $r = 1$, we have $n - r = 2$ linearly independent eigenvectors corresponding to $\lambda_2 = -1$, which matches its algebraic multiplicity.

From the first row, we have:

$$-8x + 4y + 4z = 0 \quad (6.399)$$
$$-8x + 4(y + z) = 0 \quad (6.400)$$
$$-8x + 4(y + z) = 0 \quad (6.401)$$
$$-2x + (y + z) = 0 \quad (6.402)$$
$$2x = y + z \quad (6.403)$$
$$(6.404)$$

To find two linearly independent eigenvectors, we can set:
Case 1: $y = 0, z = 1$ gives $x = \frac{1}{2}$

$$\vec{X}_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \quad (6.405)$$

Case 2: $y = 1, z = 0$ gives $x = \frac{1}{2}$

$$\vec{X}_3 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \quad (6.406)$$

Thus, the eigenvalues and corresponding eigenvectors of matrix $\mathbf{A}$ are:

$$\lambda_1 = 3 \text{ (with algebraic multiplicity 1)}, \qquad \vec{X}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad (6.407)$$

$$\lambda_2 = -1 \text{ (with algebraic multiplicity 2)}, \qquad \vec{X}_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}, \vec{X}_3 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \quad (6.408)$$

Remark 6.11. *In this example, we observed that the algebraic multiplicity of $\lambda_2 = -1$ is 2, and we were able to find exactly 2 linearly independent eigenvectors corresponding to this eigenvalue. This occurred because the rank of $[\mathbf{A} + \mathbf{I}]$ is 1, leading to $n - r = 3 - 1 = 2$ linearly independent eigenvectors.*

For asymmetric matrices, the number of linearly independent eigenvectors (geometric multiplicity) corresponding to a repeated eigenvalue can be less than its algebraic multiplicity. However, in this specific example, they are equal.

A key insight for Type II problems is to determine the rank of $[\mathbf{A} - \lambda\mathbf{I}]$ for each repeated eigenvalue λ. This rank determines how many linearly independent eigenvectors exist for that eigenvalue.

Example 2: Finding Eigenvalues and Eigenvectors for an Asymmetric Matrix with Repeated Eigenvalues

Consider the matrix $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors.

Solution:

We first observe that the given matrix is asymmetric.

Step 1: Finding the Eigenvalues

The characteristic equation for a 3×3 matrix is given by:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \tag{6.409}$$

where:

$$S_1 = \text{Trace of } A \tag{6.410}$$
$$S_2 = \text{Sum of minors of diagonal elements} \tag{6.411}$$
$$|A| = \text{Determinant of matrix } A \tag{6.412}$$

Calculate S_1, the trace of A:

$$S_1 = -2 + 1 + 0 = -1 \tag{6.413}$$

Calculate S_2, the sum of minors of diagonal elements:

$$S_2 = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} \tag{6.414}$$
$$= (1 \cdot 0 - (-6) \cdot (-2)) + ((-2) \cdot 0 - (-3) \cdot (-1)) + ((-2) \cdot 1 - 2 \cdot 2) \tag{6.415}$$
$$= (0 - 12) + (0 - 3) + (-2 - 4) \tag{6.416}$$
$$= -12 - 3 - 6 \tag{6.417}$$
$$= -21 \tag{6.418}$$

Calculate $|A|$, the determinant of matrix A:

$$|A| = -2\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ -1 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} \tag{6.419}$$
$$= -2(1 \cdot 0 - (-6) \cdot (-2)) - 2(2 \cdot 0 - (-6) \cdot (-1)) + (-3)(2 \cdot (-2) - 1 \cdot (-1)) \tag{6.420}$$
$$= -2(0 - 12) - 2(0 - 6) + (-3)(-4 + 1) \tag{6.421}$$
$$= -2(-12) - 2(-6) + (-3)(-3) \tag{6.422}$$
$$= 24 + 12 + 9 \tag{6.423}$$
$$= 45 \tag{6.424}$$

So our characteristic equation is:

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \tag{6.425}$$

To find the roots, let's try some values. Let's try $\lambda = -3$:

$$(-3)^3 + (-3)^2 - 21(-3) - 45 = -27 + 9 + 63 - 45 \tag{6.426}$$
$$= -18 + 18 \tag{6.427}$$
$$= 0 \tag{6.428}$$

