Chapter 1

Prerequisites and Mathematical Foundations

1.1 Review of Single Variable Calculus

Before delving into multivariable calculus and partial differentiation, it is essential to have a strong foundation in single variable calculus. This section provides a concise review of key concepts that will be extended to functions of several variables.

1.1.1 Functions and Their Properties

A function $f : A \to B$ assigns to each element $x \in A$ exactly one element $y \in B$, denoted as $y = f(x)$. The set A is called the domain and B is the codomain.

Important Function Properties

For a function $f : \mathbb{R} \to \mathbb{R}$:

- **Injectivity (One-to-one):** $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- **Surjectivity (Onto):** For every $y \in B$, there exists $x \in A$ such that $f(x) = y$
- **Bijection:** A function that is both injective and surjective

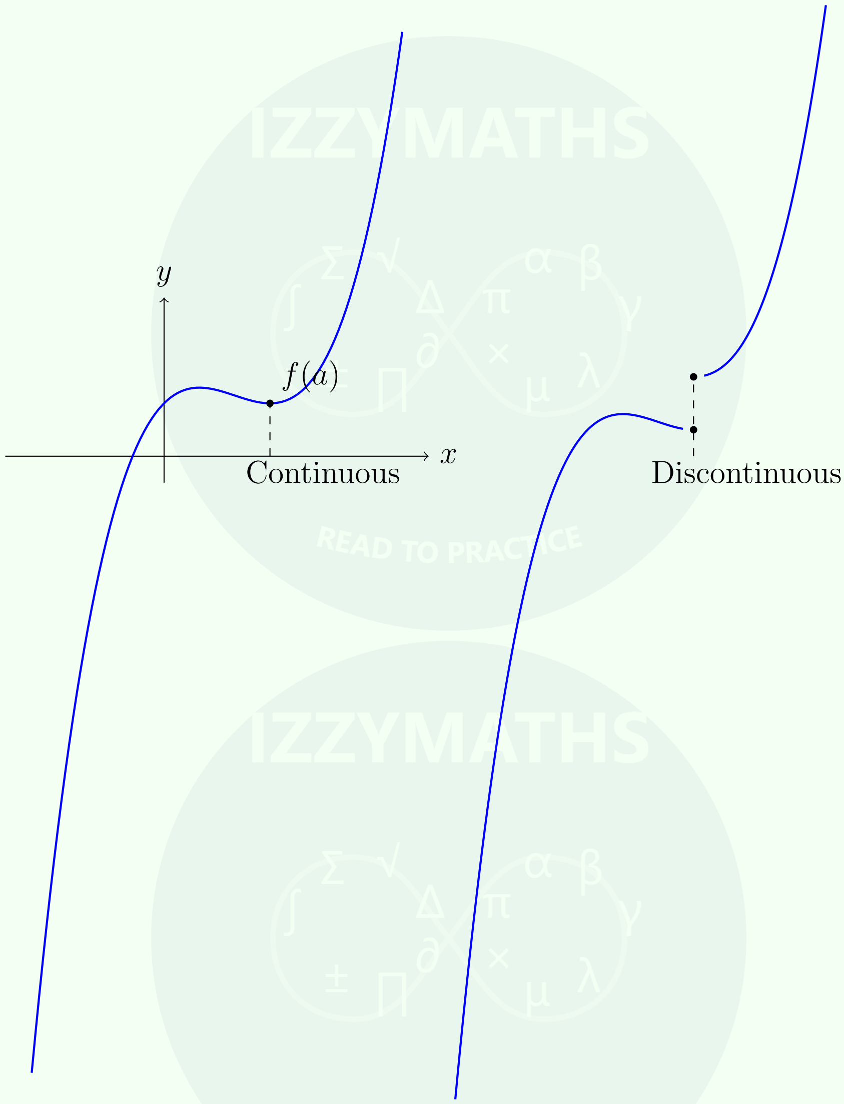
1.1.2 Limits and Continuity

Definition 1.1 (Limit). *We say* $\lim_{x\to a} f(x) = L$ *if for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that whenever* $0 < |x - a| < \delta$*, we have* $|f(x) - L| < \varepsilon$*.*

Definition 1.2 (Continuity). *A function* f *is continuous at a point* a *if:*

1. $f(a)$ *is defined*
2. $\lim_{x\to a} f(x)$ *exists*
3. $\lim_{x\to a} f(x) = f(a)$

Geometric Interpretation of Continuity



Geometrically, continuity means the graph of the function has no breaks, holes, or jumps.

