AE353: Additional Notes on Matrix Exponential

(to be treated as an appendix to the presentation)

A. Borum T. Bretl M. Ornik P. Thangeda

1 Matrix Exponential

We have seen that the solution to linear system of equations of the form

$$\dot{x} = Ax \tag{1}$$

is

$$x(t) = e^{At}x(0), \tag{2}$$

where e^{At} is the matrix exponential function. The exact expression for the matrix exponential is given as

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!}.$$

Computing the matrix exponential using the above expression is difficult and the next lecture presents an approach to simplify this calculation. You can use the MATLAB function `expm` to calculate the matrix exponential, as shown in section 1.2.

1.1 Properties of the Matrix Exponential

The matrix exponential has many useful properties, and these properties can often be proved using the series expansion of the exponential. We'll give one example here. Let's prove that

$$Ae^{At} = e^{At}A.$$

First, let's expand the left-hand side using the series expansion of e^{At}.

$$Ae^{At} = A\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots\right).$$

Distributing the A term gives

$$Ae^{At} = A + A^2t + \frac{1}{2!}A^3t^2 + \frac{1}{3!}A^4t^3 + \dots.$$

We can now factor out an A on the right side,

$$\begin{aligned} Ae^{At} &= \left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots\right)A \\ &= e^{At}A. \end{aligned}$$

Similarly, we can also easily prove other properties such as $e^0 = I$ and $\frac{d}{dt}e^{At} = Ae^{At}$.

1.2 MATLAB Snippets

This function can be evaluated in MATLAB using `expm`. Here is an example in which e^{At} is evaluated at a particular time t, for a particular choice of A:

```
>> A = [0 1; -6 -5];
>> t = 1;
>> expm(A*t)

ans =

    0.3064    0.0855
   -0.5133   -0.1213
```

It is also possible to evaluate the matrix exponential function at an arbitrary time t using MATLAB's symbolic toolbox. Here is an example:

```
1 >> A = [0 1; -6 -5];
2 >> syms t real
3 >> expm(A*t)
4
5 ans =
6
7 [ 3*exp(-2*t) - 2*exp(-3*t),     exp(-2*t) - exp(-3*t)]
8 [ 6*exp(-3*t) - 6*exp(-2*t), 3*exp(-3*t) - 2*exp(-2*t)]
```

The command "`syms t real`" on line 3 is what tells MATLAB that t is a symbolic variable, in this case a variable that is restricted to be a real number. You'll note that the result is given in terms of this symbolic variable. Given the symbolic expression for matrix exponential, you can find its value at any time instant t by using the MATLAB *subs* function.

2 Solution for Different Inputs

What if the input is non-zero?

Consider the state-space model

$$\dot{x} = Ax + Bu. \tag{3}$$

This model does not look the same as (1). Without specifying u, we cannot solve for x. However, particular choices of u allow us to simplify (3). For example, if we choose $u = 0$, then—as we have seen—we can write

$$\begin{aligned}\dot{x} &= Ax + Bu \\ &= Ax + B \cdot (0) \\ &= Ax + 0 \\ &= Ax.\end{aligned}$$

And so, we're right back at (1), and can solve for x using the matrix exponential. Another common choice of u is

$$u = Kx \tag{4}$$

for some constant matrix K. (What would the size of K have to be for us to define u in this way?) This choice of u is called state feedback, since the input depends on the state. If you plug (4) into (3), you should find—just like before—that the result simplifies to something that looks like (1), and so can be solved using the matrix exponential.

What if the initial time is non-zero?

Suppose we want to solve (1), but our initial condition is $x(t_0)$ for some $t \neq 0$ and not $x(0)$. Then, (2) isn't going to work for us. We need the following, more general, solution:

$$x(t) = e^{A(t-t_0)}x(t_0). \tag{5}$$

In this expression, "$A(t-t_0)$" is the product of the matrix A with the scalar difference $t-t_0$, while "$x(t_0)$" is the state at time t_0 (or x evaluated at t_0). Notice that (2) is a special case of (5)—the two expressions are the same when $t_0 = 0$. Notice also that the left- and right-hand side of (5) agree at the initial condition—plug in $t = t_0$ and you get

$$\begin{aligned}x(t_0) &= e^{A(t_0-t_0)}x(t_0) \\ &= e^0 x(t_0) \\ &= Ix(t_0) \\ &= x(t_0).\end{aligned}$$

The formal proof that (5) solves (1) starting from time t_0 is exactly the same as the formal proof that (2) solves (1) starting from time 0.