Chapter 1

Prerequisites and Mathematical Foundations

1.1 Review of Calculus Concepts

Before diving into differential equations, it's essential to review key calculus concepts that form the foundation of our study. This section provides a refresher on differentiation rules, integration techniques, and series expansions that will be used extensively throughout this book.

1.1.1 Differentiation Rules

Differential equations, by definition, involve derivatives of functions. Let's review the fundamental differentiation rules that will be used extensively in our study.

Basic Differentiation Rules

If $u = u(x)$ and $v = v(x)$ are differentiable functions and c is a constant, then:

$$\frac{d}{dx}(c) = 0 \quad \text{(Constant Rule)} \tag{1.1}$$

$$\frac{d}{dx}(x) = 1 \quad \text{(Identity Rule)} \tag{1.2}$$

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \text{(Power Rule)} \tag{1.3}$$

$$\frac{d}{dx}(cu) = c\frac{du}{dx} \quad \text{(Constant Multiplication Rule)} \tag{1.4}$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx} \quad \text{(Sum/Difference Rule)} \tag{1.5}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{(Product Rule)} \tag{1.6}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \text{(Quotient Rule)} \tag{1.7}$$

Chain Rule

If $y = f(u)$ and $u = g(x)$ where f and g are differentiable functions, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x) \tag{1.8}$$

Applying Differentiation Rules

Find the derivative of $f(x) = x^3 \sin(x^2)$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[x^3 \sin(x^2)] && (1.9)\\
&= x^3 \frac{d}{dx}[\sin(x^2)] + \sin(x^2)\frac{d}{dx}[x^3] \quad \text{(Product Rule)} && (1.10)\\
&= x^3 \cdot \cos(x^2) \cdot \frac{d}{dx}[x^2] + \sin(x^2) \cdot 3x^2 \quad \text{(Chain Rule \& Power Rule)} && (1.11)\\
&= x^3 \cdot \cos(x^2) \cdot 2x + 3x^2 \sin(x^2) && (1.12)\\
&= 2x^4 \cos(x^2) + 3x^2 \sin(x^2) && (1.13)
\end{aligned}$$

Derivatives of Common Functions

$$\frac{d}{dx}[\sin(x)] = \cos(x) \tag{1.14}$$
$$\frac{d}{dx}[\cos(x)] = -\sin(x) \tag{1.15}$$
$$\frac{d}{dx}[\tan(x)] = \sec^2(x) \tag{1.16}$$
$$\frac{d}{dx}[e^x] = e^x \tag{1.17}$$
$$\frac{d}{dx}[\ln(x)] = \frac{1}{x} \tag{1.18}$$
$$\frac{d}{dx}[a^x] = a^x \ln(a) \tag{1.19}$$
$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}} \tag{1.20}$$
$$\frac{d}{dx}[\cos^{-1}(x)] = -\frac{1}{\sqrt{1-x^2}} \tag{1.21}$$
$$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2} \tag{1.22}$$

Practice Problem

Find the derivative of $f(x) = \frac{e^x \ln(x)}{x^2+1}$.

Solution

Using the quotient rule:

$$f'(x) = \frac{d}{dx}\left[\frac{e^x \ln(x)}{x^2+1}\right] \tag{1.23}$$
$$= \frac{(x^2+1)\frac{d}{dx}[e^x \ln(x)] - e^x \ln(x)\frac{d}{dx}[x^2+1]}{(x^2+1)^2} \tag{1.24}$$

For the numerator's first term, we use the product rule:

$$\frac{d}{dx}[e^x \ln(x)] = e^x \ln(x) + e^x \cdot \frac{1}{x} \tag{1.25}$$

$$= e^x \ln(x) + \frac{e^x}{x} \tag{1.26}$$

For the numerator's second term:

$$\frac{d}{dx}[x^2 + 1] = 2x \tag{1.27}$$

Substituting back:

$$f'(x) = \frac{(x^2+1)\left(e^x \ln(x) + \frac{e^x}{x}\right) - e^x \ln(x) \cdot 2x}{(x^2+1)^2} \tag{1.28}$$

$$= \frac{(x^2+1)e^x \ln(x) + (x^2+1)\frac{e^x}{x} - 2xe^x \ln(x)}{(x^2+1)^2} \tag{1.29}$$

$$= \frac{e^x \ln(x)(x^2+1-2x) + \frac{e^x}{x}(x^2+1)}{(x^2+1)^2} \tag{1.30}$$

$$= \frac{e^x \ln(x)(x^2-2x+1) + e^x(x+\frac{1}{x})}{(x^2+1)^2} \tag{1.31}$$

$$= \frac{e^x \ln(x)(x-1)^2 + e^x \frac{x^2+1}{x}}{(x^2+1)^2} \tag{1.32}$$

$$= \frac{e^x \left[\ln(x)(x-1)^2 + \frac{x^2+1}{x}\right]}{(x^2+1)^2} \tag{1.33}$$

