AE353: Additional Notes on Optimization and Introduction to Optimal Control

(to be treated as an appendix to the presentation)

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1 Optimization

The following thing is called an *optimization problem*:

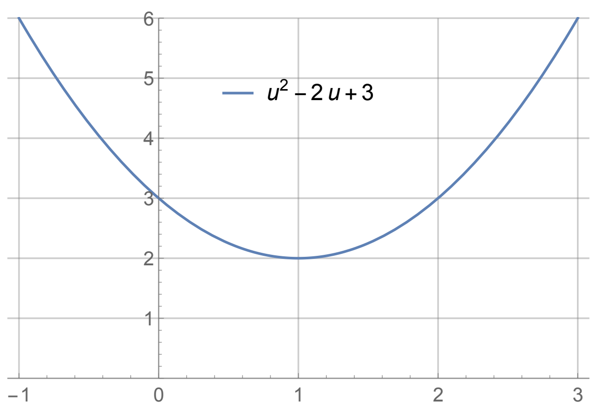
$$\underset{u}{\text{minimize}} \qquad u^2 - 2u + 3$$

The solution to this problem is the value of u that makes $u^2 - 2u + 3$ as small as possible.

- We know that we are supposed to choose a value of u because "u" appears underneath the "minimize" statement. We call u the *decision variable*.
- We know that we are supposed to minimize $u^2 - 2u + 3$ because "$u^2 - 2u + 3$" appears to the right of the "minimize" statement. We call $u^2 - 2u + 3$ the *cost function*.

In particular, the solution to this problem is $u = 1$. There are at least two different ways to arrive at this result:

- We could plot the cost function. It is clear from the plot that the minimum is at $u = 1$.



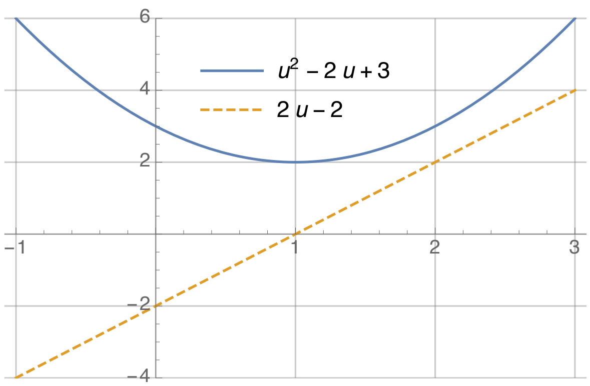
- We could apply the first derivative test. We compute the first derivative:

$$\frac{d}{du}\left(u^2 - 2u + 3\right) = 2u - 2$$

Then, we set the first derivative equal to zero and solve for u:

$$2u - 2 = 0 \qquad \Rightarrow \qquad u = 1$$

Values of u that satisfy the first derivative test are only "candidates" for optimality—they could be maxima instead of minima, or could be only one of many minima. We'll ignore this distinction for now. Here's a plot of the cost function and of it's derivative. Note that, clearly, the derivative is equal to zero when the cost function is minimized:



In general, we write optimization problems like this:

$$\underset{u}{\text{minimize}} \qquad g(u)$$

Again, u is the decision variable and $g(u)$ is the cost function. In the previous example:

$$g(u) = u^2 - 2u + 3$$

Here is another example:

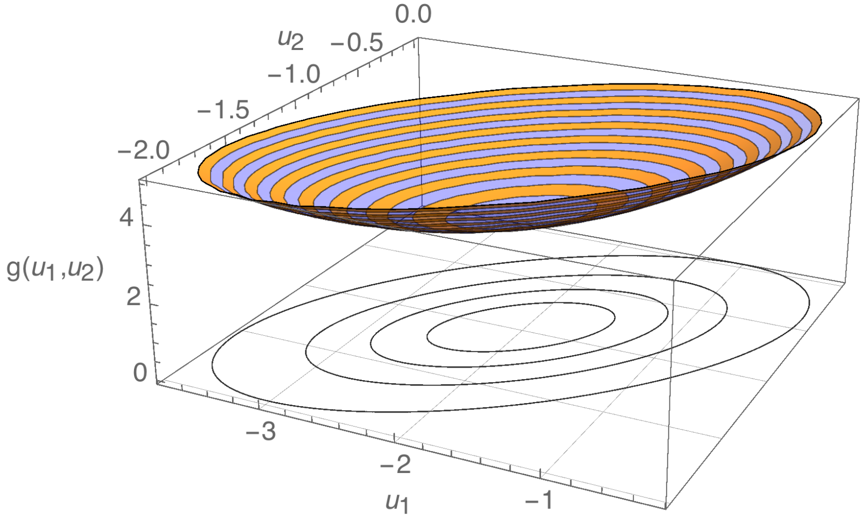
$$\underset{u_1, u_2}{\text{minimize}} \qquad u_1^2 + 3u_2^2 - 2u_1u_2 + 2u_1 + 2u_2 + 6$$

