Classification: Part 1

Statistical Analysis and Document Mining

Spring 2019

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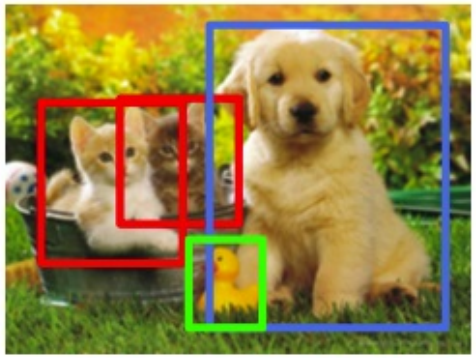
Outline

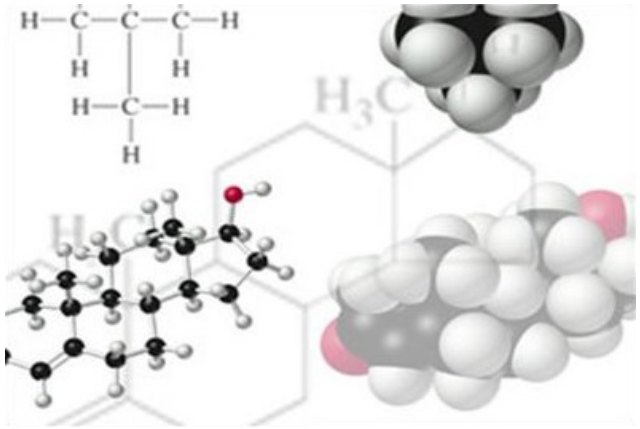
Introduction

CAT

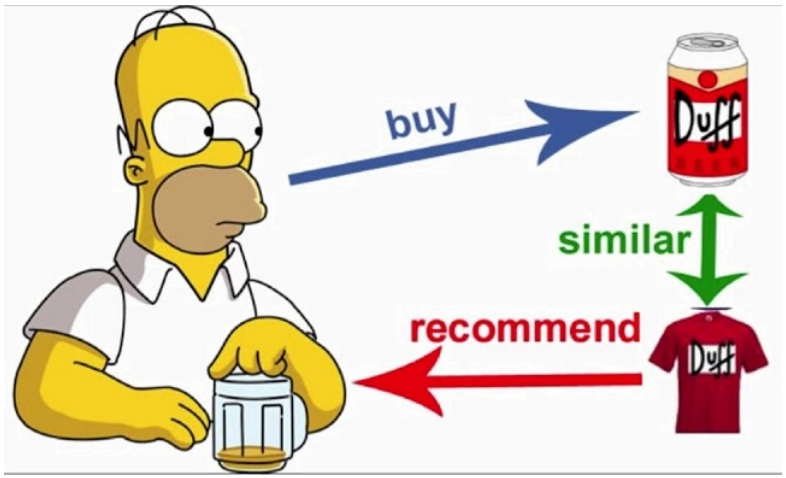


CAT, DOG, DUCK

(a) Computer Vision



(b) Bioinformatics



(c) Recommender Systems

(d) Pattern Recognition

Iris Flower Data Set

(a) Iris Setosa

(b) Iris Versicolor

(c) Iris Virginica

Figure: The data set consists of 50 samples from each of 3 species of Iris. From each sample the length and the width of the sepals and petals were measured. Based on this, the goal is to build a model that distinguishes the species from each other.[1]

[1] https://en.wikipedia.org/wiki/Iris_flower_data_set

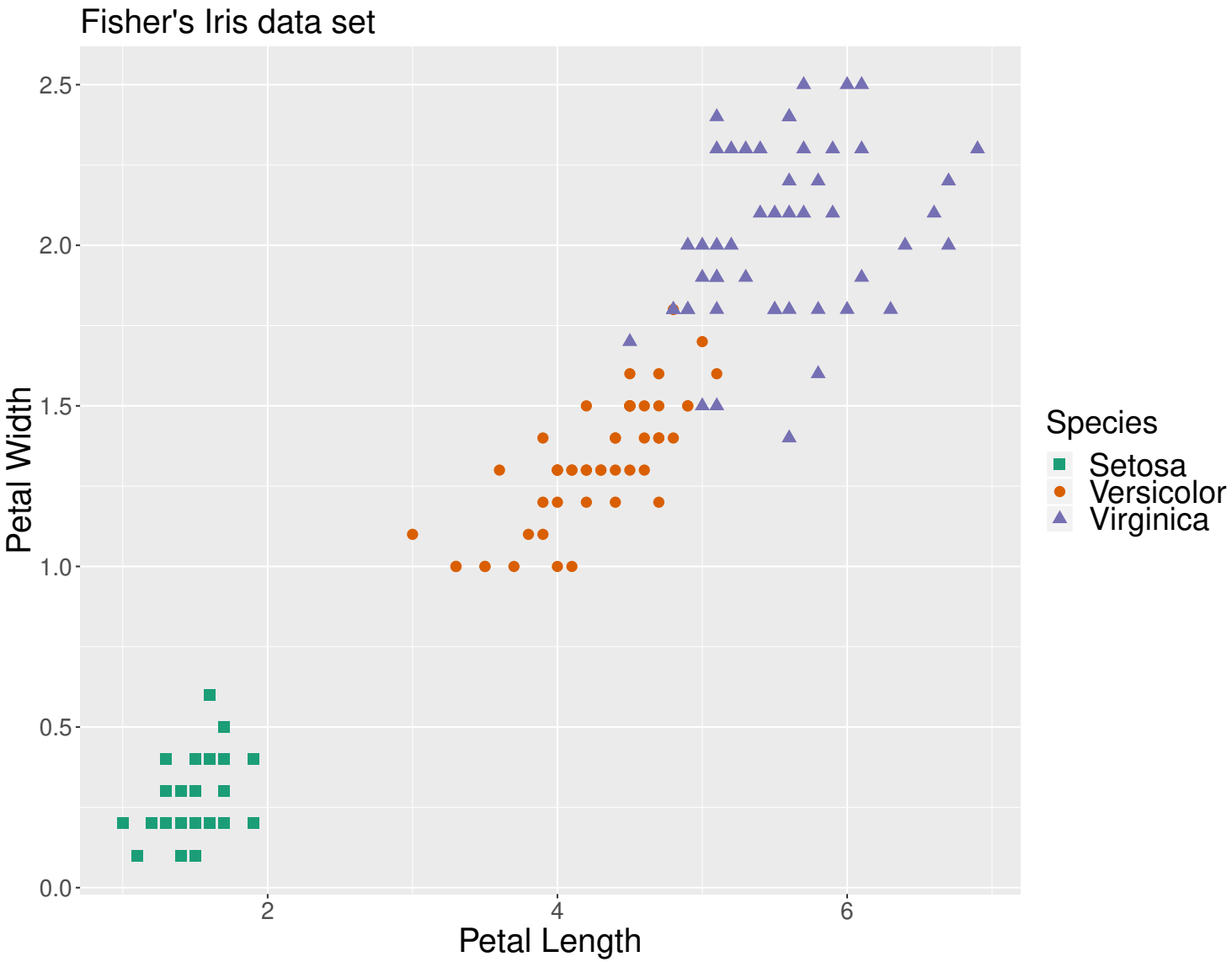
Classification: Simple Formulation

- Training set $\{\mathbf{x}_i, y_i\}_{i=1}^n$: collected samples with class labels;
- A new data point $\mathbf{x}$ without any label;
- Build a classifier $h(\mathbf{x})$ that predicts y as accurately as possible:

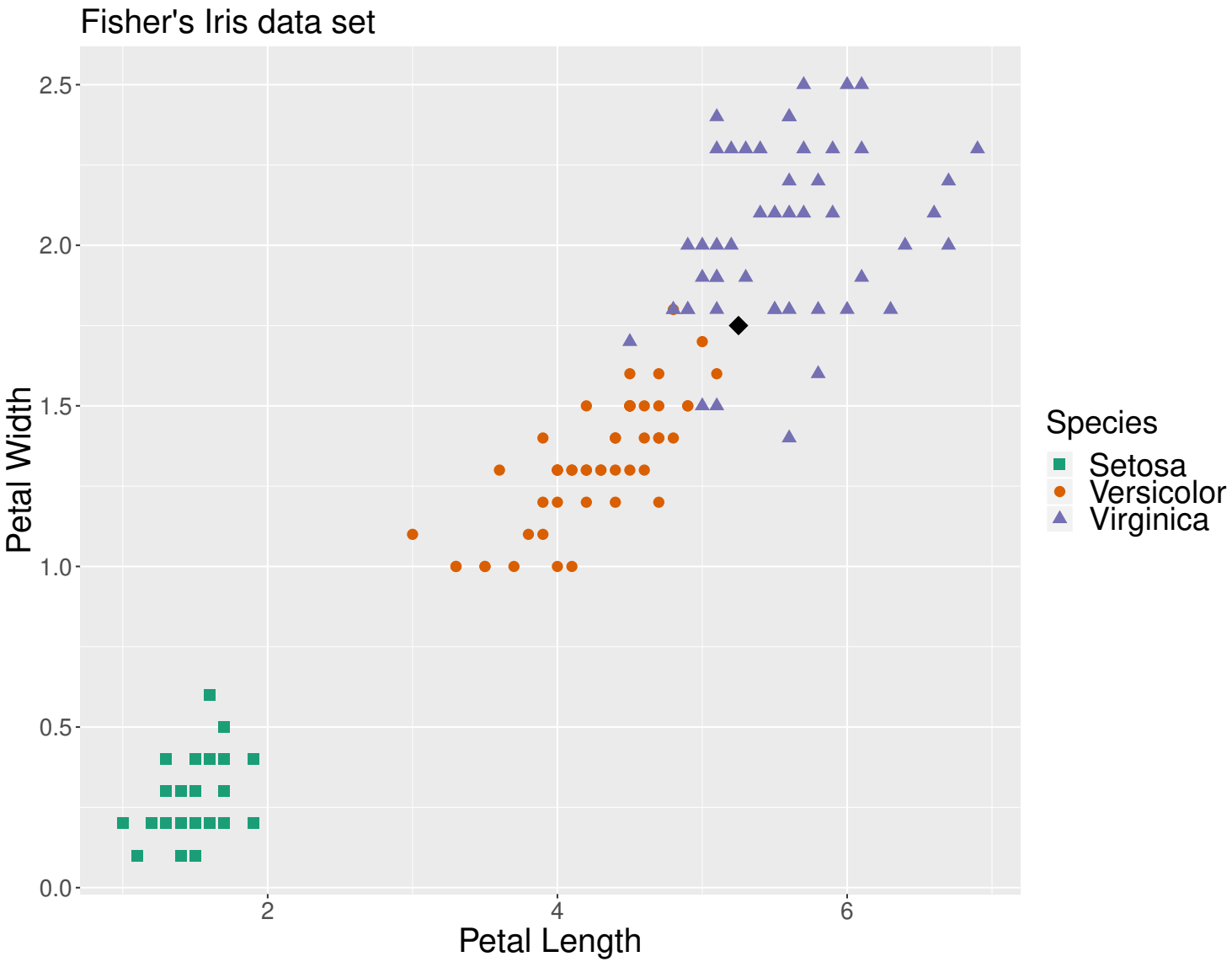
$$\mathbf{x} = \overset{\text{Sepal.L}}{(6.2,} \quad \overset{\text{Sepal.W}}{2.8,} \quad \overset{\text{Petal.L}}{5.6,} \quad \overset{\text{Petal.W}}{2.4)} \quad \overset{?}{\longrightarrow} \quad y = \texttt{"Virginica"}$$

How to do that?