Higher-Order Derivatives

The second derivative of a function $f(x)$ is denoted as:

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) \tag{1.34}$$

Similarly, the n-th derivative is denoted as:

$$f^{(n)}(x) = \frac{d^n f}{dx^n} \tag{1.35}$$

Applications in Differential Equations

Higher-order derivatives appear naturally in the mathematical modeling of physical systems. For example:

- In mechanics, the position $x(t)$ of an object relates to velocity $v(t) = \frac{dx}{dt}$ and acceleration $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$
- Newton's second law $F = ma$ can be written as $F = m\frac{d^2x}{dt^2}$, creating a second-order differential equation
- In circuit theory, the relationship between current and voltage across an inductor is $V_L = L\frac{dI}{dt}$

1.1.2 Integration Techniques

Integration is the inverse operation of differentiation, and its techniques are essential for solving differential equations.

Basic Integration Rules

If $u = u(x)$ and $v = v(x)$ are functions with continuous derivatives and c is a constant, then:

$$\int c\,dx = cx + C \quad \text{(Constant Rule)} \tag{1.36}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{(Power Rule, } n \neq -1\text{)} \tag{1.37}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \tag{1.38}$$

$$\int (u \pm v)\,dx = \int u\,dx \pm \int v\,dx \quad \text{(Linearity)} \tag{1.39}$$

$$\int cu\,dx = c\int u\,dx \quad \text{(Constant Multiple Rule)} \tag{1.40}$$

Integration by Substitution

If $u = g(x)$ is a differentiable function and f is continuous, then:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \tag{1.41}$$

where $du = g'(x)\,dx$.

Integration by Substitution

Evaluate $\int x\sin(x^2)\,dx$.
Let $u = x^2$, then $du = 2x\,dx$ or $x\,dx = \frac{du}{2}$. Substituting:

$$\int x\sin\left(x^2\right)dx = \int \sin(u)\cdot\frac{du}{2} \tag{1.42}$$

$$= \frac{1}{2}\int \sin(u)\,du \tag{1.43}$$

$$= \frac{1}{2}(-\cos(u)) + C \tag{1.44}$$

$$= -\frac{1}{2}\cos\left(x^2\right) + C \tag{1.45}$$

Integration by Parts

For differentiable functions $u(x)$ and $v(x)$:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx \tag{1.46}$$

In the form $\int u\,dv = uv - \int v\,du$, where $dv = v'(x)\,dx$ and $du = u'(x)\,dx$.

Integration by Parts

Evaluate $\int x\cos(x)\,dx$.
Let $u = x$ and $dv = \cos(x)\,dx$. Then $du = dx$ and $v = \sin(x)$.

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx \tag{1.47}$$
$$= x\sin(x) - (-\cos(x)) + C \tag{1.48}$$
$$= x\sin(x) + \cos(x) + C \tag{1.49}$$

Integration of Rational Functions

For rational functions, we use partial fraction decomposition:

$$\int \frac{P(x)}{Q(x)}\,dx \tag{1.50}$$

where $P(x)$ and $Q(x)$ are polynomials with $\deg(P) < \deg(Q)$.
The decomposition depends on the factorization of $Q(x)$, yielding forms like:

$$\frac{A}{(x-a)} \quad \text{or} \quad \frac{Ax+B}{(x^2+px+q)} \tag{1.51}$$

Partial Fraction Decomposition

Evaluate $\int \frac{3x+2}{x^2-x-2}\,dx$.
First, we factor the denominator: $x^2 - x - 2 = (x-2)(x+1)$
Then we write the partial fraction decomposition:

$$\frac{3x+2}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \tag{1.52}$$

Multiplying both sides by $(x-2)(x+1)$:

$$3x+2 = A(x+1) + B(x-2) \tag{1.53}$$
$$3x+2 = Ax + A + Bx - 2B \tag{1.54}$$
$$3x+2 = (A+B)x + (A-2B) \tag{1.55}$$

Comparing coefficients:

$$A + B = 3 \tag{1.56}$$
$$A - 2B = 2 \tag{1.57}$$

Solving: $A = \frac{8}{3}$ and $B = \frac{1}{3}$
Now we can integrate:

$$\int \frac{3x+2}{x^2-x-2}\,dx = \int \left(\frac{8/3}{x-2} + \frac{1/3}{x+1}\right)dx \tag{1.58}$$
$$= \frac{8}{3}\int \frac{1}{x-2}\,dx + \frac{1}{3}\int \frac{1}{x+1}\,dx \tag{1.59}$$
$$= \frac{8}{3}\ln|x-2| + \frac{1}{3}\ln|x+1| + C \tag{1.60}$$

Common Integration Formulas

$$\int \sin(x)\,dx = -\cos(x) + C \tag{1.61}$$
$$\int \cos(x)\,dx = \sin(x) + C \tag{1.62}$$
$$\int \sec^2(x)\,dx = \tan(x) + C \tag{1.63}$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C \tag{1.64}$$
$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C \tag{1.65}$$
$$\int e^x\,dx = e^x + C \tag{1.66}$$
$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \quad (a > 0, a \neq 1) \tag{1.67}$$

Practice Problem

Evaluate $\int \frac{x^2}{(x-1)(x^2+1)}\,dx$.

