TÉCNICO LISBOA

Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR

Inverse Problems in Interferometric Phase Imaging

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Phase estimation from interferometric measurements

Problem: given a set of observations $e^{j\phi_p} \equiv (\cos\phi_p, \sin\phi_p)$, determine ϕ_p (up to a constant) for $p \in \mathcal{V} \equiv \{1, \ldots, n\}$

$e^{j\phi_p}$ is 2π-periodic $\rightarrow$ nonlinear and ill-posed inverse problem

Continuous/discrete flavor: $\phi = \mathcal{W}(\phi) + 2k\pi \qquad \mathcal{W} : \mathbb{R} \rightarrow [\pi, \pi[$

Phase Unwrapping (PU)	**Phase Denoising (PD)**
$\downarrow$	$\downarrow$
Estimation of $k \in \mathbb{Z}$	Estimation of $\mathcal{W}(\phi) \in [\pi, \pi[$ (wrapped phase)

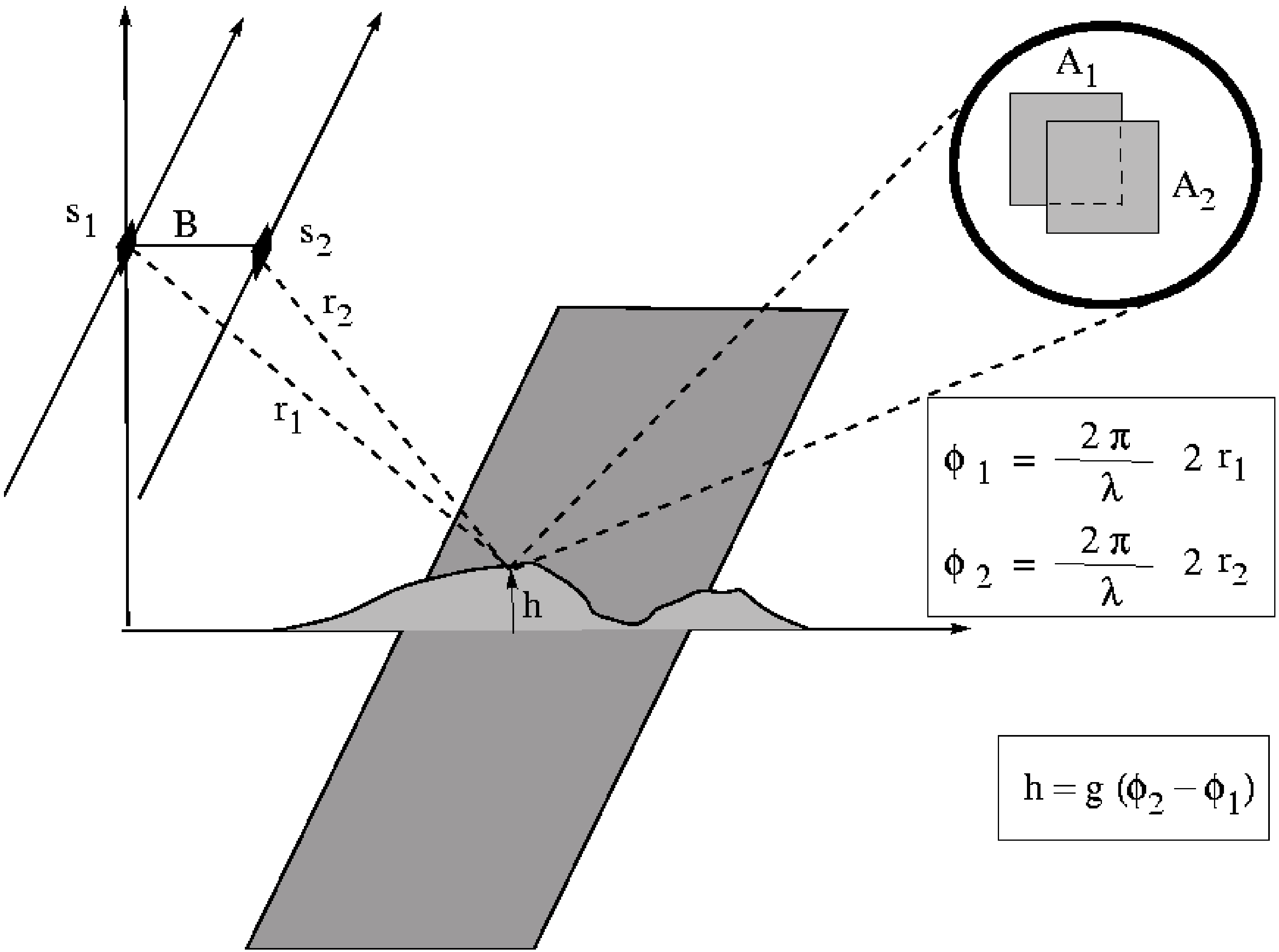
Outline

- Interferometric phase imaging. Examples
- Absolute phase estimation
- Phase unwrapping
- Interferometric phase denoising via sparse regression
- Multisource phase estimation
- Concluding remarks

Applications

- Synthetic aperture radar/sonar
- Magnetic resonance imaging
- 3D surface imaging from structured light
- High dynamic range photography
- Diffraction tomography
- Optical interferometry
- Tomographic phase microscopy
- Doppler echocardiography
- Doppler weather radar

Absolute phase estimation in InSAR (Interferometric SAR)

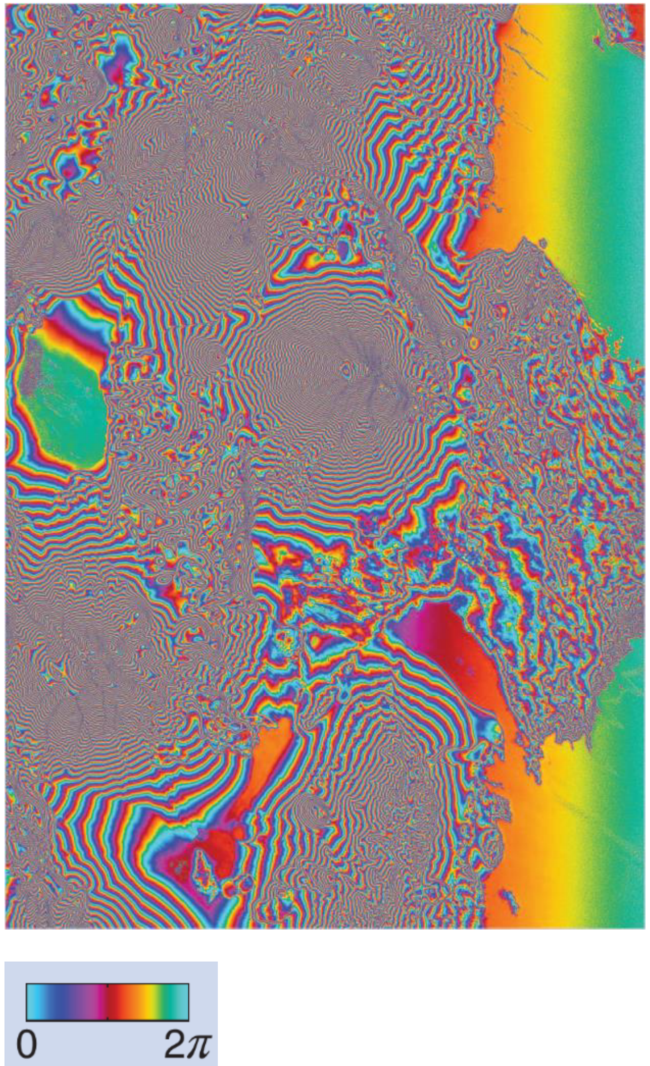


InSAR Problem: Estimate $\phi_2 - \phi_1$ from signals read by s_1 and s_2

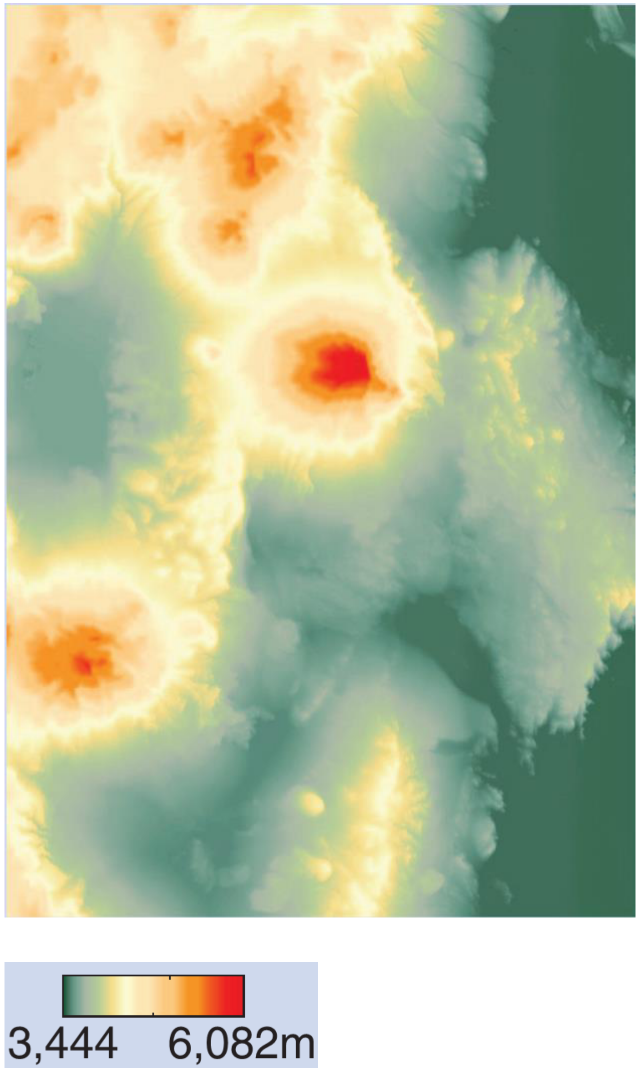
InSAR Example

Atacama desert (Chile) (from [Moreira et *al.*,13])

