AE353: Additional Notes on State Space Models

(to be treated as an appendix to the presentation)

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1 Assumptions

The method described in the presentation assumes (among other things) that:

- there is a one-to-one function that transforms $p(t)$ into $u(t)$ so that $u(t)$ appears linearly in the ODE that describes the system dynamics and the equation that describes the output,
- all elements $w(t)$, $\dot{w}(t)$, $\ldots$, $w^{(n)}(t)$ appear linearly in the ODE and in the description of the output y.

Both of these assumptions are trivially wrong even for dynamics of some simple systems. We will spend the initial part of the course discussing and trying to remove them.

2 Notation

We usually write state x, input u, and output y as column vectors:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix}.$$

We then use $\dot{x}$ to denote

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix}.$$

Since the possible values of x are real n-tuples, $\mathbb{R}^n$ is the *state space*. Sometimes we say that $\mathbb{R}^m$ is the *input space* and $\mathbb{R}^k$ is the *output space*.

3 Examples

Example 1

Consider a system with dynamics

$$\ddot{w}(t) = p(t)/m - g, \tag{1}$$

where m and g are some given constants. We are able to observe output $w(t)$ at any time.

The highest derivative of the state w present in the above ODE is the second derivative. Thus, we define $n = 2$, $x_1(t) = w(t)$, $x_2(t) = \dot{w}(t)$. The right hand side of (1) is affine, but not linear in p, so we define $u(t) = p(t)/m - g$ (this transformation is not the only possible one, but it is one way). Naturally, $m = 1$.

Equation (1) is now transformed into $\dot{x}_2(t) = u(t)$, and given that $\dot{x}_1(t) = \dot{w}(t) = x_2(t)$, we obtain

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

Hence, the system dynamics are now in a linear state space form $\dot{x} = Ax + Bu$, with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Given that the system output $y(t)$ equals $w(t) = x_1(t)$, we obtain $k = 1$ and $y = Cx + Du$ with

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad D = 0.$$

Example 2 (Slight Extension)

Consider the linear system with dynamics

$$\begin{aligned} -\dot{\eta} + 3\dot{\alpha} - 5\eta &= b, \\ -\ddot{\alpha} - \eta + 4\alpha &= a, \end{aligned}$$

where α and η are states, a and b inputs, and output

$$o = \eta + \dot{\alpha}.$$

We naturally adapt the method of the previous example. Because there are two states, we need to place both of them, and possibly the appropriate derivatives, into x. Since the highest derivative of η that appears in the above system is the first one, and the highest derivative of α is the second one, we take $n = 1 + 2 = 3$, and

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \eta \\ \alpha \\ \dot{\alpha} \end{pmatrix}.$$

Inputs and outputs are straightforward:

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \end{pmatrix} = \begin{pmatrix} \eta + \dot{\alpha} \end{pmatrix}.$$

We write $\dot{x}$ in terms of x and u:

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \dot{\eta} \\ \dot{\alpha} \\ \ddot{\alpha} \end{pmatrix} = \begin{pmatrix} -5\eta + 3\dot{\alpha} - b \\ \dot{\alpha} \\ -\eta + 4\alpha - a \end{pmatrix} = \begin{pmatrix} -5x_1 + 3x_3 - u_2 \\ x_3 \\ -x_1 + 4x_2 - u_1 \end{pmatrix}.$$

We write y it in terms of x and u:

$$y = \begin{bmatrix}\eta + \dot{\alpha}\end{bmatrix} = \begin{bmatrix}x_1 + x_3\end{bmatrix}.$$

Finally, we can find the matrices A, B, C, and D:

$$A = \begin{pmatrix} -5 & 0 & 3 \\ 0 & 0 & 1 \\ -1 & 4 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

The resulting state-space model is, as always,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

4 Code Snippets

This section provides relevant code snippets to get you started with solving problems in MATLAB. Feel free to skip this section if you are comfortable with MATLAB.

```
% enter A matrix
A = [-5 0 3; 0 0 1; -1 4 0];
% alternatively
A = [-5, 0, 3; 0, 0, 1; -1, 4, 0];

% enter vector C
C = [1 0 1];

% find size of A and C
A_size = size(A); % A_size should be [3,3]
C_size = size(C); % C_size should be [1,3]

% find transpose of A or C
A_t = A';
% alternatively using 'transpose' function
C_t = transpose(C);
```