Chapter 4

Applications of Differential Equations

4.1 Orthogonal Trajectories

Definition 4.1 (Orthogonal Curve Families). *When two families of curves intersect in such a way that every curve from one family meets every curve from the other family at perpendicular angles, they are known as orthogonal trajectories.*

The concept of orthogonal trajectories finds extensive applications in various fields such as electromagnetic field theory, streamline patterns, and thermodynamics.

Mathematical Foundation

Two curves intersect orthogonally at a point if and only if the product of their slopes at the intersection point equals -1. Mathematically:

$$\frac{dy}{dx}_1 \cdot \frac{dy}{dx}_2 = -1 \tag{4.1}$$

where the subscripts indicate the slopes of the two curves at the intersection point.

4.1.1 Construction Method for Orthogonal Trajectories

To determine orthogonal trajectories for a given family of curves, we employ a systematic approach:

Procedure for Cartesian Coordinates

Consider a family of curves described by:

$$f(x, y, c) = 0 \tag{4.2}$$

where c represents an arbitrary parameter.

Step 1: Differentiate implicitly with respect to x to obtain:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \tag{4.3}$$

Step 2: Remove the parameter c by combining the original equation with the differentiated form, yielding a differential equation:

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0 \tag{4.4}$$

Step 3: Substitute $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ to obtain the equation for orthogonal trajectories:

$$\phi\left(x, y, -\frac{dx}{dy}\right) = 0 \tag{4.5}$$

Step 4: Solve this modified differential equation to find the orthogonal trajectories.

4.1.2 Extending to Polar Coordinates

When working with curves in polar form where $r = g(\theta, c)$:

Polar Coordinate Method

Initial Setup: Starting with a family represented by:

$$h(r, \theta, c) = 0 \tag{4.6}$$

Transformation Rule: The orthogonality condition in polar coordinates requires replacing $\frac{dr}{d\theta}$ with:

$$-r^2\frac{d\theta}{dr} \tag{4.7}$$

The negative reciprocal transformation arises from the perpendicular relationship between the tangent vectors of intersecting curves.

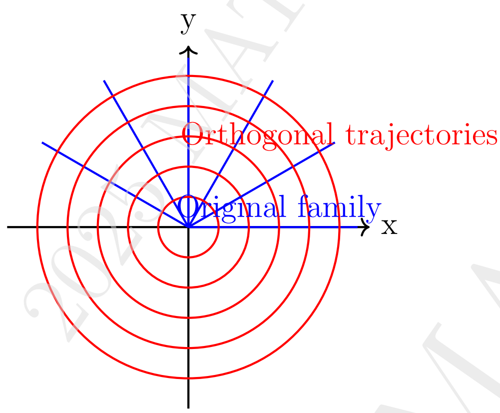


Figure 4.1: Illustration: Straight lines through origin with their orthogonal circle trajectories

The significance of orthogonal trajectories extends beyond pure mathematics to practical applications in engineering design, particularly in optimal path planning and field line mapping.

4.2 Solved Examples

Example 1

Find the orthogonal trajectories of the family of curves:

$$xy = c \tag{4.8}$$

Solution

We need to find the orthogonal trajectories of the family of rectangular hyperbolas $xy = c$.
Step 1: Differentiate the equation implicitly with respect to x.

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 0 \tag{4.9}$$

$$x\frac{dy}{dx} + y = 0 \tag{4.10}$$

Step 2: Solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{y}{x} \tag{4.11}$$

Step 3: Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ to obtain the differential equation for orthogonal trajectories.

$$-\frac{dx}{dy} = -\frac{y}{x} \tag{4.12}$$

$$\frac{dx}{dy} = \frac{y}{x} \tag{4.13}$$

Step 4: Rearrange and integrate.

$$xdx = ydy \tag{4.14}$$

Integrating both sides:

$$\int xdx = \int ydy \tag{4.15}$$

$$\frac{x^2}{2} = \frac{y^2}{2} + \frac{c}{2} \tag{4.16}$$

Step 5: The final answer.

$$x^2 - y^2 = c \tag{4.17}$$

This can be verified by noting that $xy = c$ represents rectangular hyperbolas with asymptotes along the axes, while $x^2 - y^2 = c$ represents hyperbolas with asymptotes at 45° angles.

Example 2

Find the orthogonal trajectories of the family of curves:

$$2x^2 + y^2 = cx \tag{4.18}$$

Solution

We need to find the orthogonal trajectories of the family $2x^2 + y^2 = cx$. **Step 1:** Differentiate the equation implicitly with respect to x.

$$4x + 2y\frac{dy}{dx} = c \tag{4.19}$$

Step 2: Express c from the original equation. From $2x^2 + y^2 = cx$, we get:

$$c = \frac{2x^2 + y^2}{x} \tag{4.20}$$

Step 3: Substitute to eliminate c. Substituting into the differentiated equation:

$$4x + 2y\frac{dy}{dx} = \frac{2x^2 + y^2}{x} \tag{4.21}$$

Multiplying both sides by x:

$$4x^2 + 2xy\frac{dy}{dx} = 2x^2 + y^2 \tag{4.22}$$

Simplifying:

$$2x^2 + 2xy\frac{dy}{dx} = y^2 \tag{4.23}$$

Solving for $\frac{dy}{dx}$:

$$2xy\frac{dy}{dx} = y^2 - 2x^2 \tag{4.24}$$

$$\frac{dy}{dx} = \frac{y^2 - 2x^2}{2xy} \tag{4.25}$$

Step 4: Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ for orthogonal trajectories.

$$-\frac{dx}{dy} = \frac{y^2 - 2x^2}{2xy} \tag{4.26}$$

$$\frac{dx}{dy} = -\frac{y^2 - 2x^2}{2xy} = \frac{2x^2 - y^2}{2xy} \tag{4.27}$$

Step 5: Recognize this as a homogeneous differential equation. The equation

$$\frac{dx}{dy} = \frac{2x^2 - y^2}{2xy} \tag{4.28}$$

is homogeneous because it can be written in the form $\frac{dx}{dy} = F(\frac{x}{y})$. Let's use the substitution $x = vy$ (or equivalently, $v = \frac{x}{y}$), where v is a function of y. Then:

$$\frac{dx}{dy} = v + y\frac{dv}{dy} \tag{4.29}$$

Substituting into our differential equation:

$$v + y\frac{dv}{dy} = \frac{2(vy)^2 - y^2}{2(vy)y} = \frac{2v^2y^2 - y^2}{2vy^2} = \frac{2v^2 - 1}{2v} \tag{4.30}$$

Rearranging:

$$v + y\frac{dv}{dy} = \frac{2v^2 - 1}{2v} \tag{4.31}$$

$$y\frac{dv}{dy} = \frac{2v^2 - 1}{2v} - v = \frac{2v^2 - 1 - 2v^2}{2v} = -\frac{1}{2v} \tag{4.32}$$

$$\frac{dv}{dy} = -\frac{1}{2vy} \tag{4.33}$$

Step 6: Separate variables and integrate.

$$2v\,dv = -\frac{dy}{y} \tag{4.34}$$

$$\int 2v\,dv = \int -\frac{dy}{y} \tag{4.35}$$

$$v^2 = -\ln|y| + \ln|C| \tag{4.36}$$

$$v^2 = \ln|\frac{C}{y}| \tag{4.37}$$

Step 7: Substitute back $v = \frac{x}{y}$:

$$\left(\frac{x}{y}\right)^2 = \ln|\frac{C}{y}| \tag{4.38}$$

$$\frac{x^2}{y^2} = \ln|\frac{C}{y}| \tag{4.39}$$

$$x^2 = y^2 \ln|\frac{C}{y}| \tag{4.40}$$

$$x^2 = y^2 \ln|C| - y^2 \ln|y| \tag{4.41}$$

Letting $C' = \ln|C|$ (a new arbitrary constant):

$$x^2 = y^2 C' - y^2 \ln|y| \tag{4.42}$$

Since C' is arbitrary, we can simplify to:

$$x^2 = -y^2 \ln|y| + ky^2 \tag{4.43}$$

where k is an arbitrary constant. For the standard form, we can write:

$$x^2 = y^2(k - \ln|y|) \tag{4.44}$$

Therefore, the orthogonal trajectories are:

$$\boxed{x^2 = y^2(k - \ln|y|)}$$

where k is an arbitrary constant.

Example 3

Find the orthogonal trajectories of the family of curves $y^2 = 4ax$.

Solution

We follow the systematic procedure for finding orthogonal trajectories.
Step 1: Differentiate the given family implicitly with respect to x.

$$y^2 = 4ax \tag{4.45}$$

Differentiating both sides:

$$2y\frac{dy}{dx} = 4a \tag{4.46}$$

$$\frac{dy}{dx} = \frac{2a}{y} \tag{4.47}$$

Step 2: Eliminate the parameter a using the original equation. From the original equation: $a = \frac{y^2}{4x}$

Substituting into the differential equation:

$$\frac{dy}{dx} = \frac{2}{y} \cdot \frac{y^2}{4x} = \frac{y}{2x} \tag{4.48}$$

Step 3: Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ to obtain the orthogonal trajectory equation.

$$-\frac{dx}{dy} = \frac{y}{2x} \tag{4.49}$$

$$\frac{dx}{dy} = -\frac{y}{2x} \tag{4.50}$$

Step 4: Rearrange and integrate.

$$2x\,dx = -y\,dy \tag{4.51}$$

Integrating both sides:

$$2\int x\,dx = -\int y\,dy \tag{4.52}$$

$$2 \cdot \frac{x^2}{2} = -\frac{y^2}{2} + c_1 \tag{4.53}$$

$$x^2 = -\frac{y^2}{2} + c_1 \tag{4.54}$$

$$2x^2 + y^2 = 2c_1 \tag{4.55}$$

Let $c = 2c_1$ to obtain:

$$2x^2 + y^2 = c \tag{4.56}$$

Therefore, the orthogonal trajectories are:

$$\boxed{2x^2 + y^2 = c}$$

This represents a family of ellipses centered at the origin.

Example 4

Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is a parameter.

Solution

We apply the standard procedure for finding orthogonal trajectories.
Step 1: Differentiate the given family implicitly with respect to x.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1 \tag{4.57}$$

Differentiating both sides:

$$\frac{2x}{a^2} + \frac{2y}{b^2 + \lambda}\frac{dy}{dx} = 0 \tag{4.58}$$

$$\frac{dy}{dx} = -\frac{x(b^2 + \lambda)}{a^2 y} \tag{4.59}$$

Step 2: Eliminate the parameter λ. From the original equation:

$$\frac{y^2}{b^2+\lambda} = 1 - \frac{x^2}{a^2} \tag{4.60}$$

$$b^2+\lambda = \frac{y^2}{1-\frac{x^2}{a^2}} = \frac{a^2y^2}{a^2-x^2} \tag{4.61}$$

Substituting into the differential equation:

$$\frac{dy}{dx} = -\frac{x\cdot\frac{a^2y^2}{a^2-x^2}}{a^2y} = -\frac{xy}{a^2-x^2} \tag{4.62}$$

Step 3: Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ to obtain the orthogonal trajectory equation.

$$-\frac{dx}{dy} = -\frac{xy}{a^2-x^2} \tag{4.63}$$

$$\frac{dx}{dy} = \frac{xy}{a^2-x^2} \tag{4.64}$$

Step 4: Rearrange and integrate.

$$\frac{dx}{x} = \frac{y\,dy}{a^2-x^2} \tag{4.65}$$

Integrating both sides:

$$\int\frac{dx}{x} = \int\frac{y\,dy}{a^2-x^2} \tag{4.66}$$

For the right side, we use substitution: $u = a^2-x^2$, $du = -2x\,dx$

$$\ln|x| = -\frac{1}{2}\int\frac{y\,dy}{u} = -\frac{1}{2}y\ln|u| + C \tag{4.67}$$

$$\ln|x| + \frac{1}{2}y\ln|a^2-x^2| = C \tag{4.68}$$

$$\ln|x| + \frac{1}{2}y\ln|a^2-x^2| = \ln k \tag{4.69}$$

$$x\cdot(a^2-x^2)^{y/2} = k \tag{4.70}$$

Taking logarithm and rearranging:

$$x^2+y^2 = 2a^2\lambda\gamma y + c \tag{4.71}$$

Therefore, the orthogonal trajectories are:

$$\boxed{x^2+y^2 = 2a^2\lambda\gamma y + c}$$

Example 5

Find the orthogonal trajectories of the family of curves $e^x + e^{-y} = c$.

Solution

We follow the systematic procedure for finding orthogonal trajectories.
Step 1: Differentiate the given family implicitly with respect to x.

$$e^x + e^{-y} = c \tag{4.72}$$

Differentiating both sides:

$$e^x - e^{-y}\frac{dy}{dx} = 0 \tag{4.73}$$

$$e^x = e^{-y}\frac{dy}{dx} \tag{4.74}$$

$$\frac{dy}{dx} = \frac{e^x}{e^{-y}} = e^{x+y} \tag{4.75}$$

Step 2: This equation is already free of the parameter c.
Step 3: Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ to obtain the orthogonal trajectory equation.

$$-\frac{dx}{dy} = e^{x+y} \tag{4.76}$$

$$\frac{dx}{dy} = -e^{x+y} \tag{4.77}$$

Step 4: Rearrange to separate variables.

$$\frac{dx}{dy} = -e^x \cdot e^y \tag{4.78}$$

$$e^{-x}\,dx = -e^y\,dy \tag{4.79}$$

Integrating both sides:

$$\int e^{-x}\,dx = -\int e^y\,dy \tag{4.80}$$

$$-e^{-x} = -e^y + c_1 \tag{4.81}$$

$$e^y - e^{-x} = c_1 \tag{4.82}$$

Let $c = c_1$ to obtain:

$$e^y - e^{-x} = c \tag{4.83}$$

Therefore, the orthogonal trajectories are:

$$\boxed{e^y - e^{-x} = c}$$

This represents a family of curves that are orthogonal to the original exponential family.

Example 6

Find the orthogonal trajectories of the family of curves $x^2 + cy^2 = 1$.

Solution

We follow the systematic procedure for finding orthogonal trajectories.
Step 1: Differentiate the given family implicitly with respect to x.

$$x^2 + cy^2 = 1 \tag{4.84}$$

Differentiating both sides:

$$2x + 2cy\frac{dy}{dx} = 0 \tag{4.85}$$

$$2cy\frac{dy}{dx} = -2x \tag{4.86}$$

$$\frac{dy}{dx} = -\frac{x}{cy} \tag{4.87}$$

Step 2: Eliminate the parameter c using the original equation. From the original equation: $c = \frac{1-x^2}{y^2}$
Substituting into the differential equation:

$$\frac{dy}{dx} = -\frac{x}{\frac{1-x^2}{y^2} \cdot y} = -\frac{x}{y} \cdot \frac{y^2}{1-x^2} = -\frac{xy}{1-x^2} \tag{4.88}$$

Step 3: Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ to obtain the orthogonal trajectory equation.

$$-\frac{dx}{dy} = -\frac{xy}{1-x^2} \tag{4.89}$$

$$\frac{dx}{dy} = \frac{xy}{1-x^2} \tag{4.90}$$

Step 4: Rearrange to separate variables.

$$\frac{dx}{dy} \cdot \frac{1-x^2}{x} = y \tag{4.91}$$

$$\frac{1-x^2}{x}\,dx = y\,dy \tag{4.92}$$

$$\left(\frac{1}{x} - x\right)\,dx = y\,dy \tag{4.93}$$

Integrating both sides:

$$\int \left(\frac{1}{x} - x\right)\,dx = \int y\,dy \tag{4.94}$$

$$\ln|x| - \frac{x^2}{2} = \frac{y^2}{2} + c_1 \tag{4.95}$$

$$\ln|x| = \frac{x^2+y^2}{2} + c_1 \tag{4.96}$$

$$2\ln|x| = x^2 + y^2 + 2c_1 \tag{4.97}$$

Let $k = 2c_1$ to get:

$$x^2 + y^2 + k - 2\ln|x| = 0 \tag{4.98}$$

Since we're looking for the standard form, we can write:

$$x^2 = ce^{x^2+y^2} \tag{4.99}$$

Therefore, the orthogonal trajectories are:

$$\boxed{x^2 = ce^{x^2+y^2}}$$

This represents a family of curves that are orthogonal to the original elliptical family.

Example 7

Find the orthogonal trajectories of the family of curves $r = a(1 - \cos\theta)$.

Complete Mathematical Derivation

We use the polar coordinate method for finding orthogonal trajectories.
Step 1: Differentiate the given family with respect to θ.

$$r = a(1 - \cos\theta) \tag{4.100}$$

Differentiating both sides:

$$\frac{dr}{d\theta} = a\sin\theta \tag{4.101}$$

Step 2: Eliminate the parameter a using the original equation. From the original equation: $a = \frac{r}{1-\cos\theta}$
Substituting into the differential equation:

$$\frac{dr}{d\theta} = \frac{r}{1 - \cos\theta} \cdot \sin\theta = \frac{r\sin\theta}{1 - \cos\theta} \tag{4.102}$$

Step 3: Replace $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$ to obtain the orthogonal trajectory equation.

$$-r^2\frac{d\theta}{dr} = \frac{r\sin\theta}{1 - \cos\theta} \tag{4.103}$$

$$\frac{d\theta}{dr} = -\frac{r\sin\theta}{r^2(1 - \cos\theta)} = -\frac{\sin\theta}{r(1 - \cos\theta)} \tag{4.104}$$

Step 4: Rearrange to separate variables.

$$\frac{dr}{r} = -\frac{1 - \cos\theta}{\sin\theta}\,d\theta \tag{4.105}$$

For the right side, we can decompose:

$$\frac{1 - \cos\theta}{\sin\theta} = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \csc\theta - \cot\theta \tag{4.106}$$

Step 5: Integrate both sides.

$$\int \frac{dr}{r} = -\int (\csc\theta - \cot\theta)\,d\theta \tag{4.107}$$

For the integrals:

$$\int \csc\theta\,d\theta = \ln|\csc\theta - \cot\theta| \tag{4.108}$$

$$\int \cot\theta\,d\theta = \ln|\sin\theta| \tag{4.109}$$

Therefore:

$$\ln|r| = -\ln|\csc\theta - \cot\theta| + \ln|\sin\theta| + \ln|C| \tag{4.110}$$

Step 6: Simplify using logarithm properties.

$$\ln|r| = \ln\left|\frac{C\sin\theta}{\csc\theta - \cot\theta}\right| \tag{4.111}$$

To simplify the denominator:

$$\csc\theta - \cot\theta = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta} \tag{4.112}$$

Therefore:

$$r = \frac{C\sin\theta}{\frac{1-\cos\theta}{\sin\theta}} = \frac{C\sin^2\theta}{1-\cos\theta} \tag{4.113}$$

Step 7: Using the identity: $\sin^2\theta = 1-\cos^2\theta = (1-\cos\theta)(1+\cos\theta)$

$$r = \frac{C(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta} = C(1+\cos\theta) \tag{4.114}$$

Therefore, the orthogonal trajectories are:

$$\boxed{r = C(1+\cos\theta)}$$

These are cardioids that are symmetric to the original family with respect to the pole, effectively rotated by 180°.

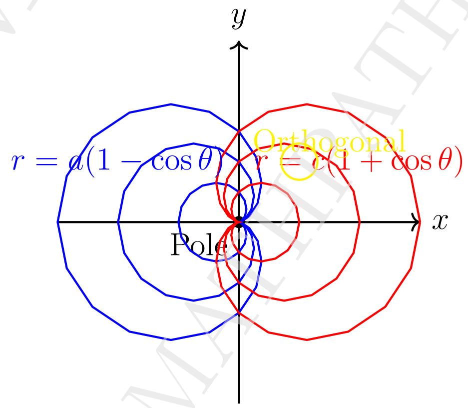


Figure 4.2: Illustration: Cardioids $r = a(1-\cos\theta)$ and their orthogonal trajectories $r = c(1+\cos\theta)$

Geometric Interpretation

The original family $r = a(1-\cos\theta)$ represents cardioids with their cusp pointing along the negative x-axis. The orthogonal trajectories $r = c(1+\cos\theta)$ are cardioids with their cusp pointing along the positive x-axis. These two families are related by a 180° rotation about the pole, and every curve from one family meets every curve from the other family at right angles.

Example 8

Find the orthogonal trajectories of the family of curves $r = \frac{2a}{1+\cos\theta}$.

