Some Mathematical Techniques

-Matrices-

Introduction to matrices

A matrix is an array of numbers

Examples:

$$\begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix} \qquad \begin{pmatrix} 1 & -3 & 4 \\ 4 & 6 & 2 \\ 0 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} \qquad \begin{pmatrix} 3 & -9 \end{pmatrix} \qquad \begin{pmatrix} 6 & -3 \\ 5 & 20 \\ 1 & 9 \end{pmatrix}$$

$$2 \times 2 \qquad 3 \times 3 \qquad 3 \times 1 \qquad 1 \times 2 \qquad 3 \times 2$$

We can denote the elements (individual numbers) by a_{ij} where i denotes the specific row and j the column numbers

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$$

If vector, $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, then the length or norm, when squared,

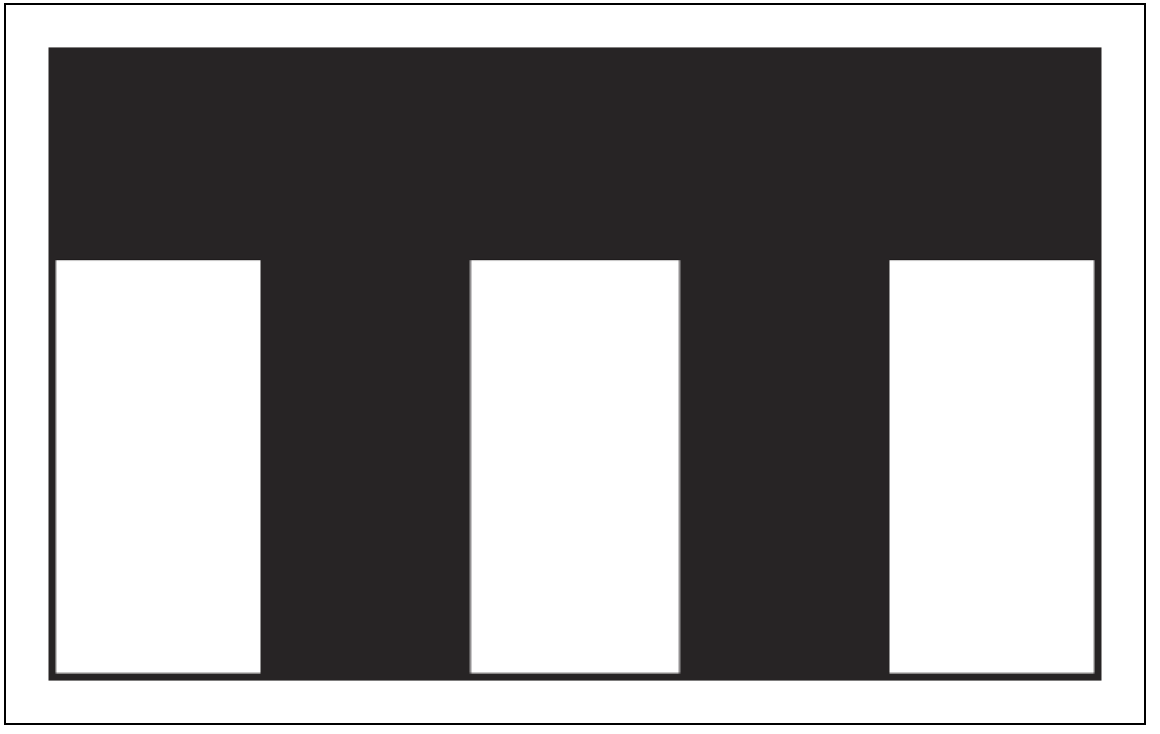
$$|\boldsymbol{v}|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = \sum_{i=1}^{3} v_i{}^2 = v_1{}^2 + v_2{}^2 + v_3{}^2$$

Dot product

Dot product of 2 different vectors of length N

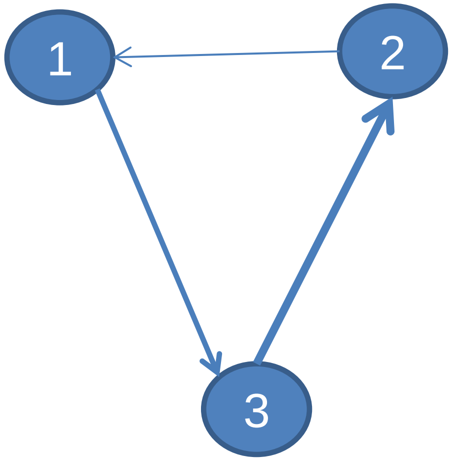
$$\boldsymbol{v} \cdot \boldsymbol{u} = \sum_{i=1}^{N} v_i u_i$$

Transpose of $\boldsymbol{v}$, $\boldsymbol{v}^{\boldsymbol{T}} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}^T = (v_1 \; v_2 \; v_3)$



Digital image processing: A simple digital representation of π

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$



We can represent the interactions of these 3 agents as $a_{ij} = a_{i \to j}$:

$$\mathbf{a} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

Used in networks (social, electrical, neural, biological, etc)

Definitions:

1. Square matrix – same number of rows as columns

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

2. Identity matrix – diagonal elements of 1 and 0's elsewhere.
$a_{ij} = \delta_{ij}$ (Kronecker delta function), i.e. zero when $i \neq j$ and one when $i = j$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. Transpose of a matrix (A^T) – swop rows and columns.
If $A = \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$ $\qquad {a_{ij}}^T = a_{ji}$

4. Trace of a matrix : Tr(A) = $\sum_{i=1}^{n} a_{ii}$ $\qquad A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}. \mathrm{Tr(A)} = 1 + (-1) = 0$

4. Symmetric matrix : $\qquad a_{ij} = a_{ji} \qquad A = \begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix}$

Adding and subtracting matrices

$$A + B = \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1+2 & 5+3 \\ 3+0 & 4+7 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 3 & 11 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1-2 & 5-3 \\ 3-0 & 4-7 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 3 & -3 \end{pmatrix}$$

Abbreviated form: $a_{ij} + b_{ij}$.
Can easily generalise to many matrices.

