Chapter 5

Vector Differential Calculus IV: Vector Identities

Vector identities provide powerful tools for manipulating and simplifying expressions in vector calculus. These identities are fundamental for solving problems in physics, engineering, and applied mathematics. This chapter presents a systematic collection of important vector identities, demonstrating their applications and providing insights into their derivations.

5.1 First-Order Vector Identities

First-order vector identities involve the differential operators ∇, $\nabla\cdot$, and $\nabla\times$ applied once to scalar and vector fields.

5.1.1 Identities Involving Scalar Fields

Let ϕ and ψ be scalar fields. The following identities hold:

Gradient Identities

$$\nabla(c\phi) = c\nabla\phi \quad \text{(where } c \text{ is a constant)} \tag{5.1}$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi \tag{5.2}$$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \quad \text{(Product rule)} \tag{5.3}$$

$$\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2} \quad \text{(Quotient rule, } \psi \neq 0\text{)} \tag{5.4}$$

Applying Gradient Identities

Calculate $\nabla(xy^2z^3)$ using the product rule.
Treating $\phi = xy^2$ and $\psi = z^3$:

$$\nabla(xy^2z^3) = xy^2\nabla(z^3) + z^3\nabla(xy^2) \tag{5.5}$$

$$= xy^2 \cdot 3z^2\hat{\mathbf{k}} + z^3[\nabla(x) \cdot y^2 + x\nabla(y^2)] \tag{5.6}$$

$$= 3xy^2z^2\hat{\mathbf{k}} + z^3[y^2\hat{\mathbf{i}} + x \cdot 2y\hat{\mathbf{j}}] \tag{5.7}$$

$$= 3xy^2z^2\hat{\mathbf{k}} + y^2z^3\hat{\mathbf{i}} + 2xyz^3\hat{\mathbf{j}} \tag{5.8}$$

5.1.2 Identities Involving Vector Fields

Let $\bar{\mathbf{F}}$ and $\bar{\mathbf{G}}$ be vector fields, and ϕ be a scalar field.

Divergence Identities

$$\nabla \cdot (c\bar{\mathbf{F}}) = c(\nabla \cdot \bar{\mathbf{F}}) \quad \text{(where } c \text{ is a constant)} \tag{5.9}$$
$$\nabla \cdot (\bar{\mathbf{F}} + \bar{\mathbf{G}}) = \nabla \cdot \bar{\mathbf{F}} + \nabla \cdot \bar{\mathbf{G}} \tag{5.10}$$
$$\nabla \cdot (\phi\bar{\mathbf{F}}) = \phi(\nabla \cdot \bar{\mathbf{F}}) + \nabla\phi \cdot \bar{\mathbf{F}} \quad \text{(Product rule)} \tag{5.11}$$
$$\nabla \cdot (\bar{\mathbf{F}} \times \bar{\mathbf{G}}) = \bar{\mathbf{G}} \cdot (\nabla \times \bar{\mathbf{F}}) - \bar{\mathbf{F}} \cdot (\nabla \times \bar{\mathbf{G}}) \tag{5.12}$$

Curl Identities

$$\nabla \times (c\bar{\mathbf{F}}) = c(\nabla \times \bar{\mathbf{F}}) \quad \text{(where } c \text{ is a constant)} \tag{5.13}$$
$$\nabla \times (\bar{\mathbf{F}} + \bar{\mathbf{G}}) = \nabla \times \bar{\mathbf{F}} + \nabla \times \bar{\mathbf{G}} \tag{5.14}$$
$$\nabla \times (\phi\bar{\mathbf{F}}) = \phi(\nabla \times \bar{\mathbf{F}}) + (\nabla\phi) \times \bar{\mathbf{F}} \quad \text{(Product rule)} \tag{5.15}$$
$$\nabla \times (\nabla\phi) = \mathbf{0} \quad \text{(Curl of a gradient is zero)} \tag{5.16}$$

Proving Curl of Gradient is Zero

Prove that $\nabla \times (\nabla\phi) = \mathbf{0}$ for any scalar field $\phi(x, y, z)$.
First, write out the components of the gradient:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}} \tag{5.17}$$

Now, compute the curl:

$$\nabla \times (\nabla\phi) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} \tag{5.18}$$
$$= \hat{\mathbf{i}}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) + \hat{\mathbf{j}}\left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) \tag{5.19}$$

Since partial derivatives commute for sufficiently smooth functions (which we assume for ϕ):

$$\frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y} \tag{5.20}$$
$$\frac{\partial^2\phi}{\partial z\partial x} = \frac{\partial^2\phi}{\partial x\partial z} \tag{5.21}$$
$$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x} \tag{5.22}$$

Therefore, all components of $\nabla \times (\nabla\phi)$ are zero, proving that $\nabla \times (\nabla\phi) = \mathbf{0}$.

