Chapter 3

Partial Derivative of Composite Functions, Total Derivative and Change of Independent Variables

3.1 Composite Functions of Several Variables

3.1.1 Basic Concepts and Notation

When working with functions of several variables, we often encounter scenarios where one function depends on other functions, creating a composition. Understanding how to differentiate such composite functions is essential in many applications.

Definition 3.1 (Composite Function). *A composite function of several variables occurs when a function depends on one or more intermediate functions, which in turn depend on the independent variables.*

The most common forms of composition include:

1. $z = f(g(x, y), h(x, y))$ where f is a function of two variables u and v, with $u = g(x, y)$ and $v = h(x, y)$
2. $z = f(x, y, g(x, y))$ where f is a function of three variables, and the third argument is itself a function of x and y
3. $z = f(g(x, y, z))$ where f depends on a single function g, which depends on three variables including z (implicit form)

3.1.2 Chain Rule for Functions of Several Variables

The chain rule extends naturally to partial derivatives of composite functions.

Chain Rule for Two Variables

If $z = f(u, v)$ where $u = g(x, y)$ and $v = h(x, y)$, then:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \tag{3.1}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} \tag{3.2}$$

Applying the Chain Rule

Consider $z = \sin(x^2 + y^2) \cdot e^{xy}$
We can view this as $z = f(u, v)$ where $f(u, v) = \sin(u) \cdot e^v$, $u = x^2 + y^2$, and $v = xy$.
First, we compute the partial derivatives of f with respect to u and v:

$$\frac{\partial f}{\partial u} = \cos(u) \cdot e^v \tag{3.3}$$

$$\frac{\partial f}{\partial v} = \sin(u) \cdot e^v \tag{3.4}$$

Next, we compute the partial derivatives of u and v with respect to x and y:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y \tag{3.5}$$

$$\frac{\partial v}{\partial x} = y, \quad \frac{\partial v}{\partial y} = x \tag{3.6}$$

Applying the chain rule:

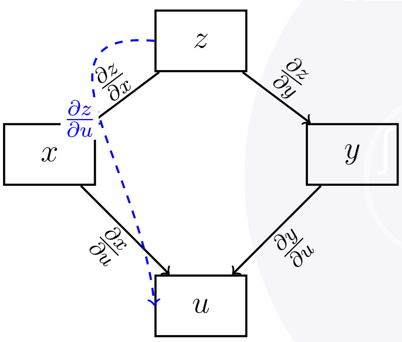
$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} && (3.7)\\
&= \cos\left(x^2 + y^2\right) \cdot e^{xy} \cdot 2x + \sin\left(x^2 + y^2\right) \cdot e^{xy} \cdot y && (3.8)\\
&= e^{xy}\left[2x \cos\left(x^2 + y^2\right) + y \sin\left(x^2 + y^2\right)\right] && (3.9)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} && (3.10)\\
&= \cos\left(x^2 + y^2\right) \cdot e^{xy} \cdot 2y + \sin\left(x^2 + y^2\right) \cdot e^{xy} \cdot x && (3.11)\\
&= e^{xy}\left[2y \cos\left(x^2 + y^2\right) + x \sin\left(x^2 + y^2\right)\right] && (3.12)
\end{aligned}$$

Sometimes a function is defined implicitly through an equation like $F(x, y, z) = 0$. To find the partial derivatives, we use the implicit function theorem.

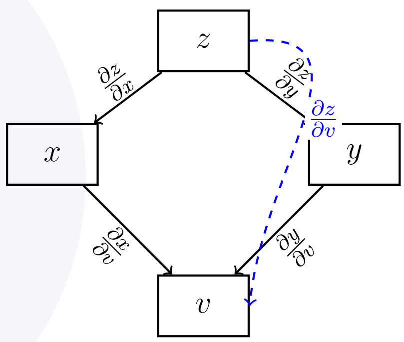
Multi-Variable Chain Rule for $z = f(x, y)$, $x = g(u, v)$, $y = h(u, v)$

For $\frac{\partial z}{\partial u}$:



$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}}$$

For $\frac{\partial z}{\partial v}$:



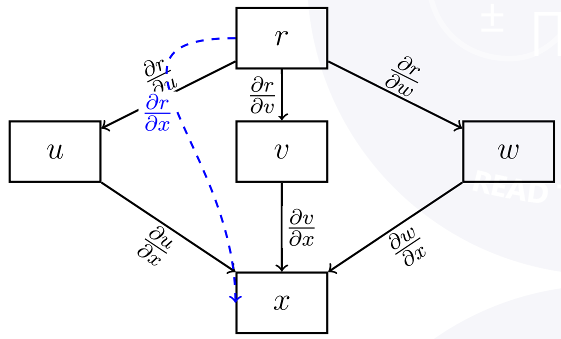
$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}}$$

The diagrams above illustrate the multi-variable chain rule when z is a function of x and y, while x and y are both functions of u and v. The left diagram shows how to calculate $\frac{\partial z}{\partial u}$, and the right diagram shows how to calculate $\frac{\partial z}{\partial v}$.

The blue dashed arrows represent the direct partial derivatives we seek to find, while the solid arrows show the path through the intermediate variables. The formulas at the bottom provide the mathematical expressions for these derivatives using the chain rule.

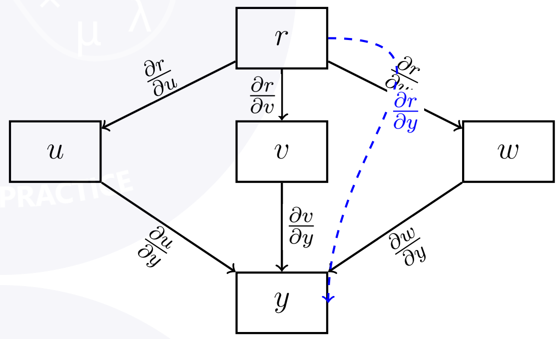
Chain Rule for $r = f(u, v, w)$ where $u = g(x, y, z)$, $v = h(x, y, z)$, $w = k(x, y, z)$

For $\frac{\partial r}{\partial x}$:



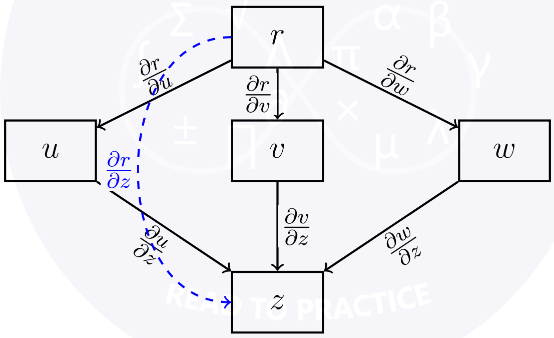
$$\boxed{\frac{\partial r}{\partial x} = \frac{\partial r}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial r}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial r}{\partial w} \cdot \frac{\partial w}{\partial x}}$$

For $\frac{\partial r}{\partial y}$:



$$\boxed{\frac{\partial r}{\partial y} = \frac{\partial r}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial r}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial r}{\partial w} \cdot \frac{\partial w}{\partial y}}$$

For $\frac{\partial r}{\partial z}$:



$$\boxed{\frac{\partial r}{\partial z} = \frac{\partial r}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial r}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial r}{\partial w} \cdot \frac{\partial w}{\partial z}}$$

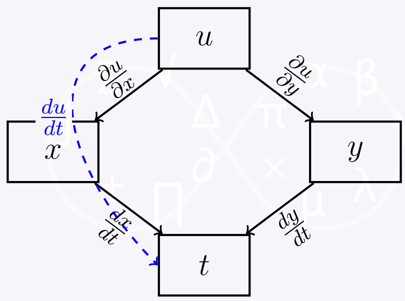
The diagrams above illustrate the chain rule when r is a function of the three variables u, v, and w, each of which is in turn a function of the three variables x, y, and z.

The top two diagrams show how to calculate $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$, while the bottom diagram shows how to calculate $\frac{\partial r}{\partial z}$.

The blue arrows represent the direct partial derivatives we seek to find, while the solid arrows show the paths through the intermediate variables. The formulas below each diagram give the mathematical expression for the derivative using the chain rule, showing how each partial derivative is the sum of three terms.

Chain Rule for $u = f(x, y)$ where $x = g(t), y = h(t)$

For $\frac{du}{dt}$:



$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}}$$

The diagram above illustrates the chain rule for finding $\frac{du}{dt}$ when u is a function of x and y, and both x and y are functions of t (usually representing time).

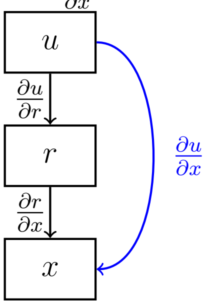
This is a common scenario in physics and engineering when tracking how a quantity changes over time while moving along a path. The right side shows a visualization of a parameter path in the xy-plane, with the velocity vector indicating the direction and rate of movement.

Example 1: Laplacian of a Radial Function

Problem: If $u = f(r)$ and $r = \sqrt{x^2 + y^2 + z^2}$, then prove that $u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r)$.

Chain Rule Visualizations for $u = f(r)$, $r = \sqrt{x^2 + y^2 + z^2}$

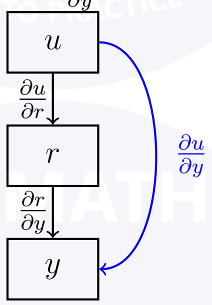
For $\frac{\partial u}{\partial x}$:



$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}}$$

$$\boxed{\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r}}$$

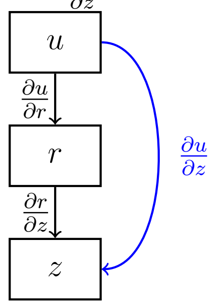
For $\frac{\partial u}{\partial y}$:



$$\boxed{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}}$$

$$\boxed{\frac{\partial u}{\partial y} = f'(r) \cdot \frac{y}{r}}$$

For $\frac{\partial u}{\partial z}$:



$$\boxed{\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}}$$

$$\boxed{\frac{\partial u}{\partial z} = f'(r) \cdot \frac{z}{r}}$$

These diagrams clearly illustrate how we apply the chain rule when differentiating a composite function. The arrows indicate the path of differentiation, and the boxes represent the intermediate variables in the composition.