Iris Classification Based on the Petal Information

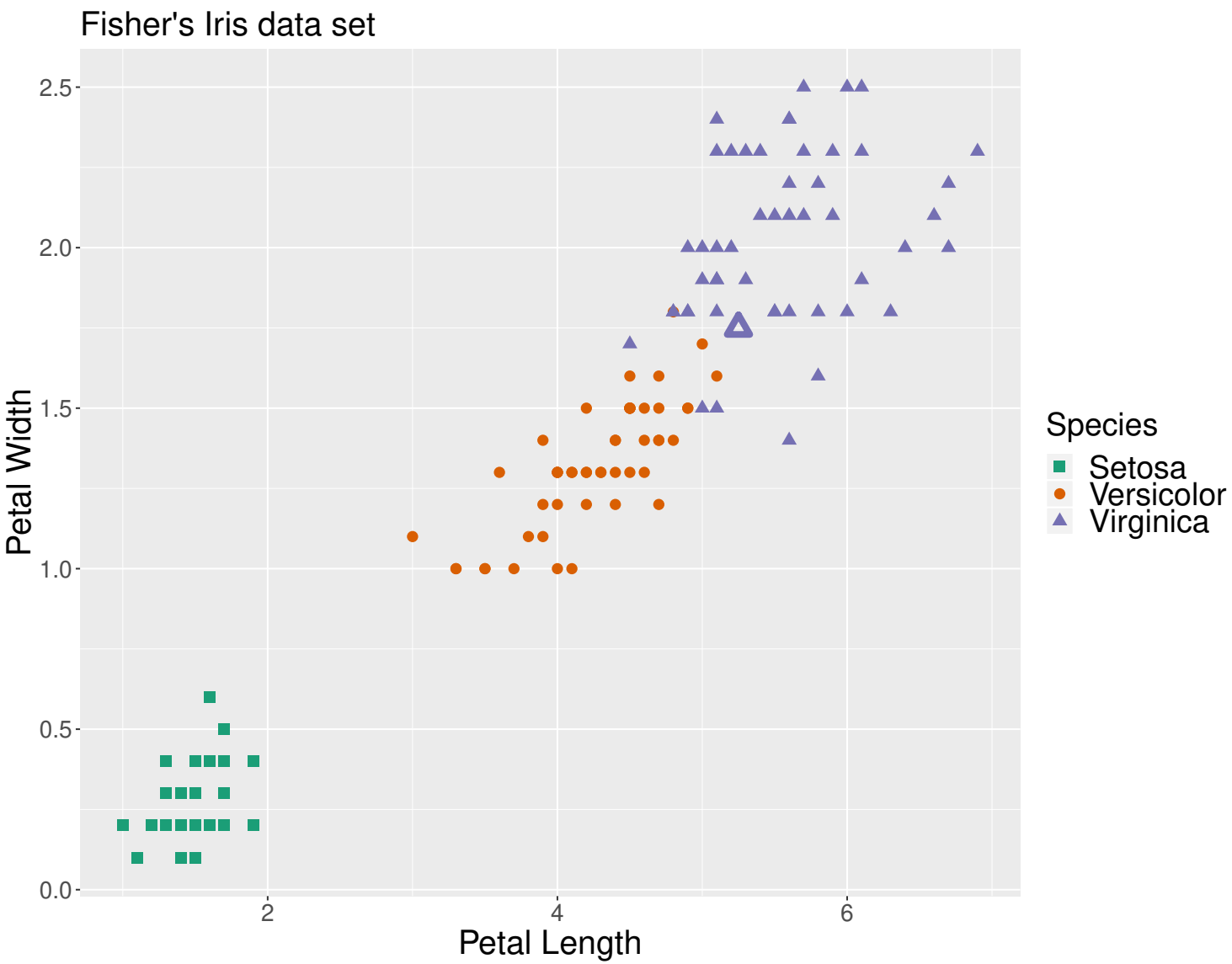


Iris: New Data Point

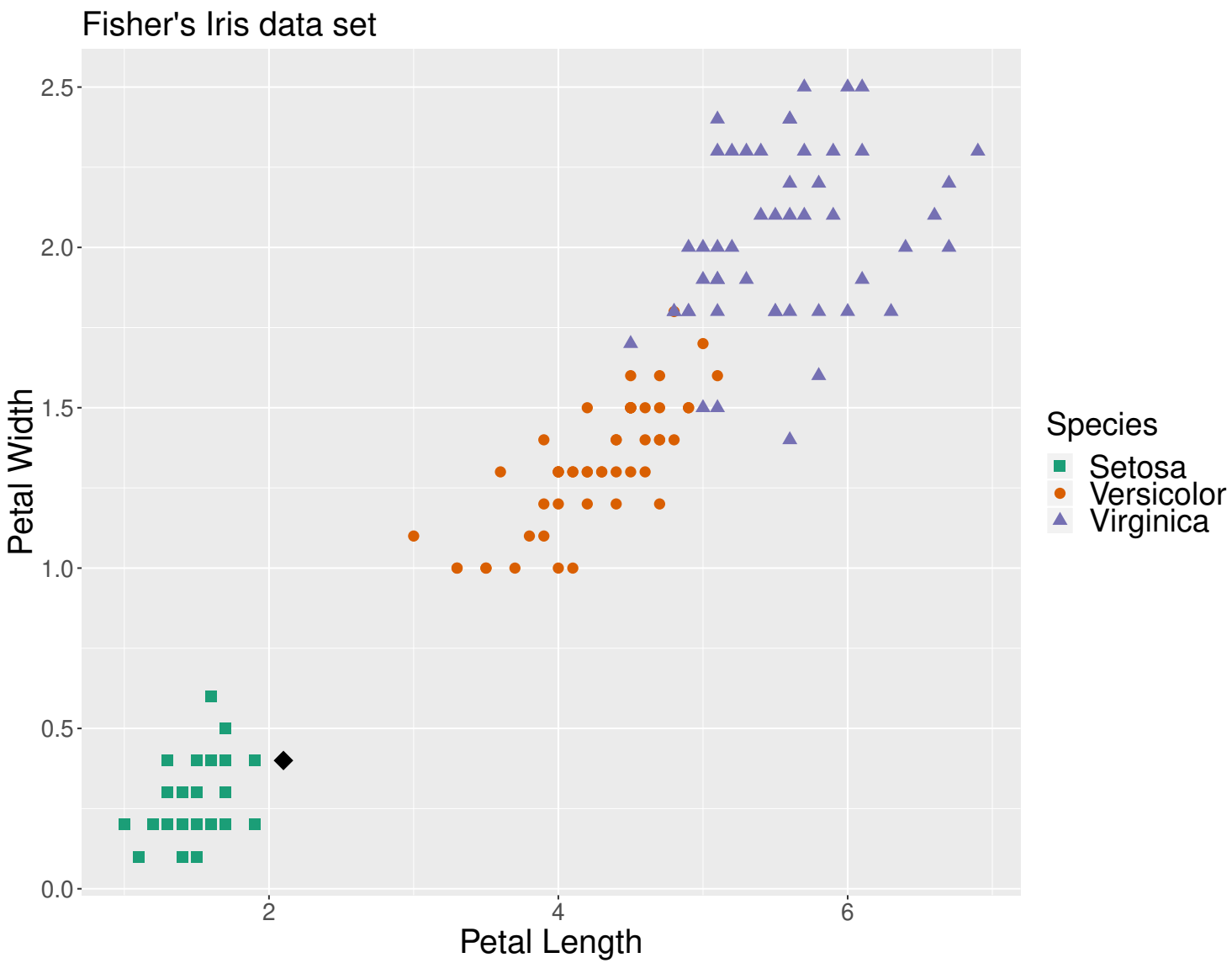


4 Nearest Neighbours

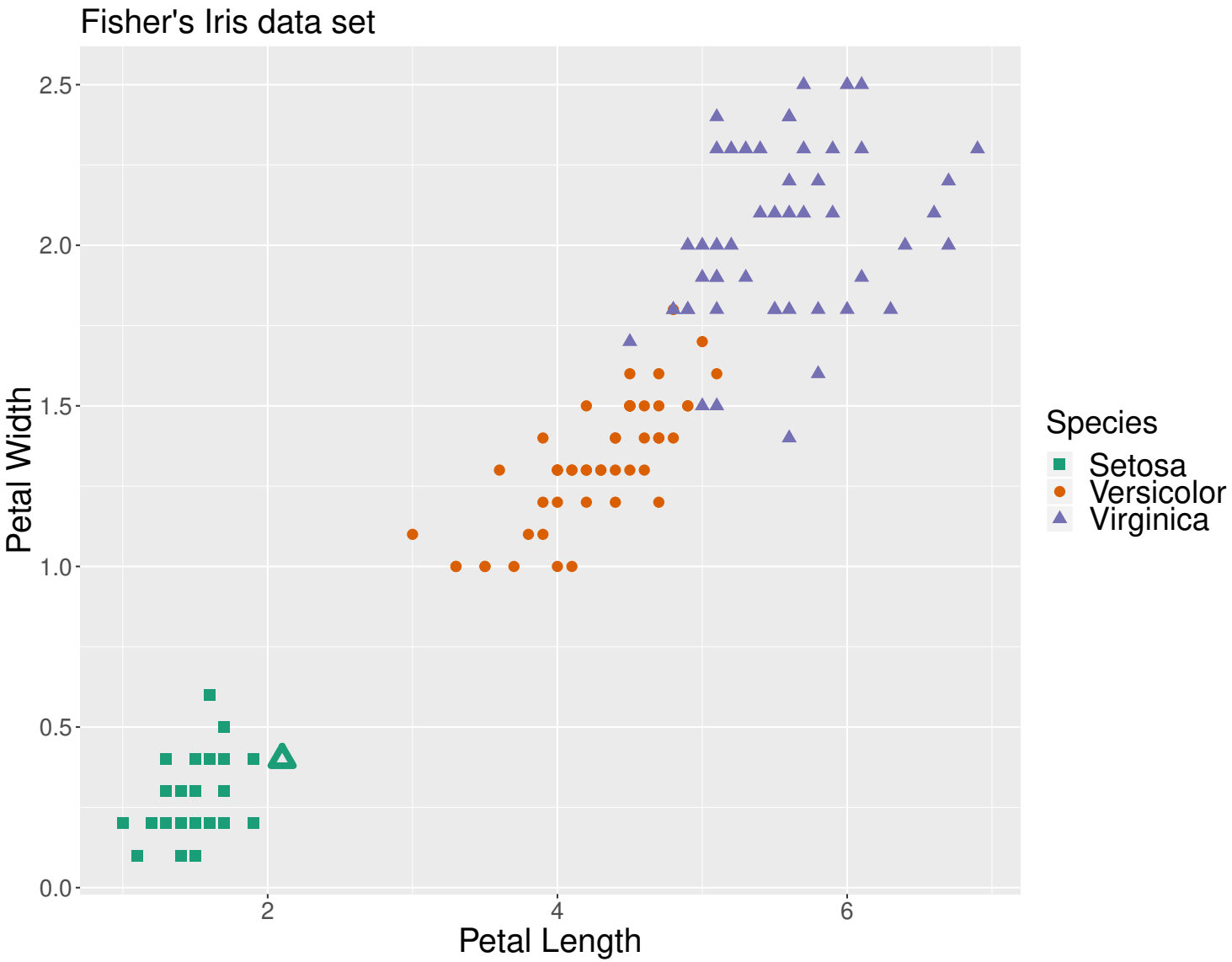
4NN Clasification



Another Data Point



And Its Prediction



One More Example

What to do in this case?

Algorithm k Nearest Neighbours (kNN)

Input: Training set $\{\mathbf{x}_i, y_i\}_{i=1}^n$;
Number of classes K;
New data point $\mathbf{x}$.

1. Compute distance $d(\mathbf{x}, \mathbf{x}_i)$ **for** $i = \{1, \ldots, n\}$.

2. Find k closest training examples to $\mathbf{x}$:

$$J \subset \{1, \ldots, n\} \quad \text{s.t.} \quad |J| = k;$$
$$\forall j \in J, \forall t \in \{1, \ldots, n\} \setminus J : d(\mathbf{x}, \mathbf{x}_j) \leq d(\mathbf{x}, \mathbf{x}_t).$$

Output: Majority class $h(\mathbf{x}) = \text{argmax}_{c=1,\ldots,K} \sum_{j \in J} \mathbb{I}(y_j = c)$.

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- What is the time complexity of the algorithm?

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- Is the nearest neighbours approach applicable for the regression task?

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- kNN is an *instance-based* learning algorithm. What is the main drawback of such methods?

Outline

Discriminant Analysis (Fisher, 1936)

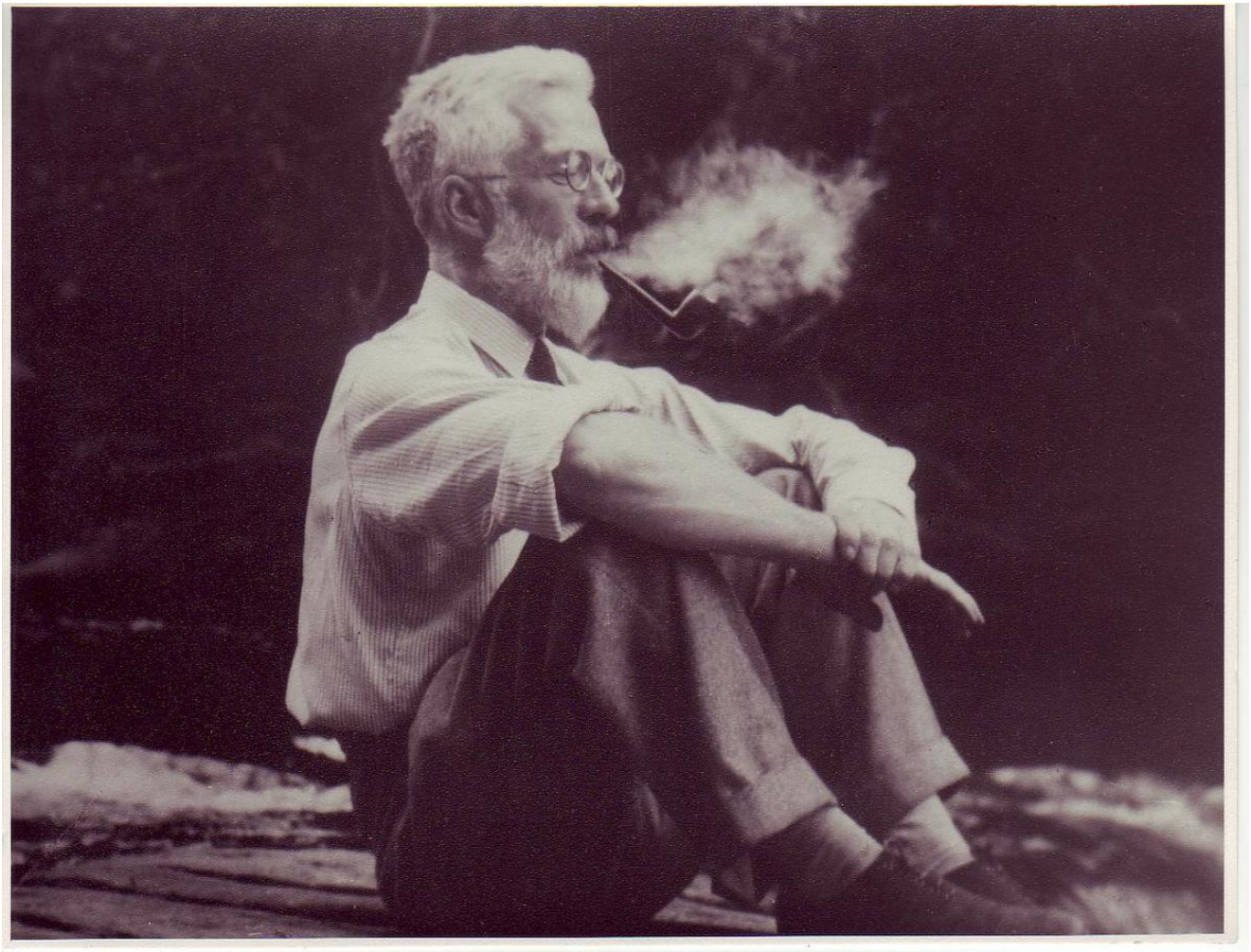

Figure: Ronald A. Fisher in 1946.

Classification: More Formally

- *Input space:* $\mathcal{X} \subseteq \mathbb{R}^d$;
- Output space: $\mathcal{Y} = \{-1, +1\}$ (binary classification),
  $\mathcal{Y} = \{1, \ldots, K\}$ (multi-class classification);
- *Assumption:* all $(\mathbf{X}, Y) \in \mathcal{X} \times \mathcal{Y}$ are **i.i.d.** from $\mathcal{D}$ with respect to a fixed unknown probability distribution $P(\mathbf{X}, Y)$;
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- *Sample Data:* we observe $S = \{\mathbf{x}_i, y_i\}_{i=1}^n$;
- *Loss Function:* $\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^+$;
- *Target*: minimise the *risk* $R^\ell(h) := \mathbb{E}_{(\mathbf{X},Y)\sim\mathcal{D}}\, \ell(h(\mathbf{X}), Y)$.

0/1 Loss Function

0/1 loss function has the following view:

$$\ell^{0/1}(h(\mathbf{x}), y) = \mathbb{I}(h(\mathbf{x}) \neq y) = \begin{cases} 1, & \text{if } h(\mathbf{x}) \neq y; \\ 0, & \text{if } h(\mathbf{x}) = y \end{cases}$$

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In this case, the risk is written as:

$$\begin{aligned} R(h) &= P(h(\mathbf{X}) \neq Y) \\ &= \sum_{c \in \{1,\dots,K\}} P(Y = c) P(h(\mathbf{X}) \neq c | Y = c) \end{aligned}$$

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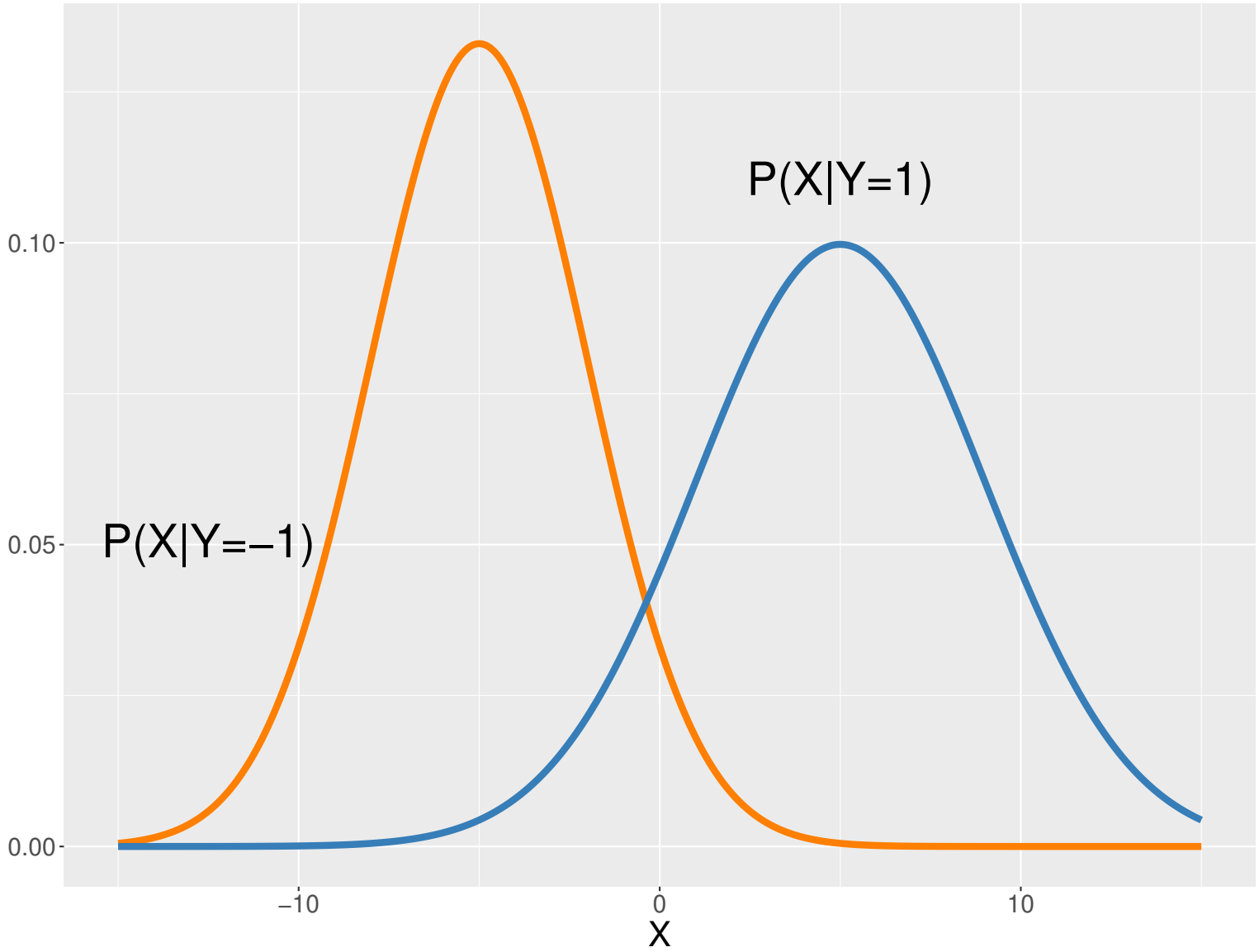
In the binary case, it is represented as:

$$R(h) = P(Y = -1)P(h(\mathbf{X}) = 1|Y = -1) + P(Y = 1)P(h(\mathbf{X}) = -1|Y = 1)$$

Classification in Terms of Distributions



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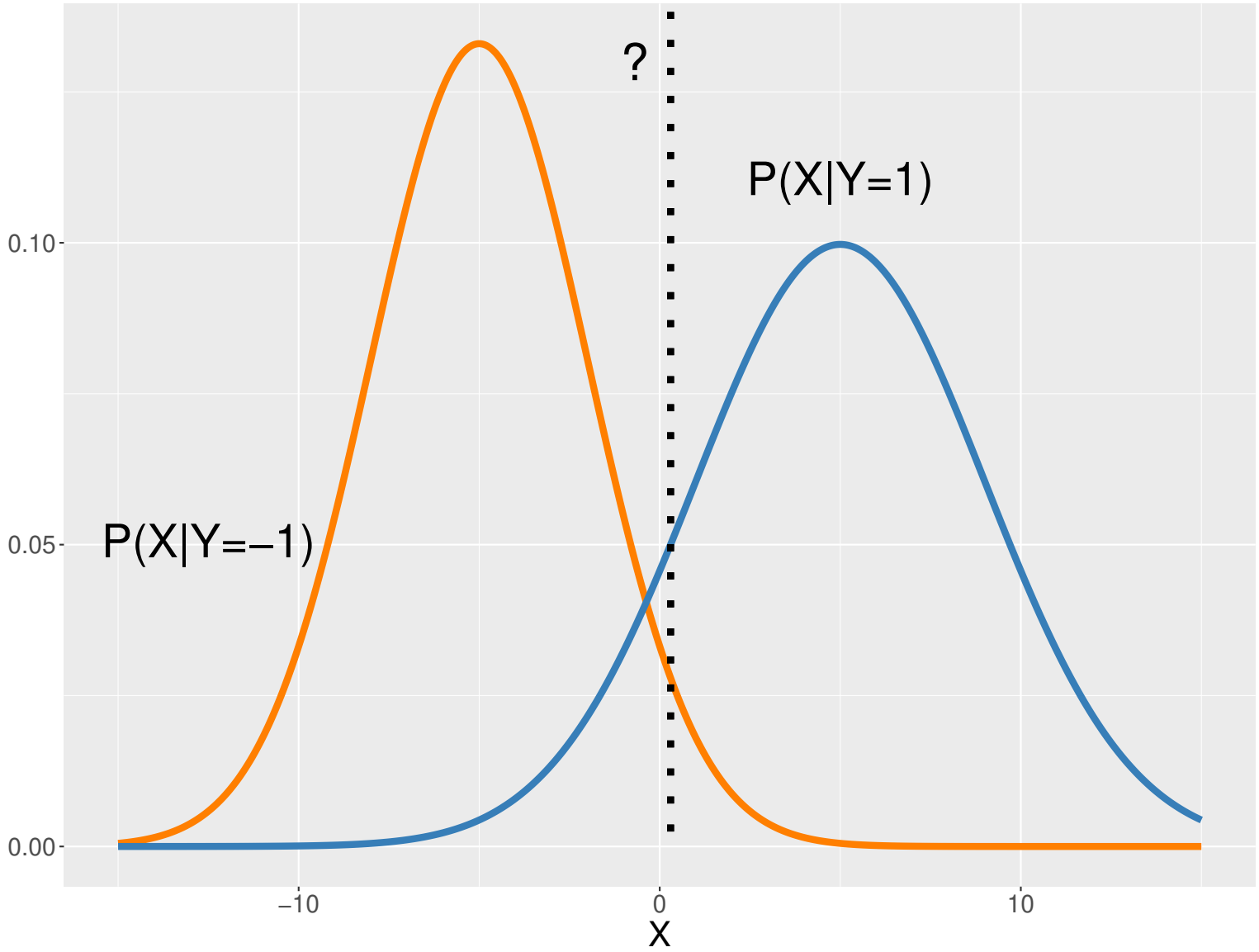


Bayes' Rule

$$\underbrace{P(Y|\mathbf{X})}_{\text{Posterior}} = \frac{\overbrace{P(\mathbf{X}|Y)}^{\text{Likelihood}}\overbrace{P(Y)}^{\text{Class Prior}}}{\underbrace{P(\mathbf{X})}_{\text{Evidence}}}$$

Maximum A Posteriori Rule

Idea: Classify $\mathbf{x}$ to a class with the highest posterior probability:

$$h_B(\mathbf{x}) := \underset{y \in \mathcal{Y}}{\operatorname{argmax}} P(Y = y | \mathbf{X} = \mathbf{x}).$$

