Overview

1) Linear algebra → vectors and matrices recap
   Systems of equations, matrix inverse
   Eigenvalues and eigenvectors.

2) Calculus → Ordinary differential equations (ODEs)
   Numerical methods
   Systems of ODEs.

3) Stochastic ODEs → Definition + comparison to ODEs
   Stochastic LIF model

After each section, we will do some exercises and go through the jupyter notebooks.

Cover page

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LINEAR

ALGEBRA

Vectors and matrices

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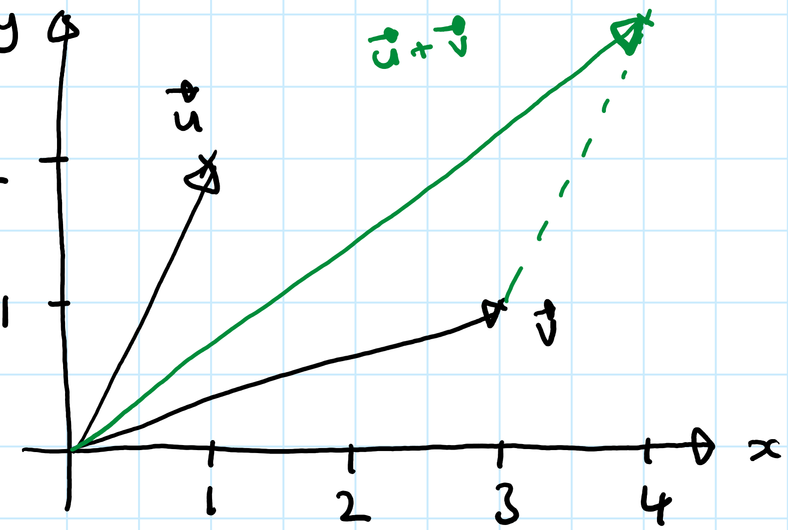
VECTORS AND MATRICES

Geometric view, $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



Add vectors, $\vec{u} + \vec{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

Dot product, $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3 + 2 = 5$

Length of a vector, $|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{1^2 + 2^2} = \sqrt{5}$

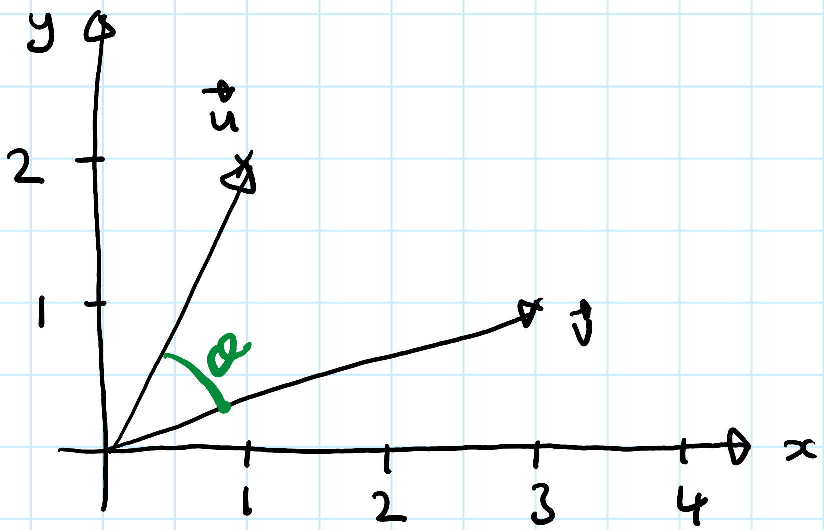
$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{3^2 + 1^2} = \sqrt{10}$

Pythagoras theorem

VECTORS AND MATRICES

Geometric view, $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



angles between vectors, $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$

perpendicular $\Rightarrow$ $\vec{a} \cdot \vec{b} = 0$, e.g.) $\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\rightarrow$ vectors are at right angles

we define unit vectors as having length 1, e.g.) $\hat{u} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Vectors and matrices

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We can use <u>matrices</u> to transform vectors.

