Chapter 3

Vector Differential Calculus II: Scalar and Vector Fields

Vector differential calculus extends from the analysis of vector functions to the study of scalar and vector fields. These fields are fundamental in describing various physical phenomena such as temperature distributions, fluid flow, electromagnetic fields, and gravitational fields.

3.1 Scalar Fields and Level Surfaces

3.1.1 Definition and Properties

Definition 3.1 (Scalar Field). *A scalar field ϕ in three-dimensional space assigns a scalar value to each point (x, y, z) in a region of space:*

$$\phi : \mathbb{R}^3 \to \mathbb{R} \tag{3.1}$$

Examples of scalar fields include:

- Temperature distribution in a room: $\phi_T(x, y, z)$
- Pressure in a fluid: $\phi_p(x, y, z)$
- Electric potential: $\phi_V(x, y, z)$
- Gravitational potential: $\phi_G(x, y, z)$

Scalar Field

The function $\phi(x, y, z) = x^2+y^2+z^2$ defines a scalar field whose value at any point (x, y, z) equals the square of the distance from that point to the origin.

3.1.2 Level Surfaces

Definition 3.2 (Level Surface). *A level surface of a scalar field $\phi(x, y, z)$ is the set of all points where ϕ has a constant value c:*

$$\phi(x, y, z) = c \tag{3.2}$$

Level surfaces provide a visual representation of scalar fields. For a two-dimensional scalar field $\phi(x, y)$, we have level curves instead of level surfaces.

Level Surfaces

For the scalar field $\phi(x, y, z) = x^2 + y^2 + z^2$, the level surfaces are spheres centered at the

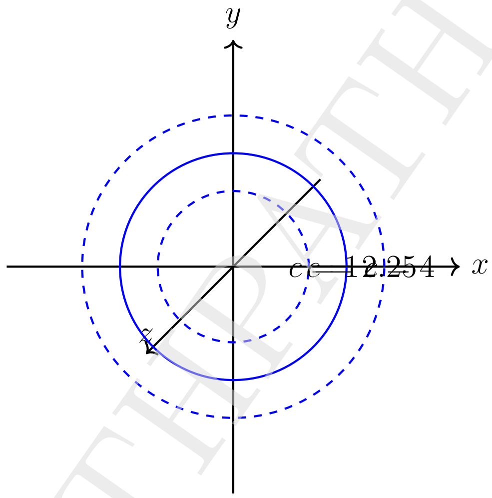

Figure 3.1: Level surfaces (shown as circles in this 2D projection) of the scalar field $\phi(x, y, z) = x^2 + y^2 + z^2$

origin:

$$x^2 + y^2 + z^2 = c \tag{3.3}$$

where $c > 0$ is a positive constant.
For the scalar field $\phi(x, y, z) = x^2 + y^2$, the level surfaces are circular cylinders aligned with the z-axis:

$$x^2 + y^2 = c \tag{3.4}$$

3.1.3 Scalar Field Operations

Operations on scalar fields include:

- Addition/subtraction: $(\phi \pm \psi)(x, y, z) = \phi(x, y, z) \pm \psi(x, y, z)$
- Multiplication/division: $(\phi \cdot \psi)(x, y, z) = \phi(x, y, z) \cdot \psi(x, y, z)$, $(\phi/\psi)(x, y, z) = \phi(x, y, z)/\psi(x, y, z)$ (where $\psi \neq 0$)
- Composition: $(\phi \circ \psi)(x, y, z) = \phi(\psi(x, y, z))$

3.2 Vector Fields

3.2.1 Definition and Properties

Definition 3.3 (Vector Field)**.** *A vector field* $\bar{\mathbf{F}}$ *in three-dimensional space assigns a vector to each point* (x, y, z) *in a region of space:*

$$\bar{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3 \tag{3.5}$$

In component form:

$$\bar{\mathbf{F}}(x, y, z) = F_1(x, y, z)\hat{\mathbf{i}} + F_2(x, y, z)\hat{\mathbf{j}} + F_3(x, y, z)\hat{\mathbf{k}} \tag{3.6}$$

Examples of vector fields include:

- Velocity field of a fluid: $\bar{\mathbf{v}}(x, y, z, t)$
- Electric field: $\bar{\mathbf{E}}(x, y, z)$

- Magnetic field: $\bar{\mathbf{B}}(x, y, z)$
- Gravitational field: $\bar{\mathbf{g}}(x, y, z)$

Vector Field

The vector field $\bar{\mathbf{F}}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ assigns to each point (x, y, z) a vector pointing away from the origin with magnitude equal to the distance from the point to the origin.

3.2.2 Visualizing Vector Fields

Vector fields can be visualized using:

- Arrow plots: Arrows represent vectors at various points
- Streamlines: Curves that are tangent to the vector field at each point
- Field line density: Represents the magnitude of the field

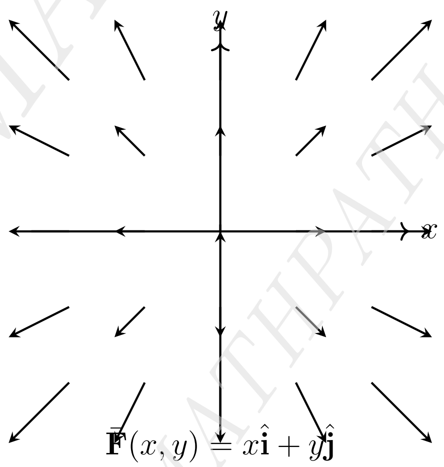

Figure 3.2: Arrow plot of the radial vector field $\bar{\mathbf{F}}(x, y) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$

3.2.3 Vector Field Operations

Basic operations on vector fields include:

- Addition/subtraction: $(\bar{\mathbf{F}} \pm \bar{\mathbf{G}})(x, y, z) = \bar{\mathbf{F}}(x, y, z) \pm \bar{\mathbf{G}}(x, y, z)$
- Scalar multiplication: $(c\bar{\mathbf{F}})(x, y, z) = c \cdot \bar{\mathbf{F}}(x, y, z)$
- Dot product: $(\bar{\mathbf{F}} \cdot \bar{\mathbf{G}})(x, y, z) = \bar{\mathbf{F}}(x, y, z) \cdot \bar{\mathbf{G}}(x, y, z)$
- Cross product: $(\bar{\mathbf{F}} \times \bar{\mathbf{G}})(x, y, z) = \bar{\mathbf{F}}(x, y, z) \times \bar{\mathbf{G}}(x, y, z)$

3.3 Vector Differential Operator

3.3.1 The Del Operator

The del operator ∇ (pronounced "del" or "nabla") is a vector differential operator that forms the foundation of vector calculus:

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \tag{3.7}$$

This symbolic operator is applied to scalar and vector fields to produce important vector calculus operations.

3.4 Gradient of a Scalar Field

3.4.1 Definition and Properties

The gradient of a scalar field $\phi(x, y, z)$, denoted $\nabla\phi$ or grad(ϕ), is a vector field that points in the direction of the maximum rate of increase of the scalar field.

Definition 3.4 (Gradient). *The gradient of a scalar field $\phi(x, y, z)$ is:*

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}} \tag{3.8}$$

Properties of the Gradient

For scalar fields ϕ and ψ and a constant c:
1. $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$ (Linearity)
2. $\nabla(c\phi) = c\nabla\phi$ (Scalar multiplication)
3. $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$ (Product rule)
4. $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$ (Quotient rule)

3.4.2 Geometric Interpretation

The gradient has several important geometric interpretations:

- The gradient $\nabla\phi$ at a point is perpendicular to the level surface of ϕ through that point
- The magnitude of the gradient $|\nabla\phi|$ gives the maximum rate of change of ϕ
- The gradient points in the direction of steepest ascent of ϕ

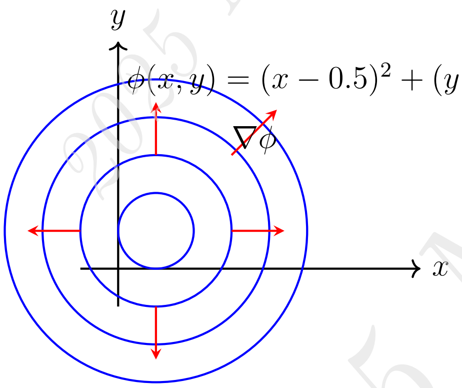

Figure 3.3: Gradient vectors (red arrows) perpendicular to level curves of the scalar field $\phi(x, y) = (x - 0.5)^2 + (y - 0.5)^2$

Computing the Gradient

Calculate the gradient of $\phi(x, y, z) = xy^2z^3$.
Applying the definition:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}} \tag{3.9}$$

$$= y^2z^3\hat{\mathbf{i}} + 2xyz^3\hat{\mathbf{j}} + 3xy^2z^2\hat{\mathbf{k}} \tag{3.10}$$

3.5 Directional Derivatives

3.5.1 Definition and Properties

While the gradient gives the maximum rate of change, the directional derivative measures the rate of change in any specified direction.

Definition 3.5 (Directional Derivative). *The directional derivative of a scalar field $\phi(x, y, z)$ in the direction of a unit vector $\hat{\mathbf{u}}$ is:*

$$D_{\hat{\mathbf{u}}}\phi = \nabla\phi \cdot \hat{\mathbf{u}} \tag{3.11}$$

Properties of Directional Derivatives

1. The directional derivative in the direction of the gradient gives the maximum rate of change: $D_{\nabla\phi/|\nabla\phi|}\phi = |\nabla\phi|$
2. The directional derivative in the direction perpendicular to the gradient is zero
3. $D_{\hat{\mathbf{u}}}\phi = |\nabla\phi| \cos\theta$, where θ is the angle between $\nabla\phi$ and $\hat{\mathbf{u}}$

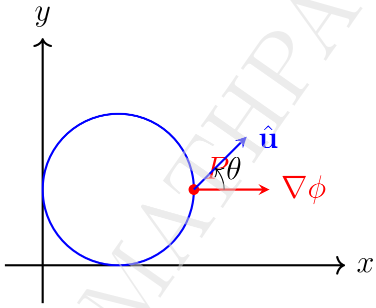

Figure 3.4: Directional derivative: gradient $\nabla\phi$ and direction vector $\hat{\mathbf{u}}$ at point P

Computing a Directional Derivative

Calculate the directional derivative of $\phi(x, y, z) = x^2 + y^2 + z^2$ at the point $(1, 2, 3)$ in the direction of $\mathbf{v} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$.

First, find the gradient:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}} \tag{3.12}$$

$$= 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}} \tag{3.13}$$

At the point $(1, 2, 3)$:

$$\nabla\phi(1, 2, 3) = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}} \tag{3.14}$$

The direction vector $\mathbf{v}$ needs to be normalized:

$$|\mathbf{v}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \tag{3.15}$$

$$\hat{\mathbf{u}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \tag{3.16}$$

Now, calculate the directional derivative:

$$D_{\hat{\mathbf{u}}}\phi = \nabla\phi \cdot \hat{\mathbf{u}} \tag{3.17}$$
$$= (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \cdot \frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \tag{3.18}$$
$$= \frac{1}{\sqrt{3}}(2 + 4 + 6) \tag{3.19}$$
$$= \frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3} \tag{3.20}$$

3.5.2 Relationship with Partial Derivatives

Partial derivatives are special cases of directional derivatives along the coordinate axes:

$$\frac{\partial\phi}{\partial x} = D_{\hat{\mathbf{i}}}\phi \tag{3.21}$$
$$\frac{\partial\phi}{\partial y} = D_{\hat{\mathbf{j}}}\phi \tag{3.22}$$
$$\frac{\partial\phi}{\partial z} = D_{\hat{\mathbf{k}}}\phi \tag{3.23}$$

3.6 Angles Between Surfaces

3.6.1 Definition and Calculation

The angle between two intersecting surfaces can be determined using their normal vectors at the point of intersection.

Definition 3.6 (Angle Between Surfaces). *If $\phi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$ are two intersecting surfaces, the angle θ between them at a point of intersection is:*

$$\cos\theta = \frac{\nabla\phi \cdot \nabla\psi}{|\nabla\phi||\nabla\psi|} \tag{3.24}$$

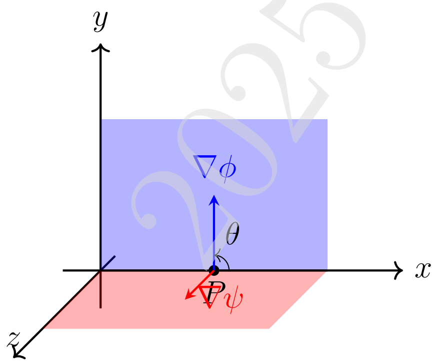

Figure 3.5: Angle between two intersecting surfaces represented by their normal vectors $\nabla\phi$ and $\nabla\psi$

Angle Between Surfaces

Find the angle between the surfaces $\phi(x,y,z) = x^2+y^2+z^2 = 4$ (a sphere) and $\psi(x,y,z) = x+y+z = 2$ (a plane) at the point $(1,1,\sqrt{2})$.
First, compute the gradients:

$$\nabla\phi = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}} \tag{3.25}$$

$$\nabla\psi = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \tag{3.26}$$

At the point $(1,1,\sqrt{2})$:

$$\nabla\phi(1,1,\sqrt{2}) = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\sqrt{2}\hat{\mathbf{k}} \tag{3.27}$$

$$\nabla\psi(1,1,\sqrt{2}) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \tag{3.28}$$

Calculate their magnitudes:

$$|\nabla\phi| = \sqrt{2^2+2^2+(2\sqrt{2})^2} = \sqrt{4+4+8} = \sqrt{16} = 4 \tag{3.29}$$

$$|\nabla\psi| = \sqrt{1^2+1^2+1^2} = \sqrt{3} \tag{3.30}$$

Now, calculate the dot product:

$$\nabla\phi\cdot\nabla\psi = 2\cdot 1 + 2\cdot 1 + 2\sqrt{2}\cdot 1 \tag{3.31}$$

$$= 2+2+2\sqrt{2} \tag{3.32}$$

$$= 4+2\sqrt{2} \tag{3.33}$$

Finally, calculate the cosine of the angle:

$$\cos\theta = \frac{\nabla\phi\cdot\nabla\psi}{|\nabla\phi||\nabla\psi|} \tag{3.34}$$

$$= \frac{4+2\sqrt{2}}{4\cdot\sqrt{3}} \tag{3.35}$$

$$= \frac{4+2\sqrt{2}}{4\sqrt{3}} \tag{3.36}$$

$$= \frac{4+2\sqrt{2}}{4\sqrt{3}} \tag{3.37}$$

Therefore:

$$\theta = \cos^{-1}\left(\frac{4+2\sqrt{2}}{4\sqrt{3}}\right) \tag{3.38}$$

3.6.2 Orthogonal Surfaces

Two surfaces are orthogonal at their point of intersection if their normal vectors (gradients) are perpendicular:

$$\nabla\phi\cdot\nabla\psi = 0 \tag{3.39}$$

3.7 Solved Examples on tangent, acceleration, angle, Divergence , Curl

Example 1: Velocity and Acceleration of a Particle

Find velocity and acceleration and their magnitudes for a particle moving along the curve $x = t^2$, $y = 2t + 1$, $z = t^3$ at $t = 1$.