5.1.3 Triple Product Identities

The following identities involve the scalar and vector triple products:

Triple Product Identities

$$(\bar{\mathbf{F}} \times \bar{\mathbf{G}}) \cdot \bar{\mathbf{H}} = (\bar{\mathbf{G}} \times \bar{\mathbf{H}}) \cdot \bar{\mathbf{F}} = (\bar{\mathbf{H}} \times \bar{\mathbf{F}}) \cdot \bar{\mathbf{G}} \quad \text{(Scalar triple product)} \tag{5.23}$$

$$\bar{\mathbf{F}} \times (\bar{\mathbf{G}} \times \bar{\mathbf{H}}) = \bar{\mathbf{G}}(\bar{\mathbf{F}} \cdot \bar{\mathbf{H}}) - \bar{\mathbf{H}}(\bar{\mathbf{F}} \cdot \bar{\mathbf{G}}) \quad \text{(Vector triple product)} \tag{5.24}$$

The vector triple product identity is often remembered with the mnemonic "BAC minus CAB," where for $\bar{\mathbf{F}} \times (\bar{\mathbf{G}} \times \bar{\mathbf{H}})$, $\bar{\mathbf{F}}$ is denoted as A, $\bar{\mathbf{G}}$ as B, and $\bar{\mathbf{H}}$ as C.

Vector Triple Product Application

Express $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}})$ using the vector triple product identity.
Using the vector triple product identity with $\bar{\mathbf{F}} = \hat{\mathbf{i}}$, $\bar{\mathbf{G}} = \hat{\mathbf{j}}$, and $\bar{\mathbf{H}} = \hat{\mathbf{k}}$:

$$\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}}) = \hat{\mathbf{j}}(\hat{\mathbf{i}} \cdot \hat{\mathbf{k}}) - \hat{\mathbf{k}}(\hat{\mathbf{i}} \cdot \hat{\mathbf{j}}) \tag{5.25}$$

$$= \hat{\mathbf{j}} \cdot 0 - \hat{\mathbf{k}} \cdot 0 \tag{5.26}$$

$$= \mathbf{0} \tag{5.27}$$

This is expected since $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$, and $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0}$.

5.2 Second-Order Vector Identities

Second-order vector identities involve the differential operators applied twice to scalar and vector fields.

5.2.1 Identities Involving the Double Application of Operators

Second-Order Identities

$$\nabla \cdot (\nabla \times \bar{\mathbf{F}}) = 0 \quad \text{(Divergence of curl is zero)} \tag{5.28}$$

$$\nabla \times (\nabla \times \bar{\mathbf{F}}) = \nabla(\nabla \cdot \bar{\mathbf{F}}) - \nabla^2 \bar{\mathbf{F}} \quad \text{(Curl of curl identity)} \tag{5.29}$$

$$\nabla(\nabla \cdot \bar{\mathbf{F}}) - \nabla \times (\nabla \times \bar{\mathbf{F}}) = \nabla^2 \bar{\mathbf{F}} \quad \text{(Vector Laplacian identity)} \tag{5.30}$$

Proving Divergence of Curl is Zero

Prove that $\nabla \cdot (\nabla \times \bar{\mathbf{F}}) = 0$ for any vector field $\bar{\mathbf{F}}(x, y, z) = F_1\hat{\mathbf{i}} + F_2\hat{\mathbf{j}} + F_3\hat{\mathbf{k}}$.
First, compute the curl:

$$\nabla \times \bar{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{5.31}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}} \tag{5.32}$$

Now, compute the divergence of this curl:

$$\nabla \cdot (\nabla \times \bar{\mathbf{F}}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \tag{5.33}$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \tag{5.34}$$

Since partial derivatives commute for sufficiently smooth functions:

$$\frac{\partial^2 F_3}{\partial x \partial y} = \frac{\partial^2 F_3}{\partial y \partial x} \tag{5.35}$$

$$\frac{\partial^2 F_2}{\partial x \partial z} = \frac{\partial^2 F_2}{\partial z \partial x} \tag{5.36}$$

$$\frac{\partial^2 F_1}{\partial y \partial z} = \frac{\partial^2 F_1}{\partial z \partial y} \tag{5.37}$$

Substituting these equalities, all terms cancel out in pairs, proving that $\nabla \cdot (\nabla \times \bar{\mathbf{F}}) = 0$.