Note that $(A + B)^T = A^T + B^T$

Multiplication of a matrix by a number

$$k\ A = k \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} k\ a_{11} & k\ a_{12} \\ k\ a_{21} & k\ a_{22} \end{pmatrix}$$

E.g. $2\ A = 2 \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 10 \\ 6 & 14 \end{pmatrix}$

E.g. $\frac{1}{2}\ A = \frac{1}{2} \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{3}{2} & \frac{7}{2} \end{pmatrix}$

Note that $(k\ A)^T = k\ A^T = k \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$

E.g. $A - 3B = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} - 3\begin{pmatrix} 1 & 4 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} 3 & 12 \\ 6 & 18 \end{pmatrix} = \begin{pmatrix} -1 & -9 \\ -1 & -12 \end{pmatrix}$

E.g. $A^T - 2\,I = \begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ 3 & 4 \end{pmatrix}$

E.g. $\begin{pmatrix} 1 & 2 \\ 3 & 5 \\ 2 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & -1 \end{pmatrix}$

Thus, a $m \times n$ matrix becomes $n \times m$ after transposed

E.g. $\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & -1 \end{pmatrix}$=?

E.g. $\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 5 & 8 \\ 0 & 9 \end{pmatrix}$=?

Multiplication of matrices

$$A \qquad B \quad = \quad C$$

$$m \times n \qquad n \times \mathrm{p} \qquad m \times \mathrm{p}$$

Note that $A\ B \neq B\ A$ *Not commutative*

$A\ A = A^2$ is possible only if it A is a square matrix

$$A \qquad B \qquad C \quad = \quad \mathrm{D}$$

$$m \times n \qquad n \times p \qquad p \times q \qquad m \times \mathrm{q}$$

Note $\mathrm{A}BC = (AB)C = A(BC)$ *Associative*

In abbreviated form: $A\ B = \sum_{j=1} a_{ij} b_{jk} = c_{ik} = C$, or simply $a_{ij} b_{jk}$

E.g. Compute AB if $\mathrm{A} = \begin{pmatrix} 3 & 1 & 4 \\ 5 & 2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ -6 & 4 \end{pmatrix}$

First check their dimensions on whether A can be multiplied to B.

$$A\ B = \begin{pmatrix} 3 \times 1 + 1 \times 4 + 4 \times (-6) & 3 \times (-1) + 1 \times 0 + 4 \times 4 \\ 5 \times 1 + 2 \times 2 + (-2) \times (-6) & 5 \times (-1) + 2 \times 0 + (-2) \times 4 \end{pmatrix}$$

$$= \begin{pmatrix} -19 & 13 \\ 21 & -13 \end{pmatrix}$$

E.g. Compute AB if $\mathrm{A} = \begin{pmatrix} 9 & -1 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 2 & -1 \end{pmatrix}$

Again, check their dimensions on whether A can be multiplied to B.

$$A\ B = \begin{pmatrix} 35 & 25 & 19 \\ 11 & 12 & 1 \end{pmatrix}$$

Matrix product with a vector

$$\boldsymbol{W} \cdot \boldsymbol{v} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} W_{11}v_1 + W_{12}v_2 \\ W_{21}v_1 + W_{22}v_2 \end{pmatrix} = \sum_{j=1}^{2} W_{ij}v_j$$

Note: There is no actual “division” for matrices. The equivalence of the division requires the inverse. See later.

Multiplication of matrices is a “linear operator”: $\mathrm{O}(\mathrm{a\,X} + \mathrm{b\,Y}) = \mathrm{a\,O}(\mathrm{X}) + \mathrm{b\,O}(\mathrm{Y})$

Note that $A\,I = I\,A = A$ where I is the identity matrix

E.g. $A\,I = \begin{pmatrix} 3 & -2 \\ 10 & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \cdots = \begin{pmatrix} 3 & -2 \\ 10 & 7 \end{pmatrix} = A$

Check $I\,A = A$ too.

E.g. Given $\begin{pmatrix} 4 & -1 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -13 \\ 13 \end{pmatrix}$, find x and y.

Multiplying out the matrices on the L.H.S.,

$$\begin{pmatrix} 4x - y \\ 6x + 5y \end{pmatrix} = \begin{pmatrix} -13 \\ 13 \end{pmatrix}$$

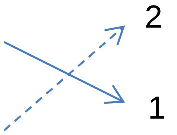
Comparing the top and bottom elements, we have to solve for the simultaneous equations:

$$4x - y = -13 \qquad\qquad 6x + 5y = 13$$

Determinants

Consider $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the <u>determinant</u> of matrix A is

det(A) = | A | = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$



E.g. $A = \begin{pmatrix} 3 & -2 \\ 10 & 7 \end{pmatrix}, \det(A) = 3 \times 7 - 10 \times (-2) = 21 + 20 = 41$

E.g. $|I|$=1x1-0x0=1

<u>Control engineering</u>: Many engineering systems can be modelled by a set of simultaneous equations. From these equations, a “state matrix”, A, can be obtained which provides important information about the system, e.g. how stable it is under perturbation. Stability of a system is determined by the “poles” of the system. The poles can be found from the equation:

$$|A - sI| = 0$$

where I is the identity matrix. A matrix that has zero determinant is called **singular**.

E.g. Given $A = \begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix}$, determine the poles of the system.

$$A - sI = \begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix} - s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-s & -2 \\ -1 & 5-s \end{pmatrix}$$

$$|A - sI| = (3-s)(5-s) - (-1)(-2) = s^2 - 8s + 13$$

Need to solve $s^2 - 8s + 13 = 0$, which turns out to be $s = 2.268, 5.732$

For a system to be **stable**, the real values of s, i.e. $\mathbf{Re}(\boldsymbol{s})$ **values have to be negative**. If Re(s) positive, then it is unstable, which is our case here.

Minors and cofactors of a 3x3 matrix

For a 3x3 matrix, $\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,the **minor** of a_{ij} is the determnant of the 2x2 matrix by crossing out the row and column that include a_{ij}.

E.g. The minor of a_{11} is det$\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

E.g. The minor of a_{23} is det$\begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}$.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

E.g. The minor of 9 in the matrix $A = \begin{pmatrix} 3 & -1 & 6 \\ 9 & -5 & 2 \\ 0 & 4 & 7 \end{pmatrix}$ is

det$\begin{pmatrix} -1 & 6 \\ 4 & 7 \end{pmatrix} = (-1 \times 7) - (4 \times 6) = -7 - 24 = -31$.

