AE353: Additional Notes on Eigenvalue Placement

(to be treated as an appendix to the presentation)

A. Borum T. Bretl M. Ornik P. Thangeda

1 Eigenvalue placement by controllable canonical form

Apply the input

$$u = -Kx$$

to the open-loop system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}$$

and you get the closed-loop system

$$\begin{aligned}\dot{x} &= (A - BK)x\\ y &= Cx.\end{aligned}$$

Suppose we want to choose K to put the eigenvalues of the closed-loop system, i.e., the eigenvalues of the matrix $A - BK$, at given locations. We will derive a formula that allows us to do this when possible, and will show how to decide when doing so is impossible.

1.1 Eigenvalues are invariant to coordinate transformation

Consider the system

$$\dot{x} = Ax + Bu. \tag{1}$$

Suppose we define a new state variable z so that

$$x = Vz$$

for some invertible matrix V, and so

$$\dot{x} = V\dot{z}$$

by differentiation. (We have called this process "coordinate transformation.") Plug these two things into (1) and we get

$$V\dot{z} = AVz + Bu$$

Solve for $\dot{z}$, and we get

$$\dot{z} = V^{-1}AVz + V^{-1}Bu \tag{2}$$

Finding a solution $z(t)$ to (2) allows us to recover a solution $x(t) = Vz(t)$ to (1). We would like to know if these two solutions "behave" the same way. In particular, we would like to know if the eigenvalues of A are the same as the eigenvalues of $V^{-1}AV$. First, let's look at the eigenvalues of A. We know that they are the roots of

$$\det(\lambda I - A)$$

Second, let's look at the eigenvalues of $V^{-1}AV$. We know that they are the roots of

$$\det\left(\lambda I - V^{-1}AV\right)$$

We can play a trick. Notice that

$$V^{-1}(\lambda I)V = \lambda V^{-1}V = \lambda I$$

so

$$\begin{aligned}
\det\left(\lambda I - V^{-1}AV\right) &= \det\left(V^{-1}(\lambda I)V - V^{-1}AV\right) && \text{because of our trick} \\
&= \det\left(V^{-1}(\lambda I - A)V\right)
\end{aligned}$$

It is a fact that

$$\det(MN) = \det(M)\det(N)$$

for any square matrices M and N. Applying this fact, we find

$$\det\left(V^{-1}(\lambda I - A)V\right) = \det(V^{-1})\det(\lambda I - A)\det(V)$$

It is another fact that

$$\det(M^{-1}) = (\det(M))^{-1} = \frac{1}{\det(M)}$$

Applying this other fact, we find

$$\det(V^{-1})\det(\lambda I - A)\det(V) = \frac{\det(\lambda I - A)\det(V)}{\det(V)} = \det(\lambda I - A)$$

In summary, we have established that

$$\det(\lambda I - A) = \det\left(\lambda I - V^{-1}AV\right)$$

and so the eigenvalues of A and $V^{-1}AV$ are the same. The consequence is, if you design state feedback for the transformed system, you'll recover the behavior you want on the original system. In particular, suppose you apply the input

$$u = -Lz$$

to the transformed system and choose L to place the eigenvalues of $V^{-1}AV$ in given locations. Applying the input

$$u = -LV^{-1}x$$

to the original system, i.e., choosing

$$K = LV^{-1},$$

will result in placing the eigenvalues of A at these same locations. The reason this is important is that it is often easier to choose L than to choose K. (The process of diagonalization was important for a similar reason.)