Solution

Given:

$$\begin{aligned} x &= t^2 \\ y &= 2t + 1 \\ z &= t^3 \end{aligned}$$

Step 1: Find the position vector $\vec{r}(t)$ of the particle.

$$\begin{aligned} \vec{r}(t) &= x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \\ &= t^2\vec{i} + (2t+1)\vec{j} + t^3\vec{k} \end{aligned}$$

Step 2: Find the velocity vector $\vec{v}(t)$ by differentiating the position vector with respect to t.

$$\begin{aligned} \vec{v}(t) &= \frac{d\vec{r}(t)}{dt} \\ &= \frac{d}{dt}(t^2\vec{i} + (2t+1)\vec{j} + t^3\vec{k}) \\ &= \frac{d}{dt}(t^2)\vec{i} + \frac{d}{dt}(2t+1)\vec{j} + \frac{d}{dt}(t^3)\vec{k} \\ &= 2t\vec{i} + 2\vec{j} + 3t^2\vec{k} \end{aligned}$$

Step 3: Evaluate the velocity vector at $t = 1$.

$$\begin{aligned} \vec{v}(1) &= 2(1)\vec{i} + 2\vec{j} + 3(1)^2\vec{k} \\ &= 2\vec{i} + 2\vec{j} + 3\vec{k} \end{aligned}$$

Step 4: Calculate the magnitude of the velocity vector at $t = 1$.

$$\begin{aligned} |\vec{v}(1)| &= \sqrt{(2)^2 + (2)^2 + (3)^2} \\ &= \sqrt{4+4+9} \\ &= \sqrt{17} \\ &\approx 4.123 \end{aligned}$$

Step 5: Find the acceleration vector $\vec{a}(t)$ by differentiating the velocity vector with respect to t.

$$\begin{aligned} \vec{a}(t) &= \frac{d\vec{v}(t)}{dt} \\ &= \frac{d}{dt}(2t\vec{i} + 2\vec{j} + 3t^2\vec{k}) \\ &= \frac{d}{dt}(2t)\vec{i} + \frac{d}{dt}(2)\vec{j} + \frac{d}{dt}(3t^2)\vec{k} \\ &= 2\vec{i} + 0\vec{j} + 6t\vec{k} \\ &= 2\vec{i} + 6t\vec{k} \end{aligned}$$

Step 6: Evaluate the acceleration vector at $t = 1$.

$$\begin{aligned}\vec{a}(1) &= 2\vec{i} + 6(1)\vec{k} \\ &= 2\vec{i} + 6\vec{k}\end{aligned}$$

Step 7: Calculate the magnitude of the acceleration vector at $t = 1$.

$$\begin{aligned}|\vec{a}(1)| &= \sqrt{(2)^2 + (0)^2 + (6)^2} \\ &= \sqrt{4 + 0 + 36} \\ &= \sqrt{40} \\ &= 2\sqrt{10} \\ &\approx 6.325\end{aligned}$$

Therefore, at $t = 1$:

- Velocity vector: $\vec{v}(1) = 2\vec{i} + 2\vec{j} + 3\vec{k}$
- Velocity magnitude: $|\vec{v}(1)| = \sqrt{17} \approx 4.123$ units/sec
- Acceleration vector: $\vec{a}(1) = 2\vec{i} + 6\vec{k}$
- Acceleration magnitude: $|\vec{a}(1)| = 2\sqrt{10} \approx 6.325$ units/sec^2

Example 2: Finding Constants for Orthogonal Surfaces

Find the constants a and b, so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at point $(1, -1, 2)$.

Solution

Given:

$$\begin{aligned}&\text{First surface: } ax^2 - byz = (a+2)x \\ &\text{Second surface: } 4x^2y + z^3 = 4 \\ &\text{Point of intersection: } (1, -1, 2)\end{aligned}$$

Step 1: Rewrite the first surface equation in the form $F(x, y, z) = 0$.

$$ax^2 - byz - (a+2)x = 0$$

Step 2: For the second surface, we have

$$G(x, y, z) = 4x^2y + z^3 - 4 = 0$$

Step 3: For two surfaces to be orthogonal at a point, their normal vectors must be perpendicular. The normal vector to a surface is given by the gradient.
The normal vector to the first surface is:

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

Calculate the partial derivatives:

$$\begin{aligned}\frac{\partial F}{\partial x} &= 2ax-(a+2)\\ \frac{\partial F}{\partial y} &= -bz\\ \frac{\partial F}{\partial z} &= -by\end{aligned}$$

Therefore, the normal vector to the first surface is:

$$\nabla F = (2ax-(a+2), -bz, -by)$$

Step 4: Find the normal vector to the second surface.

$$\nabla G = \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}\right)$$

Calculate the partial derivatives:

$$\begin{aligned}\frac{\partial G}{\partial x} &= 8xy\\ \frac{\partial G}{\partial y} &= 4x^2\\ \frac{\partial G}{\partial z} &= 3z^2\end{aligned}$$

Therefore, the normal vector to the second surface is:

$$\nabla G = (8xy, 4x^2, 3z^2)$$

Step 5: Evaluate the normal vectors at the point $(1,-1,2)$.
For the first surface:

$$\begin{aligned}\nabla F|_{(1,-1,2)} &= (2a(1)-(a+2), -b(2), -b(-1))\\ &= (2a-a-2, -2b, b)\\ &= (a-2, -2b, b)\end{aligned}$$

For the second surface:

$$\begin{aligned}\nabla G|_{(1,-1,2)} &= (8(1)(-1), 4(1)^2, 3(2)^2)\\ &= (-8, 4, 12)\end{aligned}$$

Step 6: For the surfaces to be orthogonal, the dot product of their normal vectors must be zero.

$$\begin{aligned}\nabla F|_{(1,-1,2)} \cdot \nabla G|_{(1,-1,2)} &= 0\\ (a-2, -2b, b)\cdot(-8,4,12) &= 0\\ (a-2)(-8)+(-2b)(4)+(b)(12) &= 0\\ -8a+16-8b+12b &= 0\\ -8a+4b &= -16\\ -2a+b &= -4\end{aligned}$$

Step 7: We need another equation to solve for a and b. We must verify that the point $(1, -1, 2)$ lies on both surfaces.
For the second surface:

$$\begin{aligned}4x^2y + z^3 &= 4\\4(1)^2(-1) + (2)^3 &= 4\\-4 + 8 &= 4\\4 &= 4\checkmark\end{aligned}$$

This confirms that the point $(1, -1, 2)$ does lie on the second surface. Now let's check if the point lies on the first surface:

$$\begin{aligned}ax^2 - byz &= (a + 2)x\\a(1)^2 - b(-1)(2) &= (a + 2)(1)\\a + 2b &= a + 2\\2b &= 2\\b &= 1\end{aligned}$$

Substituting $b = 1$ into our equation $-2a + b = -4$:

$$\begin{aligned}-2a + 1 &= -4\\-2a &= -5\\a &= \frac{5}{2}\end{aligned}$$

Therefore, $a = \frac{5}{2}$ and $b = 1$.
Step 8: Let's verify our solution by checking if the point $(1, -1, 2)$ lies on both surfaces and that the surfaces are orthogonal at this point:
For the first surface with $a = \frac{5}{2}$ and $b = 1$:

$$\begin{aligned}ax^2 - byz &= (a + 2)x\\\frac{5}{2}(1)^2 - (1)(-1)(2) &= (\frac{5}{2} + 2)(1)\\\frac{5}{2} + 2 &= \frac{5}{2} + 2\\\frac{9}{2} &= \frac{9}{2}\checkmark\end{aligned}$$

For orthogonality with $a = \frac{5}{2}$ and $b = 1$:

$$\begin{aligned}-2a + b &= -4\\-2(\frac{5}{2}) + 1 &= -4\\-5 + 1 &= -4\\-4 &= -4\checkmark\end{aligned}$$

Therefore, the constants are:

$$\begin{aligned}a &= \frac{5}{2}\\b &= 1\end{aligned}$$

Example 3: Finding Angle Between Tangents

A curve is given by the equations $x = t^2 + 1$, $y = 4t - 3$, $z = 2t^2 - 6t$. Find the angle between tangents at $t = 1$ and at $t = 2$.

Solution

Given:

$$\begin{aligned} x &= t^2 + 1 \\ y &= 4t - 3 \\ z &= 2t^2 - 6t \end{aligned}$$

Step 1: Find the tangent vectors at any point on the curve by differentiating the position vector with respect to t.
The position vector at any point is:

$$\vec{r}(t) = (t^2 + 1)\vec{i} + (4t - 3)\vec{j} + (2t^2 - 6t)\vec{k}$$

The tangent vector is:

$$\begin{aligned} \vec{r}'(t) &= \frac{d\vec{r}(t)}{dt} \\ &= \frac{d}{dt}(t^2 + 1)\vec{i} + \frac{d}{dt}(4t - 3)\vec{j} + \frac{d}{dt}(2t^2 - 6t)\vec{k} \\ &= 2t\vec{i} + 4\vec{j} + (4t - 6)\vec{k} \end{aligned}$$

Step 2: Find the tangent vector at $t = 1$.

$$\begin{aligned} \vec{r}'(1) &= 2(1)\vec{i} + 4\vec{j} + (4(1) - 6)\vec{k} \\ &= 2\vec{i} + 4\vec{j} + (-2)\vec{k} \\ &= 2\vec{i} + 4\vec{j} - 2\vec{k} \end{aligned}$$

Step 3: Find the tangent vector at $t = 2$.

$$\begin{aligned} \vec{r}'(2) &= 2(2)\vec{i} + 4\vec{j} + (4(2) - 6)\vec{k} \\ &= 4\vec{i} + 4\vec{j} + 2\vec{k} \end{aligned}$$

Step 4: To find the angle between these two tangent vectors, we use the dot product formula:

$$\cos\theta = \frac{\vec{r}'(1) \cdot \vec{r}'(2)}{|\vec{r}'(1)||\vec{r}'(2)|}$$

Step 5: Calculate the dot product of the two tangent vectors.

$$\begin{aligned} \vec{r}'(1) \cdot \vec{r}'(2) &= (2\vec{i} + 4\vec{j} - 2\vec{k}) \cdot (4\vec{i} + 4\vec{j} + 2\vec{k}) \\ &= 2 \cdot 4 + 4 \cdot 4 + (-2) \cdot 2 \\ &= 8 + 16 - 4 \\ &= 20 \end{aligned}$$

Step 6: Calculate the magnitudes of both tangent vectors.

$$\begin{aligned}|\vec{r}'(1)| &= \sqrt{(2)^2+(4)^2+(-2)^2}\\ &= \sqrt{4+16+4}\\ &= \sqrt{24}\\ &= 2\sqrt{6}\end{aligned}$$

$$\begin{aligned}|\vec{r}'(2)| &= \sqrt{(4)^2+(4)^2+(2)^2}\\ &= \sqrt{16+16+4}\\ &= \sqrt{36}\\ &= 6\end{aligned}$$

Step 7: Calculate the cosine of the angle.

$$\begin{aligned}\cos\theta &= \frac{\vec{r}'(1)\cdot\vec{r}'(2)}{|\vec{r}'(1)||\vec{r}'(2)|}\\ &= \frac{20}{2\sqrt{6}\cdot 6}\\ &= \frac{20}{12\sqrt{6}}\\ &= \frac{5}{3\sqrt{6}}\end{aligned}$$

Step 8: Find the angle θ.

$$\theta = \cos^{-1}\left(\frac{5}{3\sqrt{6}}\right)$$

Therefore, the angle between the tangents at $t=1$ and at $t=2$ is:

$$\theta = \cos^{-1}\left[\frac{5}{3\sqrt{6}}\right]$$

Example 4: Finding Condition for Normal Acceleration

The position vector of a particle at a time t is $\vec{r} = \cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + mt^3\hat{k}$. Find the condition imposed on m by requiring that at $t=1$, the acceleration is normal to position vector.

Solution

Given:

$$\vec{r}(t) = \cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + mt^3\hat{k}$$

Step 1: Find the position vector at $t = 1$.

$$\begin{aligned}\vec{r}(1) &= \cos(1-1)\hat{\imath} + \sinh(1-1)\hat{\jmath} + m(1)^3\hat{k} \\ &= \cos(0)\hat{\imath} + \sinh(0)\hat{\jmath} + m\hat{k} \\ &= 1\hat{\imath} + 0\hat{\jmath} + m\hat{k} \\ &= \hat{\imath} + m\hat{k}\end{aligned}$$

Step 2: Find the velocity vector by differentiating the position vector with respect to t.

$$\begin{aligned}\vec{v}(t) &= \frac{d\vec{r}(t)}{dt} \\ &= \frac{d}{dt}[\cos(t-1)\hat{\imath}] + \frac{d}{dt}[\sinh(t-1)\hat{\jmath}] + \frac{d}{dt}[mt^3\hat{k}] \\ &= -\sin(t-1)\hat{\imath} + \cosh(t-1)\hat{\jmath} + 3mt^2\hat{k}\end{aligned}$$

Step 3: Find the acceleration vector by differentiating the velocity vector with respect to t.

$$\begin{aligned}\vec{a}(t) &= \frac{d\vec{v}(t)}{dt} \\ &= \frac{d}{dt}[-\sin(t-1)\hat{\imath}] + \frac{d}{dt}[\cosh(t-1)\hat{\jmath}] + \frac{d}{dt}[3mt^2\hat{k}] \\ &= -\cos(t-1)\hat{\imath} + \sinh(t-1)\hat{\jmath} + 6mt\hat{k}\end{aligned}$$

Step 4: Find the acceleration vector at $t = 1$.

$$\begin{aligned}\vec{a}(1) &= -\cos(1-1)\hat{\imath} + \sinh(1-1)\hat{\jmath} + 6m(1)\hat{k} \\ &= -\cos(0)\hat{\imath} + \sinh(0)\hat{\jmath} + 6m\hat{k} \\ &= -1\hat{\imath} + 0\hat{\jmath} + 6m\hat{k} \\ &= -\hat{\imath} + 6m\hat{k}\end{aligned}$$

Step 5: For the acceleration to be normal to the position vector at $t = 1$, their dot product must be zero.

$$\begin{aligned}\vec{r}(1) \cdot \vec{a}(1) &= 0 \\ (\hat{\imath} + m\hat{k}) \cdot (-\hat{\imath} + 6m\hat{k}) &= 0 \\ 1 \cdot (-1) + m \cdot 6m &= 0 \\ -1 + 6m^2 &= 0 \\ 6m^2 &= 1 \\ m^2 &= \frac{1}{6} \\ m &= \pm\frac{1}{\sqrt{6}}\end{aligned}$$

Therefore, the condition imposed on m is:

$$m = \pm\frac{1}{\sqrt{6}} = \pm\frac{1}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \pm\frac{\sqrt{6}}{6}$$

Example 5: Finding Angle Between Tangents

Find angle between the tangents at $t = 1$ and $t = 2$ to the curve $\vec{r} = t^2\vec{i}+(t^3-2t)\vec{j}+(3t-4)\vec{k}$

Solution

Given:

$$\vec{r}(t) = t^2\vec{i} + (t^3 - 2t)\vec{j} + (3t - 4)\vec{k}$$

Step 1: Find the tangent vector by differentiating the position vector with respect to t.

$$\begin{aligned}\vec{T}(t) &= \frac{d\vec{r}(t)}{dt} \\ &= \frac{d}{dt}[t^2\vec{i} + (t^3 - 2t)\vec{j} + (3t - 4)\vec{k}] \\ &= 2t\vec{i} + (3t^2 - 2)\vec{j} + 3\vec{k}\end{aligned}$$

Step 2: Find the tangent vector at $t = 1$.

$$\begin{aligned}\vec{T}(1) &= 2(1)\vec{i} + (3(1)^2 - 2)\vec{j} + 3\vec{k} \\ &= 2\vec{i} + (3 - 2)\vec{j} + 3\vec{k} \\ &= 2\vec{i} + \vec{j} + 3\vec{k}\end{aligned}$$

Step 3: Find the tangent vector at $t = 2$.