5.2.2 Advanced Second-Order Identities

These identities are useful in more complex vector calculus problems:

Advanced Second-Order Identities

$$\nabla \times (\nabla\phi \times \nabla\psi) = \nabla\phi(\nabla^2\psi) - \nabla\psi(\nabla^2\phi) \tag{5.38}$$

$$\nabla \cdot (\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi \tag{5.39}$$

$$\nabla \times (\phi\nabla\psi) = \nabla\phi \times \nabla\psi \tag{5.40}$$

$$\nabla \times (\phi\bar{\mathbf{F}}) = \nabla\phi \times \bar{\mathbf{F}} + \phi\nabla \times \bar{\mathbf{F}} \tag{5.41}$$

$$\nabla(\bar{\mathbf{F}} \cdot \bar{\mathbf{G}}) = (\bar{\mathbf{F}} \cdot \nabla)\bar{\mathbf{G}} + (\bar{\mathbf{G}} \cdot \nabla)\bar{\mathbf{F}} + \bar{\mathbf{F}} \times (\nabla \times \bar{\mathbf{G}}) + \bar{\mathbf{G}} \times (\nabla \times \bar{\mathbf{F}}) \tag{5.42}$$

Green's First Identity Application

The identity $\nabla \cdot (\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi$ is used to derive Green's first identity:

$$\int_V \phi\nabla^2\psi \, dV = \oint_S \phi\nabla\psi \cdot \hat{\mathbf{n}} \, dS - \int_V \nabla\phi \cdot \nabla\psi \, dV \tag{5.43}$$

To derive this, apply the divergence theorem to $\nabla \cdot (\phi\nabla\psi)$:

$$\int_V \nabla \cdot (\phi\nabla\psi) \, dV = \oint_S \phi\nabla\psi \cdot \hat{\mathbf{n}} \, dS \tag{5.44}$$

$$\int_V [\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi] \, dV = \oint_S \phi\nabla\psi \cdot \hat{\mathbf{n}} \, dS \tag{5.45}$$

Rearranging terms:

$$\int_V \phi\nabla^2\psi \, dV = \oint_S \phi\nabla\psi \cdot \hat{\mathbf{n}} \, dS - \int_V \nabla\phi \cdot \nabla\psi \, dV \tag{5.46}$$

5.3 Laplacian Operator

The Laplacian operator ∇^2 (also written as Δ) is a second-order differential operator that appears frequently in physics and engineering.

5.3.1 Laplacian of Scalar Fields

The Laplacian of a scalar field ϕ is defined as the divergence of the gradient:

$$\nabla^2\phi = \nabla \cdot (\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \tag{5.47}$$

Laplacian of a Scalar Function

Calculate the Laplacian of $\phi(x, y, z) = x^2 + y^2 + z^2$.

$$\begin{aligned}
\nabla^2\phi &= \frac{\partial^2}{\partial x^2}(x^2 + y^2 + z^2) + \frac{\partial^2}{\partial y^2}(x^2 + y^2 + z^2) + \frac{\partial^2}{\partial z^2}(x^2 + y^2 + z^2) && (5.48)\\
&= \frac{\partial^2}{\partial x^2}(x^2) + \frac{\partial^2}{\partial y^2}(y^2) + \frac{\partial^2}{\partial z^2}(z^2) && (5.49)\\
&= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(2z) && (5.50)\\
&= 2 + 2 + 2 = 6 && (5.51)
\end{aligned}$$

5.3.2 Laplacian of Vector Fields

The Laplacian of a vector field $\bar{\mathbf{F}}$ is defined component-wise:

$$\nabla^2\bar{\mathbf{F}} = \nabla^2 F_1\hat{\mathbf{i}} + \nabla^2 F_2\hat{\mathbf{j}} + \nabla^2 F_3\hat{\mathbf{k}} \tag{5.52}$$

Alternatively, using the vector Laplacian identity:

$$\nabla^2\bar{\mathbf{F}} = \nabla(\nabla \cdot \bar{\mathbf{F}}) - \nabla \times (\nabla \times \bar{\mathbf{F}}) \tag{5.53}$$

Laplacian of a Vector Field

Calculate the Laplacian of $\bar{\mathbf{F}}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.
Using the component-wise definition:

$$\begin{aligned}
\nabla^2\bar{\mathbf{F}} &= \nabla^2(x)\hat{\mathbf{i}} + \nabla^2(y)\hat{\mathbf{j}} + \nabla^2(z)\hat{\mathbf{k}} && (5.54)\\
&= \left(\frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 x}{\partial y^2} + \frac{\partial^2 x}{\partial z^2}\right)\hat{\mathbf{i}} + \left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} + \frac{\partial^2 y}{\partial z^2}\right)\hat{\mathbf{j}} + \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial z^2}\right)\hat{\mathbf{k}} && (5.55)\\
&= (1 + 0 + 0)\hat{\mathbf{i}} + (0 + 1 + 0)\hat{\mathbf{j}} + (0 + 0 + 1)\hat{\mathbf{k}} && (5.56)\\
&= \mathbf{0} && (5.57)
\end{aligned}$$