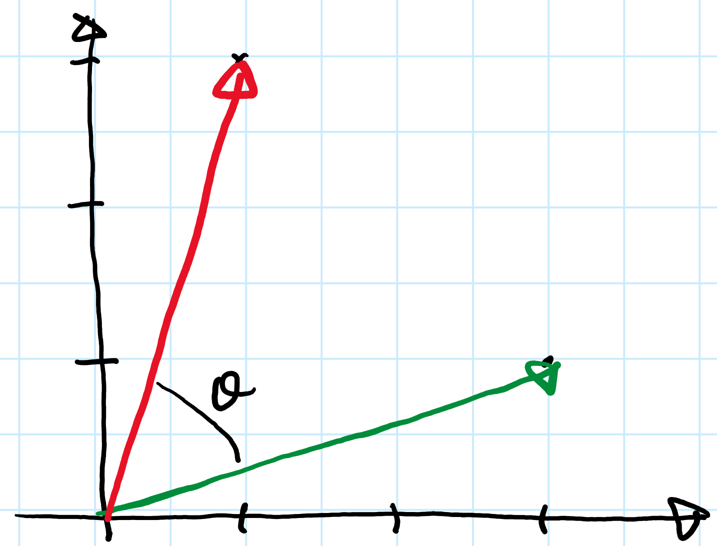
Say we have, $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, we can transform it like

$$\vec{v}' = A\vec{v}, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\vec{v}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Identity matrix: $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{1}\vec{u} = \vec{u}$

(does nothing but shows up everywhere)



Rotation matrix: $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\vec{u} = R(\theta)\vec{v}$, preserves lengths, $|\vec{u}| = |\vec{v}|$.

SYSTEMS OF EQUATIONS

Say we have, $\begin{aligned} 2x + y &= 1 \\ 5x + y &= 4 \end{aligned} \longrightarrow \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$A \quad \vec{u} = \vec{v}$$

Solve with the matrix inverse.

The matrix inverse is defined s.t. $A^{-1} A = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

If we know the inverse, then:

$$A \vec{u} = \vec{v}$$

$$\underbrace{A^{-1} A}_{\mathbb{1}} \vec{u} = A^{-1} \vec{v}$$

$$\vec{u} = A^{-1} \vec{v}$$

SYSTEMS OF EQUATIONS

Calculating the inverse, for a 2D matrix.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(M) = ad - bc$$

(determinant)

$$\rightarrow \quad M^{-1} = \frac{1}{\det(M)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \longrightarrow \text{ check } M^{-1}M = \mathbb{1}.$$

For $A = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix}$, $\det(A) = 8 - 5 = 3$, $A^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -5 & 2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \longrightarrow \quad x = 0, \; y = 1.$$

If $\det(M) = 0 \Rightarrow M$ has no inverse.

Eigenvalues and eigenvectors

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EIGENVALUES & EIGENVECTORS

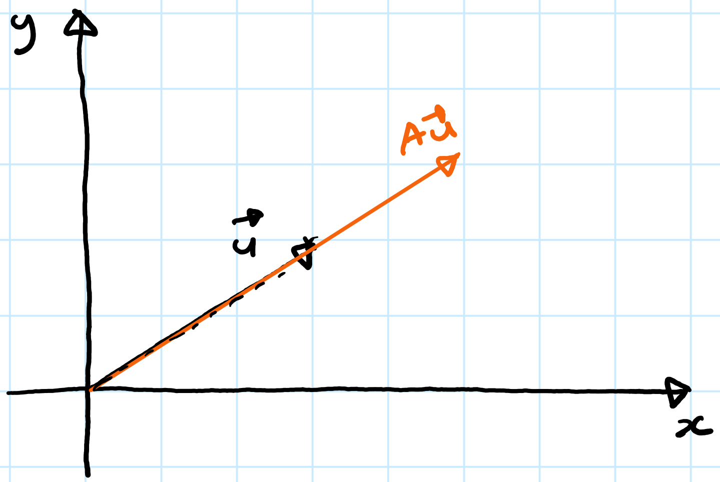
Eigenvalue equations

$$\underbrace{A}_{\text{matrix}} \; \vec{u} = \overbrace{\lambda}^{\text{eigenvalue}} \; \vec{u}$$

eigenvectors

[before, we had $A\vec{u} = \vec{v}$]

The transformation A doesn't rotate the eigenvectors, it changes their length by a factor of λ, the eigenvalue



A 2×2 matrix can have 2 eigenvalues and eigenvectors.

But not all matrices have them.