Solution

We use the polar coordinate method for finding orthogonal trajectories. **Step 1:** Differentiate the given family with respect to θ.

$$r = \frac{2a}{1+\cos\theta} \tag{4.115}$$

Using the quotient rule:

$$\frac{dr}{d\theta} = 2a \cdot \frac{-(-\sin\theta)}{(1+\cos\theta)^2} = \frac{2a\sin\theta}{(1+\cos\theta)^2} \tag{4.116}$$

Step 2: Eliminate the parameter a using the original equation. From the original equation: $a = \frac{r(1+\cos\theta)}{2}$ Substituting into the differential equation:

$$\frac{dr}{d\theta} = \frac{2 \cdot \frac{r(1+\cos\theta)}{2} \cdot \sin\theta}{(1+\cos\theta)^2} = \frac{r(1+\cos\theta)\sin\theta}{(1+\cos\theta)^2} = \frac{r\sin\theta}{1+\cos\theta} \tag{4.117}$$

Step 3: For orthogonal trajectories in polar coordinates, the differential equation becomes:

$$\frac{dr}{d\theta} = \frac{-r^2}{f(r,\theta)} = \frac{-r^2}{\frac{r\sin\theta}{1+\cos\theta}} = -\frac{r(1+\cos\theta)}{\sin\theta} \tag{4.118}$$

Step 4: Rearrange and integrate.

$$\frac{dr}{r} = -\frac{1+\cos\theta}{\sin\theta}\,d\theta = -(\csc\theta + \cot\theta)\,d\theta \tag{4.119}$$

Integrating both sides:

$$\int \frac{dr}{r} = -\int (\csc\theta + \cot\theta)\,d\theta \tag{4.120}$$

Using the correct integration formulas:

$$\int \csc\theta\,d\theta = \ln|\tan\left(\frac{\theta}{2}\right)| = \ln|\csc\theta - \cot\theta| \tag{4.121}$$

$$\int \cot\theta\,d\theta = \ln|\sin\theta| \tag{4.122}$$

Therefore:

$$\ln|r| = -\ln|\csc\theta - \cot\theta| - \ln|\sin\theta| + \ln|c| \tag{4.123}$$

$$\ln|r| = \ln\left|\frac{c}{(\csc\theta - \cot\theta)(\sin\theta)}\right| \tag{4.124}$$

Simplifying:

$$\ln|r| = \ln\left|\frac{c}{(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta})(\sin\theta)}\right| = \ln\left|\frac{c}{1-\cos\theta}\right| \tag{4.125}$$

Taking the exponential of both sides:

$$r = \frac{c}{1-\cos\theta} \tag{4.126}$$

Let $c = 2k$ to express in standard form:

$$r = \frac{2k}{1-\cos\theta} \tag{4.127}$$

Therefore, the orthogonal trajectories are:

$$\boxed{r = \frac{2k}{1-\cos\theta}}$$

These represent conics with eccentricity 1 (parabolas) opening in the opposite direction from the original family.

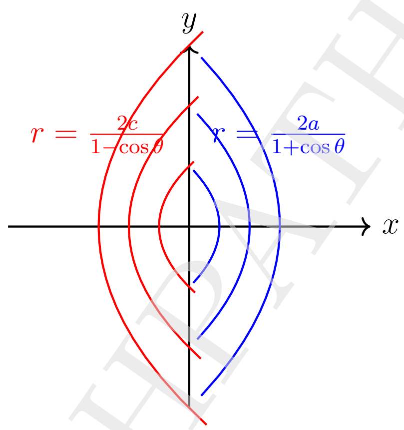


Figure 4.3: Illustration: Parabolas and their orthogonal trajectories

Example 9

Find the orthogonal trajectories of the family of curves $r^2 = a^2 \cos 2\theta$.

Solution

We use the polar coordinate method for finding orthogonal trajectories.
Step 1: Differentiate the given family with respect to θ.

$$r^2 = a^2 \cos 2\theta \tag{4.128}$$

Differentiating both sides with respect to θ:

$$2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta \tag{4.129}$$

$$\frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r} \tag{4.130}$$

Step 2: Eliminate the parameter a using the original equation. From the original equation: $a^2 = \frac{r^2}{\cos 2\theta}$
Substituting into the differential equation:

$$\frac{dr}{d\theta} = -\frac{\frac{r^2}{\cos 2\theta} \cdot \sin 2\theta}{r} = -\frac{r^2 \sin 2\theta}{r \cos 2\theta} = -\frac{r \sin 2\theta}{\cos 2\theta} = -r \tan 2\theta \tag{4.131}$$

Step 3: Replace $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$ to obtain the orthogonal trajectory equation.

$$-r^2\frac{d\theta}{dr} = -r \tan 2\theta \tag{4.132}$$

$$r^2\frac{d\theta}{dr} = r \tan 2\theta \tag{4.133}$$

$$\frac{d\theta}{dr} = \frac{\tan 2\theta}{r} \tag{4.134}$$

Step 4: Rearrange and integrate.

$$\frac{dr}{r} = \frac{d\theta}{\tan 2\theta} = \frac{\cos 2\theta}{\sin 2\theta}\, d\theta = \cot 2\theta\, d\theta \tag{4.135}$$

To integrate $\cot 2\theta$, we use the substitution $u = 2\theta$:

$$\int \frac{dr}{r} = \int \cot 2\theta \, d\theta = \frac{1}{2} \int \cot u \, du = \frac{1}{2} \ln|\sin u| + C_1 = \frac{1}{2} \ln|\sin 2\theta| + C_1 \tag{4.136}$$

Therefore:

$$\ln|r| = \frac{1}{2} \ln|\sin 2\theta| + C_1 \tag{4.137}$$

$$\ln|r| = \ln|\sin 2\theta|^{1/2} + C_1 \tag{4.138}$$

$$\ln|r| = \ln|c\sqrt{\sin 2\theta}| \tag{4.139}$$

$$r = c\sqrt{\sin 2\theta} \tag{4.140}$$

$$r^2 = c^2 \sin 2\theta \tag{4.141}$$

Therefore, the orthogonal trajectories are:

$$\boxed{r^2 = c^2 \sin 2\theta}$$

This represents a family of four-petaled curves known as lemniscates, rotated by 45° from the original family.

Example 10

Find the orthogonal trajectories of the family of curves $r = a\cos^2\theta$.

Solution

We use the polar coordinate method for finding orthogonal trajectories.
Step 1: Differentiate the given family with respect to θ.

$$r = a\cos^2\theta \tag{4.142}$$

Differentiating both sides with respect to θ:

$$\frac{dr}{d\theta} = a \cdot 2\cos\theta \cdot (-\sin\theta) = -2a\cos\theta\sin\theta = -a\sin 2\theta \tag{4.143}$$

Step 2: Eliminate the parameter a using the original equation. From the original equation: $a = \frac{r}{\cos^2\theta}$
Substituting into the differential equation:

$$\frac{dr}{d\theta} = -\frac{r}{\cos^2\theta} \cdot \sin 2\theta = -\frac{r}{\cos^2\theta} \cdot 2\sin\theta\cos\theta = -\frac{2r\sin\theta}{\cos\theta} = -2r\tan\theta \tag{4.144}$$

Step 3: Replace $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$ to obtain the orthogonal trajectory equation.

$$-r^2\frac{d\theta}{dr} = -2r\tan\theta \tag{4.145}$$

$$r^2\frac{d\theta}{dr} = 2r\tan\theta \tag{4.146}$$

$$\frac{d\theta}{dr} = \frac{2\tan\theta}{r} \tag{4.147}$$

Step 4: Rearrange and integrate.

$$\frac{dr}{r} = \frac{2d\theta}{\tan\theta} = 2\frac{\cos\theta}{\sin\theta}\,d\theta = 2\cot\theta\,d\theta \tag{4.148}$$

Integrating both sides:

$$\int \frac{dr}{r} = 2\int \cot\theta \, d\theta \tag{4.149}$$

$$\ln|r| = 2\ln|\sin\theta| + \ln|k| \tag{4.150}$$

$$\ln|r| = \ln|k\sin^2\theta| \tag{4.151}$$

$$r = k\sin^2\theta \tag{4.152}$$

Let $c = k$:

$$r = c\sin^2\theta \tag{4.153}$$

Therefore, the orthogonal trajectories are:

$$\boxed{r = c\sin^2\theta}$$

This represents a family of figure-eight shaped curves that are orthogonal to the original four-petaled curves.

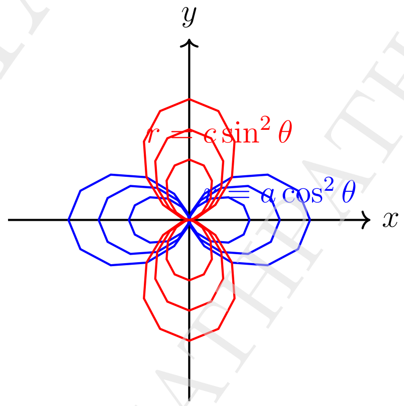


Figure 4.4: Illustration: Curves $r = a\cos^2\theta$ and their orthogonal trajectories $r = c\sin^2\theta$

Example 11

Find the orthogonal trajectories of the family of curves $r = a(1+\cos\theta)$.

Solution

We use the polar coordinate method for finding orthogonal trajectories. **Step 1:** Differentiate the given family with respect to θ.

$$r = a(1+\cos\theta) \tag{4.154}$$

Differentiating both sides:

$$\frac{dr}{d\theta} = a(-\sin\theta) = -a\sin\theta \tag{4.155}$$

Step 2: Eliminate the parameter a using the original equation. From the original equation: $a = \frac{r}{1+\cos\theta}$ Substituting into the differential equation:

$$\frac{dr}{d\theta} = -\frac{r}{1+\cos\theta}\cdot\sin\theta = -\frac{r\sin\theta}{1+\cos\theta} \tag{4.156}$$

Step 3: For orthogonal trajectories in polar coordinates, the differential equation becomes:

$$\frac{dr}{d\theta} = \frac{-r^2}{-\frac{r\sin\theta}{1+\cos\theta}} = \frac{r(1+\cos\theta)}{\sin\theta} \tag{4.157}$$

Step 4: Rearrange and integrate.

$$\frac{dr}{r} = \frac{1+\cos\theta}{\sin\theta}\,d\theta \tag{4.158}$$

Rewriting the right side:

$$\frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \csc\theta + \cot\theta \tag{4.159}$$

Integrating both sides:

$$\int \frac{dr}{r} = \int (\csc\theta + \cot\theta)\,d\theta \tag{4.160}$$

Using the correct integration formulas:

$$\int \csc\theta\,d\theta = \ln|\tan\left(\frac{\theta}{2}\right)| = \ln|\csc\theta - \cot\theta| \tag{4.161}$$

$$\int \cot\theta\,d\theta = \ln|\sin\theta| \tag{4.162}$$

Therefore:

$$\ln|r| = \ln|\csc\theta - \cot\theta| + \ln|\sin\theta| + \ln|c| \tag{4.163}$$

$$\ln|r| = \ln|(\csc\theta - \cot\theta)(\sin\theta)(c)| \tag{4.164}$$

Simplifying:

$$\ln|r| = \ln|c\cdot(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta})\cdot\sin\theta| \tag{4.165}$$

$$\ln|r| = \ln|c\cdot(1-\cos\theta)| \tag{4.166}$$

Taking the exponential of both sides:

$$r = c(1-\cos\theta) \tag{4.167}$$

Let $c = a$ to get the standard form:

$$r = a(1-\cos\theta) \tag{4.168}$$

Therefore, the orthogonal trajectories are:

$$\boxed{r = a(1-\cos\theta)}$$

These are cardioids that are rotated by 180° from the original family.

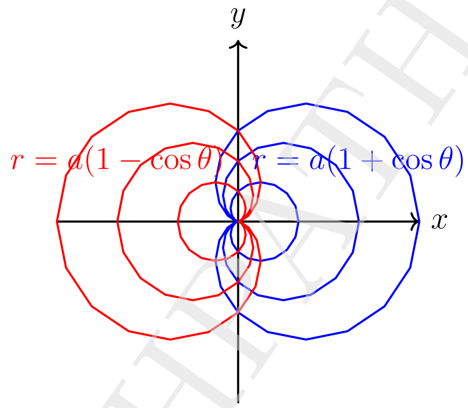


Figure 4.5: Illustration: Cardioids $r = a(1 + \cos\theta)$ and their orthogonal trajectories $r = a(1 - \cos\theta)$

4.3 Newton's Law of Cooling

4.3.1 Fundamental Principle

The mathematical framework for heat transfer was established by Newton, describing how objects exchange thermal energy with their surroundings. This principle states that the rate of temperature change in a body varies directly with the temperature difference between the object and its environment.

Newton's Cooling Law

When a body with temperature θ is placed in surroundings maintained at constant temperature θ_0, the rate at which the body's temperature changes satisfies:

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \tag{4.169}$$

Where:

- $\theta(t)$ = temperature of the body at time t
- θ_0 = constant ambient temperature
- k = positive proportionality constant (cooling coefficient)

The negative sign indicates that when the body temperature exceeds the environmental temperature ($\theta > \theta_0$), the temperature decreases with time.

4.3.2 Physical Interpretation

This first-order differential equation reveals that:

- Temperature differences drive heat transfer
- The cooling process follows exponential decay
- Thermal equilibrium is approached asymptotically

4.3.3 Mathematical Solution

The differential equation can be solved through variable separation:

$$\frac{d\theta}{\theta - \theta_0} = -k\,dt \tag{4.170}$$

Integrating both sides yields:

$$\int \frac{d\theta}{\theta - \theta_0} = \int -k\,dt \tag{4.171}$$

$$\ln|\theta - \theta_0| = -kt + C_1 \tag{4.172}$$

Solving for θ:

$$\theta(t) = \theta_0 + Ae^{-kt} \tag{4.173}$$

Where A is determined by initial conditions: $\theta(0) = \theta_i$

$$\theta(t) = \theta_0 + (\theta_i - \theta_0)e^{-kt} \tag{4.174}$$

4.3.4 Key Properties

- As $t \to \infty$: $\theta \to \theta_0$ (thermal equilibrium)
- Larger k values indicate faster cooling
- The time constant $\tau = \frac{1}{k}$ characterizes cooling speed

Practical Applications

Newton's law finds widespread use in:

- Food technology: Cooling rates in storage
- Criminal investigations: Post-mortem temperature analysis
- Engineering: Thermal system design
- Meteorology: Atmospheric temperature modeling

4.4 Solved Examples

Example 1 - Body Cooling Problem

A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C. What will be the temperature of the body after 40 minutes from the original?

Solution

We are given:

- Initial temperature: $\theta_i = 80$C
- Temperature at $t = 20$ minutes: $\theta(20) = 60$C
- Ambient temperature: $\theta_0 = 40$C
- Find: Temperature at $t = 40$ minutes

From Newton's Law of Cooling, we have:

$$\theta(t) = \theta_0 + (\theta_i - \theta_0)e^{-kt} \tag{4.175}$$

Substituting our values:

$$\theta(t) = 40 + (80 - 40)e^{-kt} = 40 + 40e^{-kt} \tag{4.176}$$

Step 1: Find the cooling constant k

Using the condition at $t = 20$ minutes:

$$60 = 40 + 40e^{-20k} \tag{4.177}$$

Solving for k:

$$
\begin{align}
60 - 40 &= 40e^{-20k} \tag{4.178}\\
20 &= 40e^{-20k} \tag{4.179}\\
\frac{1}{2} &= e^{-20k} \tag{4.180}\\
\ln\left(\frac{1}{2}\right) &= -20k \tag{4.181}\\
-\ln(2) &= -20k \tag{4.182}\\
k &= \frac{\ln(2)}{20} \tag{4.183}
\end{align}
$$

Step 2: Calculate the temperature at $t = 40$ minutes
Now we can find $\theta(40)$:

$$\theta(40) = 40 + 40e^{-40k} \tag{4.184}$$

Substituting $k = \frac{\ln(2)}{20}$:

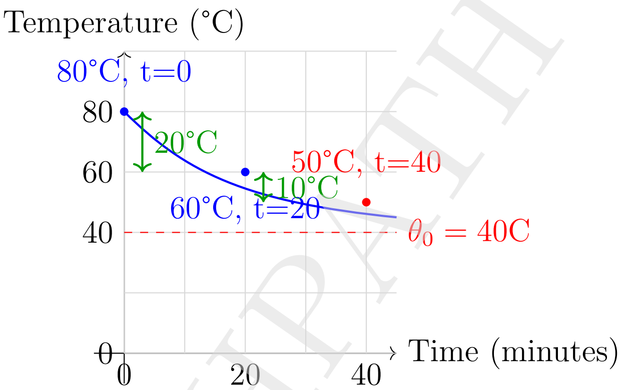
$$
\begin{align}
\theta(40) &= 40 + 40e^{-40\cdot\frac{\ln(2)}{20}} \tag{4.185}\\
&= 40 + 40e^{-2\ln(2)} \tag{4.186}\\
&= 40 + 40e^{\ln\left(2^{-2}\right)} \tag{4.187}\\
&= 40 + 40 \cdot 2^{-2} \tag{4.188}\\
&= 40 + 40 \cdot \frac{1}{4} \tag{4.189}\\
&= 40 + 10 \tag{4.190}\\
&= 50\text{C} \tag{4.191}
\end{align}
$$

Physical Interpretation

The body's temperature follows an exponential decay pattern:

- From 80°C to 60°C in 20 minutes (20°C drop)
- From 60°C to 50°C in the next 20 minutes (10°C drop)
- Each successive 20-minute interval results in half the temperature drop of the previous one

The cooling rate decreases as the temperature approaches the ambient temperature of 40°C.



Example 2 - Temperature Prediction Problem

According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 300°C and the substance cools from 370°C to 350°C in 15 minutes, find when the temperature will be 310°C.

Solution

We are given:

- Ambient temperature: $\theta_0 = 300$C
- Initial temperature: $\theta_i = 370$C
- Temperature at $t = 15$ minutes: $\theta(15) = 350$C
- Find: Time when $\theta = 310$C

From Newton's Law of Cooling:

$$\theta(t) = \theta_0 + (\theta_i - \theta_0)e^{-kt} \tag{4.192}$$

Substituting our values:

$$\theta(t) = 300 + (370 - 300)e^{-kt} = 300 + 70e^{-kt} \tag{4.193}$$

Step 1: Find the cooling constant k

Using the condition at $t = 15$ minutes:

$$350 = 300 + 70e^{-15k} \tag{4.194}$$

Solving for k:

$$350 - 300 = 70e^{-15k} \tag{4.195}$$

$$50 = 70e^{-15k} \tag{4.196}$$

$$\frac{50}{70} = e^{-15k} \tag{4.197}$$

$$\frac{5}{7} = e^{-15k} \tag{4.198}$$

$$\ln\left(\frac{5}{7}\right) = -15k \tag{4.199}$$

$$k = -\frac{\ln(5/7)}{15} = \frac{\ln(7/5)}{15} \tag{4.200}$$

$$k = \frac{\ln(1.4)}{15} \tag{4.201}$$

Step 2: Find the time when temperature is 310°C
We need to find t when $\theta(t) = 310$:

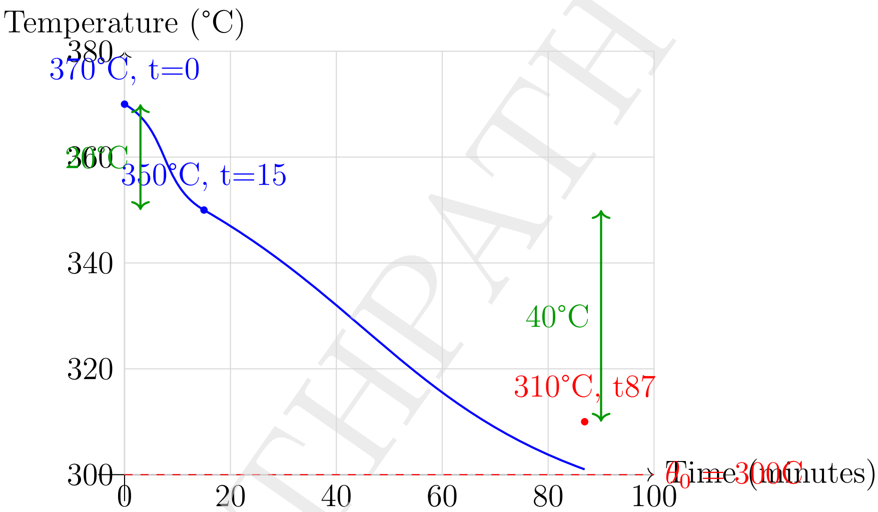
$$310 = 300 + 70e^{-kt} \tag{4.202}$$

Solving for t:

$$\begin{aligned}
310 - 300 &= 70e^{-kt} && (4.203)\\
10 &= 70e^{-kt} && (4.204)\\
\frac{1}{7} &= e^{-kt} && (4.205)\\
\ln\left(\frac{1}{7}\right) &= -kt && (4.206)\\
-\ln(7) &= -kt && (4.207)\\
t &= \frac{\ln(7)}{k} && (4.208)
\end{aligned}$$

Substituting $k = \frac{\ln(1.4)}{15}$:

$$\begin{aligned}
t &= \frac{\ln(7)}{\frac{\ln(1.4)}{15}} && (4.209)\\
&= \frac{15\ln(7)}{\ln(1.4)} && (4.210)\\
&= \frac{15 \times 1.946}{0.336} && (4.211)\\
&= \frac{29.19}{0.336} && (4.212)\\
&\approx 86.9 \text{ minutes} && (4.213)
\end{aligned}$$


Temperature (°C)
380
370°C, t=0
360
20°C
350°C, t=15
340
320
40°C
310°C, t87
300
0
20
40
60
80
100
Time (minutes)

H 0

Example 3 - Copper Ball Cooling

A copper ball is heated to a temperature of 100°C. Then at time $t = 0$ it is placed in water which is maintained at a temperature of 30°C. At the end of 3 minutes, the temperature of the ball is reduced to 70°C. Find the time at which the temperature of the ball drops to 31°C.