1.1.3 Differentiation

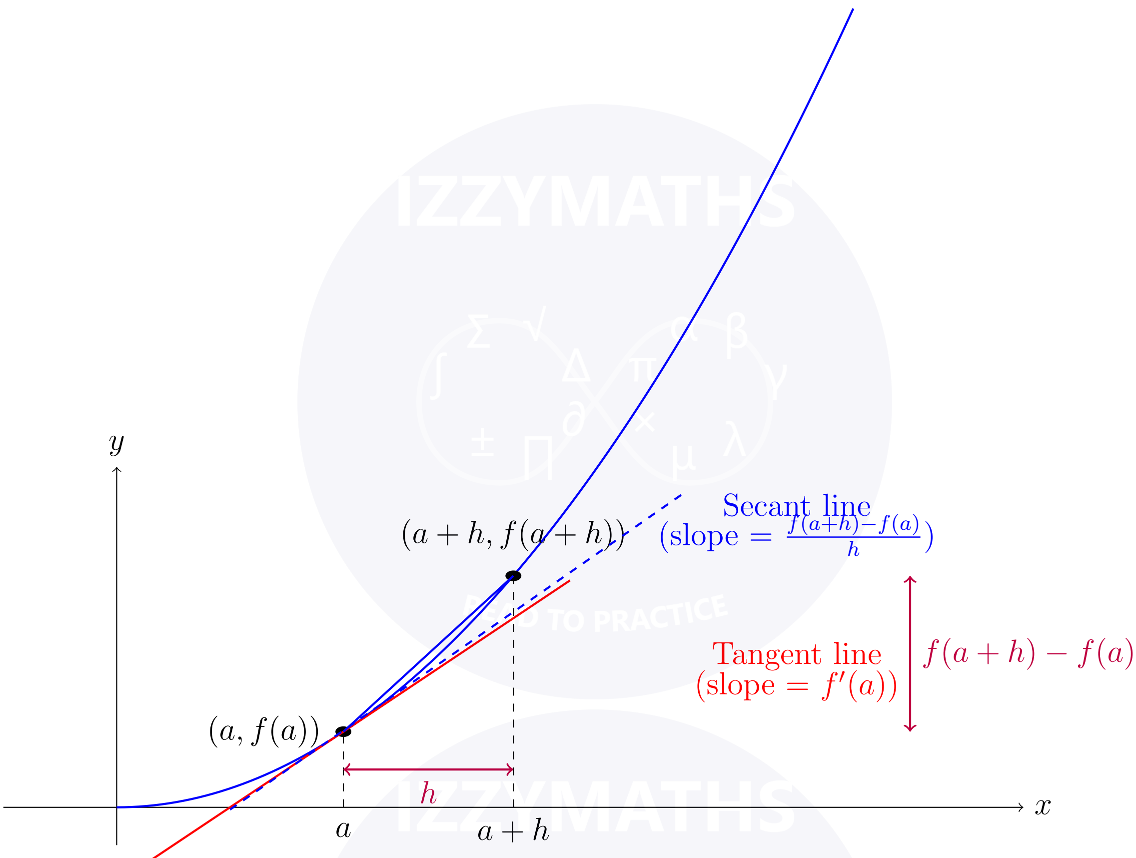
Definition 1.3 (Derivative)**.** *The derivative of a function f at point $x = a$ is defined as:*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

Geometric Interpretation of Derivative

The derivative $f'(a)$ represents the slope of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$.