Also means that
$$A A \vec{u} = \lambda^2 \vec{u}$$
$$A^3 \vec{u} = \lambda^3 \vec{u}$$
$$A^n \vec{u} = \lambda^n \vec{u}$$

Eigenvalues and Eigenvectors

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We can rearrange the matrix equation to

$$A\vec{u} = \lambda\vec{u} \longrightarrow (A - \lambda \mathbb{1})\vec{u} = 0$$

Goal: find $\vec{u}$ and λ for A.

$$\vec{u} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underbrace{\det(A - \lambda\mathbb{1}) = 0}$$

↳ solve this equation to find λ.

Say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A - \lambda\mathbb{1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

$$\det(A - \lambda\mathbb{1}) = (a-\lambda)(d-\lambda) - bc = 0.$$

→ solve to find λ_1 and λ_2

then plug them into the eigenvalue equation to find the eigenvectors.

Eigenvalues and Eigenvectors

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Example

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}, \quad A - \lambda \mathbb{1} = \begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix}$$

$$\det(A - \lambda \mathbb{1}) = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10$$

$$= (\lambda - 2)(\lambda - 5)$$

$$\rightarrow \lambda_1 = 2, \quad \lambda_2 = 5$$

eigenvectors: $\lambda_1 : (A - \lambda \mathbb{1})\vec{u} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$2x + y = 0$$

$$\lambda_1 = 2, \quad u_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 : (A - \lambda \mathbb{1})\vec{u} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x - y = 0$$

$$\lambda_2 = 5, \quad u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

DIAGONALIZATION

In general, say A is an $n \times n$ matrix $\rightarrow$ has n eigenvalues & eigenvectors.

Define a matrix $X = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{pmatrix}$

each column is an eigenvector.

$$\Rightarrow \quad AX = \begin{pmatrix} \lambda_1 \vec{u}_1 & \lambda_2 \vec{u}_2 & \cdots & \lambda_n \vec{u}_n \end{pmatrix}$$

$$= X\Lambda \quad , \quad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

With the inverse of X, we get $X^{-1} A X = \Lambda \qquad (*)$

We say that X diagonalises A.

$\rightarrow$ Also $A = X \Lambda X^{-1}$, $A^2 = X \Lambda \underline{X^{-1} X} \Lambda X^{-1} = X \Lambda^2 X^{-1}$

$$\ldots \quad A^k = X \Lambda^k X^{-1} \quad , \quad \Lambda^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots & \\ & & & \lambda_n^k \end{pmatrix}$$

Questions

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QUESTIONS

1) For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, check that $M^{-1} M = \mathbb{1}$.

2) Solve $2x + y = 5$, $x - y = 1$

3) Solve $5x + 2y = 14$, $3x - 4y = -2$

4) Solve $-2x + y = 3$, $4x + 5y = -6$

} using matrix inverses

5) Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}$

Plot the eigenvectors, and show how transforming them by A changes them.

Cover page

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CALCULUS

ORDINARY DIFFERENTIAL EQUATIONS

DEs tell us how quantities change with time or position.

Eg) If we have a function $y(t)$, $\frac{dy}{dt} = \frac{y(t+dt) - y(t)}{dt}$ → rate of change of $y(t)$.

→ the DE : $\frac{dy}{dt} = \underbrace{f(y,t)}$

this function tells us how $y(t)$ changes in time.

Goal : find the solution $y(t)$ exactly

this is not always possible → numerical approximations to the solution

→ 'simulations'

Exact solution examples

We have some population $N(t)$ that changes in time.

Described by: $\frac{dN}{dt} = aN$

We can guess the solution because we know how exponentials work.