This is equivalent to:

$$h_B(\mathbf{x}) \propto \underset{y \in \mathcal{Y}}{\operatorname{argmax}} P(\mathbf{X} = \mathbf{x} | Y = y) P(Y = y).$$

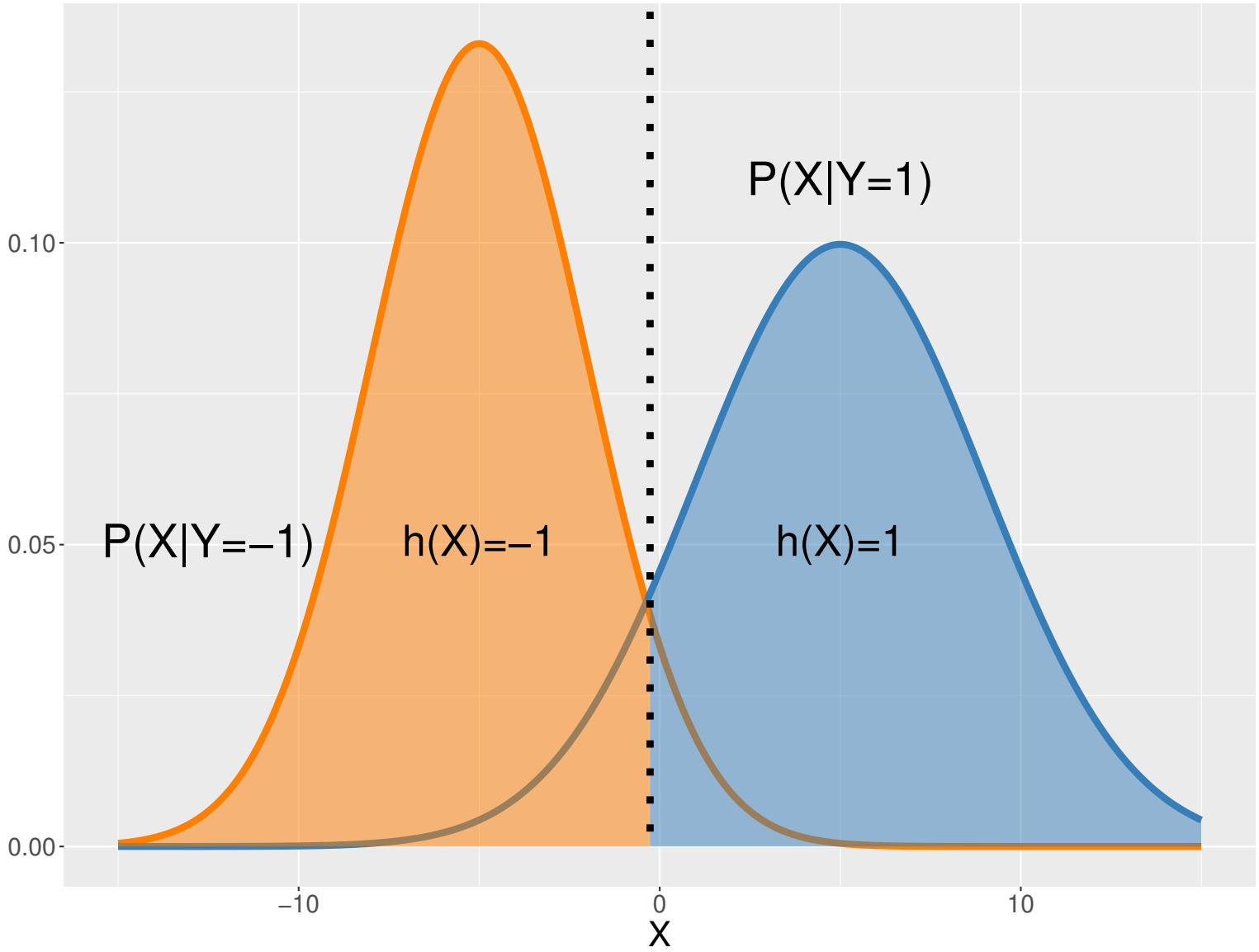
The method is often called the *Bayes* classifier.

Bayes Classifier in the Binary Case

In the binary case ($\mathcal{Y} = \{-1, +1\}$), the Bayes classifier is usually defined in the following way:

$$h_B(\mathbf{x}) = \begin{cases} +1, & \text{if } P(Y = +1 | \mathbf{X} = \mathbf{x}) \geq P(Y = -1 | \mathbf{X} = \mathbf{x}); \\ -1, & \text{if } P(Y = +1 | \mathbf{X} = \mathbf{x}) < P(Y = -1 | \mathbf{X} = \mathbf{x}). \end{cases}$$

Particular Case: Maximum Likelihood Estimator



Bayes Classifier Is Optimal

Theorem

Suppose $P(Y)$ and $P(\mathbf{X}|Y)$ are given. Then the Bayes classifier yields the minimum of the misclassification error.

Exercise: Prove this theorem. For sake of simplicity, consider the binary case ($\mathcal{Y} = \{-1, +1\}$).

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1. $P(h(\mathbf{X}) \neq Y) = \int P(h(\mathbf{X}) \neq Y|\mathbf{X} = \mathbf{x})d\mathbf{x}$. Then, what is the value of $P(h_B(\mathbf{X}) \neq Y|\mathbf{X} = \mathbf{x})$?

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2. $P(h_B(\mathbf{x}) \neq Y|\mathbf{X} = \mathbf{x}) = 1 - \max_{c \in \mathcal{Y}} P(Y = c|\mathbf{X} = \mathbf{x})$. Does this prove the theorem?

Bayes Classifier: Discussion

In practice, we don't have any information about data distribution.

- Can we estimate $P(Y)$?

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- Can we estimate $P(Y)$?
- Can we estimate $P(\mathbf{X}|Y)$?

Additional Assumptions

- *Assumption 1:* Observations from a class $c \in \mathcal{Y}$ are normally distributed $[\mathbf{X}|Y = c] \sim \mathcal{N}(\boldsymbol{\mu}_c, \boldsymbol{\Sigma}_c)$.
  - Remember the formula of the multivariate normal distribution.

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$$f(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_c)^\intercal \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}_c)}.$$

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- *Assumption 2:* The covariance matrices of the classes are equal:

$$\boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_K = \boldsymbol{\Sigma}.$$

Due to the logarithm properties, the Bayes classifier can be written as follows:

$$h_B(\mathbf{x}) = \underset{c \in \mathcal{Y}}{\operatorname{argmax}} [\ln P(\mathbf{X} = \mathbf{x} | Y = c) + \ln P(Y = c)].$$

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Taking into account Assumption 1 and Assumption 2, we derive the algorithm called *Linear Discriminant Analysis (LDA)*:

$$h_{LDA}(\mathbf{x}) = \underset{c \in \mathcal{Y}}{\operatorname{argmax}} \delta_c(\mathbf{x}),$$

$$\delta_c(\mathbf{x}) = \boldsymbol{\mu}_c^\intercal \boldsymbol{\Sigma}^{-1} \mathbf{x} - \frac{1}{2} \boldsymbol{\mu}_c^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_c + \ln P(Y = c);$$

δ_c is usually called the *discriminant function*.

LDA in the Binary Case

In the binary case, the decision rule can be also written as:

$$h_{LDA}(\mathbf{x}) = \begin{cases} +1, & \text{if } \textit{Condition} \text{ is true;} \\ -1, & \text{otherwise.} \end{cases}$$

Condition:

$$(\boldsymbol{\mu}_{+1}-\boldsymbol{\mu}_{-1})^{\intercal}\boldsymbol{\Sigma}^{-1}\mathbf{x}-\frac{1}{2}(\boldsymbol{\mu}_{+1}-\boldsymbol{\mu}_{-1})^{\intercal}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_{+1}+\boldsymbol{\mu}_{-1})-\ln\frac{P(Y=-1)}{P(Y=+1)} \geq 0.$$

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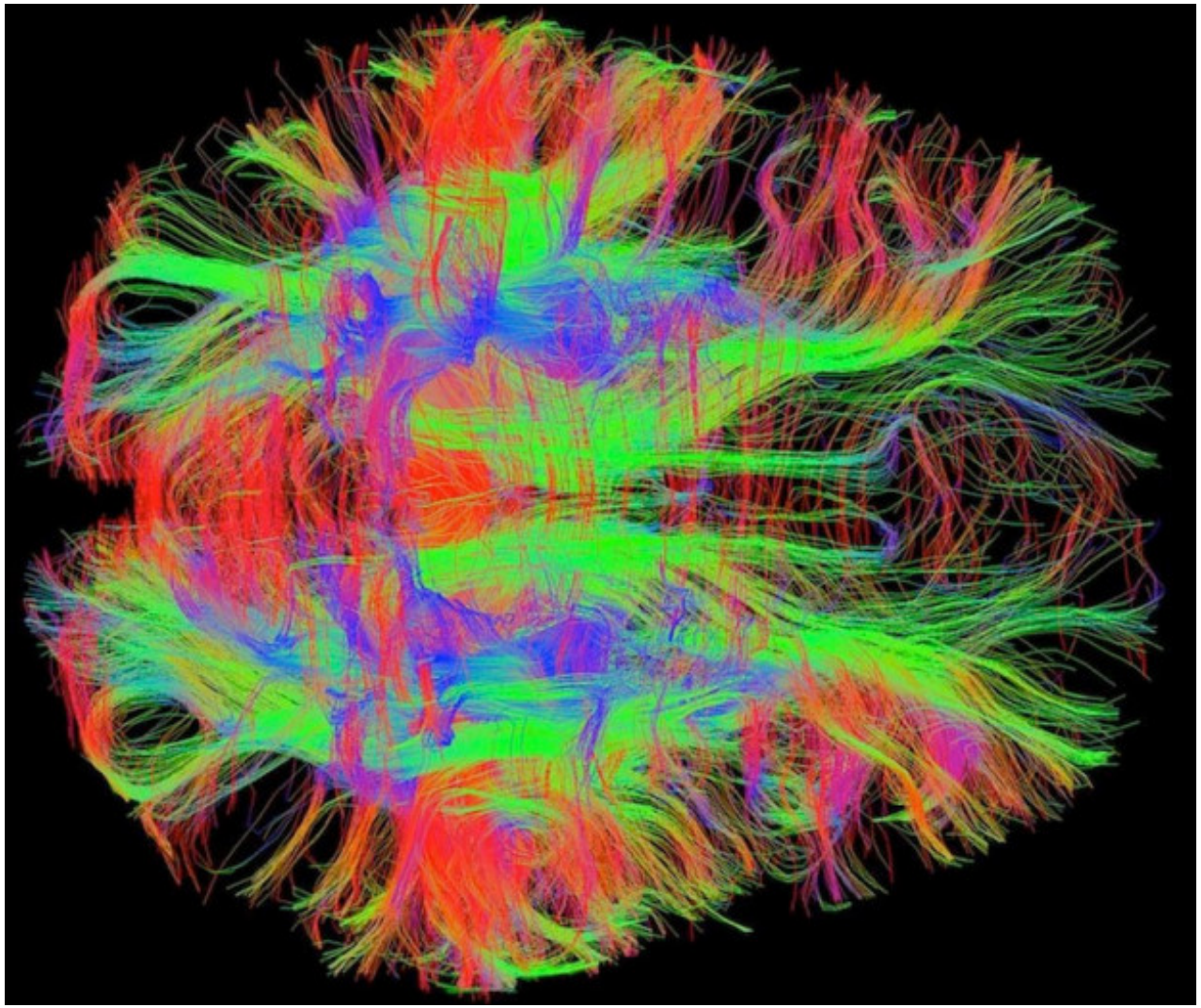
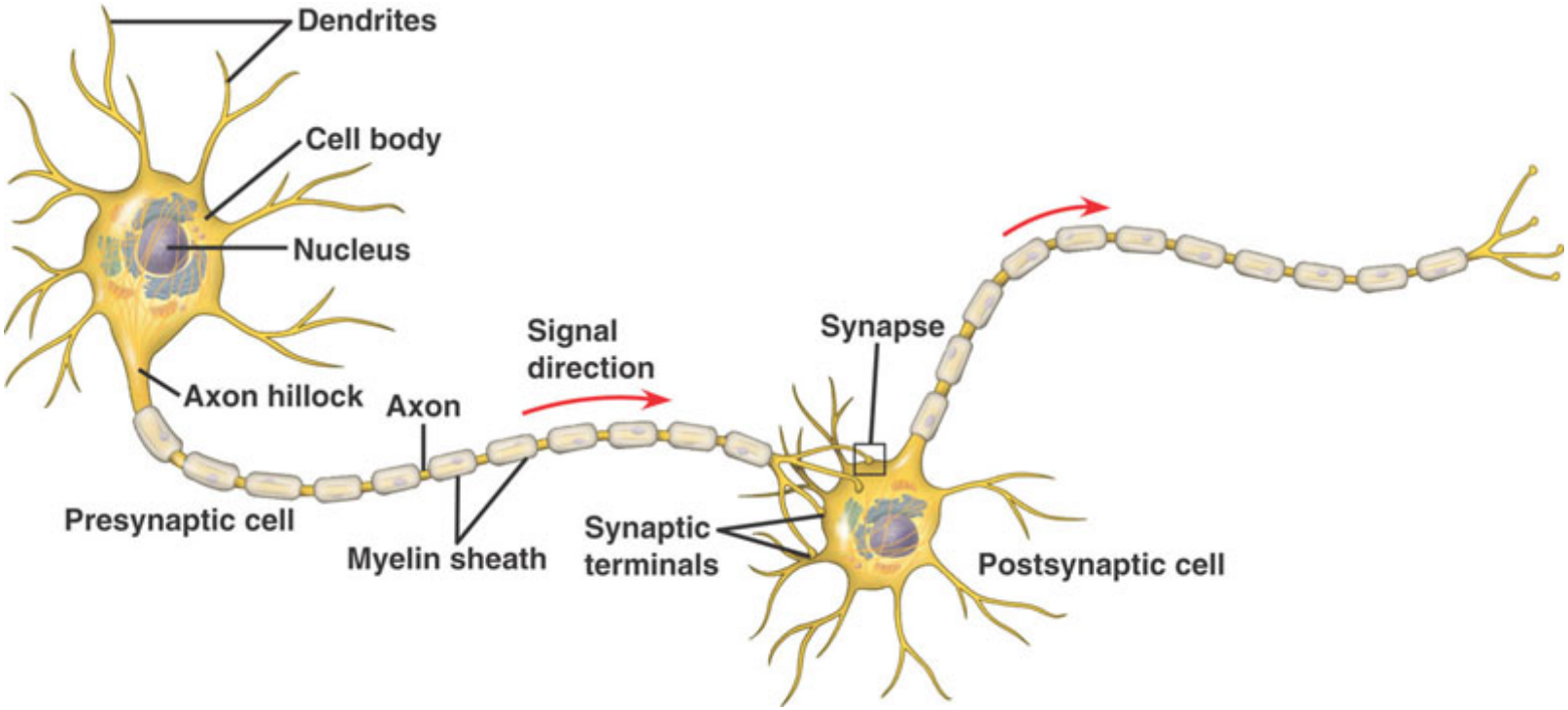
- Why it is called "linear"?
- How many parameters we need to estimate?
- What happens when Assumption 1, 2 are violated?
- What is the time complexity during training phase?
- What is the time complexity to predict a label for new $\mathbf{x}$?

Outline

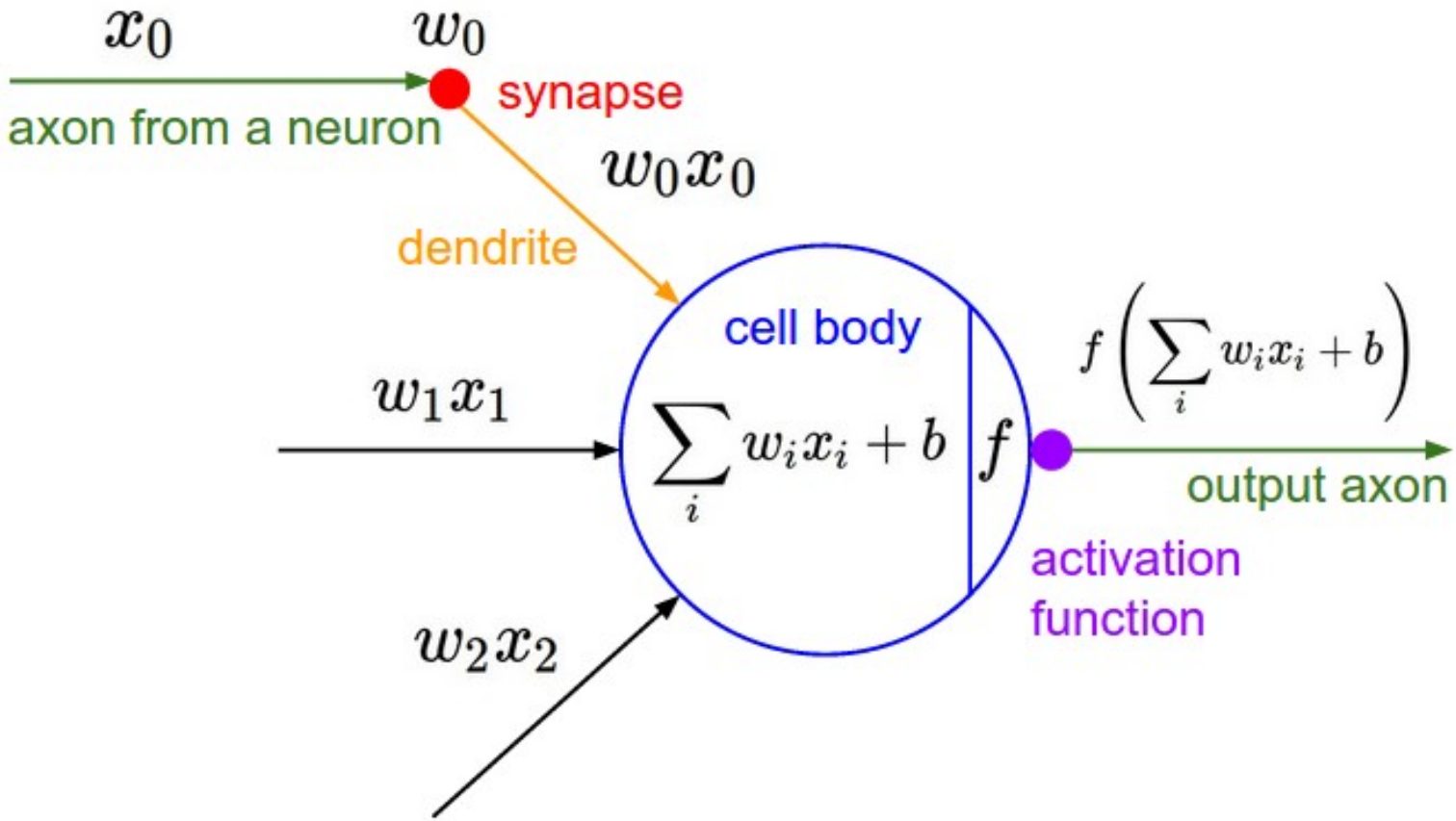
AI and Human Brain Study



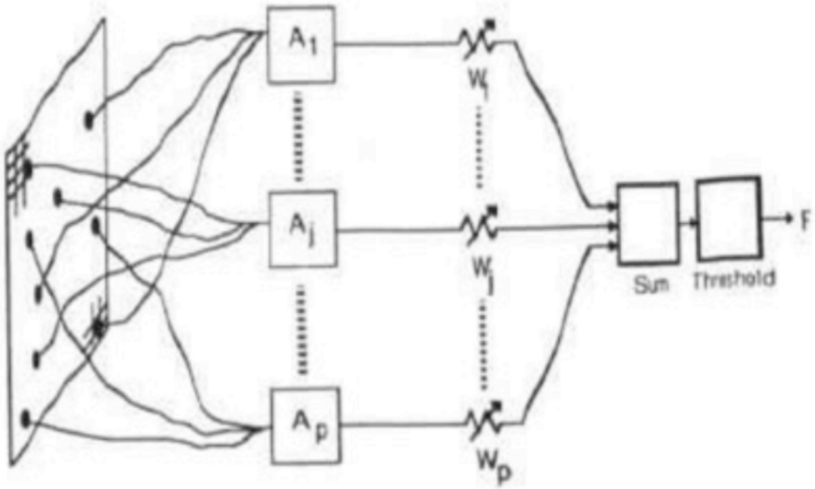
Biological Neuron

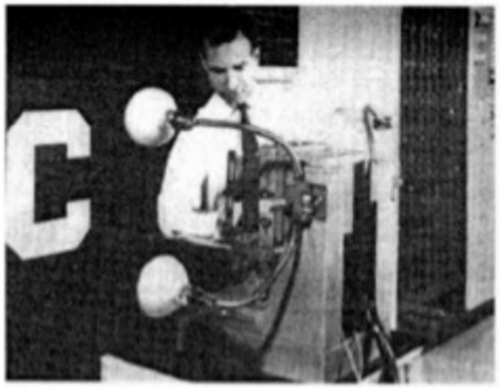


Neuron Model (MuCulloch & Pitts, 1943)