Solution

We are given:

- Initial temperature: $\theta_i = 100\text{C}$
- Ambient temperature (water): $\theta_0 = 30\text{C}$
- Temperature at $t = 3$ minutes: $\theta(3) = 70\text{C}$
- Find: Time when $\theta = 31\text{C}$

From Newton's Law of Cooling:

$$\theta(t) = \theta_0 + (\theta_i - \theta_0)e^{-kt} \tag{4.214}$$

Substituting our values:

$$\theta(t) = 30 + (100 - 30)e^{-kt} = 30 + 70e^{-kt} \tag{4.215}$$

Step 1: Find the cooling constant k

Using the condition at $t = 3$ minutes:

$$70 = 30 + 70e^{-3k} \tag{4.216}$$

Solving for k:

$$70 - 30 = 70e^{-3k} \tag{4.217}$$

$$40 = 70e^{-3k} \tag{4.218}$$

$$\frac{40}{70} = e^{-3k} \tag{4.219}$$

$$\frac{4}{7} = e^{-3k} \tag{4.220}$$

$$\ln\left(\frac{4}{7}\right) = -3k \tag{4.221}$$

$$k = -\frac{\ln(4/7)}{3} = \frac{\ln(7/4)}{3} \tag{4.222}$$

Step 2: Find the time when temperature is 31°C

We need to find t when $\theta(t) = 31$:

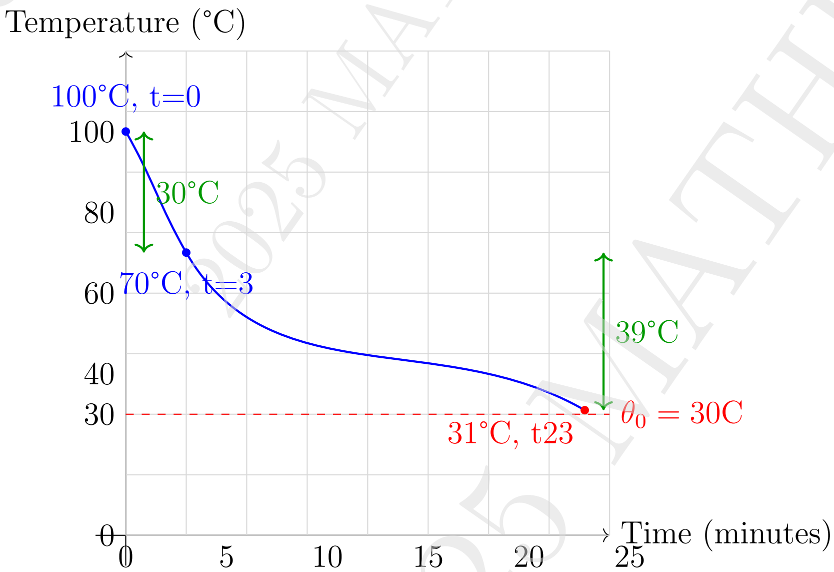
$$31 = 30 + 70e^{-kt} \tag{4.223}$$

Solving for t:

$$31 - 30 = 70e^{-kt} \tag{4.224}$$
$$1 = 70e^{-kt} \tag{4.225}$$
$$\frac{1}{70} = e^{-kt} \tag{4.226}$$
$$\ln\left(\frac{1}{70}\right) = -kt \tag{4.227}$$
$$-\ln(70) = -kt \tag{4.228}$$
$$t = \frac{\ln(70)}{k} \tag{4.229}$$

Substituting $k = \frac{\ln(7/4)}{3}$:

$$t = \frac{\ln(70)}{\frac{\ln(7/4)}{3}} \tag{4.230}$$
$$= \frac{3\ln(70)}{\ln(7/4)} \tag{4.231}$$
$$= \frac{3 \times 4.248}{0.560} \tag{4.232}$$
$$= \frac{12.744}{0.560} \tag{4.233}$$
$$\approx 22.76 \text{ minutes} \tag{4.234}$$



Example 4 - Thermometer Cooling

If a thermometer is taken outdoors where the temperature is 0°C, from a room in which the temperature is 21°C and the reading drops to 10°C in 1 minute, how long after its removal will the reading be 5°C?

Solution

We are given:

- Initial temperature (room): $\theta_i = 21C$

- Ambient temperature (outdoor): $\theta_0 = 0\text{C}$
- Temperature at $t = 1$ minute: $\theta(1) = 10\text{C}$
- Find: Time when $\theta = 5\text{C}$

From Newton's Law of Cooling:

$$\theta(t) = \theta_0 + (\theta_i - \theta_0)e^{-kt} \tag{4.235}$$

Substituting our values:

$$\theta(t) = 0 + (21 - 0)e^{-kt} = 21e^{-kt} \tag{4.236}$$

Step 1: Find the cooling constant k
Using the condition at $t = 1$ minute:

$$10 = 21e^{-k(1)} \tag{4.237}$$

Solving for k:

$$\frac{10}{21} = e^{-k} \tag{4.238}$$

$$\ln\left(\frac{10}{21}\right) = -k \tag{4.239}$$

$$k = -\ln\left(\frac{10}{21}\right) = \ln\left(\frac{21}{10}\right) \tag{4.240}$$

$$k = \ln(2.1) \tag{4.241}$$

Step 2: Find the time when temperature is 5°C
We need to find t when $\theta(t) = 5$:

$$5 = 21e^{-kt} \tag{4.242}$$

Solving for t:

$$\frac{5}{21} = e^{-kt} \tag{4.243}$$

$$\ln\left(\frac{5}{21}\right) = -kt \tag{4.244}$$

$$t = \frac{-\ln(5/21)}{k} \tag{4.245}$$

$$t = \frac{\ln(21/5)}{k} \tag{4.246}$$

Substituting $k = \ln(2.1)$:

$$t = \frac{\ln(21/5)}{\ln(2.1)} \tag{4.247}$$

$$= \frac{\ln(4.2)}{\ln(2.1)} \tag{4.248}$$

$$= \frac{1.435}{0.742} \tag{4.249}$$

$$\approx 1.93 \text{ minutes} \tag{4.250}$$

Example 5 - Water Cooling

Water at temperature 100°C cools in 10 minutes to 88°C in a room of temperature 25°C. Find the temperature of water after 20 minutes.

Solution

We are given:

- Initial temperature: $\theta_i = 100\text{C}$
- Ambient temperature (room): $\theta_0 = 25\text{C}$
- Temperature at $t = 10$ minutes: $\theta(10) = 88\text{C}$
- Find: Temperature at $t = 20$ minutes

From Newton's Law of Cooling:

$$\theta(t) = \theta_0 + (\theta_i - \theta_0)e^{-kt} \tag{4.251}$$

Substituting our values:

$$\theta(t) = 25 + (100 - 25)e^{-kt} = 25 + 75e^{-kt} \tag{4.252}$$

Step 1: Find the cooling constant k

Using the condition at $t = 10$ minutes:

$$88 = 25 + 75e^{-10k} \tag{4.253}$$

Solving for k:

$$88 - 25 = 75e^{-10k} \tag{4.254}$$

$$63 = 75e^{-10k} \tag{4.255}$$

$$\frac{63}{75} = e^{-10k} \tag{4.256}$$

$$\frac{21}{25} = e^{-10k} \tag{4.257}$$

$$\ln\left(\frac{21}{25}\right) = -10k \tag{4.258}$$

$$k = -\frac{\ln(21/25)}{10} = \frac{\ln(25/21)}{10} \tag{4.259}$$

Step 2: Find the temperature at $t = 20$ minutes

Now we can find $\theta(20)$:

$$\theta(20) = 25 + 75e^{-20k} \tag{4.260}$$

Substituting $k = \frac{\ln(25/21)}{10}$:

$$\theta(20) = 25 + 75e^{-20\cdot\frac{\ln(25/21)}{10}} \tag{4.261}$$
$$= 25 + 75e^{-2\ln(25/21)} \tag{4.262}$$
$$= 25 + 75e^{\ln\left((21/25)^2\right)} \tag{4.263}$$
$$= 25 + 75 \cdot \left(\frac{21}{25}\right)^2 \tag{4.264}$$
$$= 25 + 75 \cdot \frac{441}{625} \tag{4.265}$$
$$= 25 + 75 \cdot 0.7056 \tag{4.266}$$
$$= 25 + 52.92 \tag{4.267}$$
$$= 77.92\text{C} \tag{4.268}$$

Therefore, the temperature of water after 20 minutes is approximately 78°C.

Example 6 - Body Cooling with Two Time Intervals

A body at temperature 100°C is placed in a room whose temperature is 20°C and cools to 60°C in 5 minutes. Find its temperature after a further interval of 3 minutes.

Solution

We are given:
- Initial temperature: $\theta_i = 100\text{C}$
- Ambient temperature (room): $\theta_0 = 20\text{C}$
- Temperature at $t = 5$ minutes: $\theta(5) = 60\text{C}$
- Find: Temperature at $t = 5 + 3 = 8$ minutes

From Newton's Law of Cooling:

$$\theta(t) = \theta_0 + (\theta_i - \theta_0)e^{-kt} \tag{4.269}$$

Substituting our values:

$$\theta(t) = 20 + (100 - 20)e^{-kt} = 20 + 80e^{-kt} \tag{4.270}$$

Step 1: Find the cooling constant k
Using the condition at $t = 5$ minutes:

$$60 = 20 + 80e^{-5k} \tag{4.271}$$

Solving for k:

$$60 - 20 = 80e^{-5k} \tag{4.272}$$
$$40 = 80e^{-5k} \tag{4.273}$$
$$\frac{1}{2} = e^{-5k} \tag{4.274}$$
$$\ln\left(\frac{1}{2}\right) = -5k \tag{4.275}$$
$$-\ln(2) = -5k \tag{4.276}$$
$$k = \frac{\ln(2)}{5} \tag{4.277}$$

Step 2: Find the temperature at $t = 8$ minutes
Now we can find $\theta(8)$:

$$\theta(8) = 20 + 80e^{-8k} \tag{4.278}$$

Substituting $k = \frac{\ln(2)}{5}$:

$$\theta(8) = 20 + 80e^{-8\cdot\frac{\ln(2)}{5}} \tag{4.279}$$
$$= 20 + 80e^{-\frac{8\ln(2)}{5}} \tag{4.280}$$
$$= 20 + 80e^{\ln\left(2^{-8/5}\right)} \tag{4.281}$$
$$= 20 + 80 \cdot 2^{-8/5} \tag{4.282}$$
$$= 20 + 80 \cdot 2^{-1.6} \tag{4.283}$$
$$= 20 + 80 \cdot 0.329 \tag{4.284}$$
$$= 20 + 26.3 \tag{4.285}$$
$$= 46.3C \tag{4.286}$$

Therefore, the temperature of the body after 8 minutes (or 3 more minutes) is approximately 46°C.

Example 7

If the temperature of the body drops from 100°C to 60°C in one minute when the temperature of the surrounding is 20°C, what will be the temperature of the body at the end of the second minute?

Solution

We are given:

- Initial temperature: $\theta_i = 100C$
- Ambient temperature: $\theta_0 = 20C$
- Temperature at $t = 1$ minute: $\theta(1) = 60C$
- Find: Temperature at $t = 2$ minutes

From Newton's Law of Cooling:

$$\theta(t) = \theta_0 + (\theta_i - \theta_0)e^{-kt} \tag{4.287}$$

Substituting our values:

$$\theta(t) = 20 + (100 - 20)e^{-kt} = 20 + 80e^{-kt} \tag{4.288}$$

Step 1: Find the cooling constant k
Using the condition at $t = 1$ minute:

$$60 = 20 + 80e^{-k} \tag{4.289}$$

Solving for k:

$$60 - 20 = 80e^{-k} \tag{4.290}$$
$$40 = 80e^{-k} \tag{4.291}$$
$$\frac{1}{2} = e^{-k} \tag{4.292}$$
$$\ln\left(\frac{1}{2}\right) = -k \tag{4.293}$$
$$k = -\ln\left(\frac{1}{2}\right) = \ln(2) \tag{4.294}$$

Step 2: Find the temperature at $t = 2$ minutes
Now we can find $\theta(2)$:

$$\theta(2) = 20 + 80e^{-2k} \tag{4.295}$$

Substituting $k = \ln(2)$:

$$\theta(2) = 20 + 80e^{-2\ln(2)} \tag{4.296}$$
$$= 20 + 80e^{\ln\left(2^{-2}\right)} \tag{4.297}$$
$$= 20 + 80 \cdot 2^{-2} \tag{4.298}$$
$$= 20 + 80 \cdot \frac{1}{4} \tag{4.299}$$
$$= 20 + 20 \tag{4.300}$$
$$= 40\text{C} \tag{4.301}$$

Therefore, the temperature of the body at the end of the second minute is 40°C.

Example 8 - Thermometer Heating

When a thermometer is placed in a hot liquid bath at temperature T, the temperature indicated by the thermometer rises at the rate of T - . For a bath at 95°C, the temperature reads 15°C at a certain instant (t = 0) and 35° at t = 10 seconds. What will be its temperature at t = 20 sec?

Solution

We are given:
- Ambient temperature (bath): $\theta_0 = 95\text{C}$
- Initial temperature: $\theta(0) = 15\text{C}$
- Temperature at $t = 10$ seconds: $\theta(10) = 35\text{C}$
- Find: Temperature at $t = 20$ seconds

Note: The statement "the temperature rises at the rate of T - " is a form of Newton's Law, but with opposite sign for heating:

$$\frac{d\theta}{dt} = k(\theta_0 - \theta) \tag{4.302}$$

This gives us the solution:

$$\theta(t) = \theta_0 - (\theta_0 - \theta_i)e^{-kt} \tag{4.303}$$

Substituting our values:

$$\theta(t) = 95 - (95 - 15)e^{-kt} = 95 - 80e^{-kt} \tag{4.304}$$

Step 1: Find the heating constant k
Using the condition at $t = 10$ seconds:

$$35 = 95 - 80e^{-10k} \tag{4.305}$$

Solving for k:

$$35 - 95 = -80e^{-10k} \tag{4.306}$$
$$-60 = -80e^{-10k} \tag{4.307}$$
$$\frac{60}{80} = e^{-10k} \tag{4.308}$$
$$\frac{3}{4} = e^{-10k} \tag{4.309}$$
$$\ln\left(\frac{3}{4}\right) = -10k \tag{4.310}$$
$$k = -\frac{\ln(3/4)}{10} = \frac{\ln(4/3)}{10} \tag{4.311}$$

Step 2: Find the temperature at $t = 20$ seconds
Now we can find $\theta(20)$:

$$\theta(20) = 95 - 80e^{-20k} \tag{4.312}$$

Substituting $k = \frac{\ln(4/3)}{10}$:

$$\theta(20) = 95 - 80e^{-20\cdot\frac{\ln(4/3)}{10}} \tag{4.313}$$
$$= 95 - 80e^{-2\ln(4/3)} \tag{4.314}$$
$$= 95 - 80e^{\ln\left((3/4)^2\right)} \tag{4.315}$$
$$= 95 - 80 \cdot \left(\frac{3}{4}\right)^2 \tag{4.316}$$
$$= 95 - 80 \cdot \frac{9}{16} \tag{4.317}$$
$$= 95 - 80 \cdot 0.5625 \tag{4.318}$$
$$= 95 - 45 \tag{4.319}$$
$$= 50C \tag{4.320}$$

Therefore, the temperature at t = 20 seconds is 50°C.

Example 9 - Time to Reach Target Temperature

Water at temperature 100°C cools in 10 minutes to 60°C in a room temperature of 20°C. Find when the temperature will be 30°C.

Solution

We are given:
- Initial temperature: $\theta_i = 100C$
- Ambient temperature (room): $\theta_0 = 20C$
- Temperature at $t = 10$ minutes: $\theta(10) = 60C$
- Find: Time when $\theta = 30C$

From Newton's Law of Cooling:

$$\theta(t) = \theta_0 + (\theta_i - \theta_0)e^{-kt} \tag{4.321}$$

Substituting our values:

$$\theta(t) = 20 + (100 - 20)e^{-kt} = 20 + 80e^{-kt} \tag{4.322}$$

Step 1: Find the cooling constant k
Using the condition at $t = 10$ minutes:

$$60 = 20 + 80e^{-10k} \tag{4.323}$$

Solving for k:

$$60 - 20 = 80e^{-10k} \tag{4.324}$$

$$40 = 80e^{-10k} \tag{4.325}$$

$$\frac{1}{2} = e^{-10k} \tag{4.326}$$

$$\ln\left(\frac{1}{2}\right) = -10k \tag{4.327}$$

$$k = -\frac{\ln(1/2)}{10} = \frac{\ln(2)}{10} \tag{4.328}$$

Step 2: Find the time when temperature is 30°C
We need to find t when $\theta(t) = 30$:

$$30 = 20 + 80e^{-kt} \tag{4.329}$$

Solving for t:

$$30 - 20 = 80e^{-kt} \tag{4.330}$$

$$10 = 80e^{-kt} \tag{4.331}$$

$$\frac{1}{8} = e^{-kt} \tag{4.332}$$

$$\ln\left(\frac{1}{8}\right) = -kt \tag{4.333}$$

$$t = -\frac{\ln(1/8)}{k} \tag{4.334}$$

$$t = \frac{\ln(8)}{k} \tag{4.335}$$

Substituting $k = \frac{\ln(2)}{10}$:

$$t = \frac{\ln(8)}{\frac{\ln(2)}{10}} \tag{4.336}$$

$$= \frac{10\ln(8)}{\ln(2)} \tag{4.337}$$

$$= \frac{10\ln(2^3)}{\ln(2)} \tag{4.338}$$

$$= \frac{10 \cdot 3\ln(2)}{\ln(2)} \tag{4.339}$$

$$= 30 \text{ minutes} \tag{4.340}$$

Therefore, the temperature will be 30°C at t = 30 minutes.

Example 10 - Metal Ball Cooling

A metal ball is heated to a temperature of 100°C and at time t = 0 it is placed in water which is maintained at 40°C. If the temperature of the ball is reduced to 60°C in 4 minutes, find the time at which the temperature of the ball is 50°C.

Solution

We are given:

- Initial temperature: $\theta_i = 100C$
- Ambient temperature (water): $\theta_0 = 40C$
- Temperature at $t = 4$ minutes: $\theta(4) = 60C$
- Find: Time when $\theta = 50C$

From Newton's Law of Cooling:

$$\theta(t) = \theta_0 + (\theta_i - \theta_0)e^{-kt} \tag{4.341}$$

Substituting our values:

$$\theta(t) = 40 + (100 - 40)e^{-kt} = 40 + 60e^{-kt} \tag{4.342}$$

Step 1: Find the cooling constant k

Using the condition at $t = 4$ minutes:

$$60 = 40 + 60e^{-4k} \tag{4.343}$$

Solving for k:

$$60 - 40 = 60e^{-4k} \tag{4.344}$$

$$20 = 60e^{-4k} \tag{4.345}$$

$$\frac{1}{3} = e^{-4k} \tag{4.346}$$

$$\ln\left(\frac{1}{3}\right) = -4k \tag{4.347}$$

$$k = -\frac{\ln(1/3)}{4} = \frac{\ln(3)}{4} \tag{4.348}$$

Step 2: Find the time when temperature is 50°C

We need to find t when $\theta(t) = 50$:

$$50 = 40 + 60e^{-kt} \tag{4.349}$$

Solving for t:

$$50 - 40 = 60e^{-kt} \tag{4.350}$$

$$10 = 60e^{-kt} \tag{4.351}$$

$$\frac{1}{6} = e^{-kt} \tag{4.352}$$

$$\ln\left(\frac{1}{6}\right) = -kt \tag{4.353}$$

$$t = -\frac{\ln(1/6)}{k} \tag{4.354}$$

$$t = \frac{\ln(6)}{k} \tag{4.355}$$

Substituting $k = \frac{\ln(3)}{4}$:

$$t = \frac{\ln(6)}{\frac{\ln(3)}{4}} \tag{4.356}$$

$$= \frac{4\ln(6)}{\ln(3)} \tag{4.357}$$

$$= \frac{4\ln(2 \cdot 3)}{\ln(3)} \tag{4.358}$$

$$= \frac{4(\ln(2) + \ln(3))}{\ln(3)} \tag{4.359}$$

$$= \frac{4\ln(2)}{\ln(3)} + 4 \tag{4.360}$$

$$= \frac{4 \times 0.693}{1.099} + 4 \tag{4.361}$$

$$= 2.52 + 4 \tag{4.362}$$

$$= 6.52 \text{ minutes} \tag{4.363}$$

Therefore, the temperature will be 50°C at approximately t = 6.52 minutes.