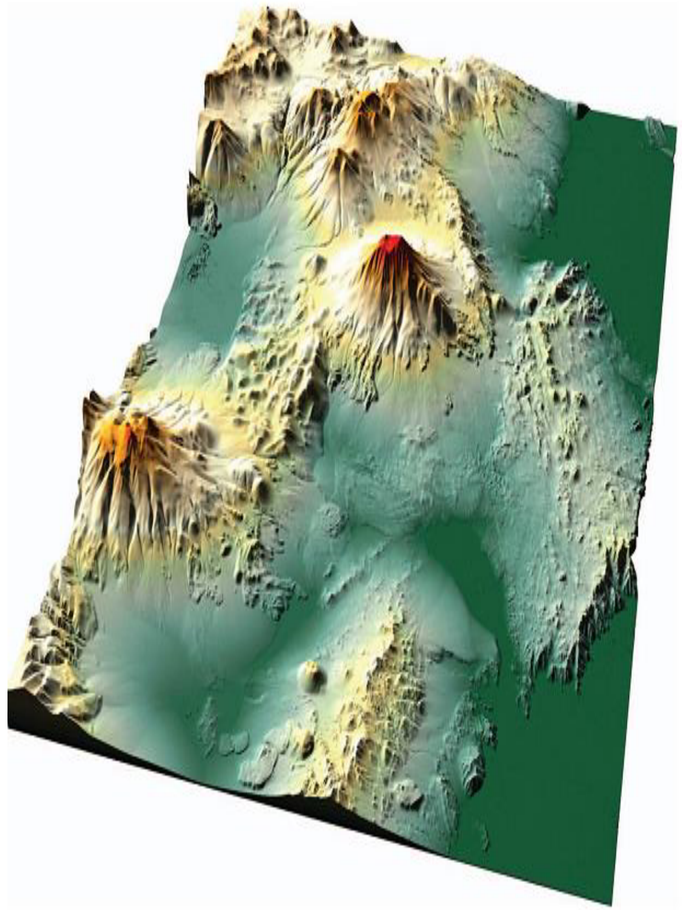
Interferogram $\mathcal{W}(\phi_1 - \phi_2)$



Unwrapped phase $\phi_1 - \phi_2$

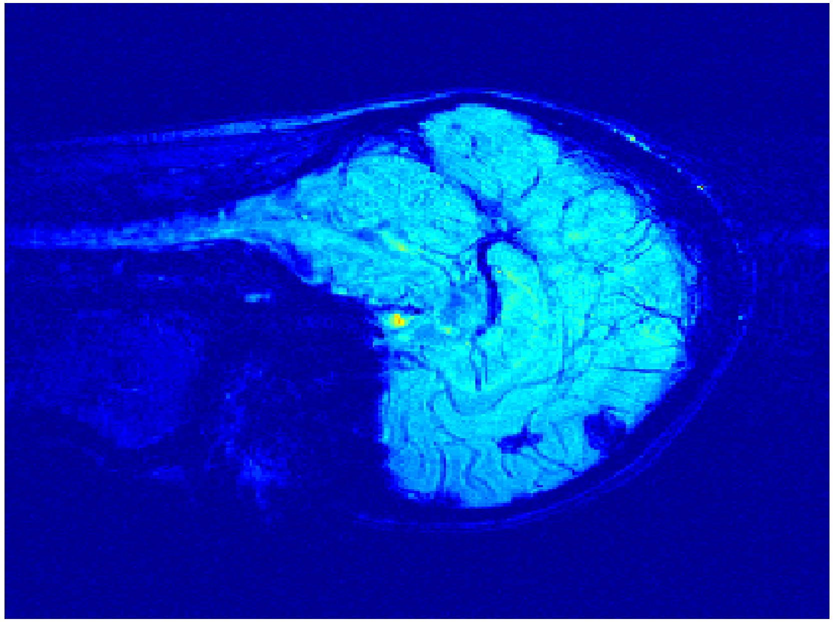


Geocoded digital elevation model (DEM)

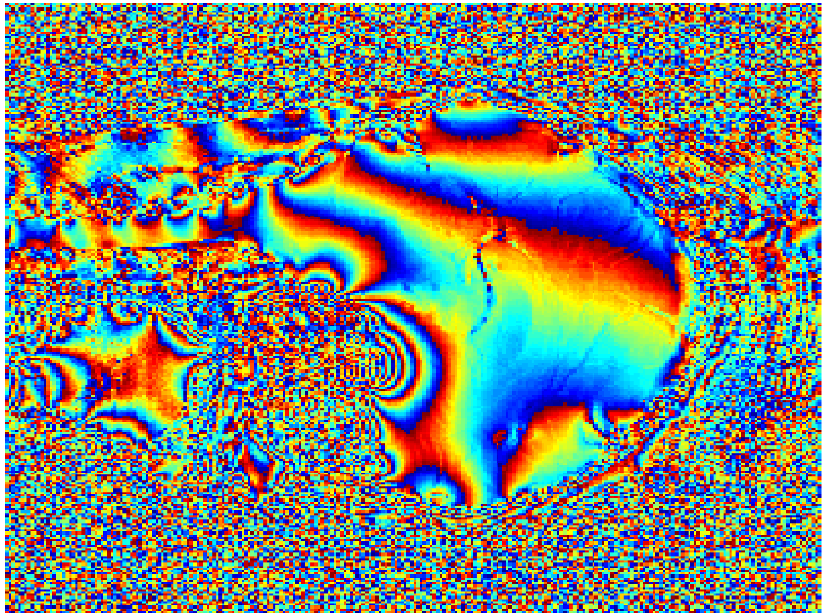


Magnetic resonance imaging (MRI)

Intensity



Interferometric phase



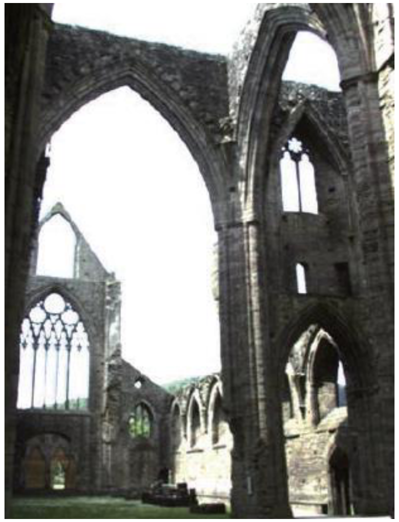
Interferomeric phase

- measure temperature
- visualize veins in tissues
- water-fat separation
- map the principal magnetic field

High dynamic range photography

(from [Zhao et *al.*, 15])

Intensity camera



Modulo camera

$$I_N = \mod\left(I, 2^N\right)$$

Unwrapped image (tone-mapped)

3D surface imaging from structured light

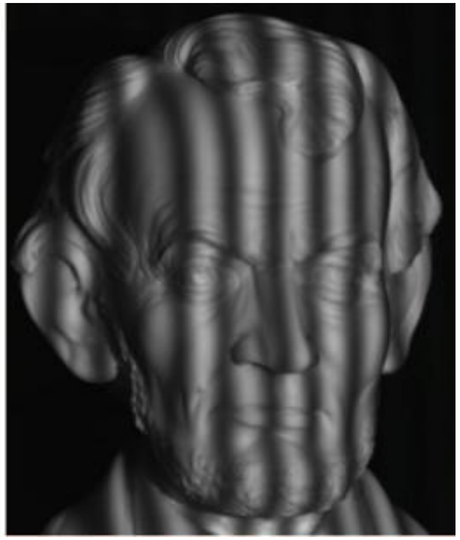
(from [Huang et *al.*, 06])

Fringe images

$\phi_{r_1} = -120^\circ$

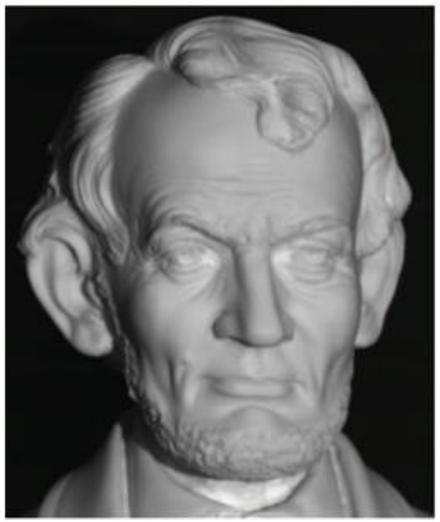
$\phi_{r_2} = 0^\circ$

$\phi_{r_3} = 0^\circ$

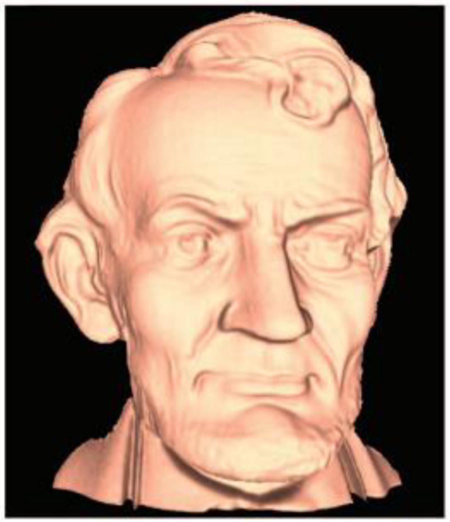


$$I_k = b_0 + b_1 \cos(\phi - \phi_{r_k})$$

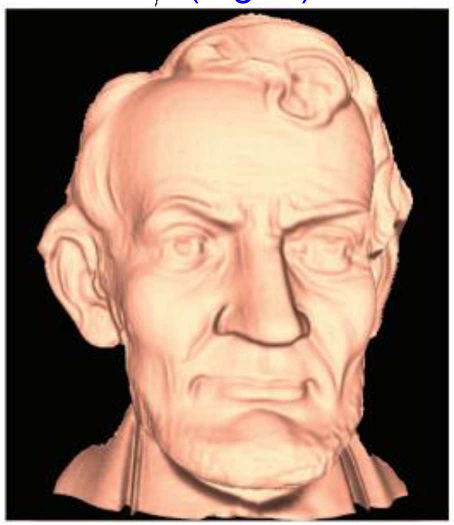
Original



$\widehat{\phi}$ (alg. 1)