The **cofactor** of an element a_{ij} in the matrix is

cofactor of $a_{ij} = (-1)^{i+j} \times$ minor of a_{ij}

where the $(-1)^{i+j}$ or + and – signs are known as the **place signs**.

Specifically, for a 3x3 matrix the place signs look like the below:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

E.g. The cofactor of 4 in the matrix $A = \begin{pmatrix} 3 & -1 & 6 \\ 9 & -5 & 2 \\ 0 & 4 & 7 \end{pmatrix}$ is

$(-1)^{3+2} \times$ det$\begin{pmatrix} 3 & 6 \\ 9 & 2 \end{pmatrix} = (-1)^5 \times (3 \times 2 - 9 \times 6)$

$= -(6 - 54) = -31 = -(-48) = 48.$

What is the cofactor of 0 in A?

Determinant of a 3x3 matrix

For a 3x3 matrix, $\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, its determinant is

$$\det(A) = |A| = a_{11} \times (its\ cofactor) + a_{12} \times (its\ cofactor) + a_{13} \times (its\ cofactor)$$

(expanding along the first row)

We could also find the determinant by expanding on any row or column, just have to keep in mind the place signs.

E.g. The determinant of the matrix $A = \begin{pmatrix} 3 & -1 & 6 \\ 9 & -5 & 2 \\ 0 & 4 & 7 \end{pmatrix}$ is

$$\det(A) = 3 \times (+1) \times \begin{vmatrix} -5 & 2 \\ 4 & 7 \end{vmatrix} + (-1) \times (-1) \times \begin{vmatrix} 9 & 2 \\ 0 & 7 \end{vmatrix} + 6 \times (+1) \times \begin{vmatrix} 9 & -5 \\ 0 & 4 \end{vmatrix} = 150$$

How about expanding it along the 2nd column?

Properties of determinants

Property 1:

Suppose A is a $n \times n$ matrix, and that matrix B is obtained by multiplying a **single** row or column of A by k, then

$$\det(B) = k \det(A)$$

If the matrix A is multiplied by k, i.e. every element in the matrix A is multipled by k, then

$$\det(kA) = k^n \det(A)$$

This can be understood from the definition of the determinant.

E.g. If $A = \begin{pmatrix} 5 & 7 \\ -2 & 3 \end{pmatrix}$ and If $\mathrm{B} = \begin{pmatrix} 5 & 21 \\ -2 & 9 \end{pmatrix}$, show that $\det(B) = k \det(A)$ where $k = 3$.

E.g. If $A = \begin{pmatrix} 5 & 7 \\ -2 & 3 \end{pmatrix}$, find $\det(A^4)$.

Property 2:

If B is obtained from A by interchanging 2 rows or 2 columns, then

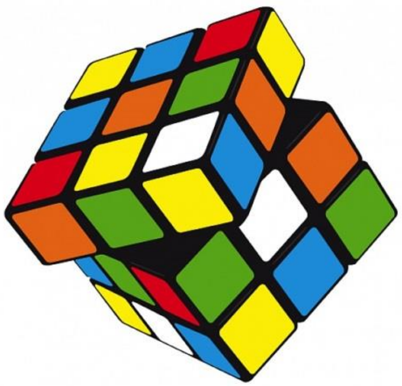
$$\det(B) = -\det(A)$$

E.g. $A = \begin{pmatrix} 3 & -1 & 6 \\ 9 & -5 & 2 \\ 0 & 4 & 7 \end{pmatrix}$, find determinant of the matrices:

(a) $\begin{pmatrix} 0 & 4 & 7 \\ 9 & -5 & 2 \\ 3 & -1 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} 9 & -5 & 2 \\ 0 & 4 & 7 \\ 3 & -1 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} -5 & 9 & 2 \\ -1 & 3 & 6 \\ 4 & 0 & 7 \end{pmatrix}$



Property 3:

Adding or subtracting a multiple of one row (or column) to another row (or column) leaves the determinant unchanged. This can be used to simplify the calculation for the determinant.

E.g. $\begin{vmatrix} 3 & -1 & 6 \\ 9 & -5 & 2 \\ 0 & 4 & 7 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 6 \\ 0 & -2 & -16 \\ 0 & 4 & 7 \end{vmatrix}$ (subtracting 3 x row 1 from row 2)

$= \begin{vmatrix} 3 & 0 & 6 \\ 0 & -2 & -16 \\ 0 & 4 & 7 \end{vmatrix}$ (adding 1/3 of column 1 to column 2)

$= \begin{vmatrix} 3 & 0 & 0 \\ 0 & -2 & -16 \\ 0 & 4 & 7 \end{vmatrix}$ (subtracting 2 x column 1 from column 3)

$= \begin{vmatrix} 3 & 0 & 0 \\ 0 & -2 & -16 \\ 0 & 0 & -25 \end{vmatrix}$ (adding 2 x row 2 to row 3)

$$= 3 \begin{vmatrix} -2 & -16 \\ 0 & -25 \end{vmatrix} = 3\,(-2 \times -25 - 0 \times -16) = 3(50) = 150$$

Property 4:

$$\det(A\,B) = \det(A)\,\det(B)$$

E.g. Verify this property for the 2 matrices $\mathrm{A} = \begin{pmatrix} -3 & 1 \\ 2 & 5 \end{pmatrix}$ and $\mathrm{B} = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$

Property 5:

If 2 rows or columns of a matrix are equal, the determinant is zero. This follows directly from Property 3 (*corollary*) and the definition of the determinant.

E.g. $A = \begin{pmatrix} 26 & 32 & 3 \\ 100 & -2 & 4 \\ 26 & 32 & 3 \end{pmatrix}$. Det(A)=0.

The inverse of a matrix

If for 2 $n \times n$ matrices, A and B, such that $A\,B = BA = I$, where I is the $n \times n$ identity matrix. Then A is the inverse of B, and vice versa. The inverse of B is denoted as B^{-1} and that for A is A^{-1}. Hence for a matrix A, $A\,A^{-1} = A^{-1}\,A = I$, provided its inverse A^{-1} ***exists***.