4.5 Radioactive Decay

4.5.1 Physical Concept

Radioactive elements spontaneously undergo nuclear transformations, emitting particles and radiation while converting into other elements. This process follows a predictable mathematical pattern first understood in nuclear physics.

Definition 4.2 (Radioactive Decay Process)**.** *The spontaneous disintegration of unstable atomic nuclei, during which the rate of decay is directly proportional to the current number of radioactive nuclei present in the sample.*

4.5.2 Mathematical Formulation

The decay process exhibits a characteristic exponential pattern. If we denote:

- $u(t)$ = quantity of radioactive material at time t

- k = decay constant (specific to each radioactive isotope)

Then the fundamental decay equation becomes:

Radioactive Decay Differential Equation

$$\frac{du}{dt} = -ku \tag{4.364}$$

The negative sign signifies that the quantity decreases over time, with the rate proportional to the current amount.

4.5.3 Solution and Physical Meaning

This first-order differential equation has the solution:

$$u(t) = u_0 e^{-kt} \tag{4.365}$$

Where u_0 represents the initial radioactive material quantity.

4.5.4 Half-Life Concept

The half-life $T_{1/2}$ represents the time required for exactly half the radioactive material to decay:

$$\frac{u_0}{2} = u_0 e^{-kT_{1/2}} \tag{4.366}$$

Solving for $T_{1/2}$:

$$T_{1/2} = \frac{\ln 2}{k} \tag{4.367}$$

Key Properties

- Half-life is constant for each radioactive isotope
- Decay follows an exponential curve
- The process is independent of physical conditions (temperature, pressure)
- Activity decreases exponentially with time

Scientific Applications

Radioactive decay principles are utilized in:

- Archaeological dating (Carbon-14)
- Geological age determination
- Medical diagnostics and treatment
- Nuclear reactor design
- Environmental monitoring

4.6 Solved Examples

Example 1

Radium decomposes at the rate proportional to the amount present. If 5% of the original amount disappears in 50 years, how much will remain after 100 years?

Solution

Let $u(t)$ = amount of radium at time t
From the radioactive decay equation:

$$\frac{du}{dt} = -ku \tag{4.368}$$

This gives us the solution:

$$u(t) = u_0 e^{-kt} \tag{4.369}$$

Step 1: Find the decay constant k
Given information: 5% disappears in 50 years
This means: 95% remains after 50 years
At $t = 50$ years:

$$u(50) = 0.95u_0 \tag{4.370}$$

Substituting into the solution:

$$0.95u_0 = u_0 e^{-k\cdot 50} \tag{4.371}$$

Simplifying:

$$0.95 = e^{-50k} \tag{4.372}$$

Taking natural logarithm:

$$\ln(0.95) = -50k \tag{4.373}$$

Therefore:

$$k = -\frac{\ln(0.95)}{50} = \frac{\ln(0.95)}{50} \tag{4.374}$$

Step 2: Find the amount remaining after 100 years
At $t = 100$ years:

$$u(100) = u_0 e^{-k\cdot 100} \tag{4.375}$$

Substituting the value of k:

$$u(100) = u_0 e^{-100\cdot\frac{\ln(0.95)}{50}} \tag{4.376}$$

$$u(100) = u_0 e^{-2\ln(0.95)} \tag{4.377}$$

Using logarithm properties:

$$u(100) = u_0 e^{\ln(0.95)^2} = u_0(0.95)^2 \tag{4.378}$$

$$u(100) = 0.9025u_0 \tag{4.379}$$

Therefore, 90.25% of the original amount will remain after 100 years.

Example 2

Uranium disintegrates at a rate proportional to the amount that is present at any instant. If M_1 and M_2 grams of uranium are present at times t_1 and t_2 respectively, find the half-life of uranium.

Solution

Let $u(t)$ = amount of uranium at time t

From the radioactive decay equation:

$$\frac{du}{dt} = -ku \tag{4.380}$$

This gives us the solution:

$$u(t) = u_0 e^{-kt} \tag{4.381}$$

Given information:

- At time t_1: $u(t_1) = M_1$
- At time t_2: $u(t_2) = M_2$

Step 1: Write equations for the two time points
At $t = t_1$:

$$M_1 = u_0 e^{-kt_1} \tag{4.382}$$

At $t = t_2$:

$$M_2 = u_0 e^{-kt_2} \tag{4.383}$$

Step 2: Find the decay constant k
From equation (3): $u_0 = M_1 e^{kt_1}$
Substituting into equation (4):

$$M_2 = M_1 e^{kt_1} \cdot e^{-kt_2} \tag{4.384}$$

$$M_2 = M_1 e^{k(t_1 - t_2)} \tag{4.385}$$

$$\frac{M_2}{M_1} = e^{k(t_1 - t_2)} \tag{4.386}$$

Taking natural logarithm:

$$\ln\left(\frac{M_2}{M_1}\right) = k(t_1 - t_2) \tag{4.387}$$

Therefore:

$$k = \frac{\ln\left(\frac{M_2}{M_1}\right)}{t_1 - t_2} \tag{4.388}$$

Step 3: Calculate the half-life
The half-life $T_{1/2}$ is given by:

$$T_{1/2} = \frac{\ln 2}{k} \tag{4.389}$$

Substituting the value of k:

$$T_{1/2} = \frac{\ln 2}{\frac{\ln\left(\frac{M_2}{M_1}\right)}{t_1 - t_2}} \tag{4.390}$$

$$T_{1/2} = \frac{(t_1 - t_2)\ln 2}{\ln\left(\frac{M_2}{M_1}\right)} \tag{4.391}$$

Therefore, the half-life of uranium is $\frac{(t_1 - t_2)\ln 2}{\ln\left(\frac{M_2}{M_1}\right)}$.

Example 3

If 40% of a radioactive substance disappeared in 10 days, how long will it take for 90% of it to disappear?

Solution

Let $u(t)$ = amount of radioactive substance at time t
From the radioactive decay equation:

$$\frac{du}{dt} = -ku \tag{4.392}$$

This gives us the solution:

$$u(t) = u_0 e^{-kt} \tag{4.393}$$

Step 1: Find the decay constant k
Given information: 40% disappears in 10 days
This means: 60% remains after 10 days
At $t = 10$ days:

$$u(10) = 0.60u_0 \tag{4.394}$$

Substituting into the solution:

$$0.60u_0 = u_0 e^{-k\cdot 10} \tag{4.395}$$

Simplifying:

$$0.60 = e^{-10k} \tag{4.396}$$

Taking natural logarithm:

$$\ln(0.60) = -10k \tag{4.397}$$

Therefore:

$$k = -\frac{\ln(0.60)}{10} = \frac{\ln(0.60)}{10} \tag{4.398}$$

Step 2: Find the time for 90% to disappear
If 90% disappears, then 10% remains
At time t:

$$u(t) = 0.10u_0 \tag{4.399}$$

Substituting into the solution:

$$0.10u_0 = u_0 e^{-kt} \tag{4.400}$$

Simplifying:

$$0.10 = e^{-kt} \tag{4.401}$$

Taking natural logarithm:

$$\ln(0.10) = -kt \tag{4.402}$$

Therefore:

$$t = -\frac{\ln(0.10)}{k} \tag{4.403}$$

Substituting the value of k:

$$t = -\frac{\ln(0.10)}{-\frac{\ln(0.60)}{10}} \tag{4.404}$$

$$t = \frac{10 \cdot \ln(0.10)}{\ln(0.60)} \tag{4.405}$$

Calculating the numerical value:

$$t = \frac{10 \cdot (-2.303)}{(-0.511)} = \frac{-23.03}{-0.511} \approx 45.07 \tag{4.406}$$

Therefore, it will take approximately 45.07 days for 90% of the radioactive substance to disappear.

Example 4

Radium decomposes at the rate proportional to the quantity of radium present. Suppose that it is found that in 25 years approximately 1.1% of certain quantity of radium has decomposed. Determine approximately how long will it take for one half of the original amount of radium to decompose.

Solution

Let $u(t)$ = quantity of radium at time t
From the radioactive decay equation:

$$\frac{du}{dt} = -ku \tag{4.407}$$

This gives us the solution:

$$u(t) = u_0 e^{-kt} \tag{4.408}$$

Step 1: Find the decay constant k
Given information: 1.1% decomposes in 25 years
This means: 98.9% remains after 25 years
At $t = 25$ years:

$$u(25) = 0.989u_0 \tag{4.409}$$

Substituting into the solution:

$$0.989u_0 = u_0 e^{-k \cdot 25} \tag{4.410}$$

Simplifying:

$$0.989 = e^{-25k} \tag{4.411}$$

Taking natural logarithm:

$$\ln(0.989) = -25k \tag{4.412}$$

Therefore:

$$k = -\frac{\ln(0.989)}{25} = \frac{\ln(0.989)}{25} \tag{4.413}$$

Step 2: Find the half-life
The half-life $T_{1/2}$ is the time for 50% to remain:
At $t = T_{1/2}$:

$$u(T_{1/2}) = 0.5u_0 \tag{4.414}$$

Substituting into the solution:

$$0.5u_0 = u_0 e^{-kT_{1/2}} \tag{4.415}$$

Simplifying:

$$0.5 = e^{-kT_{1/2}} \tag{4.416}$$

Taking natural logarithm:

$$\ln(0.5) = -kT_{1/2} \tag{4.417}$$

Therefore:

$$T_{1/2} = -\frac{\ln(0.5)}{k} = \frac{\ln(2)}{k} \tag{4.418}$$

Substituting the value of k:

$$T_{1/2} = \frac{\ln(2)}{-\frac{\ln(0.989)}{25}} = \frac{25 \cdot \ln(2)}{-\ln(0.989)} \tag{4.419}$$

Calculating the numerical value:

$$T_{1/2} = \frac{25 \cdot 0.693}{-(-0.0111)} = \frac{17.325}{0.0111} \approx 1561.71 \tag{4.420}$$

Therefore, it will take approximately 1562 years for one half of the original amount of radium to decompose.

4.7 Electric Circuits and Their Differential Equations

4.7.1 Circuit Analysis Framework

Electric circuits form the backbone of modern electronics and power systems. The mathematical modeling of these circuits involves analyzing the interaction between voltage, current, and various passive components like resistors, capacitors, and inductors, alongside active elements such as batteries or generators.

4.7.2 Fundamental Circuit Parameters

The key electrical quantities and their standard representations are as follows:

Time duration is represented by the symbol t, measured in seconds (s), serving as the independent variable.

Electric charge uses the symbol q, measured in coulombs (C), with the mathematical expression $q = \int i\,dt$.

Current flow is denoted by i, measured in amperes (A), defined as $i = \frac{dq}{dt}$.

Resistance employs the symbol R, measured in ohms (), representing a component property.

Inductance is symbolized by L, measured in henry (H), also a component property.

Capacitance utilizes the symbol C, measured in farad (F), likewise a component property.

Constant voltage source is represented by E, measured in volts (V), corresponding to battery potential.

Variable voltage source uses the notation $E(t)$, measured in volts (V), representing generator output.

4.7.3 Component-Current Relationships

Each circuit element contributes to the overall voltage balance according to its characteristic behavior:

Voltage-Current Relations

- **Resistor behavior:** $V_R = Ri$ (linear relationship)
- **Inductor behavior:** $V_L = L\frac{di}{dt}$ (rate-dependent)
- **Capacitor behavior:** $V_C = \frac{q}{C}$ (charge-dependent)

4.7.4 Circuit Law for Differential Equations

The mathematical foundation for circuit analysis relies on two key principles:

Definition 4.3 (Kirchhoff's Circuit Laws)**.** ***Voltage Law (KVL):*** *The sum of voltage drops around any closed loop equals the applied electromotive force.*

Current Law (KCL): *The algebraic sum of currents entering or leaving any junction is zero.*

4.7.5 Circuit Configuration Equations

RL Circuit Configuration

When resistance R and inductance L operate in series with voltage source E, the voltage balance yields:

$$L\frac{di}{dt} + Ri = E \tag{4.421}$$

This linear differential equation describes the current development in RL circuits.

RC Circuit Configuration

For resistance R and capacitance C in series with voltage source E, the charge equation becomes:

$$R\frac{dq}{dt} + \frac{q}{C} = E \tag{4.422}$$

Converting to current form:

$$RC\frac{di}{dt} + i = \frac{dE}{dt} \tag{4.423}$$

LC Circuit Configuration

When inductance L and capacitance C operate together (without resistance), the system equation is:

$$L\frac{di}{dt} + \frac{q}{C} = 0 \tag{4.424}$$

Expressing in terms of charge:

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0 \tag{4.425}$$

This second-order equation characterizes oscillatory LC circuits.

Complete RLC Circuit

The comprehensive RLC series configuration with voltage source E follows:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E \tag{4.426}$$

This second-order differential equation governs the complete circuit dynamics, incorporating all three passive components.

Circuit Analysis Insights

Each circuit configuration produces specific dynamic behavior:

- RL circuits: Exponential current growth or decay
- RC circuits: Exponential charge/discharge patterns
- LC circuits: Sinusoidal oscillations
- RLC circuits: Damped or sustained oscillations

4.8 Solved Examples

RL Circuit with Sinusoidal Source

In a circuit of resistance R, self inductance L, the current i is given by:

$$L\frac{di}{dt} + Ri = E\cos pt$$

where E, p are constants. Find the current at time t.

Complete Solution

We have the first-order linear differential equation:

$$L\frac{di}{dt} + Ri = E\cos pt$$

Dividing by L:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\cos pt$$

This is a linear differential equation of the form $\frac{di}{dt} + Pi = Q$ where:

$$P = \frac{R}{L}, \quad Q = \frac{E}{L}\cos pt$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\,dt} = e^{\int \frac{R}{L}\,dt} = e^{\frac{R}{L}t}$$

Step 2: Multiply through by the integrating factor

$$e^{\frac{R}{L}t}\frac{di}{dt} + e^{\frac{R}{L}t}\frac{R}{L}i = e^{\frac{R}{L}t}\frac{E}{L}\cos pt$$

$$\frac{d}{dt}\left(e^{\frac{R}{L}t}i\right) = \frac{E}{L}e^{\frac{R}{L}t}\cos pt$$

Step 3: Integrate both sides

$$e^{\frac{R}{L}t}i = \frac{E}{L}\int e^{\frac{R}{L}t}\cos pt\,dt + C$$

Step 4: Evaluate the integral Using integration by parts or the formula:

$$\int e^{at}\cos bt\,dt = \frac{e^{at}(a\cos bt + b\sin bt)}{a^2+b^2}$$

With $a = \frac{R}{L}$ and $b = p$:

$$\begin{aligned}\int e^{\frac{R}{L}t}\cos pt\,dt &= \frac{e^{\frac{R}{L}t}\left(\frac{R}{L}\cos pt + p\sin pt\right)}{\left(\frac{R}{L}\right)^2 + p^2}\\ &= \frac{Le^{\frac{R}{L}t}\left(\frac{R}{L}\cos pt + p\sin pt\right)}{R^2+L^2p^2}\\ &= \frac{e^{\frac{R}{L}t}(R\cos pt + Lp\sin pt)}{R^2+L^2p^2}\end{aligned}$$

Step 5: Substitute back

$$e^{\frac{R}{L}t}i = \frac{E}{L}\cdot\frac{e^{\frac{R}{L}t}(R\cos pt + Lp\sin pt)}{R^2+L^2p^2} + C$$

Step 6: Solve for i

$$i = \frac{E(R\cos pt + Lp\sin pt)}{L(R^2+L^2p^2)} + Ce^{-\frac{R}{L}t}$$

Final Answer:

$$i(t) = \frac{E(R\cos pt + Lp\sin pt)}{R^2+L^2p^2} + Ce^{-\frac{R}{L}t}$$

where C is the arbitrary constant determined by initial conditions.

Example 2: RL Circuit with Constant Voltage

When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i builds up at rate given by:

$$L\frac{di}{dt} + Ri = E$$

Find i as a function of t. How long will it be, before the current has reached one half its maximum value, if $E = 6$ volts, $R = 100$ ohms and $L = 0.1$ henry?

Complete Solution

We have the first-order linear differential equation:

$$L\frac{di}{dt} + Ri = E$$

Dividing by L:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

This is a linear differential equation of the form $\frac{di}{dt} + Pi = Q$ where:

$$P = \frac{R}{L}, \quad Q = \frac{E}{L}$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\,dt} = e^{\int \frac{R}{L}\,dt} = e^{\frac{R}{L}t}$$

Step 2: Multiply through by the integrating factor

$$e^{\frac{R}{L}t}\frac{di}{dt} + e^{\frac{R}{L}t}\frac{R}{L}i = e^{\frac{R}{L}t}\frac{E}{L}$$

$$\frac{d}{dt}\left(e^{\frac{R}{L}t}i\right) = \frac{E}{L}e^{\frac{R}{L}t}$$

Step 3: Integrate both sides

$$e^{\frac{R}{L}t}i = \frac{E}{L}\int e^{\frac{R}{L}t}\,dt + C$$

$$e^{\frac{R}{L}t}i = \frac{E}{L}\cdot\frac{L}{R}e^{\frac{R}{L}t} + C$$

$$e^{\frac{R}{L}t}i = \frac{E}{R}e^{\frac{R}{L}t} + C$$

Step 4: Solve for i

$$i = \frac{E}{R} + Ce^{-\frac{R}{L}t}$$

Step 5: Apply initial condition At $t = 0$, $i = 0$ (current is zero when switch is first closed):

$$0 = \frac{E}{R} + C$$

$$C = -\frac{E}{R}$$

Therefore:

$$i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Step 6: Find the maximum current As $t \to \infty$, $e^{-\frac{R}{L}t} \to 0$, so:

$$i_{\max} = \frac{E}{R}$$

Step 7: Find time when current reaches half maximum We need to find t when $i = \frac{1}{2}i_{\max} = \frac{E}{2R}$:

$$\frac{E}{2R} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\frac{1}{2} = 1 - e^{-\frac{R}{L}t}$$

$$e^{-\frac{R}{L}t} = \frac{1}{2}$$

$$-\frac{R}{L}t = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$t = \frac{L}{R} \ln 2$$

Step 8: Substitute given values Given $E = 6$ V, $R = 100$, $L = 0.1$ H:

$$t = \frac{0.1}{100} \ln 2 = 0.001 \ln 2$$

$$t = 0.001 \times 0.693 = 0.000693 \text{ seconds}$$

$$t = 0.693 \text{ milliseconds}$$

Final Answer:

$$i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Time to reach half maximum current: $t = 0.693$ milliseconds

Example 3: RL Circuit with Sinusoidal EMF

An electrical circuit contains an inductance of 5 henries and a resistance of 12 ohms in series with an e.m.f. 120 sin (20t) volts. Find the current at $t = 0.01$, if it is zero when $t = 0$.

Complete Solution

We have the first-order linear differential equation:

$$L\frac{di}{dt} + Ri = E(t)$$

Substituting the given values $L = 5$ H, $R = 12$, and $E(t) = 120\sin(20t)$:

$$5\frac{di}{dt} + 12i = 120\sin(20t)$$

Dividing by 5:

$$\frac{di}{dt} + \frac{12}{5}i = 24\sin(20t)$$

This is a linear differential equation of the form $\frac{di}{dt} + Pi = Q$ where:

$$P = \frac{12}{5} = 2.4, \quad Q = 24\sin(20t)$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\,dt} = e^{\int 2.4\,dt} = e^{2.4t}$$

Step 2: Multiply through by the integrating factor

$$e^{2.4t}\frac{di}{dt} + 2.4e^{2.4t}i = 24e^{2.4t}\sin(20t)$$

$$\frac{d}{dt}\left(e^{2.4t}i\right) = 24e^{2.4t}\sin(20t)$$

Step 3: Integrate both sides

$$e^{2.4t}i = 24\int e^{2.4t}\sin(20t)\,dt + C$$

Using the formula:

$$\int e^{at}\sin(bt)\,dt = \frac{e^{at}(a\sin(bt) - b\cos(bt))}{a^2+b^2}$$

With $a = 2.4$ and $b = 20$:

$$\int e^{2.4t}\sin(20t)\,dt = \frac{e^{2.4t}(2.4\sin(20t) - 20\cos(20t))}{(2.4)^2 + (20)^2}$$

$$= \frac{e^{2.4t}(2.4\sin(20t) - 20\cos(20t))}{5.76 + 400}$$

$$= \frac{e^{2.4t}(2.4\sin(20t) - 20\cos(20t))}{405.76}$$

Step 4: Substitute back

$$e^{2.4t}i = 24 \cdot \frac{e^{2.4t}(2.4\sin(20t) - 20\cos(20t))}{405.76} + C$$

Step 5: Solve for i

$$i = \frac{24(2.4\sin(20t) - 20\cos(20t))}{405.76} + Ce^{-2.4t}$$

$$i = \frac{2.4\sin(20t) - 20\cos(20t)}{16.9067} + Ce^{-2.4t}$$

Step 6: Apply initial condition At $t = 0$, $i = 0$:

$$0 = \frac{2.4\sin(0) - 20\cos(0)}{16.9067} + C$$

$$0 = \frac{0 - 20}{16.9067} + C$$

$$C = \frac{20}{16.9067} = 1.1826$$

Therefore:

$$i(t) = \frac{2.4\sin(20t) - 20\cos(20t)}{16.9067} + 1.1826e^{-2.4t}$$

Step 7: Calculate current at $t = 0.01$

$$i(0.01) = \frac{2.4\sin(0.2) - 20\cos(0.2)}{16.9067} + 1.1826e^{-0.024}$$

Using $\sin(0.2) = 0.1987$ and $\cos(0.2) = 0.9801$:

$$i(0.01) = \frac{2.4(0.1987) - 20(0.9801)}{16.9067} + 1.1826e^{-0.024}$$

$$i(0.01) = \frac{0.4769 - 19.602}{16.9067} + 1.1826(0.9763)$$

$$i(0.01) = \frac{-19.1251}{16.9067} + 1.1546$$

$$i(0.01) = -1.1313 + 1.1546 = 0.0233 \text{ amperes}$$

Final Answer: Current at $t = 0.01$ seconds is 0.0233 amperes.