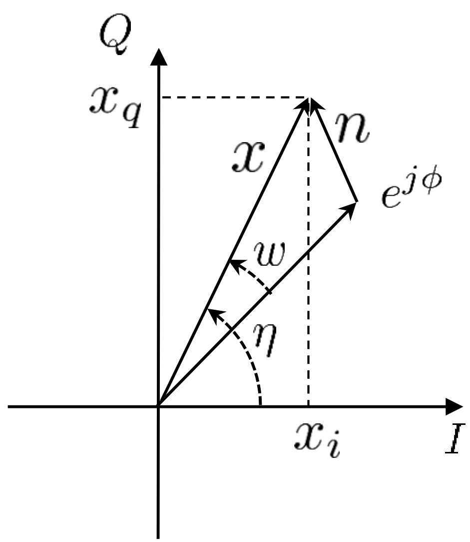
$\widehat{\phi}$ (alg. 2)



Forward problem: sensor model

$$x_i = \cos\phi + n_i \qquad n = (n_i, n_q)$$

$$x_q = \sin\phi + n_q \qquad x = (x_i, x_q)$$



$$\eta = \mathcal{W}(\phi) + w$$

$$\mathcal{W}(\phi), w \in [-\pi, \pi[$$

$$n_i, n_q \sim \mathcal{N}(0, \sigma^2)$$ independent

Data likelihood

$$p(x|\phi) \propto c\, e^{\lambda \cos(\phi - \eta)}$$

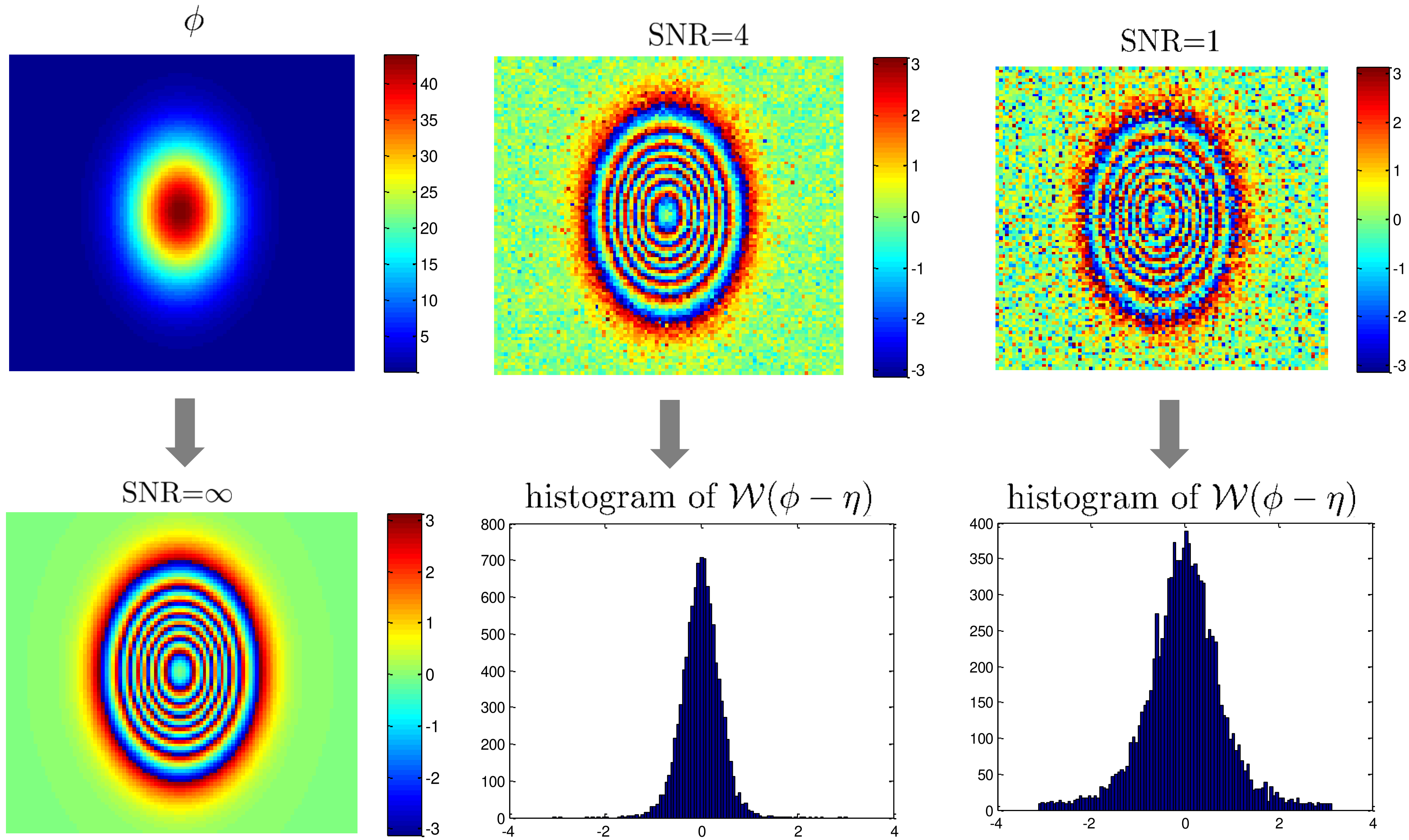
$$\eta = \arg(x) \quad \lambda = \frac{2|x|}{\sigma^2}$$

$$\widehat{\phi}_{ML} = \eta + 2k\pi$$

Simulated Interferograms

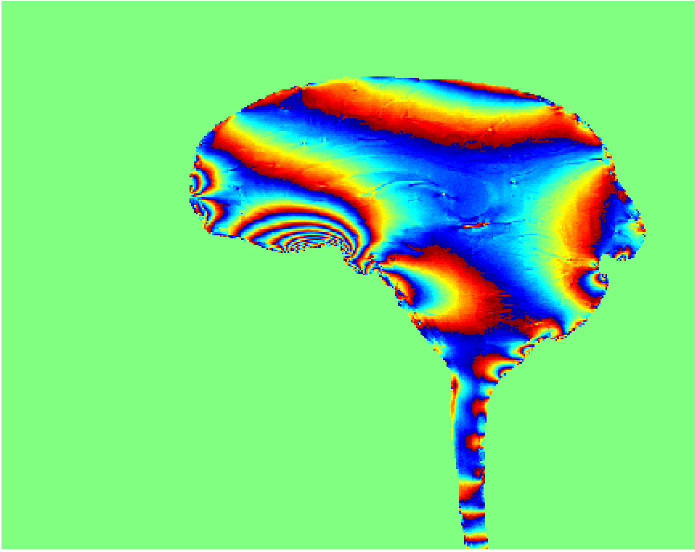
Images: $\eta = \arg(e^{j\phi} + n)$

$$\boxed{\mathrm{SNR} = \frac{1}{2\sigma^2}}$$

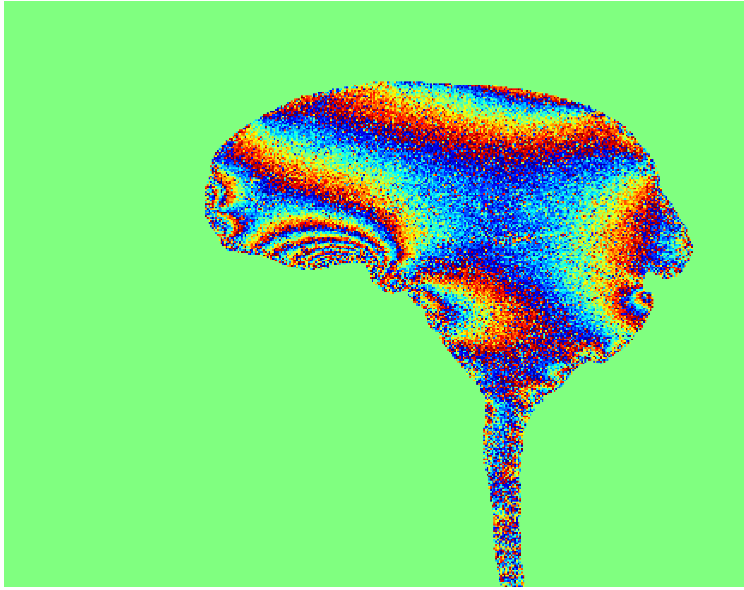


Real interferograms

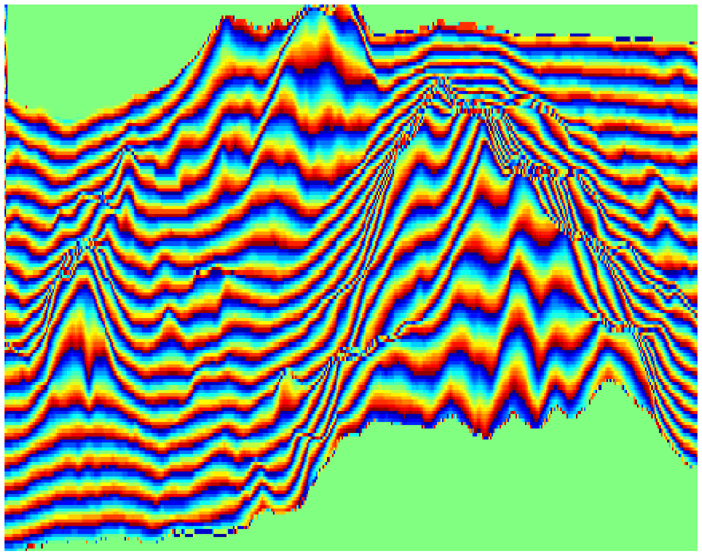
MRI



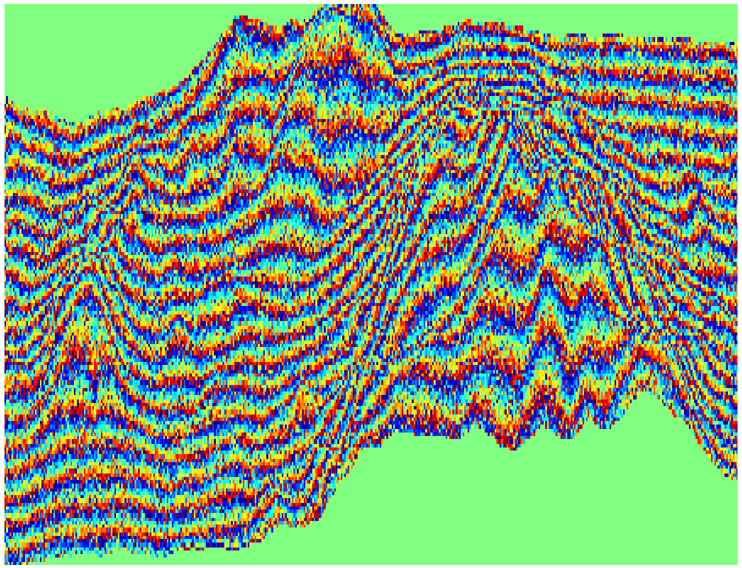
MRI



InSAR



InSAR



Bayesian absolute phase estimation

Data term: $p(\mathbf{x}|\boldsymbol{\phi}) = \prod_{p\in\mathcal{V}} p(x_p|\phi_p)$