E.g. Show $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ and $\mathrm{B} = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$ are inverse of each other.

Notes:

(i) A^{-1} does ***not*** mean its reciprocal $\frac{1}{A}$.

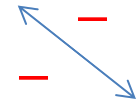
(ii) If det(A)=0, A does ***not*** have an inverse.

(iii) If det(A)≠0, A does have an inverse.

(iv) If AB=C, then multiplying both sides by A^{-1} gives B= A^{-1}C (equivalent to a "division"). Or if AC=sC, for some constant s, then C^{-1}AC=s, a mere number.

Finding the inverse of a 2 x 2 matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then



$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $\det(\mathrm{A}) = (ad - bc)$

E.g. Find the inverse of $A = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$, if it exists.

E.g. Find the inverse of $A = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}$, if it exists.

E.g. Find the inverse of $A = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, if it exists.

E.g. Find the inverse of $A = \begin{pmatrix} 4 & 8 \\ 5 & 10 \end{pmatrix}$, if it exists.

Finding the inverse of a n x n matrix

$$A^{-1} = \frac{adj(A)}{\det(A)}$$

where the adjoint of A, adj(A), is obtained by replacing each element of A^T by its cofactor.

E.g. $A = \begin{pmatrix} 3 & 1 & 0 \\ 5 & 2 & -1 \\ 1 & 4 & -2 \end{pmatrix}$. Its transpose is $A^T = \begin{pmatrix} 3 & 5 & 1 \\ 1 & 2 & 4 \\ 0 & -1 & -2 \end{pmatrix}$. Then its adjoint is

$adj(A) = \begin{pmatrix} 0 & 2 & -1 \\ 9 & -6 & 3 \\ 18 & -11 & 1 \end{pmatrix}$. Also, its determinant, det(A)=9. Thus, the inverse of A is

$$A^{-1} = \frac{adj(A)}{\det(A)} = \frac{1}{9}\begin{pmatrix} 0 & 2 & -1 \\ 9 & -6 & 3 \\ 18 & -11 & 1 \end{pmatrix}$$

Using the inverse matrix to solve simultaneous equations

$$a_1 x + b_1 y = k_1$$
$$a_2 x + b_2 y = k_2$$

2 equations, 2 unknowns

Can be rewritten in matrix form:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

check

or $$A\,X = B$$

where $$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, B = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}$$

Now times A^{-1} both sides $$A^{-1}A\,X = A^{-1}B$$

i.e. $$X = A^{-1}B$$

Generalizable to n x n matrix

E.g. Solve
$$7x + 2y = 12$$
$$3x + y = 5$$

$$\begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

$$A = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}, B = \begin{pmatrix} 12 \\ 5 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A^{-1} = \frac{adj(A)}{\det(A)} = \frac{1}{\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}} \begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix}$$

Therefore
$$X = A^{-1}B = \begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 12 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

i.e. x=2 and y=-1

Note: Similar methods for 3 x 3 matrix. Of course, finding A^{-1} for a 3 x 3 matrix is not as easy.

E.g. Solve the simultaneous equations

$$\begin{aligned} 3x + y &= 1 \\ 5x + 2y - z &= 0 \\ x + 4y - 2z &= 0 \end{aligned}$$

Set $A = \begin{pmatrix} 3 & 1 & 0 \\ 5 & 2 & -1 \\ 1 & 4 & -2 \end{pmatrix}$

$$A^T = \begin{pmatrix} 3 & 5 & 1 \\ 1 & 2 & 4 \\ 0 & -1 & -2 \end{pmatrix}$$

$$adj(A) = \begin{pmatrix} 0 & 2 & -1 \\ 9 & -6 & 3 \\ 18 & -11 & 1 \end{pmatrix}$$

det(A)=9

$$A^{-1} = \frac{adj(A)}{\det(A)} = \frac{1}{9}\begin{pmatrix} 0 & 2 & -1 \\ 9 & -6 & 3 \\ 18 & -11 & 1 \end{pmatrix}$$

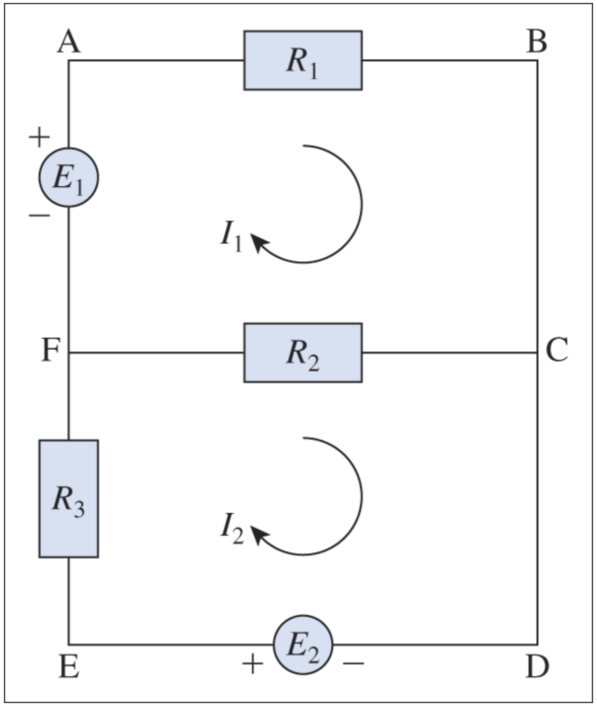
Hence, for A X = B, the solutions are

$$X = A^{-1}B = \frac{1}{9}\begin{pmatrix} 0 & 2 & -1 \\ 9 & -6 & 3 \\ 18 & -11 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 0 \\ 9 \\ 18 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Check the solutions by substituting them back into the original 3 equations.

L.H.S.= $3(0) + (1) = 1$=R.H.S.
L.H.S.= $5(0) + 2(1) - (2) = 0$ = R.H.S.
L.H.S. = $(0) + 4(1) - 2(2) = 0$ = R.H.S.

An application: electrical networks (block 6.2)



There are 2 meshes here.