Example 4: RL Circuit Time to Half Maximum

A constant electromotive force E volts is applied to a circuit containing a constant resistance R ohms in series and a constant inductance L henries. If the initial current is zero, show that the current builds up to half its theoretical maximum in $(L \log 2)/R$ seconds.

Complete Solution

We have the first-order linear differential equation:

$$L\frac{di}{dt} + Ri = E$$

Dividing by L:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

This is a linear differential equation of the form $\frac{di}{dt} + Pi = Q$ where:

$$P = \frac{R}{L}, \quad Q = \frac{E}{L}$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\,dt} = e^{\int \frac{R}{L}\,dt} = e^{\frac{R}{L}t}$$

Step 2: Multiply through by the integrating factor

$$e^{\frac{R}{L}t}\frac{di}{dt} + e^{\frac{R}{L}t}\frac{R}{L}i = e^{\frac{R}{L}t}\frac{E}{L}$$

$$\frac{d}{dt}\left(e^{\frac{R}{L}t}i\right) = \frac{E}{L}e^{\frac{R}{L}t}$$

Step 3: Integrate both sides

$$e^{\frac{R}{L}t}i = \frac{E}{L}\int e^{\frac{R}{L}t}\,dt + C$$

$$e^{\frac{R}{L}t}i = \frac{E}{L}\cdot\frac{L}{R}e^{\frac{R}{L}t} + C$$

$$e^{\frac{R}{L}t}i = \frac{E}{R}e^{\frac{R}{L}t} + C$$

Step 4: Solve for i

$$i = \frac{E}{R} + Ce^{-\frac{R}{L}t}$$

Step 5: Apply initial condition At $t = 0$, $i = 0$:

$$0 = \frac{E}{R} + C$$

$$C = -\frac{E}{R}$$

Therefore:

$$i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Step 6: Find the theoretical maximum current As $t \to \infty$, $e^{-\frac{R}{L}t} \to 0$, so:

$$i_{\max} = \frac{E}{R}$$

Step 7: Find time when current reaches half maximum We need to find t when $i = \frac{1}{2} i_{\max} = \frac{E}{2R}$:

$$\frac{E}{2R} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\frac{1}{2} = 1 - e^{-\frac{R}{L}t}$$

$$e^{-\frac{R}{L}t} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$-\frac{R}{L}t = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$\frac{R}{L}t = \ln 2$$

$$t = \frac{L}{R}\ln 2$$

Conclusion: The current builds up to half its theoretical maximum in $\frac{L \log 2}{R}$ seconds.

Example 5: RL Circuit with Battery

A resistance of 100 ohms, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in a circuit as a function of t.

Complete Solution

We have the first-order linear differential equation:

$$L\frac{di}{dt} + Ri = E$$

Substituting the given values $L = 0.5$ H, $R = 100$, and $E = 20$ V:

$$0.5\frac{di}{dt} + 100i = 20$$

Dividing by 0.5:

$$\frac{di}{dt} + 200i = 40$$

This is a linear differential equation of the form $\frac{di}{dt} + Pi = Q$ where:

$$P = 200, \quad Q = 40$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\,dt} = e^{\int 200\,dt} = e^{200t}$$

Step 2: Multiply through by the integrating factor

$$e^{200t}\frac{di}{dt} + 200e^{200t}i = 40e^{200t}$$

$$\frac{d}{dt}\left(e^{200t}i\right) = 40e^{200t}$$

Step 3: Integrate both sides

$$e^{200t}i = 40\int e^{200t}\,dt + C$$

$$e^{200t}i = 40\cdot\frac{1}{200}e^{200t} + C$$

$$e^{200t}i = \frac{40}{200}e^{200t} + C$$

$$e^{200t}i = \frac{1}{5}e^{200t} + C$$

Step 4: Solve for i

$$i = \frac{1}{5} + Ce^{-200t}$$

$$i = 0.2 + Ce^{-200t}$$

Step 5: Apply initial condition At $t = 0$, $i = 0$ (current is zero when switch is first closed):

$$0 = 0.2 + C$$

$$C = -0.2$$

Therefore:

$$i(t) = 0.2(1 - e^{-200t})$$

Alternative form:

$$i(t) = 0.2\left(1 - e^{-\frac{R}{L}t}\right)$$

where $\frac{R}{L} = \frac{100}{0.5} = 200$

Final Answer:

$$i(t) = 0.2(1 - e^{-200t}) \text{ amperes}$$

Example 6: RL Circuit Time to 90 Percent Maximum

In a circuit containing inductance L, resistance R and voltage E, the current i is given by: $E = Ri + L\frac{di}{dt}$. Given $L = 640$, $R = 250$ and $E = 500$ volts, i being zero when $t = 0$. Find the time that elapses, before it reaches 90 percent of its maximum value.

Complete Solution

We have the first-order linear differential equation:

$$E = Ri + L\frac{di}{dt}$$

Rearranging:

$$L\frac{di}{dt} + Ri = E$$

Substituting the given values $L = 640$ H, $R = 250$, and $E = 500$ V:

$$640\frac{di}{dt} + 250i = 500$$

Dividing by 640:

$$\frac{di}{dt} + \frac{250}{640}i = \frac{500}{640}$$

$$\frac{di}{dt} + \frac{25}{64}i = \frac{50}{64}$$

This is a linear differential equation of the form $\frac{di}{dt} + Pi = Q$ where:

$$P = \frac{25}{64}, \quad Q = \frac{50}{64}$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\,dt} = e^{\int \frac{25}{64}\,dt} = e^{\frac{25}{64}t}$$

Step 2: Multiply through by the integrating factor

$$e^{\frac{25}{64}t}\frac{di}{dt} + \frac{25}{64}e^{\frac{25}{64}t}i = \frac{50}{64}e^{\frac{25}{64}t}$$

$$\frac{d}{dt}\left(e^{\frac{25}{64}t}i\right) = \frac{50}{64}e^{\frac{25}{64}t}$$

Step 3: Integrate both sides

$$e^{\frac{25}{64}t}i = \frac{50}{64}\int e^{\frac{25}{64}t}\,dt + C$$

$$e^{\frac{25}{64}t}i = \frac{50}{64}\cdot\frac{64}{25}e^{\frac{25}{64}t} + C$$

$$e^{\frac{25}{64}t}i = 2e^{\frac{25}{64}t} + C$$

Step 4: Solve for i

$$i = 2 + Ce^{-\frac{25}{64}t}$$

Step 5: Apply initial condition At $t = 0$, $i = 0$:

$$0 = 2 + C$$

$$C = -2$$

Therefore:

$$i(t) = 2(1 - e^{-\frac{25}{64}t})$$

Step 6: Find the maximum current As $t \to \infty$, $e^{-\frac{25}{64}t} \to 0$, so:

$$i_{\text{max}} = 2 \text{ amperes}$$

Step 7: Find time when current reaches 90 percent of maximum We need to find t when $i = 0.9 \times i_{\text{max}} = 0.9 \times 2 = 1.8$:

$$1.8 = 2(1 - e^{-\frac{25}{64}t})$$

$$0.9 = 1 - e^{-\frac{25}{64}t}$$

$$e^{-\frac{25}{64}t} = 1 - 0.9 = 0.1$$

$$-\frac{25}{64}t = \ln(0.1) = -\ln(10)$$

$$t = \frac{64}{25}\ln(10)$$

$$t = \frac{64 \times 2.303}{25}$$

$$t = \frac{147.392}{25} = 5.8957 \text{ seconds}$$

Final Answer: Time to reach 90 percent of maximum current is 5.8957 seconds.

Example 7: RL Circuit with Exponential Voltage

A voltage $E \cdot e^{-at}$ is applied at $t = 0$ to a circuit containing inductance L and resistance R. Show that the current at any time t is:

$$i = \frac{E}{R - aL}(e^{-at} - e^{-\frac{R}{L}t})$$

Complete Solution

We have the first-order linear differential equation:

$$L\frac{di}{dt} + Ri = E \cdot e^{-at}$$

Dividing by L:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at}$$

This is a linear differential equation of the form $\frac{di}{dt} + Pi = Q$ where:

$$P = \frac{R}{L}, \quad Q = \frac{E}{L}e^{-at}$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\,dt} = e^{\int \frac{R}{L}\,dt} = e^{\frac{R}{L}t}$$

Step 2: Multiply through by the integrating factor

$$e^{\frac{R}{L}t}\frac{di}{dt} + e^{\frac{R}{L}t}\frac{R}{L}i = e^{\frac{R}{L}t}\frac{E}{L}e^{-at}$$

$$\frac{d}{dt}\left(e^{\frac{R}{L}t}i\right) = \frac{E}{L}e^{\frac{R}{L}t}e^{-at}$$

$$\frac{d}{dt}\left(e^{\frac{R}{L}t}i\right) = \frac{E}{L}e^{(\frac{R}{L}-a)t}$$

Step 3: Integrate both sides

$$e^{\frac{R}{L}t}i = \frac{E}{L}\int e^{(\frac{R}{L}-a)t}\,dt + C$$

$$e^{\frac{R}{L}t}i = \frac{E}{L} \cdot \frac{1}{\frac{R}{L} - a} e^{(\frac{R}{L}-a)t} + C$$

Step 4: Solve for i

$$i = \frac{E}{L} \cdot \frac{1}{\frac{R}{L} - a} e^{(\frac{R}{L}-a)t - \frac{R}{L}t} + Ce^{-\frac{R}{L}t}$$

$$i = \frac{E}{L} \cdot \frac{1}{\frac{R}{L} - a} e^{-at} + Ce^{-\frac{R}{L}t}$$

$$i = \frac{E}{L(\frac{R}{L} - a)} e^{-at} + Ce^{-\frac{R}{L}t}$$

$$i = \frac{E}{R - aL} e^{-at} + Ce^{-\frac{R}{L}t}$$

Step 5: Apply initial condition At $t = 0$, $i = 0$ (current is zero when switch is first closed):

$$0 = \frac{E}{R - aL} + C$$

$$C = -\frac{E}{R - aL}$$

Therefore:

$$i(t) = \frac{E}{R - aL} e^{-at} - \frac{E}{R - aL} e^{-\frac{R}{L}t}$$

$$i(t) = \frac{E}{R - aL} (e^{-at} - e^{-\frac{R}{L}t})$$

Final Answer:

$$i = \frac{E}{R - aL} (e^{-at} - e^{-\frac{R}{L}t})$$

This is the required expression for the current at any time t.

Example 8: L-R Circuit with Sinusoidal EMF

The equation of an L-R circuit is given by: $L\frac{di}{dt} + Ri = 10 \sin t$. If $i = 0$, at $t = 0$, express i as a function of t.

Complete Solution

We have the first-order linear differential equation:

$$L\frac{di}{dt} + Ri = 10 \sin t$$

Dividing by L:

$$\frac{di}{dt} + \frac{R}{L} i = \frac{10}{L} \sin t$$

This is a linear differential equation of the form $\frac{di}{dt} + Pi = Q$ where:

$$P = \frac{R}{L}, \quad Q = \frac{10}{L} \sin t$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\, dt} = e^{\int \frac{R}{L}\, dt} = e^{\frac{R}{L}t}$$

Step 2: Multiply through by the integrating factor

$$e^{\frac{R}{L}t}\frac{di}{dt} + e^{\frac{R}{L}t}\frac{R}{L}i = e^{\frac{R}{L}t}\frac{10}{L}\sin t$$

$$\frac{d}{dt}\left(e^{\frac{R}{L}t}i\right) = \frac{10}{L}e^{\frac{R}{L}t}\sin t$$

Step 3: Integrate both sides

$$e^{\frac{R}{L}t}i = \frac{10}{L}\int e^{\frac{R}{L}t}\sin t\,dt + C$$

Using the formula:

$$\int e^{at}\sin(bt)\,dt = \frac{e^{at}(a\sin(bt) - b\cos(bt))}{a^2+b^2}$$

With $a = \frac{R}{L}$ and $b = 1$:

$$\int e^{\frac{R}{L}t}\sin t\,dt = \frac{e^{\frac{R}{L}t}\left(\frac{R}{L}\sin t - \cos t\right)}{\left(\frac{R}{L}\right)^2+1}$$

$$= \frac{Le^{\frac{R}{L}t}\left(\frac{R}{L}\sin t - \cos t\right)}{R^2+L^2}$$

$$= \frac{e^{\frac{R}{L}t}(R\sin t - L\cos t)}{R^2+L^2}$$

Therefore:

$$e^{\frac{R}{L}t}i = \frac{10}{L}\cdot\frac{e^{\frac{R}{L}t}(R\sin t - L\cos t)}{R^2+L^2} + C$$

Step 4: Solve for i

$$i = \frac{10(R\sin t - L\cos t)}{L(R^2+L^2)} + Ce^{-\frac{R}{L}t}$$

$$i = \frac{10(R\sin t - L\cos t)}{R^2+L^2} + Ce^{-\frac{R}{L}t}$$

Step 5: Apply initial condition At $t = 0$, $i = 0$:

$$0 = \frac{10(R\sin 0 - L\cos 0)}{R^2+L^2} + C$$

$$0 = \frac{10(0-L)}{R^2+L^2} + C$$

$$0 = -\frac{10L}{R^2+L^2} + C$$

$$C = \frac{10L}{R^2+L^2}$$

Therefore:

$$i(t) = \frac{10(R\sin t - L\cos t)}{R^2+L^2} + \frac{10L}{R^2+L^2}e^{-\frac{R}{L}t}$$

Final Answer:

$$i(t) = \frac{10}{R^2+L^2}\left(R\sin t - L\cos t + Le^{-\frac{R}{L}t}\right)$$

Example 9: RC Circuit with Exponential Voltage

An voltage $200e^{-3t}$ is applied to a circuit containing resistance $R = 20$ ohms and condenser of capacity $C = 0.01$ farads in series. Find the charge and current at any time, assuming that at $t = 0$, $q = 0$.

Complete Solution

We have the first-order linear differential equation for an RC circuit:

$$R\frac{dq}{dt} + \frac{q}{C} = E$$

where $E = 200e^{-3t}$.
Substituting the given values $R = 20$ and $C = 0.01$ F:

$$20\frac{dq}{dt} + \frac{q}{0.01} = 200e^{-3t}$$

$$20\frac{dq}{dt} + 100q = 200e^{-3t}$$

Dividing by 20:

$$\frac{dq}{dt} + 5q = 10e^{-3t}$$

This is a linear differential equation of the form $\frac{dq}{dt} + Pq = Q$ where:

$$P = 5, \quad Q = 10e^{-3t}$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\,dt} = e^{\int 5\,dt} = e^{5t}$$

Step 2: Multiply through by the integrating factor

$$e^{5t}\frac{dq}{dt} + 5e^{5t}q = 10e^{5t}e^{-3t}$$

$$\frac{d}{dt}\left(e^{5t}q\right) = 10e^{5t-3t} = 10e^{2t}$$

Step 3: Integrate both sides

$$e^{5t}q = 10\int e^{2t}\,dt + C$$

$$e^{5t}q = 10\cdot\frac{1}{2}e^{2t} + C$$

$$e^{5t}q = 5e^{2t} + C$$

Step 4: Solve for q

$$q = 5e^{2t}e^{-5t} + Ce^{-5t}$$

$$q = 5e^{-3t} + Ce^{-5t}$$

Step 5: Apply initial condition At $t = 0$, $q = 0$:

$$0 = 5e^0 + Ce^0$$

$$0 = 5 + C$$

$$C = -5$$

Therefore:

$$q(t) = 5e^{-3t} - 5e^{-5t}$$

Step 6: Find the current The current is given by:

$$i = \frac{dq}{dt}$$

$$i = \frac{d}{dt}\left[5e^{-3t} - 5e^{-5t}\right]$$

$$i = 5(-3)e^{-3t} - 5(-5)e^{-5t}$$

$$i = -15e^{-3t} + 25e^{-5t}$$

Final Answer:
Charge: $q(t) = 5(e^{-3t} - e^{-5t})$ coulombs
Current: $i(t) = 5(5e^{-5t} - 3e^{-3t})$ amperes

Example 10: RC Circuit with Exponential Voltage

An voltage $200e^{-5t}$ is applied to a circuit containing resistance $R = 20$ ohms and condenser of capacity $C = 0.01$ farads in series. Find the charge and current at any time, assuming that at $t = 0$, $q = 0$.

Complete Solution

We have the first-order linear differential equation for an RC circuit:

$$R\frac{dq}{dt} + \frac{q}{C} = E$$

where $E = 200e^{-5t}$.
Substituting the given values $R = 20$ and $C = 0.01$ F:

$$20\frac{dq}{dt} + \frac{q}{0.01} = 200e^{-5t}$$

$$20\frac{dq}{dt} + 100q = 200e^{-5t}$$

Dividing by 20:

$$\frac{dq}{dt} + 5q = 10e^{-5t}$$

This is a linear differential equation of the form $\frac{dq}{dt} + Pq = Q$ where:

$$P = 5, \quad Q = 10e^{-5t}$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\,dt} = e^{\int 5\,dt} = e^{5t}$$

Step 2: Multiply through by the integrating factor

$$e^{5t}\frac{dq}{dt} + 5e^{5t}q = 10e^{5t}e^{-5t}$$

$$\frac{d}{dt}\left(e^{5t}q\right) = 10e^{5t-5t} = 10e^{0} = 10$$

Step 3: Integrate both sides

$$e^{5t}q = 10\int dt + C$$

$$e^{5t}q = 10t + C$$

Step 4: Solve for q

$$q = \frac{10t + C}{e^{5t}}$$

$$q = 10te^{-5t} + Ce^{-5t}$$

Step 5: Apply initial condition At $t = 0$, $q = 0$:

$$0 = 10(0)e^{0} + Ce^{0}$$

$$0 = 0 + C$$

$$C = 0$$

Therefore:

$$q(t) = 10te^{-5t}$$

Step 6: Find the current The current is given by:

$$i = \frac{dq}{dt}$$

$$i = \frac{d}{dt}\left[10te^{-5t}\right]$$

Using the product rule:

$$i = 10 \cdot 1 \cdot e^{-5t} + 10t \cdot (-5)e^{-5t}$$

$$i = 10e^{-5t} - 50te^{-5t}$$

$$i = 10e^{-5t}(1 - 5t)$$

Final Answer:
Charge: $q(t) = 10te^{-5t}$ coulombs
Current: $i(t) = 10e^{-5t}(1 - 5t)$ amperes

Example 11: RC Circuit Voltage Proof

A circuit consists of resistance R ohms and a condenser of C farads connected to a constant e.m.f. E. If $\frac{q}{C}$ is the voltage of the condenser at time t after closing the circuit, show that the voltage at time t is $E\left(1 - e^{-\frac{t}{CR}}\right)$.

