Chapter 1

Prerequisites and Mathematical Foundations

1.1 Essential Formulas from Elementary Calculus

Before exploring advanced calculus techniques for engineering applications, it is important to have a strong command of the fundamental rules and formulas. This section serves as a comprehensive reference for essential formulas from elementary calculus that will be frequently used throughout this book.

1.1.1 Differentiation Rules

The process of differentiation is foundational in studying rates of change, optimization, and dynamic systems. The following rules provide the framework for differentiating functions of various forms.

Basic Differentiation Rules

For functions $f(x)$ and $g(x)$ that are differentiable at x, and constants a and n:

1. **Constant Rule:** $\frac{d}{dx}(c) = 0$
2. **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$
3. **Constant Multiple Rule:** $\frac{d}{dx}[cf(x)] = c\frac{df(x)}{dx}$
4. **Sum Rule:** $\frac{d}{dx}[f(x) \pm g(x)] = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$
5. **Product Rule:** $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$
6. **Quotient Rule:** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{[g(x)]^2}$
7. **Chain Rule:** $\frac{d}{dx}[f(g(x))] = \frac{df}{dg} \cdot \frac{dg}{dx}$

Application of Differentiation Rules

Find the derivative of $h(x) = x^3 \sin(x^2) + \frac{e^x}{\ln(x)}$.
Solution: We break down the function and apply appropriate rules:
For $x^3 \sin(x^2)$, we use the product rule:

$$
\begin{aligned}
\frac{d}{dx}[x^3 \sin(x^2)] &= \frac{d}{dx}(x^3) \cdot \sin(x^2) + x^3 \cdot \frac{d}{dx}[\sin(x^2)] && (1.1)\\
&= 3x^2 \cdot \sin(x^2) + x^3 \cdot \cos(x^2) \cdot \frac{d}{dx}(x^2) && (1.2)\\
&= 3x^2 \cdot \sin(x^2) + x^3 \cdot \cos(x^2) \cdot 2x && (1.3)\\
&= 3x^2 \sin(x^2) + 2x^4 \cos(x^2) && (1.4)
\end{aligned}
$$

For $\frac{e^x}{\ln(x)}$, we use the quotient rule:

$$\frac{d}{dx}\left[\frac{e^x}{\ln(x)}\right] = \frac{\ln(x) \cdot \frac{d}{dx}(e^x) - e^x \cdot \frac{d}{dx}[\ln(x)]}{[\ln(x)]^2} \tag{1.5}$$

$$= \frac{\ln(x) \cdot e^x - e^x \cdot \frac{1}{x}}{[\ln(x)]^2} \tag{1.6}$$

$$= \frac{e^x \ln(x) - \frac{e^x}{x}}{[\ln(x)]^2} \tag{1.7}$$

$$= \frac{e^x[x\ln(x) - 1]}{x[\ln(x)]^2} \tag{1.8}$$

Therefore:

$$\frac{dh}{dx} = 3x^2 \sin(x^2) + 2x^4 \cos(x^2) + \frac{e^x[x\ln(x) - 1]}{x[\ln(x)]^2} \tag{1.9}$$

Derivatives of Common Functions

For all applicable domains:

$$\frac{d}{dx}(\sin x) = \cos x \tag{1.10}$$
$$\frac{d}{dx}(\cos x) = -\sin x \tag{1.11}$$
$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{1.12}$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x \tag{1.13}$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \tag{1.14}$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x \tag{1.15}$$
$$\frac{d}{dx}(e^x) = e^x \tag{1.16}$$
$$\frac{d}{dx}(a^x) = a^x \ln a \tag{1.17}$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x} \tag{1.18}$$
$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \tag{1.19}$$
$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \tag{1.20}$$
$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \tag{1.21}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \tag{1.22}$$
$$\frac{d}{dx}(\text{arccot}\, x) = -\frac{1}{1+x^2} \tag{1.23}$$
$$\frac{d}{dx}(\sinh x) = \cosh x \tag{1.24}$$
$$\frac{d}{dx}(\cosh x) = \sinh x \tag{1.25}$$

1.1.2 Integration Formulas

Integration is the inverse operation of differentiation and is crucial for calculating areas, volumes, work, and many other engineering quantities. Below are the fundamental integration formulas that form the basis for solving complex integration problems.