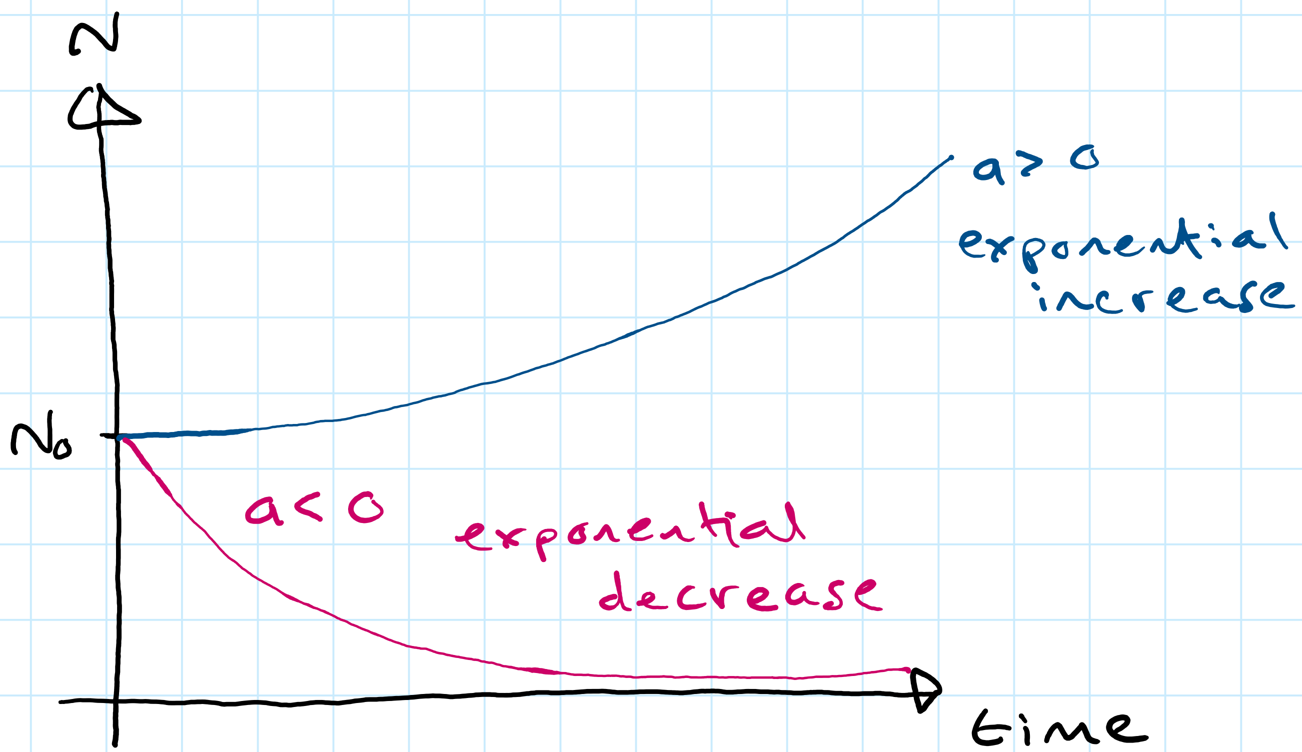
$$\frac{1}{aN}\,dN = dt$$

$$\int \frac{1}{aN}\,dN = \int dt$$

$$\frac{1}{a}\ln(N) = t + C$$

$$N = N_0 e^{at}$$

population at $t=0$.



this example is linear — no N^2 or higher terms

and first-order — no $\frac{d^2N}{dt^2}$ or higher terms

Exact solution examples

At second order we can get periodic behaviour.

$y(t)$: $$\frac{d^2y}{dt^2} = a y$$

if $a>0$, we have another exponential solution:

$$y(t) = A e^{\sqrt{a}\, t} + B e^{-\sqrt{a}\, t}$$

if $a<0$, same solution but we have $\sqrt{\text{negative number}}$.

$\rightarrow$ imaginary numbers.

we define $i = \sqrt{-1}$ as the imaginary unit, $i^2 = -1$

Complex numbers have a real and imaginary component.

$$z = a + ib$$ a and b are real.

ODEs

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Exact solution examples

At second order we can get periodic behaviour.

Rewrite ; $\frac{d^2y}{dt^2} = a y \rightarrow \frac{d^2y}{dt} = -a^2 y$, a is real

Then the solution is :

$$y(t) = Ae^{iat} + Be^{-iat}$$

Euler's theorem : $e^{i\theta} = \cos(\theta) + i\sin(\theta) \rightarrow$ complex number

We always take the real part of the solution at the end.

$Re[e^{i\theta}] = \cos\theta$, $Im[e^{i\theta}] = \sin\theta$ $\rightarrow$ check differentiation rules.

ODEs

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Exact solution examples $\longrightarrow$ Leaky integrate & fire model

LIF is a model for how neurons spike when a current is applied.