1.2 Controllable canonical form

In the previous section, we showed that eigenvalues are invariant to coordinate transformation. The next question is what coordinates are useful for control design. The answer to that question turns out to be something called "controllable canonical form." A system is in this form if it looks like:

$$\dot{z} = A_{\text{ccf}} z + B_{\text{ccf}} u$$

where

$$A_{\text{ccf}} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_{\text{ccf}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for some integer n (so A_{ccf} is a matrix of size $n \times n$ and B_{ccf} is a matrix of size $n \times 1$). Now, it is a fact that the characteristic equation of this system is given by

$$\det(\lambda I - A_{\text{ccf}}) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

It is easy to see that this formula is true for $n = 2$ and $n = 3$. In particular:

- If $n = 2$, then:

$$\begin{aligned} \det(\lambda I - A_{\text{ccf}}) &= \det \begin{bmatrix} \lambda + a_1 & a_2 \\ -1 & \lambda \end{bmatrix} \\ &= (\lambda + a_1)\lambda + a_2 = \lambda^2 + a_1\lambda + a_2 \end{aligned}$$

- If $n = 3$, then:

$$\begin{aligned} \det(\lambda I - A_{\text{ccf}}) &= \det \begin{bmatrix} \lambda + a_1 & a_2 & a_3 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{bmatrix} \\ &= (\lambda + a_1)\lambda^2 + a_3 - (-a_2\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \end{aligned}$$

There are a variety of ways to prove that this same formula is true in general. Applying the general formula to compute the matrix determinant, for example, we would find:

$$\begin{aligned} \det(\lambda I - A_{\text{ccf}}) &= \det \begin{bmatrix} \lambda + a_1 & a_2 & \cdots & a_{n-1} & a_n \\ -1 & \lambda & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda \end{bmatrix} \\ &= (\lambda + a_1)\det(T_1) - a_2 \det(T_2) + a_3 \det(T_3) - \cdots \end{aligned}$$

where each matrix T_i is upper-triangular with -1 in $i-1$ diagonal entries and λ in $n-i$ diagonal entries. Since the determinant of an upper-triangular matrix is the product of its diagonal entries, we have

$$\det(T_i) = \begin{cases} \lambda^{n-i} & \text{when } i \text{ is odd} \\ -\lambda^{n-i} & \text{when } i \text{ is even} \end{cases}$$

Plug this in, and our result follows. Now, the reason that controllable canonical form is useful is that if we choose the input

$$u = -Lz$$

for some choice of gains

$$L = \begin{bmatrix} \ell_1 & \cdots & \ell_n \end{bmatrix}$$

then the "A matrix" of the closed-loop system is

$$A_{\text{ccf}} - B_{\text{ccf}}L = \begin{bmatrix} -a_1 - \ell_1 & -a_2 - \ell_2 & \cdots & -a_{n-1} - \ell_{n-1} & -a_n - \ell_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

The characteristic equation of this closed-loop system, computed in the same way as for A_{ccf}, is

$$\lambda^n + (a_1 + \ell_1)\lambda^{n-1} + \cdots + (a_{n-1} + \ell_{n-1})\lambda + (a_n + \ell_n).$$

If you want this characteristic equation to look like

$$\lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$$

then it's obvious what gains you should choose:

$$\ell_1 = \alpha_1 - a_1 \qquad \ell_2 = \alpha_2 - a_2 \qquad \cdots \qquad \ell_n = \alpha_n - a_n.$$

So, if you have a system in controllable canonical form, then it is extremely easy to choose gains that make the characteristic equation of the closed-loop system look like anything you want (i.e., to put the closed-loop eigenvalues anywhere you want). In other words, it is extremely easy to do control design.

1.3 Putting a system in controllable canonical form

We have seen that controllable canonical form is useful. Now we'll see how to put a system in this form. Suppose we have a system

$$\dot{x} = Ax + Bu$$

and we want to choose an invertible matrix V so that if we define a new state variable z by

$$x = Vz$$

then we can rewrite the system as

$$\dot{z} = A_{\text{ccf}} z + B_{\text{ccf}} u$$

where

$$A_{\text{ccf}} = V^{-1} A V \qquad \text{and} \qquad B_{\text{ccf}} = V^{-1} B$$

are in controllable canonical form. The trick is to look at the so-called *controllability matrix* that is associated with the transformed system:

$$W_{\text{ccf}} = \begin{bmatrix} B_{\text{ccf}} & A_{\text{ccf}} B_{\text{ccf}} & \cdots & A_{\text{ccf}}^{n-1} B_{\text{ccf}} \end{bmatrix}.$$