Prior term: $p(\boldsymbol{\phi}) = \frac{1}{Z}e^{-U(\boldsymbol{\phi})}$

Ex: pairwise interactions $U(\boldsymbol{\phi}) = \sum_{\{p,q\}\in\mathcal{E}} U_{pq}(\phi_p - \phi_q)$

- $\mathcal{E} = \{\{p,q\} : p \sim q\}$ clique set
- U_{pq} clique potential

U_{pq} convex ↓ Enforces smoothness	U_{pq} non-convex ↓ Enforces piecewise smoothness (discontinuity preserving)

Estimation criteria

Maximum a posteriori (MAP) $\widehat{\phi} \in \arg\max_{\boldsymbol{\phi}\in\mathbb{R}^n} \; p(\mathbf{x}|\boldsymbol{\phi})p(\boldsymbol{\phi}) = \arg\min_{\boldsymbol{\phi}\in\mathbb{R}^n} E(\boldsymbol{\phi})$

$$\boxed{E(\boldsymbol{\phi}) = \sum_{p\in\mathcal{V}} -\lambda_p \cos(\phi_p - \eta_p) + U(\boldsymbol{\phi})}$$

E is hard to optimize due to the sinusoidal data terms

Popular approaches to absolute phase estimation

- Reformulation as linear observations in non-Gaussian noise
- Interferometric phase denoising + phase unwrapping

Phase differences

Wrapped difference of wrapped phases: $\eta = \mathcal{W}(\phi) + w$

$$\mathcal{W}(\eta_p - \eta_q) = (\phi_q - \phi_q) + \boxed{(w_p - w_q)} + \boxed{2\pi l_{p,q}}$$

additive noise distributed in $[-2\pi, 2\pi[$

wrap errors due to discontinuities, high phase rate, and noise

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- In the absence of noise, $l_{p,q} = 0$ if $|\phi_q - \phi_q| < \pi$ (Itoh condition)
- In most applications $P(|\phi_q - \phi_q| \geq \pi)$ is small but positive
- $l_{p,q} = 0 \text{ for } \{p,q\} \in \mathcal{E}$ if $\max_{\{p,q\}\in\mathcal{E}} |\phi_p - \phi_q| + \max_{\{p,q\}\in\mathcal{E}} |w_p - w_q| < \pi$
- Number of wrap errors increases with σ. If $w_p \sim \mathcal{N}(0,\sigma^2)$, then

$$\mathbb{E}[\max_{\{p,q\}\in\varepsilon} |w_p - w_q|]) \geq \mathbb{E}[\max_{\{p,q\}\in\varepsilon} (w_p - w_q)] = O\left(\sigma\sqrt{\log|\mathcal{E}|}\right)$$

Absolute phase estimation: linear observations in non-Gaussian noise

$\mathbf{y} = \mathcal{W}(\mathbf{D}\boldsymbol{\eta})$ $\quad \mathbf{D} : \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ – discrete gradient

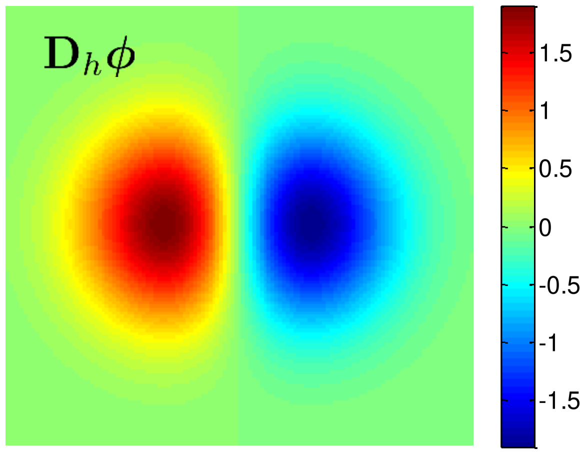
$\mathbf{w}_\eta$ – interferometric noise

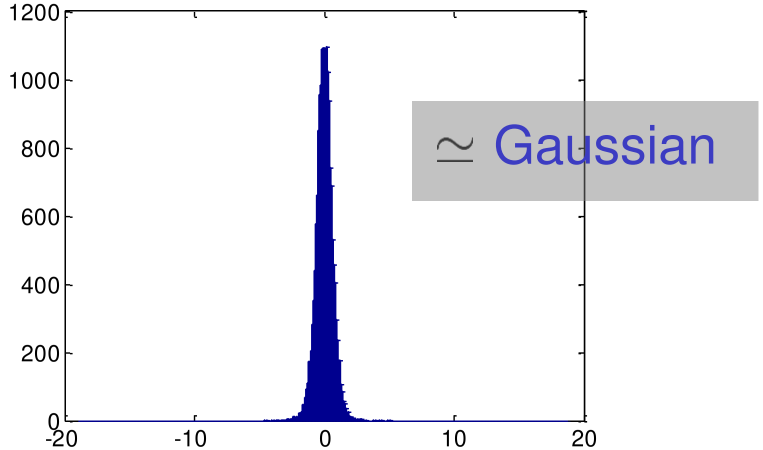
$\mathbf{w}_\pi$ – wrap errors

$$\boxed{\mathbf{y} = \mathbf{D}\boldsymbol{\phi} + \mathbf{w}_\eta + \mathbf{w}_\pi}$$

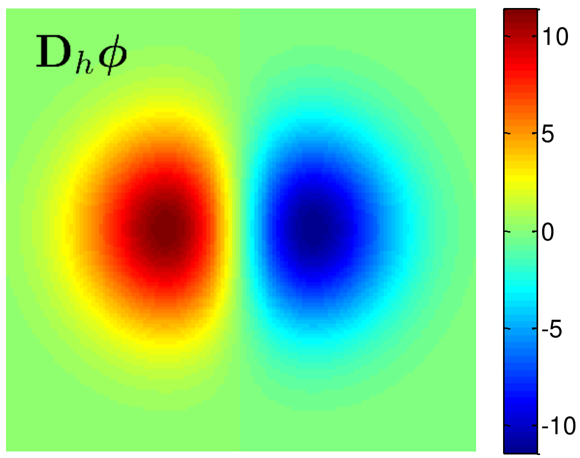
Histograms of $\mathbf{y} - \mathbf{D}\boldsymbol{\phi} = \mathbf{w}_\eta + \mathbf{w}_\pi$ for a Gaussian phase surface

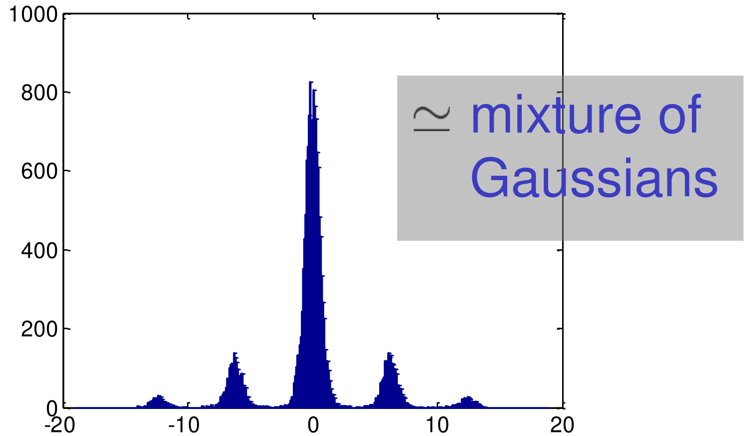
$|\phi_q - \phi_q| < \pi$





$|\phi_q - \phi_q| \geq \pi$





Formulation based on the linear observation model (LOM)

Minimum ℓ_p norm $0 < p < 2$ [Ghiglia & Pritt, 98]

$$\min_{\phi \in \mathbb{R}^n} \|\mathbf{y} - \mathbf{D}\phi\|_{p,Q} \quad \text{s.t.} \quad \mathbf{A}\phi = \mathbf{b}$$

Regularized ℓ_1 norm (convex) [Gonzalez & Jacques, 15]

$$\min_{\phi, \mathbf{u} \in \mathbb{R}^n} \|\mathbf{W}\phi\|_1 \text{ s.t.} \begin{cases} \|\mathbf{y} - \mathbf{D}(\phi + \mathbf{u})\|_1 \leq \varepsilon_\pi \\ \|\mathbf{u}\|_2 \leq \varepsilon_w \\ \mathbf{A}\phi = \mathbf{b} \end{cases}$$

Adaptive regularized ℓ_2 norm [Kamilov et al.,15]

$$\min_{\phi \in \mathbb{R}^n} \sum_{i=1}^{n} q_i^t \|\mathbf{y}_i - \mathbf{D}_i\phi\|_2 + \tau \boxed{\|\mathbf{H}_i\phi\|_*} \quad \text{s.t.} \quad \mathbf{A}\phi = \mathbf{b}$$

$\mathbf{D}_i : \mathbb{R}^n \to \mathbb{R}^2$ – discrete gradient

$\mathbf{y}_i = (y_{h,i}, y_{v,i})$

Nuclear norm

$\mathbf{H}_i : \mathbb{R}^n \to \mathbb{R}^4$ – discrete Hessian

Algorithms

IRLS, MM
[Lange & Fessler., 95]