Using *Kirchhoff's voltage law*, and the equation $V = R\,I$:

Mesh ABCF $\quad R_1 I_1 + (I_1 - I_2)R_2 = E_1$

Mesh CDEF $\quad -(I_1 - I_2)R_2 + I_2 R_3 = E_2$

Determine the currents I_1 and I_2

$$R_1 I_1 + (I_1 - I_2)R_2 = E_1$$

$$(I_2 - I_1)R_2 + I_2 R_3 = E_2$$

Can be rewritten as:

$$(R_1 + R_2)I_1 - R_2 I_2 = E_1$$

$$-R_2 I_1 + (R_2 + R_3)I_2 = E_2$$

For $E_1 = 3, E_2 = 6, R_1 = 1, R_2 = 4, R_3 = 2$, n matrix form:

$$\begin{pmatrix} 5 & -4 \\ -4 & 6 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 6 & 4 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 18 + 24 \\ 12 + 30 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Eigenvalues and eigenvectors

Consider
$$5x - 3y = 0$$
$$10x - 2y = 0$$

The solution x=0, y=0 is the only solution (trivial solution).

Consider
$$5x - 3y = 0$$
$$10x - 6y = 0$$

The solution y=(5/3) x, or x=t, y=(5/3) t ; infinite non-trivial solutions

A system of (linear) equations in matrix representation:

$$A\,X = \mathbf{0}$$

$\mathbf{0}$ is a matrix with all zeros

for any square matrix A, has non-trivial solutions if det(A)=0, and trivial solution if det(A)≠0.

E.g.

$$\begin{aligned} 4x - y &= 0 \\ 2x - 3y &= 0 \end{aligned}$$

E.g.

$$\begin{aligned} x + 2y &= 0 \\ 3x + 6y &= 0 \end{aligned}$$

E.g.

$$\begin{aligned} 3x - y + z &= 0 \\ x + 2y + 2z &= 0 \\ 4x + y + 3z &= 0 \end{aligned}$$

Eigenvalues

E.g.

$$2x + y = \lambda x$$
$$3x + 4y = \lambda y$$

Can be rewritten as: $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

or $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \lambda \begin{pmatrix} x \\ y \end{pmatrix} = 0$

or $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

or $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

Hence, in the form: $(A - \lambda\, \mathbb{I})\, X = 0$

which has non-trivial solutions if $\det(A - \lambda\, \mathbb{I}) \;\; = 0$

$$\det(A) = \begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix}$$
$$= (2-\lambda)(4-\lambda) - 3$$
$$= \lambda^2 - 6\lambda + 5$$
$$= (\lambda - 1)(\lambda - 5)$$

Setting det(A)=0, $\lambda^2 - 6\lambda + 5 = 0$

$\lambda = 1, \lambda = 5$.

The λ values are called the **eigenvalues**, and the quadratic equation $\lambda^2 - 6\lambda + 5 = 0$ is the **characteristic equation**.

Note: Sometimes the eigenvalues are repeated (e.g. $\lambda = 2, \lambda = 2$) or can be complex numbers (e.g. $\lambda = 2j, \lambda = -2 + 5$j), or complex conjugate of one another.

E.g. Find the characteristic equation and hence, the eigenvalues of

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix}$$

… …

Characteristic equation turns out to be: $\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$

… …

and eigenvalues are $\lambda = -2, -1, 1$

Why are we doing all these?? Recall the example on control engineering. Just replace s by λ. Other applications: vibration analysis, tensor moment of inertia, stress tensor, nonlinear and chaotic dynamics, data analysis (pattern recognition, principal component analysis), quantum mechanics, chemistry, geology, and many others! Extremely important!

Eigenvectors

For each eigenvalue of A (or the system), $(A - \lambda\,\mathbb{I})\,X = 0$

there exists an **eigenvector**.

E.g. $$\begin{aligned} 2x + y &= \lambda x \\ 3x + 4y &= \lambda y \end{aligned}$$

From previous example, we have found that the eigenvalues are:
$\lambda = 5, \lambda = 1$.

For $\lambda = 1$: $$\begin{aligned} 2x + y &= x \\ 3x + 4y &= y \end{aligned}$$ $$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} x + y &= 0 \\ 3x + 3y &= 0 \end{aligned}$$

Hence $x = t, y = -t$. Then the eigenvector corresponding to the eigenvalue of $\lambda = 1$ is $\begin{pmatrix} t \\ -t \end{pmatrix} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with an arbitrary scaling constant t.

For $\lambda = 5$:

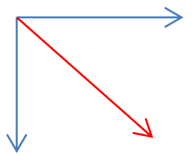
$$\begin{aligned} 2x + y &= 5x \\ 3x + 4y &= 5y \end{aligned} \qquad \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} -3x + y &= 0 \\ 3x - y &= 0 \end{aligned}$$

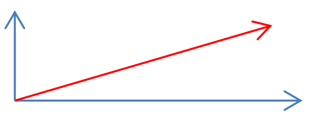
Hence $x = t, y = 3t$. Then the eigenvector corresponding to the eigenvalue of $\lambda = 5$ is $\begin{pmatrix} t \\ 3t \end{pmatrix} = t \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ with an arbitrary scaling constant t.

It is sometimes useful to **normalize** the eigenvectors such that their magnitudes are 1:

$$\frac{1}{\sqrt{1^2 + (-1)^2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



$$\frac{1}{\sqrt{1^2 + 3^2}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



The method for calculating the eigenvectors can be generalized to larger square matrices.

