Ensimag 2nd year **Statistical Analysis and Document Mining**
MSIAM 1st year

Worksheet n°4

Exercise 1

Table 1 gives the measures of 6 economic indicators in 12 countries in 1991.
Interpret the results performed by a normalized PCA applied on this table. Results are summarized in figure 1. We also give the eigenvalues of the variance/covariance matrix.

i	1	2	3	4	5	6
λ_i	2.67	1.50	1.16	0.53	0.08	0.06

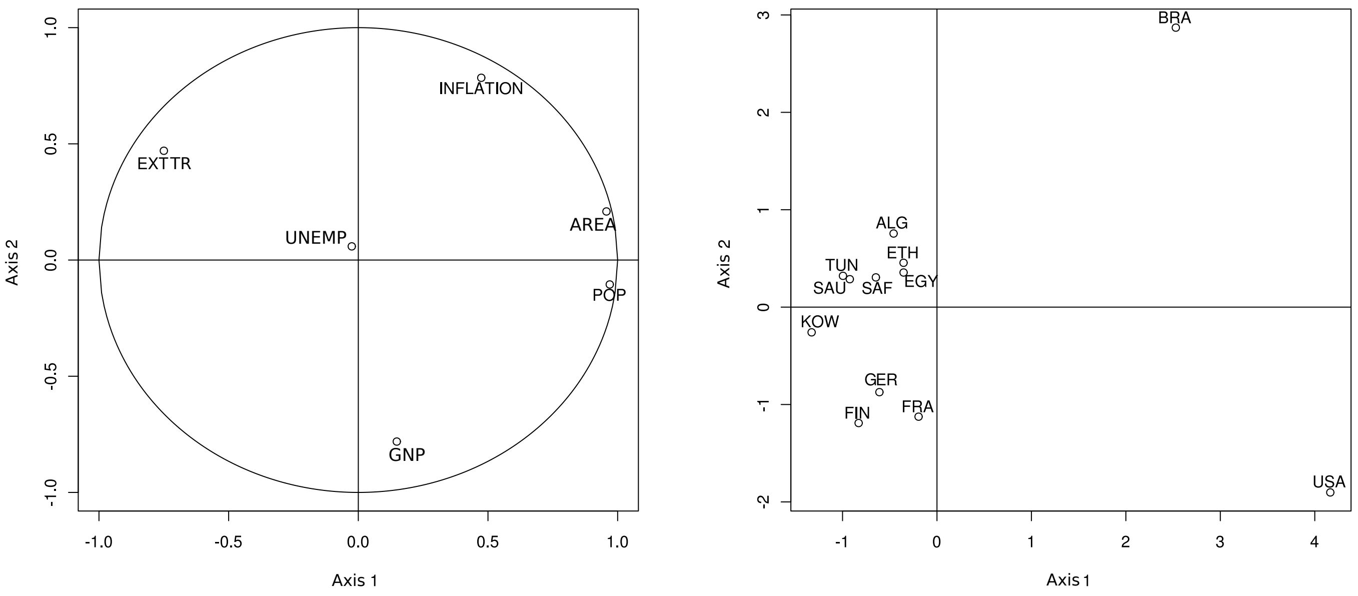
Country	Per capita GNP	Inflation (%)	Unemployment rate (%)	External trade (M $)	Population (M cap.)	Area (M km^2)
South Africa	2810	14,7	0,0	6,3	36,8	1,22
Algeria	1540	50,0	24,3	4,2	26,1	2,38
Germany	24130	3,5	5,1	23,5	81,0	0,35
Saudi Arabia	7328	4,4	0,0	22,1	14,8	2,15
Bresil	2400	440,8	4,8	10,5	153,0	8,51
Egypt	620	19,8	17,5	-5,9	56,1	1,00
USA	21890	4,2	6,7	-73,4	255,0	9,36
Ethiopia	110	35,7	0,0	-0,6	54,6	1,22
Finland	25800	4,1	7,7	2,2	5,0	0,33
France	21030	3,2	9,4	-10,1	57,2	0,55
Koweit	14000	3,3	0,0	20,0	1,3	0,02
Tunisia	1350	8,2	15,0	-0,9	8,6	0,16

Table 1: Economic data 1991

Indications.