Complete Solution with Chain Rule Approach

Part 2: Computing the First Partial Derivatives
For $r = \sqrt{x^2+y^2+z^2}$, we compute:

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) \tag{3.13}$$
$$= \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2x \tag{3.14}$$
$$= \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} \tag{3.15}$$

Similarly:

$$\frac{\partial r}{\partial y} = \frac{y}{r} \tag{3.16}$$
$$\frac{\partial r}{\partial z} = \frac{z}{r} \tag{3.17}$$

Using the chain rule, we find:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} = f'(r) \cdot \frac{x}{r} \tag{3.18}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} = f'(r) \cdot \frac{y}{r} \tag{3.19}$$
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} = f'(r) \cdot \frac{z}{r} \tag{3.20}$$

Part 3: Computing the Second Partial Derivatives
Now we need to find the second derivatives. For $\frac{\partial^2 u}{\partial x^2}$, we differentiate $\frac{\partial u}{\partial x} = \frac{x}{r} f'(r)$ with respect to x:
Using the product rule:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{r} f'(r)\right) \tag{3.21}$$
$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) \cdot f'(r) + \frac{x}{r} \cdot \frac{\partial}{\partial x}(f'(r)) \tag{3.22}$$

First, we compute $\frac{\partial}{\partial x}\left(\frac{x}{r}\right)$ using the quotient rule:

$$\frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{r \cdot \frac{\partial x}{\partial x} - x \cdot \frac{\partial r}{\partial x}}{r^2} \tag{3.23}$$
$$= \frac{r \cdot 1 - x \cdot \frac{x}{r}}{r^2} \tag{3.24}$$
$$= \frac{r - \frac{x^2}{r}}{r^2} \tag{3.25}$$
$$= \frac{1}{r} - \frac{x^2}{r^3} \tag{3.26}$$

Next, we compute $\frac{\partial}{\partial x}(f'(r))$ using the chain rule:

$$\frac{\partial}{\partial x}(f'(r)) = f''(r) \cdot \frac{\partial r}{\partial x} \tag{3.27}$$
$$= f''(r) \cdot \frac{x}{r} \tag{3.28}$$

Combining these results:

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{1}{r} - \frac{x^2}{r^3}\right) \cdot f'(r) + \frac{x}{r} \cdot f''(r) \cdot \frac{x}{r} \tag{3.29}$$

$$= \frac{f'(r)}{r} - \frac{x^2 f'(r)}{r^3} + \frac{x^2 f''(r)}{r^2} \tag{3.30}$$

By similar calculations:

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r} - \frac{y^2 f'(r)}{r^3} + \frac{y^2 f''(r)}{r^2} \tag{3.31}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{f'(r)}{r} - \frac{z^2 f'(r)}{r^3} + \frac{z^2 f''(r)}{r^2} \tag{3.32}$$

Part 4: Computing the Laplacian $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{3.33}$$

$$= \left(\frac{f'(r)}{r} - \frac{x^2 f'(r)}{r^3} + \frac{x^2 f''(r)}{r^2}\right) + \left(\frac{f'(r)}{r} - \frac{y^2 f'(r)}{r^3} + \frac{y^2 f''(r)}{r^2}\right) + \left(\frac{f'(r)}{r} - \frac{z^2 f'(r)}{r^3} + \frac{z^2 f''(r)}{r^2}\right) \tag{3.34}$$

Grouping similar terms:

$$\nabla^2 u = \frac{3f'(r)}{r} - \frac{(x^2+y^2+z^2) f'(r)}{r^3} + \frac{(x^2+y^2+z^2) f''(r)}{r^2} \tag{3.35}$$

Since $r^2 = x^2 + y^2 + z^2$, we can simplify:

$$\nabla^2 u = \frac{3f'(r)}{r} - \frac{r^2 f'(r)}{r^3} + \frac{r^2 f''(r)}{r^2} \tag{3.36}$$

$$= \frac{3f'(r)}{r} - \frac{f'(r)}{r} + f''(r) \tag{3.37}$$

$$= \frac{2f'(r)}{r} + f''(r) \tag{3.38}$$

$$= f''(r) + \frac{2}{r} f'(r) \tag{3.39}$$

Therefore, we have proven that $u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r)$ for $u = f(r)$ with $r = \sqrt{x^2 + y^2 + z^2}$.

Note: This result is important in mathematical physics and represents the radial part of the Laplacian operator in spherical coordinates. The expression $f''(r) + \frac{2}{r} f'(r)$ can also be written as $\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{df}{dr}\right)$, which appears in many physical equations including the radial Schrödinger equation.

Example 2

Problem: If $u = f(x - y, y - z, z - x)$ then find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

Complete Solution with Chain Rule Diagram

To solve this problem, we'll use the chain rule for multivariable functions with a change of variables.

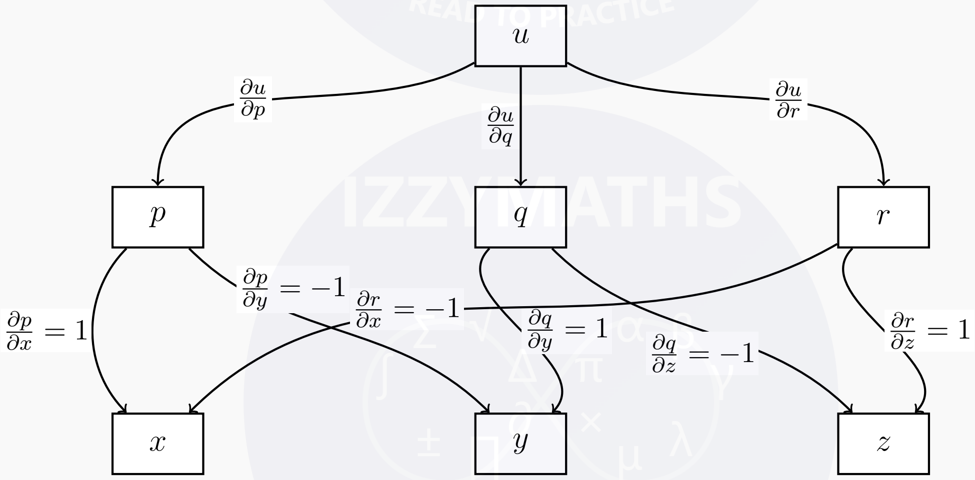
Step 1: Let's introduce new variables to simplify our work:

$$p = x - y \tag{3.40}$$
$$q = y - z \tag{3.41}$$
$$r = z - x \tag{3.42}$$

Note that $u = f(p, q, r)$ is a function of these three new variables.

Step 2: Let's visualize the functional dependencies using a chain rule diagram:

Chain Rule for $u = f(p, q, r)$ where $p = x - y, q = y - z, r = z - x$



$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot 1 + \frac{\partial u}{\partial q} \cdot 0 + \frac{\partial u}{\partial r} \cdot (-1) = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}}$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot (-1) + \frac{\partial u}{\partial q} \cdot 1 + \frac{\partial u}{\partial r} \cdot 0 = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}}$$

$$\boxed{\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot 0 + \frac{\partial u}{\partial q} \cdot (-1) + \frac{\partial u}{\partial r} \cdot 1 = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}}$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$$

Step 3: Calculate the partial derivatives of the intermediate variables with respect to x, y, and z.

For the partial derivatives with respect to x:

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1 \tag{3.43}$$
$$\frac{\partial q}{\partial x} = \frac{\partial}{\partial x}(y-z) = 0 \tag{3.44}$$
$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(z-x) = -1 \tag{3.45}$$

For the partial derivatives with respect to y:

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1 \tag{3.46}$$
$$\frac{\partial q}{\partial y} = \frac{\partial}{\partial y}(y-z) = 1 \tag{3.47}$$
$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(z-x) = 0 \tag{3.48}$$

For the partial derivatives with respect to z:

$$\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}(x-y) = 0 \tag{3.49}$$
$$\frac{\partial q}{\partial z} = \frac{\partial}{\partial z}(y-z) = -1 \tag{3.50}$$
$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}(z-x) = 1 \tag{3.51}$$

Step 4: Using the chain rule, we can find the partial derivatives of u with respect to x, y, and z.
For $\frac{\partial u}{\partial x}$:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} \tag{3.52}$$
$$= \frac{\partial u}{\partial p}\cdot 1 + \frac{\partial u}{\partial q}\cdot 0 + \frac{\partial u}{\partial r}\cdot(-1) \tag{3.53}$$
$$= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \tag{3.54}$$

For $\frac{\partial u}{\partial y}$:

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} \tag{3.55}$$
$$= \frac{\partial u}{\partial p}\cdot(-1) + \frac{\partial u}{\partial q}\cdot 1 + \frac{\partial u}{\partial r}\cdot 0 \tag{3.56}$$
$$= -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \tag{3.57}$$

For $\frac{\partial u}{\partial z}$:

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} \tag{3.58}$$
$$= \frac{\partial u}{\partial p}\cdot 0 + \frac{\partial u}{\partial q}\cdot(-1) + \frac{\partial u}{\partial r}\cdot 1 \tag{3.59}$$
$$= -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \tag{3.60}$$

Step 5: Now we can find the sum of the partial derivatives:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \left(\frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}\right) + \left(-\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}\right) + \left(-\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}\right) \tag{3.61}$$

Grouping the terms with the same derivatives:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial q} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial r} \tag{3.62}$$

$$= 0 \tag{3.63}$$

Conclusion: The value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
This result is independent of the specific form of the function f. It holds for any differentiable function of the form $f(x-y, y-z, z-x)$.

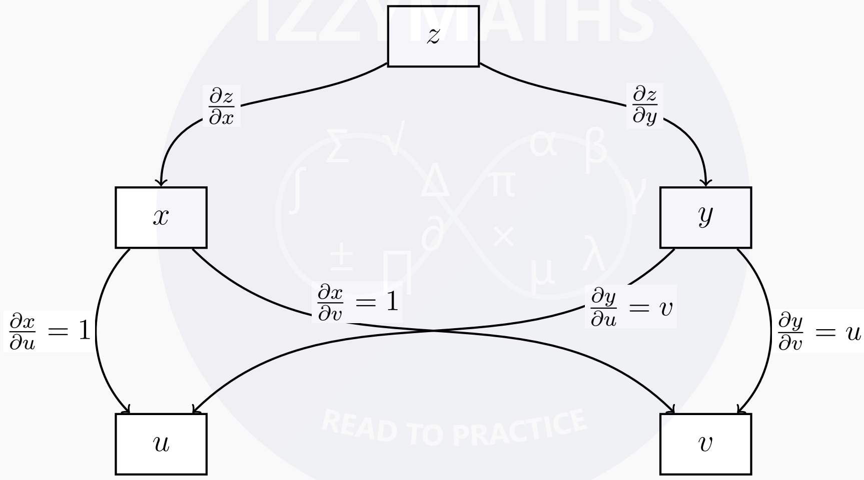
Example 3

Problem: If $z = f(x, y)$ and $x = u + v$, $y = uv$ then prove that $u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}$

Detailed Solution with Chain Rule Diagram

Step 1: We'll use the chain rule to establish the relationship between the derivatives with respect to (u, v) and those with respect to (x, y).
Here's a visualization of the functional dependencies:

Chain Rule for $z = f(x,y)$ where $x = u + v, y = uv$



$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \cdot 1 + \frac{\partial z}{\partial y} \cdot v}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \cdot 1 + \frac{\partial z}{\partial y} \cdot u}$$

$$\boxed{u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}}$$

Step 2: Calculate the partial derivatives of the intermediate variables (x, y) with respect to (u, v).
From our given equations:

$$x = u + v \tag{3.64}$$
$$y = uv \tag{3.65}$$

Taking partial derivatives:

$$\frac{\partial x}{\partial u} = 1 \tag{3.66}$$
$$\frac{\partial x}{\partial v} = 1 \tag{3.67}$$
$$\frac{\partial y}{\partial u} = v \tag{3.68}$$
$$\frac{\partial y}{\partial v} = u \tag{3.69}$$