V(t) is the membrane potential.

$$\underbrace{C_m}_{\text{capacitance}} \frac{dV(t)}{dt} = \underbrace{-g_L(V(t) - E_L)}_{\text{leak current}} + \underbrace{I(t)}_{\text{input current}}$$

The potential $V(t)$ evolves in time

when $V(t) > V_{threshold}$, neuron spikes and $V(t) \xrightarrow{\text{resets}} V_{reset}$.
(fires)

ODEs

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Exact solution examples → Leaky integrate & fire model

LIF is a model for how neurons spike when a current is applied.

$$C_m \frac{dV(t)}{dt} = -g_L(V(t) - E_L) + I(t)$$

To solve, assume a constant input potential, $I(t) = I_0$, and rewrite:

$$\frac{dV}{dt} + \frac{g_L}{C_m} V = \frac{g_L}{C_m} E_L + \frac{1}{C_m} I_0$$

$$\frac{d}{dt}\left(e^{\frac{g_L}{C_m} t} V\right) = \left(\frac{g_L}{C_m} E_L + \frac{1}{C_m} I_0\right) e^{\frac{g_L}{C_m} t}$$

Now integrate:

$$V(t) = E_L + \tau_m R_m I_0 + C e^{-t/\tau_m}, \quad \tau_m = C_m / g_L$$

$$R_m = 1/g_L$$

$$= V_\infty + (V_0 - V_\infty) e^{-t/\tau_m}$$

NUMERICAL METHODS

Euler's method

First order ODE $\rightarrow$ $$\frac{dy}{dt} = f(y, t)$$

In infinitesimal form, we write $$\frac{dy}{dt} = \frac{y(t+dt) - y(t)}{dt}$$

dt is a very small time-step

So we can re-write the ODE as $$y(t+dt) = y(t) + f(y,t)dt$$

$\Rightarrow$ we can evolve the solution from an initial time t_0.

1) Choose t_0, $y(t_0)$, and dt.

2) $y(t_0 + dt) = y(t_0) + f(y, t_0)\,dt$

3) $y(t_0 + 2dt) = y(t_0 + dt) + f(y, t_0 + dt)\,dt$

4) $y(t_0 + 3dt) = y(t_0 + 2dt) + f(y, t_0 + 2dt)\,dt$

$\vdots$

$y(t + Ndt)$

NUMERICAL METHODS

RUNGE-KUTTA METHOD

We can Taylor expand the solution about $t = t_0$

$$y(t) = \underbrace{\underbrace{y(t_0) + dt\, f(y, t_0)}_{\text{Euler's method}} + \frac{1}{2!} dt^2 \frac{d}{dt} f(y, t_0) + \dots \quad O(dt^4)}_{\text{Runge-Kutta method}} \qquad dt = t - t_0$$

The update rule is :

$$y(t + dt) = y(t) + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = dt\, f(y, t)$$

$$k_2 = dt\, f\left(y + \tfrac{1}{2}k_1,\; t + \tfrac{1}{2}dt\right)$$

$$k_3 = dt\, f\left(y + \tfrac{1}{2}k_2,\; t + \tfrac{1}{2}dt\right)$$

$$k_4 = dt\, f\left(y + dt\, k_3,\; t + dt\right)$$

SYSTEMS OF ODEs

Say we have two coupled ODEs :

$$\frac{dx}{dt} - y = 0$$

$$\frac{dy}{dt} + 2x + 3y = 0$$

We can write this as a matrix equation

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

where $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, and $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$

If we write $\vec{u} = e^{\lambda_1 t}\vec{v}_1$, where $\vec{v}_1$ is constant in time we get

$$\frac{d\vec{u}}{dt} = \lambda_1 \vec{u} \implies A\vec{v}_1 = \lambda_1 \vec{v}_1 \rightarrow \text{eigenvalue equation}$$

A 2×2 matrix has 2 eigenvalues and eigenvectors

→ the general solution is : $\vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda_1 t}\vec{v}_1 + e^{\lambda_2 t}\vec{v}_2$

SYSTEMS OF ODEs

Say we have two coupled ODEs :

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$$\frac{dy}{dt} + 2x + 3y = 0$$

We can write this as a matrix equation

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

where $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, and $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$

integration constants

The solution :

$$\vec{u}(t) = c_1 e^{-t} \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\vec{v}_1} + c_2 e^{-2t} \underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{\vec{v}_2}$$

$\lambda_1 = -1$

$\lambda_2 = -2$

Asymptotic behaviour : As $t \to \infty$, $\vec{u} \to c_1\vec{v}_1 + c_2\vec{v}_2$

$\frac{dx}{dt} = 0$ & $\frac{dy}{dt} = 0$ $\longrightarrow$ equilibrium points.