- a): If the orthogonal projection of a variable on one principal axis (here, one of the first two principal axes) is close to 1 (respectively -1), i.e., close to the edge of the correlation circle, this variable is highly positively (respectively negatively) correlated with this axis.
- In a): Positively correlated variables are close; negatively correlated variables are at antipodal positions; orthogonal variables are uncorrelated.

  Caution: the correlation between variables must be investigated in a factorial plane (or subspace) only when they are "well represented" (close to the edge of the correlation circle) in this factorial plane (here the first factorial plane given by the first two principal axes 1 and 2). For example, the variable UNEMP (Unemployment rate) is not well represented in the first factorial plane, while the variable POP is well represented.



a) Variable space

b) Samples space

Figure 1: Principal component representation in the first plane of the variable and of the sample spaces.

- b): Similarity/dissimilarity between individuals (in the first plane) is given by their proximity/distance in the factorial plane b). This similarity/dissimilarity is mainly supported by Axis 1, then Axis 2.

  Caution: Let's not look for any proximity between the individuals and the variables. Only the directions are important here.

- a): When a variable is not well represented (close to the origin of the first factorial plane like UNEMP in a)), that means the main difference between individuals is not explained by this variable (in the first plan).

- b): Similarly, when an individual is not well represented (close to the origin of the factorial plane like EGY (Egypt) in b)), that means the main difference between the values taken by the variables is not explained by this individual (in the first plan).

- a): Group of variables (highly) correlated with a given principal axis can be considered in order to give an "interpretation" to this axis. This interpretation must correspond to a notion that is common to all of these variables. Here, for example, variables AREA (Area) and POP (Population) are highly (positively) correlated with the Axis 1 (EXTTR (External Trade) is also (negatively) correlated with this axis but not as much as AREA and POP). Furthermore, both AREA and POP can be related to the notion of "size".

  But we must be careful with this type of interpretation which sometimes requires a good knowledge of the concerned variables.

- The first plane is "enough" to explain the similarity/dissimilarity between individuals (and the correlation/un-correlation between variables) if the percentage of variance explained by the axes 1 and 2 is large "enough". This percentage of explained variance is here equal to $\frac{\sum_1^2 \lambda_i}{\sum_1^6 \lambda_i} = 0.695 \approx 0.70 = 70\%$

Exercice 2

We are interested by some cars models from 2004. Each car is described by 11 variables in table 2.

Variable	Meaning
Retail	Builder recommended price(US$)
Dealer	Seller price (US$)
Engine	Motor capacity (liters)
Cylinders	Number of cylinders in the motor
Horsepower	Engine power
CityMPG	Consumption in city (Miles or gallon; proportional to km/liter)
HighwayMPG	Consumption on roadway (Miles or gallon)
Weight	Weight (pounds)
Wheelbase	Distance between front and rear wheels (inches)
Length	Length (inches)
Width	Width (inches)

Table 2: Meaning of cars variabless

The aim of this exercise is to summarize and interpret the data using PCA.

Question 1

Using PCA performed by `R` we obtain:

```
>cars04.pca <- prcomp(cars04[,8:18], scale=TRUE)
>summary(cars04.pca)
                         PC1  PC2  PC3  PC4   PC5   PC6   PC7  PC8  PC9   PC10  PC11
Standard deviation      2.66 1.37 0.92 0.59  0.52  0.44  0.37 0.29 0.25  0.19  0.02
Proportion of Variance  0.64 0.17 0.07 0.03  0.02  0.01  0.01 0.00 0.00  0.00  0.00
Cumulative Proportion   0.64 0.81 0.89 0.92  0.95  0.96  0.98 0.99 0.99  0.99  1.00
```

a) What does the argument scale=TRUE do?

b) Does the representation in the first two principal components give a good idea of dataset variations?

Indications.

a) scale=TRUE indicates that the variables are scaled to have unit variance before the analysis.

b) Line "Cumulative Proportion" indicates that 0.81 = 81% of the dataset variations are explained by the first two principal components. This represents a fairly large proportion of the total variations.