Step 3: Use the chain rule to express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ in terms of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

For $\frac{\partial z}{\partial u}$:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \tag{3.70}$$
$$= \frac{\partial z}{\partial x} \cdot 1 + \frac{\partial z}{\partial y} \cdot v \tag{3.71}$$
$$= \frac{\partial z}{\partial x} + v \cdot \frac{\partial z}{\partial y} \tag{3.72}$$

For $\frac{\partial z}{\partial v}$:

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \tag{3.73}$$
$$= \frac{\partial z}{\partial x} \cdot 1 + \frac{\partial z}{\partial y} \cdot u \tag{3.74}$$
$$= \frac{\partial z}{\partial x} + u \cdot \frac{\partial z}{\partial y} \tag{3.75}$$

Step 4: Now let's evaluate the left side of the equation we need to prove: $u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = u\left(\frac{\partial z}{\partial x} + v \cdot \frac{\partial z}{\partial y}\right) + v\left(\frac{\partial z}{\partial x} + u \cdot \frac{\partial z}{\partial y}\right) \tag{3.76}$$
$$= u \cdot \frac{\partial z}{\partial x} + uv \cdot \frac{\partial z}{\partial y} + v \cdot \frac{\partial z}{\partial x} + uv \cdot \frac{\partial z}{\partial y} \tag{3.77}$$
$$= (u+v) \cdot \frac{\partial z}{\partial x} + 2uv \cdot \frac{\partial z}{\partial y} \tag{3.78}$$

Step 5: Substitute the original variable definitions to obtain our result.
Since $x = u + v$ and $y = uv$, we have:

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = (u+v) \cdot \frac{\partial z}{\partial x} + 2uv \cdot \frac{\partial z}{\partial y} \tag{3.79}$$
$$= x \cdot \frac{\partial z}{\partial x} + 2y \cdot \frac{\partial z}{\partial y} \tag{3.80}$$

Therefore, we have proven that:

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}$$

Example 4

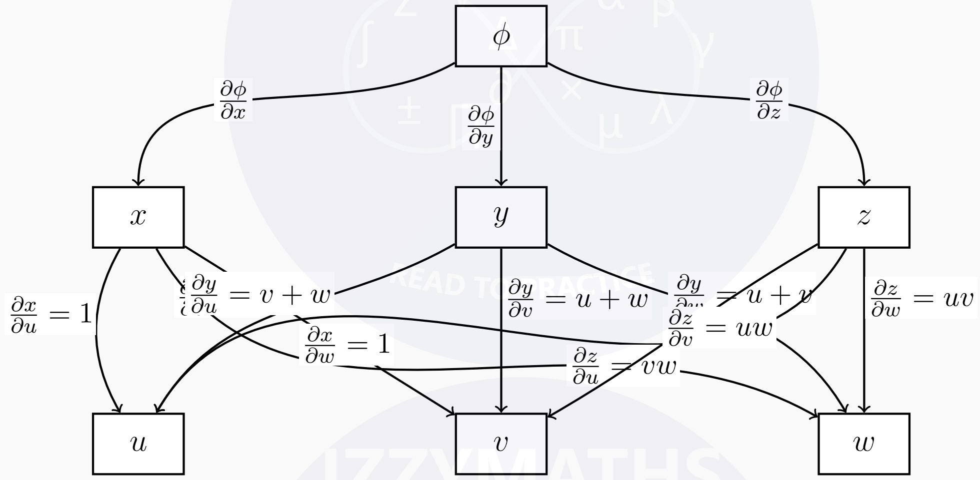
Problem: If $x = u + v + w$, $y = uv + vw + uw$ and $z = uvw$ and ϕ is a function of x, y, z, then prove that:

$$u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w} = x\frac{\partial \phi}{\partial x} + 2y\frac{\partial \phi}{\partial y} + 3z\frac{\partial \phi}{\partial z}$$

Detailed Solution with Chain Rule Diagram

Step 1: Let's use the chain rule to find the relationship between the derivatives with respect to (u, v, w) and those with respect to (x, y, z).
Here is a visualization of the functional dependencies:

Chain Rule for $\phi(x, y, z)$ where $x = u + v + w$, $y = uv + vw + uw$, $z = uvw$



Step 2: Calculate the partial derivatives of the intermediate variables (x, y, z) with respect to (u, v, w).
From our given equations:

$$x = u + v + w \tag{3.81}$$
$$y = uv + vw + uw \tag{3.82}$$
$$z = uvw \tag{3.83}$$

Taking partial derivatives with respect to u:

$$\frac{\partial x}{\partial u} = 1 \tag{3.84}$$
$$\frac{\partial y}{\partial u} = v + w \tag{3.85}$$
$$\frac{\partial z}{\partial u} = vw \tag{3.86}$$

Taking partial derivatives with respect to v:

$$\frac{\partial x}{\partial v} = 1 \tag{3.87}$$
$$\frac{\partial y}{\partial v} = u + w \tag{3.88}$$
$$\frac{\partial z}{\partial v} = uw \tag{3.89}$$

Taking partial derivatives with respect to w:

$$\frac{\partial x}{\partial w} = 1 \tag{3.90}$$
$$\frac{\partial y}{\partial w} = u + v \tag{3.91}$$
$$\frac{\partial z}{\partial w} = uv \tag{3.92}$$

Step 3: Use the chain rule to express $\frac{\partial \phi}{\partial u}$, $\frac{\partial \phi}{\partial v}$, and $\frac{\partial \phi}{\partial w}$ in terms of $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$, and $\frac{\partial \phi}{\partial z}$.
For $\frac{\partial \phi}{\partial u}$:

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial u} \tag{3.93}$$
$$= \frac{\partial \phi}{\partial x} \cdot 1 + \frac{\partial \phi}{\partial y} \cdot (v+w) + \frac{\partial \phi}{\partial z} \cdot vw \tag{3.94}$$
$$= \frac{\partial \phi}{\partial x} + (v+w)\frac{\partial \phi}{\partial y} + vw\frac{\partial \phi}{\partial z} \tag{3.95}$$

For $\frac{\partial \phi}{\partial v}$:

$$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial v} \tag{3.96}$$
$$= \frac{\partial \phi}{\partial x} \cdot 1 + \frac{\partial \phi}{\partial y} \cdot (u+w) + \frac{\partial \phi}{\partial z} \cdot uw \tag{3.97}$$
$$= \frac{\partial \phi}{\partial x} + (u+w)\frac{\partial \phi}{\partial y} + uw\frac{\partial \phi}{\partial z} \tag{3.98}$$

For $\frac{\partial \phi}{\partial w}$:

$$\frac{\partial \phi}{\partial w} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial w} \tag{3.99}$$
$$= \frac{\partial \phi}{\partial x} \cdot 1 + \frac{\partial \phi}{\partial y} \cdot (u+v) + \frac{\partial \phi}{\partial z} \cdot uv \tag{3.100}$$
$$= \frac{\partial \phi}{\partial x} + (u+v)\frac{\partial \phi}{\partial y} + uv\frac{\partial \phi}{\partial z} \tag{3.101}$$

Step 4: Now evaluate the left side of the equation we need to prove: $u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$

$$u\frac{\partial \phi}{\partial u} = u\left[\frac{\partial \phi}{\partial x} + (v+w)\frac{\partial \phi}{\partial y} + vw\frac{\partial \phi}{\partial z}\right] \tag{3.102}$$
$$= u\frac{\partial \phi}{\partial x} + u(v+w)\frac{\partial \phi}{\partial y} + uvw\frac{\partial \phi}{\partial z} \tag{3.103}$$

$$v\frac{\partial \phi}{\partial v} = v\left[\frac{\partial \phi}{\partial x} + (u+w)\frac{\partial \phi}{\partial y} + uw\frac{\partial \phi}{\partial z}\right] \tag{3.104}$$
$$= v\frac{\partial \phi}{\partial x} + v(u+w)\frac{\partial \phi}{\partial y} + uvw\frac{\partial \phi}{\partial z} \tag{3.105}$$

$$w\frac{\partial \phi}{\partial w} = w\left[\frac{\partial \phi}{\partial x} + (u+v)\frac{\partial \phi}{\partial y} + uv\frac{\partial \phi}{\partial z}\right] \tag{3.106}$$
$$= w\frac{\partial \phi}{\partial x} + w(u+v)\frac{\partial \phi}{\partial y} + uvw\frac{\partial \phi}{\partial z} \tag{3.107}$$

Adding these three equations:

$$u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w} = (u+v+w)\frac{\partial \phi}{\partial x} + [u(v+w) + v(u+w) + w(u+v)]\frac{\partial \phi}{\partial y} + 3uvw\frac{\partial \phi}{\partial z} \tag{3.108}$$

Step 5: Simplify the coefficients using the original variable definitions.
First, note that $u + v + w = x$.
For the coefficient of $\frac{\partial \phi}{\partial y}$:

$$u(v+w) + v(u+w) + w(u+v) = uv + uw + uv + vw + uw + vw \tag{3.109}$$
$$= 2uv + 2uw + 2vw \tag{3.110}$$
$$= 2(uv + uw + vw) \tag{3.111}$$
$$= 2y \tag{3.112}$$

And for the coefficient of $\frac{\partial \phi}{\partial z}$, we have $uvw = z$, so $3uvw = 3z$.
Therefore:

$$u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w} = x\frac{\partial \phi}{\partial x} + 2y\frac{\partial \phi}{\partial y} + 3z\frac{\partial \phi}{\partial z} \tag{3.113}$$

Thus, we have proven that:

$$u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w} = x\frac{\partial \phi}{\partial x} + 2y\frac{\partial \phi}{\partial y} + 3z\frac{\partial \phi}{\partial z}$$

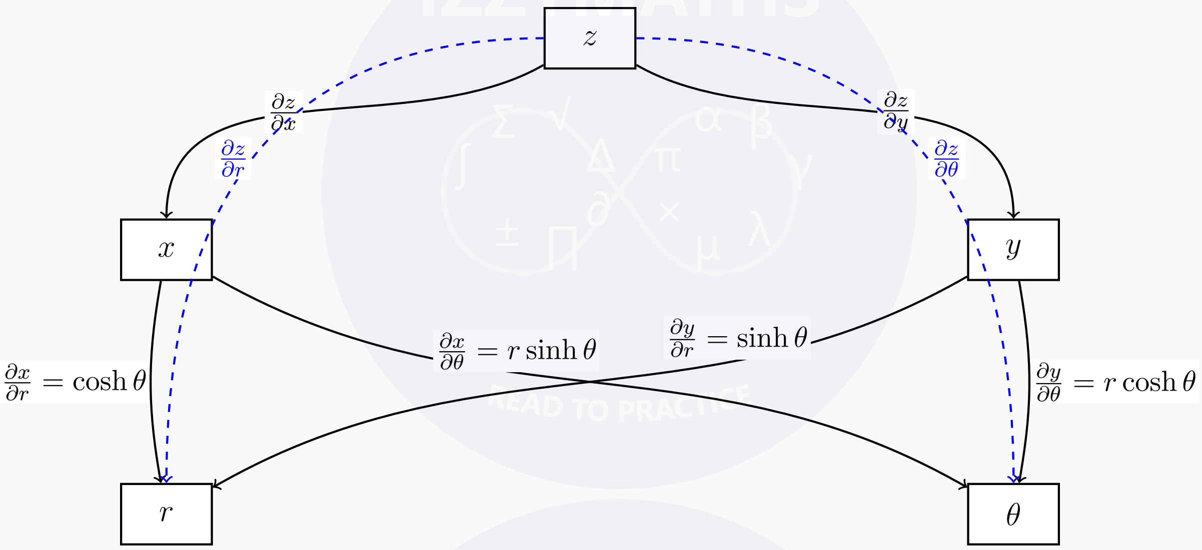
Example 5

Problem: If $z = f(x, y)$ and $x = r\cosh\theta$, $y = r\sinh\theta$ then show that $(x-y)(z_x - z_y) = rz_r - z_\theta$