E.g. $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix}$

Based on previous example, the eigenvalues for A are found to be:

$$\lambda = -2, -1, 1$$

Substitute these values one by one into the equation:

$$(A - \lambda\, \mathbb{I})\, X = 0$$

$$\begin{pmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 3 & 2 & -2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

For $\lambda = -2$, $\quad 3x + 2y = 0, \quad -x + y + z = 0, \quad 3x + 2y = 0$

$$x = t, \qquad y = -\frac{3}{2}t, \qquad z = t - \left(-\frac{3}{2}t\right) = \frac{5}{2}t$$

Hence the eigenvector of A corresponding to the eigenvalue $\lambda = -2$ is $t\begin{pmatrix} 1 \\ -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix}$

For $\lambda = -1$, $\begin{pmatrix} 1-(-1) & 2 & 0 \\ -1 & -1-(-1) & 1 \\ 3 & 2 & -2-(-1) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$$2x + 2y = 0, \qquad -x + z = 0, \qquad 3x + 2y - z = 0$$

$$x = t, \qquad y = -t, \qquad z = t$$

Hence the eigenvector of A corresponding to the eigenvalue $\lambda = -1$ is $t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

For λ =1, $$\begin{pmatrix} 1-1 & 2 & 0 \\ -1 & -1-1 & 1 \\ 3 & 2 & -2-1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$y = 0, \qquad -x - 2y + z = 0, \qquad 3x + 2y - 3z = 0$$

$$x = t, \qquad y = 0, \qquad z = t$$

Hence the eigenvector of A corresponding to the eigenvalue $\lambda = 1$ is $t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Note that we can also normalise the 3 eigenvectors:

$$\frac{1}{\sqrt{1^2 + (-\frac{3}{2})^2 + (\frac{5}{2})^2}}\begin{pmatrix} 1 \\ -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix} = \frac{4}{\sqrt{38}}\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \qquad \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

We can form a new (**modal**) matrix M whose columns are the eigenvectors X_i's:

$$\begin{pmatrix} a_{11} & a_{12} & \\ \vdots & \vdots & \dots \\ a_{n1} & a_{n2} & \end{pmatrix} = (X_1 \; X_2 \; \cdots)$$

E.g. In the previous 2 examples, the modal matrices are:

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \qquad\qquad M = \begin{pmatrix} 2 & 1 & 1 \\ -3 & -1 & 0 \\ 5 & 1 & 1 \end{pmatrix}$$

We can show that $M^{-1}A\,M = D$, where the matrix D is a **diagonal matrix** with the eigenvalues of A along as diagonal elements (in the right order). D is called the **spectral** matrix corresponding to the modal matrix M. The matrix D is called **similar** to A.

E.g. $M^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$. Then $M^{-1}A\,M = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

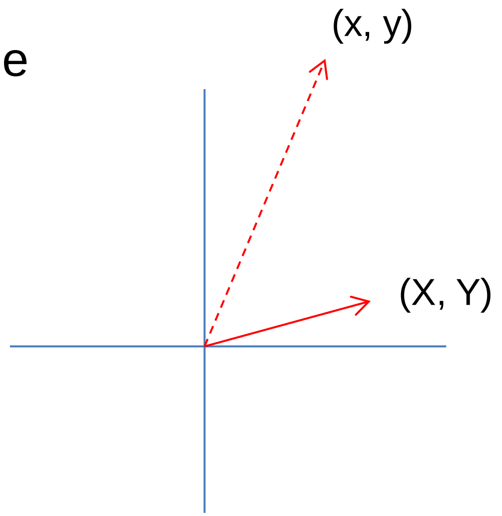
$$= \frac{1}{4}\begin{pmatrix} 3 & -1 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 20 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Check for the other example (multiplication of three 3 x 3 matrices).

Linear transformations

A linear transformation is one in which each new variable is some linear combination of the old variables.

$$X = ax + by$$
$$Y = cx + dy$$



or $$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or $$V = A\,v\,, \qquad A^{-1}V = v$$

A matrix with the property $A^T = A^{-1}$ or $A\,A^T = \mathbb{I}$ is an **orthogonal** matrix, where $\mathbb{I}$ is the identity matrix with elements of 1's along the main diagonal and 0's otherwise.

.

E.g. $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Check property.

If the transformation is orthogonal, then it is a pure rotation without changing the magnitude. In the example, it is an anti-clockwise rotation of angle θ.

Now, suppose that in a linear transformation, we can find the eigenvectors and eigenvalues of A such that v is an eigenvector, then

$$\begin{aligned} V &= A\,v \\ &= \lambda\,v \end{aligned}$$

Since λ is just a number (scalar), the transformation is just a mere change in magnitude without affecting the original direction. Hence, the eigenvectors are the vectors not affected by the transformation.

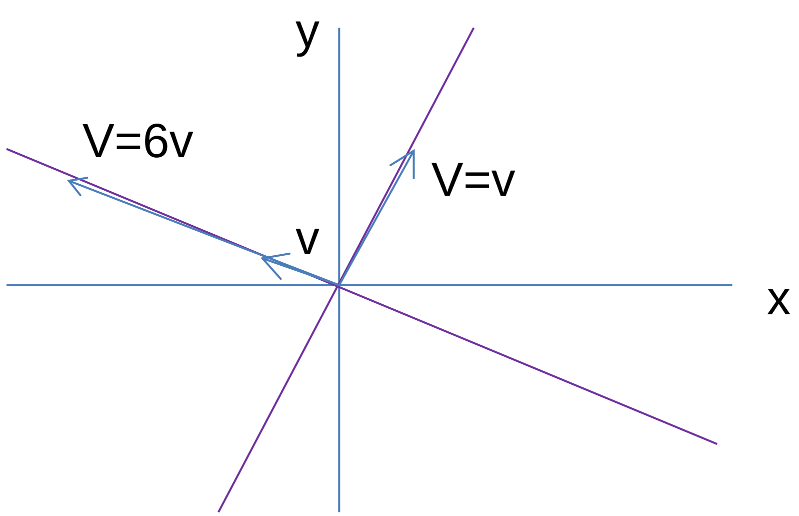
Recall $M^{-1}A\,M = D$. When we obtain D given A, we say that we have **diagonalized** A by a **similarity transformation**. This amounts physically to a simplification of the problem by a better choice of variables.

E.g. $$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues of $A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$ are λ=1 and 6, and the corresponding normalized eigenvectors are

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Hence $D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$ and $M = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$



Degeneracy: If 2 eigenvectors have the same eigenvalues.
If a N x N matrix **W** has non-degenerate eigenvalues, then then eigenvectors $\boldsymbol{e_1}, \boldsymbol{e_2}, \dots, \boldsymbol{e_N}$ are linearly independent such that any arbitrary N-component vector $\boldsymbol{v}$ can be represented in the form:

$$\boldsymbol{v} = \sum_{i=1}^{N} c_i \boldsymbol{e_i}$$

$\{\boldsymbol{e_1}, \boldsymbol{e_2}, \dots, \boldsymbol{e_N}\}$ forms a "basis" for the set of vectors $\boldsymbol{v}$.