Solution

First, we use partial fraction decomposition:

$$\frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \tag{1.68}$$

Multiplying both sides by $(x-1)(x^2+1)$:

$$x^2 = A(x^2+1) + (Bx+C)(x-1) \tag{1.69}$$
$$x^2 = Ax^2 + A + Bx^2 - Bx + Cx - C \tag{1.70}$$
$$x^2 = (A+B)x^2 + (C-B)x + (A-C) \tag{1.71}$$

Comparing coefficients:

$$A + B = 1 \tag{1.72}$$
$$C - B = 0 \tag{1.73}$$
$$A - C = 0 \tag{1.74}$$

This gives us $A = C$, $B = C$, and $A + B = 1$. Therefore, $A + A = 1$, so $A = \frac{1}{2}$, which means $B = \frac{1}{2}$ and $C = \frac{1}{2}$.
Now we can integrate:

$$\int \frac{x^2}{(x-1)(x^2+1)}\,dx = \int \left(\frac{1/2}{x-1} + \frac{(1/2)x + (1/2)}{x^2+1} \right) dx \tag{1.75}$$
$$= \frac{1}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx \tag{1.76}$$

For the first integral: $\int \frac{1}{x-1}\,dx = \ln|x-1| + C_1$
For the second integral: $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C_2$

For the third integral: $\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C_3$
Combining these results:

$$\int \frac{x^2}{(x-1)(x^2+1)}\,dx = \frac{1}{2}\ln|x-1| + \frac{1}{4}\ln\left(x^2+1\right) + \frac{1}{2}\tan^{-1}(x) + C \tag{1.77}$$

(1.78)

Applications in Differential Equations

Integration is central to solving differential equations. For instance:

- The general solution to a first-order separable differential equation involves integration
- The method of integrating factors for linear first-order equations relies on integration techniques
- Finding the particular solution to higher-order differential equations often requires integration

1.1.3 Power Series and Taylor Series

Power series and Taylor series provide powerful tools for representing functions and solving differential equations that may not have elementary solutions.

Power Series

A power series centered at $x = a$ has the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \tag{1.79}$$

where c_n are constants. When $a = 0$, it's called a Maclaurin series.

Taylor Series

For a function $f(x)$ that is infinitely differentiable at $x = a$, its Taylor series is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \tag{1.80}$$

Taylor Series Expansion

Find the Taylor series for $f(x) = e^x$ centered at $a = 0$ (Maclaurin series).
We know that $f^{(n)}(x) = e^x$ for all $n \geq 0$. So $f^{(n)}(0) = e^0 = 1$ for all n.
Therefore:

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{1.81}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}x^n \tag{1.82}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{1.83}$$

Common Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{1.84}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{1.85}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{1.86}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad \text{for } |x| < 1 \tag{1.87}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1 \tag{1.88}$$

Operations on Power Series

Given power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, with radii of convergence R_a and R_b respectively:

1. **Addition/Subtraction**: $\sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ with radius of convergence $R = \min(R_a, R_b)$
2. **Multiplication**: $\sum_{n=0}^{\infty} c_n x^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ with radius of convergence $R = \min(R_a, R_b)$
3. **Differentiation**: $\frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}$ with radius of convergence $R = R_a$
4. **Integration**: $\int \sum_{n=0}^{\infty} a_n x^n dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ with radius of convergence $R = R_a$

Differentiation of Power Series

Find the derivative of $f(x) = \sin(x)$ using its Maclaurin series.
The Maclaurin series for $\sin(x)$ is:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{1.89}$$

Differentiating term by term:

$$\frac{d}{dx} \sin(x) = \frac{d}{dx} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right] \tag{1.90}$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \cdots \tag{1.91}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{1.92}$$

This is the Maclaurin series for $\cos(x)$, confirming that $\frac{d}{dx} \sin(x) = \cos(x)$.

Practice Problem

Find the first four terms of the Taylor series for $f(x) = \ln(x)$ centered at $a = 1$.