Basic Integration Formulas

For continuous functions $f(x)$ and $g(x)$, and constants a, b, and $n \neq -1$:

1. **Constant Rule:** $\int c\, dx = cx + C$
2. **Power Rule:** $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$
3. **Integration of $\frac{1}{x}$:** $\int \frac{1}{x}\, dx = \ln|x| + C$
4. **Linearity:** $\int [af(x) \pm bg(x)]\, dx = a\int f(x)\, dx \pm b\int g(x)\, dx$
5. **Integration by Parts:** $\int u\, dv = uv - \int v\, du$
6. **Integration by Substitution:** $\int f(g(x))g'(x)\, dx = \int f(u)\, du$ where $u = g(x)$

Integrals of Common Functions

For all applicable domains:

$$\int \sin x\,dx = -\cos x + C \tag{1.26}$$

$$\int \cos x\,dx = \sin x + C \tag{1.27}$$

$$\int \tan x\,dx = -\ln|\cos x| + C \tag{1.28}$$

$$\int \sec^2 x\,dx = \tan x + C \tag{1.29}$$

$$\int \sec x \tan x\,dx = \sec x + C \tag{1.30}$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C \tag{1.31}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C \tag{1.32}$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{|x|}{a} + C \tag{1.33}$$

$$\int e^x\,dx = e^x + C \tag{1.34}$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \tag{1.35}$$

$$\int \ln x\,dx = x\ln x - x + C \tag{1.36}$$

$$\int \sinh x\,dx = \cosh x + C \tag{1.37}$$

$$\int \cosh x\,dx = \sinh x + C \tag{1.38}$$

Application of Integration Techniques

Evaluate the integral $\int x^2 \ln(x)\,dx$.
Solution: We use integration by parts with $u = \ln(x)$ and $dv = x^2\,dx$.
This gives:

$$du = \frac{1}{x}\,dx \tag{1.39}$$

$$v = \int x^2\,dx = \frac{x^3}{3} \tag{1.40}$$

Applying the integration by parts formula $\int u\,dv = uv - \int v\,du$:

$$\int x^2 \ln(x)\,dx = \ln(x) \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x}\,dx \tag{1.41}$$

$$= \frac{x^3 \ln(x)}{3} - \frac{1}{3} \int x^2\,dx \tag{1.42}$$

$$= \frac{x^3 \ln(x)}{3} - \frac{1}{3} \cdot \frac{x^3}{3} + C \tag{1.43}$$

$$= \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + C \tag{1.44}$$

1.1.3 Limits and Continuity

Limits form the foundation of calculus and are essential for understanding the behavior of functions at critical points and defining derivatives and integrals.

Properties of Limits

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ exist, then:

1. **Sum Rule:** $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$
2. **Product Rule:** $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M$
3. **Quotient Rule:** $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ if $M \neq 0$
4. **Power Rule:** $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n = L^n$ for any real number n
5. **Root Rule:** $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ if $L > 0$ for even n
6. **Composition Rule:** If g is continuous at L and $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} g(f(x)) = g(L)$

Definition of Continuity

A function f is continuous at a point $x = a$ if and only if:

1. $f(a)$ is defined (i.e., a is in the domain of f)
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

Evaluating Limits

Evaluate $\lim_{x\to 0} \frac{\sin(3x)}{x}$.