Detailed Solution with Chain Rule Diagram

Step 1: Let's use the chain rule to establish the relationship between derivatives in the (x, y) and (r, θ) coordinate systems.
Here is a visualization of the functional dependencies:

Chain Rule for $z = f(x, y)$ where $x = r\cosh\theta$, $y = r\sinh\theta$



$$\boxed{\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cosh\theta + \frac{\partial z}{\partial y}\sinh\theta}$$

$$\boxed{\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}r\sinh\theta + \frac{\partial z}{\partial y}r\cosh\theta}$$

$$\boxed{(x-y)(z_x - z_y) = rz_r - z_\theta}$$

Step 2: Calculate the partial derivatives of the intermediate variables (x, y) with respect to (r, θ).
From our given equations:

$$x = r\cosh\theta \tag{3.114}$$
$$y = r\sinh\theta \tag{3.115}$$

Taking partial derivatives with respect to r:

$$\frac{\partial x}{\partial r} = \cosh\theta \tag{3.116}$$
$$\frac{\partial y}{\partial r} = \sinh\theta \tag{3.117}$$

Taking partial derivatives with respect to θ:

$$\frac{\partial x}{\partial \theta} = r\sinh\theta \tag{3.118}$$

$$\frac{\partial y}{\partial \theta} = r\cosh\theta \tag{3.119}$$

Step 3: Use the chain rule to express $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ in terms of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
For $\frac{\partial z}{\partial r}$:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} \tag{3.120}$$

$$= \frac{\partial z}{\partial x}\cdot\cosh\theta + \frac{\partial z}{\partial y}\cdot\sinh\theta \tag{3.121}$$

$$\tag{3.122}$$

For $\frac{\partial z}{\partial \theta}$:

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} \tag{3.123}$$

$$= \frac{\partial z}{\partial x}\cdot r\sinh\theta + \frac{\partial z}{\partial y}\cdot r\cosh\theta \tag{3.124}$$

$$\tag{3.125}$$

Step 4: Now we'll work with the left side of the equation to be proved: $(x-y)(z_x-z_y)$
Substituting the expressions for x and y:

$$x - y = r\cosh\theta - r\sinh\theta \tag{3.126}$$

$$= r(\cosh\theta - \sinh\theta) \tag{3.127}$$

Step 5: Compute $rz_r - z_\theta$ using our expressions from Step 3:

$$rz_r - z_\theta = r\left(\frac{\partial z}{\partial x}\cdot\cosh\theta + \frac{\partial z}{\partial y}\cdot\sinh\theta\right) - \left(\frac{\partial z}{\partial x}\cdot r\sinh\theta + \frac{\partial z}{\partial y}\cdot r\cosh\theta\right) \tag{3.128}$$

$$= r\frac{\partial z}{\partial x}\cosh\theta + r\frac{\partial z}{\partial y}\sinh\theta - r\frac{\partial z}{\partial x}\sinh\theta - r\frac{\partial z}{\partial y}\cosh\theta \tag{3.129}$$

$$= r\frac{\partial z}{\partial x}(\cosh\theta - \sinh\theta) - r\frac{\partial z}{\partial y}(\cosh\theta - \sinh\theta) \tag{3.130}$$

$$= r(\cosh\theta - \sinh\theta)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) \tag{3.131}$$

$$= (x-y)(z_x - z_y) \tag{3.132}$$

Therefore, we have proven that:

$$(x-y)(z_x - z_y) = rz_r - z_\theta$$

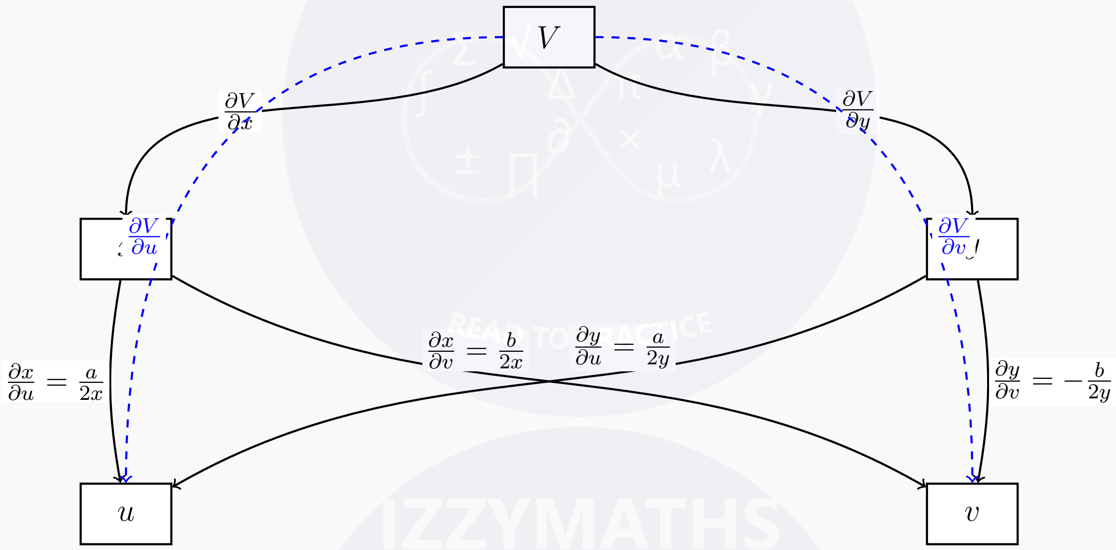
Example 6

Problem: If $x^2 = au + bv$, $y^2 = au - bv$ and $V = f(x, y)$, then show that $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = 2\left(u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v}\right)$.

Detailed Solution with Chain Rule Diagram

Step 1: Let's visualize the chain rule relationships with a diagram.

Chain Rule for $V = f(x, y)$ where $x^2 = au + bv, y^2 = au - bv$



Step 2: Calculate the partial derivatives using the chain rule.
First, we need to compute $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, and $\frac{\partial y}{\partial v}$.
From $x^2 = au + bv$, we get:

$$\frac{\partial x^2}{\partial u} = a \tag{3.133}$$

$$2x\frac{\partial x}{\partial u} = a \tag{3.134}$$

$$\frac{\partial x}{\partial u} = \frac{a}{2x} \tag{3.135}$$

Similarly:

$$\frac{\partial x}{\partial v} = \frac{b}{2x} \tag{3.136}$$

$$\frac{\partial y}{\partial u} = \frac{a}{2y} \tag{3.137}$$

$$\frac{\partial y}{\partial v} = -\frac{b}{2y} \tag{3.138}$$

Step 3: Express $\frac{\partial V}{\partial u}$ and $\frac{\partial V}{\partial v}$ in terms of $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ using the chain rule.

$$\frac{\partial V}{\partial u} = \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial V}{\partial y} \cdot \frac{\partial y}{\partial u} \tag{3.139}$$

$$= \frac{\partial V}{\partial x} \cdot \frac{a}{2x} + \frac{\partial V}{\partial y} \cdot \frac{a}{2y} \tag{3.140}$$

$$\frac{\partial V}{\partial v} = \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial V}{\partial y} \cdot \frac{\partial y}{\partial v} \tag{3.141}$$

$$= \frac{\partial V}{\partial x} \cdot \frac{b}{2x} + \frac{\partial V}{\partial y} \cdot \left(-\frac{b}{2y}\right) \tag{3.142}$$

$$= \frac{\partial V}{\partial x} \cdot \frac{b}{2x} - \frac{\partial V}{\partial y} \cdot \frac{b}{2y} \tag{3.143}$$

Step 4: Now we compute the expression $u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v}$.

$$u\frac{\partial V}{\partial u} = u\left(\frac{\partial V}{\partial x} \cdot \frac{a}{2x} + \frac{\partial V}{\partial y} \cdot \frac{a}{2y}\right) \tag{3.144}$$

$$= \frac{ua}{2x}\frac{\partial V}{\partial x} + \frac{ua}{2y}\frac{\partial V}{\partial y} \tag{3.145}$$

$$v\frac{\partial V}{\partial v} = v\left(\frac{\partial V}{\partial x} \cdot \frac{b}{2x} - \frac{\partial V}{\partial y} \cdot \frac{b}{2y}\right) \tag{3.146}$$

$$= \frac{vb}{2x}\frac{\partial V}{\partial x} - \frac{vb}{2y}\frac{\partial V}{\partial y} \tag{3.147}$$

Adding these expressions:

$$u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v} = \frac{ua}{2x}\frac{\partial V}{\partial x} + \frac{ua}{2y}\frac{\partial V}{\partial y} + \frac{vb}{2x}\frac{\partial V}{\partial x} - \frac{vb}{2y}\frac{\partial V}{\partial y} \tag{3.148}$$

$$= \frac{1}{2x}(ua + vb)\frac{\partial V}{\partial x} + \frac{1}{2y}(ua - vb)\frac{\partial V}{\partial y} \tag{3.149}$$

Step 5: Use the original equations to simplify.
From the given equations, we have:

$$x^2 = au + bv \tag{3.150}$$

$$y^2 = au - bv \tag{3.151}$$

Rearranging:

$$au + bv = x^2 \tag{3.152}$$

$$au - bv = y^2 \tag{3.153}$$

Substituting these into our expression:

$$u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v} = \frac{1}{2x}(x^2)\frac{\partial V}{\partial x} + \frac{1}{2y}(y^2)\frac{\partial V}{\partial y} \tag{3.154}$$

$$= \frac{x}{2}\frac{\partial V}{\partial x} + \frac{y}{2}\frac{\partial V}{\partial y} \tag{3.155}$$

$$= \frac{1}{2}\left(x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}\right) \tag{3.156}$$

Therefore:

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = 2\left(u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v}\right) \tag{3.157}$$

which is what we wanted to prove.