Perceptron Machine (Rosenblatt, 1958)





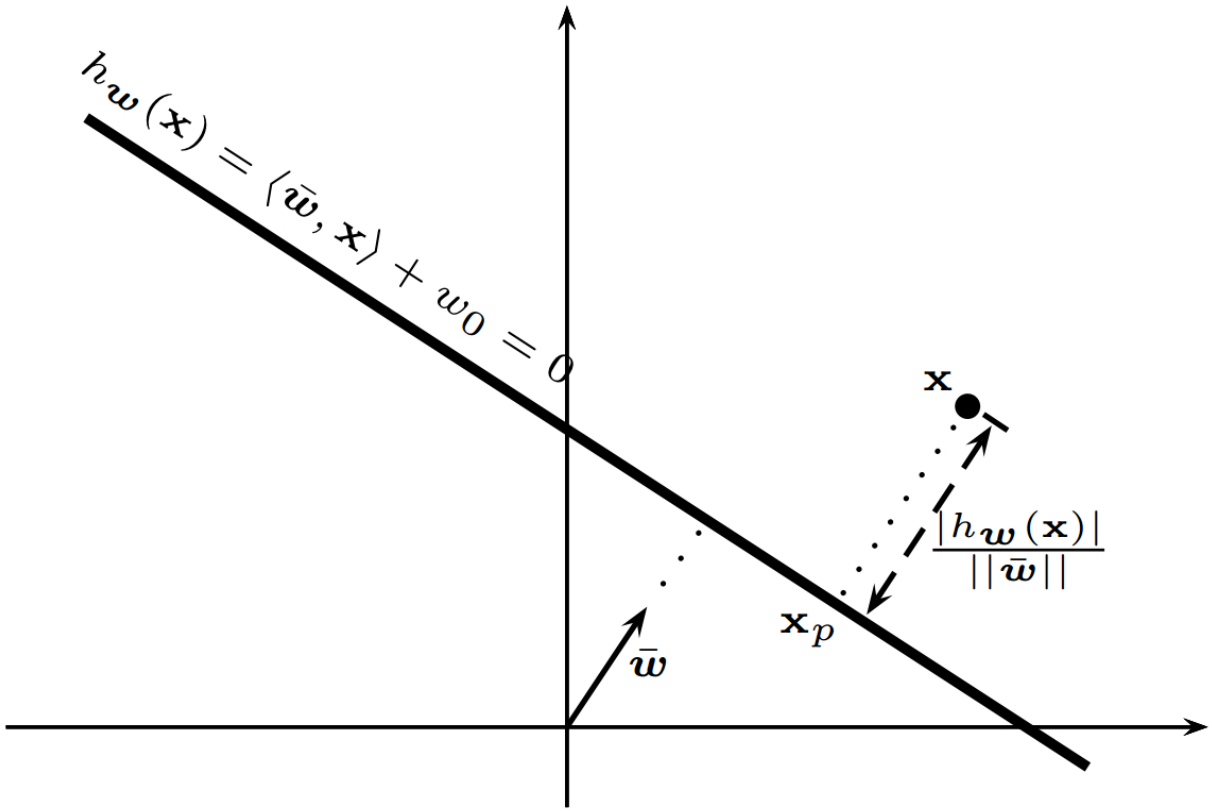
- **Summation:**

$$h_w(\mathbf{x}) := \langle \mathbf{w}, \mathbf{x} \rangle + w_0 = \sum_{j=1}^{d} w_j x_j + w_0.$$

- **Activation:**

$$a(\mathbf{x}) = \begin{cases} +1, & \text{if } y h_w(\mathbf{x}) \geq 0; \\ -1, & \text{if } y h_w(\mathbf{x}) < 0. \end{cases}$$

We want to find parameters $(w_0, \mathbf{w})$ such that the distance between the misclassified examples and the decision boundary is minimised.



Perceptron Error Minimisation

- We would like to minimise the *perceptron error*:

$$\hat{L}(\mathbf{w}, w_0) = -\sum_{i' \in \mathcal{I}} y_{i'}(\langle \mathbf{w}, \mathbf{x}_{i'} \rangle + w_0).$$

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- If we take derivatives with respect to the weights:

$$\frac{\partial \hat{L}(\mathbf{w})}{\partial w_0} = -\sum_{i' \in \mathcal{I}} y_{i'},$$

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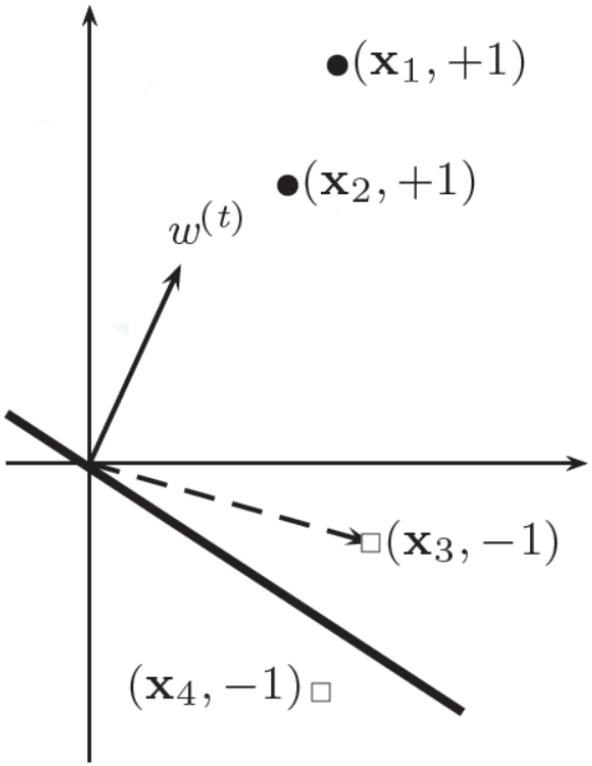
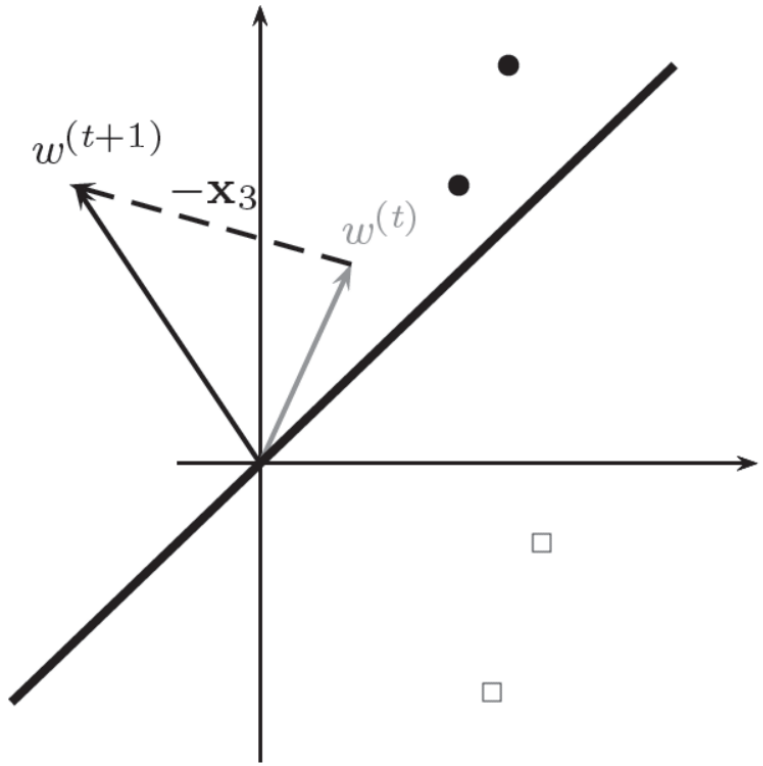
- Minimisation of the perceptron error is done by stochastic gradient descent algorithm:

$$\text{if } y(\langle \mathbf{w}, \mathbf{x} \rangle + w_0) < 0, \text{then } \begin{pmatrix} w_0 \\ \mathbf{w} \end{pmatrix} \leftarrow \begin{pmatrix} w_0 \\ \mathbf{w} \end{pmatrix} + \eta \begin{pmatrix} y \\ y\mathbf{x} \end{pmatrix}.$$

Update Rule: Illustration





Algorithm Perceptron

Input: Training set $\mathrm{S} = \{\mathbf{x}_i, y_i\}_{i=1}^n$.
Max number of iterations T.

Initialisation: Weights $\mathbf{w}^{(0)} \leftarrow \mathbf{0}, w_0 \leftarrow 0$;
Counter $t \leftarrow 0$, learning rate $\eta > 0$.

repeat
Choose randomly an example $(\mathbf{x}^{(t)}, y^{(t)}) \in \mathrm{S}$
if $y \cdot (\langle \mathbf{w}^{(t)}, \mathbf{x}^{(t)} \rangle + w_0) < 0$ **then**

$$w_0^{(t+1)} \leftarrow w_0^{(t)} + \eta\, y^{(t)}$$
$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + \eta\, y^{(t)}\, \mathbf{x}^{(t)}$$

end if
$t \leftarrow t + 1$
until $t > T$
Output: $(w_0^{(T)}, \mathbf{w}^{(T)})$.

Perceptron: Discussion

- What is the time complexity during training phase?

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- What is the time complexity during training phase?
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- How to tune learning rate η?
- Does the algorithm converge to an exact solution?
- Under what condition the algorithm does not converge?

AI Winter

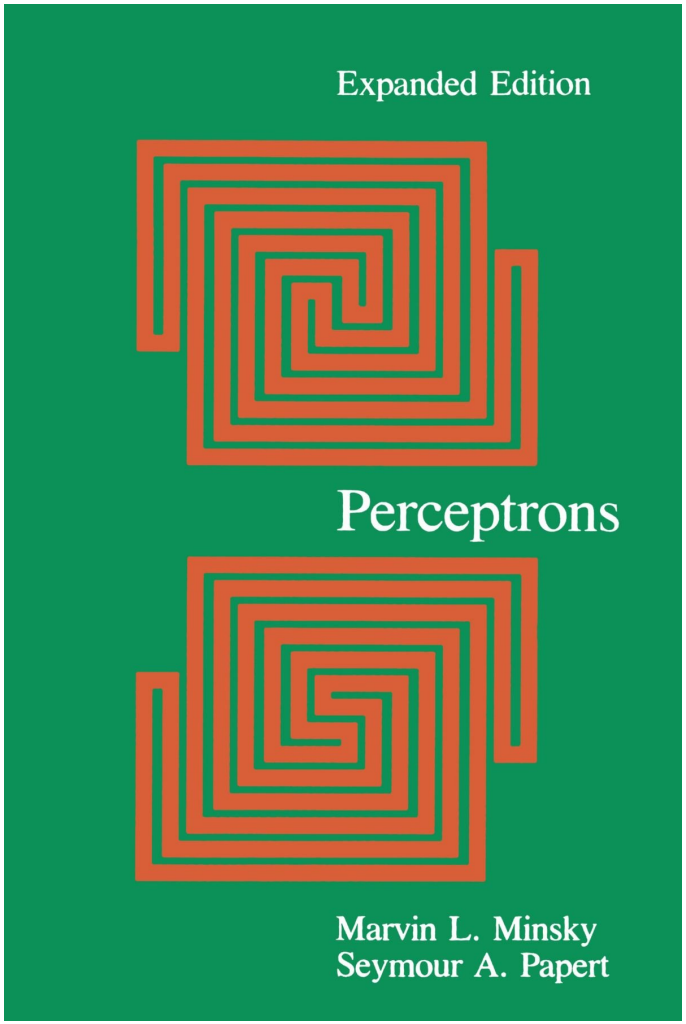


Figure: (Minsky & Papert, 1969)

Figure: Minsky and Papert in 1971.

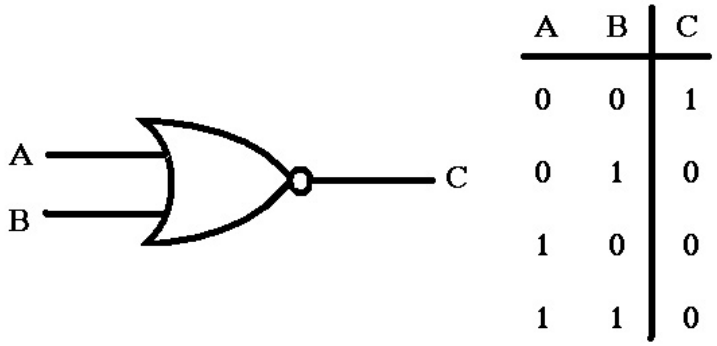
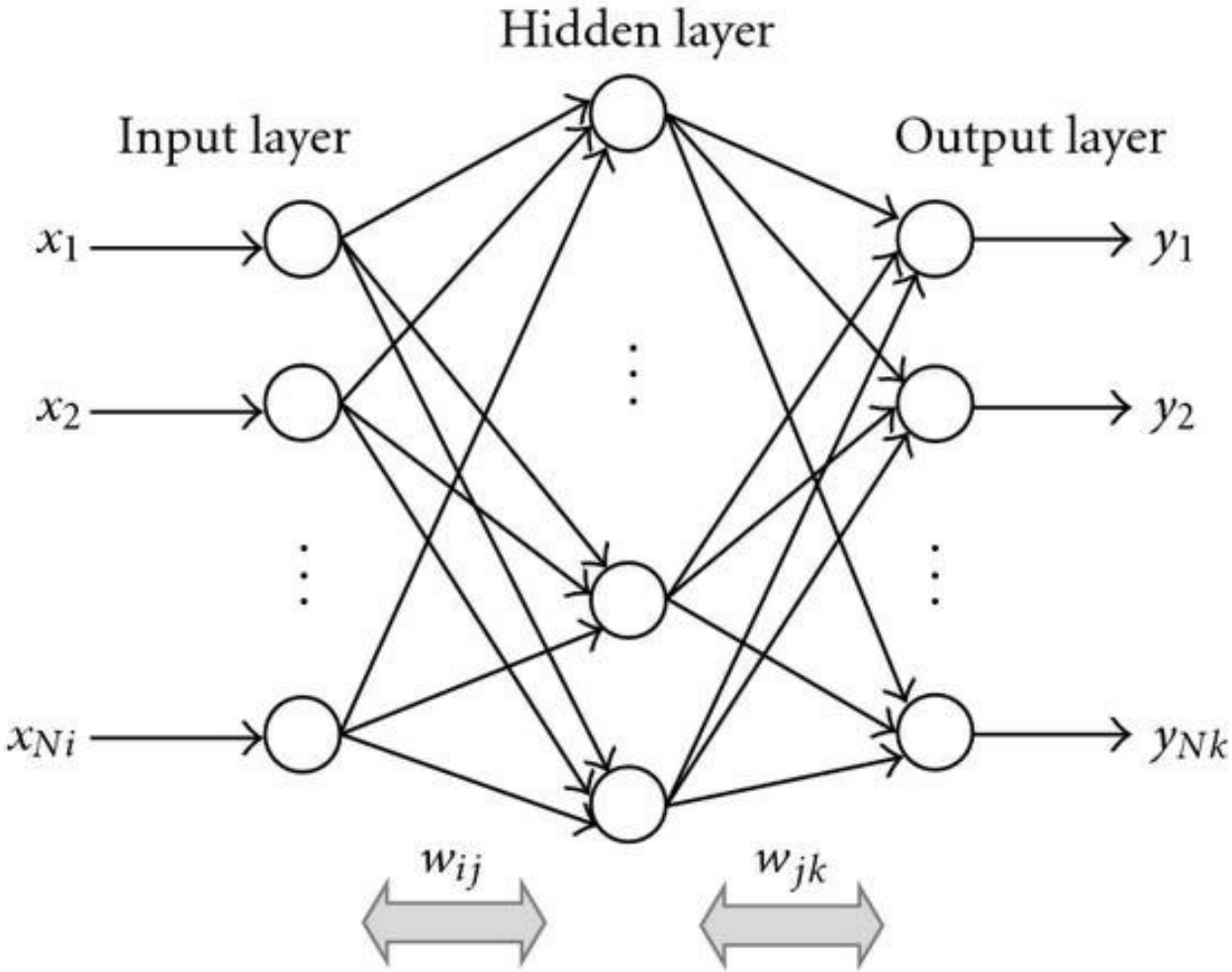
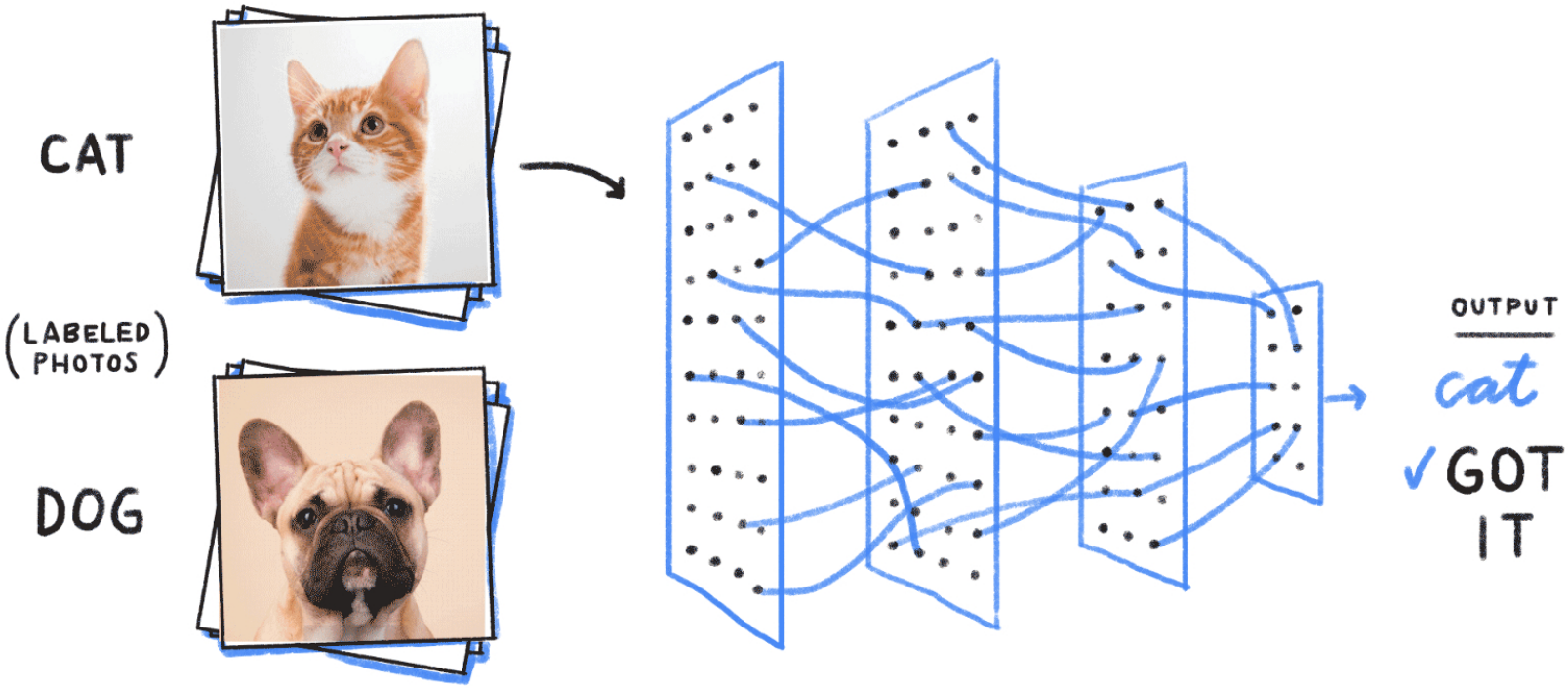