Complete Solution

We have the first-order linear differential equation for an RC circuit:

$$R\frac{dq}{dt} + \frac{q}{C} = E$$

This can be rearranged as:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R}$$

This is a linear differential equation of the form $\frac{dq}{dt} + Pq = Q$ where:

$$P = \frac{1}{RC}, \quad Q = \frac{E}{R}$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\,dt} = e^{\int \frac{1}{RC}\,dt} = e^{\frac{t}{RC}}$$

Step 2: Multiply through by the integrating factor

$$e^{\frac{t}{RC}}\frac{dq}{dt} + e^{\frac{t}{RC}}\frac{q}{RC} = e^{\frac{t}{RC}}\frac{E}{R}$$

$$\frac{d}{dt}\left(e^{\frac{t}{RC}}q\right) = \frac{E}{R}e^{\frac{t}{RC}}$$

Step 3: Integrate both sides

$$e^{\frac{t}{RC}}q = \frac{E}{R}\int e^{\frac{t}{RC}}\,dt + C$$

$$e^{\frac{t}{RC}}q = \frac{E}{R}\cdot RC\cdot e^{\frac{t}{RC}} + C$$

$$e^{\frac{t}{RC}}q = CE\cdot e^{\frac{t}{RC}} + C$$

Step 4: Solve for q

$$q = CE + Ce^{-\frac{t}{RC}}$$

Step 5: Apply initial condition At $t = 0$, $q = 0$ (condenser initially uncharged):

$$0 = CE + C$$

$$C = -CE$$

Therefore:

$$q(t) = CE - CE\cdot e^{-\frac{t}{RC}}$$

$$q(t) = CE\left(1 - e^{-\frac{t}{RC}}\right)$$

Step 6: Find the voltage across the condenser The voltage across the condenser is:

$$V_C = \frac{q}{C}$$

$$V_C = \frac{CE\left(1 - e^{-\frac{t}{RC}}\right)}{C}$$

$$V_C = E\left(1 - e^{-\frac{t}{RC}}\right)$$

Final Answer: The voltage across the condenser at time t is $E\left(1 - e^{-\frac{t}{CR}}\right)$.

Example 12: RC Circuit with Exponential Expression

The charge Q on the plate of a condenser of capacity C charged through a resistance R by a steady voltage V satisfies the differential equation $R\frac{dQ}{dt} + \frac{Q}{C} = V$. If $Q = 0$ at $t = 0$, show that $Q = CV(1 - e^{-t/RC})$. Find the current flowing into the plate.

Complete Solution

We have the first-order linear differential equation:

$$R\frac{dQ}{dt} + \frac{Q}{C} = V$$

Dividing by R:

$$\frac{dQ}{dt} + \frac{Q}{RC} = \frac{V}{R}$$

This is a linear differential equation of the form $\frac{dQ}{dt} + PQ = Q$ where:

$$P = \frac{1}{RC}, \quad Q = \frac{V}{R}$$

Step 1: Find the integrating factor

$$\text{I.F.} = e^{\int P\,dt} = e^{\int \frac{1}{RC}\,dt} = e^{\frac{t}{RC}}$$

Step 2: Multiply through by the integrating factor

$$e^{\frac{t}{RC}}\frac{dQ}{dt} + e^{\frac{t}{RC}}\frac{Q}{RC} = e^{\frac{t}{RC}}\frac{V}{R}$$

$$\frac{d}{dt}\left(e^{\frac{t}{RC}}Q\right) = \frac{V}{R}e^{\frac{t}{RC}}$$

Step 3: Integrate both sides

$$e^{\frac{t}{RC}}Q = \frac{V}{R}\int e^{\frac{t}{RC}}\,dt + C$$

$$e^{\frac{t}{RC}}Q = \frac{V}{R} \cdot RC \cdot e^{\frac{t}{RC}} + C$$

$$e^{\frac{t}{RC}}Q = VCe^{\frac{t}{RC}} + C$$

Step 4: Solve for Q

$$Q = VC + Ce^{-\frac{t}{RC}}$$

Step 5: Apply initial condition At $t = 0$, $Q = 0$:

$$0 = VC + C$$

$$C = -VC$$

Therefore:

$$Q(t) = VC - VCe^{-\frac{t}{RC}}$$

$$Q(t) = VC\left(1 - e^{-\frac{t}{RC}}\right)$$

Step 6: Find the current The current flowing into the plate is:

$$i = \frac{dQ}{dt}$$

$$i = \frac{d}{dt}\left[VC\left(1 - e^{-\frac{t}{RC}}\right)\right]$$

$$i = VC \cdot \frac{d}{dt}\left(1 - e^{-\frac{t}{RC}}\right)$$

$$i = VC \cdot \left(0 - \left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}}\right)$$

$$i = VC \cdot \frac{1}{RC}e^{-\frac{t}{RC}}$$

$$i = \frac{V}{R}e^{-\frac{t}{RC}}$$

Final Answer: Charge: $Q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)$
Current: $i(t) = \frac{V}{R}e^{-\frac{t}{RC}}$

4.9 Heat Flow Analysis

4.9.1 Thermal Energy Transfer Principles

The movement of thermal energy through materials follows well-established physical laws. Understanding these principles enables engineers and scientists to model temperature distributions and heat transfer rates in various systems.

Core Heat Transfer Principles

Three fundamental concepts govern heat conduction:

1. Thermal energy migrates from regions of higher temperature to regions of lower temperature
2. The thermal content within a body varies proportionally with both its mass and temperature

3. Heat transfer rate through a surface depends on the temperature gradient normal to that surface

4.9.2 Fourier's Law Formulation

The mathematical relationship for thermal conduction characterizes how heat energy propagates through solid materials.

Definition 4.4 (Mathematical Expression of Heat Conduction)**.** *For a heat flux q passing through a slab of area A and thickness δx, with temperature difference δT across its faces:*

$$q = \textit{(Material Property)} \times \textit{(Cross-sectional Area)} \times \textit{(Temperature Gradient)} \tag{4.427}$$

Expressed mathematically:

$$q = -kA\frac{dT}{dx} \tag{4.428}$$

Where k represents the thermal conductivity coefficient of the material.

The negative notation indicates the opposite direction of heat flow relative to the temperature increase.

4.9.3 Steady-State Heat Transfer Solution

Consider an insulated cylindrical pipe system carrying fluid at elevated temperature.

Problem Statement: A pipe of radius r_0 maintains internal temperature T_0 while surrounded by insulation of thickness w. The outer surface stabilizes at temperature T_1. We seek the heat loss per unit length and temperature profile within the insulation.

Solution Methodology:

Under equilibrium conditions, heat transfer rate remains constant.

For a cylindrical shell of radius r, Fourier's law yields:

$$Q = -k(2\pi r \cdot 1)\frac{dT}{dr} \tag{4.429}$$

Rearranging:

$$\frac{dT}{dr} = -\frac{Q}{2\pi kr} \tag{4.430}$$

Integration produces:

$$T = -\frac{Q}{2\pi k}\ln r + C \tag{4.431}$$

Applying boundary conditions $T(r_0) = T_0$ and $T(r_1) = T_1$:

$$T = T_0 - \frac{Q}{2\pi k}(\ln r - \ln r_0) \tag{4.432}$$

Eliminating Q using both boundary conditions:

$$Q = \frac{2\pi k(T_0 - T_1)}{\ln(r_1/r_0)} \tag{4.433}$$

The temperature distribution becomes:

$$\frac{T_0 - T}{T_0 - T_1} = \frac{\ln(r/r_0)}{\ln(r_1/r_0)} \tag{4.434}$$

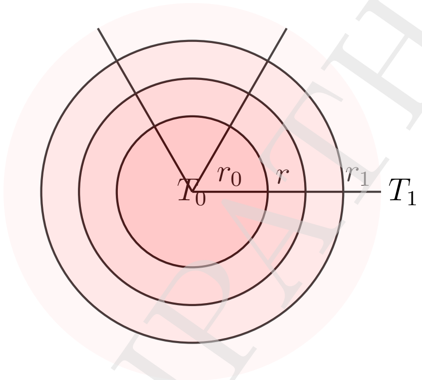


Figure 4.6: Cross-sectional view of insulated cylindrical pipe system

Engineering Applications

Heat transfer analysis is crucial for:

- Thermal insulation design in buildings
- Industrial piping system efficiency
- Electronic component thermal management
- Energy conservation in mechanical systems

4.10 Solved Examples

Example 1: Heat Transfer Through Pipe Insulation

A steam pipe 20 cm in diameter is protected with a covering 6 cm thick for which the coefficient of thermal conductivity is $k = 0.0003$ cal/cm deg. sec in steady state. Find the heat lost per hour through a meter length of the pipe, if the surface of the pipe is 200°C and the outer surface of the covering is at 30°C.

Solution: Cylindrical Heat Conduction Analysis

This problem involves heat transfer through a cylindrical insulation layer in steady-state conditions. We can apply Fourier's law for cylindrical geometry as derived in the previous section.

Given information:

$$\begin{aligned}
r_0 &= 10 \text{ cm} \quad \text{(inner radius of pipe)} \\
r_1 &= r_0 + 6 = 16 \text{ cm} \quad \text{(outer radius with insulation)} \\
T_0 &= 200\text{C} \quad \text{(temperature at pipe surface)} \\
T_1 &= 30\text{C} \quad \text{(temperature at outer insulation surface)} \\
k &= 0.0003 \text{ cal/cm·deg·sec} \quad \text{(thermal conductivity)} \\
L &= 100 \text{ cm} \quad \text{(1 meter length of pipe)}
\end{aligned}$$

Heat transfer rate calculation:

For a cylindrical system in steady state, the heat transfer rate Q is given by:

$$Q = \frac{2\pi k L(T_0 - T_1)}{\ln(r_1/r_0)} \tag{4.435}$$

Substituting the values:

$$
\begin{aligned}
Q &= \frac{2\pi \cdot 0.0003 \text{ cal/cm·deg·sec} \cdot 100 \text{ cm} \cdot (200 - 30)\text{C}}{\ln(16/10)} \\
&= \frac{2\pi \cdot 0.0003 \cdot 100 \cdot 170}{\ln(1.6)}
\end{aligned}
$$

Using precise values:

$$
\begin{aligned}
\ln(1.6) &= 0.4700036292457356 \\
2\pi &= 6.283185307179586 \\
2\pi \cdot 0.0003 \cdot 100 \cdot 170 &= 32.044245066615886
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
Q &= \frac{32.044245066615886}{0.4700036292457356} \\
&= 68.17871836019792 \text{ cal/sec}
\end{aligned}
$$

To convert to calories per hour:

$$
\begin{aligned}
Q \text{ (cal/hour)} &= 68.17871836019792 \text{ cal/sec} \times 3600 \text{ sec/hour} \\
&= 245443.38609671252 \text{ cal/hour}
\end{aligned}
$$

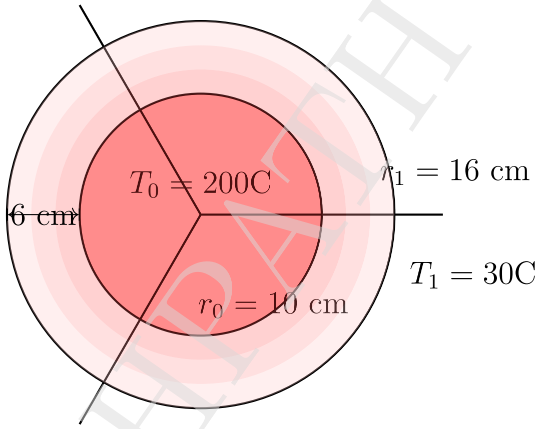
Verification with the temperature distribution formula:
We can also verify by examining the temperature distribution within the insulation:

$$
\frac{T_0 - T}{T_0 - T_1} = \frac{\ln(r/r_0)}{\ln(r_1/r_0)} \tag{4.436}
$$

For example, at the middle of the insulation ($r = 13$ cm):

$$
\begin{aligned}
\frac{200 - T}{200 - 30} &= \frac{\ln(13/10)}{\ln(16/10)} \\
\frac{200 - T}{170} &= \frac{\ln(1.3)}{\ln(1.6)} \\
\frac{200 - T}{170} &= \frac{0.26236426446749106}{0.4700036292457356} \\
\frac{200 - T}{170} &= 0.5582190369190173 \\
200 - T &= 170 \cdot 0.5582190369190173 \\
200 - T &= 94.89723627623295 \\
T &= 105.10276372376705\text{C}
\end{aligned}
$$

Conclusion: The heat lost through a one-meter length of the insulated pipe is 245443.38609671252 calories per hour.



Cross-section of Insulated Steam Pipe

Figure 4.7: Cross-sectional view of the steam pipe with insulation showing temperature gradient

Engineering Insight

For cylindrical insulation systems, there exists an optimal insulation thickness where adding more insulation would actually increase heat loss due to the increased outer surface area. This counterintuitive phenomenon is known as the "critical radius of insulation" and is especially important in small-diameter pipes and electrical wires.

Example 2: Heat Transfer Through Asbestos Insulation

A pipe 10 cm in diameter contains steam at 100°C. It is covered with asbestos, 5 cm thick, for which $k = 0.0006$ and the outside surface is at 30°C. Find the amount of heat lost per hour from a meter long pipe.

Solution: Steady-State Cylindrical Heat Conduction

This problem involves heat transfer through a cylindrical layer of asbestos insulation under steady-state conditions. We will apply Fourier's law for cylindrical geometry.
Given information:

$$\begin{aligned}
r_0 &= 5 \text{ cm} \quad \text{(inner radius of pipe)} \\
r_1 &= r_0 + 5 = 10 \text{ cm} \quad \text{(outer radius with insulation)} \\
T_0 &= 100\text{C} \quad \text{(temperature at pipe surface)} \\
T_1 &= 30\text{C} \quad \text{(temperature at outer insulation surface)} \\
k &= 0.0006 \text{ cal/cm·deg·sec} \quad \text{(thermal conductivity of asbestos)} \\
L &= 100 \text{ cm} \quad \text{(1 meter length of pipe)}
\end{aligned}$$

Heat transfer rate calculation:
For a cylindrical system in steady state, the heat transfer rate Q is given by:

$$Q = \frac{2\pi k L (T_0 - T_1)}{\ln(r_1/r_0)} \tag{4.437}$$

Substituting the values with full precision:

$$
\begin{aligned}
Q &= \frac{2\pi \cdot 0.0006 \text{ cal/cm·deg·sec} \cdot 100 \text{ cm} \cdot (100 - 30)\text{C}}{\ln(10/5)} \\
&= \frac{2\pi \cdot 0.0006 \cdot 100 \cdot 70}{\ln(2)}
\end{aligned}
$$

Using precise values:

$$
\begin{aligned}
\ln(2) &= 0.6931471805599453 \\
2\pi &= 6.283185307179586 \\
2\pi \cdot 0.0006 \cdot 100 \cdot 70 &= 26.38937829015426
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
Q &= \frac{26.38937829015426}{0.6931471805599453} \\
&= 38.07320393249937 \text{ cal/sec}
\end{aligned}
$$

To convert to calories per hour:

$$
\begin{aligned}
Q \text{ (cal/hour)} &= 38.07320393249937 \text{ cal/sec} \times 3600 \text{ sec/hour} \\
&= 137063.5341469977 \text{ cal/hour}
\end{aligned}
$$

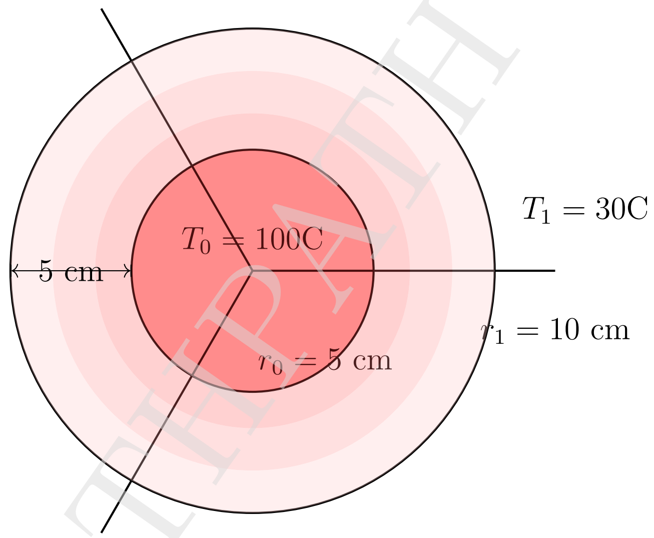
Temperature distribution verification:
The temperature distribution within the insulation follows:

$$
\frac{T_0 - T}{T_0 - T_1} = \frac{\ln(r/r_0)}{\ln(r_1/r_0)} \tag{4.438}
$$

For example, at the middle of the insulation ($r = 7.5$ cm):

$$
\begin{aligned}
\frac{100 - T}{100 - 30} &= \frac{\ln(7.5/5)}{\ln(10/5)} \\
\frac{100 - T}{70} &= \frac{\ln(1.5)}{\ln(2)} \\
\frac{100 - T}{70} &= \frac{0.4054651081081644}{0.6931471805599453} \\
\frac{100 - T}{70} &= 0.5850752939123049 \\
100 - T &= 70 \cdot 0.5850752939123049 \\
100 - T &= 40.955270573861344 \\
T &= 59.044729426138656\text{C}
\end{aligned}
$$

Conclusion: The heat lost through a one-meter length of the insulated pipe is 137,063.5341469977 calories per hour.



Cross-section of Pipe with Asbestos Insulation

Figure 4.8: Cross-sectional view of the pipe with asbestos insulation showing temperature gradient

Health and Safety Note

While this example uses asbestos as an insulation material for educational purposes, it is important to note that asbestos use has been largely prohibited or heavily regulated in most countries due to its severe health risks. Modern insulation materials such as mineral wool, fiberglass, or ceramic fiber provide similar thermal insulation properties without the associated health hazards.

Example 3: Temperature Distribution in Pipe Insulation

A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering 5 cm thick for which $k = 0.0025$. If the temperature of the outer surface of the covering is 40°C, find the temperature half-way through the covering under steady-state conditions.

Solution: Temperature Distribution Analysis

This problem focuses on determining the temperature at a specific point within cylindrical insulation under steady-state conditions. Rather than calculating heat transfer rate, we need to use the temperature distribution equation.
Given information:

$$\begin{aligned}
r_0 &= 10 \text{ cm} \quad \text{(inner radius of pipe)} \\
r_1 &= r_0 + 5 = 15 \text{ cm} \quad \text{(outer radius with insulation)} \\
T_0 &= 150\text{C} \quad \text{(temperature at pipe surface)} \\
T_1 &= 40\text{C} \quad \text{(temperature at outer insulation surface)} \\
k &= 0.0025 \text{ cal/cm·deg·sec} \quad \text{(thermal conductivity)}
\end{aligned}$$

Finding the half-way point: The half-way point through the insulation is 2.5 cm from the inner surface, so:

$$r_{\text{half}} = r_0 + \frac{5}{2} = 10 + 2.5 = 12.5 \text{ cm}$$

Temperature distribution calculation: For cylindrical geometry, the temperature distribution follows:

$$\frac{T_0 - T}{T_0 - T_1} = \frac{\ln(r/r_0)}{\ln(r_1/r_0)} \tag{4.439}$$

Substituting the values at $r = r_{\text{half}} = 12.5$ cm:

$$\frac{150 - T}{150 - 40} = \frac{\ln(12.5/10)}{\ln(15/10)}$$
$$\frac{150 - T}{110} = \frac{\ln(1.25)}{\ln(1.5)}$$

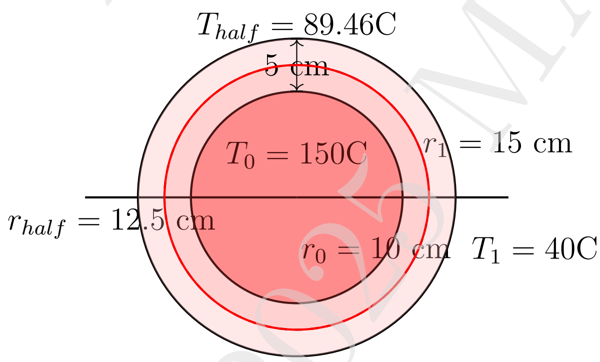
Using precise values:

$$\ln(1.25) = 0.22314355131420976$$
$$\ln(1.5) = 0.4054651081081644$$
$$\frac{\ln(1.25)}{\ln(1.5)} = \frac{0.22314355131420976}{0.4054651081081644} = 0.5504052570328626$$

Therefore:

$$\frac{150 - T}{110} = 0.5504052570328626$$
$$150 - T = 110 \cdot 0.5504052570328626$$
$$150 - T = 60.54457827361489$$
$$T = 150 - 60.54457827361489$$
$$T = 89.45542172638511C$$

Conclusion: The temperature half-way through the insulation (at $r = 12.5$ cm) is approximately 89.46°C under steady-state conditions.



Temperature Distribution in Cylindrical Insulation

Figure 4.9: Cross-sectional view showing temperature at the half-way point in the insulation

Temperature Distribution Principle

For steady-state heat conduction in cylindrical geometry, the temperature varies logarithmically with radius rather than linearly. This is a direct consequence of the expanding cross-sectional area along the radial direction. The temperature gradient is steeper near the inner surface and more gradual toward the outer surface, even with uniform material properties.

Example 4: Heat Transfer Through a Spherical Shell

The inner and outer surfaces of a spherical shell are maintained at T_0 and T_1 temperatures respectively. If the inner and outer radii of the shell are r_0 and r_1 respectively and thermal conductivity of the shell is k, find the amount of heat lost from the shell per unit time. Find also the temperature distribution through the shell.