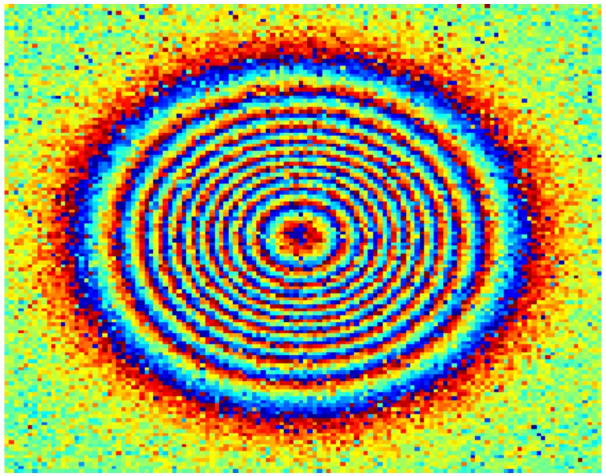
PD
[Chambolle, Pock, 11]

Seq. of ADMM subproblems (q_i^t)

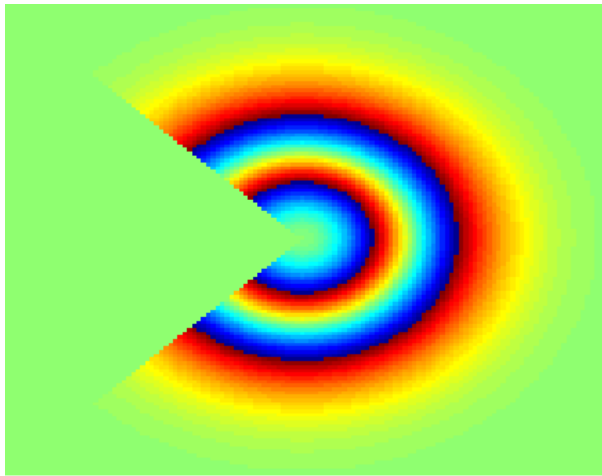
SALSA
[Afonso, B-D, Fig., 11]

Example: IRTV ([Kamilov et al.,15]) (SALSA implementation) $n = 128 \times 128$

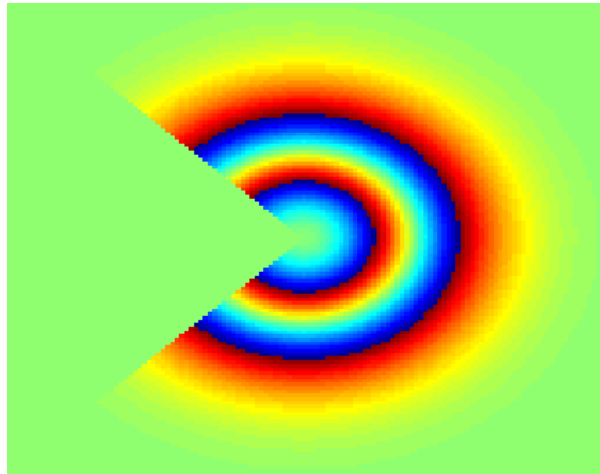
$\max \phi_p = 20\pi \quad \sigma = 0.5$



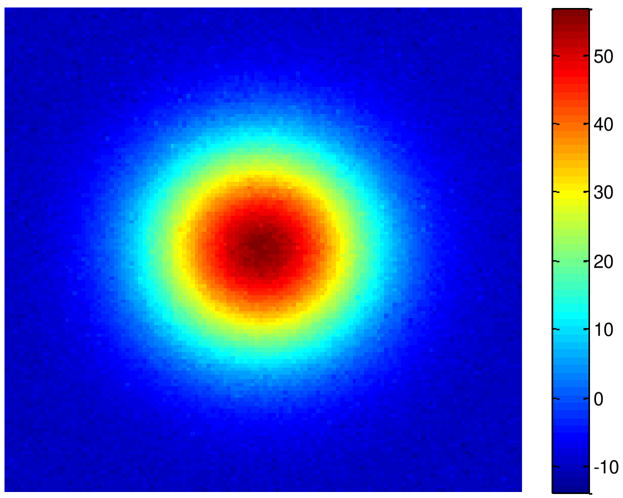
$\max \phi_p = 4\pi$



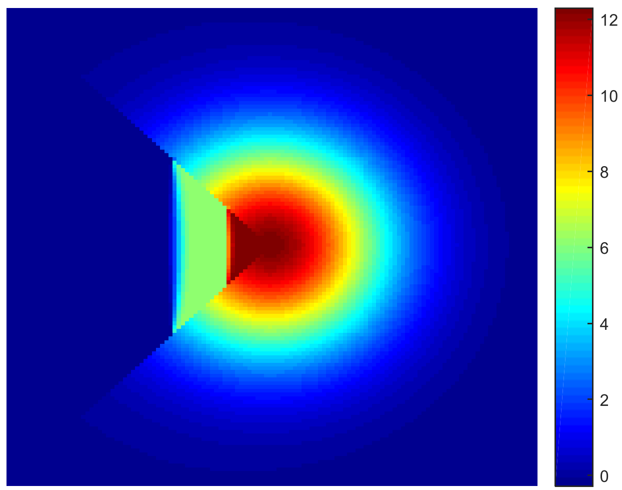
$\max \phi_p = 4\pi$



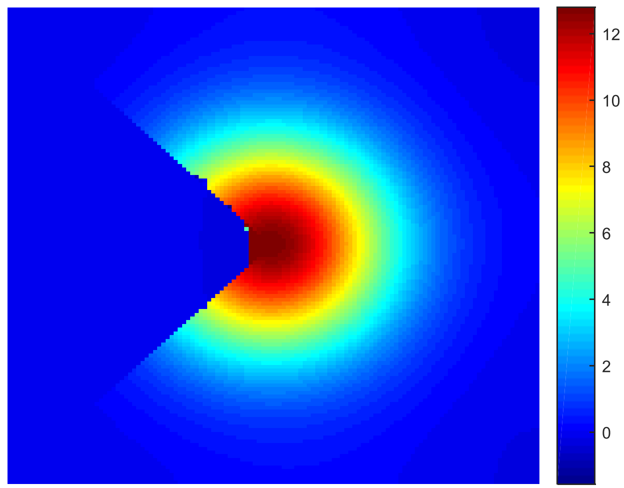
$\tau = 10^{-3}$



$\tau = 10^{-3}$



$\tau = 10^{-3}$



$$\mathrm{ISNR} = \frac{2n\sigma^2}{\|\widehat{\phi} - \phi\|_F^2}$$

$\mathrm{ISNR} = (1.4, 1.5, -16.4)\,\mathrm{dB}$

$\tau = (10^{-4}, 10^{-2}, 10^{0})$

1 iter (fixed weights)

time $= 20\,\mathrm{s}$

10 iters (adaptive weights)

time $= 200\,\mathrm{s}$

A few comments on the LOM-based phase estimation

Observation model: $\mathbf{y} = \mathbf{D}\boxed{\phi} + \boxed{\mathbf{w}_\eta} + \boxed{\mathbf{w}_\pi}$ $\qquad \mathbf{w}_\eta = \mathbf{D}\mathbf{w}$

self-similar highpass sparse and ϕ-dependent

- Regularization is challenging. Ex: Tickhonov regularization using $\|\mathbf{D}\phi\|^2$

$$\widehat{\phi} = \frac{1}{1+\tau}\left(\phi + \boxed{\mathbf{w}} + \boxed{\mathbf{D}^\dagger \mathbf{w}_\pi}\right)$$

original interferometric

$\longrightarrow$ wrap errors are amplified

- The wrap errors $\mathbf{w}_\pi$ due to phase discontinuities tend to be sparse and thus well modeled by ℓ_p norms with $p \leq 1$

- ℓ_1 norm (and ℓ_1 on the gradient) yields convex programs but has limited power to cope with wrap errors

1) Denoise (filter out $\mathbf{w}$)

2) (Use ℓ_p with $p < 1$) or ($p \geq 1$ and detect the discontinuities)

Interferometric phase denoising + phase unwrapping

Back to MAP estimate

$$\widehat{\phi} \in \arg\min_{\boldsymbol{\phi}\in\mathbb{R}^n} E(\boldsymbol{\phi}) \qquad E(\boldsymbol{\phi}) = \sum_{p\in\mathcal{V}} -\lambda_p \cos(\phi_p - \eta_p) + U(\boldsymbol{\phi})$$

Assume that: $\boldsymbol{\phi} = \{\phi_p | \phi_p = \eta_p + 2k_p\pi, p \in \mathcal{V}, k_p \in \mathbb{Z}\} \qquad (\Leftrightarrow \lambda_p \to \infty)$

Then: $$\widehat{\mathbf{k}} \in \arg\min_{\mathbf{k}\in\mathbb{Z}^n} E(\boldsymbol{\eta}, \mathbf{k}) = \arg\min_{\mathbf{k}\in\mathbb{Z}^n} U(\boldsymbol{\eta}, \mathbf{k})$$

Integer optimization

Pairwise interactions: $U(\boldsymbol{\eta}, \mathbf{k}) = \sum_{\{p,q\}\in\mathcal{E}} V_{pq}(k_p - k_q)$

$$V_{pq}(k_p - k_q) = U_{pq}(\eta_p - \eta_q + 2\pi(k_p - k_q))$$

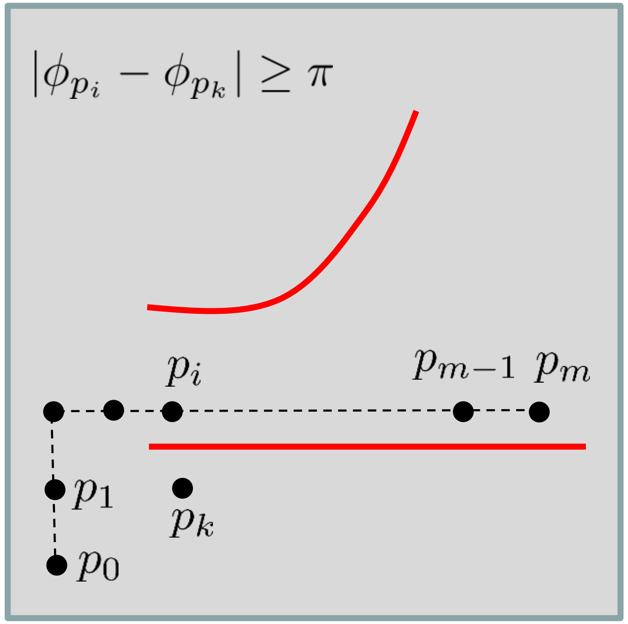
Phase unwrapping: path following methods

Assume that $|\phi_p - \phi_q| < \pi$ (Itoh condition) $\phi_p = \eta_p + 2k_p\pi$

Then $\phi_p - \phi_q = \mathcal{W}(\phi_p - \phi_q) = \mathcal{W}(\eta_p - \eta_q)$

PU $\Leftrightarrow$ summing $\mathcal{W}(\eta_p - \eta_q)$ over walks

$$\phi_{p_m} = \phi_{p_0} + \sum_{i=1}^{m} \mathcal{W}(\eta_{p_i} - \eta_{p_{i-1}})$$



Why isn't PU a trivial problem?