So $\lambda = -3$ is a root. Let's use synthetic division with $\lambda = -3$:

-3	1	1	-21	-45
		-3	6	45
	1	-2	-15	0

The quotient is $\lambda^2 - 2\lambda - 15$, which can be factored as $(\lambda - 5)(\lambda + 3)$.
So we have:

$$\begin{aligned}
\lambda^3 + \lambda^2 - 21\lambda - 45 &= (\lambda + 3)(\lambda^2 - 2\lambda - 15) && (6.429)\\
&= (\lambda + 3)(\lambda - 5)(\lambda + 3) && (6.430)\\
&= (\lambda + 3)^2(\lambda - 5) && (6.431)
\end{aligned}$$

Therefore, the eigenvalues of matrix A are:

$$\begin{aligned}
\lambda_1 &= 5 \text{ (with algebraic multiplicity 1)} && (6.432)\\
\lambda_2 &= -3 \text{ (with algebraic multiplicity 2)} && (6.433)
\end{aligned}$$

Step 2: Finding the Eigenvectors
(i) For eigenvalue $\lambda_1 = 5$:
For $\lambda_1 = 5$, we form the system $(\mathbf{A} - \lambda_1\mathbf{I})\vec{x} = \vec{0}$:

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (6.434)$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (6.435)$$

The eigenvalue $\lambda_1 = 5$ is not repeated (algebraic multiplicity 1), so we need to consider any two linearly independent rows. Let's consider the first and third rows:

$$\begin{aligned}
-7x + 2y - 3z &= 0 && (6.436)\\
-x - 2y - 5z &= 0 && (6.437)
\end{aligned}$$

Using Cramer's Rule to find the proportional relationships:

$$\frac{x}{\begin{vmatrix} 2 & -3 \\ -2 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -7 & -3 \\ -1 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & 2 \\ -1 & -2 \end{vmatrix}} \quad (6.438)$$

Computing these determinants:

$$\begin{aligned}
\begin{vmatrix} 2 & -3 \\ -2 & -5 \end{vmatrix} &= 2 \cdot (-5) - (-3) \cdot (-2) = -10 - 6 = -16 && (6.439)\\
\begin{vmatrix} -7 & -3 \\ -1 & -5 \end{vmatrix} &= (-7) \cdot (-5) - (-3) \cdot (-1) = 35 - 3 = 32 && (6.440)\\
\begin{vmatrix} -7 & 2 \\ -1 & -2 \end{vmatrix} &= (-7) \cdot (-2) - 2 \cdot (-1) = 14 + 2 = 16 && (6.441)
\end{aligned}$$

Therefore:

$$\frac{x}{-16} = \frac{-y}{32} = \frac{z}{16} \quad (6.442)$$

Simplifying:

$$\frac{x}{-1} = \frac{-y}{2} = \frac{z}{1} \quad (6.443)$$

Further simplifying:

$$\frac{x}{-1} = \frac{y}{-2} = \frac{z}{1} \tag{6.444}$$

Setting $z = 1$ for simplicity, we get $x = -1$ and $y = -2$.
Therefore, an eigenvector corresponding to $\lambda_1 = 5$ is:

$$\vec{X}_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \tag{6.445}$$

(ii) For eigenvalue $\lambda_2 = -3$ (with algebraic multiplicity 2):
For $\lambda_2 = -3$, we form the system $(\mathbf{A} - \lambda_2 \mathbf{I})\vec{x} = \vec{0}$:

$$\begin{bmatrix} -2-(-3) & 2 & -3 \\ 2 & 1-(-3) & -6 \\ -1 & -2 & 0-(-3) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.446}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.447}$$

We observe that the second row is 2 times the first row, and the third row is -1 times the first row. This means the rank of $[\mathbf{A} - (-3)\mathbf{I}] = [\mathbf{A} + 3\mathbf{I}]$ is 1.
Since $n = 3$ and $r = 1$, we have $n - r = 2$ linearly independent eigenvectors corresponding to $\lambda_2 = -3$, which matches its algebraic multiplicity.
From the first row, we have:

$$x + 2y - 3z = 0 \tag{6.448}$$

$$x + 2y = 3z \tag{6.449}$$

To find two linearly independent eigenvectors, we can set:
Case 1: $y = 0, z = 1$ gives $x = 3$

$$\vec{X}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \tag{6.450}$$

Case 2: $y = 1, z = 1$ gives $x = 1$

$$\vec{X}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{6.451}$$

Verification: Let's verify that $\vec{X}_2$ and $\vec{X}_3$ are linearly independent by checking if one is a scalar multiple of the other:

$$\vec{X}_3 = k\vec{X}_2 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = k \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \tag{6.452}$$

This equation has no solution for k since $1 \neq k \cdot 0$, confirming that $\vec{X}_2$ and $\vec{X}_3$ are linearly independent.