We will talk later about the controllability matrix—why we define it this way, what it means, etc. For now, notice:

$$\begin{aligned} B_{\text{ccf}} &= V^{-1} B \\ A_{\text{ccf}} B_{\text{ccf}} &= V^{-1} A V V^{-1} B = V^{-1} A B \\ A_{\text{ccf}}^2 B_{\text{ccf}} &= A_{\text{ccf}} (A_{\text{ccf}} B_{\text{ccf}}) = V^{-1} A V V^{-1} A B = V^{-1} A^2 B \\ \vdots \quad &= \quad \vdots \end{aligned}$$

You see the pattern here, I'm sure. The result is:

$$\begin{aligned} W_{\text{ccf}} &= \begin{bmatrix} V^{-1} B & V^{-1} A B & \cdots & V^{-1} A^{n-1} B \end{bmatrix} \\ &= V^{-1} \begin{bmatrix} B & A B & \cdots & A^{n-1} B \end{bmatrix} \\ &= V^{-1} W \end{aligned}$$

where

$$W = \begin{bmatrix} B & A B & \cdots & A^{n-1} B \end{bmatrix}$$

is the controllability matrix associated with the original system. Note that A and B are things that you know (because you have a description of the original system, as always), so you can compute W. Note that A_{ccf} and B_{ccf} are also things that you know (because they are written in terms of the eigenvalues of the original system, which you can find from the matrix A), so you can compute W_{ccf}. As a consequence, you can solve for the matrix V^{-1}:

$$V^{-1} = W_{\text{ccf}} W^{-1}.$$

Now, suppose you design a control policy for the transformed system:

$$u = -L z.$$

Remember, you can do this easily, because the transformed system is in controllable canonical form. We can compute the equivalent control policy, that would be applied to the original system:

$$u = -L z = -L V^{-1} x = -L W_{\text{ccf}} W^{-1} x.$$

In particular, if we choose

$$K = L W_{\text{ccf}} W^{-1}$$

then we get the behavior that we want. Note that this only works if W is invertible, and that W is only invertible if $\det(W) \neq 0$. This is why we say that: "a system is controllable if and only if $\det(W) \neq 0$." Remember that what we mean by "controllable" is that it's possible to design a control policy that puts the closed-loop eigenvalues anywhere we want.

1.4 A systematic process for control design

Apply the input

$$u = -Kx$$

to the open-loop system

$$\dot{x} = Ax + Bu$$

and you get the closed-loop system

$$\dot{x} = (A - BK)x$$

Suppose we want to choose K to put the eigenvalues of the closed-loop system at $\Sigma_1, \ldots, \Sigma_n$. Using the results of the previous sections, we know we can do this as follows:

- Compute the characteristic equation that we want:

$$(\lambda - \Sigma_1) \cdots (\lambda - \Sigma_n) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$$

- Compute the characteristic equation that we have:

$$\det(\lambda I - A) = \lambda^n + \beta_1 \lambda^{n-1} + \cdots + \beta_{n-1}\lambda + \beta_n$$

- Compute the controllability matrix of the original system (and check that $\det(W) \neq 0$):

$$W = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

- Compute the controllability matrix of the transformed system:

$$W_{\text{ccf}} = \begin{bmatrix} B_{\text{ccf}} & A_{\text{ccf}}B_{\text{ccf}} & \cdots & A_{\text{ccf}}^{n-1}B_{\text{ccf}} \end{bmatrix}$$

  where

$$A_{\text{ccf}} = \begin{bmatrix} -\beta_1 & -\beta_2 & \cdots & -\beta_{n-1} & -\beta_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_{\text{ccf}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Compute the gains for the transformed system:

$$L = \begin{bmatrix} \alpha_1 - \beta_1 & \cdots & \alpha_n - \beta_n \end{bmatrix}$$

- Compute the gains for the original system:

$$K = L W_{\text{ccf}} W^{-1}$$

And we're done! This process is easy to implement, without any symbolic computation.