$$\begin{aligned}\vec{T}(2) &= 2(2)\vec{i} + (3(2)^2 - 2)\vec{j} + 3\vec{k} \\ &= 4\vec{i} + (12 - 2)\vec{j} + 3\vec{k} \\ &= 4\vec{i} + 10\vec{j} + 3\vec{k}\end{aligned}$$

Step 4: To find the angle between these two tangent vectors, we use the dot product formula:

$$\cos\theta = \frac{\vec{T}(1) \cdot \vec{T}(2)}{|\vec{T}(1)||\vec{T}(2)|}$$

Step 5: Calculate the dot product of the two tangent vectors.

$$\begin{aligned}\vec{T}(1) \cdot \vec{T}(2) &= (2\vec{i} + \vec{j} + 3\vec{k}) \cdot (4\vec{i} + 10\vec{j} + 3\vec{k}) \\ &= 2 \cdot 4 + 1 \cdot 10 + 3 \cdot 3 \\ &= 8 + 10 + 9 \\ &= 27\end{aligned}$$

Step 6: Calculate the magnitudes of both tangent vectors.

$$\begin{aligned}|\vec{T}(1)| &= \sqrt{(2)^2 + (1)^2 + (3)^2} \\ &= \sqrt{4 + 1 + 9} \\ &= \sqrt{14}\end{aligned}$$

$$\begin{aligned}|\vec{T}(2)| &= \sqrt{(4)^2 + (10)^2 + (3)^2} \\ &= \sqrt{16 + 100 + 9} \\ &= \sqrt{125} \\ &= 5\sqrt{5}\end{aligned}$$

Step 7: Calculate the cosine of the angle.

$$\begin{aligned}\cos\theta &= \frac{\vec{T}(1)\cdot\vec{T}(2)}{|\vec{T}(1)||\vec{T}(2)|}\\ &= \frac{27}{\sqrt{14}\cdot 5\sqrt{5}}\\ &= \frac{27}{5\sqrt{70}}\\ &= \frac{27}{5\sqrt{70}}\cdot\frac{\sqrt{70}}{\sqrt{70}}\\ &= \frac{27\sqrt{70}}{5\cdot 70}\\ &= \frac{27\sqrt{70}}{350}\end{aligned}$$

Step 8: Find the angle θ.

$$\theta = \cos^{-1}\left(\frac{27\sqrt{70}}{350}\right)$$

Therefore, the angle between the tangents at $t = 1$ and at $t = 2$ is:

$$\theta = \cos^{-1}\left(\frac{27\sqrt{70}}{350}\right)$$

Example 6: Finding Gradients of Scalar Functions

Find Gradient of scalar functions
i) $\phi(x, y, z) = x^2yz^3$ at point P (-1, -2, 3)
ii) $\phi(x, y, z) = xy^2 + yz^2 + zx^2$ at point P (3, -2, 1)

Solution

The gradient of a scalar function $\phi(x, y, z)$ is defined as:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Part (i): Find $\nabla\phi$ where $\phi(x, y, z) = x^2yz^3$ at point P(-1, -2, 3)
Step 1: Calculate the partial derivatives of ϕ.

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= 2xyz^3\\ \frac{\partial\phi}{\partial y} &= x^2z^3\\ \frac{\partial\phi}{\partial z} &= 3x^2yz^2\end{aligned}$$

Step 2: Evaluate these partial derivatives at the point P(-1, -2, 3).

$$\begin{aligned}\left.\frac{\partial \phi}{\partial x}\right|_P &= 2(-1)(-2)(3)^3 \\ &= 2(1)(-2)(27) \\ &= 2(-2)(27) \\ &= -108\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial \phi}{\partial y}\right|_P &= (-1)^2(3)^3 \\ &= 1(27) \\ &= 27\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial \phi}{\partial z}\right|_P &= 3(-1)^2(-2)(3)^2 \\ &= 3(1)(-2)(9) \\ &= 3(-2)(9) \\ &= -54\end{aligned}$$

Step 3: Construct the gradient vector.

$$\begin{aligned}\nabla\phi|_P &= \left.\frac{\partial \phi}{\partial x}\right|_P \hat{i} + \left.\frac{\partial \phi}{\partial y}\right|_P \hat{j} + \left.\frac{\partial \phi}{\partial z}\right|_P \hat{k} \\ &= -108\hat{i} + 27\hat{j} - 54\hat{k}\end{aligned}$$

Part (ii): Find $\nabla\phi$ where $\phi(x, y, z) = xy^2 + yz^2 + zx^2$ at point P(3, -2, 1)
Step 1: Calculate the partial derivatives of ϕ.

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= y^2 + 2zx \\ \frac{\partial \phi}{\partial y} &= 2xy + z^2 \\ \frac{\partial \phi}{\partial z} &= 2yz + x^2\end{aligned}$$

Step 2: Evaluate these partial derivatives at the point P(3, -2, 1).

$$\begin{aligned}\left.\frac{\partial \phi}{\partial x}\right|_P &= (-2)^2 + 2(1)(3) \\ &= 4 + 6 \\ &= 10\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial \phi}{\partial y}\right|_P &= 2(3)(-2) + (1)^2 \\ &= 2(3)(-2) + 1 \\ &= -12 + 1 \\ &= -11\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial \phi}{\partial z}\right|_P &= 2(-2)(1) + (3)^2 \\ &= -4 + 9 \\ &= 5\end{aligned}$$

Step 3: Construct the gradient vector.

$$\begin{aligned}\nabla\phi|_P &= \left.\frac{\partial \phi}{\partial x}\right|_P \hat{i} + \left.\frac{\partial \phi}{\partial y}\right|_P \hat{j} + \left.\frac{\partial \phi}{\partial z}\right|_P \hat{k} \\ &= 10\hat{i} - 11\hat{j} + 5\hat{k}\end{aligned}$$

Example 7: Finding Divergence and Curl of Vector Functions

Find Divergence and Curl of vector functions.
i) $\vec{F} = (x^2y)\hat{i} + (y^2z)\hat{j} + (z^2x)\hat{k}$ at point P (2, 2, 1)
ii) $\vec{F} = (xyz)\hat{i} + (xy + yz)\hat{j} + (x + y + z)\hat{k}$ at point P (2, -2, -1)

Solution

For a vector field $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, the divergence and curl are defined as:
Divergence:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Which expands to:

$$\nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

Part (i): For $\vec{F} = (x^2y)\hat{i} + (y^2z)\hat{j} + (z^2x)\hat{k}$ at point P (2, 2, 1)

Step 1: Identify the components of $\vec{F}$.

$$F_1 = x^2 y$$
$$F_2 = y^2 z$$
$$F_3 = z^2 x$$

Step 2: Calculate the partial derivatives needed for divergence.

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(x^2 y) = 2xy$$
$$\frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}(y^2 z) = 2yz$$
$$\frac{\partial F_3}{\partial z} = \frac{\partial}{\partial z}(z^2 x) = 2zx$$

Step 3: Calculate the divergence.

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= 2xy + 2yz + 2zx\end{aligned}$$

Step 4: Evaluate the divergence at point P(2, 2, 1).

$$\begin{aligned}\nabla \cdot \vec{F}|_P &= 2(2)(2) + 2(2)(1) + 2(1)(2) \\ &= 8 + 4 + 4 \\ &= 16\end{aligned}$$

Step 5: Calculate the partial derivatives needed for curl.

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(x^2 y) = x^2$$
$$\frac{\partial F_1}{\partial z} = \frac{\partial}{\partial z}(x^2 y) = 0$$
$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(y^2 z) = 0$$
$$\frac{\partial F_2}{\partial z} = \frac{\partial}{\partial z}(y^2 z) = y^2$$
$$\frac{\partial F_3}{\partial x} = \frac{\partial}{\partial x}(z^2 x) = z^2$$
$$\frac{\partial F_3}{\partial y} = \frac{\partial}{\partial y}(z^2 x) = 0$$

Step 6: Calculate the curl.

$$\begin{aligned}\nabla \times \vec{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k} \\ &= (0 - y^2)\hat{\imath} + (0 - z^2)\hat{\jmath} + (0 - x^2)\hat{k} \\ &= -y^2\hat{\imath} - z^2\hat{\jmath} - x^2\hat{k}\end{aligned}$$

Step 7: Evaluate the curl at point P(2, 2, 1).

$$\begin{aligned}\nabla \times \vec{F}|_P &= -(2)^2\hat{\imath} - (1)^2\hat{\jmath} - (2)^2\hat{k} \\ &= -4\hat{\imath} - 1\hat{\jmath} - 4\hat{k} \\ &= -4\hat{\imath} - \hat{\jmath} - 4\hat{k}\end{aligned}$$

Part (ii): For $\vec{F} = (xyz)\hat{i} + (xy + yz)\hat{j} + (x + y + z)\hat{k}$ at point P (2, -2, -1)
Step 1: Identify the components of $\vec{F}$.

$$\begin{aligned} F_1 &= xyz \\ F_2 &= xy + yz \\ F_3 &= x + y + z \end{aligned}$$

Step 2: Calculate the partial derivatives needed for divergence.

$$\begin{aligned} \frac{\partial F_1}{\partial x} &= \frac{\partial}{\partial x}(xyz) = yz \\ \frac{\partial F_2}{\partial y} &= \frac{\partial}{\partial y}(xy + yz) = x + z \\ \frac{\partial F_3}{\partial z} &= \frac{\partial}{\partial z}(x + y + z) = 1 \end{aligned}$$

Step 3: Calculate the divergence.

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= yz + x + z + 1 \\ &= yz + x + z + 1 \end{aligned}$$

Step 4: Evaluate the divergence at point P(2, -2, -1).

$$\begin{aligned} \nabla \cdot \vec{F}|_P &= (-2)(-1) + 2 + (-1) + 1 \\ &= 2 + 2 - 1 + 1 \\ &= 4 \end{aligned}$$

Step 5: Calculate the partial derivatives needed for curl.

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}(xyz) = xz \\ \frac{\partial F_1}{\partial z} &= \frac{\partial}{\partial z}(xyz) = xy \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}(xy + yz) = y \\ \frac{\partial F_2}{\partial z} &= \frac{\partial}{\partial z}(xy + yz) = y \\ \frac{\partial F_3}{\partial x} &= \frac{\partial}{\partial x}(x + y + z) = 1 \\ \frac{\partial F_3}{\partial y} &= \frac{\partial}{\partial y}(x + y + z) = 1 \end{aligned}$$

Step 6: Calculate the curl.

$$\begin{aligned} \nabla \times \vec{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k} \\ &= (1 - y)\hat{i} + (xy - 1)\hat{j} + (y - xz)\hat{k} \end{aligned}$$

Step 7: Evaluate the curl at point P(2, -2, -1).

$$\begin{aligned}\nabla \times \vec{F}|_P &= (1-(-2))\hat{\imath} + ((2)(-1)-1)\hat{\jmath} + ((-2)-(2)(-1))\hat{k} \\ &= (1+2)\hat{\imath} + (-2-1)\hat{\jmath} + (-2-(-2))\hat{k} \\ &= 3\hat{\imath} - 3\hat{\jmath} + 0\hat{k} \\ &= 3\hat{\imath} - 3\hat{\jmath}\end{aligned}$$

3.8 Solved Examples on Directional Derivative

In this section, we'll solve various examples related to directional derivatives to illustrate their applications and calculation techniques.

Example 1: Directional Derivative along a Curve

Find the directional derivative of $\phi = e^{2x}\cos yz$ at the origin in the direction tangent to the curve $x = a\sin t$, $y = a\cos t$, $z = at$ at $t = \pi/4$.

Solution

Given:

$$\begin{aligned}\phi &= e^{2x}\cos yz \\ \text{Curve: } \vec{C} &= x\vec{\imath} + y\vec{\jmath} + z\vec{k} \\ &= a\sin t\ \vec{\imath} + a\cos t\ \vec{\jmath} + at\ \vec{k}\end{aligned}$$

Step 1: Find the gradient of ϕ.

$$\begin{aligned}\nabla\phi &= \vec{\imath}\frac{\partial\phi}{\partial x} + \vec{\jmath}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\ &= \vec{\imath}(2e^{2x}\cos yz) + \vec{\jmath}(-ze^{2x}\sin yz) + \vec{k}(-ye^{2x}\sin yz)\end{aligned}$$

Step 2: Evaluate $\nabla\phi$ at the origin (0,0,0).

$$\begin{aligned}\nabla\phi|_{(0,0,0)} &= \vec{\imath}(2e^0\cos(0\cdot 0)) + \vec{\jmath}(-0\cdot e^0\sin(0\cdot 0)) + \vec{k}(-0\cdot e^0\sin(0\cdot 0)) \\ &= 2\vec{\imath} + 0\vec{\jmath} + 0\vec{k}\end{aligned}$$

Step 3: Find the tangent to the curve at $t = \pi/4$.

$$\vec{a} = \frac{d\vec{C}}{dt} = a\cos t\ \vec{\imath} - a\sin t\ \vec{\jmath} + a\vec{k}$$

At $t = \pi/4$:

$$\begin{aligned}\vec{a}|_{t=\pi/4} &= a\cos(\pi/4)\ \vec{\imath} - a\sin(\pi/4)\ \vec{\jmath} + a\vec{k} \\ &= \frac{a}{\sqrt{2}}\vec{\imath} - \frac{a}{\sqrt{2}}\vec{\jmath} + a\vec{k}\end{aligned}$$

Step 4: Find the unit vector in the direction of $\vec{a}$.

$$\begin{aligned}
\hat{a} = \frac{\vec{a}}{|\vec{a}|} &= \frac{\frac{a}{\sqrt{2}}\vec{i} - \frac{a}{\sqrt{2}}\vec{j} + a\vec{k}}{\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + \left(-\frac{a}{\sqrt{2}}\right)^2 + a^2}} \\
&= \frac{\frac{a}{\sqrt{2}}\vec{i} - \frac{a}{\sqrt{2}}\vec{j} + a\vec{k}}{\sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2}} \\
&= \frac{\frac{a}{\sqrt{2}}\vec{i} - \frac{a}{\sqrt{2}}\vec{j} + a\vec{k}}{\sqrt{2a^2}} \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\frac{a}{\sqrt{2}}\vec{i} - \frac{a}{\sqrt{2}}\vec{j} + a\vec{k}}{a} \\
&= \frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}
\end{aligned}$$

Step 5: Calculate the directional derivative.

$$\begin{aligned}
D \cdot D = \nabla\phi \cdot \hat{a} & \\
&= (2\vec{i} + 0\vec{j} + 0\vec{k}) \cdot \left(\frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}\right) \\
&= 2 \cdot \frac{1}{2} + 0 \cdot \left(-\frac{1}{2}\right) + 0 \cdot \frac{1}{\sqrt{2}} \\
&= 1
\end{aligned}$$

Therefore, the directional derivative of $\phi = e^{2x}\cos yz$ at the origin in the direction tangent to the given curve at $t = \pi/4$ is 1.