The solution to this problem is the value of both u_1 and u_2 that, together, make $u_1^2 + 3u_2^2 - 2u_1u_2 + 2u_1 + 2u_2 + 6$ as small as possible. There are two differences between this optimization problem and the previous one. First, there is a different cost function:

$$g(u_1, u_2) = u_1^2 + 3u_2^2 - 2u_1u_2 + 2u_1 + 2u_2 + 6$$

Second, there are two decision variables instead of one. But again, there are at least two ways of finding the solution to this problem:

- We could plot the cost function. The plot is now 3D—the "x" and "y" axes are u_1 and u_2, and the "z" axis is $g(u_1, u_2)$. The shape of the plot is a bowl. It's hard to tell where the minimum is from looking at the bowl, so I've also plotted contours of the cost function underneath. "Contours" are like the lines on a topographic map. From the contours, it looks like the minimum is at $(u_1, u_2) = (-2, -1)$.



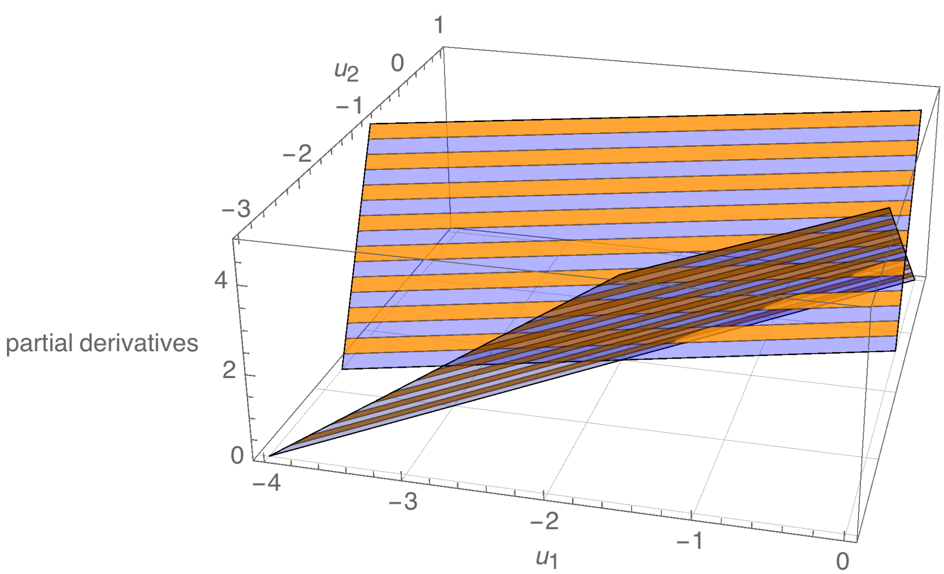
- We could apply the first derivative test. We compute the partial derivative of $g(u_1, u_2)$ with respect to both u_1 and u_2:

$$\frac{\partial}{\partial u_1} g(u_1, u_2) = 2u_1 - 2u_2 + 2$$
$$\frac{\partial}{\partial u_2} g(u_1, u_2) = 6u_2 - 2u_1 + 2$$

Then, we set both partial derivatives equal to zero and solve for u_1 and u_2:

$$\begin{aligned} 2u_1 - 2u_2 + 2 &= 0 \\ 6u_2 - 2u_1 + 2 &= 0 \end{aligned} \qquad \Rightarrow \qquad (u_1, u_2) = (-2, -1)$$

As before, we would have to apply a further test in order to verify that this choice of (u_1, u_2) is actually a minimum. But it is certainly consistent with what we observed above. Here is a plot of each partial derivative as a function of u_1 and u_2. The shape of each plot is a plane (i.e., a flat surface). Both planes are zero at $(-2, -1)$:



An equivalent way of stating this same optimization problem would have been as follows:

$$\underset{u_1,u_2}{\text{minimize}} \qquad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}^T \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + 6$$

You can check that the cost function shown above is the same as the cost function we saw before (e.g., by multiplying it out). We could have gone farther and stated the problem as follows:

$$\underset{u}{\text{minimize}} \qquad u^T \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} u + \begin{bmatrix} 2 \\ 2 \end{bmatrix}^T u + 6$$

We have returned to having just one decision variable u, as in the first example, but this variable is now a 2×1 matrix—i.e., it has two elements, which we would normally write as u_1 and u_2. The point here is that the "decision variable" in an optimization problem can be a variety of different things: a scalar, a vector (i.e., an $n \times 1$ matrix), and—as we will see—even a function of time. Before proceeding, however, let's look at one more example of an optimization problem:

$$\begin{aligned} \underset{u,x}{\text{minimize}} \qquad & u^2 + 3x^2 - 2ux + 2u + 2x + 6 \\ \text{subject to} \qquad & u + x = 3 \end{aligned}$$

This example is exactly the same as the previous example, except that the two decision variables (now renamed u and x) are subject to a constraint: $u + x = 3$. We are no longer free to choose u and x arbitrarily. We are restricted to choices that satisfy the constraint. The solution to this optimization problem is the value (u, x) that minimizes the cost function, chosen from among all values (u, x) that satisfy the constraint. Again, there are a variety of ways to solve this problem. One way is to eliminate the constraint. First, we solve the constraint equation:

$$u + x = 3 \qquad \Rightarrow \qquad x = 3 - u$$

Then, we plug this result into the cost function:

$$\begin{aligned} u^2 + 3x^2 - 2ux + 2u + 2x + 6 &= u^2 + 3(3-u)^2 - 2u(3-u) + 2u + 2(3-u) + 6 \\ &= 6u^2 - 24u + 39 \end{aligned}$$

By doing so, we have shown that solving the constrained optimization problem

$$\begin{aligned} \underset{u,x}{\text{minimize}} \qquad & u^2 + 3x^2 - 2ux + 2u + 2x + 6 \\ \text{subject to} \qquad & u + x = 3 \end{aligned}$$

is equivalent to solving the unconstrained optimization problem

$$\underset{u}{\text{minimize}} \qquad 6u^2 - 24u + 39$$

and then taking $x = 3 - u$. We can do so easily by taking the first derivative and setting it equal to zero, as we did in the first example:

$$0 = \frac{d}{du}\left(6u^2 - 24u + 39\right) = 12u - 24 \qquad \Rightarrow \qquad u = 2 \qquad \Rightarrow \qquad x = 3 - u = 1$$

The point here was not to show how to solve constrained optimization problems in general, but rather to identify the different parts of a problem of this type. As a quick note, you will sometimes see the example optimization problem we've been considering written as

$$\begin{array}{ll} \underset{u}{\text{minimize}} & u^2 + 3x^2 - 2ux + 2u + 2x + 6 \\ \text{subject to} & u + x = 3 \end{array}$$

The meaning is exactly the same, but x isn't listed as one of the decision variables under "minimize." The idea here is that x is an "extra variable" that we don't really care about. This optimization problem is trying to say the following:

> "Among all choices of u for which there exists an x satisfying $u + x = 3$, find the one that minimizes $u^2 + 3x^2 - 2ux + 2u + 2x + 6$."

2 Optimum vs. Optimizer

We have seen three example problems. In each case, we were looking for the optimizer (minimizer in this case), i.e., the choice of decision variable that made the cost function as small as possible:

- The solution to
$$\underset{u}{\text{minimize}} \qquad u^2 - 2u + 3$$
was $u = 1$.
- The solution to
$$\underset{u_1,u_2}{\text{minimize}} \qquad u_1^2 + 3u_2^2 - 2u_1u_2 + 2u_1 + 2u_2 + 6$$
was $(u_1, u_2) = (-2, -1)$.
- The solution to
$$\begin{array}{ll} \underset{u}{\text{minimize}} & u^2 + 3x^2 - 2ux + 2u + 2x + 6 \\ \text{subject to} & u + x = 3 \end{array}$$
was $(u, x) = (2, 1)$.

