Distribution of Random Walks

October 17, 2014

Convergence of Binomial Probabilities[1]

We all intuitively understand that when a fair coin (50% heads, 50% tails) is flipped many times. the probability of heads versus tails should start to converge. For example, let's say we get 1 point for every head and 0 points for every tail. What will the *average* number of points we get at any point in time? Let's compute and plot this in R with `n=5000` coin flips!

[1] These examples are based on: `http://nvenkataraman1.github.io/random-walks/`

```r
n <- 5000
flips <- rbinom(n, 1, 0.5)  # a fair coin follows a simple binomial distribution
means <- c(0)
for (i in 1:n) {
    means[i] <- mean(flips[1:i])
}

plot(means, type = "l", ylim = c(0, 1))
abline(h = 0.5, col = "red")
```

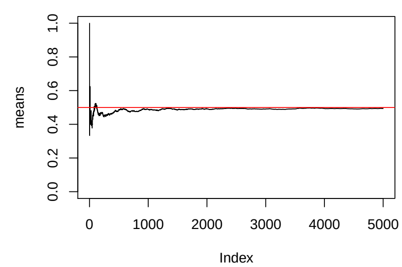

Figure 1: Coin flipping probabilities

We see that the average number of points quickly converges to 0.5 (a *half*), just as we suspected. This is our common intuition about how probabilities average out over time.

Now, instead of average probabilities, let's make a game where we gain 1 point for heads and lose one point for tails. How many *total points* would we make over time? Will it converge?

```r
n <- 5000
flip <- 2 * rbinom(n, 1, 0.5) - 1
total <- c(0)
for (i in 2:n) {
    total[i] <- total[i - 1] + flip[i]
}

plot(total, type = "l", ylim = c(-300, 300))
abline(h = 0)
```

Figure 2: A random (binomial) walk

When we plot out our total number of points, rather than converging, it seems to just randomly walk away! *Random walks* are one way of modeling many everyday phenomena that are affected by so many variables that they appear random in their progress: stock markets, weather fluctuations, ant foraging, brownian motion, and more[2].

[2] `http://en.wikipedia.org/wiki/Random_walk`

BUT HOW FAR FROM THEIR STARTING POINTS do random walks go *on average*? It isn't easy to tell, so let's simulate random walks and find out!

We first create a 'function' that creates binomial walks for us. It reuses the code we saw earlier. This function takes the number of 'steps' to walk as a parameter.

```
# Random binomial walk function
random_binomial_walk <- function(steps) {
    flip <- 2 * rbinom(n, 1, 0.5) - 1
    total <- c(0)
    for (i in 2:steps) {
        total[i] <- total[i - 1] + flip[i]
    }
    return(total)
}
```

We first create a 'data.frame' that is the size of a 2x3 matrix. The we use a loop to run our random binomial walk function six times. The result of our six function calls are stored in the data.frame

We then plot the six walks. The 'mfrow' function lets us put plots side-by-side in a grid pattern! At the end, we have to be careful to call mfrow again to return the graphics setting to plot 1x1 figures at a time.

```
# Simulate six random walks of 5000 steps each
walks <- data.frame(matrix(0, nrow = 5000, ncol = 6))
for (i in 1:6) {
    walks[i] <- random_binomial_walk(5000)
}

# Plot out our six random walks
par(mfrow = c(2, 3))  # setup plotting to 2rows x 3 columns
for (i in 1:6) {
    plot(walks[[i]], type = "l", ylim = c(-200,
        200))
    abline(h = 0, col = "red")
}
```

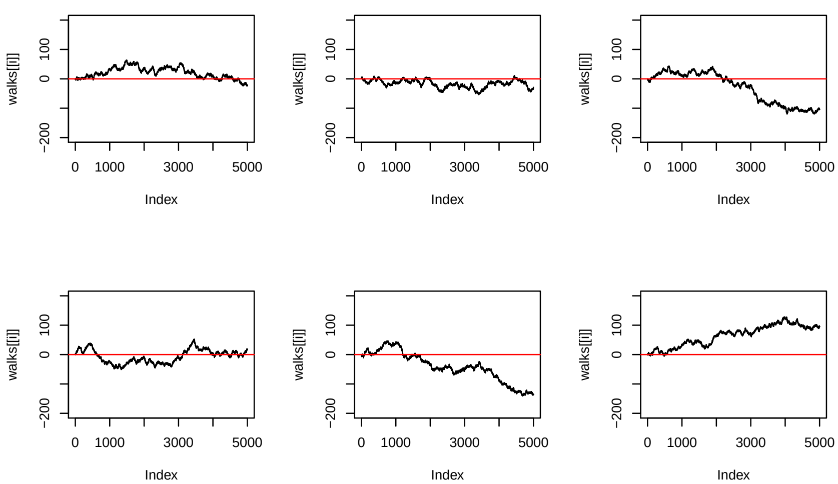

Figure 3: Six Random Binomial Walks

```
par(mfrow = c(1, 1))  # return graphics settings to 1 x 1
```

Even with a small number of random walks, we see they can go anywhere. It will be hard to plot a large number of walks. So let's simulate a large number of walks and see what distance from zero points each walk ended up. We will use m=3000 walks, each having n=2000 steps.

```
m <- 3000
n <- 2000
walks <- data.frame(matrix(0, nrow = n, ncol = m))
for (i in 1:m) {
    walks[i] = random_binomial_walk(n)
}
final.points <- c(0)
for (i in 1:m) {
    final.points[i] <- walks[n, i]
}
```

We can now plot the distribution of how many points away from zero (positive or negative) each each of the 3000 walks finally got to, after 2000 steps.

```
hist(final.points, breaks = 20, prob = TRUE)
lines(density(final.points), lwd = 3, col = "blue")
```

Does this bell-shaped symmetric distribution look familiar? It is our old friend the *normal distribution* again!

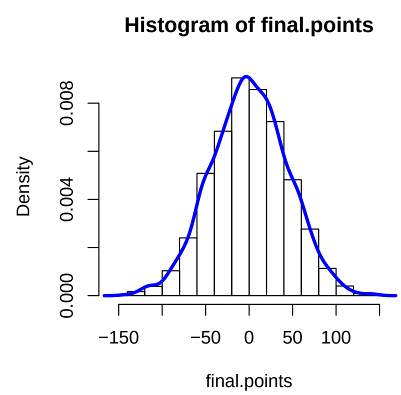

Figure 4: Distribution of binomial walks