Example 2: Directional Derivatives in Different Directions

Find the directional derivative of the function $\phi = xy^2 + yz^3$ at $(1, -1, 1)$:

1. Along the direction normal to the surface $2x^2 + y^2 + 2z^2 = 9$ at $(1, 2, 1)$
2. Along the vector $\vec{i} + 2\vec{j} + 2\vec{k}$
3. Towards the point $(2, 1, -1)$

Solution

Given: $\phi = xy^2 + yz^3$

Step 1: Find the gradient of ϕ.

$$\begin{aligned}
\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\
&= \vec{i}(y^2) + \vec{j}(2xy + z^3) + \vec{k}(3yz^2)
\end{aligned}$$

Step 2: Evaluate $\nabla\phi$ at the point $(1, -1, 1)$.

$$\begin{aligned}
\nabla\phi|_{(1,-1,1)} &= \vec{i}((-1)^2) + \vec{j}(2(1)(-1) + (1)^3) + \vec{k}(3(-1)(1)^2) \\
&= \vec{i}(1) + \vec{j}(-2 + 1) + \vec{k}(-3) \\
&= \vec{i} - \vec{j} - 3\vec{k}
\end{aligned}$$

Part (i): Along the direction normal to the surface $2x^2 + y^2 + 2z^2 = 9$ at $(1, 2, 1)$.
To find the normal vector to the surface, we take the gradient of the surface equation:

$$\begin{aligned}\nabla u &= \nabla(2x^2 + y^2 + 2z^2)\\ &= \vec{i}(4x) + \vec{j}(2y) + \vec{k}(4z)\end{aligned}$$

At the point $(1, 2, 1)$:

$$\begin{aligned}\nabla u|_{(1,2,1)} &= \vec{i}(4(1)) + \vec{j}(2(2)) + \vec{k}(4(1))\\ &= 4\vec{i} + 4\vec{j} + 4\vec{k}\end{aligned}$$

The unit normal vector is:

$$\begin{aligned}\hat{a} = \frac{\nabla u}{|\nabla u|} &= \frac{4\vec{i} + 4\vec{j} + 4\vec{k}}{\sqrt{4^2 + 4^2 + 4^2}}\\ &= \frac{4\vec{i} + 4\vec{j} + 4\vec{k}}{\sqrt{48}}\\ &= \frac{4\vec{i} + 4\vec{j} + 4\vec{k}}{4\sqrt{3}}\\ &= \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})\end{aligned}$$

Now calculating the directional derivative:

$$\begin{aligned}D \cdot D &= \nabla\phi \cdot \hat{a}\\ &= (\vec{i} - \vec{j} - 3\vec{k}) \cdot \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})\\ &= \frac{1}{\sqrt{3}}(1 \cdot 1 + (-1) \cdot 1 + (-3) \cdot 1)\\ &= \frac{1}{\sqrt{3}}(1 - 1 - 3)\\ &= \frac{1}{\sqrt{3}}(-3)\\ &= -\frac{3}{\sqrt{3}} = -\sqrt{3}\end{aligned}$$

Part (ii): Along the vector $\vec{i} + 2\vec{j} + 2\vec{k}$.
First, we need to normalize the given vector:

$$\begin{aligned}|\vec{i} + 2\vec{j} + 2\vec{k}| &= \sqrt{1^2 + 2^2 + 2^2}\\ &= \sqrt{1 + 4 + 4}\\ &= \sqrt{9} = 3\end{aligned}$$

The unit vector in this direction is:

$$\begin{aligned}\hat{a} &= \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}\\ &= \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\end{aligned}$$

Now calculating the directional derivative:

$$
\begin{aligned}
D \cdot D &= \nabla\phi \cdot \hat{a} \\
&= (\vec{i} - \vec{j} - 3\vec{k}) \cdot \left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right) \\
&= \frac{1}{3} - \frac{2}{3} - 3 \cdot \frac{2}{3} \\
&= \frac{1}{3} - \frac{2}{3} - \frac{6}{3} \\
&= \frac{1-2-6}{3} \\
&= \frac{-7}{3}
\end{aligned}
$$

Part (iii): Towards the point $(2, 1, -1)$.
The vector from $(1, -1, 1)$ to $(2, 1, -1)$ is:

$$
\begin{aligned}
\vec{a} &= (2-1)\vec{i} + (1-(-1))\vec{j} + (-1-1)\vec{k} \\
&= \vec{i} + 2\vec{j} - 2\vec{k}
\end{aligned}
$$

The magnitude of this vector is:

$$
\begin{aligned}
|\vec{a}| &= \sqrt{1^2 + 2^2 + (-2)^2} \\
&= \sqrt{1+4+4} \\
&= \sqrt{9} = 3
\end{aligned}
$$

The unit vector in this direction is:

$$
\begin{aligned}
\hat{a} &= \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} + 2\vec{j} - 2\vec{k}}{3} \\
&= \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}
\end{aligned}
$$

Now calculating the directional derivative:

$$
\begin{aligned}
D \cdot D &= \nabla\phi \cdot \hat{a} \\
&= (\vec{i} - \vec{j} - 3\vec{k}) \cdot \left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}\right) \\
&= \frac{1}{3} - \frac{2}{3} - 3 \cdot \left(-\frac{2}{3}\right) \\
&= \frac{1}{3} - \frac{2}{3} + \frac{6}{3} \\
&= \frac{1-2+6}{3} \\
&= \frac{5}{3}
\end{aligned}
$$

Therefore:

1. The directional derivative along the normal to the surface is $-\sqrt{3}$.
2. The directional derivative along the vector $\vec{i} + 2\vec{j} + 2\vec{k}$ is $-\frac{7}{3}$.
3. The directional derivative towards the point $(2, 1, -1)$ is $\frac{5}{3}$.

Example 3: Directional Derivative for Maximum Value

If the directional derivative of $\phi = axy + byz + czx$ at $(1, 1, 1)$ has maximum magnitude 4 in the direction parallel to the x-axis, find the values of a, b, and c.

Solution

Given:

$$\phi = axy + byz + czx$$

Step 1: Find the gradient of ϕ.

$$\begin{aligned}\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\\ &= \vec{i}(ay + cz) + \vec{j}(ax + bz) + \vec{k}(by + cx)\end{aligned}$$

Step 2: Evaluate $\nabla\phi$ at the point $(1, 1, 1)$.

$$\begin{aligned}\nabla\phi|_{(1,1,1)} &= \vec{i}(a\cdot 1 + c\cdot 1) + \vec{j}(a\cdot 1 + b\cdot 1) + \vec{k}(b\cdot 1 + c\cdot 1)\\ &= \vec{i}(a + c) + \vec{j}(a + b) + \vec{k}(b + c)\end{aligned}$$

Step 3: Find the direction parallel to the x-axis. The unit vector in the direction of the x-axis is $\hat{a} = \vec{i}$.
Step 4: Calculate the directional derivative.

$$\begin{aligned}D\cdot D &= \nabla\phi\cdot\hat{a}\\ &= [(a + c)\vec{i} + (a + b)\vec{j} + (b + c)\vec{k}]\cdot\vec{i}\\ &= a + c\end{aligned}$$

We are told that this has a maximum magnitude of 4, so:

$$|a + c| = 4$$

This gives us:

$$a + c = 4 \quad \text{(Equation 1)}$$

Step 5: Apply additional constraints. Since $\nabla\phi$ is in the direction of the x-axis, it should be perpendicular to the y and z axes. This gives us:

$$\begin{aligned}\nabla\phi\cdot\vec{j} &= 0\\ (a + b) &= 0\\ \Rightarrow a + b &= 0 \quad \text{(Equation 2)}\end{aligned}$$

And:

$$\begin{aligned}\nabla\phi\cdot\vec{k} &= 0\\ (b + c) &= 0\\ \Rightarrow b + c &= 0 \quad \text{(Equation 3)}\end{aligned}$$

Step 6: Solve the system of equations. From Equation 2:

$$\begin{aligned}a + b &= 0\\ \Rightarrow b &= -a \quad \text{(Equation 4)}\end{aligned}$$

From Equation 3:

$$\begin{aligned} b + c &= 0 \\ \Rightarrow c &= -b \quad \text{(Equation 5)} \end{aligned}$$

Substituting Equation 4 into Equation 5:

$$\begin{aligned} c &= -b \\ c &= -(-a) \\ c &= a \quad \text{(Equation 6)} \end{aligned}$$

Substituting Equation 6 into Equation 1:

$$\begin{aligned} a + c &= 4 \\ a + a &= 4 \\ 2a &= 4 \\ a &= 2 \end{aligned}$$

Now we can find the other values:

$$\begin{aligned} c &= a = 2 \\ b &= -a = -2 \end{aligned}$$

Therefore, $a = 2$, $b = -2$, and $c = 2$.
To verify our answer, let's check if $\nabla\phi$ at $(1, 1, 1)$ is indeed parallel to the x-axis:

$$\begin{aligned} \nabla\phi|_{(1,1,1)} &= \vec{i}(a + c) + \vec{j}(a + b) + \vec{k}(b + c) \\ &= \vec{i}(2 + 2) + \vec{j}(2 + (-2)) + \vec{k}((-2) + 2) \\ &= 4\vec{i} + 0\vec{j} + 0\vec{k} \end{aligned}$$

This confirms that $\nabla\phi$ is indeed parallel to the x-axis with magnitude 4.

Example 4: Directional Derivative along a Parameterized Curve

Find the directional derivative of the function $\phi = x^2y + xyz + z^3$ at $(1, 2, -1)$ along the normal to the surface $x^2y^3 = 4xy + y^2z$ at $(1, 2, 0)$.

Solution

Given:

$$\begin{aligned} \phi &= x^2y + xyz + z^3 \\ \text{Surface equation:} \quad u &= x^2y^3 - 4xy - y^2z = 0 \end{aligned}$$

Step 1: Find the gradient of ϕ.

$$\begin{aligned} \nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\ &= \vec{i}(2xy + yz) + \vec{j}(x^2 + xz) + \vec{k}(xy + 3z^2) \end{aligned}$$

Step 2: Evaluate $\nabla\phi$ at the point $(1, 2, -1)$.

$$\begin{aligned}
\nabla\phi|_{(1,2,-1)} &= \vec{i}(2\cdot 1\cdot 2 + 2\cdot(-1)) + \vec{j}(1^2 + 1\cdot(-1)) + \vec{k}(1\cdot 2 + 3\cdot(-1)^2) \\
&= \vec{i}(4-2) + \vec{j}(1-1) + \vec{k}(2+3) \\
&= 2\vec{i} + 0\vec{j} + 5\vec{k}
\end{aligned}$$

Step 3: Find the normal to the surface at the point $(1, 2, 0)$.
To find the normal vector to the surface, we take the gradient of u:

$$\begin{aligned}
\nabla u &= \vec{i}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z} \\
&= \vec{i}(2xy^3 - 4y) + \vec{j}(3x^2y^2 - 4x - 2yz) + \vec{k}(-y^2)
\end{aligned}$$

Evaluating at the point $(1, 2, 0)$:

$$\begin{aligned}
\nabla u|_{(1,2,0)} &= \vec{i}(2\cdot 1\cdot 2^3 - 4\cdot 2) + \vec{j}(3\cdot 1^2\cdot 2^2 - 4\cdot 1 - 2\cdot 2\cdot 0) + \vec{k}(-2^2) \\
&= \vec{i}(2\cdot 8 - 8) + \vec{j}(3\cdot 4 - 4 - 0) + \vec{k}(-4) \\
&= \vec{i}(16-8) + \vec{j}(12-4) + \vec{k}(-4) \\
&= 8\vec{i} + 8\vec{j} - 4\vec{k}
\end{aligned}$$

Step 4: Normalize the normal vector to get the unit vector.

$$\begin{aligned}
|\nabla u| &= \sqrt{8^2 + 8^2 + (-4)^2} \\
&= \sqrt{64 + 64 + 16} \\
&= \sqrt{144} = 12
\end{aligned}$$

The unit normal vector is:

$$\begin{aligned}
\hat{a} &= \frac{\nabla u}{|\nabla u|} = \frac{8\vec{i} + 8\vec{j} - 4\vec{k}}{12} \\
&= \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}
\end{aligned}$$

Step 5: Calculate the directional derivative.

$$\begin{aligned}
D\cdot D &= \nabla\phi\cdot\hat{a} \\
&= (2\vec{i} + 0\vec{j} + 5\vec{k})\cdot\left(\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}\right) \\
&= 2\cdot\frac{2}{3} + 0\cdot\frac{2}{3} + 5\cdot\left(-\frac{1}{3}\right) \\
&= \frac{4}{3} - \frac{5}{3} \\
&= -\frac{1}{3}
\end{aligned}$$

Therefore, the directional derivative of the function $\phi = x^2y + xyz + z^3$ at the point $(1, 2, -1)$ along the normal to the given surface at the point $(1, 2, 0)$ is $-\frac{1}{3}$.

Example 5: Finding Coefficients for Given Directional Derivative Maximum

Find the value of the constants a, b, c so that the directional derivative of $f = ax^2y^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum of magnitude 64 in the direction parallel to the z-axis.

Solution

Given: $f = ax^2y^2 + byz + cz^2x^3$
Step 1: Find the gradient of f.

$$\begin{aligned}\nabla f &= \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} \\ &= \vec{i}(2ax^1y^2 + 3cz^2x^2) + \vec{j}(2ax^2y + bz) + \vec{k}(by + 2czx^3)\end{aligned}$$

Step 2: Evaluate ∇f at the point $(1, 2, -1)$.

$$\begin{aligned}\nabla f|_{(1,2,-1)} &= \vec{i}[2a(1)(2)^2 + 3c(-1)^2(1)^2] + \vec{j}[2a(1)^2(2) + b(-1)] + \vec{k}[b(2) + 2c(-1)(1)^3] \\ &= \vec{i}(8a + 3c) + \vec{j}(4a - b) + \vec{k}(2b - 2c)\end{aligned}$$

Step 3: Find the unit vector in the direction of the z-axis. The unit vector along the z-axis is $\hat{a} = \vec{k}$.
Step 4: Calculate the condition for directional derivative. The directional derivative in the direction of the z-axis is:

$$\begin{aligned}D \cdot D &= \nabla f \cdot \hat{a} \\ &= [(8a + 3c)\vec{i} + (4a - b)\vec{j} + (2b - 2c)\vec{k}] \cdot \vec{k} \\ &= 2b - 2c\end{aligned}$$

We are told that this has a maximum magnitude of 64. Since we want the directional derivative to be maximized along the z-axis, we need:

$$\begin{aligned}2b - 2c &= 64 \\ \Rightarrow b - c = 32 &\quad \text{(Equation 1)}\end{aligned}$$

Step 5: Apply additional constraints. Since the maximum is in the direction of the z-axis, the gradient should be parallel to the z-axis, which means its x and y components must be zero:

$$\begin{aligned}8a + 3c = 0 &\quad \text{(Equation 2)} \\ 4a - b = 0 &\quad \text{(Equation 3)}\end{aligned}$$

Step 6: Solve the system of equations. From Equation 2:

$$\begin{aligned}8a + 3c &= 0 \\ \Rightarrow a = -\frac{3c}{8} &\quad \text{(Equation 4)}\end{aligned}$$

From Equation 3:

$$\begin{aligned}4a - b &= 0 \\ \Rightarrow b = 4a &\quad \text{(Equation 5)}\end{aligned}$$

Substituting Equation 4 into Equation 5:

$$b = 4a$$
$$b = 4\left(-\frac{3c}{8}\right)$$
$$b = -\frac{12c}{8}$$
$$b = -\frac{3c}{2} \quad \text{(Equation 6)}$$

Now, substituting Equation 6 into Equation 1:

$$b - c = 32$$
$$-\frac{3c}{2} - c = 32$$
$$-\frac{3c}{2} - \frac{2c}{2} = 32$$
$$-\frac{5c}{2} = 32$$
$$c = -\frac{64}{5}$$

With this value of c, we can find b using Equation 6:

$$b = -\frac{3c}{2}$$
$$b = -\frac{3}{2}\left(-\frac{64}{5}\right)$$
$$b = \frac{3 \cdot 64}{2 \cdot 5}$$
$$b = \frac{192}{10} = \frac{96}{5}$$

And we can find a using Equation 4:

$$a = -\frac{3c}{8}$$
$$a = -\frac{3}{8}\left(-\frac{64}{5}\right)$$
$$a = \frac{3 \cdot 64}{8 \cdot 5}$$
$$a = \frac{192}{40} = \frac{24}{5}$$

Therefore, $a = \frac{24}{5}$, $b = \frac{96}{5}$, and $c = -\frac{64}{5}$.