Thus, the eigenvalues and corresponding eigenvectors of matrix **A** are:

$$\lambda_1 = 5 \text{ (with algebraic multiplicity 1)}, \qquad \vec{X}_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \tag{6.453}$$

$$\lambda_2 = -3 \text{ (with algebraic multiplicity 2)}, \qquad \vec{X}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \vec{X}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{6.454}$$

Remark 6.12. *In this example, the eigenvalue $\lambda_2 = -3$ has an algebraic multiplicity of 2, and we were able to find exactly 2 linearly independent eigenvectors. This occurred because the rank of $[\mathbf{A} + 3\mathbf{I}]$ is 1, leading to $n - r = 3 - 1 = 2$ linearly independent eigenvectors.*

Example 3: Finding Eigenvalues and Eigenvectors

Consider the matrix $\mathbf{A} = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors.

Solution:

Step 1: Finding the Eigenvalues

The characteristic equation for a 3×3 matrix is given by:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \tag{6.455}$$

where:

$$S_1 = \text{Trace of } A \tag{6.456}$$

$$S_2 = \text{Sum of minors of diagonal elements} \tag{6.457}$$

$$|A| = \text{Determinant of matrix } A \tag{6.458}$$

Calculate S_1, the trace of A:

$$S_1 = 4 + 3 + (-2) = 5 \tag{6.459}$$

Calculate S_2, the sum of minors of diagonal elements:

$$S_2 = \begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix} \tag{6.460}$$

$$= (3 \cdot (-2) - 2 \cdot (-5)) + (4 \cdot (-2) - 6 \cdot (-1)) + (4 \cdot 3 - 6 \cdot 1) \tag{6.461}$$

$$= (-6 + 10) + (-8 + 6) + (12 - 6) \tag{6.462}$$

$$= 4 - 2 + 6 \tag{6.463}$$

$$= 8 \tag{6.464}$$

Calculate $|A|$, the determinant of matrix A:

$$|A| = 4\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} + 6\begin{vmatrix} 1 & 3 \\ -1 & -5 \end{vmatrix} \tag{6.465}$$

$$= 4(3 \cdot (-2) - 2 \cdot (-5)) - 6(1 \cdot (-2) - 2 \cdot (-1)) + 6(1 \cdot (-5) - 3 \cdot (-1)) \tag{6.466}$$

$$= 4(-6 + 10) - 6(-2 + 2) + 6(-5 + 3) \tag{6.467}$$

$$= 4(4) - 6(0) + 6(-2) \tag{6.468}$$

$$= 16 - 0 - 12 \tag{6.469}$$

$$= 4 \tag{6.470}$$

So our characteristic equation is:

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0 \tag{6.471}$$

Let's try some values to see if we can find a root:
For $\lambda = 1$:

$$1^3 - 5(1)^2 + 8(1) - 4 = 1 - 5 + 8 - 4 = 0 \tag{6.472}$$

$\lambda = 1$ is a root. Now let's use synthetic division:

$$\begin{array}{c|cccc} 1 & 1 & -5 & 8 & -4 \\ & & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

So we have:

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda^2 - 4\lambda + 4) \tag{6.473}$$

$$= (\lambda - 1)(\lambda - 2)^2 \tag{6.474}$$

Therefore, the eigenvalues are:

$$\lambda_1 = 1 \text{ (with algebraic multiplicity 1)} \tag{6.475}$$

$$\lambda_2 = 2 \text{ (with algebraic multiplicity 2)} \tag{6.476}$$

Step 2: Finding the Eigenvector for $\lambda_1 = 1$
For $\lambda_1 = 1$, we form the system $(A - \lambda_1 I)X = 0$:

$$\begin{bmatrix} 4-1 & 6 & 6 \\ 1 & 3-1 & 2 \\ -1 & -5 & -2-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.477}$$