Solution: We can use the standard limit $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$:

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot 3 \tag{1.45}$$

$$= \lim_{u\to 0} \frac{\sin(u)}{u} \cdot 3 \quad \text{(substituting } u = 3x\text{)} \tag{1.46}$$

$$= 1 \cdot 3 \tag{1.47}$$

$$= 3 \tag{1.48}$$

One-sided Limits

Evaluate the following one-sided limits:

1. $\lim_{x\to 0^+} \ln(x)$

2. $\lim_{x\to 2^-} \frac{|x-2|}{x-2}$
3. $\lim_{x\to 1^+} \frac{\sqrt{x-1}}{x^2-1}$

Solution

1. $\lim_{x\to 0^+} \ln(x) = -\infty$ since $\ln(x)$ approaches negative infinity as x approaches zero from the right.
2. $\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = -1$ since $x-2<0$ when $x<2$.
3. $\lim_{x\to 1^+} \frac{\sqrt{x-1}}{x^2-1} = \lim_{x\to 1^+} \frac{\sqrt{x-1}}{(x-1)(x+1)} = \lim_{x\to 1^+} \frac{1}{\sqrt{x-1}(x+1)} = \lim_{x\to 1^+} \frac{1}{2\sqrt{x-1}} = \infty$

1.2 Essential Trigonometric Identities

Trigonometric functions and their identities are fundamental tools in engineering mathematics. They appear in various applications including signal processing, mechanical vibrations, electrical circuits, and wave phenomena. This section provides a comprehensive reference of trigonometric identities that will be used throughout this book.

1.2.1 Fundamental Trigonometric Relations

The fundamental trigonometric functions—sine, cosine, tangent, cotangent, secant, and cosecant—are interrelated through several basic identities.

Pythagorean Identities

For any angle θ:

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.49}$$
$$\tan^2\theta + 1 = \sec^2\theta \tag{1.50}$$
$$1 + \cot^2\theta = \csc^2\theta \tag{1.51}$$

Reciprocal Identities

For any angle θ where the functions are defined:

$$\sec\theta = \frac{1}{\cos\theta} \tag{1.52}$$
$$\csc\theta = \frac{1}{\sin\theta} \tag{1.53}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} \tag{1.54}$$

Quotient Identities

For any angle θ where $\cos\theta \neq 0$ and $\sin\theta \neq 0$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{1.55}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{1.56}$$

Simplifying Trigonometric Expressions

Simplify the expression $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta}$.
Solution:

$$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta(1+\cos\theta)} \tag{1.57}$$
$$= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)} \tag{1.58}$$
$$= \frac{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)} \tag{1.59}$$
$$= \frac{1 + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)} \tag{1.60}$$
$$= \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} \tag{1.61}$$
$$= \frac{2}{\sin\theta} \tag{1.62}$$
$$= 2\csc\theta \tag{1.63}$$

Negative Angle Identities

For any angle θ:

$$\sin(-\theta) = -\sin\theta \tag{1.64}$$
$$\cos(-\theta) = \cos\theta \tag{1.65}$$
$$\tan(-\theta) = -\tan\theta \tag{1.66}$$
$$\cot(-\theta) = -\cot\theta \tag{1.67}$$
$$\sec(-\theta) = \sec\theta \tag{1.68}$$
$$\csc(-\theta) = -\csc\theta \tag{1.69}$$

Periodicity

For any integer n and angle θ:

$$\sin(\theta + 2\pi n) = \sin\theta \tag{1.70}$$
$$\cos(\theta + 2\pi n) = \cos\theta \tag{1.71}$$
$$\tan(\theta + \pi n) = \tan\theta \tag{1.72}$$
$$\cot(\theta + \pi n) = \cot\theta \tag{1.73}$$
$$\sec(\theta + 2\pi n) = \sec\theta \tag{1.74}$$
$$\csc(\theta + 2\pi n) = \csc\theta \tag{1.75}$$

1.2.2 Sum and Difference Formulas

These formulas express trigonometric functions of the sum or difference of two angles in terms of trigonometric functions of the individual angles.

Sum and Difference Formulas for Sine

For any angles α and β:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{1.76}$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{1.77}$$

Sum and Difference Formulas for Cosine

For any angles α and β:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{1.78}$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{1.79}$$

Sum and Difference Formulas for Tangent

For any angles α and β where $\tan(\alpha+\beta)$ and $\tan(\alpha-\beta)$ are defined:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{1.80}$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \tag{1.81}$$

Applying Sum Formula

Evaluate $\sin(75°)$ using the sum formula.
Solution: We can express $75°$ as $45° + 30°$, angles whose trigonometric values are known.

$$\sin(75°) = \sin(45° + 30°) \tag{1.82}$$
$$= \sin(45°)\cos(30°) + \cos(45°)\sin(30°) \tag{1.83}$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \tag{1.84}$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \tag{1.85}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \tag{1.86}$$
$$= \frac{\sqrt{3}+1}{2} \cdot \frac{1}{\sqrt{2}} \tag{1.87}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \tag{1.88}$$

Practice Problem

Prove that $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$ using the sum and difference formulas.