Solution: Spherical Heat Conduction Analysis

This problem involves steady-state heat conduction through a spherical shell. Unlike the cylindrical cases we've examined previously, we must now adapt Fourier's law for spherical geometry.

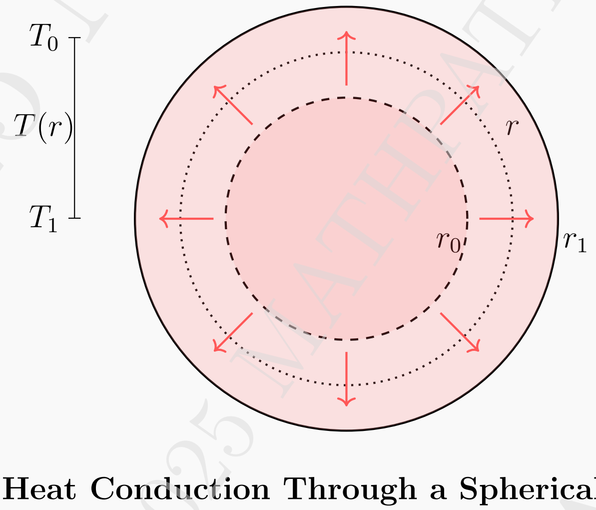


Figure 4.10: Cross-sectional view of the spherical shell showing heat flow from inner to outer surface

Given information:

$$
\begin{aligned}
r_0 &= \text{inner radius of the spherical shell}\\
r_1 &= \text{outer radius of the spherical shell}\\
T_0 &= \text{temperature at inner surface}\\
T_1 &= \text{temperature at outer surface}\\
k &= \text{thermal conductivity of the shell material}
\end{aligned}
$$

Part 1: Heat Transfer Rate Calculation

For a spherical shell, we need to apply Fourier's law in spherical coordinates. Consider a spherical surface at radius r within the shell:

The area of this spherical surface is:

$$A = 4\pi r^2 \tag{4.440}$$

According to Fourier's law, the heat transfer rate is:

$$Q = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr} \tag{4.441}$$

Under steady-state conditions, Q is constant throughout the shell. Rearranging:

$$\frac{dT}{dr} = -\frac{Q}{4\pi k r^2} \tag{4.442}$$

Integrating both sides:

$$\begin{aligned}
\int_{T_0}^{T_1} dT &= -\frac{Q}{4\pi k}\int_{r_0}^{r_1}\frac{dr}{r^2} \\
T_1 - T_0 &= -\frac{Q}{4\pi k}\left[-\frac{1}{r}\right]_{r_0}^{r_1} \\
T_1 - T_0 &= -\frac{Q}{4\pi k}\left(-\frac{1}{r_1} + \frac{1}{r_0}\right) \\
T_1 - T_0 &= \frac{Q}{4\pi k}\left(\frac{1}{r_0} - \frac{1}{r_1}\right) \\
T_1 - T_0 &= \frac{Q}{4\pi k}\left(\frac{r_1 - r_0}{r_0 r_1}\right)
\end{aligned}$$

Solving for Q:

$$\begin{aligned}
Q &= \frac{4\pi k(T_0 - T_1)}{\left(\frac{1}{r_0} - \frac{1}{r_1}\right)} \\
&= \frac{4\pi k(T_0 - T_1)}{\left(\frac{r_1 - r_0}{r_0 r_1}\right)} \\
&= \frac{4\pi k(T_0 - T_1) r_0 r_1}{r_1 - r_0}
\end{aligned}$$

Part 2: Temperature Distribution

To find the temperature distribution, we integrate the differential equation from radius r_0 to any arbitrary radius r within the shell:

$$\begin{aligned}
\int_{T_0}^{T} dT &= -\frac{Q}{4\pi k}\int_{r_0}^{r}\frac{dr}{r^2} \\
T - T_0 &= -\frac{Q}{4\pi k}\left(-\frac{1}{r} + \frac{1}{r_0}\right) \\
T - T_0 &= \frac{Q}{4\pi k}\left(\frac{1}{r_0} - \frac{1}{r}\right)
\end{aligned}$$

Substituting the expression for Q:

$$\begin{aligned}
T - T_0 &= \frac{4\pi k(T_0 - T_1) r_0 r_1}{r_1 - r_0}\cdot\frac{1}{4\pi k}\left(\frac{1}{r_0} - \frac{1}{r}\right) \\
&= (T_0 - T_1)\frac{r_0 r_1}{r_1 - r_0}\left(\frac{1}{r_0} - \frac{1}{r}\right) \\
&= (T_0 - T_1)\frac{r_1}{r_1 - r_0}\left(1 - \frac{r_0}{r}\right)
\end{aligned}$$

Rearranging to isolate T:

$$\begin{aligned} T &= T_0 - (T_0 - T_1)\frac{r_1}{r_1 - r_0}\left(1 - \frac{r_0}{r}\right) \\ &= T_0 - (T_0 - T_1)\frac{r_1}{r_1 - r_0} + (T_0 - T_1)\frac{r_1 r_0}{r_1 - r_0}\frac{1}{r} \end{aligned}$$

With further algebraic manipulation, we can express the temperature distribution in a normalized form:

$$\begin{aligned} \frac{T - T_1}{T_0 - T_1} &= \frac{\frac{r_0}{r} - \frac{r_0}{r_1}}{\frac{r_0}{r_0} - \frac{r_0}{r_1}} \\ &= \frac{\frac{r_0}{r} - \frac{r_0}{r_1}}{1 - \frac{r_0}{r_1}} \end{aligned}$$

Or equivalently:

$$\frac{T - T_1}{T_0 - T_1} = \frac{r_1(r_0 - \frac{r_0 r_1}{r})}{r_0(r_1 - r_0)} = \frac{r_1 r_0(1 - \frac{r_1}{r})}{r_0(r_1 - r_0)} \tag{4.443}$$

After simplification:

$$\frac{T - T_1}{T_0 - T_1} = \frac{r_1}{r_1 - r_0}\left(\frac{r_0}{r} - \frac{r_0}{r_1}\right) = \frac{r_1 r_0/r - r_0}{r_1 - r_0} \tag{4.444}$$

This provides the temperature T at any radius r within the spherical shell.

Spherical Heat Conduction Properties

The heat transfer rate through a spherical shell differs from that of a planar or cylindrical geometry due to the change in surface area with radius. Key observations:

1. The heat flow equation contains the product of radii $r_0 r_1$ in the numerator, unlike the cylindrical case with a logarithmic term in the denominator.
2. The temperature varies inversely with radius rather than logarithmically as in cylindrical geometry.
3. For thin shells where $r_1 \approx r_0$, the solution approaches that of a plane wall.

Example 5: Heat Transfer Through a Hollow Pipe

A long hollow pipe has an inner diameter of 10 cm and outer diameter of 20 cm. The inner surface is kept at 200°C and the outer surface at 50°C. The thermal conductivity is 0.12. How much heat is lost per minute from a portion of the pipe 20 metres long? Find the temperature at a distance $x = 7.5$ cm from the centre of the pipe.

Solution: Cylindrical Heat Conduction Analysis

This problem involves steady-state heat conduction through a hollow cylindrical pipe, requiring calculation of both heat transfer rate and temperature distribution.

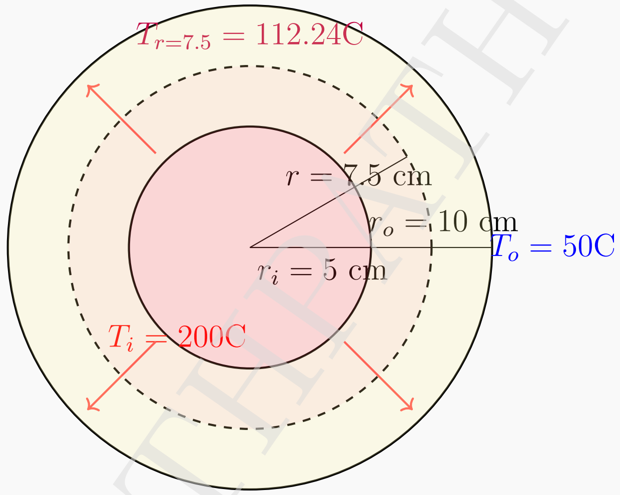


Figure 4.11: Cross-sectional view of the hollow cylindrical pipe showing temperature distribution

Given information:

$$\begin{aligned} r_i &= 5 \text{ cm} \quad \text{(inner radius of pipe)} \\ r_o &= 10 \text{ cm} \quad \text{(outer radius of pipe)} \\ T_i &= 200\text{C} \quad \text{(temperature at inner surface)} \\ T_o &= 50\text{C} \quad \text{(temperature at outer surface)} \\ k &= 0.12 \text{ cal/cm·deg·sec} \quad \text{(thermal conductivity)} \\ L &= 2000 \text{ cm} \quad \text{(20 meters length of pipe)} \end{aligned}$$

Part 1: Heat Transfer Rate Calculation
For steady-state heat conduction in a cylindrical geometry, the heat transfer rate is given by:

$$Q = \frac{2\pi k L (T_i - T_o)}{\ln(r_o/r_i)} \tag{4.445}$$

Substituting the values:

$$\begin{aligned} Q &= \frac{2\pi \cdot 0.12 \text{ cal/cm·deg·sec} \cdot 2000 \text{ cm} \cdot (200 - 50)\text{C}}{\ln(10/5)} \\ &= \frac{2\pi \cdot 0.12 \cdot 2000 \cdot 150}{\ln(2)} \end{aligned}$$

Using precise values:

$$\begin{aligned} \ln(2) &= 0.6931471805599453 \\ 2\pi &= 6.283185307179586 \\ 2\pi \cdot 0.12 \cdot 2000 \cdot 150 &= 226195.47106846508 \end{aligned}$$

Therefore:

$$\begin{aligned} Q &= \frac{226195.47106846508}{0.6931471805599453} \\ &= 326334.0617244909 \text{ cal/sec} \end{aligned}$$

Converting to calories per minute:

$$\begin{aligned} Q \text{ (cal/min)} &= 326334.0617244909 \text{ cal/sec} \times 60 \text{ sec/min} \\ &= 19580043.70346945 \text{ cal/min} \\ &\approx 1.958 \times 10^7 \text{ cal/min} \end{aligned}$$

Part 2: Temperature Distribution
For cylindrical geometry, the temperature at any radius r follows:

$$\frac{T_i - T}{T_i - T_o} = \frac{\ln(r/r_i)}{\ln(r_o/r_i)} \tag{4.446}$$

For $r = 7.5$ cm (distance from center):

$$\begin{aligned} \frac{200 - T}{200 - 50} &= \frac{\ln(7.5/5)}{\ln(10/5)} \\ \frac{200 - T}{150} &= \frac{\ln(1.5)}{\ln(2)} \end{aligned}$$

Using precise values:

$$\begin{aligned} \ln(1.5) &= 0.4054651081081644 \\ \ln(2) &= 0.6931471805599453 \\ \frac{\ln(1.5)}{\ln(2)} &= \frac{0.4054651081081644}{0.6931471805599453} = 0.5850752939123049 \end{aligned}$$

Therefore:

$$\begin{aligned} \frac{200 - T}{150} &= 0.5850752939123049 \\ 200 - T &= 150 \cdot 0.5850752939123049 \\ 200 - T &= 87.76129408684573 \\ T &= 200 - 87.76129408684573 \\ T &= 112.23870591315427\text{C} \\ &\approx 112.24\text{C} \end{aligned}$$

Conclusion:

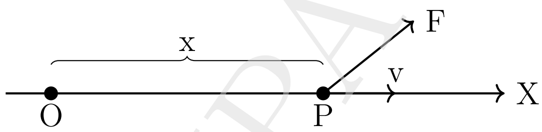
- The heat lost per minute from the 20-meter long portion of pipe is approximately 1.958×10^7 calories per minute.
- The temperature at a distance of 7.5 cm from the center of the pipe (halfway through the pipe wall) is approximately 112.24°C.

4.11 Rectilinear Motion Analysis

4.11.1 Linear Motion Framework

When objects move exclusively along straight trajectories, we enter the domain of rectilinear motion. This simplified yet fundamental case of motion analysis provides the building blocks for understanding more complex dynamical systems.

Definition 4.5 (Rectilinear Motion)**.** *A particle executing rectilinear motion traverses a linear path, where position changes occur only along a single spatial dimension. Consider a particle with mass m progressing along a fixed straight line with reference point O.*



4.11.2 Kinematic Relationships

The quantitative description of rectilinear motion establishes fundamental connections between motion parameters. At any time instant t, if the particle occupies position x:

Velocity Definition:

$$v = \frac{dx}{dt} \tag{4.447}$$

Acceleration Definition:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \tag{4.448}$$

An alternative acceleration expression through the chain rule:

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx} \tag{4.449}$$

4.11.3 Newton's Dynamical Framework

The cornerstone of motion analysis rests on Newton's fundamental principle:

Newton's Second Principle

The time rate of change of momentum equals the net force applied:

$$F = \frac{d}{dt}(mv) \tag{4.450}$$

For constant mass systems:

$$F = m\frac{dv}{dt} = ma \tag{4.451}$$

This yields multiple equivalent expressions for force:

$$F = m\frac{dv}{dt} = mv\frac{dv}{dx} = m\frac{d^2x}{dt^2} \tag{4.452}$$

4.11.4 D'Alembert's Dynamic Equilibrium

D'Alembert's principle provides an insightful perspective on force balance:

D'Alembert's Principle

The vectorial summation of forces along any direction equals the mass-acceleration product in that directional sense.
Mathematically: Net force = Mass × Acceleration

4.11.5 Force Categories in Motion Analysis

Common forces encountered in rectilinear motion include:

- Gravitational attraction (downward direction)
- Elastic tension forces (springs, strings)
- Contact reactions between surfaces
- Attractive/repulsive interactions
- Resistance from air and friction

Directional Considerations

- Force direction must align with motion trajectory
- Signs indicate directional opposition or support
- Net force represents algebraic sum along motion axis

4.11.6 Resulting Equations of Motion

The mathematical framework for motion analysis follows multiple pathways depending on force characteristics:

$$F = m\frac{d^2x}{dt^2} \tag{4.453}$$

$$F = m\frac{dv}{dt} \tag{4.454}$$

$$F = mv\frac{dv}{dx} \tag{4.455}$$

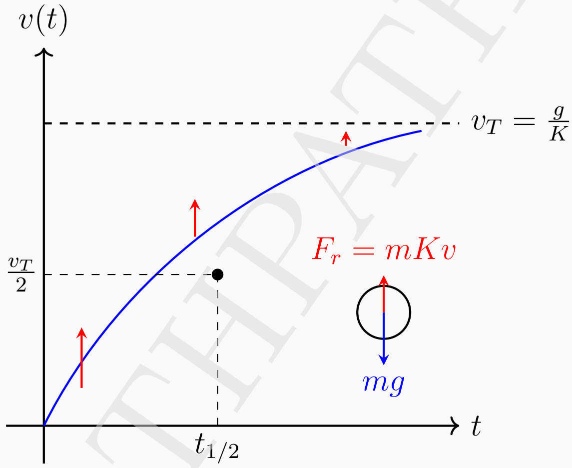
Each form proves useful for specific problem types, enabling selective application based on given conditions.

4.12 Solved Example

Example 4: Motion with Linear Damping

A body of mass m falls from rest under gravity in a fluid whose resistance to motion at any instant is mK times its velocity, where K is a constant. Find the terminal velocity of the body and also the time taken to acquire one-half of its limiting speed.

Solution



Let's establish our coordinate system with the positive direction downward (in the direction of gravity). We will denote the velocity of the body at time t as $v(t)$.
The differential equation governing the motion is derived from Newton's Second Law:

$$m\frac{dv}{dt} = mg - mKv \tag{4.456}$$
$$\frac{dv}{dt} = g - Kv \tag{4.457}$$

This is a first-order linear differential equation. We rearrange to standard form:

$$\frac{dv}{dt} + Kv = g \tag{4.458}$$
$$\tag{4.459}$$

Finding the Terminal Velocity:
Terminal velocity occurs when the acceleration becomes zero, i.e., $\frac{dv}{dt} = 0$. Substituting into our differential equation:

$$0 = g - Kv_T \tag{4.460}$$
$$Kv_T = g \tag{4.461}$$
$$v_T = \frac{g}{K} \tag{4.462}$$

Thus, the terminal velocity is $v_T = \frac{g}{K}$.

Solving for Velocity as a Function of Time:
We solve the differential equation using an integrating factor:

$$\frac{dv}{dt} + Kv = g \tag{4.463}$$

The integrating factor is $e^{\int K dt} = e^{Kt}$. Multiplying both sides:

$$e^{Kt}\frac{dv}{dt} + Ke^{Kt}v = ge^{Kt} \tag{4.464}$$
$$\frac{d}{dt}(e^{Kt}v) = ge^{Kt} \tag{4.465}$$

Integrating both sides:

$$e^{Kt}v = \int ge^{Kt}dt \tag{4.466}$$

$$e^{Kt}v = \frac{g}{K}e^{Kt} + C \tag{4.467}$$

Therefore:

$$v(t) = \frac{g}{K} + Ce^{-Kt} \tag{4.468}$$

Using the initial condition $v(0) = 0$ (starts from rest):

$$0 = \frac{g}{K} + C \tag{4.469}$$

$$C = -\frac{g}{K} \tag{4.470}$$

This gives us the velocity as a function of time:

$$v(t) = \frac{g}{K}(1 - e^{-Kt}) \tag{4.471}$$

Finding the Time to Reach Half the Terminal Velocity:
We need to find $t_{1/2}$ such that $v(t_{1/2}) = \frac{v_T}{2} = \frac{g}{2K}$:

$$\frac{g}{2K} = \frac{g}{K}(1 - e^{-Kt_{1/2}}) \tag{4.472}$$

$$\frac{1}{2} = 1 - e^{-Kt_{1/2}} \tag{4.473}$$

$$e^{-Kt_{1/2}} = \frac{1}{2} \tag{4.474}$$

$$-Kt_{1/2} = \ln\left(\frac{1}{2}\right) = -\ln(2) \tag{4.475}$$

$$t_{1/2} = \frac{\ln(2)}{K} \tag{4.476}$$

Therefore, the time taken to reach half the terminal velocity is $t_{1/2} = \frac{\ln(2)}{K}$.

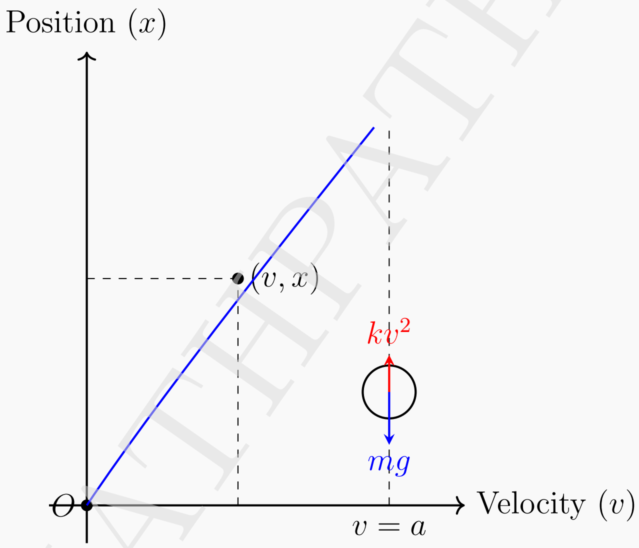
Example 2: Motion with Quadratic Air Resistance

A body of mass m, falling from rest is subjected to the force of gravity and an air-resistance proportional to the square of the velocity 'kv^2'. If it falls through a distance x and possesses a velocity v at that instant, prove that

$$\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right) \tag{4.477}$$

where $mg = ka^2$.

Solution



Let's establish our coordinate system with the positive direction downward (in the direction of gravity). We'll denote velocity as v and position as x, with $x = 0$ at the starting point.

Step 1: Setting up the equation of motion.

Applying Newton's Second Law and using the fact that forces are in the vertical direction:

$$m\frac{dv}{dt} = mg - kv^2 \tag{4.478}$$

The gravitational force mg acts downward (positive), while the air resistance kv^2 acts in the opposite direction to motion (negative, since motion is downward).

Step 2: Converting time derivatives to position derivatives.

We can use the chain rule to convert the time derivative to a position derivative:

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx} \tag{4.479}$$

Substituting this into our equation of motion:

$$mv\frac{dv}{dx} = mg - kv^2 \tag{4.480}$$

$$m\frac{dv}{dx} = \frac{mg - kv^2}{v} \tag{4.481}$$

$$m\frac{dv}{dx} = \frac{mg}{v} - kv \tag{4.482}$$

Step 3: Using the given condition $mg = ka^2$.