Special case (symmetric matrix): eigenvalues are real and eigenvectors are real and orthogonal ($\boldsymbol{e_i} \cdot \boldsymbol{e_j} = \boldsymbol{0}$ if $i \neq j$).

MATLAB functions for matrices

```
>> A=[1 2; 3 4];
>> A

A =

     1     2
     3     4

>> eig(A)

ans =

   -0.3723
    5.3723

>> [V,D]=eig(A)

V =

   -0.8246   -0.4160
    0.5658   -0.9094


D =

   -0.3723         0
         0    5.3723

>>
```

```
>> inv(A)

ans =

   -2.0000    1.0000
    1.5000   -0.5000

>>


>> A.'

ans =

     1     3
     2     4

>>
```

```
>> det(A)

ans =

    -2

>> Tr(A)

ans =

     5

>> v=[1 3]

v =

     1     3

>> A*v.'

ans =

     7
    15

>>
```

```
>> B=[2 3; 7 6]

B =

     2     3
     7     6

>> A*B

ans =

    16    15
    34    33

>>
```

Some Mathematical Techniques

-Differential Equations-

Ordinary differential equations (ODEs)

A differential equation with respect to time *(dynamical system)* can be expressed in a general form:

$$\frac{dx}{dt} = f(x, t)$$

Essentially describing a rate of change of a quantity x with respect to time.

E.g. Kinematics: ds/dt = u – g t

E.g. Force and acceleration: F = m dv/dt, or dv/dt = F/m

This is sometimes call a *dynamical system*.

Very useful in modelling a variety of systems: engineering, physics, biology, chemistry, …

More generally, $$\frac{dy}{dx} = f(x, y)$$

x: independent variable
y: dependent variable

Definitions:

1. The <u>order</u> of a DE is the order of its highest derivative.
2. We shall only deal with first order DEs as higher order ones can be made into first order DEs through redefining the variables.
3. A DE is <u>linear</u> if the dependent variable and its derivative occur to the first power only, and no “mixed” terms.
4. A linear DE can have <u>constant coefficients</u>.
5. A <u>solution</u> of a DE is found if the dependent variable can be expressed as a function of the independent variable and satisfies the DE.

E.g. Direct integration $$\frac{dx}{dt} = -t + 5$$

$$\int \frac{dx}{dt}\, dt = \int (-t+5)\, dt$$

$$\int dx = x = -\frac{t^2}{2} + 5t + C$$

If $x = 1$ when $t = 0$ (*initial conditions*), then $x = x(t) = -\frac{t^2}{2} + 5t + 1$

Suppose a dynamical equation is of the form:

$$\frac{dx}{dt} = -x + F(p)$$

for some constant parameter p , then $F(p)$ is the “steady state” of x, i.e. as time t gets large (after sufficiently long time), $x \approx F(p)$ in an exponential fashion. The solution is:

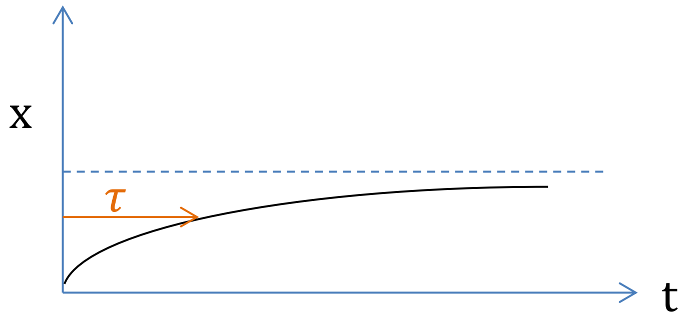
$$x(t) = F(p) - (F(p) - x(0))\, e^{-t}$$

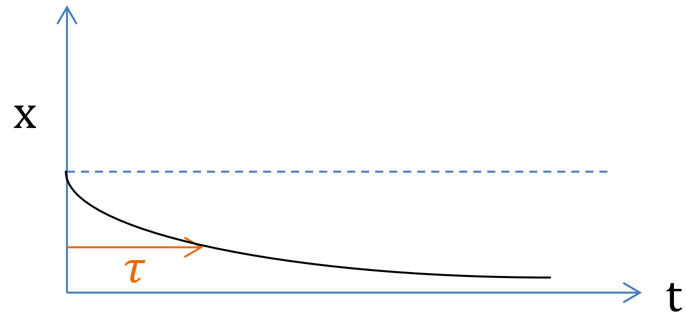
E.g. $\frac{dx}{dt} = -x + 5$

$$x(t) = 5 + (5 - x(0))e^{-\int dt} = 5 + (5 - x(0))e^{-t}$$

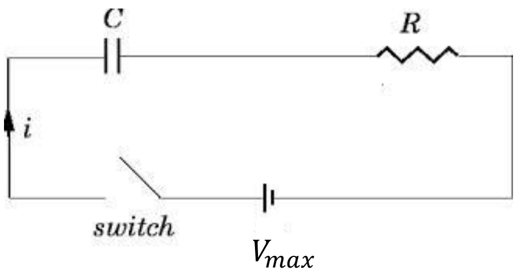
More generally, $\frac{dx}{dt} = \frac{-(x - x_{ss})}{\tau}$ with constant x_{ss} such that $x(t) = x_{ss} - (x_{ss} - x(0))e^{-t/\tau}$

Hence, we can think of τ as the time constant of the system, i.e. the characteristic time to reach towards its steady state x_{ss} .





E.g. Charging an electrical RC (with constant resistance R and capacitance C) circuit with a battery of voltage V_{max} :

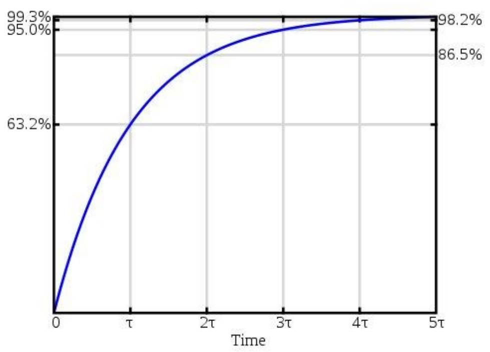


Recall Ohm's law, $\frac{V_{max}}{R} = I$,

$I = \frac{dQ}{dt}$ and $Q = C\ V$

Based on Kirchoff's law: $V_{max} - RI - \frac{Q}{C} = 0$

Rewriting the equation: $\mathrm{I} = \frac{dQ}{dt} = \frac{V_{max}}{R} - \frac{Q}{RC}$



with solution $Q(t) = CV_{max}\left(1 - e^{-\frac{t}{RC}}\right) = Q_{max}\left(1 - e^{-\frac{t}{RC}}\right)$

Or $V(t) = \frac{Q(t)}{C} = V_{max}\left(1 - e^{-\frac{t}{RC}}\right)$

Hence RC acts like a time constant (τ).