2 Example and MATLAB Snippets

In this section, we look at an example related to eigenvalue placement. We also provide relevant MATLAB snippets for solving this problem. Consider the system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where given

```
>> A = [0.40 -0.70 -0.60 -0.90; -0.80 0.20 0.40 -0.40; -0.50 -0.40 -0.50
    -0.90; -0.40 0.20 0.60 0.70];
>> B = [0.60; 0.20; 0.30; -0.90];
>> C = [-0.80 -0.20 0.40 -0.20];
>> D = [0.00];
```

Consider the input

$$u = -Kx.$$

We are supposed to find the values of K such that the eigenvalues of the closed-loop system are placed at the following values: $p = [-2.97 + 0.00j, -7.79 - 3.93j, -7.79 + 3.93j, -3.25 + 0.00j]$. We use the steps described in 1.4 to design this controller in MATLAB. First, we compute the characteristic equation we want:

```
% Find desired characteristic polynomial
>> p = [-2.97+0.00j -7.79-3.93j -7.79+3.93j -3.25+0.00j];
>> charpoly_desired = poly(p)

% returns an array containing coefficients of the desired characteristic
    polynomial
charpoly_desired =

    1.0000   21.8000  182.6891  623.9083  734.8352
```

Now find the characteristic polynomial of A:

```
% Find characteristic polynomial of A
>> charpoly_given = charpoly(A)

% returns an array containing coefficients of the characteristic
    polynomial of A
charpoly_given =

    1.0000   -0.8000   -0.5900   -0.0820    0.1876
```

The next step would be to calculate the controllability matrix of the original system:

```
% Find controllability matrix
>> W = ctrb(A,B)

W =

    0.6000    0.7300    0.6810    0.8907
    0.2000    0.0400   -0.2040   -0.3316
    0.3000    0.2800    0.0640    0.2230
   -0.9000   -0.6500   -0.5710   -0.6745
```

We then proceed to find the controllable canonical form of the system its corresponding controllability matrix:

```
% Find controllable canonical form
>> A_ccf = [-charpoly_given(2:end); [eye(3,3), zeros(3,1)]]

A_ccf =

    0.8000    0.5900    0.0820   -0.1876
    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000         0

>> B_ccf = [1; zeros(3,1)]

B_ccf =

     1
     0
     0
     0

% Find the controllability matrix of the transformed system
>> W_ccf = ctrb(A_ccf, B_ccf)

W_ccf =

    1.0000    0.8000    1.2300    1.5380
         0    1.0000    0.8000    1.2300
         0         0    1.0000    0.8000
         0         0         0    1.0000
```

and then find the gains for the transformed system:

```
% Gains for transformed system
>> L = charpoly_desired(2:end) - charpoly_given(2:end)

L =

   22.6000  183.2791  623.9903  734.6476
```

Finally, we find the gains of the original system K using L obtained above:

```
% Find gains for original system
>> K = L*W_ccf*inv(W)

K =

   1.0e+03 *

   -1.2680   -4.2364    1.8750   -1.1869
```

Alternatively, we can directly find the gains K using MATLAB built-in functions *acker* and *place*:

```
% Enter the desired eigenvalues
>> p = [-2.97+0.00j -7.79-3.93j -7.79+3.93j -3.25+0.00j];

% Use Akermann's method to get K (single-input systems only)
>> K = acker(A,B,p)

K =

   1.0e+03 *

   -1.2680   -4.2364    1.8750   -1.1869

% Alternative Option - use place function (works for both single-input and multi-input systems)
>> K = place(A,B,p)

K =

   1.0e+03 *

   -1.2680   -4.2364    1.8750   -1.1869
```

The K values we obtain above will ensure that the closed-loop system has the desired eigenvalues.