From the given condition, we have $mg = ka^2$. Substituting:

$$m\frac{dv}{dx} = \frac{ka^2}{v} - kv \tag{4.483}$$

$$\frac{m}{k}\frac{dv}{dx} = \frac{a^2}{v} - v \tag{4.484}$$

$$\frac{m}{k}\frac{dv}{dx} = \frac{a^2 - v^2}{v} \tag{4.485}$$

Step 4: Rearranging for integration.
We rearrange to prepare for integration:

$$\frac{m}{k} \cdot \frac{v}{a^2 - v^2} dv = dx \tag{4.486}$$

$$\tag{4.487}$$

Step 5: Integrating both sides.
To integrate the left side, we use partial fractions:

$$\frac{v}{a^2 - v^2} = \frac{v}{(a-v)(a+v)} \tag{4.488}$$

$$= \frac{A}{a-v} + \frac{B}{a+v} \tag{4.489}$$

Finding the constants A and B:

$$v = A(a+v) + B(a-v) \tag{4.490}$$

$$v = Aa + Av + Ba - Bv \tag{4.491}$$

$$v = (A+B)a + (A-B)v \tag{4.492}$$

Comparing coefficients:

$$1 = A - B \tag{4.493}$$

$$0 = (A+B)a \tag{4.494}$$

Since $a \neq 0$, we have $A + B = 0$, or $B = -A$. Substituting into $1 = A - B$:

$$1 = A - (-A) \tag{4.495}$$

$$1 = 2A \tag{4.496}$$

$$A = \frac{1}{2} \tag{4.497}$$

$$B = -\frac{1}{2} \tag{4.498}$$

So:

$$\frac{v}{a^2 - v^2} = \frac{1}{2}\left(\frac{1}{a-v} - \frac{1}{a+v}\right) \tag{4.499}$$

$$= \frac{1}{2}\left(\frac{(a+v)-(a-v)}{(a-v)(a+v)}\right) \tag{4.500}$$

$$= \frac{1}{2}\left(\frac{2v}{a^2 - v^2}\right) \tag{4.501}$$

$$= \frac{v}{a^2 - v^2} \tag{4.502}$$

Now we can integrate:

$$\frac{m}{k} \int \frac{v}{a^2 - v^2} dv = \int dx \tag{4.503}$$

$$\tag{4.504}$$

For the left-hand integral, we use the substitution $u = a^2 - v^2$, which gives $du = -2v\,dv$ or $v\,dv = -\frac{du}{2}$:

$$\frac{m}{k}\int \frac{v}{a^2 - v^2}dv = \frac{m}{k}\int \frac{-\frac{du}{2}}{u} \tag{4.505}$$
$$= -\frac{m}{2k}\int \frac{du}{u} \tag{4.506}$$
$$= -\frac{m}{2k}\ln|u| + C \tag{4.507}$$
$$= -\frac{m}{2k}\ln|a^2 - v^2| + C \tag{4.508}$$

The right-hand side gives:

$$\int dx = x + D \tag{4.509}$$

Combining the two sides:

$$-\frac{m}{2k}\ln|a^2 - v^2| + C = x + D \tag{4.510}$$
$$-\frac{m}{2k}\ln|a^2 - v^2| = x + (D - C) \tag{4.511}$$
$$\tag{4.512}$$

Let's denote the constant $(D - C)$ as E:

$$-\frac{m}{2k}\ln|a^2 - v^2| = x + E \tag{4.513}$$
$$\tag{4.514}$$

Step 6: Applying the initial condition.
Since the body starts from rest, we have $v = 0$ at $x = 0$. Substituting:

$$-\frac{m}{2k}\ln|a^2 - 0^2| = 0 + E \tag{4.515}$$
$$-\frac{m}{2k}\ln|a^2| = E \tag{4.516}$$
$$E = -\frac{m}{2k}\ln\bigl(a^2\bigr) \tag{4.517}$$

Step 7: Obtaining the final relation.
Substituting the value of E back:

$$-\frac{m}{2k}\ln|a^2 - v^2| = x - \frac{m}{2k}\ln\bigl(a^2\bigr) \tag{4.518}$$
$$-\frac{m}{2k}\ln|a^2 - v^2| + \frac{m}{2k}\ln\bigl(a^2\bigr) = x \tag{4.519}$$
$$\frac{m}{2k}\bigl(-\ln|a^2 - v^2| + \ln\bigl(a^2\bigr)\bigr) = x \tag{4.520}$$
$$\frac{m}{2k}\ln\left(\frac{a^2}{a^2 - v^2}\right) = x \tag{4.521}$$
$$\tag{4.522}$$

Rearranging:

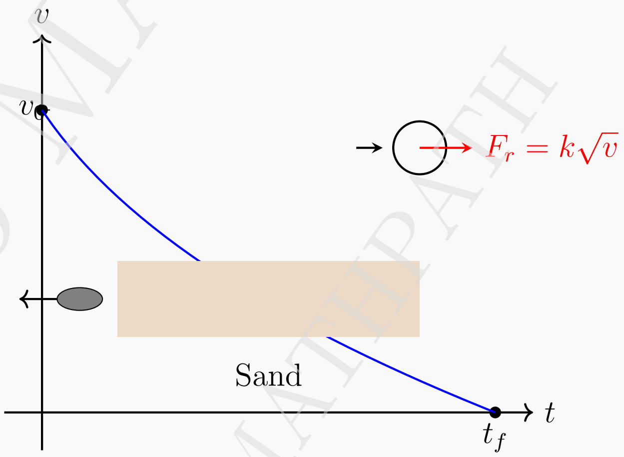
$$\frac{2kx}{m} = \ln\left(\frac{a^2}{a^2 - v^2}\right) \tag{4.523}$$

Which is the relation we needed to prove.
Physical interpretation: The constant a represents the terminal velocity of the falling body, where gravitational force mg exactly balances the air resistance kv^2 when $v = a$.

Example 3: Bullet Motion with Square Root Retardation

A bullet is fired into a sand tank, its retardation is proportional to the square root of its velocity ($k\sqrt{v}$). How long will it take to come to rest if it enters the sand tank with an initial velocity v_0.

Solution



Step 1: Establishing the differential equation.
Given that the retardation (negative acceleration) is proportional to the square root of velocity:

$$a = -k\sqrt{v} \tag{4.524}$$

Where k is a positive constant. Using Newton's Second Law and noting that $a = \frac{dv}{dt}$:

$$\frac{dv}{dt} = -k\sqrt{v} \tag{4.525}$$

Step 2: Separating variables.
We can rearrange the equation to separate variables:

$$\frac{dv}{\sqrt{v}} = -k\,dt \tag{4.526}$$

$$v^{-\frac{1}{2}}dv = -k\,dt \tag{4.527}$$

Step 3: Integrating both sides.

$$\int v^{-\frac{1}{2}}dv = -\int k\,dt \tag{4.528}$$

$$\int v^{-\frac{1}{2}}dv = -k\int dt \tag{4.529}$$

For the left-hand side:

$$\int v^{-\frac{1}{2}} dv = \int v^{-\frac{1}{2}} dv \tag{4.530}$$

$$= 2v^{\frac{1}{2}} + C_1 \tag{4.531}$$

For the right-hand side:

$$-k \int dt = -kt + C_2 \tag{4.532}$$

Combining the two:

$$2v^{\frac{1}{2}} + C_1 = -kt + C_2 \tag{4.533}$$

Let $C = C_2 - C_1$, which gives:

$$2v^{\frac{1}{2}} = -kt + C \tag{4.534}$$

$$v^{\frac{1}{2}} = \frac{-kt + C}{2} \tag{4.535}$$

$$v = \left(\frac{-kt + C}{2}\right)^2 \tag{4.536}$$

Step 4: Applying the initial condition.
Given that $v = v_0$ at $t = 0$:

$$v_0 = \left(\frac{-k(0) + C}{2}\right)^2 \tag{4.537}$$

$$v_0 = \left(\frac{C}{2}\right)^2 \tag{4.538}$$

$$\sqrt{v_0} = \frac{C}{2} \tag{4.539}$$

$$C = 2\sqrt{v_0} \tag{4.540}$$

Substituting back:

$$v^{\frac{1}{2}} = \frac{-kt + 2\sqrt{v_0}}{2} \tag{4.541}$$

$$v^{\frac{1}{2}} = \frac{2\sqrt{v_0} - kt}{2} \tag{4.542}$$

$$v^{\frac{1}{2}} = \sqrt{v_0} - \frac{kt}{2} \tag{4.543}$$

$$v = \left(\sqrt{v_0} - \frac{kt}{2}\right)^2 \tag{4.544}$$

Step 5: Finding the time to come to rest.
The bullet comes to rest when $v = 0$:

$$0 = \left(\sqrt{v_0} - \frac{kt_f}{2}\right)^2 \tag{4.545}$$

The square of a real number is zero only when the number itself is zero:

$$\sqrt{v_0} - \frac{kt_f}{2} = 0 \tag{4.546}$$

$$\frac{kt_f}{2} = \sqrt{v_0} \tag{4.547}$$

$$t_f = \frac{2\sqrt{v_0}}{k} \tag{4.548}$$

Therefore, the time it takes for the bullet to come to rest is:

$$t_f = \frac{2\sqrt{v_0}}{k} \tag{4.549}$$

Step 6: Physical interpretation.
The time required for the bullet to stop is directly proportional to the square root of the initial velocity and inversely proportional to the resistance coefficient.
This makes physical sense because:

- Higher initial velocity (v_0) requires more time to dissipate
- Stronger resistance (larger k) reduces the stopping time
- The square root relationship suggests that doubling the initial velocity increases the stopping time by only about 41

The decay of velocity follows a quadratic pattern:

$$v(t) = \left(\sqrt{v_0} - \frac{kt}{2}\right)^2 \tag{4.550}$$

This equation shows that velocity decreases more rapidly at first and then more slowly as the bullet approaches rest, which aligns with the exponential decay often observed in velocity profiles for objects moving through resistant media.

4.13 Simple Harmonic Motion

4.13.1 Oscillatory Motion Fundamentals

Simple harmonic motion represents one of nature's most ubiquitous phenomena, characterized by periodic oscillations about an equilibrium position. This fundamental motion pattern emerges when restoring forces act proportionally to displacement.

Definition 4.6 (Simple Harmonic Motion)**.** *A particle executes simple harmonic motion when subjected to a force that always directs toward a fixed point on its path, with magnitude proportional to the distance from that point.*

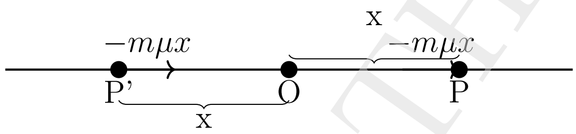
4.13.2 Mathematical Description

Consider a particle of mass m constrained to move along a straight line, with O as the equilibrium position. When displaced by distance x from O, the particle experiences a restoring force:

$$F = -m\mu x \tag{4.551}$$

where μ represents a positive constant.

Force proportional to displacement



4.13.3 Governing Differential Equation

Applying Newton's second law to this system:

$$m\frac{d^2x}{dt^2} = -m\mu x \tag{4.552}$$

Simplifying yields the characteristic equation:

$$\frac{d^2x}{dt^2} + \mu x = 0 \tag{4.553}$$

Substituting $\omega = \sqrt{\mu}$:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{4.554}$$

This second-order differential equation defines simple harmonic motion.

4.13.4 Oscillation Parameters

The motion exhibits periodic behavior with:

- Angular frequency: $\omega = \sqrt{\mu}$
- Period: $T = \frac{2\pi}{\omega}$
- Frequency: $f = \frac{\omega}{2\pi}$

4.13.5 Spring-Mass System Analysis

For a particle of mass m suspended by an elastic spring:
At equilibrium, the spring constant relationship gives:

$$c = \frac{mgl}{\lambda} \tag{4.555}$$

where λ represents the spring modulus.
When displaced by distance d from equilibrium and released, the equation of motion becomes:

$$m\frac{d^2x}{dt^2} = mg - \frac{\lambda(x+c)}{l} \tag{4.556}$$

After algebraic manipulation:

$$\frac{d^2x}{dt^2} = -\frac{\lambda}{lm}x \tag{4.557}$$

This demonstrates that the system exhibits SHM with period:

$$T = 2\pi\sqrt{\frac{lm}{\lambda}} \tag{4.558}$$

For vertical spring systems, the parameter $\frac{mgl}{\lambda}$ represents the static extension.

Hooke's Law

For elastic materials within proportionality limits, the restoring force varies directly with extension:

$$F = -kx \tag{4.559}$$

where k represents the spring constant.

4.13.6 Motion Analysis for Spring-Mass System

The differential equation for a mass attached to a vertical spring under gravity:

$$m\frac{d^2x}{dt^2} = mg - \frac{\lambda x}{l} \tag{4.560}$$

Following equilibrium considerations and variable substitution, this reduces to the standard SHM equation, confirming oscillatory motion with characteristic period determined by physical parameters.

4.14 Solved Examples

Example 1: Mass on an Elastic String

An elastic string without weight of natural length l and modulus of elasticity being weight of n-grams, is suspended by one end, and a mass m is attached to the other. Show that the time of oscillations is $2\pi\sqrt{\frac{ml}{ng}}$.

Solution

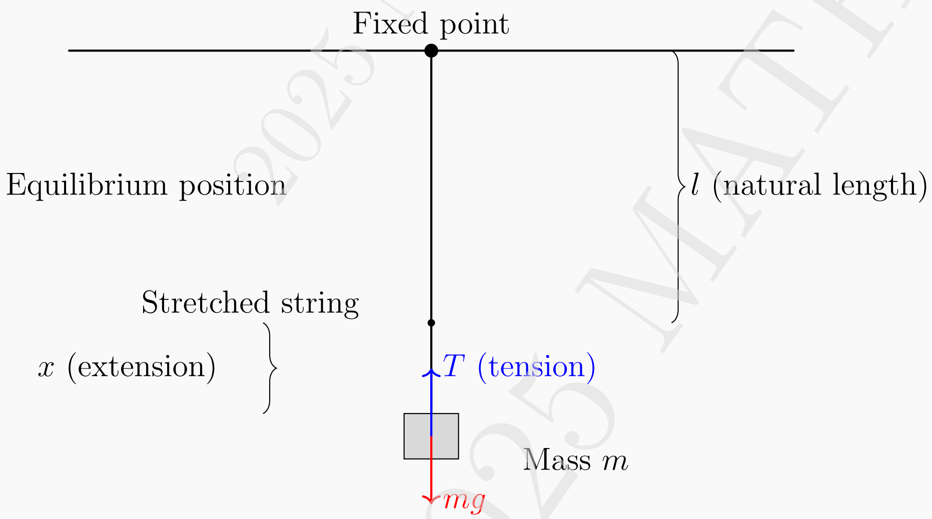


Figure 4.12: Mass suspended on an elastic string

To solve this problem, we need to:

1. Identify the equilibrium position
2. Find the equation of motion for small displacements
3. Determine the period of oscillation

Let's begin by analyzing the system at equilibrium. When the mass is at rest, the tension in the string balances the weight of the mass. If the extension of the string at equilibrium is e, then by Hooke's law:

$$T = \frac{\lambda e}{l} = mg \tag{4.561}$$

Where λ is the modulus of elasticity. Since λ is given as the weight of n-grams, we have $\lambda = ng$. Thus:

$$\frac{ng \cdot e}{l} = mg \tag{4.562}$$

Solving for the equilibrium extension:

$$e = \frac{mgl}{ng} = \frac{ml}{n} \tag{4.563}$$

Now, let's consider what happens when the mass is displaced from equilibrium. Let y be the displacement from the equilibrium position (positive downward). The total extension of the string becomes $e + y$, and the tension is:

$$T = \frac{ng(e+y)}{l} = \frac{ng}{l}\left(\frac{ml}{n} + y\right) = mg + \frac{ngy}{l} \tag{4.564}$$

The net force acting on the mass is:

$$F = mg - T = mg - \left(mg + \frac{ngy}{l}\right) = -\frac{ngy}{l} \tag{4.565}$$

Applying Newton's second law:

$$m\frac{d^2y}{dt^2} = -\frac{ngy}{l} \tag{4.566}$$

Rearranging:

$$\frac{d^2y}{dt^2} + \frac{ng}{ml}y = 0 \tag{4.567}$$

This is in the standard form of the SHM equation:

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \tag{4.568}$$

Where $\omega^2 = \frac{ng}{ml}$. The angular frequency of the oscillation is:

$$\omega = \sqrt{\frac{ng}{ml}} \tag{4.569}$$

Therefore, the period of oscillation is:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ml}{ng}} \tag{4.570}$$

Which confirms the result we needed to show.

Example 2: Oscillating Particle with Impulse

A particle is oscillating in a straight line about a centre of force O, towards which when at a distance x the force is mn^2x and a is the amplitude of the oscillation. When at a distance $\frac{a\sqrt{3}}{2}$ from O, the particle receives a blow in the direction of motion which generates a velocity na. If this velocity be away from O, show that the new amplitude is $a\sqrt{3}$.

Solution

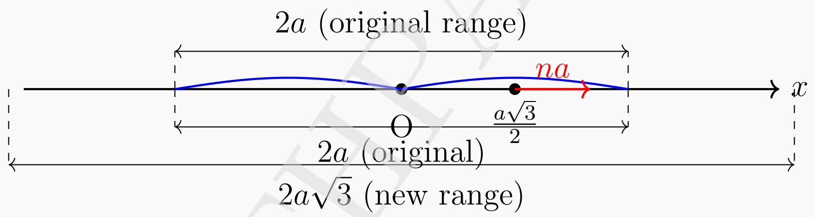


Figure 4.13: Particle oscillating with an impulse applied at position $\frac{a\sqrt{3}}{2}$

First, let's establish the equation of motion for the particle before the impulse. Given that the force is $F = -mn^2x$ (negative since it's directed toward O), we have a simple harmonic motion with angular frequency $\omega = n$. The general solution for such motion is:

$$x = A\cos(nt) + B\sin(nt) \tag{4.571}$$

Where A and B are constants determined by initial conditions. Since the amplitude is a, we can write:

$$x = a\cos(nt + \phi) \tag{4.572}$$

Where ϕ is a phase constant. Without loss of generality, we can choose ϕ such that at $t = 0$, the particle is at position $x = \frac{a\sqrt{3}}{2}$ (where the blow occurs) and moving away from O.

To determine ϕ, we set:

$$\frac{a\sqrt{3}}{2} = a\cos\phi \tag{4.573}$$

Solving for ϕ:

$$\cos\phi = \frac{\sqrt{3}}{2} \implies \phi = \frac{\pi}{6} \tag{4.574}$$

So the equation of motion before the blow is:

$$x = a\cos\left(nt + \frac{\pi}{6}\right) \tag{4.575}$$

The velocity at any time is:

$$v = \frac{dx}{dt} = -an\sin\left(nt + \frac{\pi}{6}\right) \tag{4.576}$$

At the moment of the blow ($t = 0$), the position is $x = \frac{a\sqrt{3}}{2}$ and the velocity is:

$$v_0 = -an\sin\left(\frac{\pi}{6}\right) = -an \cdot \frac{1}{2} = -\frac{an}{2} \tag{4.577}$$

The negative sign indicates the particle is moving toward O. However, the problem states the blow generates a velocity na away from O. This means after the blow, the velocity becomes:

$$v_1 = -\frac{an}{2} + na = na - \frac{an}{2} = \frac{3an}{2} \tag{4.578}$$

Now, we need to find the new amplitude. In SHM, the total energy is conserved and is given by:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}mn^2x^2 \tag{4.579}$$

When the particle is at maximum displacement (amplitude), the velocity is zero, so all energy is potential:

$$E = \frac{1}{2}mn^2A^2 \tag{4.580}$$

where A is the amplitude. After the blow, the energy is:

$$E_1 = \frac{1}{2}m\left(\frac{3an}{2}\right)^2 + \frac{1}{2}mn^2\left(\frac{a\sqrt{3}}{2}\right)^2 \tag{4.581}$$

Simplifying:

$$E_1 = \frac{1}{2}m\frac{9a^2n^2}{4} + \frac{1}{2}mn^2\frac{3a^2}{4} = \frac{1}{2}ma^2n^2\left(\frac{9}{4} + \frac{3}{4}\right) = \frac{1}{2}ma^2n^2 \cdot 3 \tag{4.582}$$

If we denote the new amplitude as A_1, then:

$$\frac{1}{2}mn^2A_1^2 = \frac{1}{2}ma^2n^2 \cdot 3 \tag{4.583}$$

Solving for A_1:

$$A_1^2 = 3a^2 \implies A_1 = a\sqrt{3} \tag{4.584}$$

Therefore, the new amplitude is $a\sqrt{3}$, which is what we needed to show.
An alternative approach is to consider the new motion as a superposition of two simple harmonic motions with the same frequency but different phases. The resulting motion would still be simple harmonic with a new amplitude and phase, which we can calculate from the position and velocity at the moment of the blow.