ENSIMAG – Grenoble INP – UGA

Introduction to Statistical Learning and Applications

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TD 1: Exercises on multivariate statistics

▶ Exercise 1

Let U and V be two independent random variables with uniform distribution over $[0, 1]$.

Let $X = U + V$ and $Y = U - V$.

(a) Compute the expectation and covariance matrix of $Z = \begin{pmatrix} X & Y \end{pmatrix}^T$.

(b) Prove that X and Y are uncorrelated but not independent.

▶ Exercise 2

Let X be a random vector in $\mathbb{R}^n$ and A be a deterministic $m \times n$ matrix.

(a) Prove that

$$K_X = \mathbb{E}\left[(X - \mathbb{E}[X])\ (X - \mathbb{E}[X])^T\right] = \mathbb{E}[XX^T] - \mathbb{E}[X]\ \mathbb{E}[X]^T\ .$$

(b) Prove that

$$K_{AX} = A\ K_X\ A^T\ .$$

(c) Use the result obtained in (b) to derive again the results of Exercise 1.

▶ Exercise 3

Let $Z = \begin{pmatrix} X & Y \end{pmatrix}^T$ be a Gaussian vector with mean $\mu = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$ and covariance $\Sigma = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

(a) Compute the probability density function of Z.

(b) Using

$$f_{Y|X=x}(y) = \frac{f_{(X,Y)}(x, y)}{f_X(x)}$$

compute the distribution of Y given $X = x$.

(c) What is the best prediction of Y given $X = x$?

▶ Exercise 4

Let $\begin{pmatrix} X & Y \end{pmatrix}^T$ be a Gaussian vector in $\mathbb{R}^2$. Let $Z = Y - \mathbb{E}[Y] - \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}[X]}\,(X - \mathbb{E}[X])$.

(a) Compute $\mathbb{E}[Z]$ and $\mathrm{Var}[Z]$.

(b) Prove that X and Z are independent.

(c) Derive the distribution of Y given $X = x$.

(d) Use (c) to derive again the result of Exercise 3.

► Exercise 5

Consider the Gaussian simple linear regression model presented in class

$$Y = \beta_0 + \beta_1 X_1 + \varepsilon \quad \text{with} \quad \varepsilon \sim \mathcal{N}(0, \sigma^2)$$

The estimates for the parameters of the model, $\hat{\beta}_0$ and $\hat{\beta}_1$, are obtained N paired samples (x_i, y_i).

(a) Show that the estimated parameters are unbiased.

(b) Show that

$$\mathrm{Var}(\hat{\beta}_1) = \frac{\sigma^2}{N}\frac{1}{s_X^2} \quad \text{and} \quad \mathrm{Var}(\hat{\beta}_0) = \frac{\sigma^2}{N}\left(1 + \frac{\bar{X}^2}{s_X^2}\right)$$

where $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$ and $s_X^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{X})^2$.

Using the estimated parameters, we can predict that for a given arbitrary value of X, say x (sometimes called the operation point), we have that on average Y will be

$$\hat{m}(x) = \hat{\beta}_0 + \hat{\beta}_1 x$$

(c) Show that

$$\mathbb{E}\Big[\hat{m}(x)\Big] = \beta_0 + \beta_1 x$$

(d) Show that

$$\mathrm{Var}\Big(\hat{m}(x)\Big) = \frac{\sigma^2}{N}\left(1 + \frac{(x - \bar{X})^2}{s_X^2}\right)$$

Describe how the variance changes for different choices of the operation point.