Step 7: Verify the solution. Let's check if ∇f at $(1, 2, -1)$ is indeed parallel to the z-axis and gives the directional derivative of 64:

$$\begin{aligned}
\nabla f|_{(1,2,-1)} &= \vec{i}(8a + 3c) + \vec{j}(4a - b) + \vec{k}(2b - 2c) \\
&= \vec{i}\left(8 \cdot \frac{24}{5} + 3 \cdot \left(-\frac{64}{5}\right)\right) + \vec{j}\left(4 \cdot \frac{24}{5} - \frac{96}{5}\right) + \vec{k}\left(2 \cdot \frac{96}{5} - 2 \cdot \left(-\frac{64}{5}\right)\right) \\
&= \vec{i}\left(\frac{192}{5} - \frac{192}{5}\right) + \vec{j}\left(\frac{96}{5} - \frac{96}{5}\right) + \vec{k}\left(\frac{192}{5} + \frac{128}{5}\right) \\
&= \vec{i}(0) + \vec{j}(0) + \vec{k}\left(\frac{320}{5}\right) \\
&= \vec{k}(64)
\end{aligned}$$

This confirms that our solution is correct. The gradient at the given point is parallel to the z-axis with a magnitude of 64, giving a directional derivative of 64 in that direction.

Example 6: Finding Coefficients for Given Directional Derivative Maximum

Find the value of the constants a, b, c so that the directional derivative of $f = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum of magnitude 64 in the direction parallel to the z-axis.

Solution

Given: $f = axy^2 + byz + cz^2x^3$
Step 1: Find the gradient of f.

$$\begin{aligned}
\nabla f &= \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} \\
&= \vec{i}(ay^2 + 3cz^2x^2) + \vec{j}(2axy + bz) + \vec{k}(by + 2czx^3)
\end{aligned}$$

Step 2: Evaluate ∇f at the point $(1, 2, -1)$.

$$\begin{aligned}
\nabla f|_{(1,2,-1)} &= \vec{i}[a(2)^2 + 3c(-1)^2(1^2)] + \vec{j}[2a(1)(2) + b(-1)] + \vec{k}[b(2) + 2c(-1)(1)^3] \\
&= \vec{i}(4a + 3c) + \vec{j}(4a - b) + \vec{k}(2b - 2c)
\end{aligned}$$

Step 3: Find the unit vector in the direction of the z-axis. The unit vector along the z-axis is $\hat{a} = \vec{k}$.
Step 4: Calculate the condition for directional derivative. The directional derivative in the direction of the z-axis is:

$$\begin{aligned}
D \cdot D &= \nabla f \cdot \hat{a} \\
&= [(4a + 3c)\vec{i} + (4a - b)\vec{j} + (2b - 2c)\vec{k}] \cdot \vec{k} \\
&= 2b - 2c
\end{aligned}$$

We are told that this has a maximum magnitude of 64. Since we want the directional derivative to be maximized along the z-axis, we need:

$$\begin{aligned}
2b - 2c &= 64 \\
\Rightarrow b - c &= 32 \quad \text{(Equation 1)}
\end{aligned}$$

Step 5: Apply additional constraints. Since the maximum is in the direction of the z-axis, the gradient should be parallel to the z-axis, which means its x and y components must be zero:

$$\begin{aligned} 4a + 3c &= 0 \quad \text{(Equation 2)} \\ 4a - b &= 0 \quad \text{(Equation 3)} \end{aligned}$$

Step 6: Solve the system of equations. From Equation 2:

$$\begin{aligned} &4a + 3c = 0 \\ &\Rightarrow a = -\frac{3c}{4} \quad \text{(Equation 4)} \end{aligned}$$

From Equation 3:

$$\begin{aligned} &4a - b = 0 \\ &\Rightarrow b = 4a \quad \text{(Equation 5)} \end{aligned}$$

Substituting Equation 4 into Equation 5:

$$\begin{aligned} b &= 4a \\ b &= 4\left(-\frac{3c}{4}\right) \\ b &= -3c \quad \text{(Equation 6)} \end{aligned}$$

Now, substituting Equation 6 into Equation 1:

$$\begin{aligned} b - c &= 32 \\ -3c - c &= 32 \\ -4c &= 32 \\ c &= -8 \end{aligned}$$

With this value of c, we can find b using Equation 6:

$$\begin{aligned} b &= -3c \\ b &= -3(-8) \\ b &= 24 \end{aligned}$$

And we can find a using Equation 4:

$$\begin{aligned} a &= -\frac{3c}{4} \\ a &= -\frac{3(-8)}{4} \\ a &= \frac{24}{4} \\ a &= 6 \end{aligned}$$

Therefore, $a = 6$, $b = 24$, and $c = -8$.

Step 7: Verify the solution. Let's check if ∇f at $(1, 2, -1)$ is indeed parallel to the z-axis and gives the directional derivative of 64:

$$\begin{aligned}
\nabla f|_{(1,2,-1)} &= \vec{i}(4a+3c) + \vec{j}(4a-b) + \vec{k}(2b-2c) \\
&= \vec{i}(4\cdot 6 + 3\cdot(-8)) + \vec{j}(4\cdot 6 - 24) + \vec{k}(2\cdot 24 - 2\cdot(-8)) \\
&= \vec{i}(24-24) + \vec{j}(24-24) + \vec{k}(48+16) \\
&= \vec{i}(0) + \vec{j}(0) + \vec{k}(64) \\
&= 64\vec{k}
\end{aligned}$$

This confirms that our solution is correct. The gradient at the given point is parallel to the z-axis with a magnitude of 64, giving a directional derivative of 64 in that direction.

Example 7: Directional Derivative in Specified Direction

Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2z^2x$ at the point $(1, 1, 1)$ in the direction of the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

Solution

Given:

$$\phi = 5x^2y - 5y^2z + 2z^2x$$

$$\text{Direction: } \frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

Step 1: Find the gradient of ϕ.

$$\begin{aligned}
\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\
&= \vec{i}(10xy + 2z^2) + \vec{j}(5x^2 - 10yz) + \vec{k}(-5y^2 + 4zx)
\end{aligned}$$

Step 2: Evaluate $\nabla\phi$ at the point $(1, 1, 1)$.

$$\begin{aligned}
\nabla\phi|_{(1,1,1)} &= \vec{i}(10\cdot 1\cdot 1 + 2\cdot 1^2) + \vec{j}(5\cdot 1^2 - 10\cdot 1\cdot 1) + \vec{k}(-5\cdot 1^2 + 4\cdot 1\cdot 1) \\
&= \vec{i}(10+2) + \vec{j}(5-10) + \vec{k}(-5+4) \\
&= 12\vec{i} - 5\vec{j} - \vec{k}
\end{aligned}$$

Step 3: Find the direction vector from the given line equation.
From the line equation $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, we can identify the direction vector by taking the denominators:

$$\vec{a} = 2\vec{i} - 2\vec{j} + \vec{k}$$

Step 4: Normalize the direction vector to get the unit vector.

$$\begin{aligned}
|\vec{a}| &= \sqrt{2^2 + (-2)^2 + 1^2} \\
&= \sqrt{4+4+1} \\
&= \sqrt{9} = 3
\end{aligned}$$

The unit vector in this direction is:

$$\begin{aligned}\hat{a} &= \frac{\vec{a}}{|\vec{a}|} = \frac{2\vec{i} - 2\vec{j} + \vec{k}}{3} \\ &= \frac{2}{3}\vec{i} - \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}\end{aligned}$$

Step 5: Calculate the directional derivative.

$$\begin{aligned}D \cdot D &= \nabla\phi \cdot \hat{a} \\ &= (12\vec{i} - 5\vec{j} - \vec{k}) \cdot \left(\frac{2}{3}\vec{i} - \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}\right) \\ &= 12 \cdot \frac{2}{3} + (-5) \cdot \left(-\frac{2}{3}\right) + (-1) \cdot \frac{1}{3} \\ &= \frac{24}{3} + \frac{10}{3} - \frac{1}{3} \\ &= \frac{24 + 10 - 1}{3} \\ &= \frac{33}{3} = 11\end{aligned}$$

Therefore, the directional derivative of the function $\phi = 5x^2y - 5y^2z + 2z^2x$ at the point $(1, 1, 1)$ in the direction of the given line is 11.

Example 8: Directional Derivative with Specified Vector

Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2, -1, 2)$ towards the point $\vec{i} + \vec{j} - \vec{k}$.

Solution

Given:

$$\phi = 4xz^3 - 3x^2y^2z$$

Step 1: Find the gradient of ϕ.

$$\begin{aligned}\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\ &= \vec{i}(4z^3 - 6xy^2z) + \vec{j}(-6x^2yz) + \vec{k}(12xz^2 - 3x^2y^2)\end{aligned}$$

Step 2: Evaluate $\nabla\phi$ at the point $(2, -1, 2)$.

$$\begin{aligned}\nabla\phi|_{(2,-1,2)} &= \vec{i}[4(2)^3 - 6(2)(-1)^2(2)] + \vec{j}[-6(2)^2(-1)(2)] + \vec{k}[12(2)(2)^2 - 3(2)^2(-1)^2] \\ &= \vec{i}[4 \cdot 8 - 6 \cdot 2 \cdot 1 \cdot 2] + \vec{j}[-6 \cdot 4 \cdot (-1) \cdot 2] + \vec{k}[12 \cdot 2 \cdot 4 - 3 \cdot 4 \cdot 1] \\ &= \vec{i}[32 - 24] + \vec{j}[48] + \vec{k}[96 - 12] \\ &= 8\vec{i} + 48\vec{j} + 84\vec{k}\end{aligned}$$

Step 3: Find the direction vector towards the point $\vec{i} + \vec{j} - \vec{k}$.
The direction vector from the point $(2, -1, 2)$ to the point $(1, 1, -1)$ is:

$$\begin{aligned}\vec{a} &= (1 - 2)\vec{i} + (1 - (-1))\vec{j} + (-1 - 2)\vec{k} \\ &= -\vec{i} + 2\vec{j} - 3\vec{k}\end{aligned}$$

Step 4: Normalize the direction vector to get the unit vector.

$$\begin{aligned} |\vec{a}| &= \sqrt{(-1)^2 + 2^2 + (-3)^2} \\ &= \sqrt{1+4+9} \\ &= \sqrt{14} \end{aligned}$$

The unit vector in this direction is:

$$\begin{aligned} \hat{a} = \frac{\vec{a}}{|\vec{a}|} &= \frac{-\vec{i} + 2\vec{j} - 3\vec{k}}{\sqrt{14}} \\ &= -\frac{1}{\sqrt{14}}\vec{i} + \frac{2}{\sqrt{14}}\vec{j} - \frac{3}{\sqrt{14}}\vec{k} \end{aligned}$$

Step 5: Calculate the directional derivative.

$$\begin{aligned} D \cdot D &= \nabla\phi \cdot \hat{a} \\ &= (8\vec{i} + 48\vec{j} + 84\vec{k}) \cdot \left(-\frac{1}{\sqrt{14}}\vec{i} + \frac{2}{\sqrt{14}}\vec{j} - \frac{3}{\sqrt{14}}\vec{k}\right) \\ &= 8 \cdot \left(-\frac{1}{\sqrt{14}}\right) + 48 \cdot \frac{2}{\sqrt{14}} + 84 \cdot \left(-\frac{3}{\sqrt{14}}\right) \\ &= \frac{1}{\sqrt{14}}(-8 + 96 - 252) \\ &= \frac{1}{\sqrt{14}}(-164) \\ &= -\frac{164}{\sqrt{14}} \end{aligned}$$

To simplify:

$$\begin{aligned} -\frac{164}{\sqrt{14}} &= -\frac{164}{\sqrt{14}} \cdot \frac{\sqrt{14}}{\sqrt{14}} \\ &= -\frac{164\sqrt{14}}{14} \\ &= -\frac{41\sqrt{14}}{3.5} \\ &\approx -11.7\sqrt{14} \end{aligned}$$

Therefore, the directional derivative of the function $\phi = 4xz^3 - 3x^2y^2z$ at the point $(2, -1, 2)$ towards the point $(1, 1, -1)$ is $-\frac{164}{\sqrt{14}}$ or approximately -43.85.

Example 9: Directional Derivative for Specified Function

Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at $(1, 2, 3)$ in the direction of $4\vec{i} - 2\vec{j} + \vec{k}$.

Solution

Given:

$$\phi = x^2 - y^2 + 2z^2$$
$$\text{Direction vector: } \vec{v} = 4\vec{i} - 2\vec{j} + \vec{k}$$

Step 1: Find the gradient of ϕ.

$$\begin{aligned}\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\ &= \vec{i}(2x) + \vec{j}(-2y) + \vec{k}(4z)\end{aligned}$$

Step 2: Evaluate $\nabla\phi$ at the point $(1, 2, 3)$.

$$\begin{aligned}\nabla\phi|_{(1,2,3)} &= \vec{i}(2\cdot 1) + \vec{j}(-2\cdot 2) + \vec{k}(4\cdot 3) \\ &= 2\vec{i} - 4\vec{j} + 12\vec{k}\end{aligned}$$

Step 3: Normalize the direction vector to get the unit vector.

$$\begin{aligned}|\vec{v}| &= \sqrt{4^2 + (-2)^2 + 1^2} \\ &= \sqrt{16 + 4 + 1} \\ &= \sqrt{21}\end{aligned}$$

The unit vector in this direction is:

$$\begin{aligned}\hat{v} &= \frac{\vec{v}}{|\vec{v}|} = \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}} \\ &= \frac{4}{\sqrt{21}}\vec{i} - \frac{2}{\sqrt{21}}\vec{j} + \frac{1}{\sqrt{21}}\vec{k}\end{aligned}$$

Step 4: Calculate the directional derivative.

$$\begin{aligned}D\cdot D &= \nabla\phi\cdot\hat{v} \\ &= (2\vec{i} - 4\vec{j} + 12\vec{k})\cdot\left(\frac{4}{\sqrt{21}}\vec{i} - \frac{2}{\sqrt{21}}\vec{j} + \frac{1}{\sqrt{21}}\vec{k}\right) \\ &= 2\cdot\frac{4}{\sqrt{21}} + (-4)\cdot\left(-\frac{2}{\sqrt{21}}\right) + 12\cdot\frac{1}{\sqrt{21}} \\ &= \frac{8}{\sqrt{21}} + \frac{8}{\sqrt{21}} + \frac{12}{\sqrt{21}} \\ &= \frac{28}{\sqrt{21}}\end{aligned}$$

To simplify:

$$\begin{aligned}\frac{28}{\sqrt{21}} &= \frac{28}{\sqrt{21}}\cdot\frac{\sqrt{21}}{\sqrt{21}} \\ &= \frac{28\sqrt{21}}{21} \\ &= \frac{4\sqrt{21}}{3} \approx 6.11\end{aligned}$$

Therefore, the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point $(1, 2, 3)$ in the direction of $4\vec{i} - 2\vec{j} + \vec{k}$ is $\frac{28}{\sqrt{21}}$ or approximately 6.11.

Example 10: Directional Derivative along Vector Direction

Find the directional derivative of $\phi = xy^3 + yz^3$ at the point $(2, -1, 1)$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$.