It is sometimes useful to focus on the optimum or optimal value (minimum in this case) instead of on the optimizer, i.e., in this case, what the "smallest value" was that we were able to achieve. When focusing on the minimum, we often use the following "set notation" instead:

- The problem
$$\underset{u}{\text{minimize}} \qquad u^2 - 2u + 3$$

is rewritten

$$\underset{u}{\text{minimum}} \left\{ u^2 - 2u + 3 \right\}.$$

The meaning is—find the minimum value of $u^2 - 2u + 3$ over all choices of u. The solution to this problem can be found by plugging in what we already know is the minimizer, $u = 1$. In particular, we find that the solution is 2.

- The problem

  $$\underset{u_1,u_2}{\text{minimize}} \qquad u_1^2 + 3u_2^2 - 2u_1u_2 + 2u_1 + 2u_2 + 6$$

  is rewritten

  $$\underset{u_1,u_2}{\text{minimum}} \left\{ u_1^2 + 3u_2^2 - 2u_1u_2 + 2u_1 + 2u_2 + 6 \right\}.$$

  Again, the meaning is—find the minimum value of $u_1^2 + 3u_2^2 - 2u_1u_2 + 2u_1 + 2u_2 + 6$ over all choices of u_1 and u_2. We plug in what we already know is the minimizer $(u_1, u_2) = (-2, -1)$ to find the solution—it is 3.

- The problem

  $$\begin{array}{ll} \underset{u}{\text{minimize}} & u^2 + 3x^2 - 2ux + 2u + 2x + 6 \\ \text{subject to} & u + x = 3 \end{array}$$

  is rewritten

  $$\underset{u}{\text{minimum}} \left\{ u^2 + 3x^2 - 2ux + 2u + 2x + 6 \;:\; u + x = 3 \right\}.$$

  And again, the meaning is—find the minimum value of $u^2 + 3x^2 - 2ux + 2u + 2x + 6$ over all choices of u for which there exists x satisfying $u + x = 3$. Plug in the known minimizer, $(u, x) = (2, 1)$, and we find that the solution is 15.

The important thing here is to understand the notation and to understand the difference between an "optimum" and an "optimizer."

3 Optimal Control

3.1 Statement of the problem

The following thing is called an *optimal control problem*:

$$\begin{array}{ll} \underset{u_{[t_0,t_1]}}{\text{minimize}} & h(x(t_1)) + \displaystyle\int_{t_0}^{t_1} g(x(t), u(t))dt \\ \text{subject to} & \dfrac{dx(t)}{dt} = f(x(t), u(t)), \quad x(t_0) = x_0 \end{array} \tag{1}$$

Let's try to understand what it means.

- The statement
$$\underset{u_{[t_0,t_1]}}{\text{minimize}}$$
says that we are being asked to choose an input trajectory u that minimizes something. Unlike in the optimization problems we saw before, the decision variable u in this problem is a function of time. The notation $u_{[t_0,t_1]}$ is one way of indicating this. Given an initial time t_0 and a final time t_1, we are being asked to choose the value of $u(t)$ at all times in between, i.e., for all $t \in [t_0, t_1]$.

- The statement
$$\frac{dx(t)}{dt} = f(x(t), u(t)), \quad x(t_0) = x_0$$
is a constraint. It implies that we are restricted to choices of u for which there exists an x satisfying a given initial condition
$$x(t_0) = x_0$$
and satisfying the ordinary differential equation
$$\frac{dx(t)}{dt} = f(x(t), u(t)).$$
One example of an ordinary differential equation that looks like this is our usual description of a system in state-space form:
$$\dot{x} = Ax + Bu,$$

- The statement
$$h(x(t_1)) + \int_{t_0}^{t_1} g(x(t), u(t))dt$$
says what we are trying to minimize—it is the cost function in this problem. Notice that the cost function depends on both x and u. Part of it—$g(\cdot)$—is integrated (i.e., "added up") over time. Part of it—$h(\cdot)$—is applied only at the final time. One example of a cost function that looks like this is
$$x(t_1)^T M x(t_1) + \int_{t_0}^{t_1} \left(x(t)^T Q x(t) + u(t)^T R u(t) \right) dt$$