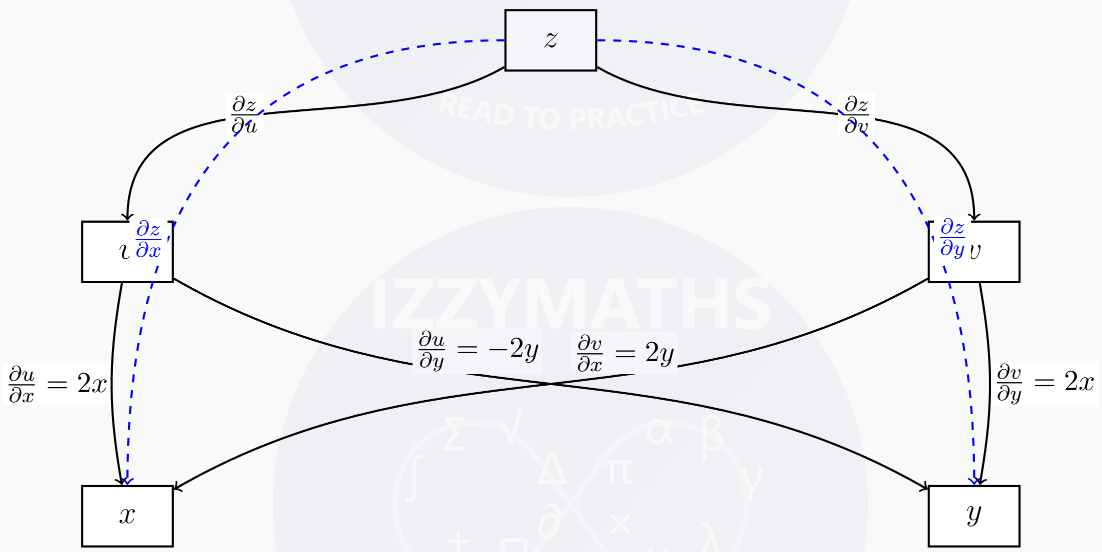
Example 7

Problem: If $z = f(u, v)$, $u = x^2 - y^2$, $v = 2xy$, prove that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(u^2 + v^2)^{1/2}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$.

Detailed Solution with Chain Rule Diagram

Step 1: We'll first visualize the chain rule relationships between the variables.

Chain Rule for $z = f(u, v)$ where $u = x^2 - y^2$, $v = 2xy$



$$\boxed{\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \cdot 2x + \frac{\partial z}{\partial v} \cdot 2y}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = -\frac{\partial z}{\partial u} \cdot 2y + \frac{\partial z}{\partial v} \cdot 2x}$$

Step 2: Calculate the partial derivatives using the chain rule.
From the diagram above, we have:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \tag{3.158}$$

$$= \frac{\partial z}{\partial u} \cdot 2x + \frac{\partial z}{\partial v} \cdot 2y \tag{3.159}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} \tag{3.160}$$

$$= \frac{\partial z}{\partial u} \cdot (-2y) + \frac{\partial z}{\partial v} \cdot 2x \tag{3.161}$$

Step 3: Square each expression and add them.

$$\left(\frac{\partial z}{\partial x}\right)^2 = \left(\frac{\partial z}{\partial u}\cdot 2x + \frac{\partial z}{\partial v}\cdot 2y\right)^2 \tag{3.162}$$
$$= 4x^2\left(\frac{\partial z}{\partial u}\right)^2 + 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + 4y^2\left(\frac{\partial z}{\partial v}\right)^2 \tag{3.163}$$

$$\left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\cdot(-2y) + \frac{\partial z}{\partial v}\cdot 2x\right)^2 \tag{3.164}$$
$$= 4y^2\left(\frac{\partial z}{\partial u}\right)^2 - 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + 4x^2\left(\frac{\partial z}{\partial v}\right)^2 \tag{3.165}$$

Adding these two expressions:

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4x^2\left(\frac{\partial z}{\partial u}\right)^2 + 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + 4y^2\left(\frac{\partial z}{\partial v}\right)^2 \tag{3.166}$$
$$+ 4y^2\left(\frac{\partial z}{\partial u}\right)^2 - 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + 4x^2\left(\frac{\partial z}{\partial v}\right)^2 \tag{3.167}$$
$$= 4(x^2+y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] \tag{3.168}$$

Step 4: Express the result in terms of u and v.
Note that in our case, $u = x^2 - y^2$ and $v = 2xy$. To express $x^2 + y^2$ in terms of u and v, we need to look at the relation between these variables.

$$u^2 + v^2 = (x^2 - y^2)^2 + (2xy)^2 \tag{3.169}$$
$$= x^4 - 2x^2y^2 + y^4 + 4x^2y^2 \tag{3.170}$$
$$= x^4 + 2x^2y^2 + y^4 \tag{3.171}$$
$$= (x^2 + y^2)^2 \tag{3.172}$$

Therefore, $x^2 + y^2 = \sqrt{u^2 + v^2}$.
Substituting this into our previous result:

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x^2+y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] \tag{3.173}$$
$$= 4\sqrt{u^2+v^2}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] \tag{3.174}$$

Step 5: Simplify the final expression.
Therefore, we have proven that:

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(u^2+v^2)^{1/2}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] \tag{3.175}$$

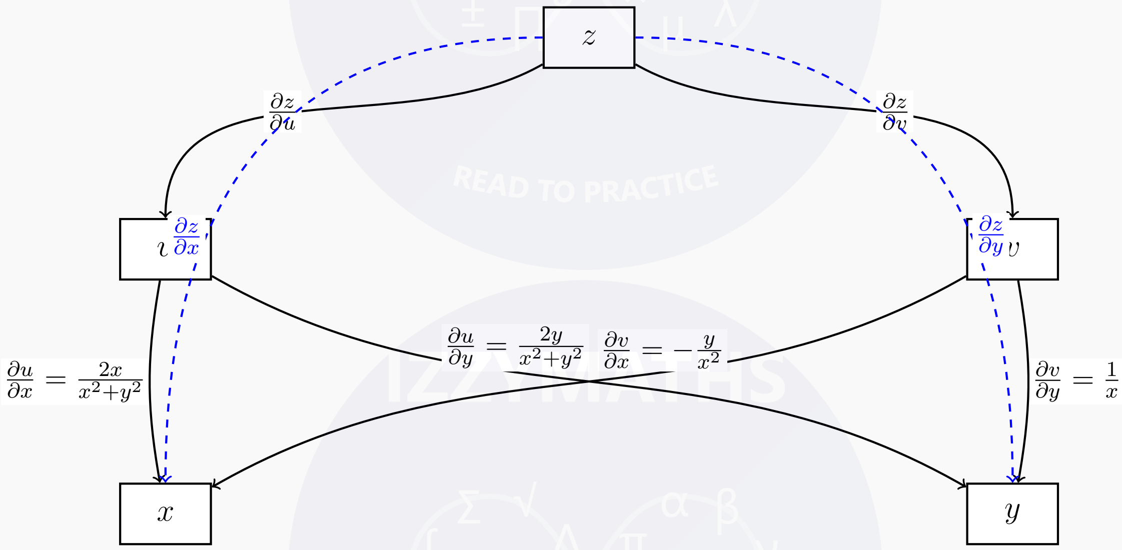
Example 8

Problem: If $u = \log(x^2 + y^2)$ and $v = \frac{y}{x}$, show that $x\frac{\partial z}{\partial y} - y\frac{dz}{dx} = (1 + v^2)\frac{\partial z}{\partial v}$.

Detailed Solution with Chain Rule Diagram

Step 1: Let's visualize the chain rule relationships with a diagram.

Chain Rule for $z = f(u, v)$ where $u = \log\left(x^2 + y^2\right)$, $v = \frac{y}{x}$



Step 2: Calculate the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.
From the given transformations:

$$u = \log\left(x^2 + y^2\right) \tag{3.176}$$

$$v = \frac{y}{x} \tag{3.177}$$

Taking the partial derivatives:

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial x}(x^2 + y^2) = \frac{1}{x^2 + y^2} \cdot 2x = \frac{2x}{x^2 + y^2} \tag{3.178}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial y}(x^2 + y^2) = \frac{1}{x^2 + y^2} \cdot 2y = \frac{2y}{x^2 + y^2} \tag{3.179}$$

For the function $v = \frac{y}{x}$, we use the quotient rule:

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{0 \cdot x - y \cdot 1}{x^2} = -\frac{y}{x^2} \tag{3.180}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1 \cdot x - y \cdot 0}{x^2} = \frac{1}{x} \tag{3.181}$$

Step 3: Apply the chain rule to express $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \tag{3.182}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{2x}{x^2 + y^2} + \frac{\partial z}{\partial v} \cdot \left(-\frac{y}{x^2}\right) \tag{3.183}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} \tag{3.184}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{2y}{x^2 + y^2} + \frac{\partial z}{\partial v} \cdot \frac{1}{x} \tag{3.185}$$

Step 4: Form the expression $x\frac{\partial z}{\partial y} - y\frac{dz}{dx}$.

$$x\frac{\partial z}{\partial y} = x\left(\frac{\partial z}{\partial u} \cdot \frac{2y}{x^2 + y^2} + \frac{\partial z}{\partial v} \cdot \frac{1}{x}\right) \tag{3.186}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{2xy}{x^2 + y^2} + \frac{\partial z}{\partial v} \cdot 1 \tag{3.187}$$

$$y\frac{dz}{dx} = y\left(\frac{\partial z}{\partial u} \cdot \frac{2x}{x^2 + y^2} + \frac{\partial z}{\partial v} \cdot \left(-\frac{y}{x^2}\right)\right) \tag{3.188}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{2xy}{x^2 + y^2} - \frac{\partial z}{\partial v} \cdot \frac{y^2}{x^2} \tag{3.189}$$

Subtracting:

$$x\frac{\partial z}{\partial y} - y\frac{dz}{dx} = \left(\frac{\partial z}{\partial u} \cdot \frac{2xy}{x^2 + y^2} + \frac{\partial z}{\partial v} \cdot 1\right) - \left(\frac{\partial z}{\partial u} \cdot \frac{2xy}{x^2 + y^2} - \frac{\partial z}{\partial v} \cdot \frac{y^2}{x^2}\right) \tag{3.190}$$

$$= \frac{\partial z}{\partial v} \cdot 1 + \frac{\partial z}{\partial v} \cdot \frac{y^2}{x^2} \tag{3.191}$$

$$= \frac{\partial z}{\partial v} \cdot \left(1 + \frac{y^2}{x^2}\right) \tag{3.192}$$

Step 5: Simplify the result using the definition of v.
Since $v = \frac{y}{x}$, we have $\frac{y^2}{x^2} = v^2$. Therefore:

$$x\frac{\partial z}{\partial y} - y\frac{dz}{dx} = \frac{\partial z}{\partial v} \cdot \left(1 + \frac{y^2}{x^2}\right) \tag{3.193}$$

$$= \frac{\partial z}{\partial v} \cdot (1 + v^2) \tag{3.194}$$

Therefore, we have proven that:

$$x\frac{\partial z}{\partial y} - y\frac{dz}{dx} = (1 + v^2)\frac{\partial z}{\partial v} \tag{3.195}$$

3.2 Implicit Functions

When a function is given in the form of an equation where one variable cannot be isolated and expressed directly in terms of the others, we call this an implicit function. Implicit functions are common in many applications, and mastering how to find their derivatives is essential.