Discontinuities
High phase rate
Noise

$\rightarrow$ $|\phi_p - \phi_q| \geq \pi$

Phase unwrapping algorithms

$$\boxed{E(\mathbf{k}) = \sum_{\{p,q\}\in\mathcal{E}} V_{pq}(k_p - k_q)}$$

- $V_{pq}(\cdot) = |\cdot|_{2\pi-\text{quantized}}$

  [Flynn, 97] (exact) sequence of positive cycles on a graph

  [Costantini, 98] (exact) min-cost flow on a graph $(|\mathcal{V}| = n, |\mathcal{E}| = 4n)$

- $V_{pq}(\cdot) = (\cdot)^2$

  [B-D & Leitao, 01] (exact) sequence of positive cycles on a graph $(|\mathcal{V}| = n, |\mathcal{E}| = 4n)$

  [Frey et *al*., 01] (approx) belief propagation on a 1st order MRF

- $V_{pq}(\cdot)$ convex

  [B-D & Valadao, 07,09,11] (exact) ∫equence of *K* min cuts ($KT(n, 6n)$)

- $V_{pq}(\cdot)$ non-convex

  [Ghiglia, 96] LPN0 (continuous relaxation)

  [B-D & G. Valadao, 07,09,11] sequence of min cuts ($KT(n, 6n)$)

PUMA (Phase Unwrapping MAx-flow) [B-D & Valadao, 07,09,11]

Algorithm 1: PUMA

$\boldsymbol{\phi} := \boldsymbol{\eta}, \quad succes == false$

while $succes == false$ **do**

$\quad \boldsymbol{\delta} := \arg \min_{\mathbf{x} \in \{0,1\}^{|\nu|}} E(\boldsymbol{\phi} + 2\mathbf{x}\pi)$

$\quad$ **if** $E(\boldsymbol{\phi} + 2\mathbf{x}\pi) < E(\boldsymbol{\phi})$ **then**

$\quad\quad \boldsymbol{\phi} := \boldsymbol{\phi} + 2\boldsymbol{\delta}\pi$

$\quad$ **else**

$\quad\quad succes == false$

return $\boldsymbol{\phi}$

PUMA finds a sequence of steepest descent binary images

Convex priors $E(\mathbf{k}) = \sum V_{pq}(k_p - k_q)$

- A local minimum is a global minimum
- Takes at most *K* (range of k) iterations
- E is submodular: $2V_{pq}(0) \leq V_{pq}(1) + V_{pq}(-1)$

$\Rightarrow$ each binary optimization has the complexity of a min cut $T(n, 6n)$

PUMA: convex priors

$$E(\mathbf{k}) = \sum V_{pq}(k_p - k_q)$$

- Let $\boldsymbol{\phi}$ be a smooth surface in the Itoh sense. That is $|\phi_p - \phi_q| < \pi$ for $\{p,q\} \in \mathcal{E}$. If $U_{pq}(x)$ is convex and strictly increasing of $|x|$, then

$$\boldsymbol{\phi} = \boldsymbol{\eta} + \widehat{\mathbf{k}} + c$$

  where $\widehat{\mathbf{k}}$ is the PUMA solution

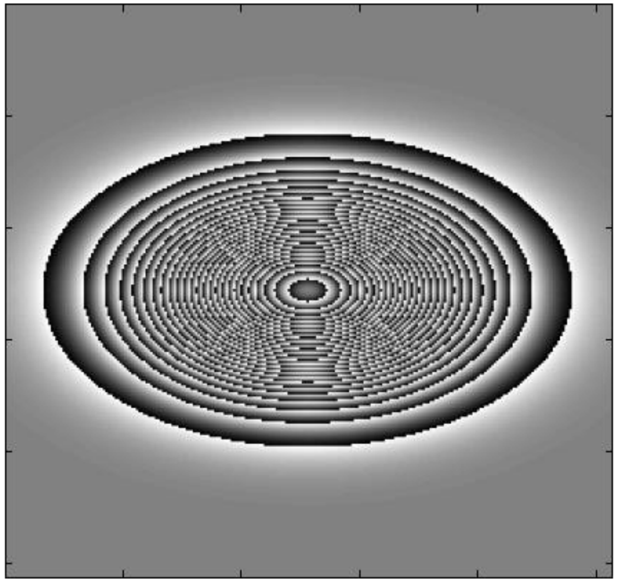
- Related algorithms

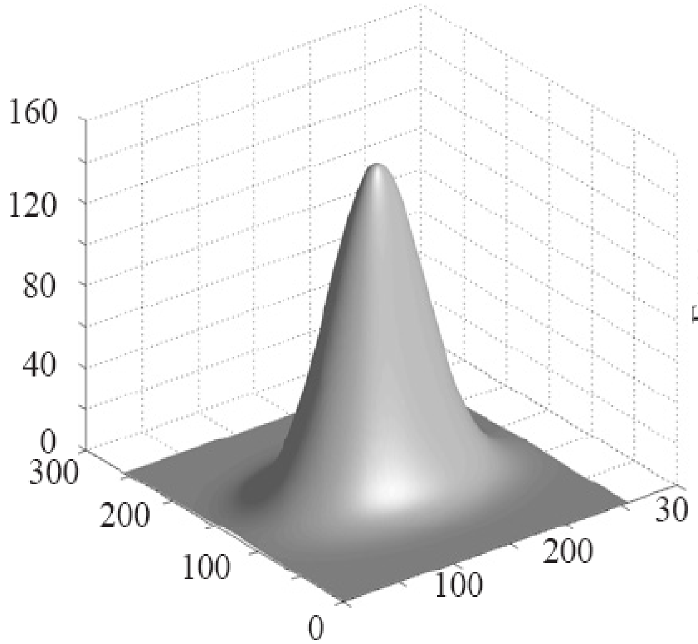
$$E(\mathbf{k}) = \sum_{p \in \mathcal{V}} D_p(k_p) + \sum_{\{p,q\} \in \mathcal{E}} V_{pq}(k_p - k_q)$$

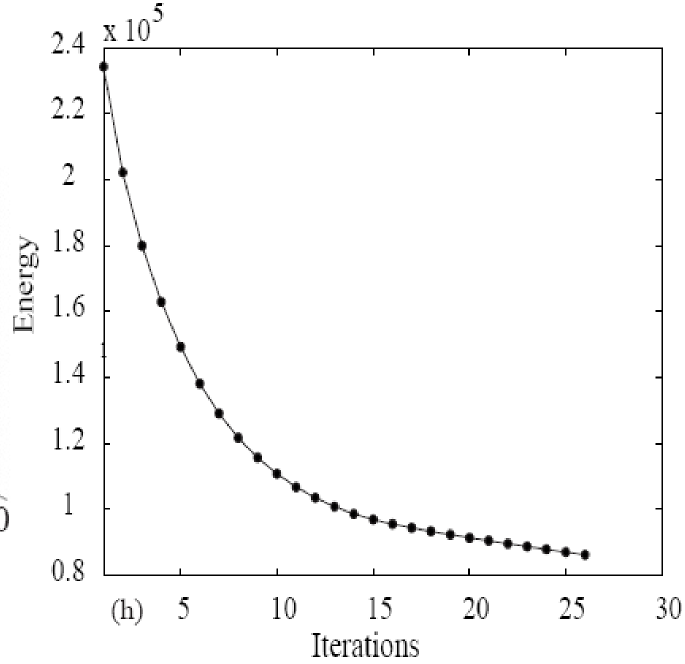
[Veksler, 99] (1-jump moves)
[Murota, 03] (steepest descent algorithm for L-convex functions)
[Ishikawa, 03] (MRFs with convex priors)
[Kolmogorov & Shioura, 05,09], [Darbon, 05] (Include unary terms)
[Ahuja, Hochbaum, Orlin, 03] (convex dual network flow problem)