Solution

To find the Taylor series of $f(x) = \ln(x)$ centered at $a = 1$, we need the derivatives evaluated at $x = 1$:
$f(x) = \ln(x)$, so $f(1) = \ln(1) = 0$
$f'(x) = \frac{1}{x}$, so $f'(1) = 1$
$f''(x) = -\frac{1}{x^2}$, so $f''(1) = -1$
$f'''(x) = \frac{2}{x^3}$, so $f'''(1) = 2$
$f^{(4)}(x) = -\frac{6}{x^4}$, so $f^{(4)}(1) = -6$
Using the Taylor series formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1.93}$$

$$= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \cdots \tag{1.94}$$

$$= 0 + 1 \cdot (x-1) + \frac{-1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 + \frac{-6}{24}(x-1)^4 + \cdots \tag{1.95}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots \tag{1.96}$$

Therefore, the first four terms of the Taylor series for $\ln(x)$ centered at $a = 1$ are:

$$\ln(x) \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} \tag{1.97}$$

This can be recognized as the beginning of the series $\ln(x) = \ln(1 + (x-1)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$.

Error Estimation in Taylor Series

Estimate the error in approximating $e^{0.2}$ using the third-degree Taylor polynomial of e^x at $a = 0$.
The third-degree Taylor polynomial is:

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \tag{1.98}$$

The error is bounded by:

$$|R_3(0.2)| = \left|\frac{f^{(4)}(\xi)}{4!}(0.2)^4\right| \quad \text{for some } \xi \in [0, 0.2] \tag{1.99}$$

$$= \left|\frac{e^{\xi}}{24}(0.2)^4\right| \tag{1.100}$$

Since e^ξ is increasing, for $\xi \in [0, 0.2]$, we have $e^\xi \leq e^{0.2} < e^{0.3} < 1.35$.

$$|R_3(0.2)| < \frac{1.35}{24}(0.2)^4 \tag{1.101}$$
$$= \frac{1.35}{24} \cdot 0.0016 \tag{1.102}$$
$$= \frac{1.35 \cdot 0.0016}{24} \tag{1.103}$$
$$< \frac{1.35 \cdot 0.002}{24} \tag{1.104}$$
$$= \frac{0.0027}{24} \tag{1.105}$$
$$\approx 0.0001125 \tag{1.106}$$

Therefore, the error in approximating $e^{0.2}$ using $P_3(0.2)$ is less than 0.00012.
The actual value is:

$$P_3(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} \tag{1.107}$$
$$= 1 + 0.2 + 0.02 + \frac{0.008}{6} \tag{1.108}$$
$$= 1.2 + 0.02 + \frac{0.008}{6} \tag{1.109}$$
$$\approx 1.22 + 0.00133 \tag{1.110}$$
$$\approx 1.22133 \tag{1.111}$$

While $e^{0.2} \approx 1.2214$, confirming our error estimate.

Radius of Convergence

For a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, the radius of convergence R is defined as:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \tag{1.112}$$

The series converges absolutely for $|x-a| < R$, diverges for $|x-a| > R$, and may converge or diverge when $|x-a| = R$.

Finding the Radius of Convergence

Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n}{2^n} x^n$.

Using the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{n+1}{2^{n+1}}x^{n+1}}{\frac{n}{2^n}x^n}\right| \tag{1.113}$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{1}{2}\cdot x\right| \tag{1.114}$$

$$= \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{|x|}{2} \tag{1.115}$$

$$= 1\cdot\frac{|x|}{2} \tag{1.116}$$

$$= \frac{|x|}{2} \tag{1.117}$$

For convergence, we need $\frac{|x|}{2} < 1$, which gives $|x| < 2$. Therefore, the radius of convergence is $R = 2$.

1.1.4 Convergence and Divergence Tests

Understanding when a series converges is crucial for the application of power series methods in differential equations.

Common Convergence Tests

For a series $\sum_{n=1}^{\infty} a_n$:

1. **Divergence Test**: If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.
2. **Comparison Test**: If $0 \leq a_n \leq b_n$ for all $n \geq N$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $a_n \geq b_n > 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
3. **Ratio Test**: If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, then:
   - If $L < 1$, the series converges absolutely.
   - If $L > 1$ or $L = \infty$, the series diverges.
   - If $L = 1$, the test is inconclusive.
4. **Root Test**: If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$, then:
   - If $L < 1$, the series converges absolutely.
   - If $L > 1$ or $L = \infty$, the series diverges.
   - If $L = 1$, the test is inconclusive.
5. **Integral Test**: If $f(x)$ is a positive, continuous, decreasing function for $x \geq 1$ with $f(n) = a_n$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Applications to Differential Equations

Series convergence is vital when solving differential equations using power series methods. For example:

- The interval of convergence defines where a series solution to a differential equation is valid
- Understanding the convergence properties helps in analyzing the behavior of solutions near singular points
- Convergence tests allow us to determine when asymptotic series approximations are valid in perturbation methods