We can verify this using the vector Laplacian identity. First:

$$\begin{aligned}
\nabla \cdot \bar{\mathbf{F}} &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3 && (5.58)\\
\nabla(\nabla \cdot \bar{\mathbf{F}}) &= \nabla(3) = \mathbf{0} && (5.59)
\end{aligned}$$

Next:

$$\nabla \times \bar{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \tag{5.60}$$

$$= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\hat{\mathbf{k}} \tag{5.61}$$

$$= (0-0)\hat{\mathbf{i}} + (0-0)\hat{\mathbf{j}} + (0-0)\hat{\mathbf{k}} = \mathbf{0} \tag{5.62}$$

Therefore:

$$\nabla \times (\nabla \times \bar{\mathbf{F}}) = \nabla \times \mathbf{0} = \mathbf{0} \tag{5.63}$$

And finally:

$$\nabla^2 \bar{\mathbf{F}} = \nabla(\nabla \cdot \bar{\mathbf{F}}) - \nabla \times (\nabla \times \bar{\mathbf{F}}) \tag{5.64}$$

$$= \mathbf{0} - \mathbf{0} = \mathbf{0} \tag{5.65}$$

This confirms our component-wise calculation.

5.3.3 Properties of Harmonic Functions

A scalar field ϕ is called harmonic if it satisfies Laplace's equation:

$$\nabla^2 \phi = 0 \tag{5.66}$$

Harmonic functions have several important properties:

Properties of Harmonic Functions

Let ϕ be a harmonic function. Then:
1. ϕ satisfies the mean value property: the value of ϕ at any point equals the average of ϕ over any sphere centered at that point.
2. ϕ satisfies the maximum principle: if ϕ is harmonic in a bounded domain, its maximum and minimum values occur on the boundary.
3. ϕ is infinitely differentiable (smooth).
4. ϕ is analytic: it can be represented by its Taylor series.

5.4 Solved Examples

Example: Proving Cross Product is Solenoidal

If $\vec{F_1} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and $\vec{F_2} = (\vec{a} \cdot \vec{r})\vec{a}$ then Show that $\vec{F_1} \times \vec{F_2}$ is solenoidal

Solution

Given:

$$\vec{F_1} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$
$$\vec{F_2} = (\vec{a} \cdot \vec{r})\vec{a}$$

To prove that $\vec{F_1} \times \vec{F_2}$ is solenoidal, we need to show that $\nabla \cdot (\vec{F_1} \times \vec{F_2}) = 0$.
We will use the vector identity:

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Step 1: First, prove that $\vec{F_1}$ is irrotational.
Calculate $\nabla \times \vec{F_1}$:

$$\begin{aligned}
\nabla \times \vec{F_1} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} \\
&= \hat{i}\left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right) - \hat{j}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right) + \hat{k}\left(\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right) \\
&= \hat{i}(x - x) - \hat{j}(y - y) + \hat{k}(z - z) \\
&= 0\hat{i} + 0\hat{j} + 0\hat{k} \\
&= \vec{0}
\end{aligned}$$

Therefore, $\vec{F_1}$ is irrotational.
Step 2: Next, prove that $\vec{F_2} = (\vec{a} \cdot \vec{r})\vec{a}$ is irrotational.
Since $\vec{F_2}$ is the gradient of a scalar function, it is always irrotational. We can write: $\vec{F_2} = \nabla\phi$ where $\phi = \frac{1}{2}(\vec{a} \cdot \vec{r})^2$.
By a fundamental theorem of vector calculus: $\nabla \times (\nabla\phi) = \vec{0}$.
Therefore, $\vec{F_2}$ is irrotational.
Step 3: Calculate $\nabla \cdot (\vec{F_1} \times \vec{F_2})$ using the vector identity.
Since both $\vec{F_1}$ and $\vec{F_2}$ are irrotational (i.e., $\nabla \times \vec{F_1} = \vec{0}$ and $\nabla \times \vec{F_2} = \vec{0}$):

$$\begin{aligned}
\nabla \cdot (\vec{F_1} \times \vec{F_2}) &= \vec{F_2} \cdot (\nabla \times \vec{F_1}) - \vec{F_1} \cdot (\nabla \times \vec{F_2}) \\
&= \vec{F_2} \cdot \vec{0} - \vec{F_1} \cdot \vec{0} \\
&= 0
\end{aligned}$$

Therefore, $\vec{F_1} \times \vec{F_2}$ is solenoidal.