For a constant applied current I injected into a <u>neuron</u> (with passive membrane), the dynamical change for the membrane potential, V looks like:

$$C\frac{dV}{dt} = \frac{E - V}{R} + I = -\frac{(V - V_{SS})}{R}$$ with solutions

$$V(t) = V_{SS} - (V_{SS} - V(0))\left(1 - e^{-\frac{t}{RC}}\right)$$

Numerical integration methods to differential equations

Euler's method: *Finite difference method*

$$dx/dt = f(x, t)$$

$$\Delta x/\Delta t \approx f(x, t)$$

$\Delta x = x(t + \Delta t) - x(t)$

$$x(t + \Delta t) - x(t) = \Delta t\, f(x(t), t)$$

$$X(t + \Delta t) = x(t) + \Delta t\, f(x(t), t)$$

Δt (step size) has to be sufficiently small to avoid numerical errors!

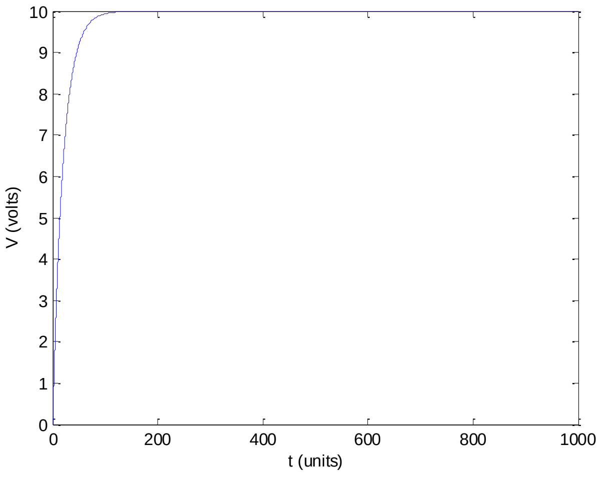
E.g. $V(t + \Delta t) = V(t) + \Delta t\, f(V(t))$ $\quad f(V(t), t) = -\dfrac{(V(t) - V_{ss})}{RC}$

Check out more accurate (e.g. with adaptive time step) numerical algorithms: e.g. Runge-Kutta 2, Runge Kutta 4.

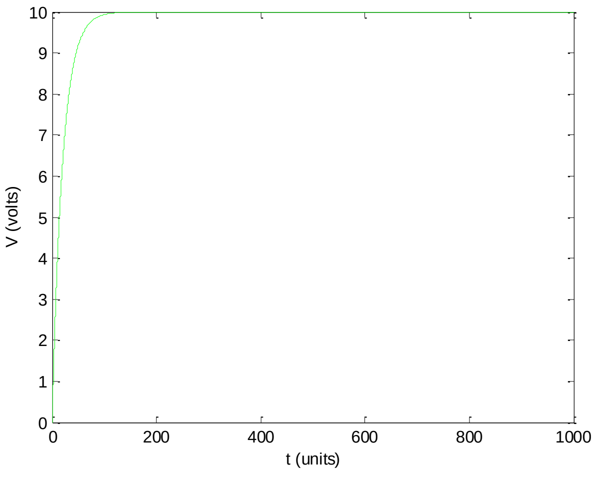
Numerical integrations with MATLAB

Brute force Euler's numerical integration method:

```
>> Vss=10

Vss =

    10

>> tau=20

tau =

    20

>> T_total=1000; dt=0.1; V=zeros(1,10*T_total);
>> for t=1:T_total/dt
V(t+1)=V(t)+dt*(-(V(t)-Vss)/tau);
end;
>> plot(V)
>> xlabel('t (units)'); ylabel('V (volts)');
```




```
>> figure; plot([0:dt:T_total],Vss-(Vss-0)*exp(-[0:dt:T_total]/tau),'g');
>> xlabel('t (units)'); ylabel('V (volts)');
```



The Matlab function “ode45” implements a Runge-Kutta method with a variable time step for efficient computation.

First, create a function file and save it as F.m, e.g.:

```
function f_V=F(t,V)
f_V=zeros(1,T_total);
f_V=-(V-Vss)/tau;
```

Then:

```
>> [t,V]=ode45('F',[0,T_total],0);
>> plot(V,‘r‘)
```

Computational Modeling Methods for Neuroscientists (MIT, 2009). See Chapter 1 on Differential Equations.

http://f3.tiera.ru/2/B_Biology/BH_Human/De%20Schutter%20E.%20%28ed.%29%20Computational%20Modeling%20Methods%20for%20Neuroscientists%20%28MIT,%202009%29%28ISBN%200262013274%29%28O%29%28433s%29_BH_.pdf

Principles of Computational Modelling in Neuroscience, Sterratt et al., Appendix B

Theoretical Neuroscience, Dayan and Abbott, Mathematical Appendix.

Computational Modeling Methods for Neuroscientists, Erik De Schutter, Chapter 1. Written by Bard Ermentrout and John Rinzel.

Mathematics for Neuroscientists, by Fabrizio Gabbiani and Steven J. Cox

https://www.khanacademy.org/math/differential-equations/first-order-differential-equations

https://www.khanacademy.org/math/algebra/algebra-matrices

www.youtube.com/watch?v=8UX82qVJzYI
www.youtube.com/watch?v=pZ6mMVEE89g
www.youtube.com//watch?v=11dNggWC4HI
www.youtube.com/watch?v=IXNXrLcoerU&list=SPE7DDD91010BC51F8

XPPAUT

http://www.math.pitt.edu/~bard/xpp/xpp.html

http://www.math.pitt.edu/~bard/bardware/tut/start.html