As $h \to 0$, the secant line approaches the tangent line at point $(a, f(a))$

1.1.4 Key Differentiation Rules

Differentiation Rules

For functions $f(x)$ and $g(x)$:

$$\text{Sum Rule: } (f+g)' = f' + g' \tag{1.1}$$

$$\text{Product Rule: } (f \cdot g)' = f' \cdot g + f \cdot g' \tag{1.2}$$

$$\text{Quotient Rule: } \left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}, \quad g \neq 0 \tag{1.3}$$

$$\text{Chain Rule: } (f \circ g)' = (f' \circ g) \cdot g' \tag{1.4}$$

1.1.5 Integration

Definition 1.4 (Definite Integral)**.** *The definite integral of $f(x)$ from a to b is defined as:*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

where $\Delta x_i = \frac{b-a}{n}$ and x_i^* is any point in the subinterval $[x_{i-1}, x_i]$.

Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is an antiderivative of f, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

1.2 Essential Formulae for Multivariable Calculus

1.2.1 Basic Functions and Their Properties

Elementary Functions

$$\text{Linear: } f(x) = mx + b \tag{1.5}$$
$$\text{Quadratic: } f(x) = ax^2 + bx + c \tag{1.6}$$
$$\text{Polynomial: } f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \tag{1.7}$$
$$\text{Rational: } f(x) = \frac{P(x)}{Q(x)} \text{ where } P \text{ and } Q \text{ are polynomials} \tag{1.8}$$
$$\text{Exponential: } f(x) = a^x, \quad \text{especially } e^x \tag{1.9}$$
$$\text{Logarithmic: } f(x) = \log_a x, \quad \text{especially } \ln x \tag{1.10}$$

1.2.2 Trigonometric Identities

Fundamental Identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.11}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{1.12}$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{1.13}$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{1.14}$$

1.2.3 Common Derivatives

Important Derivatives

$$\frac{d}{dx}(x^n) = nx^{n-1} \tag{1.15}$$
$$\frac{d}{dx}(e^x) = e^x \tag{1.16}$$
$$\frac{d}{dx}(a^x) = a^x \ln a \tag{1.17}$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x} \tag{1.18}$$
$$\frac{d}{dx}(\sin x) = \cos x \tag{1.19}$$
$$\frac{d}{dx}(\cos x) = -\sin x \tag{1.20}$$
$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{1.21}$$

Differentiation Rules

For functions $f(x)$ and $g(x)$:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \tag{1.22}$$

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}f(x) \quad (\text{constant } c) \tag{1.23}$$

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f(x) \tag{1.24}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot \frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x)}{[g(x)]^2}, \quad g(x) \neq 0 \tag{1.25}$$

$$\frac{d}{dx}[f(g(x))] = \frac{df}{dg} \cdot \frac{dg}{dx} \quad (\text{Chain Rule}) \tag{1.26}$$

1.2.4 Common Integrals

Important Indefinite Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \tag{1.27}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \tag{1.28}$$

$$\int e^x\,dx = e^x + C \tag{1.29}$$

$$\int \sin x\,dx = -\cos x + C \tag{1.30}$$

$$\int \cos x\,dx = \sin x + C \tag{1.31}$$

$$\int \sec^2 x\,dx = \tan x + C \tag{1.32}$$

1.2.5 Coordinate Systems

Coordinate Transformations

Polar Coordinates (2D):

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2} \tag{1.33}$$

$$y = r\sin\theta \qquad \theta = \arctan\left(\frac{y}{x}\right) \tag{1.34}$$

Cylindrical Coordinates (3D):

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2} \tag{1.35}$$

$$y = r\sin\theta \qquad \theta = \arctan\left(\frac{y}{x}\right) \tag{1.36}$$

$$z = z \qquad z = z \tag{1.37}$$

Spherical Coordinates (3D):

$$x = \rho \sin\phi \cos\theta \qquad \rho = \sqrt{x^2 + y^2 + z^2} \tag{1.38}$$

$$y = \rho \sin\phi \sin\theta \qquad \phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \tag{1.39}$$

$$z = \rho \cos\phi \qquad \theta = \arctan\left(\frac{y}{x}\right) \tag{1.40}$$