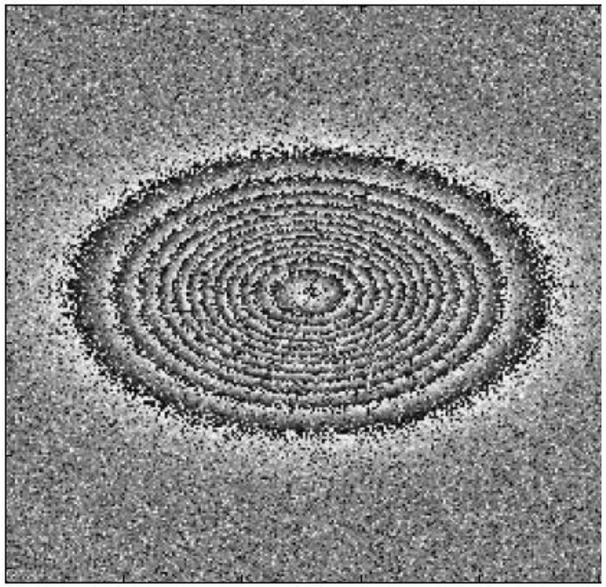
Results

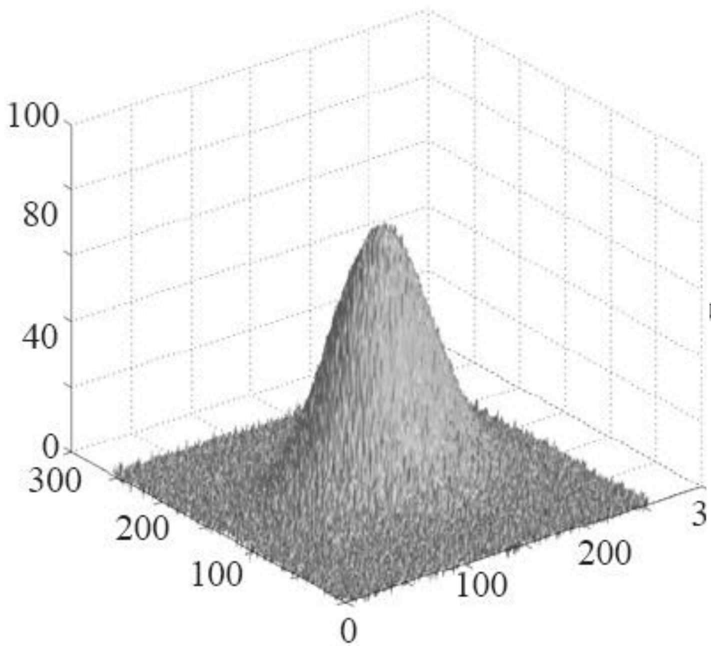
$$U_{pq}(\cdot) = (\cdot)^2$$

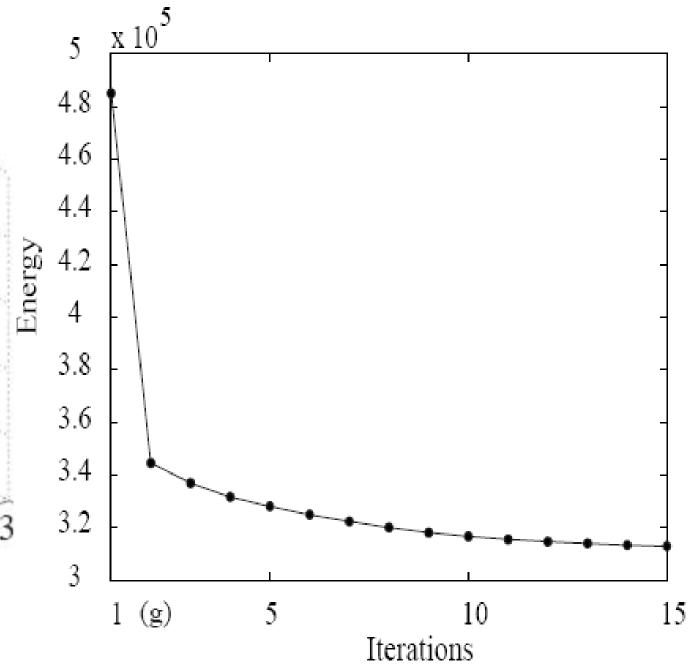






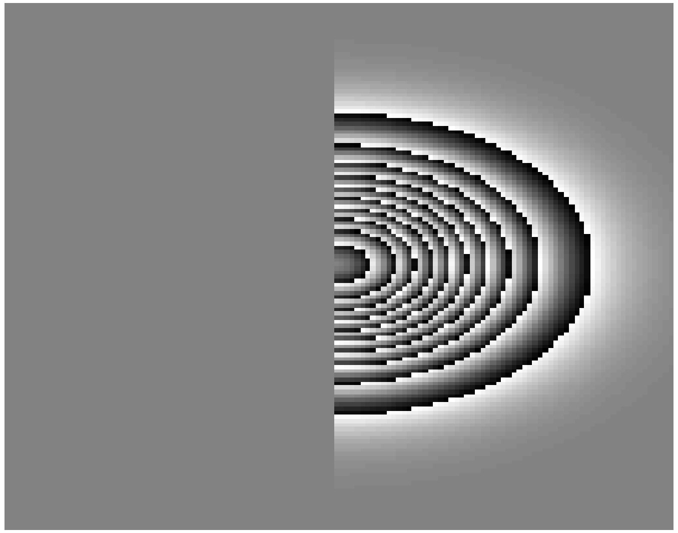




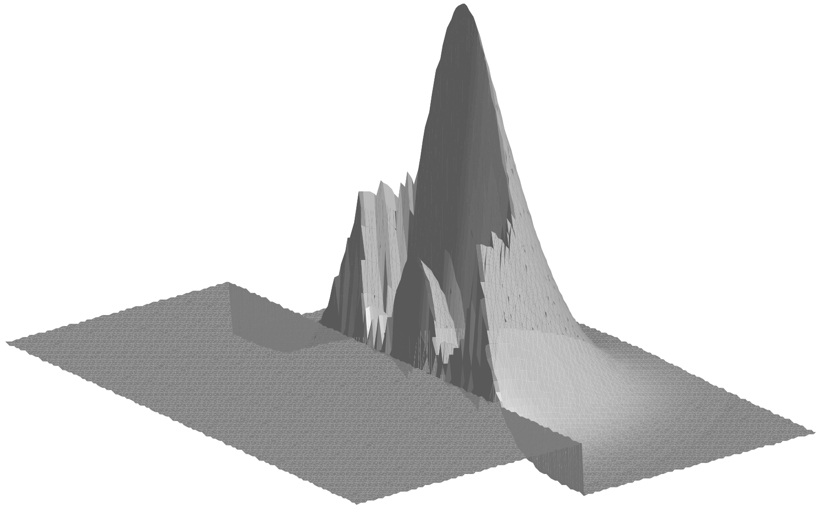


Results

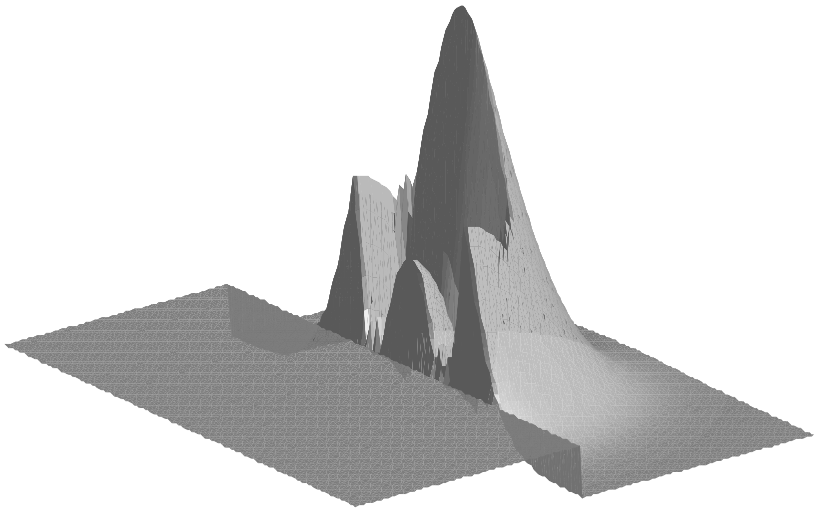
Convex priors do not preserve discontinuities



$$U_{pq}(x) = (x)^2$$

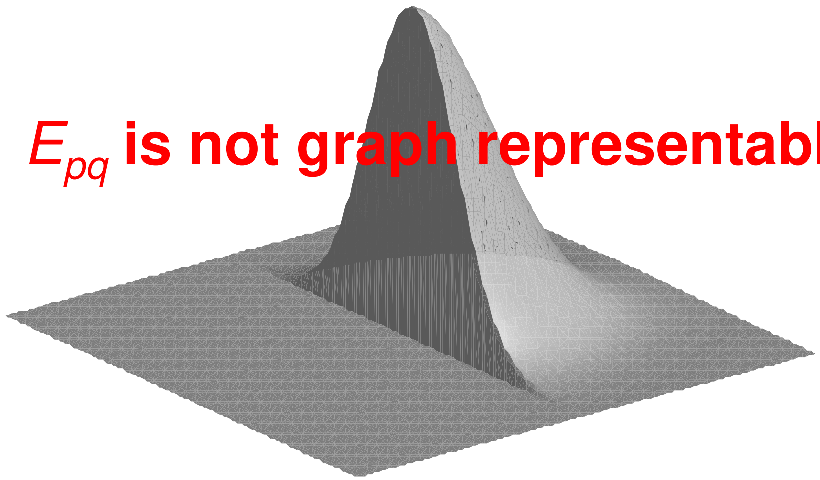


$$U_{pq}(x) = |x|$$



$$U_{pq}(x) = \begin{cases} x^2 & |x| \le \pi \\ \pi^2 |x/\pi|^{0.5} & |x| > \pi \end{cases}$$