Solution

Given:

$$\phi = xy^3 + yz^3$$
$$\text{Point: } (2, -1, 1)$$
$$\text{Direction vector: } \vec{v} = \vec{i} + 2\vec{j} + 2\vec{k}$$

Step 1: Find the gradient of ϕ.

$$\begin{aligned}\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\ &= \vec{i}(y^3) + \vec{j}(3xy^2 + z^3) + \vec{k}(3yz^2)\end{aligned}$$

Step 2: Evaluate $\nabla\phi$ at the point $(2, -1, 1)$.

$$\begin{aligned}\nabla\phi|_{(2,-1,1)} &= \vec{i}((-1)^3) + \vec{j}(3 \cdot 2 \cdot (-1)^2 + 1^3) + \vec{k}(3 \cdot (-1) \cdot 1^2) \\ &= \vec{i}(-1) + \vec{j}(3 \cdot 2 \cdot 1 + 1) + \vec{k}(3 \cdot (-1) \cdot 1) \\ &= \vec{i}(-1) + \vec{j}(6 + 1) + \vec{k}(-3) \\ &= -\vec{i} + 7\vec{j} - 3\vec{k}\end{aligned}$$

Step 3: Normalize the direction vector to get the unit vector.

$$\begin{aligned}|\vec{v}| &= \sqrt{1^2 + 2^2 + 2^2} \\ &= \sqrt{1 + 4 + 4} \\ &= \sqrt{9} = 3\end{aligned}$$

The unit vector in this direction is:

$$\begin{aligned}\hat{v} &= \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3} \\ &= \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\end{aligned}$$

Step 4: Calculate the directional derivative.

$$\begin{aligned}D \cdot D &= \nabla\phi \cdot \hat{v} \\ &= (-\vec{i} + 7\vec{j} - 3\vec{k}) \cdot \left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right) \\ &= (-1) \cdot \frac{1}{3} + 7 \cdot \frac{2}{3} + (-3) \cdot \frac{2}{3} \\ &= -\frac{1}{3} + \frac{14}{3} - \frac{6}{3} \\ &= \frac{-1 + 14 - 6}{3} \\ &= \frac{7}{3}\end{aligned}$$

Therefore, the directional derivative of the function $\phi = xy^3 + yz^3$ at the point $(2, -1, 1)$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$ is $\frac{7}{3}$.

Example 11: Directional Derivative Towards a Point

Find the directional derivative of $\phi(x, y, z) = x^2y + y^2z + z^2x$ at $(1, 1, 1)$ towards point $(3, 2, 2)$.

Solution

Given:

$$\phi = x^2y + y^2z + z^2x$$
$$\text{Point: } (1, 1, 1)$$
$$\text{Target point: } (3, 2, 2)$$

Step 1: Find the gradient of ϕ.

$$\begin{aligned}\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\ &= \vec{i}(2xy + z^2) + \vec{j}(x^2 + 2yz) + \vec{k}(y^2 + 2zx)\end{aligned}$$

Step 2: Evaluate $\nabla\phi$ at the point $(1, 1, 1)$.

$$\begin{aligned}\nabla\phi|_{(1,1,1)} &= \vec{i}(2\cdot 1\cdot 1 + 1^2) + \vec{j}(1^2 + 2\cdot 1\cdot 1) + \vec{k}(1^2 + 2\cdot 1\cdot 1) \\ &= \vec{i}(2 + 1) + \vec{j}(1 + 2) + \vec{k}(1 + 2) \\ &= 3\vec{i} + 3\vec{j} + 3\vec{k}\end{aligned}$$

Step 3: Find the direction vector from $(1, 1, 1)$ to $(3, 2, 2)$.

$$\begin{aligned}\vec{v} &= (3 - 1)\vec{i} + (2 - 1)\vec{j} + (2 - 1)\vec{k} \\ &= 2\vec{i} + \vec{j} + \vec{k}\end{aligned}$$

Step 4: Normalize the direction vector to get the unit vector.

$$\begin{aligned}|\vec{v}| &= \sqrt{2^2 + 1^2 + 1^2} \\ &= \sqrt{4 + 1 + 1} \\ &= \sqrt{6}\end{aligned}$$

The unit vector in this direction is:

$$\begin{aligned}\hat{v} &= \frac{\vec{v}}{|\vec{v}|} = \frac{2\vec{i} + \vec{j} + \vec{k}}{\sqrt{6}} \\ &= \frac{2}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}\end{aligned}$$

Step 5: Calculate the directional derivative.

$$\begin{aligned}D\cdot D &= \nabla\phi\cdot\hat{v} \\ &= (3\vec{i} + 3\vec{j} + 3\vec{k})\cdot\left(\frac{2}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}\right) \\ &= 3\cdot\frac{2}{\sqrt{6}} + 3\cdot\frac{1}{\sqrt{6}} + 3\cdot\frac{1}{\sqrt{6}} \\ &= \frac{6}{\sqrt{6}} + \frac{3}{\sqrt{6}} + \frac{3}{\sqrt{6}} \\ &= \frac{12}{\sqrt{6}}\end{aligned}$$

To simplify:

$$\begin{aligned}\frac{12}{\sqrt{6}} &= \frac{12}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} \\ &= \frac{12\sqrt{6}}{6} \\ &= 2\sqrt{6} \approx 4.90\end{aligned}$$

Therefore, the directional derivative of the function $\phi = x^2y + y^2z + z^2x$ at the point $(1,1,1)$ towards the point $(3,2,2)$ is $\frac{12}{\sqrt{6}}$ or $2\sqrt{6}$.

Example 12: Directional Derivative Along a Parametrized Curve

Find the directional derivative of $\varphi = x^2 + 2y^2 - 3z^2$ at $(1,2,1)$ in the direction tangent to the curve $x = t^2 + t$, $y = 2t$, $z = 2 - t$ at $t = 1$.

Solution

Given:

$$\begin{aligned}\varphi &= x^2 + 2y^2 - 3z^2 \\ \text{Curve: } x &= t^2 + t \\ y &= 2t \\ z &= 2 - t\end{aligned}$$

Step 1: Find the gradient of φ.

$$\begin{aligned}\nabla\varphi &= \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z} \\ &= \vec{i}(2x) + \vec{j}(4y) + \vec{k}(-6z)\end{aligned}$$

Step 2: Evaluate $\nabla\varphi$ at the point $(1,2,1)$.

$$\begin{aligned}\nabla\varphi|_{(1,2,1)} &= \vec{i}(2\cdot 1) + \vec{j}(4\cdot 2) + \vec{k}(-6\cdot 1) \\ &= 2\vec{i} + 8\vec{j} - 6\vec{k}\end{aligned}$$

Step 3: Find the tangent vector to the curve at $t = 1$.
The tangent vector is the derivative of the position vector with respect to t:

$$\begin{aligned}\vec{r}(t) &= (t^2+t)\vec{i} + (2t)\vec{j} + (2-t)\vec{k} \\ \vec{v}(t) &= \frac{d\vec{r}}{dt} = (2t+1)\vec{i} + 2\vec{j} - \vec{k}\end{aligned}$$

At $t = 1$:

$$\begin{aligned}\vec{v}(1) &= (2\cdot 1 + 1)\vec{i} + 2\vec{j} - \vec{k} \\ &= 3\vec{i} + 2\vec{j} - \vec{k}\end{aligned}$$

Step 4: Normalize the tangent vector to get the unit vector.

$$\begin{aligned}|\vec{v}(1)| &= \sqrt{3^2 + 2^2 + (-1)^2} \\ &= \sqrt{9+4+1} \\ &= \sqrt{14}\end{aligned}$$

The unit tangent vector is:

$$\hat{v} = \frac{\vec{v}(1)}{|\vec{v}(1)|} = \frac{3\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{14}}$$
$$= \frac{3}{\sqrt{14}}\vec{i} + \frac{2}{\sqrt{14}}\vec{j} - \frac{1}{\sqrt{14}}\vec{k}$$

Step 5: Calculate the directional derivative.

$$D_{\hat{v}}\varphi = \nabla\varphi \cdot \hat{v}$$
$$= (2\vec{i} + 8\vec{j} - 6\vec{k}) \cdot \left(\frac{3}{\sqrt{14}}\vec{i} + \frac{2}{\sqrt{14}}\vec{j} - \frac{1}{\sqrt{14}}\vec{k}\right)$$
$$= 2 \cdot \frac{3}{\sqrt{14}} + 8 \cdot \frac{2}{\sqrt{14}} + (-6) \cdot \left(-\frac{1}{\sqrt{14}}\right)$$
$$= \frac{6}{\sqrt{14}} + \frac{16}{\sqrt{14}} + \frac{6}{\sqrt{14}}$$
$$= \frac{28}{\sqrt{14}}$$

To simplify:

$$\frac{28}{\sqrt{14}} = \frac{28}{\sqrt{14}} \cdot \frac{\sqrt{14}}{\sqrt{14}}$$
$$= \frac{28\sqrt{14}}{14}$$
$$= 2\sqrt{14} \approx 7.48$$

Therefore, the directional derivative of the function $\varphi = x^2 + 2y^2 - 3z^2$ at the point $(1, 2, 1)$ in the direction tangent to the given curve at $t = 1$ is $\frac{28}{\sqrt{14}} = 2\sqrt{14}$.

Example 13: Directional Derivative Along a Line

Find the directional derivative of $\phi = xy^2 + yz^3$ at $(2, -1, 1)$ along the line $2(x - 2) = y + 1 = z - 1$

Solution

Given:

$$\phi = xy^2 + yz^3$$
$$\text{Point: } (2, -1, 1)$$
$$\text{Line equation: } 2(x - 2) = y + 1 = z - 1$$

Step 1: Find the gradient of ϕ.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$
$$= \vec{i}(y^2) + \vec{j}(2xy + z^3) + \vec{k}(3yz^2)$$

Step 2: Evaluate $\nabla\phi$ at the point $(2, -1, 1)$.

$$\begin{aligned}
\nabla\phi|_{(2,-1,1)} &= \vec{i}((-1)^2) + \vec{j}(2 \cdot 2 \cdot (-1) + 1^3) + \vec{k}(3 \cdot (-1) \cdot 1^2) \\
&= \vec{i}(1) + \vec{j}(-4 + 1) + \vec{k}(-3) \\
&= \vec{i} - 3\vec{j} - 3\vec{k}
\end{aligned}$$

Step 3: Find the direction vector from the line equation.
From the line equation $2(x - 2) = y + 1 = z - 1$, we can express it in parametric form by introducing a parameter t:

$$\begin{aligned}
2(x - 2) &= t \\
y + 1 &= t \\
z - 1 &= t
\end{aligned}$$

Solving for x, y, and z:

$$\begin{aligned}
x &= 2 + \frac{t}{2} \\
y &= t - 1 \\
z &= t + 1
\end{aligned}$$

The direction vector of this line is:

$$\begin{aligned}
\vec{v} &= \frac{d}{dt}(x, y, z) = \left(\frac{1}{2}, 1, 1\right) \\
&= \frac{1}{2}\vec{i} + \vec{j} + \vec{k}
\end{aligned}$$

Step 4: Normalize the direction vector to get the unit vector.

$$\begin{aligned}
|\vec{v}| &= \sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + 1^2} \\
&= \sqrt{\frac{1}{4} + 1 + 1} \\
&= \sqrt{\frac{9}{4}} = \frac{3}{2}
\end{aligned}$$

The unit vector in this direction is:

$$\begin{aligned}
\hat{v} &= \frac{\vec{v}}{|\vec{v}|} = \frac{\frac{1}{2}\vec{i} + \vec{j} + \vec{k}}{\frac{3}{2}} \\
&= \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}
\end{aligned}$$

Step 5: Calculate the directional derivative.

$$
\begin{aligned}
D_{\hat{v}}\phi &= \nabla\phi \cdot \hat{v} \\
&= (\vec{i} - 3\vec{j} - 3\vec{k}) \cdot \left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right) \\
&= 1 \cdot \frac{1}{3} + (-3) \cdot \frac{2}{3} + (-3) \cdot \frac{2}{3} \\
&= \frac{1}{3} - \frac{6}{3} - \frac{6}{3} \\
&= \frac{1-6-6}{3} \\
&= -\frac{11}{3}
\end{aligned}
$$

Therefore, the directional derivative of the function $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ along the given line is $-\frac{11}{3}$.

Example 14: Directional Derivative Along a Curve Tangent

Find the directional derivative of $\phi = e^{2x-y-z}$ at $(1, 1, 1)$ in the direction of the tangent to the curve $x = e^{-t}$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t = 0$.

Solution

Given:

$$
\begin{aligned}
\phi &= e^{2x-y-z} \\
\text{Point: } &(1, 1, 1) \\
\text{Curve: } x &= e^{-t} \\
y &= 2\sin t + 1 \\
z &= t - \cos t
\end{aligned}
$$

Step 1: Find the gradient of ϕ.

$$
\begin{aligned}
\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\
&= \vec{i}(2e^{2x-y-z}) + \vec{j}(-e^{2x-y-z}) + \vec{k}(-e^{2x-y-z}) \\
&= e^{2x-y-z}(2\vec{i} - \vec{j} - \vec{k})
\end{aligned}
$$

Step 2: Evaluate $\nabla\phi$ at the point $(1, 1, 1)$.

$$
\begin{aligned}
\nabla\phi|_{(1,1,1)} &= e^{2(1)-(1)-(1)}(2\vec{i} - \vec{j} - \vec{k}) \\
&= e^0(2\vec{i} - \vec{j} - \vec{k}) \\
&= 1 \cdot (2\vec{i} - \vec{j} - \vec{k}) \\
&= 2\vec{i} - \vec{j} - \vec{k}
\end{aligned}
$$

Step 3: Find the tangent vector to the curve at $t = 0$.

We need to calculate the derivative of the position vector with respect to t:

$$\begin{aligned}\vec{r}(t) &= e^{-t}\vec{i} + (2\sin t + 1)\vec{j} + (t - \cos t)\vec{k} \\ \vec{v}(t) &= \frac{d\vec{r}}{dt} \\ &= -e^{-t}\vec{i} + 2\cos t\vec{j} + (1 + \sin t)\vec{k}\end{aligned}$$

At $t = 0$:

$$\begin{aligned}\vec{v}(0) &= -e^{0}\vec{i} + 2\cos(0)\vec{j} + (1 + \sin(0))\vec{k} \\ &= -1\vec{i} + 2 \cdot 1\vec{j} + (1 + 0)\vec{k} \\ &= -\vec{i} + 2\vec{j} + \vec{k}\end{aligned}$$

Step 4: Normalize the tangent vector to get the unit vector.

$$\begin{aligned}|\vec{v}(0)| &= \sqrt{(-1)^2 + 2^2 + 1^2} \\ &= \sqrt{1 + 4 + 1} \\ &= \sqrt{6}\end{aligned}$$

The unit tangent vector is:

$$\begin{aligned}\hat{v} &= \frac{\vec{v}(0)}{|\vec{v}(0)|} = \frac{-\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{6}} \\ &= -\frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}\end{aligned}$$

Step 5: Calculate the directional derivative.

$$\begin{aligned}D_{\hat{v}}\phi &= \nabla\phi \cdot \hat{v} \\ &= (2\vec{i} - \vec{j} - \vec{k}) \cdot \left(-\frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}\right) \\ &= 2 \cdot \left(-\frac{1}{\sqrt{6}}\right) + (-1) \cdot \frac{2}{\sqrt{6}} + (-1) \cdot \frac{1}{\sqrt{6}} \\ &= -\frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} \\ &= -\frac{5}{\sqrt{6}}\end{aligned}$$

To simplify:

$$\begin{aligned}-\frac{5}{\sqrt{6}} &= -\frac{5}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} \\ &= -\frac{5\sqrt{6}}{6} \\ &\approx -2.04\end{aligned}$$

Therefore, the directional derivative of the function $\phi = e^{2x-y-z}$ at the point $(1, 1, 1)$ in the direction of the tangent to the given curve at $t = 0$ is $-\frac{5}{\sqrt{6}}$.