3.2.1 Implicit Functions of Two Variables

Implicit Function of Two Variables

If $f(x,y)=0$ defines y implicitly as a function of x, and $\frac{\partial f}{\partial y}\neq 0$ at a point (x_0,y_0), then y can be expressed locally as a function of x.
Using the notation $p=\frac{\partial f}{\partial x}$ and $q=\frac{\partial f}{\partial y}$, the first derivative is:

$$\frac{dy}{dx}=-\frac{p}{q}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \tag{3.196}$$

This is often remembered as the negative ratio of the partial derivatives.

Finding the First Derivative

Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
We rewrite this as $f(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ and find:

$$p=f_x=\frac{2x}{a^2} \tag{3.197}$$

$$q=f_y=\frac{2y}{b^2} \tag{3.198}$$

Using the formula $\frac{dy}{dx}=-\frac{p}{q}$:

$$\frac{dy}{dx}=-\frac{p}{q}=-\frac{\frac{2x}{a^2}}{\frac{2y}{b^2}}=-\frac{b^2x}{a^2y} \tag{3.199}$$

At any point (x,y) on the ellipse where $y\neq 0$, the slope of the tangent line is $-\frac{b^2x}{a^2y}$.

3.2.2 Second Derivative of Implicit Functions

For an implicit function $f(x,y)=0$, the second derivative formula can be expressed using a convenient notation that makes it easier to remember and apply.

Second Derivative Using p

For an implicit function $f(x,y)=0$, we introduce the notation:

$$p=\frac{\partial f}{\partial x} \qquad q=\frac{\partial f}{\partial y} \tag{3.200}$$

$$r=\frac{\partial^2 f}{\partial x^2} \qquad s=\frac{\partial^2 f}{\partial x\partial y} \qquad t=\frac{\partial^2 f}{\partial y^2} \tag{3.201}$$

The first derivative is $\frac{dy}{dx}=-\frac{p}{q}$
The second derivative is:

$$\frac{d^2y}{dx^2}=-\frac{q^2r-2pqs+p^2t}{q^3}=-\frac{1}{q^3}\begin{vmatrix} r & s & p\\ s & t & q\\ p & q & 0\end{vmatrix} \tag{3.202}$$

The determinant form provides a convenient way to memorize this formula.

Derivation of the Second Derivative Formula

To derive the formula for the second derivative, we start with $\frac{dy}{dx} = -\frac{p}{q}$ and differentiate with respect to x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{p}{q}\right) \tag{3.203}$$
$$= -\frac{q\frac{dp}{dx} - p\frac{dq}{dx}}{q^2} \tag{3.204}$$

Since p and q are functions of both x and y, and y depends on x, we use the chain rule:

$$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\frac{dy}{dx} \tag{3.205}$$
$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dx} \tag{3.206}$$
$$= r + s \cdot \left(-\frac{p}{q}\right) \tag{3.207}$$
$$= r - \frac{ps}{q} \tag{3.208}$$

Similarly:

$$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\frac{dy}{dx} \tag{3.209}$$
$$= \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dx} \tag{3.210}$$
$$= s + t \cdot \left(-\frac{p}{q}\right) \tag{3.211}$$
$$= s - \frac{pt}{q} \tag{3.212}$$

Substituting these expressions:

$$\frac{d^2y}{dx^2} = -\frac{q\left(r - \frac{ps}{q}\right) - p\left(s - \frac{pt}{q}\right)}{q^2} \tag{3.213}$$
$$= -\frac{qr - ps - ps + \frac{p^2t}{q}}{q^2} \tag{3.214}$$
$$= -\frac{qr - 2ps + \frac{p^2t}{q}}{q^2} \tag{3.215}$$
$$= -\frac{q^2r - 2pqs + p^2t}{q^3} \tag{3.216}$$

This can also be written in determinant form as:

$$\frac{d^2y}{dx^2} = -\frac{1}{q^3}\begin{vmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{vmatrix} \tag{3.217}$$

This determinant form can be helpful for memorization and quick application.

Finding the Second Derivative

Consider the circle $x^2 + y^2 = a^2$, written as $f(x, y) = x^2 + y^2 - a^2 = 0$.
Step 1: Calculate all the necessary derivatives:

$$p = f_x = 2x \qquad q = f_y = 2y \tag{3.218}$$

$$r = f_{xx} = 2 \qquad s = f_{xy} = 0 \qquad t = f_{yy} = 2 \tag{3.219}$$

Step 2: Find the first derivative:

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{2x}{2y} = -\frac{x}{y} \tag{3.220}$$

Step 3: Apply the second derivative formula:

$$\frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3} \tag{3.221}$$

$$= -\frac{(2y)^2 \cdot 2 - 2(2x)(2y) \cdot 0 + (2x)^2 \cdot 2}{(2y)^3} \tag{3.222}$$

$$= -\frac{8y^2 + 0 + 8x^2}{8y^3} \tag{3.223}$$

$$= -\frac{y^2 + x^2}{y^3} \tag{3.224}$$

Using the equation of the circle $x^2 + y^2 = a^2$:

$$\frac{d^2y}{dx^2} = -\frac{a^2}{y^3} \tag{3.225}$$

This shows that the curvature varies inversely with the cube of the y-coordinate.

3.2.3 Practical Steps for Finding Derivatives of Implicit Functions

To help students master the technique, here's a step-by-step approach:

Step-by-Step Procedure

For an implicit function $f(x, y) = 0$:
To find $\frac{dy}{dx}$: 1. Calculate $p = \frac{\partial f}{\partial x}$ and $q = \frac{\partial f}{\partial y}$ 2. Apply the formula: $\frac{dy}{dx} = -\frac{p}{q}$
To find $\frac{d^2y}{dx^2}$: 1. Calculate the five partial derivatives: - $p = \frac{\partial f}{\partial x}$ - $q = \frac{\partial f}{\partial y}$ - $r = \frac{\partial^2 f}{\partial x^2}$ - $s = \frac{\partial^2 f}{\partial x \partial y}$ - $t = \frac{\partial^2 f}{\partial y^2}$ 2. Apply the formula: $\frac{d^2y}{dx^2} = -\frac{q^2r-2pqs+p^2t}{q^3}$
Alternatively, you can use the determinant form:

$$\frac{d^2y}{dx^2} = -\frac{1}{q^3} \begin{vmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{vmatrix} \tag{3.226}$$

3.2.4 Implicit Functions of Three Variables

The p-q notation extends naturally to functions of three variables.

Implicit Function of Three Variables

If $F(x, y, z) = 0$ defines z implicitly as a function of x and y, and $\frac{\partial F}{\partial z} \neq 0$, then using the notation:

$$p = \frac{\partial F}{\partial x}, \quad q = \frac{\partial F}{\partial y}, \quad r = \frac{\partial F}{\partial z} \tag{3.227}$$

The partial derivatives of z are:

$$\frac{\partial z}{\partial x} = -\frac{p}{r} \tag{3.228}$$

$$\frac{\partial z}{\partial y} = -\frac{q}{r} \tag{3.229}$$

Partial Derivatives of a Surface

For the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, or $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$:
Calculate:

$$p = F_x = \frac{2x}{a^2} \tag{3.230}$$

$$q = F_y = \frac{2y}{b^2} \tag{3.231}$$

$$r = F_z = \frac{2z}{c^2} \tag{3.232}$$

The partial derivatives are:

$$\frac{\partial z}{\partial x} = -\frac{p}{r} = -\frac{\frac{2x}{a^2}}{\frac{2z}{c^2}} = -\frac{c^2 x}{a^2 z} \tag{3.233}$$

$$\frac{\partial z}{\partial y} = -\frac{q}{r} = -\frac{\frac{2y}{b^2}}{\frac{2z}{c^2}} = -\frac{c^2 y}{b^2 z} \tag{3.234}$$

3.2.5 Mixed Partial Derivatives for Functions of Three Variables

For a function $z = f(x, y)$ defined implicitly by $F(x, y, z) = 0$, we can find the mixed partial derivatives using the p-q-r notation.

Mixed Partial Derivatives

If $F(x, y, z) = 0$ defines z as a function of x and y, and we define:

$$p = \frac{\partial F}{\partial x} \qquad q = \frac{\partial F}{\partial y} \qquad r = \frac{\partial F}{\partial z} \tag{3.235}$$

$$p_x = \frac{\partial^2 F}{\partial x^2} \qquad p_y = \frac{\partial^2 F}{\partial x \partial y} \qquad p_z = \frac{\partial^2 F}{\partial x \partial z} \tag{3.236}$$

$$q_x = \frac{\partial^2 F}{\partial y \partial x} \qquad q_y = \frac{\partial^2 F}{\partial y^2} \qquad q_z = \frac{\partial^2 F}{\partial y \partial z} \tag{3.237}$$

$$r_x = \frac{\partial^2 F}{\partial z \partial x} \qquad r_y = \frac{\partial^2 F}{\partial z \partial y} \qquad r_z = \frac{\partial^2 F}{\partial z^2} \tag{3.238}$$

Then:

$$\frac{\partial^2 z}{\partial x^2} = -\frac{r^2 p_x - 2prp_z + p^2 r_z - pr_x z_x + p^2 r_z z_x}{r^3} \tag{3.239}$$
$$\frac{\partial^2 z}{\partial y \partial x} = -\frac{r^2 p_y - prq_z - qrp_z + pqr_z - pr_y z_x - qr_x z_y + pqr_z z_x}{r^3} \tag{3.240}$$

The formulas for higher derivatives can be quite complex, but they follow from consistent application of the chain rule.

3.2.6 Memory Aids and Shortcuts

To help remember these formulas:

Memory Aids

1. **First Derivative Rule:** "Negative p over q" - $\frac{dy}{dx} = -\frac{p}{q}$
2. **Second Derivative Mnemonic:** "Negative hessian over q-cubed" - The numerator follows the pattern of a 2×2 determinant from the Hessian matrix:

$$\frac{d^2 y}{dx^2} = -\frac{q^2 r - 2pqs + p^2 t}{q^3} \tag{3.241}$$

3. **The Determinant Form:** For visual learners, the determinant form can be easier to remember:

$$\frac{d^2 y}{dx^2} = -\frac{1}{q^3} \begin{vmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{vmatrix} \tag{3.242}$$

4. **Three-Variable Rule:** "Negative derivatives over r" - $\frac{\partial z}{\partial x} = -\frac{p}{r}$, $\frac{\partial z}{\partial y} = -\frac{q}{r}$

3.3 The Total Differential and Total Derivative

3.3.1 Total Differential

The total differential represents the approximate change in a function due to small changes in all its independent variables.

Definition 3.2 (Total Differential)**.** *For a function $z = f(x, y)$, the total differential dz is:*

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Computing the Total Differential

For $f(x, y) = x^2 y + \sin(xy)$:
First, we find the partial derivatives:

$$\frac{\partial f}{\partial x} = 2xy + y\cos(xy) \tag{3.243}$$
$$\frac{\partial f}{\partial y} = x^2 + x\cos(xy) \tag{3.244}$$

The total differential is:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{3.245}$$

$$= [2xy + y\cos(xy)]dx + [x^2 + x\cos(xy)]dy \tag{3.246}$$

This expression gives the approximate change in f when (x, y) changes by (dx, dy).