Figure: The XOR operation.

Backpropagation Learning (Rumelhart, 1985)



Nowadays: Deep Learning



Classification: Part 2

Statistical Analysis and Document Mining

Spring 2019

Vasilii Feofanov

Université Grenoble Alpes
vasilii.feofanov@univ-grenoble-alpes.fr

Outline

Classification Task

- *Input space:* $\mathcal{X} \subseteq \mathbb{R}^d$;
- Output space: $\mathcal{Y} = \{-1, +1\}$ (binary classification),
  $\mathcal{Y} = \{1, \ldots, K\}$ (multi-class classification);

- *Assumption:* all $(\mathbf{X}, Y) \in \mathcal{X} \times \mathcal{Y}$ are **i.i.d.** from $\mathcal{D}$ with respect to a fixed unknown probability distribution $P(\mathbf{X}, Y)$;
- *Sample Data:* we observe $S = \{\mathbf{x}_i, y_i\}_{i=1}^n$;

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- *Sample Data:* we observe $S = \{\mathbf{x}_i, y_i\}_{i=1}^n$;

- *Loss Function:*

$$\ell^{0/1}(h(\mathbf{x}), y) = \mathbb{I}(h(\mathbf{x}) \neq y) = \begin{cases} 1, & \text{if } h(\mathbf{x}) \neq y; \\ 0, & \text{if } h(\mathbf{x}) = y. \end{cases}$$

- *Target*: minimise the misclassification error:

$$P(h(\mathbf{X}) \neq Y) = \sum_{c \in \{1,\ldots,K\}} P(Y = c) P(h(\mathbf{X}) \neq c | Y = c).$$

Bayes' Rule

$$\underbrace{P(Y|\mathbf{X})}_{\text{Posterior}} = \frac{\overbrace{P(\mathbf{X}|Y)}^{\text{Likelihood}}\overbrace{P(Y)}^{\text{Class Prior}}}{\underbrace{P(\mathbf{X})}_{\text{Evidence}}}$$

The Bayes classifier predicts a class with the highest posterior probability:

$$h_B(\mathbf{x}) := \underset{y \in \mathcal{Y}}{\operatorname{argmax}} P(Y = y | \mathbf{X} = \mathbf{x}).$$

This is equivalent to:

$$\begin{aligned} h_B(\mathbf{x}) &\propto \underset{y \in \mathcal{Y}}{\operatorname{argmax}} P(\mathbf{X} = \mathbf{x} | Y = y) P(Y = y) \\ &\propto \underset{y \in \mathcal{Y}}{\operatorname{argmax}} \log P(\mathbf{X} = \mathbf{x} | Y = y) + \log P(Y = y). \end{aligned}$$

Naive Bayes Classifier

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- Thus, the *naive Bayes classifier* is defined in the following way:

$$h_B(\mathbf{x}) := \operatorname*{argmax}_{c \in \mathcal{Y}} P(Y = c) \prod_{j=1}^{d} P(X_j = x_j|Y = c)$$

$$\propto \operatorname*{argmax}_{c \in \mathcal{Y}} \log P(Y = c) + \sum_{j=1}^{d} \log P(X_j = x_j|Y = c).$$

Gaussian Naive Bayes Classifier

- We assume that the j-th feature of observations from the class $c \in \mathcal{Y}$ is normally distributed:

$$[X_j | Y = c] \sim \mathcal{N}(\mu_{j,c}, \sigma_{j,c}).$$

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- Then, the distribution of the class $c \in \mathcal{Y}$ is defined as:

$$P(\mathbf{X} = \mathbf{x}|Y = c) = \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi}\sigma_{j,c}} e^{-\frac{(x_j - \mu_{j,c})^2}{2\sigma_{j,c}^2}}.$$

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- Finally, the Gaussian naive Bayes classifier is defined as:

$$h_B(\mathbf{x}) := \underset{c \in \mathcal{Y}}{\operatorname{argmax}} \left[\ln P(Y = c) - \sum_{j=1}^{d} \ln \sigma_{j,c} - \sum_{j=1}^{d} \frac{(x_j - \mu_{j,c})^2}{2\sigma_{j,c}^2} \right].$$

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- Then, the Bernoulli naive Bayes classifier is defined as:

$$h_B(\mathbf{x}) := \underset{c \in \mathcal{Y}}{\operatorname{argmax}} P(Y = c) \prod_{j=1}^{d} p_{j,c}^{x_j} (1 - p_{j,c})^{1 - x_j}$$

$$\propto \underset{c \in \mathcal{Y}}{\operatorname{argmax}} \log P(Y = c) + \sum_{j=1}^{d} x_j \log p_{j,c} + \sum_{j=1}^{d} (1 - x_j) \log(1 - p_{j,c}).$$

Multinomial Distribution

Multinomial Distribution

- Let $T_1, \ldots, T_m$ be i.i.d. random variables that represent trials with the output $\in \{1, \ldots, d\}$ and are distributed as follows:

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- Then, the *multinomial* distribution $P(X_1, \ldots, X_d)$ is defined as:

$$P(X_1 = x_1, \ldots, X_d = x_d) = \frac{m!}{x_1! \cdots x_d!} p_1^{x_1} \cdots p_d^{x_d}.$$

Multinomial Bayes Classifier

- Suppose $\mathbf{X}|Y = c$ is distributed according to the multinomial distribution with parameters $\mathbf{p}_c = (p_{1,c}, \ldots, p_{d,c})$. Denoting $\mathbf{X} = (X_1, \ldots, X_d)$, $\mathbf{x} = (x_1, \ldots, x_d)$, we obtain:

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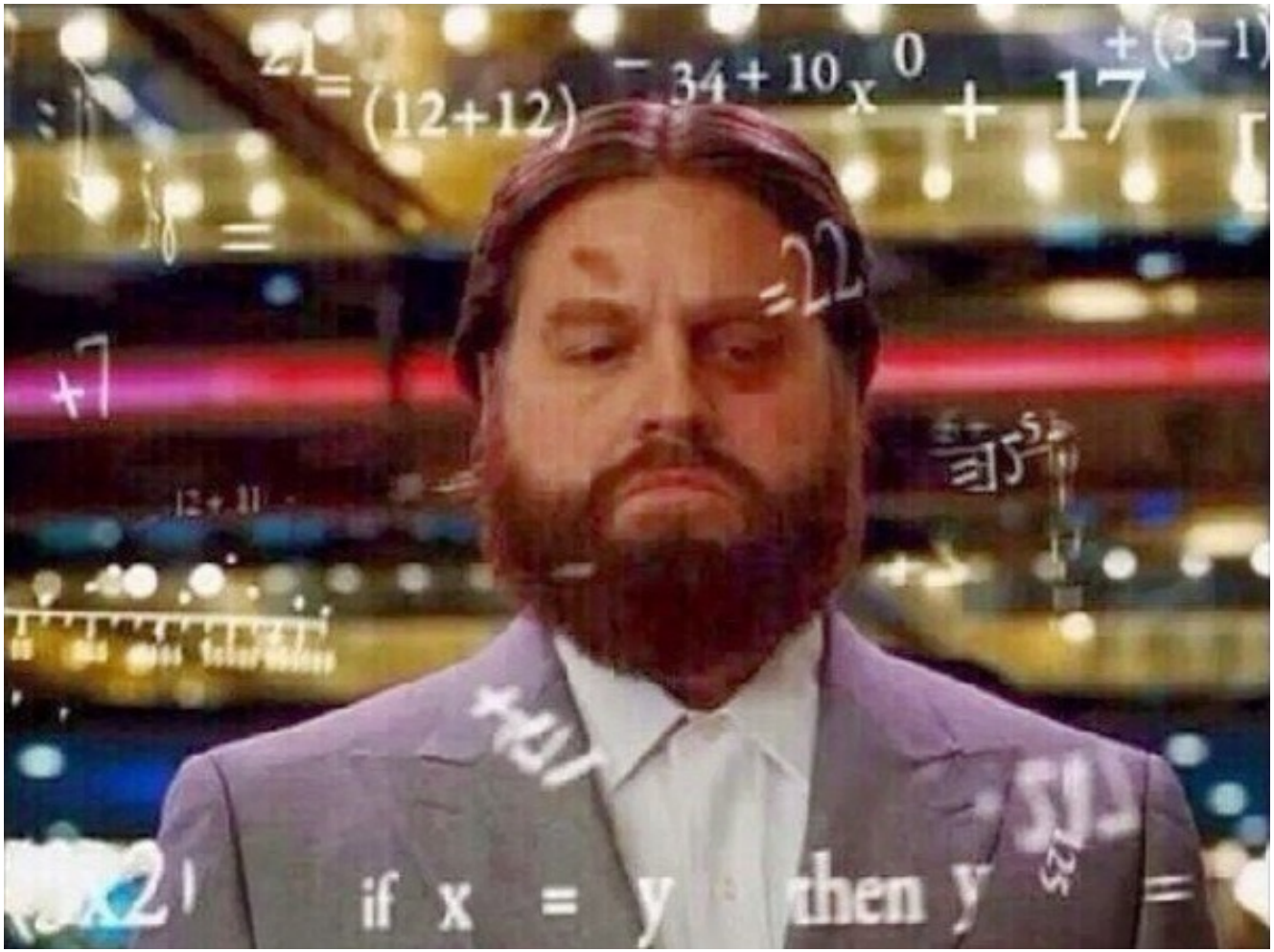
$$P(\mathbf{X} = \mathbf{x}|Y = c) \propto \prod_{j=1}^{d} p_{j,c}^{x_j}.$$

- Then, we classify by the following rule:

$$h_B(\mathbf{x}) := \operatorname*{argmax}_{c \in \mathcal{Y}} P(Y = c) \prod_{j=1}^{d} p_{j,c}^{x_j}$$

$$\propto \operatorname*{argmax}_{c \in \mathcal{Y}} P(Y = c) \sum_{j=1}^{d} x_j \log p_{j,c}.$$

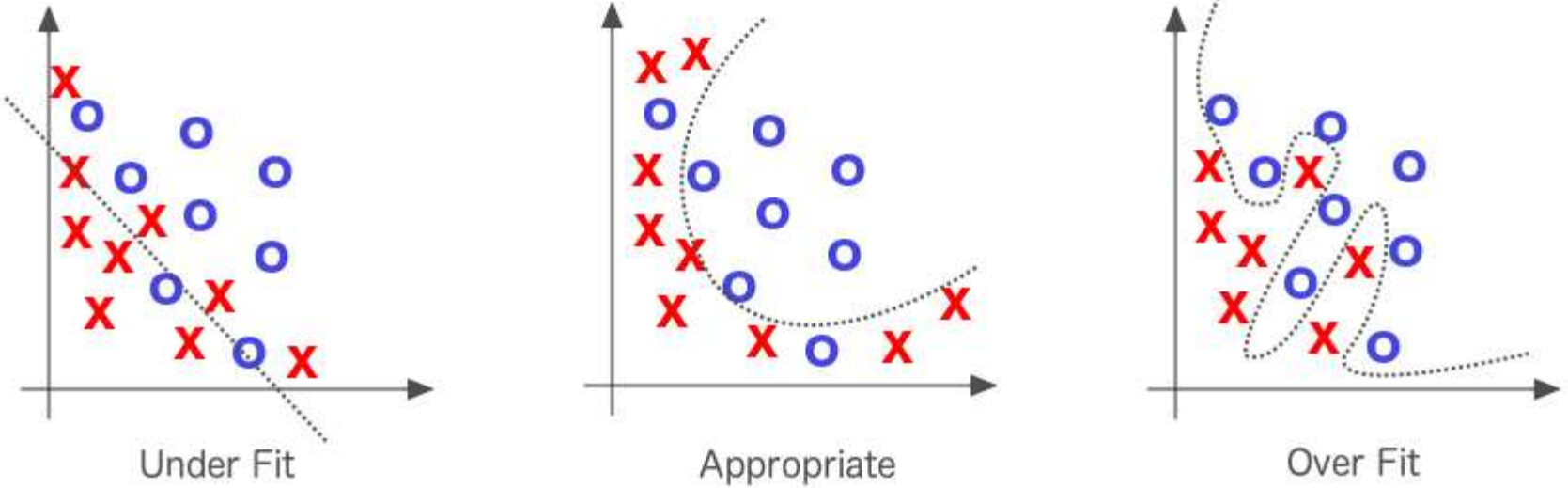
Why These Distributions?



Next Lecture

Outline

Performance of an Algorithm



Estimation of the Error Value

Question: How to estimate the error value (e.g. $P(h(\mathbf{X}) \neq Y)$) in practice when the sample data $S = \{\mathbf{x}_i, y_i\}_{i=1}^n$ is available only?

- Error on the training set?

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- Train/test split?

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- Cross-validation?