Question 2

Principal components are linear combinations of the 11 centered and scaled variables. The coefficients of the first 2 principal components on these 11 variables are:

```
> round(cbind(cars04.pca$rotation[,1:2]%*%diag(cars04.pca$sdev[1:2]),
             cars04.pca$rotation[,1:2]),2)
             ???   ???   PC1   PC2
Retail     -0.79 -0.56 -0.28 -0.45
Dealer     -0.78 -0.56 -0.28 -0.46
Engine     -0.94  0.03 -0.34  0.02
Cylinders  -0.90 -0.18 -0.32 -0.15
Horsepower -0.89 -0.36 -0.32 -0.29
CityMPG     0.90 -0.05  0.32 -0.04
HighwayMPG  0.84 -0.02  0.30 -0.02
Weight     -0.90  0.19 -0.32  0.15
Wheelbase  -0.75  0.51 -0.27  0.42
Length     -0.74  0.50 -0.26  0.41
Width      -0.83  0.41 -0.30  0.34
```

Can you give an interpretation of each of these new variables?

Indications.

The last two columns PC1 and PC2 (or $\mathrm{PC}_j, j = 1, 2$) are the first two columns of the matrix containing the eigenvectors of the covariance matrix in the PCA analysis. The first two columns ???, which can be renamed $\mathrm{COS}^2_{k,j}$, $k = 1, \ldots, p = 11$, and $j = 1, 2$, contain the square of the cosine between the variable V_k and, respectively, the first (PC_1) and second (PC_2) principal axes in the correlation circle. They are computed as $\mathrm{COS}^2_{k,j} = \sqrt{\lambda_j}\mathrm{PC}_{kj}$, where PC_{kj} represents the kth coordinate of the jth column (eigenvector) PC_j, and $\sqrt{\lambda_j}$ is given in line " Standard deviation" in the PCA results (**Question 1**) and corresponds to the square root of the jth eigenvalue.
For the interpretation: $\mathrm{COS}^2_{k,j}$ is used to assess the quality of representation of the variable V_k on the component j. The closer $\mathrm{COS}^2_{k,j}$ is to 1, the better the representation of the variable V_k on the component (PC_j).

Question 3

On Figure 2, the projection on the first 2 principal components of 50 cars models is plotted.

a) Interpret each quadrant of the Figure.

Can you describe which kind of car Audi RS 6, Ford Expedition 4.6 XLT and Nissan Sentra 1.8 are?

Indications.

See **Exercise 1**.