3.3.2 Total Derivative

When a function depends on multiple variables that in turn depend on a single parameter, the total derivative gives the rate of change with respect to that parameter.

Definition 3.3 (Total Derivative for Functions of Two Variables)**.** *If $z = f(x, y)$ where $x = x(t)$ and $y = y(t)$ are functions of a parameter t, then the total derivative of z with respect to t is:*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Definition 3.4 (Total Derivative for Functions of Three Variables)**.** *If $w = f(x, y, z)$ where $x = x(t)$, $y = y(t)$, and $z = z(t)$ are functions of a parameter t, then the total derivative of w with respect to t is:*

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

General Form of Total Derivative

For a function $F(x_1, x_2, \ldots, x_n)$ where each $x_i = x_i(t)$ depends on a parameter t, the total derivative is:

$$\frac{dF}{dt} = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \cdot \frac{dx_i}{dt}$$

This is often written using the chain rule notation:

$$\frac{dF}{dt} = \nabla F \cdot \frac{d\vec{x}}{dt}$$

where ∇F is the gradient of F and $\frac{d\vec{x}}{dt}$ is the vector of derivatives of each variable with respect to t.

Calculating the Total Derivative - Two Variables

Consider $z = x^2y$ where $x = \cos(t)$ and $y = \sin(t)$. We have:

$$\frac{\partial z}{\partial x} = 2xy = 2\cos(t)\sin(t) \tag{3.247}$$

$$\frac{\partial z}{\partial y} = x^2 = \cos^2(t) \tag{3.248}$$

$$\frac{dx}{dt} = -\sin(t) \tag{3.249}$$

$$\frac{dy}{dt} = \cos(t) \tag{3.250}$$

The total derivative is:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \tag{3.251}$$
$$= 2\cos(t)\sin(t) \cdot (-\sin(t)) + \cos^2(t) \cdot \cos(t) \tag{3.252}$$
$$= -2\cos(t)\sin^2(t) + \cos^3(t) \tag{3.253}$$
$$= \cos(t)[\cos^2(t) - 2\sin^2(t)] \tag{3.254}$$
$$= \cos(t)[1 - \sin^2(t) - 2\sin^2(t)] \tag{3.255}$$
$$= \cos(t)[1 - 3\sin^2(t)] \tag{3.256}$$

This gives the rate of change of z as the point $(x, y) = (\cos(t), \sin(t))$ moves along the unit circle.

Calculating the Total Derivative - Three Variables

Consider $w = xyz$ where $x = t$, $y = t^2$, and $z = e^t$. We have:

$$\frac{\partial w}{\partial x} = yz = t^2 e^t \tag{3.257}$$
$$\frac{\partial w}{\partial y} = xz = te^t \tag{3.258}$$
$$\frac{\partial w}{\partial z} = xy = t \cdot t^2 = t^3 \tag{3.259}$$
$$\frac{dx}{dt} = 1 \tag{3.260}$$
$$\frac{dy}{dt} = 2t \tag{3.261}$$
$$\frac{dz}{dt} = e^t \tag{3.262}$$

The total derivative is:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt} \tag{3.263}$$
$$= t^2 e^t \cdot 1 + te^t \cdot 2t + t^3 \cdot e^t \tag{3.264}$$
$$= t^2 e^t + 2t^2 e^t + t^3 e^t \tag{3.265}$$
$$= e^t(t^2 + 2t^2 + t^3) \tag{3.266}$$
$$= e^t(3t^2 + t^3) \tag{3.267}$$
$$= t^2 e^t(3 + t) \tag{3.268}$$

This gives the rate of change of w as the point $(x, y, z) = (t, t^2, e^t)$ moves along the parametric curve in 3D space.

3.4 Solved Examples

Application of Total Derivative

Problem: Find $\frac{du}{dx}$ if $u = x \cdot \log(xy)$ and $x^3 + y^3 + 3xy = 0$.

Solution Using Total Derivative Concept

This problem requires applying the concept of total derivative, where y is implicitly a function of x.
Step 1: Apply the total derivative concept.
Since u is a function of both x and y, and y is implicitly a function of x, we use the total derivative formula:

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \tag{3.269}$$

Step 2: Calculate the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.
For $u = x \cdot \log(xy)$:

$$\frac{\partial u}{\partial x} = \log(xy) + x \cdot \frac{\partial}{\partial x}[\log(xy)] \tag{3.270}$$

$$= \log(xy) + x \cdot \frac{1}{xy} \cdot y \tag{3.271}$$

$$= \log(xy) + 1 \tag{3.272}$$

Similarly:

$$\frac{\partial u}{\partial y} = x \cdot \frac{\partial}{\partial y}[\log(xy)] \tag{3.273}$$

$$= x \cdot \frac{1}{xy} \cdot x \tag{3.274}$$

$$= \frac{x^2}{xy} = \frac{x}{y} \tag{3.275}$$

Step 3: Find $\frac{dy}{dx}$ from the constraint $x^3 + y^3 + 3xy = 0$.
Using the implicit function rule:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \tag{3.276}$$

$$= -\frac{3x^2 + 3y}{3y^2 + 3x} \tag{3.277}$$

$$= -\frac{x^2 + y}{y^2 + x} \tag{3.278}$$

Step 4: Substitute into the total derivative formula.

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \tag{3.279}$$

$$= \log(xy) + 1 + \frac{x}{y} \cdot \left(-\frac{x^2 + y}{y^2 + x}\right) \tag{3.280}$$

$$= \log(xy) + 1 - \frac{x(x^2 + y)}{y(y^2 + x)} \tag{3.281}$$

Step 5: Simplify using the constraint relation.
From $x^3 + y^3 + 3xy = 0$, we know $y^3 = -x^3 - 3xy$.
We can simplify the denominator $y(y^2 + x)$:

$$y(y^2 + x) = y^3 + xy \tag{3.282}$$
$$= -x^3 - 3xy + xy \tag{3.283}$$
$$= -x^3 - 2xy \tag{3.284}$$

Therefore:

$$\frac{du}{dx} = \log(xy) + 1 - \frac{x(x^2 + y)}{-x^3 - 2xy} \tag{3.285}$$
$$= \log(xy) + 1 + \frac{x(x^2 + y)}{x^3 + 2xy} \tag{3.286}$$

Factoring out x from numerator and denominator:

$$\frac{du}{dx} = \log(xy) + 1 + \frac{x(x^2 + y)}{x(x^2 + 2y)} \tag{3.287}$$
$$= \log(xy) + 1 + \frac{x^2 + y}{x^2 + 2y} \tag{3.288}$$

Final Answer:

$$\boxed{\frac{du}{dx} = \log(xy) + 1 + \frac{x^2 + y}{x^2 + 2y}} \tag{3.289}$$

Conceptual Understanding: This problem illustrates the application of the total derivative concept where the rate of change of u with respect to x has two components: 1. The direct effect of changing x while keeping y constant ($\frac{\partial u}{\partial x}$) 2. The indirect effect through the change in y as x changes ($\frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$)
The constraint equation creates an implicit relationship between x and y, requiring us to first find $\frac{dy}{dx}$ and then apply the total derivative formula.

Application of Total Derivative with Implicit Relation

Problem: Find $\frac{dz}{dx}$ if $z = x^2y$ and $x^2 + xy + y^2 = 1$.

Solution Using Total Derivative Concept

This is a typical problem where we need to apply the total derivative concept since y is implicitly related to x through the constraint equation.
Step 1: Apply the total derivative formula.
Since $z = z(x, y)$ where y is implicitly a function of x, we use:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \tag{3.290}$$

Step 2: Calculate the partial derivatives of z.

For $z = x^2y$:

$$\frac{\partial z}{\partial x} = 2xy \tag{3.291}$$

$$\frac{\partial z}{\partial y} = x^2 \tag{3.292}$$

Step 3: Find $\frac{dy}{dx}$ from the constraint $x^2 + xy + y^2 = 1$.
Let $f(x, y) = x^2 + xy + y^2 - 1 = 0$
Then:

$$\frac{\partial f}{\partial x} = 2x + y \tag{3.293}$$

$$\frac{\partial f}{\partial y} = x + 2y \tag{3.294}$$

Using the implicit function formula:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \tag{3.295}$$

$$= -\frac{2x + y}{x + 2y} \tag{3.296}$$

Step 4: Substitute into the total derivative formula.

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \tag{3.297}$$

$$= 2xy + x^2 \cdot \left(-\frac{2x + y}{x + 2y}\right) \tag{3.298}$$

$$= 2xy - \frac{x^2(2x + y)}{x + 2y} \tag{3.299}$$

This can be written as:

$$\frac{dz}{dx} = 2xy - \frac{2x^3 + x^2y}{x + 2y} \tag{3.300}$$

The answer demonstrates how the total derivative combines both the direct effect of x on z and the indirect effect through the change in y.

Implicit Differentiation of a Three-Variable Equation

Problem: If $z^3 - zx - y = 4$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Detailed Solution

This problem requires using implicit differentiation to find the partial derivatives of z with respect to x and y.
Step 1: Set up notation.
Let's define $F(x, y, z) = z^3 - zx - y - 4 = 0$.

Then according to the implicit function theorem, we have:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \tag{3.301}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \tag{3.302}$$

Using the p, q, r notation from our earlier work:

$$p = \frac{\partial F}{\partial x} \tag{3.303}$$

$$q = \frac{\partial F}{\partial y} \tag{3.304}$$

$$r = \frac{\partial F}{\partial z} \tag{3.305}$$

So we need:

$$\frac{\partial z}{\partial x} = -\frac{p}{r} \tag{3.306}$$

$$\frac{\partial z}{\partial y} = -\frac{q}{r} \tag{3.307}$$

Step 2: Compute the partial derivatives of F.

$$p = \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(z^3 - zx - y - 4) \tag{3.308}$$

$$= -z \tag{3.309}$$

$$q = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(z^3 - zx - y - 4) \tag{3.310}$$

$$= -1 \tag{3.311}$$

$$r = \frac{\partial F}{\partial z} = \frac{\partial}{\partial z}(z^3 - zx - y - 4) \tag{3.312}$$

$$= 3z^2 - x \tag{3.313}$$

Step 3: Substitute to find $\frac{\partial z}{\partial x}$.

$$\frac{\partial z}{\partial x} = -\frac{p}{r} \tag{3.314}$$

$$= -\frac{-z}{3z^2 - x} \tag{3.315}$$

$$= \frac{z}{3z^2 - x} \tag{3.316}$$

Step 4: Substitute to find $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial y} = -\frac{q}{r} \tag{3.317}$$
$$= -\frac{-1}{3z^2 - x} \tag{3.318}$$
$$= \frac{1}{3z^2 - x} \tag{3.319}$$

Final Answers:

$$\frac{\partial z}{\partial x} = \frac{z}{3z^2 - x} \tag{3.320}$$
$$\frac{\partial z}{\partial y} = \frac{1}{3z^2 - x} \tag{3.321}$$

Implicit Differentiation with Exponential Functions

Problem: Find $\frac{dy}{dx}$, given $(\cos x)^y = (\sin y)^x$.