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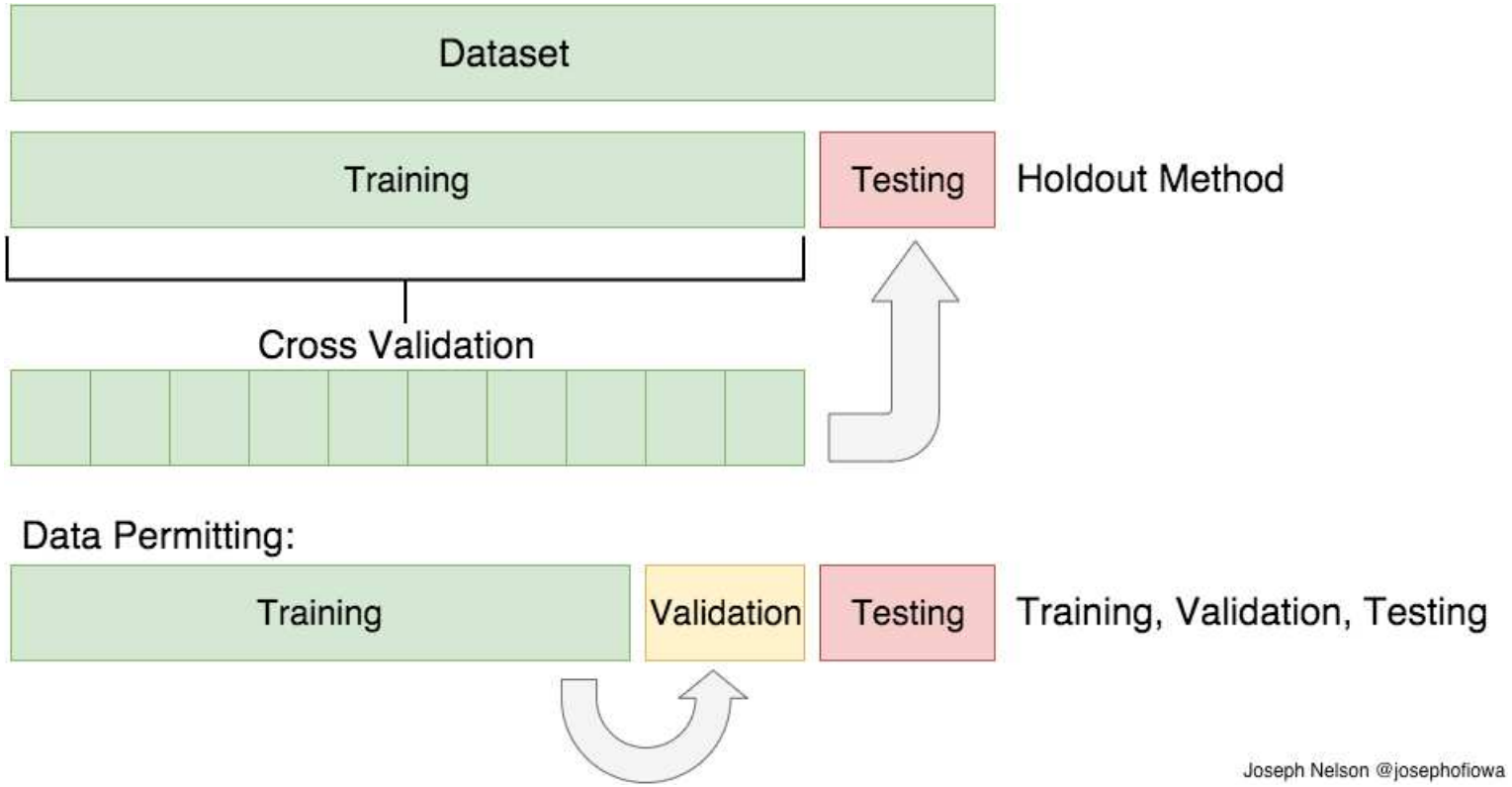
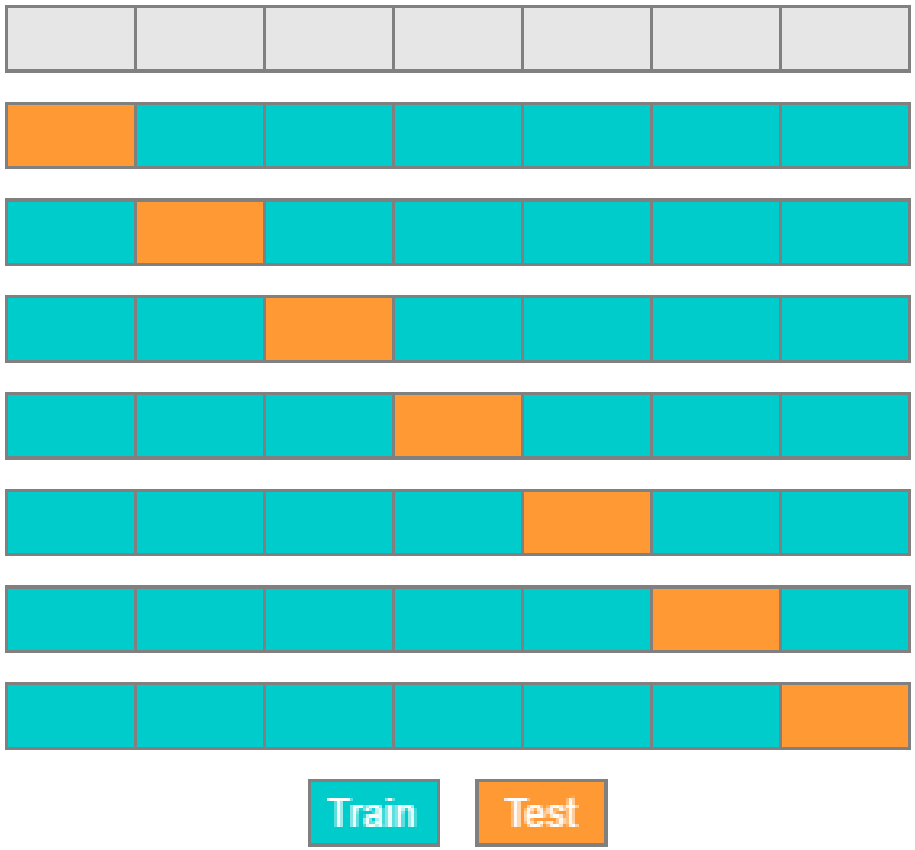
- Error on the training set? $\Rightarrow$ Overfitting.
- Train/test split? $\Rightarrow$ We should have enough data.
- Cross-validation? What is an impact of the number of folds?

Model Calibration + Performance Estimation



Leave-One-Out



Confusion Matrix

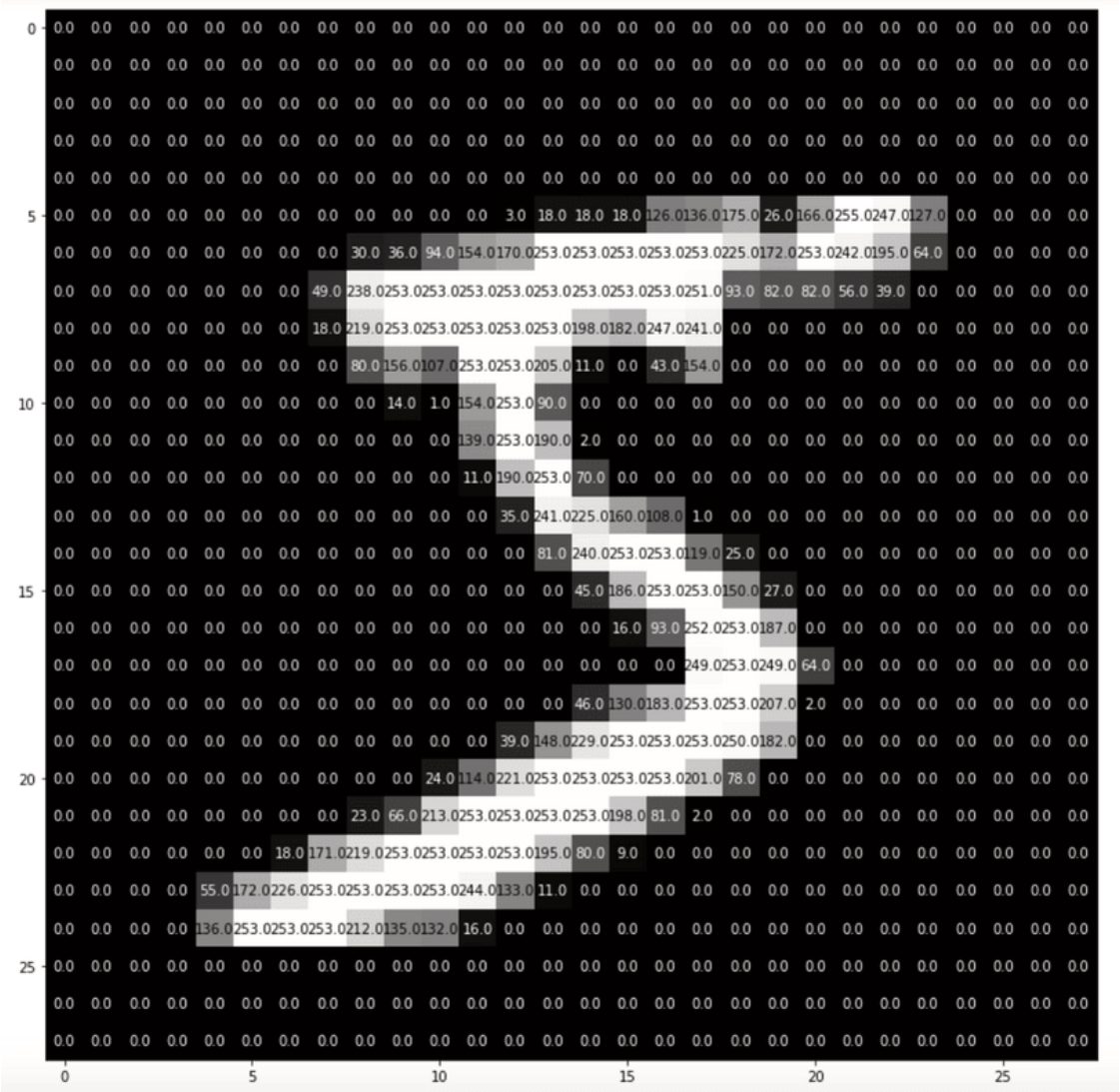
n=165	**Predicted: NO**	**Predicted: YES**
Actual: NO	50	10
Actual: YES	5	100

Script

Please follow the link:

`https://chamilo.grenoble-inp.fr/courses/ENSIMAG4MMSADM/document/DemoR/script_classif_part_I.html`

MNIST Features



Document Classification

Statistical Analysis and Document Mining

Spring 2019

Vasilii Feofanov, Massih-Reza Amini

Université Grenoble Alpes
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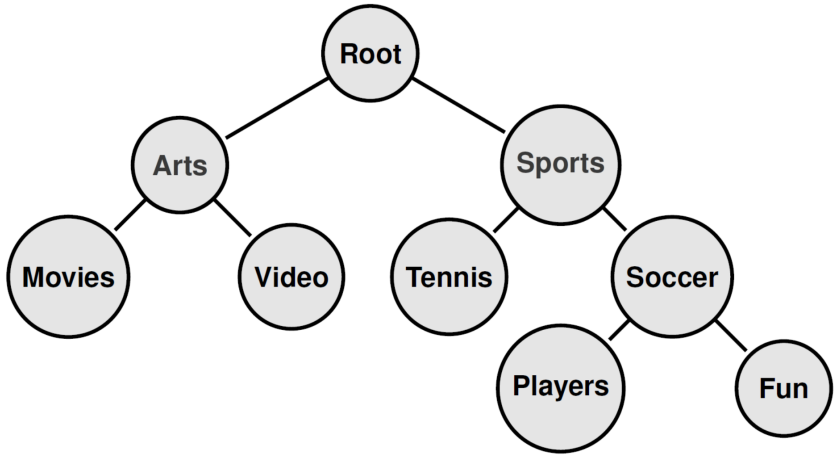
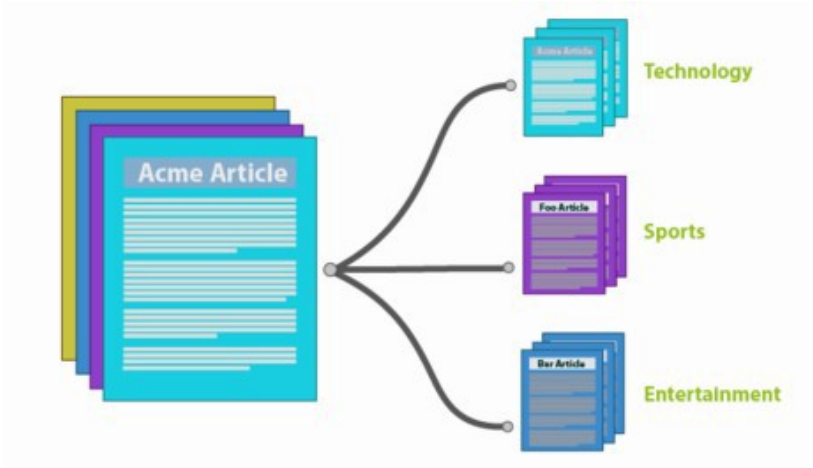
Outline