Solution

$$\sin(A+B)\sin(A-B) = [\sin A\cos B + \cos A\sin B][\sin A\cos B - \cos A\sin B] \tag{1.89}$$
$$= (\sin A\cos B)^2 - (\cos A\sin B)^2 \tag{1.90}$$
$$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B \tag{1.91}$$
$$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B \tag{1.92}$$
$$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B \tag{1.93}$$
$$= \sin^2 A - \sin^2 B \tag{1.94}$$

1.2.3 Double and Half-Angle Formulas

Double-angle formulas express trigonometric functions of twice an angle in terms of functions of the original angle. Half-angle formulas do the reverse, expressing functions of half an angle in terms of the original angle.

Double-Angle Formulas

For any angle θ:

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{1.95}$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \tag{1.96}$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \tag{1.97}$$

Half-Angle Formulas

For any angle θ where the expressions are defined:

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{1.98}$$
$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}} \tag{1.99}$$
$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \tag{1.100}$$

The sign depends on the quadrant in which $\frac{\theta}{2}$ lies.

Using Double-Angle Formula

Evaluate $\int \sin^2 x\,dx$ using a double-angle formula.

Solution: We can use the identity $\sin^2 x = \frac{1-\cos(2x)}{2}$:

$$\int \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} \, dx \tag{1.101}$$

$$= \frac{1}{2} \int (1 - \cos(2x)) \, dx \tag{1.102}$$

$$= \frac{1}{2} \left(x - \frac{\sin(2x)}{2} \right) + C \tag{1.103}$$

$$= \frac{x}{2} - \frac{\sin(2x)}{4} + C \tag{1.104}$$

1.2.4 Product-to-Sum and Sum-to-Product Identities

These identities allow the conversion between products of trigonometric functions and sums or differences, and vice versa. They are particularly useful in integration and in analyzing modulated signals.

Product-to-Sum Formulas

For any angles α and β:

$$\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \tag{1.105}$$

$$\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \tag{1.106}$$

$$\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \tag{1.107}$$

$$\cos\alpha \sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)] \tag{1.108}$$

Sum-to-Product Formulas

For any angles α and β:

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.109}$$

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{1.110}$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.111}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.112}$$

Integration Using Product-to-Sum Identity

Evaluate $\int \sin(3x)\sin(5x)\,dx$.

Solution: Using the product-to-sum identity for sine functions:

$$\int \sin(3x)\sin(5x)\,dx = \int \frac{1}{2}[\cos(3x-5x)-\cos(3x+5x)]\,dx \tag{1.113}$$
$$= \int \frac{1}{2}[\cos(-2x)-\cos(8x)]\,dx \tag{1.114}$$
$$= \int \frac{1}{2}[\cos(2x)-\cos(8x)]\,dx \tag{1.115}$$
$$= \frac{1}{2}\left[\frac{\sin(2x)}{2}-\frac{\sin(8x)}{8}\right]+C \tag{1.116}$$
$$= \frac{\sin(2x)}{4}-\frac{\sin(8x)}{16}+C \tag{1.117}$$

Practice Problem

Express $\sin(4x)+\sin(2x)$ as a product using the sum-to-product formula.

Solution

Using the sum-to-product formula $\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$:
Let $\alpha = 4x$ and $\beta = 2x$. Then:

$$\sin(4x)+\sin(2x) = 2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) \tag{1.118}$$
$$= 2\sin\left(\frac{6x}{2}\right)\cos\left(\frac{2x}{2}\right) \tag{1.119}$$
$$= 2\sin(3x)\cos(x) \tag{1.120}$$