Stability

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SYSTEMS OF ODEs

Take the general case : $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$
(possibly non-linear)

Say we have an equilibrium point at $(\bar{x}, \bar{y})$ $\Rightarrow \frac{d\bar{x}}{dt} = \frac{d\bar{y}}{dt} = 0$

Goal: study / characterize it.

$\rightarrow$ study small perturbations around $(\bar{x}, \bar{y})$

Step 1 Taylor expand around $(\bar{x}, \bar{y})$, $x = \bar{x} + p$, $y = \bar{y} + q$

$$\frac{d(\bar{x}+p)}{dt} = \frac{dp}{dt} = f(\bar{x},\bar{y}) + \frac{\partial f}{\partial x} p + \frac{\partial g}{\partial y} q$$

$$\frac{d(\bar{y}+q)}{dt} = \frac{dq}{dt} = g(\bar{x},\bar{y}) + \frac{\partial g}{\partial x} p + \frac{\partial g}{\partial y} q$$

Stability

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SYSTEMS OF ODEs

Take the general case : $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$

Step 2 write the DE's for the perturbations p and q as a matrix equation

$$\vec{u} = \begin{pmatrix} p \\ q \end{pmatrix} , \quad \frac{d\vec{u}}{dt} = J\vec{u} , \quad J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

The Jacobian

Step 3 Find the solutions , i.e. eigenvalues & eigenvectors of J.

$$\vec{u} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

λ_1 and $\lambda_2 < 0$ → stable , small perturbations near equilibrium fade away in time.

$\lambda_1 < 0$ and $\lambda_2 > 0$ → unstable saddle point

λ_1 and $\lambda_2 > 0$ → unstable

SYSTEMS OF ODEs

So far, we have considered changes with time.

But in a physical system, other things can change as well.

For example, in the LIF model, the current I_0 can change.

→ this can cause changes in the type and number of equilibrium points.

We call this a bifurcation.

Eg 1) Saddle-node bifurcation → two equilibria are created or destroyed as a parameter in the system is changed.

Eg 2) Hopf-bifurcation → the eigenvalues of J become complex as a parameter in the system is changed.
⇒ periodic behaviour in the growth of the perturbations.

Questions

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QUESTIONS

1) Calculate $\sqrt{i}$

2) Write $\cos(3x)$ and $\sin(3x)$ in terms of exponentials

3) Solve $\frac{dy}{dt} - 4y = 0$, $y(0) = 3$

4) Solve $\frac{dy}{dt} - 2y = e^t$, $y(0) = 0$

5) Using the LIF solution $V(t)$, derive the time-to-spike relation for a threshold potential $V_{threshold}$. (calculate how long the neuron takes to spike)

6) Calculate the Runge-Kutta update for exponential growth, $\frac{dN}{dt} = aN$.
→ then, verify that it is accurate to 4th order in the Taylor expansion.

7) Solve this system of ODEs using matrix methods:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -6x - 5y$$

next page →

Questions

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QUESTIONS

8) Nonlinear case: $\frac{dx}{dt} = \mu - x^2$ → a) find all equilibria as a function of μ

$\frac{dy}{dt} = -y$

b) compute the Jacobian + find eigenvalues

c) classify the equilibria

d) discuss bifurcation in the system.

Cover page

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STOCHASTIC

DIFFERENTIAL EQUATIONS

SDEs

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ODEs

$$\frac{dy}{dt} = f(y,t)$$

$$dy = f(y,t)\,dt$$

$$y(t+dt) = y(t) + f(y,t)\,dt$$

Then we can simulate:

$$y(0) \xrightarrow{f(y,0)dt} y(dt) \xrightarrow{f(y,dt)dt} y(2dt) \to \cdots$$

Stochastic DEs (SDEs)