Automatic Document Classification

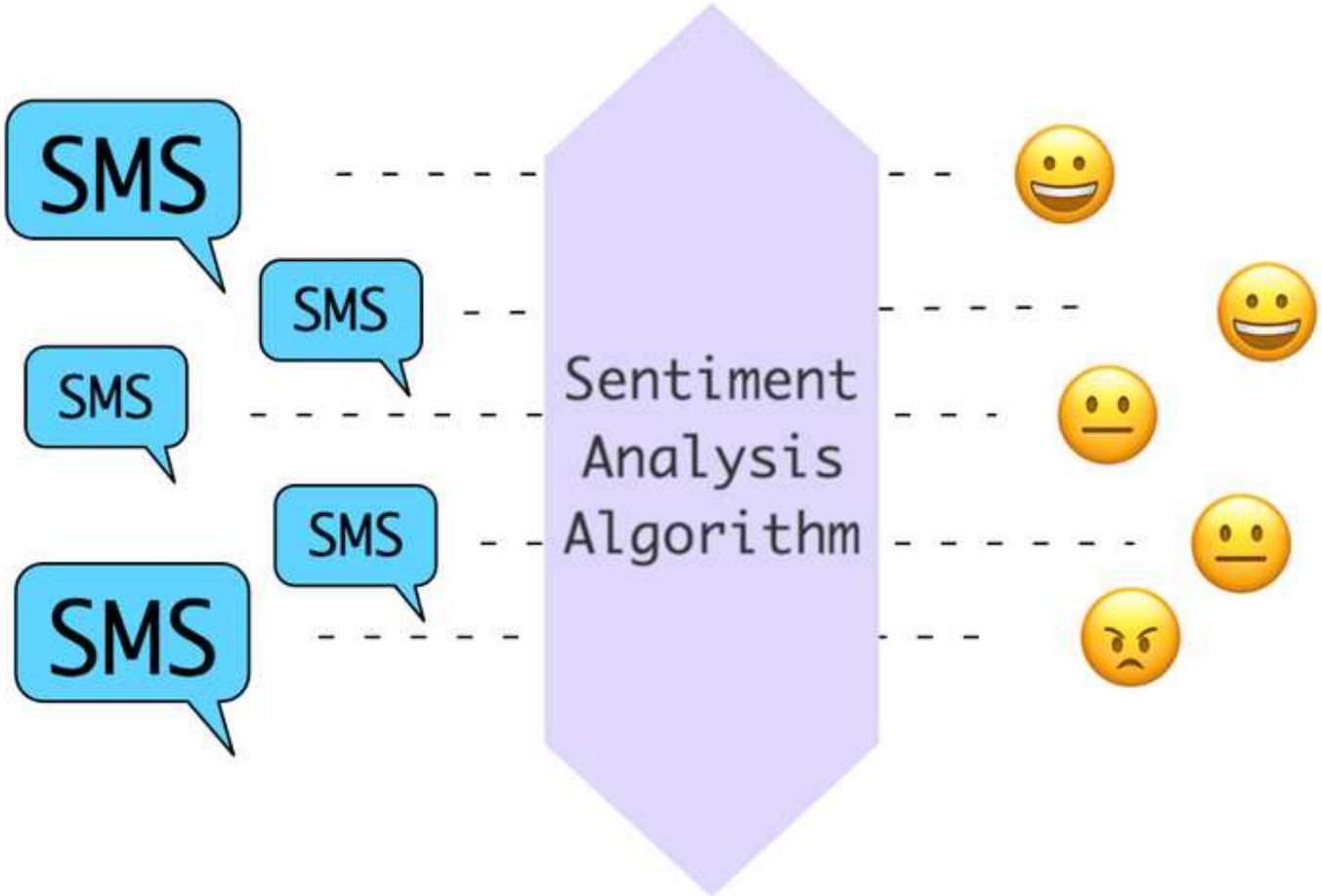
Application 1: Spam Detection

Application 2: Genre Classification



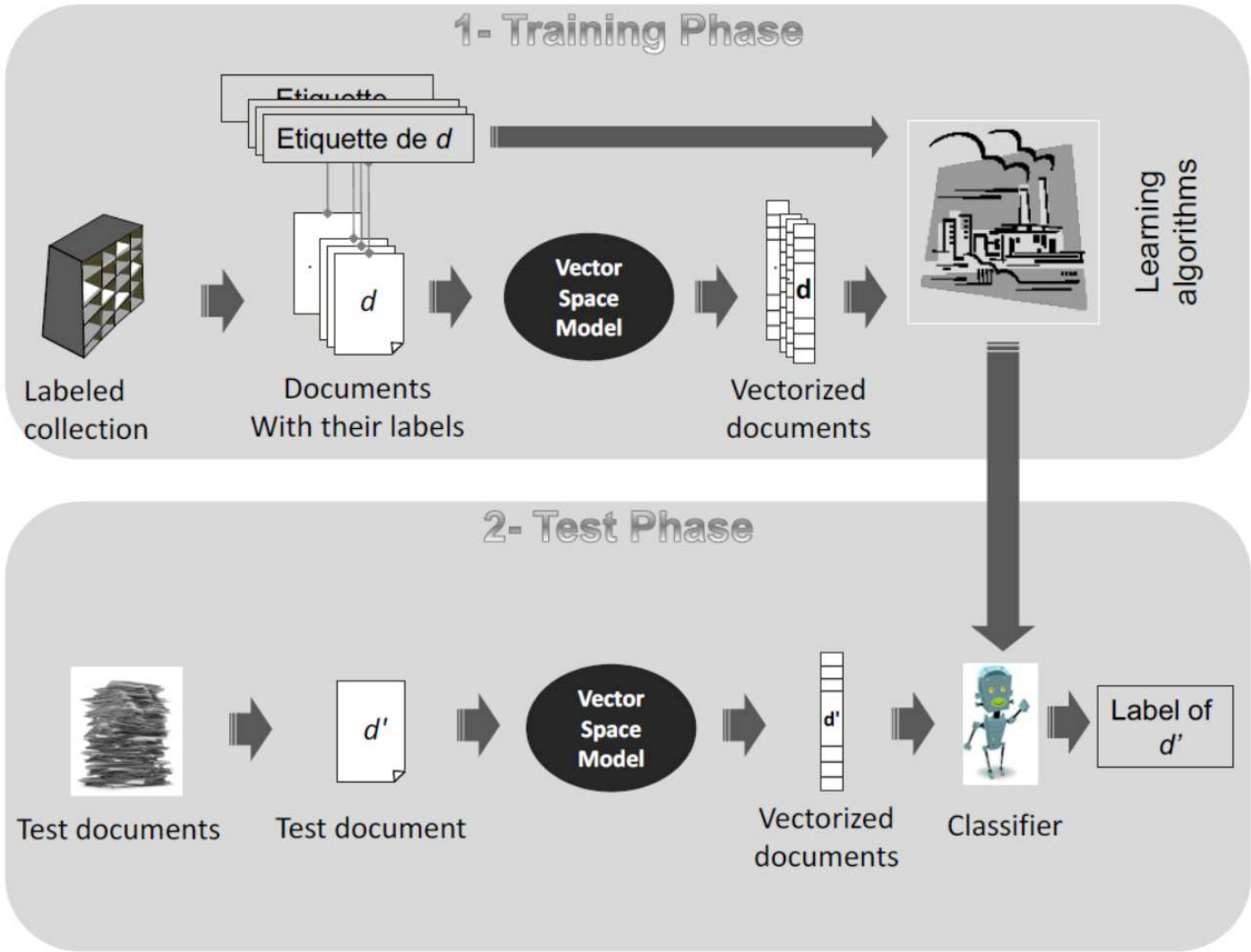


Application 3: Sentiment Analysis

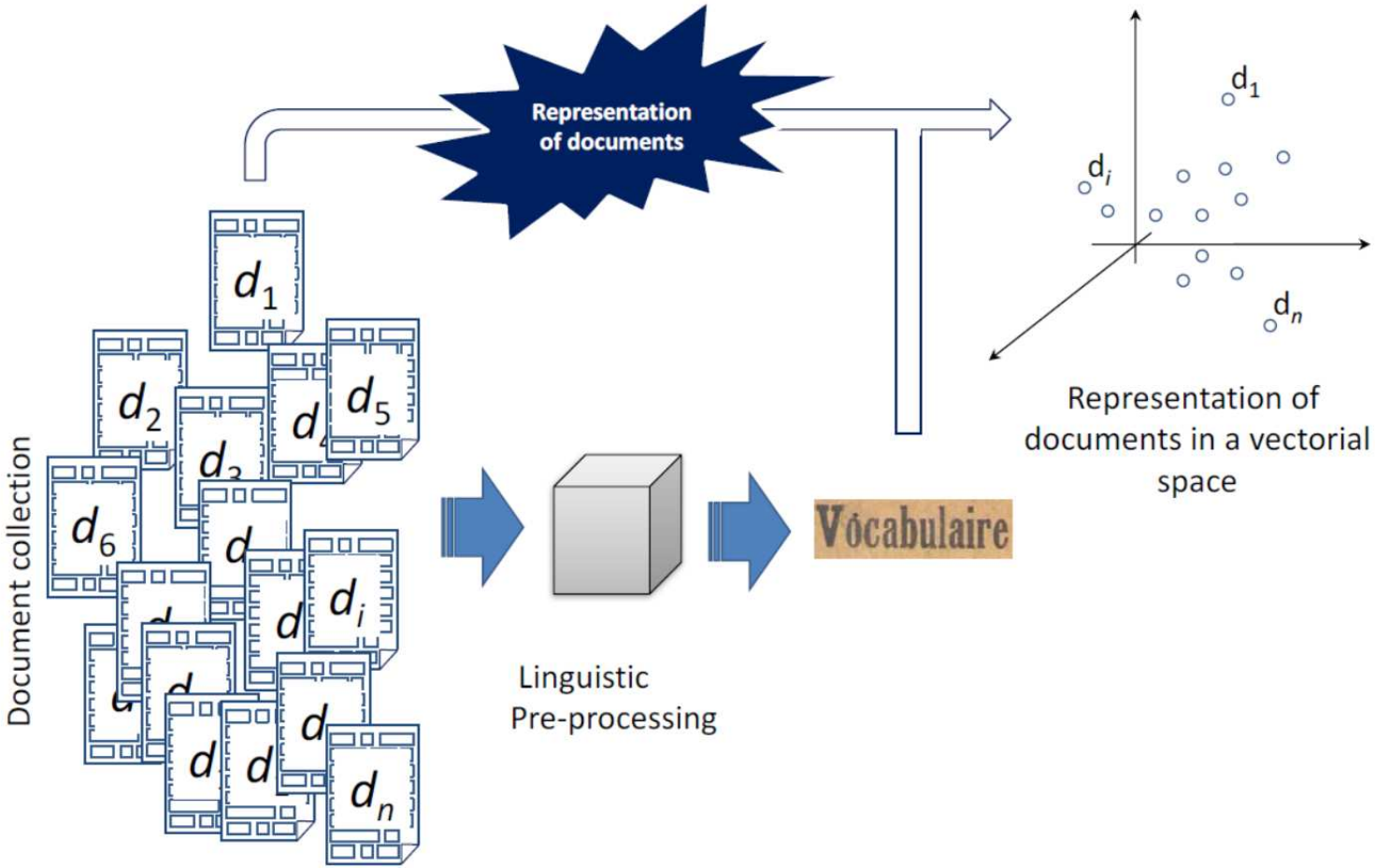


Application 4: Language Identification

Classification Scheme



From Document to Vector



Outline

Segmentation

- *Segmentation* (tokenization): separate a sequence of characters into semantic elements, or *words*.
- *Term* (type of words): class of all words having the same sequence of characters.
- *Example*:
  "The cat sat on the mat."
  Words: The, cat, sat, on, the, mat
  Terms: the, cat, sat, on, mat
- *Dificulty*: Tokenization is language specific.

Example: Segmentation In French

In French, the following issues may arise during the segmentation process:

- Lexical components with hyphens:
  chassé-croisé, peut-être, rendez-vous
- Lexical components with an apostrophe:
  jusqu'où, aujourd'hui, prud'homme
- Idiomatic expressions:
  au fait, poser un lapin, tomber dans les pommes
- Contracted forms:
  j', M'sieur, Gad'zarts (les gars des Arts et Métiers)
- Acronyms:
  K7, A.R., CV, càd, P.-V.

Normalization

1. *Textual normalization*: consists in reducing the words of a same family to their canonical forms.
   - Punctuation: suppression of points and hyphens;
   - Lower-upper case: transform all upper cases to lower cases;
   - Accents: suppression of accents.

2. *Linguistic normalization* consists in
   - Rooting: replace each word by its root;
   - Stemming: replace each word by its canonical form.

Examples of Preprocessing

Non-spam message before prepocessing

Subject: Re: 5.1344 Native speaker intuitions The discussion on native speaker

intuitions has been extremely interesting, but I worry that my brief intervention may have muddied the waters. I take it that there are a number of separable issues. The first is the extent to which a native speaker is likely to judge a lexical string as grammatical or ungrammatical per se. The second is concerned with the relationships between syntax and interpretation (although even here the distinction may not be entirely clear cut).

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Non-spam message after prepocessing

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Preprocessed Documents from Different Classes

Non-spam message after prepocessing

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Spam message after prepocessing

financial freedom follow financial freedom work ethic extraordinary desire earn least per month work home special skills experience required train personal support need ensure success legitimate homebased income opportunity put back control finance life ve try opportunity past fail live promise

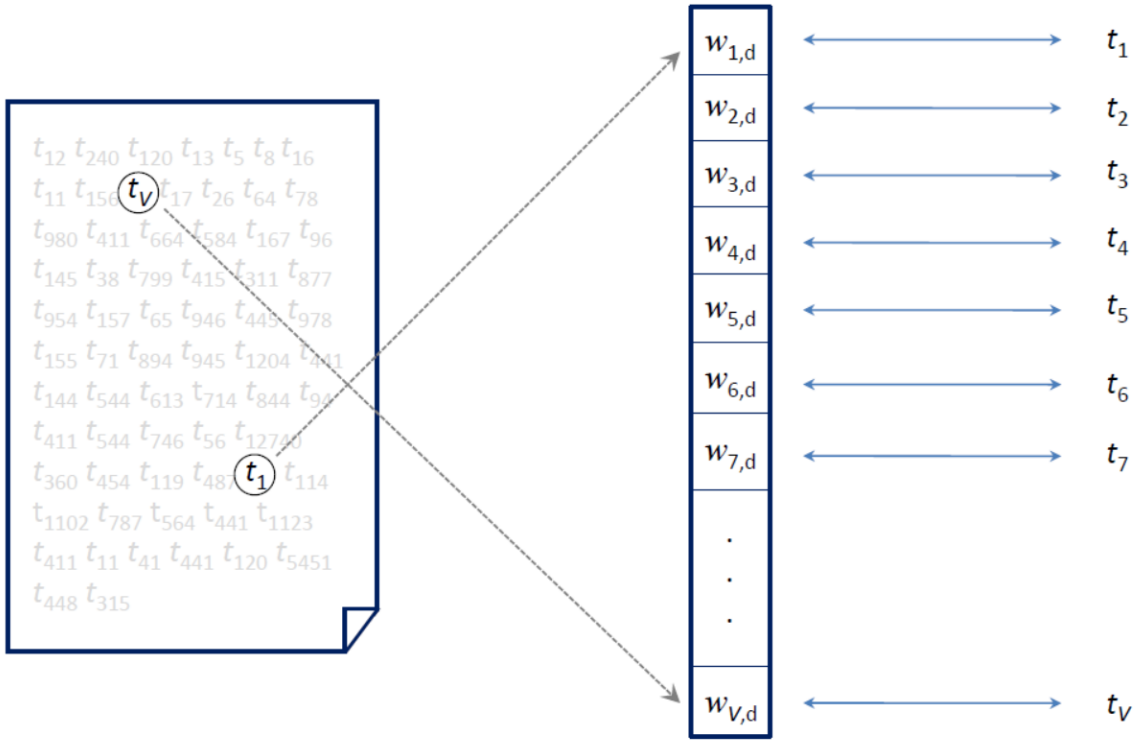
The Bag of Words Representation

I love this movie! It's sweet, but with satirical humor. The dialogue is great and the adventure scenes are fun... It manages to be whimsical and romantic while laughing at the conventions of the fairy tale genre. I would recommend it to just about anyone. I've seen it several times, and I'm always happy to see it again whenever I have a friend who hasn't seen it yet!

it	6
I	5
the	4
to	3
and	3
seen	2
yet	1
would	1
whimsical	1
times	1
sweet	1
satirical	1
adventure	1
genre	1
fairy	1
humor	1
have	1
great	1
...	...

Vector Space Model (Salton & Lesk, 1965)

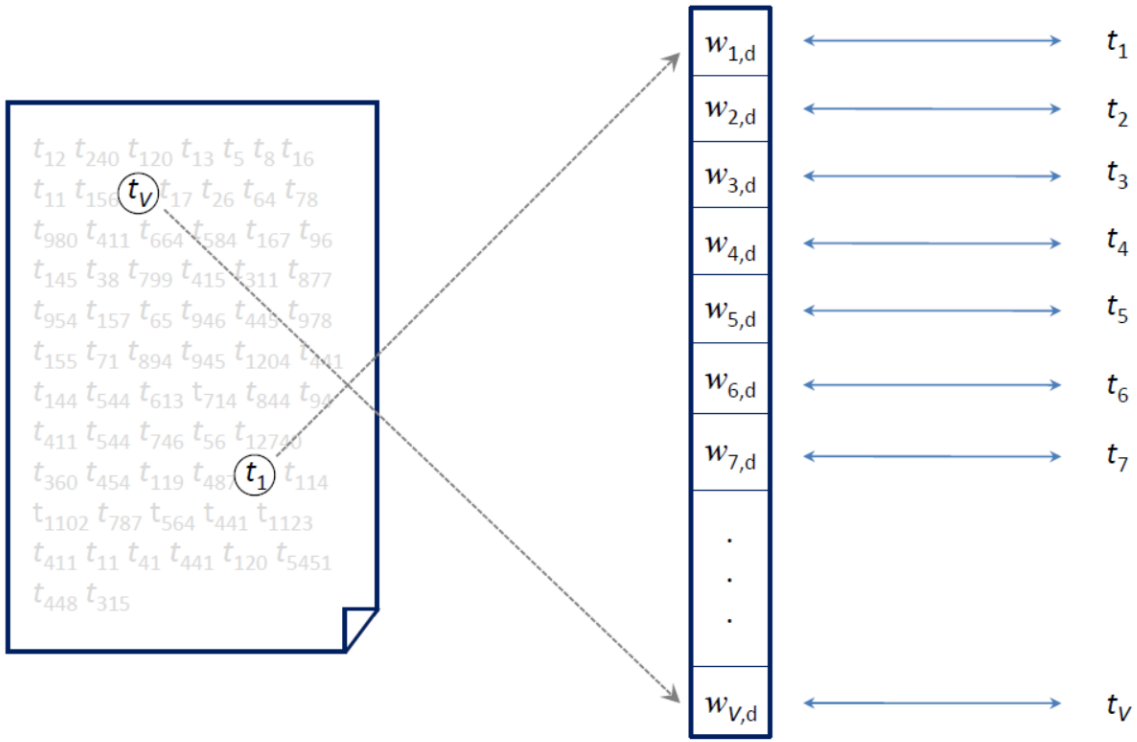
In the bag-of-words the word order in a sentence is ignored. Given a document, to each term t_j a specific value $w_{j,d}$ is assigned.



Vector Space Model (Salton & Lesk, 1965)

In the bag-of-words the word order in a sentence is ignored. Given a document, to each term t_j a specific value $w_{j,d}$ is assigned.

$\Rightarrow$ *A document can be represented as a vector:* $\mathbf{d} = (w_{j,d})_{j=1}^{V}$.



Document Classification Task

- Observation is a document $\mathbf{d} = (w_{j,d})_{i=1}^{V}$, where
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  - $\mathbf{d}_i \in \mathbb{R}^V$;
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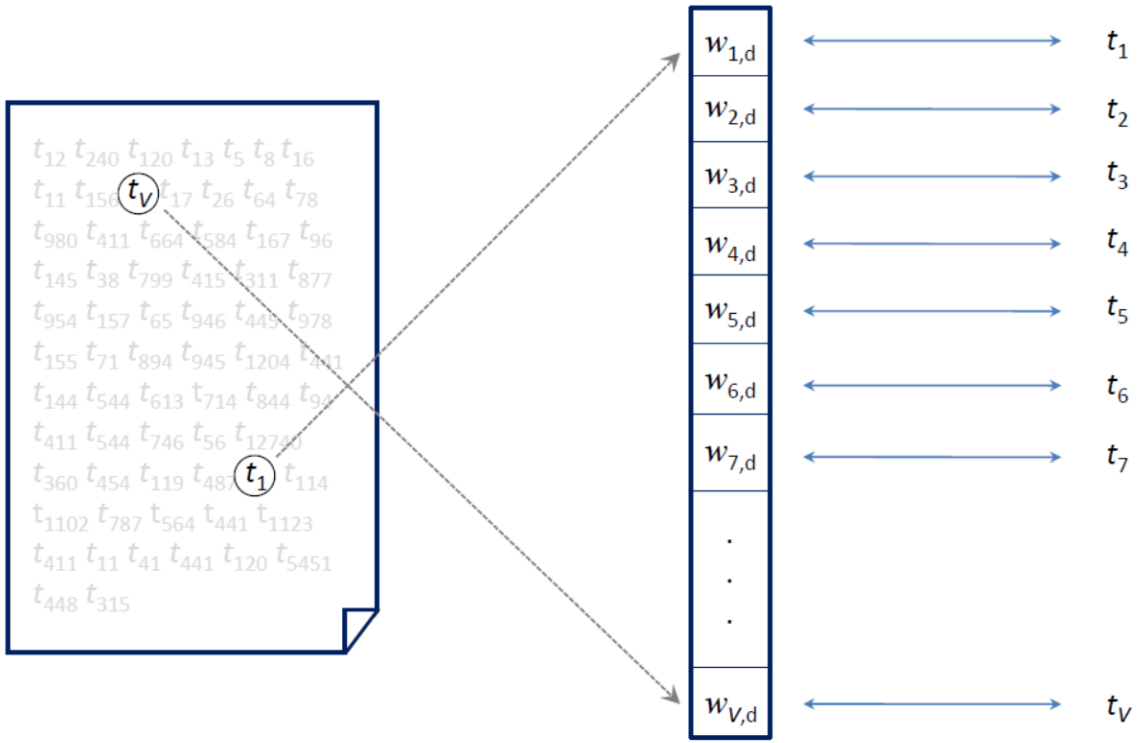
- *Target*: minimise the misclassification error:

$$P(h(\mathbf{D}) \neq Y) = \sum_{c \in \{1,\ldots,K\}} P(Y = c) P(h(\mathbf{D}) \neq c | Y = c).$$

How to Construct a Vector Space?

A document is represented as a vector: $\mathbf{d} = (w_{j,d})_{j=1}^{V}$.

How we define importance $w_{j,d}$, $j \in \{1, \ldots, V\}$?



First Approach: Binary Values

- If term t_j is present in the document, then $w_{j,d} = 1$. Otherwise, $w_{j,d} = 0$.

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- Suppose conditional independence of terms given a class.
- *Question:* How then the term t_j contributes into classification?
  - For each class, we can compute the probability that the term is present in a document from this class.
    $\Rightarrow$ Bernoulli distribution.

Bernoulli Naive Bayes for Documents

- Let $W_{j,d}|Y = c$ be distributed acc. to *Bernoulli*. Then, probability that t_j is present in a document $\mathbf{d}$ is:

$$P(W_{j,d}|Y = c) = p_{t_j|c}.$$

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$$h_{NB}(\mathbf{d}) := \operatorname*{argmax}_{c \in \mathcal{Y}} P(Y = c) \prod_{j=1}^{V} P(W_{j,d} = w_{j,d}|Y = c)$$

$$= \operatorname*{argmax}_{c \in \mathcal{Y}} P(Y = c) \prod_{j=1}^{V} p_{t_j|c}^{w_{j,d}} (1 - p_{t_j|c})^{1-w_{j,d}}.$$

Bernoulli NB: Train/Prediction

- Estimation of priors:

$$P(Y = c) \leftarrow \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(y_i = c) =: \frac{n_c}{n}.$$

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$$\hat{p}_{t_j|c} = \frac{\sum_{i=1}^{n} \mathbb{I}(w_{j,d_i} = 1 \wedge y_i = c)}{\sum_{i=1}^{n} \mathbb{I}(y_i = c)} =: \frac{\mathrm{df}_{t_j}(c)}{n_c}.$$

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$$h_{NB}(\mathbf{d}) := \operatorname*{argmax}_{c \in \mathcal{Y}} \ln \frac{n_c}{n} + \sum_{\substack{j=\{1,\ldots,V\} \\ t_j \in d}} \ln \hat{p}_{t_j|c} + \sum_{\substack{j=\{1,\ldots,V\} \\ t_j \notin d}} \ln(1 - \hat{p}_{t_j|c}).$$

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- *Naive* assumption: all N_d words are *i.i.d.* by the following law:

$$P(\text{word} = t_j) = p_{t_j|c},\ j = \{1, \dots, V\},\ \sum_{j=1}^{V} p_{t_j|c} = 1.$$

$\Rightarrow$ Then, document $\mathbf{d}$ is distributed acc. to multinomial distribution with parameters $(p_{t_j|c})_{j=1}^{V}$.

Multinomial Naive Bayes for Documents

- Let $\mathbf{D}|Y = c$ be distributed acc. to the *multinomial law*. Then, the likelihood of $\mathbf{D}$ to be from the class c is:

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Outline

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  - Around 16 GB!

French Wikipedia Collection: Statistics

Variables	Values
# of documents in the collection	**1,349,539**
Total # of occurrences of words	696,668,157
Average # of words per document	416
Size of the pre-processed collection on the disk	4.6 GB
Total # of types of words	757,476
Total # of types of words after rooting	604,244
Size of the vocabulary	**604,244**
Average # of terms per document	**225**
Size of the collection after removing a stop-list	2.8 GB

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- *LibSVM format:* the observation is written in the following format:

y	index-value		index-value
2	5:0.356	...	9:1000
3	2:10.2	...	15:0.01

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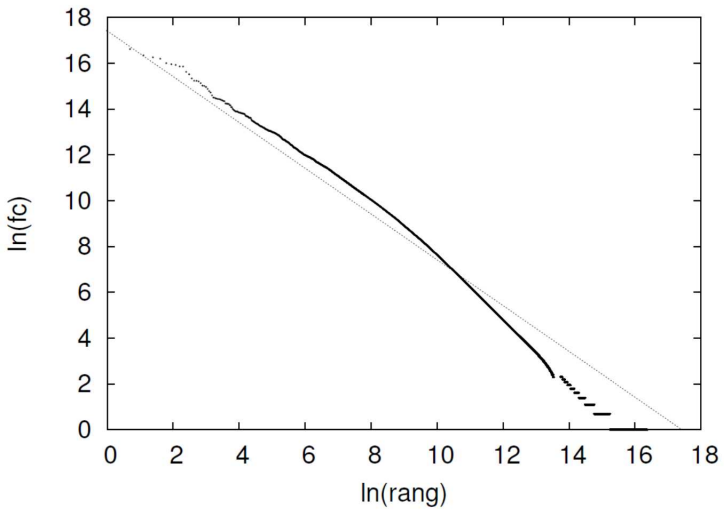
- When the collection was filtered by removing a stop-list ("a", "the", "of", etc.) of size 200 words, the average number of terms was reduced in documents from 416 to 225 (around 45% reduction). Why?
  In addition, their filtering reduces the space on the disk of about 39% (from 4.6 GB to 2.8 GB).

