AE353: Additional Notes on Asymptotic Stability

(to be treated as an appendix to the presentation)

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1 Asymptotic Stability

We've seen that the solution of the linear system $\dot{x} = Ax$ is given by

$$x(t) = e^{At}x(0).$$

We've also seen that if V is an invertible matrix and $F = V^{-1}AV$, then the solution for x is given by

$$x(t) = Ve^{Ft}V^{-1}x(0). \tag{1}$$

Finally, we've seen that if we choose the matrix V so that its columns are the eigenvectors of A, then F is sometimes diagonal. This is important because computing the exponential of a diagonal matrix is easy. Suppose A is a matrix such that $F = V^{-1}AV$ is diagonal. The diagonal entries of F are the eigenvalues of A. Let's denote them by s_1, s_2, ..., s_n, so that

$$F = \begin{bmatrix} s_1 & 0 & \dots & 0 \\ 0 & s_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & s_n \end{bmatrix}.$$

The exponential of Ft is then

$$e^{Ft} = \begin{bmatrix} e^{s_1 t} & 0 & \dots & 0 \\ 0 & e^{s_2 t} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & e^{s_n t} \end{bmatrix}.$$

In the expression (1) for $x(t)$, the terms V, V^{-1}, and $x(0)$ are constant. The only term in the expression (1) that depends upon t is e^{Ft}. Therefore, we can infer the behavior of $x(t)$ based on the entries in the matrix e^{Ft}.

If s_1 is a positive real number, then $e^{s_1 t}$ grows exponentially as t increases. If s_1 is a negative real number, then $e^{s_1 t}$ decays exponentially to zero as t increases. Therefore, if all of the eigenvalues of A are negative and real, then e^{Ft} and $x(t)$ decay to zero. In this case, we say that x is asymptotically stable. If at least one of the eigenvalues of A is a positive real number, then e^{Ft} and $x(t)$ can grow exponentially. In this case, we say that x is unstable.

We've covered the case when the eigenvalues of A are real numbers, but we know that eigenvalues can be complex. There is a formula, called Euler's formula, that tells us how to compute the exponential of a complex number. Euler's formula tells us that

$$e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$$

Note that if a, the real part of $a + bi$, is positive, then $e^{(a+bi)t}$ grows as t increases. If a is negative, then $e^{(a+bi)t}$ decays to zeros as t increases. From these facts, we can reach the same conclusions regarding stability as we did in the case when the eigenvalues of A are real. Furthermore, it's possible to show that these stability results are true even when A is not diagonalizable. Based on these observations, we can state the following results:

- The system $\dot{x} = Ax$ is asymptotically stable if and only if all the eigenvalues of A have negative real part.
- The system $\dot{x} = Ax$ is unstable if at least one eigenvalue of A has positive real part.

2 Examples

We now provide an example of verifying if a given system and controller is asymptotically stable and also provide some MATLAB snippets used for solving the problem. Consider the system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}$$

where given

$$\begin{aligned}&A = [-0.2, 0.4, 0.9, 0.3; 0.9, 0.0, -0.9, -0.9; 0.7, -0.2, -0.8, -0.4; -0.7, 0.0, 0.6, -0.8],\\ &B = [0.4; 0.0; 0.7; 0.7],\\ &C = [0.6, 0.8, -0.7, -0.7],\\ &D = [0.0].\end{aligned}$$

Consider the input

$$u = -Kx$$

where given $K = [271.8, 281.7, -86.5, -53.1]$. To check if this system is asymptotically stable, we first have to calculate the eigenvalues of the matrix $A - BK$. We use the following MATLAB code to do it:

```
>> A = [-0.2, 0.4, 0.9, 0.3; 0.9, 0.0, -0.9, -0.9; 0.7, -0.2, -0.8, -0.4;
    -0.7, 0.0, 0.6, -0.8];
>> B = [0.4; 0.0; 0.7; 0.7];
>> K = [271.8, 281.7, -86.5, -53.1];
```

```
>> eig(A-B*K)

ans =

  -8.6878 + 0.0000i
  -0.4009 + 2.2977i
  -0.4009 - 2.2977i
  -3.3105 + 0.0000i
```

We see that all the eigenvalues have a negative real part. Therefore, the system is asymptotically stable.