E_{pq} **is not graph representable**

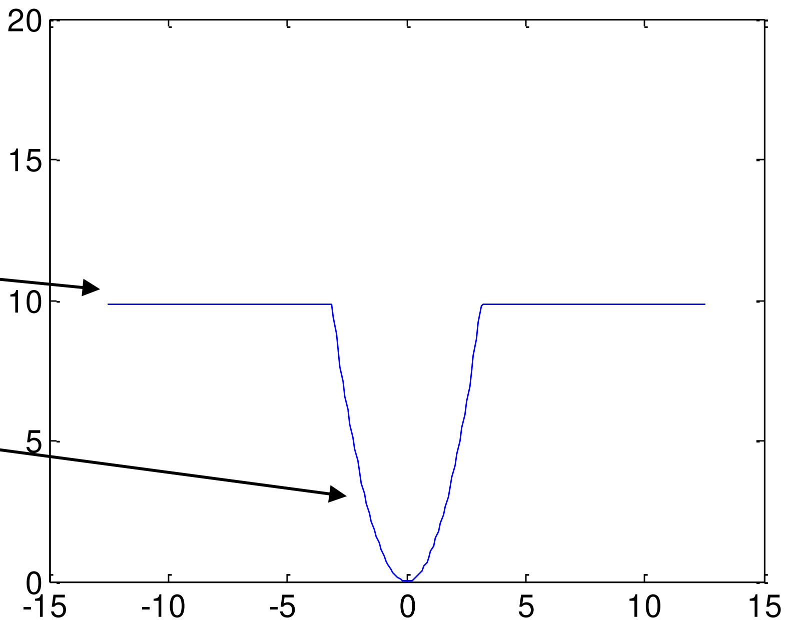


PUMA: non-convex priors

Ex: $U_{pq}(x) = \min(x^2, \pi^2)$

Models discontinuities

Models Gaussian noise

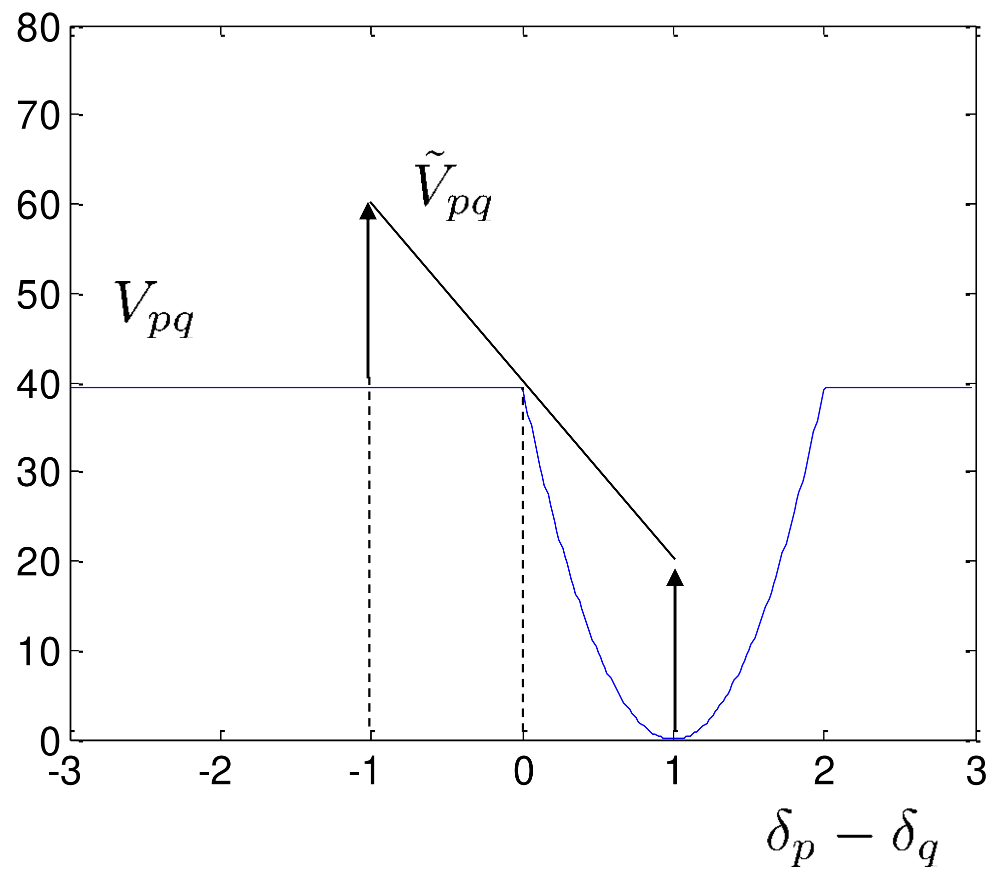


Shortcomings

- ❑ Local minima are no more global minima
- ❑ Energy contains nonsubmodular terms (NP-hard)

Proposed suboptimal solution: majorization minimization applied PUMA binary sub-problems

Majorizing nonsubmodular terms



Majorization Minimization (MM)
[Lange & Fessler, 95]

$$\begin{cases} \tilde{U}(\mathbf{k}) = U(\mathbf{k}) \\ \tilde{U}(\mathbf{k}+\boldsymbol{\delta}) \geq U(\mathbf{k}+\boldsymbol{\delta}) \end{cases}$$

$$\boldsymbol{\delta}' = \arg\min_{\boldsymbol{\delta}} \tilde{U}(\mathbf{k}+\boldsymbol{\delta})$$

Non-increasing property

$$\boxed{U(\mathbf{k}+\boldsymbol{\delta}') \leq U(\mathbf{k})}$$

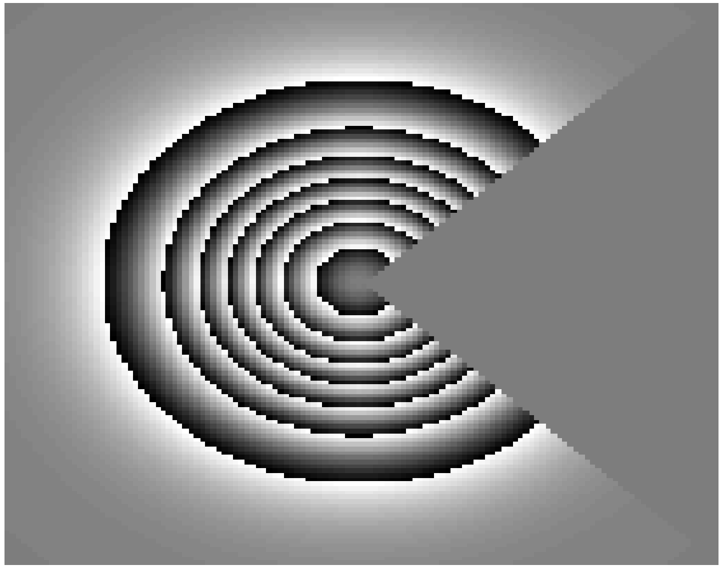
Other suboptimal approaches

- Quadratic Pseudo Boolean Optimization (Probing [Boros et *al.*, 2006], Improving [Rother et *al.*, 2007])
- Sequencial Tree-Reweighted Message Passing (TRW-S) [Kolmogorov, 2006]
- Dual decomposition (DD) [Komodakis et al., 2011]
- DD + Augmented Lagrangian [Martins et al., 2015]

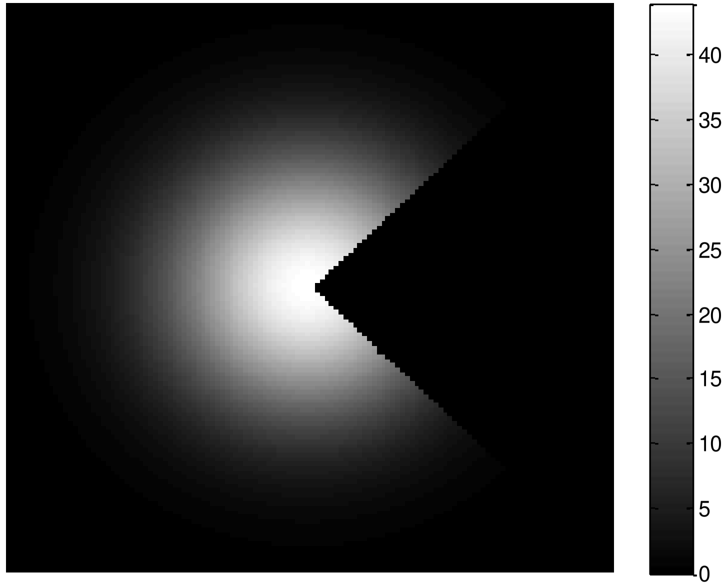
Results with PUMA (MM)

$(n = 128 \times 128, 2^{nd}$ order neighborhood, $p = 0.2, th = 0.1)$

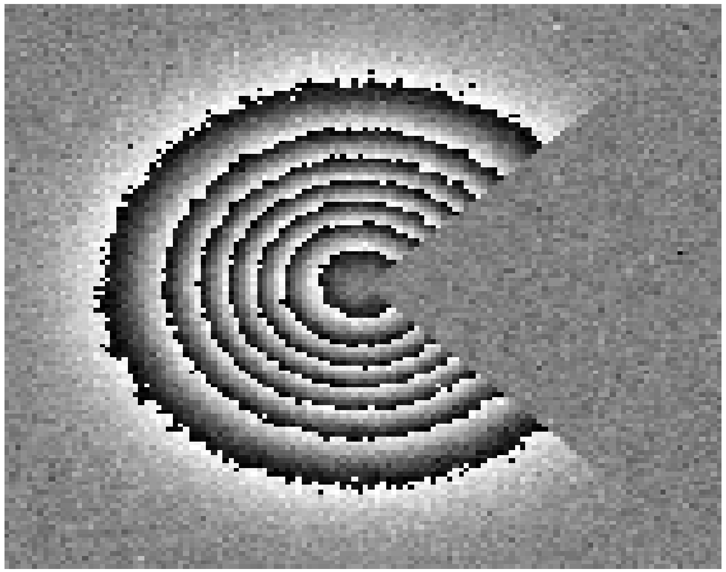
$\boldsymbol{\eta}$



$\widehat{\phi}$(MM) (8 iter)

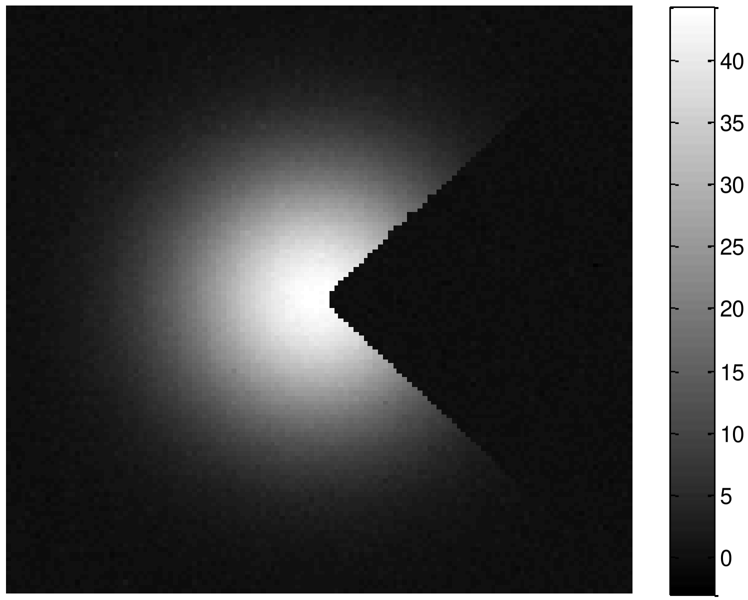


$\boldsymbol{\eta}$



Time = 1s

$\widehat{\phi}$(MM) (8 iter)



PUMA/IRTV in a HDRP example $\phi \in [0, \rho]$ $n = 256 \times 256$

PUMA: 1^{st} order neighborhood, $p = 0.2$ $th = 0.1$

$\phi \bmod 2\pi$ $(\rho = 6)$

$\phi \bmod 2\pi$ $(\rho = 8)$

ρ	SNR (dB) PUMA	SNR (dB) IRTV
4	∞	∞
5	∞	25.65
6	25.2	19.98
7	17.34	16.09
8	13.68	0.92
9	1.82	2.17
T(sec)	1	350

[Kamilov et al.,15]