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.478}$$

Let's select the first and second rows to find the eigenvector:

$$3x + 6y + 6z = 0 \tag{6.479}$$

$$x + 2y + 2z = 0 \tag{6.480}$$

Using Cramer's rule to find the proportional relationships:

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ 2 & 2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix}} \tag{6.481}$$

Computing these determinants:

$$\begin{vmatrix} 6 & 6 \\ 2 & 2 \end{vmatrix} = 6 \cdot 2 - 6 \cdot 2 = 0 \tag{6.482}$$

Since the first determinant is zero, we need to use different rows. Let's try the first and third rows:

$$3x + 6y + 6z = 0 \tag{6.483}$$

$$-x - 5y - 3z = 0 \tag{6.484}$$

Using Cramer's rule:

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ -5 & -3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 6 \\ -1 & -3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 6 \\ -1 & -5 \end{vmatrix}} \tag{6.485}$$

Computing these determinants:

$$\begin{vmatrix} 6 & 6 \\ -5 & -3 \end{vmatrix} = 6 \cdot (-3) - 6 \cdot (-5) = -18 + 30 = 12 \tag{6.486}$$

$$\begin{vmatrix} 3 & 6 \\ -1 & -3 \end{vmatrix} = 3 \cdot (-3) - 6 \cdot (-1) = -9 + 6 = -3 \tag{6.487}$$

$$\begin{vmatrix} 3 & 6 \\ -1 & -5 \end{vmatrix} = 3 \cdot (-5) - 6 \cdot (-1) = -15 + 6 = -9 \tag{6.488}$$

Therefore:

$$\frac{x}{12} = \frac{-y}{-3} = \frac{z}{-9} \tag{6.489}$$

Simplifying:

$$\frac{x}{4} = \frac{y}{1} = \frac{z}{-3} \tag{6.490}$$

Setting $y = -1$, we get $x = 4$ and $z = 3$.
Therefore, an eigenvector corresponding to $\lambda_1 = 1$ is:

$$\vec{X}_1 = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \tag{6.491}$$

Step 3: Finding the Eigenvectors for $\lambda_2 = 2$ (with algebraic multiplicity 2)
For $\lambda_2 = 2$, we form the system $(A - \lambda_2 I)X = 0$:

$$\begin{bmatrix} 4-2 & 6 & 6 \\ 1 & 3-2 & 2 \\ -1 & -5 & -2-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.492}$$

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.493}$$

Let's reduce this matrix to row echelon form:

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \tag{6.494}$$

$$\xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \\ -1 & -5 & -4 \end{bmatrix} \tag{6.495}$$

$$\xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \tag{6.496}$$

$$\xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & -2 & -1 \end{bmatrix} \tag{6.497}$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \tag{6.498}$$

The row echelon form clearly shows that the rank of $A - 2I$ is 2, as we have exactly 2 pivots. According to the Rank-Nullity Theorem, the dimension of the null space (the eigenspace) is $n - \text{rank} = 3 - 2 = 1$. Therefore, there is only one linearly independent eigenvector corresponding to $\lambda_2 = 2$.
From the reduced form, we can see that z is a free variable, and:

$$y = -\frac{1}{2}z \tag{6.499}$$

$$x = -3y - 3z = -3\left(-\frac{1}{2}z\right) - 3z = \frac{3}{2}z - 3z = -\frac{3}{2}z \tag{6.500}$$

Setting $z = 2$ for simplicity, we get $y = -1$ and $x = -3$, giving us the eigenvector:

$$\vec{X}_2 = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} \tag{6.501}$$

This confirms that despite the algebraic multiplicity of $\lambda_2 = 2$ being 2, its geometric multiplicity is only 1. This situation occurs when the matrix is not diagonalizable and is common in asymmetric matrices.
Therefore, an eigenvector corresponding to $\lambda_2 = 2$ is:

$$\vec{X}_2 = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} \tag{6.502}$$

Therefore, the eigenvalues and corresponding eigenvectors of matrix A are:

$$\lambda_1 = 1 \text{ (with algebraic multiplicity 1)}, \qquad \vec{X}_1 = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \tag{6.503}$$

$$\lambda_2 = 2 \text{ (with algebraic multiplicity 2, geometric multiplicity 1)}, \qquad \vec{X}_2 = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} \tag{6.504}$$