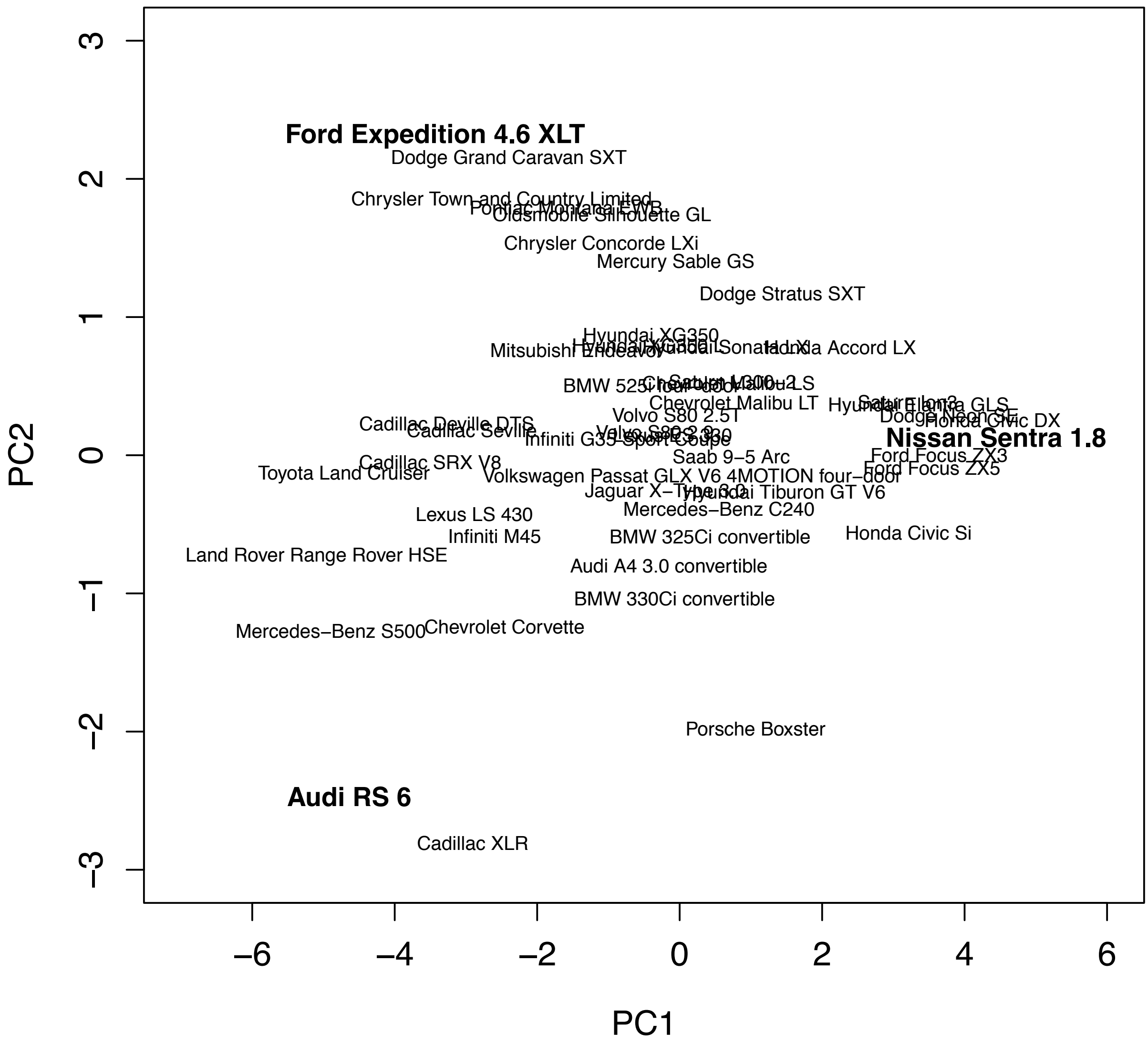


Figure 2: Projection of 50 cars models on the 2 first axes

Exercise 3

Prove Proposition 1. As a bonus, prove its Corollaries.

Proposition 1 *Let $x_i' = x_i - \bar{x}_i$ (for $i = 1, \ldots, n$) be some centered sample in dimension p with covariance matrix Σ. Then the canonical inertia of these points $\frac{1}{n}\sum_{i=1}^{n} \|x_i'\|^2$ is $tr(\Sigma)$.*

Let Π some orthogonal projection (with the canonical dot product). Then the inertia of the projected points is $tr(\Sigma\Pi)$.

Corollary 1 *As a consequence, for standardized samples, $\frac{1}{n}\sum_{i=1}^{n} \|x_i'\|^2 = p$.*

Corollary 2 *The projected inertia on the sum of two orthogonal subspaces is the sum of the projected inertia on each subspace.*

Hint: for any $a \in \mathbb{R}$, $a = \mathrm{tr}(a)$.

Exercise 4

Prove Proposition 2.

Proposition 2 *We use the notations in Proposition 1.*
Let a_1 some vector with norm 1 such that $\Sigma a_1 = \lambda_1 a_1$, λ_1 being (one of) the highest eigenvalue of Σ. Then the projected inertia on the line $D_1 = Span(a_1)$ is maximal over projected inertia on all other possible lines.
Moreover, the projected inertia on D_1 is λ_1.

Hints:

- Use the results from multiple linear regression to prove that for any matrix X with linearly independent columns $X^{(1)}, \dots, X^{(p)}$, the matrix of the orthogonal projection on $\mathrm{Span}\left(\left\{X^{(1)}, \dots, X^{(p)}\right\}\right)$ is $X(X^T X)^{-1} X^T$ (or admit this result if it does not seem obvious).

- Write the maximization problem as

$$\max_{\substack{a \\ \|a\|=1}} \mathrm{tr}(\Sigma a a^T).$$

- Introduce a Lagrange multiplier ξ and cancel the gradient of

$$(a, \xi) \to a^T \Sigma a - \xi(a^T a - 1).$$