Zipf's Law

The number of occurrences $fc(m)$ of a word m in a document collection is inversely proportional to its rank:

$$\forall m: \quad fc(m) \approx \frac{\lambda}{\text{rang}(m)}.$$

$\Rightarrow$ The k-th most frequent word is approximately k times less present than the most frequent one.



Rank	Word	Freq.	%
1	the	22,038,615	4.9%
2	be	12,545,825	2.79%
3	and	10,741,073	2.39%
4	of	10,343,885	2.3%
5	a	10,144,200	2.25%
6	in	6,996,437	1.56%
7	to (i.m.)	6,332,195	1.41%
8	have	4,303,955	0.96%
9	to (p.)	3,856,916	0.86%
10	it	3,872,477	0.86%

Top 10 frequent word from the 450 million word corpus (`https://www.wordfrequency.info`).

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- How many frequent words we should suppress?

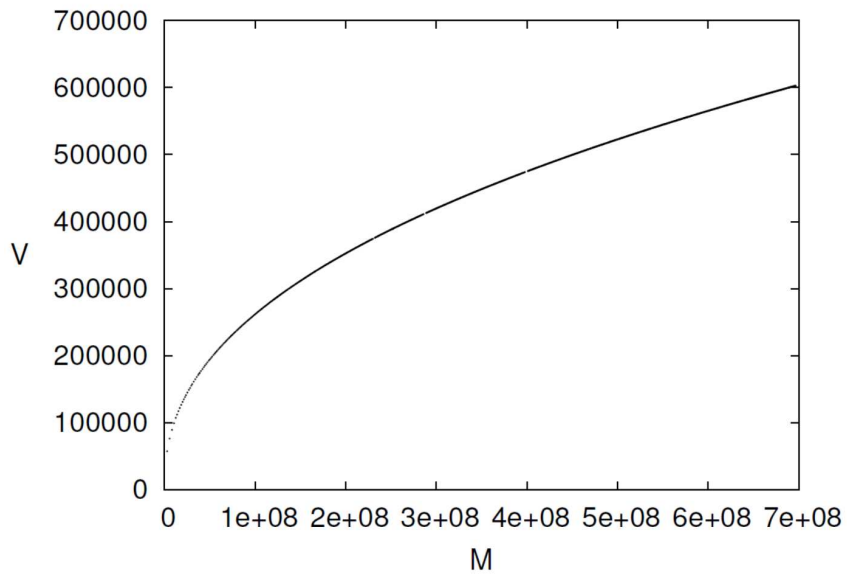
Heaps' Law

The size of the vocabulary V increases sub-linearly with respect to the number of words present in a collection M:

$$V = k \cdot M^{\beta},$$

where k and β are parameters that are dependent on the collection. Typically, in English text corpora $k \in [10, 100]$, and $\beta \in [0.4, 0.6]$.

$\Rightarrow$ Larger the collection size, larger the vocabulary size.



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You are willing to analyse a document containing 1,000,000 words.

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- It was found that the 7% of all words are "the" article. Following the Zipf's law, estimate the value of λ. What is the frequency of the second most frequent word? The third most frequent one?

Term Frequency vs Document Frequency

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  Example: Medical Prescription vs Recipe.

take	water	glass	eat	wait	...	paracetamol	sugar	stomach
7	6	4	4	4	...	0	2	0
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- We want to diminish the weight of terms that occur frequently in general and increase the weight of terms that occur rarely in average.

tf-idf weighting is a trade-off between term frequency and document frequency:

- Normalised term frequency (tf part):

$$\frac{\mathrm{tf}_{t_j,d}}{\sum_{j=1}^{V} \mathrm{tf}_{t_j,d}} = \frac{\mathrm{tf}_{t_j,d}}{N_d}.$$

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$$\frac{\mathrm{tf}_{t_j,d}}{\sum_{j=1}^{V} \mathrm{tf}_{t_j,d}} = \frac{\mathrm{tf}_{t_j,d}}{N_d}.$$

- Inverse document frequency (idf part):

$$\ln \frac{n}{\mathrm{df}_{t_j}} := \ln \frac{n}{\sum_{i=1}^{n} \mathbb{I}(\mathrm{tf}_{t_j,d_i} \neq 0)}.$$

tf-idf weighting is a trade-off between term frequency and document frequency:

- Normalised term frequency (tf part):

$$\frac{\mathrm{tf}_{t_j,d}}{\sum_{j=1}^{V} \mathrm{tf}_{t_j,d}} = \frac{\mathrm{tf}_{t_j,d}}{N_d}.$$

- Inverse document frequency (idf part):

$$\ln \frac{n}{\mathrm{df}_{t_j}} := \ln \frac{n}{\sum_{i=1}^{n} \mathbb{I}(\mathrm{tf}_{t_j,d_i} \neq 0)}.$$

- Then, the tf-idf weight is defined as:

$$w_{t_j,d} = \frac{\mathrm{tf}_{t_j,d}}{N_d} \ln \frac{n}{\mathrm{df}_{t_j}}.$$

Exercise 2

- We have the following training documents:

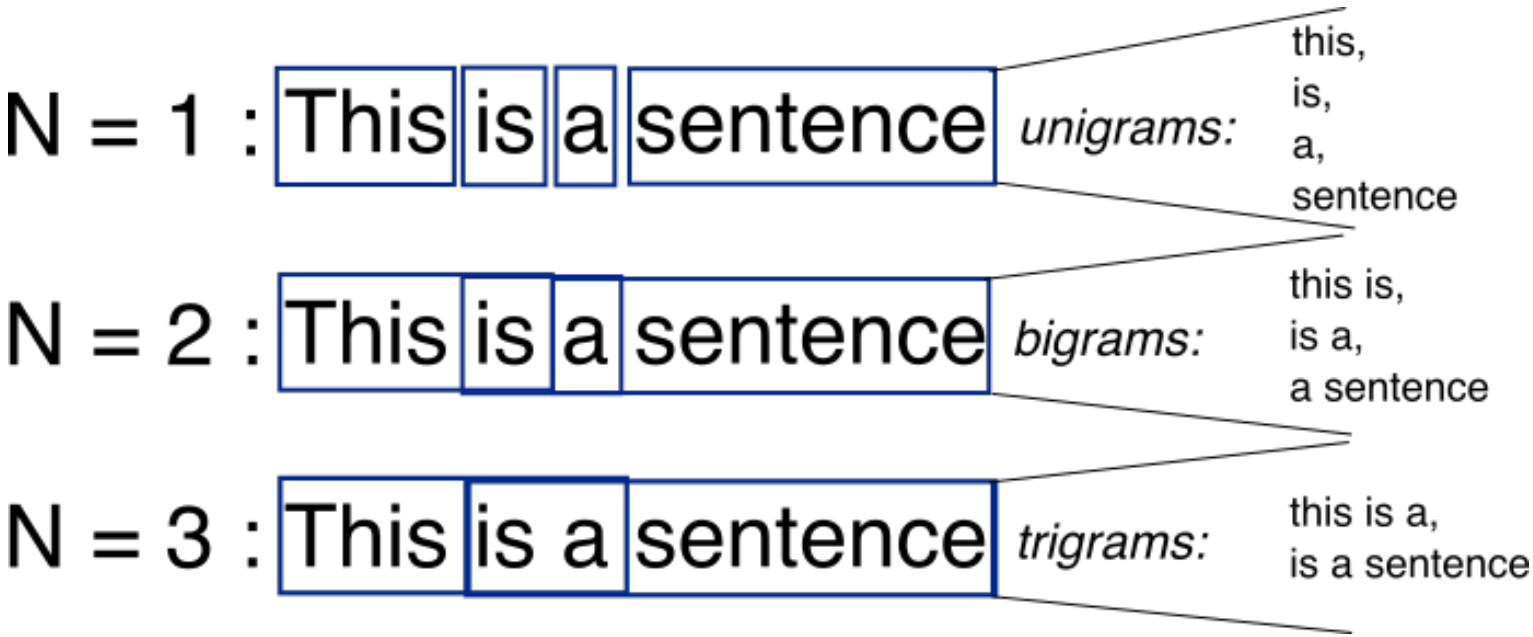
  d_1 : {"cat", "sit, "mat", "cat", "jump", "bed", "cat", "good", "sit"}

  d_2 : {"cat", "dog", "jump", "sit", "dog", "cat", "animal"}

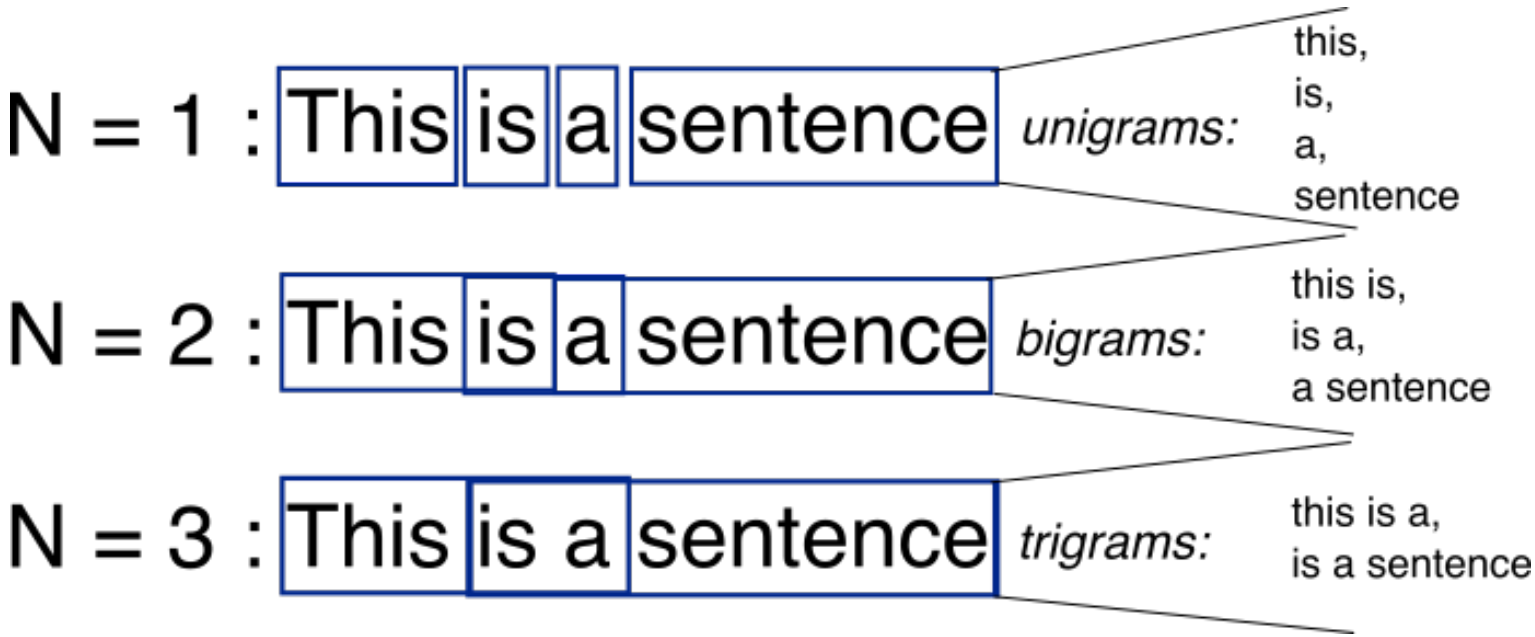
  d_3 : {"table", "sit", "write", "think", "good", "book", "dog", "sit"}

- What is the tf-idf representation for this dataset?

N-gram Representation

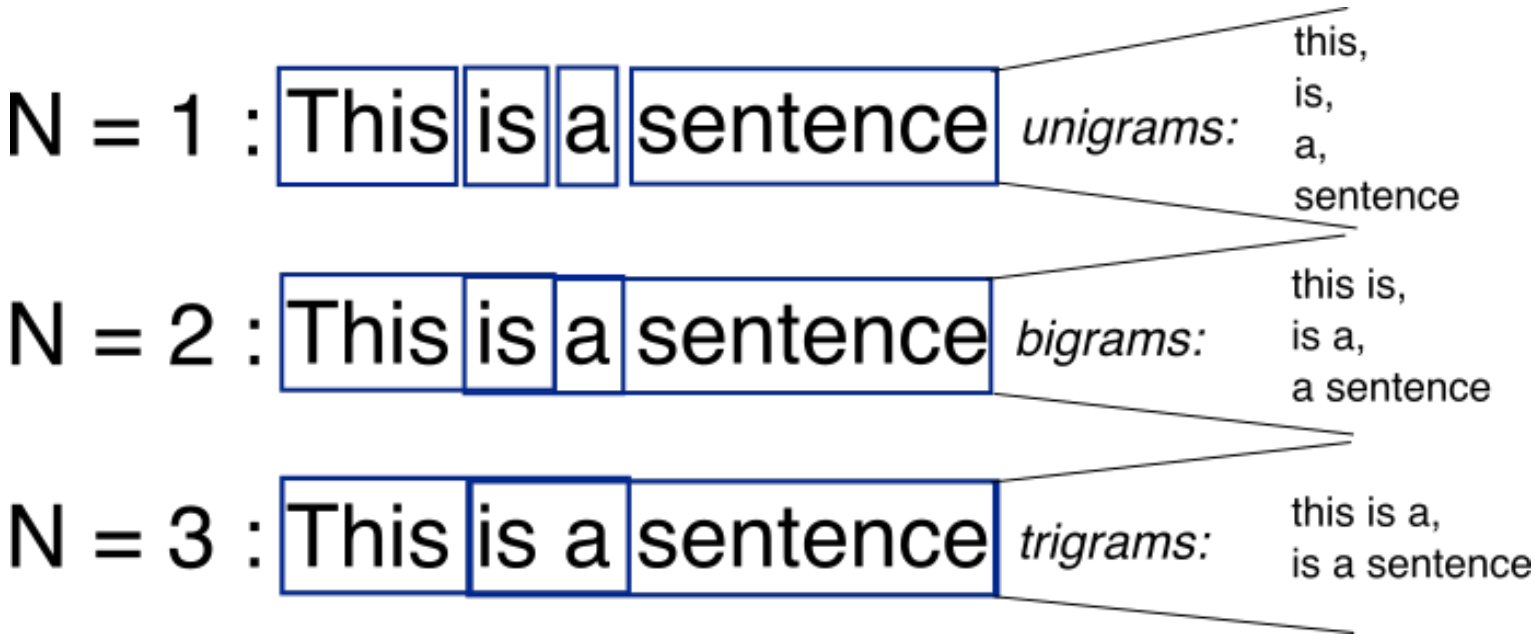


N-gram Representation



- Several words could be more important when they appear together.

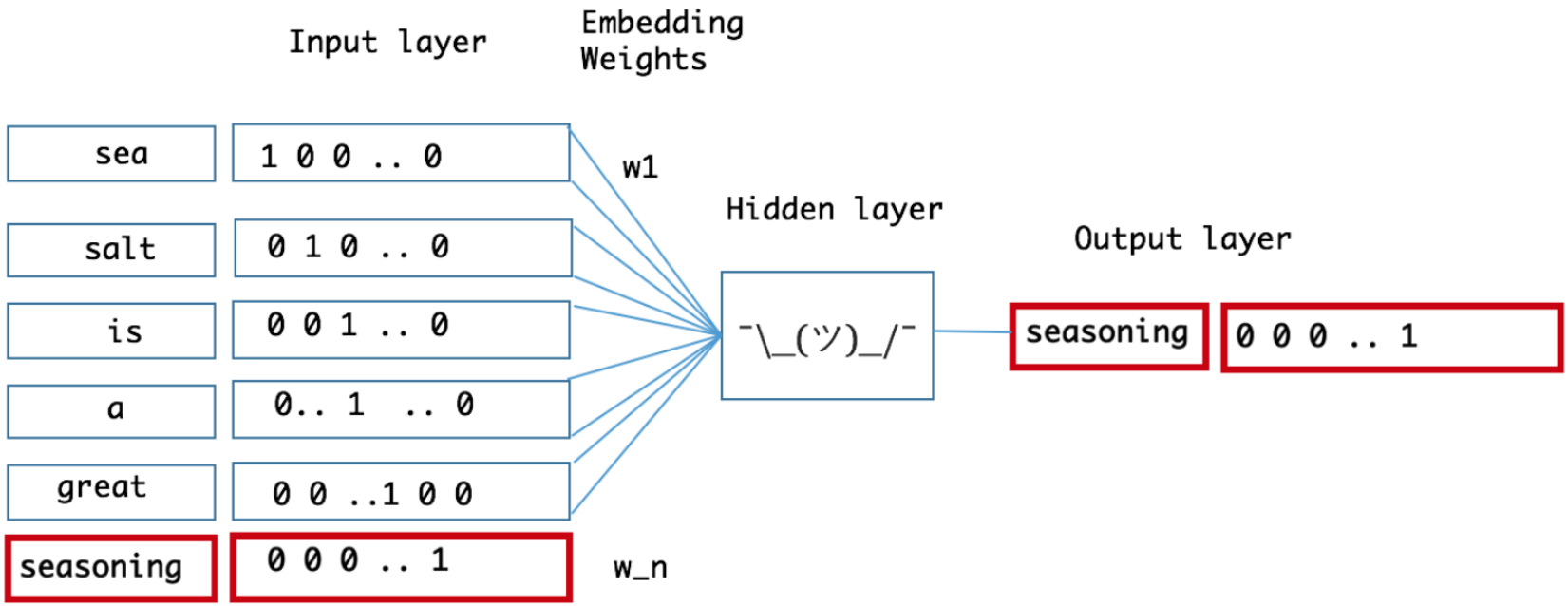
N-gram Representation



- Several words could be more important when they appear together.
- The approach is very costly when V is large.

...ATTACACGGTGACCAACCTATT...

Gram	Frequency
ATTA	4
GACC	3
CGGT	5
...	...


Input layer
Embedding Weights
sea
1 0 0 .. 0
w1
salt
0 1 0 .. 0
is
0 0 1 .. 0
a
0.. 1 .. 0
great
0 0 ..1 0 0
seasoning
0 0 0 .. 1
w_n
Hidden layer
¯_(ツ)_/¯
Output layer
seasoning
0 0 0 .. 1