Example 15: Finding Parameters for Maximum Directional Derivative

If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at $(1, 1, 1)$ has maximum magnitude 15 in the direction parallel to $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, find a, b, and c.

Solution

Given:

$$\phi = ax^2y + by^2z + cz^2x$$
$$\text{Point: } (1, 1, 1)$$
$$\text{Direction: } \frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

Step 1: Find the gradient of ϕ.

$$\begin{aligned}\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\ &= \vec{i}(2axy + cz^2) + \vec{j}(ax^2 + 2byz) + \vec{k}(by^2 + 2czx)\end{aligned}$$

Step 2: Evaluate $\nabla\phi$ at the point $(1, 1, 1)$.

$$\begin{aligned}\nabla\phi|_{(1,1,1)} &= \vec{i}(2a \cdot 1 \cdot 1 + c \cdot 1^2) + \vec{j}(a \cdot 1^2 + 2b \cdot 1 \cdot 1) + \vec{k}(b \cdot 1^2 + 2c \cdot 1 \cdot 1) \\ &= \vec{i}(2a + c) + \vec{j}(a + 2b) + \vec{k}(b + 2c)\end{aligned}$$

Step 3: Find the direction vector from the given direction.
From $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, we can identify the direction vector by taking the denominators:

$$\vec{v} = 2\vec{i} - 2\vec{j} + \vec{k}$$

Step 4: Normalize the direction vector to get the unit vector.

$$\begin{aligned}|\vec{v}| &= \sqrt{2^2 + (-2)^2 + 1^2} \\ &= \sqrt{4 + 4 + 1} \\ &= \sqrt{9} = 3\end{aligned}$$

The unit vector in this direction is:

$$\begin{aligned}\hat{v} &= \frac{\vec{v}}{|\vec{v}|} = \frac{2\vec{i} - 2\vec{j} + \vec{k}}{3} \\ &= \frac{2}{3}\vec{i} - \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}\end{aligned}$$

Step 5: Set up equation for the directional derivative.
For the directional derivative to have maximum magnitude in the given direction, the gradient must be parallel to the direction vector. This means:

$$\nabla\phi = \lambda\vec{v}$$
$$(2a + c)\vec{i} + (a + 2b)\vec{j} + (b + 2c)\vec{k} = \lambda(2\vec{i} - 2\vec{j} + \vec{k})$$

Equating components:

$$\begin{aligned}2a + c &= 2\lambda \quad \text{(Equation 1)} \\ a + 2b &= -2\lambda \quad \text{(Equation 2)} \\ b + 2c &= \lambda \quad \text{(Equation 3)}\end{aligned}$$

Step 6: Use the condition that the directional derivative has magnitude 15.
The directional derivative is the dot product of the gradient and the unit vector in the direction:

$$\begin{aligned}
D_{\hat{v}}\phi &= \nabla\phi \cdot \hat{v} \\
&= (2a+c)\cdot\frac{2}{3} + (a+2b)\cdot\left(-\frac{2}{3}\right) + (b+2c)\cdot\frac{1}{3} \\
&= \frac{2(2a+c) - 2(a+2b) + (b+2c)}{3} \\
&= \frac{4a+2c-2a-4b+b+2c}{3} \\
&= \frac{2a-3b+4c}{3}
\end{aligned}$$

Given that the directional derivative has magnitude 15:

$$\begin{aligned}
\left|\frac{2a-3b+4c}{3}\right| &= 15 \\
|2a-3b+4c| &= 45
\end{aligned}$$

Since we're looking for the maximum value (not minimum), we can assume:

$$2a - 3b + 4c = 45 \quad \text{(Equation 4)}$$

Step 7: Solve the system of equations.
From Equation 1:

$$\begin{aligned}
2a + c &= 2\lambda \\
c &= 2\lambda - 2a \quad \text{(Equation 5)}
\end{aligned}$$

From Equation 2:

$$\begin{aligned}
a + 2b &= -2\lambda \\
b &= \frac{-2\lambda - a}{2} \\
b &= -\lambda - \frac{a}{2} \quad \text{(Equation 6)}
\end{aligned}$$

From Equation 3:

$$b + 2c = \lambda$$

Substituting Equation 5:

$$\begin{aligned}
b + 2(2\lambda - 2a) &= \lambda \\
b + 4\lambda - 4a &= \lambda \\
b + 3\lambda - 4a &= 0 \\
b &= 4a - 3\lambda \quad \text{(Equation 7)}
\end{aligned}$$

Equations 6 and 7 must be equal:

$$\begin{aligned}
-\lambda - \frac{a}{2} &= 4a - 3\lambda \\
-\lambda - \frac{a}{2} + 3\lambda &= 4a \\
2\lambda - \frac{a}{2} &= 4a \\
2\lambda &= 4a + \frac{a}{2} \\
2\lambda &= \frac{8a + a}{2} \\
2\lambda &= \frac{9a}{2} \\
\lambda &= \frac{9a}{4} \quad \text{(Equation 8)}
\end{aligned}$$

Substituting Equation 8 back into Equation 6:

$$\begin{aligned}
b &= -\lambda - \frac{a}{2} \\
&= -\frac{9a}{4} - \frac{a}{2} \\
&= -\frac{9a}{4} - \frac{2a}{4} \\
&= -\frac{11a}{4} \quad \text{(Equation 9)}
\end{aligned}$$

Substituting Equation 8 back into Equation 5:

$$\begin{aligned}
c &= 2\lambda - 2a \\
&= 2 \cdot \frac{9a}{4} - 2a \\
&= \frac{18a}{4} - 2a \\
&= \frac{18a}{4} - \frac{8a}{4} \\
&= \frac{10a}{4} \\
&= \frac{5a}{2} \quad \text{(Equation 10)}
\end{aligned}$$

Now, we can substitute Equations 9 and 10 into Equation 4:

$$\begin{aligned}
2a - 3b + 4c &= 45 \\
2a - 3\left(-\frac{11a}{4}\right) + 4\left(\frac{5a}{2}\right) &= 45 \\
2a + \frac{33a}{4} + \frac{20a}{2} &= 45 \\
2a + \frac{33a}{4} + \frac{40a}{4} &= 45 \\
\frac{8a}{4} + \frac{33a}{4} + \frac{40a}{4} &= 45 \\
\frac{81a}{4} &= 45 \\
a &= \frac{45 \cdot 4}{81} \\
a &= \frac{180}{81} \\
a &= \frac{20}{9}
\end{aligned}$$

Now we can find b using Equation 9:

$$\begin{aligned}
b &= -\frac{11a}{4} \\
&= -\frac{11}{4} \cdot \frac{20}{9} \\
&= -\frac{220}{36} \\
&= -\frac{55}{9}
\end{aligned}$$

And we can find c using Equation 10:

$$\begin{aligned}
c &= \frac{5a}{2} \\
&= \frac{5}{2} \cdot \frac{20}{9} \\
&= \frac{100}{18} \\
&= \frac{50}{9}
\end{aligned}$$

Therefore, $a = \frac{20}{9}$, $b = -\frac{55}{9}$, and $c = \frac{50}{9}$.
Step 8: Verify the solution.
Let's verify that with these values, the gradient at $(1, 1, 1)$ is parallel to the direction vector and the directional derivative has magnitude 15.

The gradient at $(1, 1, 1)$ is:

$$\begin{aligned}
\nabla\phi &= \vec{i}(2a+c) + \vec{j}(a+2b) + \vec{k}(b+2c) \\
&= \vec{i}\left(2\cdot\frac{20}{9} + \frac{50}{9}\right) + \vec{j}\left(\frac{20}{9} + 2\cdot\left(-\frac{55}{9}\right)\right) + \vec{k}\left(-\frac{55}{9} + 2\cdot\frac{50}{9}\right) \\
&= \vec{i}\left(\frac{40}{9} + \frac{50}{9}\right) + \vec{j}\left(\frac{20}{9} - \frac{110}{9}\right) + \vec{k}\left(-\frac{55}{9} + \frac{100}{9}\right) \\
&= \vec{i}\left(\frac{90}{9}\right) + \vec{j}\left(-\frac{90}{9}\right) + \vec{k}\left(\frac{45}{9}\right) \\
&= 10\vec{i} - 10\vec{j} + 5\vec{k}
\end{aligned}$$

This is proportional to the direction vector $2\vec{i} - 2\vec{j} + \vec{k}$, as required.
The directional derivative is:

$$\begin{aligned}
D_{\hat{v}}\phi &= \nabla\phi\cdot\hat{v} \\
&= (10\vec{i} - 10\vec{j} + 5\vec{k})\cdot\left(\frac{2}{3}\vec{i} - \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}\right) \\
&= 10\cdot\frac{2}{3} + (-10)\cdot\left(-\frac{2}{3}\right) + 5\cdot\frac{1}{3} \\
&= \frac{20}{3} + \frac{20}{3} + \frac{5}{3} \\
&= \frac{45}{3} \\
&= 15
\end{aligned}$$

This confirms that the directional derivative has magnitude 15, as required.
Therefore, $a = \frac{20}{9}$, $b = -\frac{55}{9}$, and $c = \frac{50}{9}$ are the correct values for the given conditions.

Example 16: Finding Direction of Maximum Directional Derivative

In what direction from the point $(2, 1, -1)$ is the directional derivative of $\phi = x^2yz^3$ maximum? What is the magnitude of this maximum?

Solution

Given:

$$\phi = x^2yz^3$$
$$\text{Point: } (2, 1, -1)$$

Step 1: Find the gradient of ϕ.

$$\begin{aligned}
\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\
&= \vec{i}(2xyz^3) + \vec{j}(x^2z^3) + \vec{k}(3x^2yz^2)
\end{aligned}$$

Step 2: Evaluate $\nabla\phi$ at the point $(2, 1, -1)$.

$$\begin{aligned}
\nabla\phi|_{(2,1,-1)} &= \vec{i}(2 \cdot 2 \cdot 1 \cdot (-1)^3) + \vec{j}(2^2 \cdot (-1)^3) + \vec{k}(3 \cdot 2^2 \cdot 1 \cdot (-1)^2) \\
&= \vec{i}(2 \cdot 2 \cdot 1 \cdot (-1)) + \vec{j}(4 \cdot (-1)) + \vec{k}(3 \cdot 4 \cdot 1 \cdot 1) \\
&= \vec{i}(-4) + \vec{j}(-4) + \vec{k}(12) \\
&= -4\vec{i} - 4\vec{j} + 12\vec{k}
\end{aligned}$$

Step 3: Find the direction of maximum directional derivative.
The directional derivative is maximum in the direction of the gradient. Therefore, the direction of maximum directional derivative is the unit vector in the direction of the gradient.

$$\begin{aligned}
|\nabla\phi| &= \sqrt{(-4)^2 + (-4)^2 + 12^2} \\
&= \sqrt{16 + 16 + 144} \\
&= \sqrt{176} = 4\sqrt{11}
\end{aligned}$$

The unit vector in the direction of the gradient is:

$$\begin{aligned}
\hat{v} = \frac{\nabla\phi}{|\nabla\phi|} &= \frac{-4\vec{i} - 4\vec{j} + 12\vec{k}}{4\sqrt{11}} \\
&= \frac{-1\vec{i} - 1\vec{j} + 3\vec{k}}{\sqrt{11}} \\
&= -\frac{1}{\sqrt{11}}\vec{i} - \frac{1}{\sqrt{11}}\vec{j} + \frac{3}{\sqrt{11}}\vec{k}
\end{aligned}$$

Step 4: Calculate the maximum directional derivative.
The maximum directional derivative is the magnitude of the gradient:

$$\begin{aligned}
\max(D_{\hat{v}}\phi) &= |\nabla\phi| \\
&= 4\sqrt{11} \\
&\approx 13.266
\end{aligned}$$

Therefore, the directional derivative of $\phi = x^2yz^3$ at the point $(2, 1, -1)$ is maximum in the direction $-\frac{1}{\sqrt{11}}\vec{i} - \frac{1}{\sqrt{11}}\vec{j} + \frac{3}{\sqrt{11}}\vec{k}$, and the magnitude of this maximum is $4\sqrt{11}$.

Example 17: Directional Derivative Using Multiple Points

The directional derivative of $\phi(x, y)$ at the point $A(3, 2)$ towards the point $B(2, 3)$ is $3\sqrt{2}$ and towards the point $C(1, 0)$ is $\sqrt{8}$. Find the directional derivative at the point A towards the point $D(2, 4)$.

Solution

Given:

$$\begin{aligned}\text{Point } A &= (3,2)\\ \text{Point } B &= (2,3)\\ \text{Point } C &= (1,0)\\ \text{Point } D &= (2,4)\\ \text{Directional derivative towards } B &= 3\sqrt{2}\\ \text{Directional derivative towards } C &= \sqrt{8} = 2\sqrt{2}\end{aligned}$$

Step 1: Find the direction vectors and corresponding unit vectors.
Direction vector from A to B:

$$\begin{aligned}\overrightarrow{AB} &= (2-3, 3-2)\\ &= (-1, 1)\end{aligned}$$

Magnitude of $\overrightarrow{AB}$:

$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{(-1)^2 + 1^2}\\ &= \sqrt{2}\end{aligned}$$

Unit vector in the direction from A to B:

$$\begin{aligned}\hat{u}_{AB} &= \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}\\ &= \frac{(-1,1)}{\sqrt{2}}\\ &= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\end{aligned}$$

Direction vector from A to C:

$$\begin{aligned}\overrightarrow{AC} &= (1-3, 0-2)\\ &= (-2, -2)\end{aligned}$$

Magnitude of $\overrightarrow{AC}$:

$$\begin{aligned}|\overrightarrow{AC}| &= \sqrt{(-2)^2 + (-2)^2}\\ &= \sqrt{8} = 2\sqrt{2}\end{aligned}$$

Unit vector in the direction from A to C:

$$\begin{aligned}\hat{u}_{AC} &= \frac{\overrightarrow{AC}}{|\overrightarrow{AC}|}\\ &= \frac{(-2,-2)}{2\sqrt{2}}\\ &= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\end{aligned}$$

Direction vector from A to D:

$$\begin{aligned}\overrightarrow{AD} &= (2-3, 4-2)\\ &= (-1, 2)\end{aligned}$$

Magnitude of $\overrightarrow{AD}$:

$$\begin{aligned}|\overrightarrow{AD}| &= \sqrt{(-1)^2 + 2^2}\\ &= \sqrt{5}\end{aligned}$$

Unit vector in the direction from A to D:

$$\begin{aligned}\hat{u}_{AD} &= \frac{\overrightarrow{AD}}{|\overrightarrow{AD}|}\\ &= \frac{(-1, 2)}{\sqrt{5}}\\ &= \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\end{aligned}$$

Step 2: Find the gradient of ϕ at point A.
Let's denote the gradient of ϕ at point A as $\nabla\phi(A) = (a, b)$.
We know that the directional derivative in a direction $\hat{u}$ is given by:

$$D_{\hat{u}}\phi = \nabla\phi \cdot \hat{u}$$

Using the directional derivative towards B:

$$\begin{aligned}D_{\hat{u}_{AB}}\phi &= \nabla\phi(A) \cdot \hat{u}_{AB}\\ 3\sqrt{2} &= (a, b) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\\ 3\sqrt{2} &= -\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}}\\ 3\sqrt{2} \cdot \sqrt{2} &= -a + b\\ 6 &= -a + b\\ b &= 6 + a \quad \text{(Equation 1)}\end{aligned}$$

Using the directional derivative towards C:

$$\begin{aligned}D_{\hat{u}_{AC}}\phi &= \nabla\phi(A) \cdot \hat{u}_{AC}\\ \sqrt{8} = 2\sqrt{2} &= (a, b) \cdot \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\\ 2\sqrt{2} &= -\frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}}\\ 2\sqrt{2} \cdot \sqrt{2} &= -a - b\\ 4 &= -a - b\\ -a - b &= 4\\ -b &= 4 + a\\ b &= -4 - a \quad \text{(Equation 2)}\end{aligned}$$

Solving for a and b using Equations 1 and 2:

$$b = 6 + a \quad \text{(Equation 1)}$$
$$b = -4 - a \quad \text{(Equation 2)}$$

Equation 1 = Equation 2:

$$\begin{aligned} 6 + a &= -4 - a \\ 6 + a + a &= -4 \\ 6 + 2a &= -4 \\ 2a &= -4 - 6 \\ 2a &= -10 \\ a &= -5 \end{aligned}$$

Substituting back into Equation 1:

$$\begin{aligned} b &= 6 + a \\ b &= 6 + (-5) \\ b &= 1 \end{aligned}$$

Therefore, $\nabla\phi(A) = (-5, 1)$.
Step 3: Calculate the directional derivative towards point D.
Using the gradient and the unit vector in the direction from A to D:

$$\begin{aligned} D_{\hat{u}_{AD}}\phi &= \nabla\phi(A) \cdot \hat{u}_{AD} \\ &= (-5, 1) \cdot \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\ &= (-5) \cdot \left(-\frac{1}{\sqrt{5}}\right) + 1 \cdot \frac{2}{\sqrt{5}} \\ &= \frac{5}{\sqrt{5}} + \frac{2}{\sqrt{5}} \\ &= \frac{7}{\sqrt{5}} \\ &= \frac{7\sqrt{5}}{5} \end{aligned}$$

Therefore, the directional derivative of $\phi(x, y)$ at the point $A(3, 2)$ towards the point $D(2, 4)$ is $\frac{7\sqrt{5}}{5}$.