Implicit Differentiation with Exponential Functions

Problem: Find $\frac{dy}{dx}$, given $(\cos x)^y = (\sin y)^x$.
Solution: We will use the implicit function theorem. First, let's define:

$$f(x, y) = (\cos x)^y - (\sin y)^x = 0$$

To find $\frac{dy}{dx}$, we need to compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, then use the formula:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Step 1: Let's take the natural logarithm of both sides of the original equation to simplify our work.

$$(\cos x)^y = (\sin y)^x$$
$$\ln[(\cos x)^y] = \ln[(\sin y)^x]$$
$$y \ln(\cos x) = x \ln(\sin y)$$

Step 2: Now, let's redefine our function as:

$$f(x, y) = y \ln(\cos x) - x \ln(\sin y) = 0$$

Step 3: Compute $\frac{\partial f}{\partial x}$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[y \ln(\cos x) - x \ln(\sin y)] \tag{3.322}$$
$$= y \frac{\partial}{\partial x}[\ln(\cos x)] - \ln(\sin y) \tag{3.323}$$
$$= y \cdot \frac{1}{\cos x} \cdot \frac{\partial}{\partial x}[\cos x] - \ln(\sin y) \tag{3.324}$$
$$= y \cdot \frac{1}{\cos x} \cdot (-\sin x) - \ln(\sin y) \tag{3.325}$$
$$= -y \frac{\sin x}{\cos x} - \ln(\sin y) \tag{3.326}$$
$$= -y \tan x - \ln(\sin y) \tag{3.327}$$

Step 4: Compute $\frac{\partial f}{\partial y}$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[y\ln(\cos x) - x\ln(\sin y)] \tag{3.328}$$
$$= \ln(\cos x) - x\frac{\partial}{\partial y}[\ln(\sin y)] \tag{3.329}$$
$$= \ln(\cos x) - x\cdot\frac{1}{\sin y}\cdot\frac{\partial}{\partial y}[\sin y] \tag{3.330}$$
$$= \ln(\cos x) - x\cdot\frac{1}{\sin y}\cdot\cos y \tag{3.331}$$
$$= \ln(\cos x) - x\frac{\cos y}{\sin y} \tag{3.332}$$
$$= \ln(\cos x) - x\cot y \tag{3.333}$$

Step 5: Apply the formula for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \tag{3.334}$$
$$= -\frac{-y\tan x - \ln(\sin y)}{\ln(\cos x) - x\cot y} \tag{3.335}$$
$$= \frac{y\tan x + \ln(\sin y)}{\ln(\cos x) - x\cot y} \tag{3.336}$$

Step 6: We can simplify this by substituting back our original equation. Since $(\cos x)^y = (\sin y)^x$, we have $y\ln(\cos x) = x\ln(\sin y)$, which means:

$$\ln(\sin y) = \frac{y\ln(\cos x)}{x}$$

Substituting this:

$$\frac{dy}{dx} = \frac{y\tan x + \ln(\sin y)}{\ln(\cos x) - x\cot y} \tag{3.337}$$
$$= \frac{y\tan x + \frac{y\ln(\cos x)}{x}}{\ln(\cos x) - x\cot y} \tag{3.338}$$
$$= \frac{y\tan x\cdot x + y\ln(\cos x)}{x\ln(\cos x) - x^2\cot y} \tag{3.339}$$
$$= \frac{xy\tan x + y\ln(\cos x)}{x\ln(\cos x) - x^2\cot y} \tag{3.340}$$

This is our final expression for $\frac{dy}{dx}$ given the implicit equation $(\cos x)^y = (\sin y)^x$.

Implicit Differentiation with Multiple Transcendental Terms

Problem: If $(\tan x)^y + y^{\cot x} = a$, find $\frac{dy}{dx}$.

Solution

Step 1: Let's define the implicit function .

$$f(x,y) = (\tan x)^y + y^{\cot x} - a = 0 \tag{3.341}$$

Step 2: Find the partial derivatives of f with respect to x and y.
For $\frac{\partial f}{\partial x}$, we need to differentiate each term separately:
(a) For $(\tan x)^y$, we use the formula for differentiating with respect to the base:

$$\frac{\partial}{\partial x}[(\tan x)^y] = y(\tan x)^{y-1} \cdot \frac{\partial}{\partial x}[\tan x] \tag{3.342}$$

$$= y(\tan x)^{y-1} \cdot \sec^2 x \tag{3.343}$$

(b) For $y^{\cot x}$, we use logarithmic differentiation:

$$\frac{\partial}{\partial x}[y^{\cot x}] = y^{\cot x} \cdot \frac{\partial}{\partial x}[\cot x] \cdot \ln y \tag{3.344}$$

$$= y^{\cot x} \cdot (-\csc^2 x) \cdot \ln y \tag{3.345}$$

$$= -y^{\cot x} \cdot \csc^2 x \cdot \ln y \tag{3.346}$$

Therefore:

$$\frac{\partial f}{\partial x} = y(\tan x)^{y-1} \sec^2 x - y^{\cot x} \cdot \csc^2 x \cdot \ln y \tag{3.347}$$

$$= y(\tan x)^{y-1} \sec^2 x + y^{\cot x} \cdot \ln y \cdot (-\csc^2 x) \tag{3.348}$$

This matches the textbook's expression:

$$\frac{\partial f}{\partial x} = y(\tan x)^{y-1} \sec^2 x + y^{\cot x} \cdot \ln y \cdot (-\csc^2 x) \tag{3.349}$$

Step 3: Now find $\frac{\partial f}{\partial y}$.
(a) For $(\tan x)^y$, we use the formula for differentiating with respect to the exponent:

$$\frac{\partial}{\partial y}[(\tan x)^y] = (\tan x)^y \cdot \ln(\tan x) \cdot \frac{\partial}{\partial y}[y] \tag{3.350}$$

$$= (\tan x)^y \cdot \ln(\tan x) \tag{3.351}$$

(b) For $y^{\cot x}$, we use the formula for differentiating with respect to the base:

$$\frac{\partial}{\partial y}[y^{\cot x}] = \cot x \cdot y^{\cot x - 1} \cdot \frac{\partial}{\partial y}[y] \tag{3.352}$$

$$= \cot x \cdot y^{\cot x - 1} \tag{3.353}$$

Therefore:

$$\frac{\partial f}{\partial y} = (\tan x)^y \cdot \ln(\tan x) + \cot x \cdot y^{\cot x - 1} \tag{3.354}$$

This matches the textbook's expression:

$$\frac{\partial f}{\partial y} = (\tan x)^y \cdot \ln \tan x + (\cot x) y^{\cot x - 1} \tag{3.355}$$

Step 4: Apply the implicit function formula to find $\frac{dy}{dx}$.
From the implicit function theorem, we have:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \tag{3.356}$$

$$= -\frac{y(\tan x)^{y-1}\sec^2 x - y^{\cot x}\cdot \ln y \cdot \csc^2 x}{(\tan x)^y \cdot \ln \tan x + (\cot x) y^{\cot x - 1}} \tag{3.357}$$

Final Answer:

$$\boxed{\frac{dy}{dx} = -\frac{y(\tan x)^{y-1}\sec^2 x - y^{\cot x}\cdot \ln y \cdot \csc^2 x}{(\tan x)^y \cdot \ln \tan x + (\cot x) y^{\cot x - 1}}} \tag{3.358}$$

3.5 Change of Variables in Partial Derivatives

3.5.1 Basic Principles

In many problems, it's advantageous to transform from one coordinate system to another. This requires understanding how partial derivatives transform under such changes.

Definition 3.5 (Change of Variables)**.** *A change of variables from (x, y) to (u, v) is defined by the functions:*

$$x = x(u, v) \tag{3.359}$$

$$y = y(u, v) \tag{3.360}$$

The inverse transformation, if it exists, is given by:

$$u = u(x, y) \tag{3.361}$$

$$v = v(x, y) \tag{3.362}$$

3.5.2 Transforming Partial Derivatives

To find how partial derivatives transform, we apply the chain rule.

Transformation of First Partial Derivatives

If $z = f(x, y)$ and we change variables to (u, v) where $x = x(u, v)$ and $y = y(u, v)$, then:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \tag{3.363}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \tag{3.364}$$

Conversely, if we have $z = g(u, v)$ where $u = u(x, y)$ and $v = v(x, y)$, we can express the original partial derivatives in terms of the new ones:

Inverse Transformation of Partial Derivatives

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \tag{3.365}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} \tag{3.366}$$

Transforming to Polar Coordinates

In polar coordinates, the transformation equations are:

$$x = r\cos\theta \tag{3.367}$$

$$y = r\sin\theta \tag{3.368}$$

The inverse transformation is:

$$r = \sqrt{x^2 + y^2} \tag{3.369}$$

$$\theta = \arctan\left(\frac{y}{x}\right) \tag{3.370}$$

To find $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, we compute the necessary partial derivatives:

$$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \tag{3.371}$$

$$\frac{\partial y}{\partial r} = \sin\theta, \qquad \frac{\partial y}{\partial \theta} = r\cos\theta \tag{3.372}$$

Using the transformation formulas:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} \tag{3.373}$$

$$= \frac{\partial z}{\partial x} \cdot \cos\theta + \frac{\partial z}{\partial y} \cdot \sin\theta \tag{3.374}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta} \tag{3.375}$$

$$= \frac{\partial z}{\partial x} \cdot (-r\sin\theta) + \frac{\partial z}{\partial y} \cdot r\cos\theta \tag{3.376}$$

$$= r\left(-\frac{\partial z}{\partial x} \cdot \sin\theta + \frac{\partial z}{\partial y} \cdot \cos\theta\right) \tag{3.377}$$

Conversely, to express $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of polar derivatives, we compute:

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} = \cos\theta \tag{3.378}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r} = \sin\theta \tag{3.379}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r} \tag{3.380}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{x}{r^2} = \frac{\cos\theta}{r} \tag{3.381}$$

Using these values:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} \tag{3.382}$$

$$= \frac{\partial z}{\partial r} \cdot \cos\theta + \frac{\partial z}{\partial \theta} \cdot \left(-\frac{\sin\theta}{r}\right) \tag{3.383}$$

$$= \frac{\partial z}{\partial r} \cdot \cos\theta - \frac{1}{r}\frac{\partial z}{\partial \theta} \cdot \sin\theta \tag{3.384}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} \tag{3.385}$$

$$= \frac{\partial z}{\partial r} \cdot \sin\theta + \frac{\partial z}{\partial \theta} \cdot \frac{\cos\theta}{r} \tag{3.386}$$

$$= \frac{\partial z}{\partial r} \cdot \sin\theta + \frac{1}{r}\frac{\partial z}{\partial \theta} \cdot \cos\theta \tag{3.387}$$

In this chapter, we have explored the theory and applications of partial derivatives of composite functions, the chain rule, the total derivative, and change of variables. These concepts are fundamental in understanding how functions of several variables behave under various transformations and are essential tools in solving complex problems in mathematics, physics, engineering, and other fields.