$$dy = f(y,t)\,dt + g(y,t)\,\underline{dW(t)}$$

$dW(t)$ is a random noise contribution to the equation

→ Brownian motion, or, Wiener process

So, $y(t+dt) = y(t) + f(y,t)\,dt$
$+ g(y,t)\,dW(t)$

f: drift term
g: diffusion term

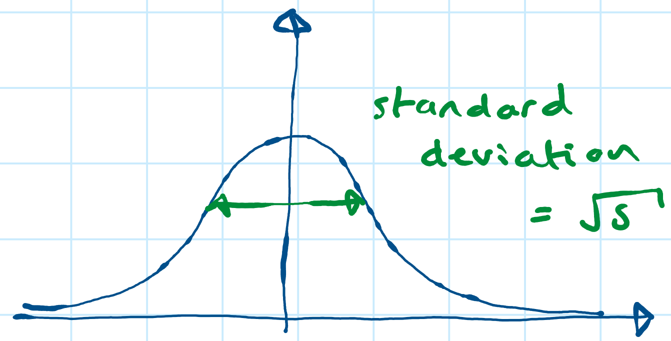
a random kick to the trajectory at each time step.

BROWNIAN MOTION

What is $W(t)$? A real-valued, continuous-time stochastic process.

means 'sampled from'

- At $t=0$, $W(0) = 0$
- $W(t)$ has Gaussian increments: $W(t+s) - W(t) \sim \mathcal{N}(0, s)$



- $W(t)$ is the accumulation of noise from $t=0$ to some time $t>0$.
- Note: $W(t)$ is <u>not</u> a distribution
  At every t, $W(t)$ is a single number
- We're doing calculus, so we need the infinitesimal increment of $W(t)$:

$$dW(t) = W(t+dt) - W(t) \sim \mathcal{N}(0, dt)$$

$$\int_0^t dW(r) = W(t)$$

BROWNIAN MOTION

A simple example, the SDE for Brownian motion:

$$dX_t = \mu\, dt + \sigma\, dW_t$$

we often use the subscript dX_t instead of $dX(t)$ with SDEs.

μ: a constant drift term

σ: a constant noise co-efficient

To solve, simply integrate both sides:

$$\int dX_t = \mu \int dt + \sigma \int dW_t$$

$$\rightarrow \quad X_t = x_0 + \mu t + \sigma W_t$$

integration constant → starting position

1) particle starts at x_0
2) it moves with velocity μ
3) there is a noise contribution to the motion, tracked by σW_t.

EULER - MARUYAMA

simulating SDEs

A general SDE looks like: $dX_t = a(X_t, t)\,dt + b(X_t, t)\,dW_t$

generic functions for drift and noise

- Like the Euler method, we write

$$dX_t = X_{t+1} - X_t$$

- Then the update rule is:

$$X_{t+1} = X_t + a(X_t, t)\,dt + b(X_t, t)\,dW_t$$

So we start at $X_t = X_0$, choose a dt, and generate / simulate the solution.

- The noise term $dW_t \sim \mathcal{N}(0, dt) \equiv dW_t \sim \sqrt{dt}\,\mathcal{N}(0,1)$

$\rightarrow$ the update: $X_{t+1} = X_t + a(X_t, t)\,dt + b(X_t, t)\sqrt{dt}\,\xi_0$

where $\xi_0 \sim \mathcal{N}(0,1)$.

STOCHASTIC CALCULUS

In calculus, the chain rule let's us change variables.

Eg if we have $\frac{dx}{dt} = g(x,t)$, and change variables $y = f(x(t))$

chain rule: $\frac{dx}{dt} = \left(\frac{df}{dx}\right)^{-1} \frac{dy}{dt} \rightarrow \frac{dy}{dt} = \left(\frac{df}{dx}\right) g(y,t)$

g is re-written in terms of y(t)

Stochastic calculus is different.

Starting with $dX_t = a(X_t, t)\,dt + b(X_t, t)\,dW_t$

if we change variables: $Y_t = f(X_t)$

we get: $dY_t = \left(a(X_t,t)\frac{df}{dX_t} + \underbrace{\frac{1}{2} b(X_t,t)\frac{d^2 f}{dx^2}}\right) dt + \frac{df}{dX_t} b(X_t,t)\,dW_t$

this term is new

it arises because $dW_t \sim \sqrt{t}$

this is called Ito's calculus.

Questions

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QUESTIONS

1) The SDE for geometric Brownian motion is :

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

Change variables, re-write the SDE in terms of $Y_t = \ln X_t$.

2) Solve the SDE, and put the solution back in terms of X_t.