Example 18: Finding the Directional Derivative

Find the directional derivative of $\phi = 4x^2 + 9x^2y^2$ at P(2, -1, 2):
(i) In the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$ (ii) Towards the point Q(5, 1, -1) (iii) Along the tangent to the curve $x = e^t \cos t, y = e^t \sin t, z = e^t$ at $t = 0$ (iv) Along a line equally inclined with co-ordinate axes.

Solution

Given: $\phi = 4x^2 + 9x^2y^2$ at point P(2, -1, 2).
The directional derivative of ϕ in the direction of unit vector $\hat{u}$ is given by:

$$D_{\hat{u}}\phi = \nabla\phi \cdot \hat{u}$$

Step 1: Find the gradient of ϕ.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$
$$\frac{\partial\phi}{\partial x} = 8x + 18xy^2$$
$$\frac{\partial\phi}{\partial y} = 18x^2y$$
$$\frac{\partial\phi}{\partial z} = 0$$

Therefore:

$$\nabla\phi = (8x + 18xy^2)\hat{i} + 18x^2y\hat{j} + 0\hat{k}$$

Step 2: Evaluate the gradient at point P(2, -1, 2).

$$\begin{aligned}\nabla\phi|_P &= (8(2) + 18(2)(-1)^2)\hat{i} + 18(2)^2(-1)\hat{j} + 0\hat{k}\\ &= (16 + 18 \cdot 2)\hat{i} + 18 \cdot 4 \cdot (-1)\hat{j} + 0\hat{k}\\ &= (16 + 36)\hat{i} - 72\hat{j} + 0\hat{k}\\ &= 52\hat{i} - 72\hat{j}\end{aligned}$$

Part (i): In the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$
Step 3: Convert to unit vector.

$$\begin{aligned}\vec{v} &= 2\hat{i} - 3\hat{j} + 6\hat{k}\\ |\vec{v}| &= \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7\\ \hat{u} &= \frac{\vec{v}}{|\vec{v}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7} = \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\end{aligned}$$

Step 4: Calculate directional derivative.

$$\begin{aligned}D_{\hat{u}}\phi &= \nabla\phi|_P \cdot \hat{u}\\ &= (52\hat{i} - 72\hat{j}) \cdot \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)\\ &= 52 \cdot \frac{2}{7} + (-72) \cdot \left(-\frac{3}{7}\right) + 0 \cdot \frac{6}{7}\\ &= \frac{104}{7} + \frac{216}{7}\\ &= \frac{320}{7}\end{aligned}$$

Part (ii): Towards the point Q(5, 1, -1)

Step 5: Find direction vector from P to Q.

$$\begin{aligned}\vec{PQ} &= Q - P = (5, 1, -1) - (2, -1, 2) \\ &= (3, 2, -3)\end{aligned}$$

Step 6: Convert to unit vector.

$$\begin{aligned}|\vec{PQ}| &= \sqrt{3^2 + 2^2 + (-3)^2} = \sqrt{9 + 4 + 9} = \sqrt{22} \\ \hat{u} &= \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{(3, 2, -3)}{\sqrt{22}}\end{aligned}$$

Step 7: Calculate directional derivative.

$$\begin{aligned}D_{\hat{u}}\phi &= \nabla\phi|_P \cdot \hat{u} \\ &= (52\hat{i} - 72\hat{j}) \cdot \frac{(3\hat{i} + 2\hat{j} - 3\hat{k})}{\sqrt{22}} \\ &= \frac{52 \cdot 3 + (-72) \cdot 2 + 0 \cdot (-3)}{\sqrt{22}} \\ &= \frac{156 - 144}{\sqrt{22}} \\ &= \frac{12}{\sqrt{22}}\end{aligned}$$

Part (iii): Along the tangent to the curve at $t = 0$
Step 8: Find the position vector of the curve.

$$\vec{r}(t) = e^t \cos t\hat{i} + e^t \sin t\hat{j} + e^t\hat{k}$$

Step 9: Find the tangent vector.

$$\begin{aligned}\vec{r}'(t) &= \frac{d}{dt}[e^t \cos t]\hat{i} + \frac{d}{dt}[e^t \sin t]\hat{j} + \frac{d}{dt}[e^t]\hat{k} \\ &= (e^t \cos t - e^t \sin t)\hat{i} + (e^t \sin t + e^t \cos t)\hat{j} + e^t\hat{k} \\ &= e^t(\cos t - \sin t)\hat{i} + e^t(\sin t + \cos t)\hat{j} + e^t\hat{k}\end{aligned}$$

Step 10: Evaluate at $t = 0$.

$$\begin{aligned}\vec{r}'(0) &= e^0(\cos 0 - \sin 0)\hat{i} + e^0(\sin 0 + \cos 0)\hat{j} + e^0\hat{k} \\ &= (1 - 0)\hat{i} + (0 + 1)\hat{j} + 1\hat{k} \\ &= \hat{i} + \hat{j} + \hat{k}\end{aligned}$$

Step 11: Convert to unit vector.

$$\begin{aligned}|\vec{r}'(0)| &= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \\ \hat{u} &= \frac{\vec{r}'(0)}{|\vec{r}'(0)|} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}\end{aligned}$$

Step 12: Calculate directional derivative.

$$\begin{aligned} D_{\hat{u}}\phi &= \nabla\phi|_P \cdot \hat{u} \\ &= (52\hat{i} - 72\hat{j}) \cdot \frac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}} \\ &= \frac{52 - 72 + 0}{\sqrt{3}} \\ &= \frac{-20}{\sqrt{3}} \end{aligned}$$

Part (iv): Along a line equally inclined with coordinate axes
Step 13: For a line equally inclined to all three coordinate axes, the direction cosines are equal: $\cos\alpha = \cos\beta = \cos\gamma$.
Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we have:

$$\begin{aligned} 3\cos^2\alpha &= 1 \\ \cos\alpha &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$

Therefore, the unit vector is:

$$\hat{u} = \pm\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

Step 14: Calculate directional derivative.

$$\begin{aligned} D_{\hat{u}}\phi &= \nabla\phi|_P \cdot \hat{u} \\ &= (52\hat{i} - 72\hat{j}) \cdot \left(\pm\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})\right) \\ &= \pm\frac{1}{\sqrt{3}}[52 - 72 + 0] \\ &= \pm\frac{-20}{\sqrt{3}} \end{aligned}$$

Since we have two possible directions (equally inclined), there are two possible values:

$$D_{\hat{u}}\phi = \frac{-20}{\sqrt{3}} \text{ or } \frac{20}{\sqrt{3}}$$

Example 19: Finding the Directional Derivative

Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at P(2, -1, 2):
(i) In the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$ (ii) Towards the point Q(1, 1, -1) (iii) Along the tangent to the curve $x = e^t\cos t, y = e^t\sin t, z = e^t$ at $t = 0$ (iv) Along a line equally inclined with co-ordinate axes. (v) Along a line making angle in the ratio 1:2:3 with co-ordinate axes.

Solution

Given: $\phi = 4xz^3 - 3x^2y^2z$ at point P(2, -1, 2).
The directional derivative of ϕ in the direction of unit vector $\hat{u}$ is given by:

$$D_{\hat{u}}\phi = \nabla\phi \cdot \hat{u}$$

Step 1: Find the gradient of ϕ.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$
$$\frac{\partial\phi}{\partial x} = 4z^3 - 6xy^2z$$
$$\frac{\partial\phi}{\partial y} = -6x^2yz$$
$$\frac{\partial\phi}{\partial z} = 12xz^2 - 3x^2y^2$$

Therefore:

$$\nabla\phi = (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

Step 2: Evaluate the gradient at point P(2, -1, 2).

$$\begin{aligned}
\nabla\phi|_P &= (4(2)^3 - 6(2)(-1)^2(2))\hat{i} + (-6(2)^2(-1)(2))\hat{j} + (12(2)(2)^2 - 3(2)^2(-1)^2)\hat{k} \\
&= (4\cdot 8 - 6\cdot 2\cdot 1\cdot 2)\hat{i} + (-6\cdot 4\cdot(-1)\cdot 2)\hat{j} + (12\cdot 2\cdot 4 - 3\cdot 4\cdot 1)\hat{k} \\
&= (32 - 24)\hat{i} + 48\hat{j} + (96 - 12)\hat{k} \\
&= 8\hat{i} + 48\hat{j} + 84\hat{k}
\end{aligned}$$

Part (i): In the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$
Step 3: Convert to unit vector.

$$\vec{v} = 2\hat{i} - 3\hat{j} + 6\hat{k}$$
$$|\vec{v}| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$
$$\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7} = \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

Step 4: Calculate directional derivative.

$$\begin{aligned}
D_{\hat{u}}\phi &= \nabla\phi|_P \cdot \hat{u} \\
&= (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) \\
&= 8\cdot\frac{2}{7} + 48\cdot\left(-\frac{3}{7}\right) + 84\cdot\frac{6}{7} \\
&= \frac{16}{7} - \frac{144}{7} + \frac{504}{7} \\
&= \frac{376}{7}
\end{aligned}$$

Part (ii): Towards the point Q(1, 1, -1)
Step 5: Find direction vector from P to Q.

$$\begin{aligned}
\vec{PQ} &= Q - P = (1, 1, -1) - (2, -1, 2) \\
&= (-1, 2, -3)
\end{aligned}$$

Step 6: Convert to unit vector.

$$|\vec{PQ}| = \sqrt{(-1)^2 + 2^2 + (-3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$
$$\hat{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{(-1, 2, -3)}{\sqrt{14}}$$

Step 7: Calculate directional derivative.

$$
\begin{aligned}
D_{\hat{u}}\phi &= \nabla\phi|_P \cdot \hat{u} \\
&= (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \frac{(-1\hat{i} + 2\hat{j} - 3\hat{k})}{\sqrt{14}} \\
&= \frac{8 \cdot (-1) + 48 \cdot 2 + 84 \cdot (-3)}{\sqrt{14}} \\
&= \frac{-8 + 96 - 252}{\sqrt{14}} \\
&= \frac{-164}{\sqrt{14}}
\end{aligned}
$$

Part (iii): Along the tangent to the curve at $t = 0$
Step 8: Find the position vector of the curve.

$$\vec{r}(t) = e^t \cos t\hat{i} + e^t \sin t\hat{j} + e^t\hat{k}$$

Step 9: Find the tangent vector.

$$
\begin{aligned}
\vec{r}'(t) &= \frac{d}{dt}[e^t \cos t]\hat{i} + \frac{d}{dt}[e^t \sin t]\hat{j} + \frac{d}{dt}[e^t]\hat{k} \\
&= (e^t \cos t - e^t \sin t)\hat{i} + (e^t \sin t + e^t \cos t)\hat{j} + e^t\hat{k} \\
&= e^t(\cos t - \sin t)\hat{i} + e^t(\sin t + \cos t)\hat{j} + e^t\hat{k}
\end{aligned}
$$

Step 10: Evaluate at $t = 0$.

$$
\begin{aligned}
\vec{r}'(0) &= e^0(\cos 0 - \sin 0)\hat{i} + e^0(\sin 0 + \cos 0)\hat{j} + e^0\hat{k} \\
&= (1 - 0)\hat{i} + (0 + 1)\hat{j} + 1\hat{k} \\
&= \hat{i} + \hat{j} + \hat{k}
\end{aligned}
$$

Step 11: Convert to unit vector.

$$
\begin{aligned}
|\vec{r}'(0)| &= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \\
\hat{u} &= \frac{\vec{r}'(0)}{|\vec{r}'(0)|} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}
\end{aligned}
$$

Step 12: Calculate directional derivative.

$$
\begin{aligned}
D_{\hat{u}}\phi &= \nabla\phi|_P \cdot \hat{u} \\
&= (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \frac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}} \\
&= \frac{8 + 48 + 84}{\sqrt{3}} \\
&= \frac{140}{\sqrt{3}}
\end{aligned}
$$

Part (iv): Along a line equally inclined with coordinate axes
Step 13: For a line equally inclined to all three coordinate axes, the direction cosines are equal: $\cos\alpha = \cos\beta = \cos\gamma$.

Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we have:

$$3\cos^2\alpha = 1$$
$$\cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Therefore, the unit vector is:

$$\hat{u} = \pm\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

Step 14: Calculate directional derivative.

$$\begin{aligned}
D_{\hat{u}}\phi &= \nabla\phi|_P \cdot \hat{u} \\
&= (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \left(\pm\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})\right) \\
&= \pm\frac{1}{\sqrt{3}}[8 + 48 + 84] \\
&= \pm\frac{140}{\sqrt{3}}
\end{aligned}$$

Part (v): Along a line making angle in the ratio 1:2:3 with coordinate axes.
Step 15: If the angle ratios are 1:2:3, we can express direction cosines as:

$$\cos\alpha : \cos\beta : \cos\gamma = 1 : 2 : 3$$

Let $\cos\alpha = k$, $\cos\beta = 2k$, $\cos\gamma = 3k$, where k is a constant.
Using $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$:

$$\begin{aligned}
k^2 + (2k)^2 + (3k)^2 &= 1 \\
k^2 + 4k^2 + 9k^2 &= 1 \\
14k^2 &= 1 \\
k &= \frac{1}{\sqrt{14}}
\end{aligned}$$

Therefore, the unit vector is:

$$\hat{u} = \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$

Step 16: Calculate directional derivative.

$$\begin{aligned}
D_{\hat{u}}\phi &= \nabla\phi|_P \cdot \hat{u} \\
&= (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \left(\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}\right) \\
&= \frac{8 \cdot 1 + 48 \cdot 2 + 84 \cdot 3}{\sqrt{14}} \\
&= \frac{8 + 96 + 252}{\sqrt{14}} \\
&= \frac{356}{\sqrt{14}}
\end{aligned}$$