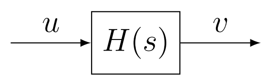
AE353: Notes on Block Diagrams

A. Borum T. Bretl

1 What is a block?

A "block" is nothing more than a picture of the relationship between an input and an output. For example, this block says that the transfer function from u to v is $H(s)$:



Remember what this means. If $u(t) = e^{st}$, then

$$\begin{aligned} v(t) &= (\text{something}) + H(s)e^{st} \\ &= (\text{something}) + H(s)u(t). \end{aligned}$$

If $H(s)$ is the transfer function of a stable system, then the "something" in this expression is a transient that converges to zero. If $H(s)$ is the transfer function of an unstable system, then the "something" may never go away, and indeed may diverge to infinity. In either case, the second part of the response—the sinusoidal part—will remain. It is customary to focus *only* on that part of the response when considering block diagrams. So, in the case of the block shown above, we would simply write

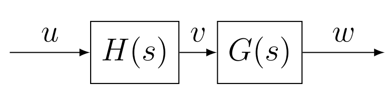
$$v(t) = H(s)u(t).$$

It is important to remember that this *only* describes the sinusoidal part—again, this is often the steady-state part—of the response to a sinusoidal input. Because of this, when working with block diagrams, we usually suppress dependence on t, and so—in this case—would simply write

$$v = H(s)u.$$

2 What if the output of one block is the input to another?

Block diagrams make it easy to build complex systems out of simple systems. For example, consider this diagram:



It says that $H(s)$ is the transfer function from u to v and that $G(s)$ is the transfer function from v to w. Equivalently, it says that

$$v = H(s)u$$

and that

$$w = G(s)v.$$

What is the transfer function from u to w? Easy. We find

$$\begin{aligned} w &= G(s)v \\ &= G(s)H(s)u \end{aligned}$$

and so it must be the case that the product $G(s)H(s)$ is the transfer function from u to w. It's just algebra!

3 Open-loop vs. closed-loop transfer functions

Figure 1 shows a *standard block diagram.* This is a canonical way to describe a control system as a composition of blocks. The signals r, d, and n are the reference, disturbance, and noise, respectively. The signals e, u, and y the error, input, and output, respectively. $F(s)$, $G(s)$, and $H(s)$ are transfer functions. In particular, $H(s)$ describes the open-loop system and is often called the "plant," while $F(s)$ and $G(s)$ together describe the controller and observer. $F(s)$ is often called the "feedforward" part and $G(s)$ the "feedback" part of the control system.

3.1 Open-loop transfer functions

We will often be asked to find either the open-loop or the closed-loop transfer function from one signal to another signal. Let's first look at open-loop transfer functions. The open-loop transfer function from one signal, say a, to another signal, say b, is simply the product of the transfer functions that appear between a and b in the block diagram. In other words,

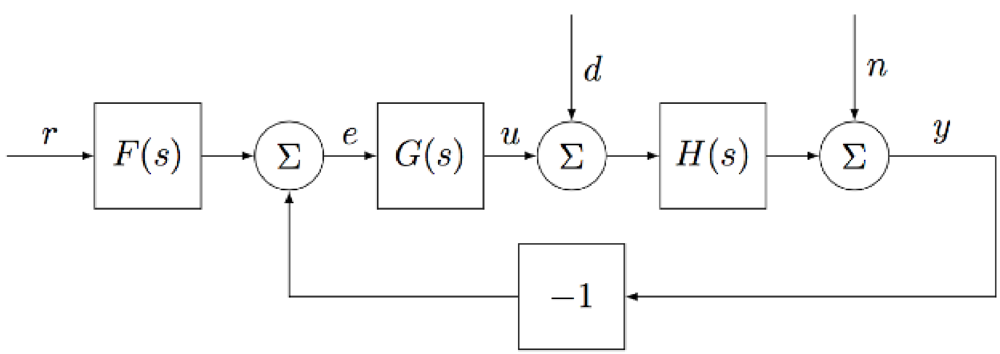


Figure 1: A standard block diagram.

start at a, follow the connections in the block diagram until you reach b, and keep track of the transfer functions you pass through.

For example, from Figure 1, we see that the open-loop transfer function from r to e is $F(s)$, the open-loop transfer function from r to u is $G(s)F(s)$, and the open-loop transfer function from r to y is $H(s)G(s)F(s)$. Note that to compute these open-loop transfer functions, we began at r and worked our way forward to the signal we are interested in, either e, u, or y. For this reason, the open-loop transfer function is sometimes called the forward transfer function.

3.2 Closed-loop transfer functions

Now let's look at closed-loop transfer functions. The method for finding closed-loop transfer functions is opposite of that for open-loop transfer functions. To find the closed-loop transfer function from one signal, say a, to another signal, say b, we begin at b and work our way backwards until we can write b in terms of itself, a, and any other input signals. (The input signals in Figure 1 are r, d, and n.) The final step is to solve the expression for b. Then the open-loop transfer function is the coefficient of a in this expression.

Let's compute the closed-loop transfer function from r to e. Starting at e, we have

$$\begin{aligned} e &= F(s)r - y \\ y &= n + H(s)(d+u) \\ u &= G(s)e. \end{aligned} \tag{1}$$

Combining these expressions, we have

$$e = F(s)r - y = F(s)r - (n + H(s)(d+u)) = F(s)r - (n + H(s)(d + (G(s)e))). \tag{2}$$

In the above expression, we successively plugged in the expressions from (1). The expression (2) can be rearranged to give

$$e = F(s)r - H(s)d - n - H(s)G(s)e. \tag{3}$$

We have written the signal e in terms of itself, r, and the other input signals. Solving for e gives

$$e = \frac{F(s)}{1+H(s)G(s)}r + \frac{-H(s)}{1+H(s)G(s)}d + \frac{-1}{1+H(s)G(s)}n \tag{4}$$

The closed-loop transfer function from r to e is therefore $\frac{F(s)}{1+H(s)G(s)}$.

4 Stability of transfer functions

If we are given expressions for the transfer functions $F(s)$, $G(s)$, and $H(s)$, how can we determine if a closed-loop transfer function is stable?

> **FACT.** The system described by a transfer function $T(s)$ is stable if and only if all roots of the denominator of $T(s)$ have negative real part.

Why would this be true? Consider the state-space system

$$\dot{x} = Ax + Bu$$
$$y = Cx.$$

You know that the transfer function $T(s)$ from u to y is

$$T(s) = C(sI - A)^{-1}B$$

The inverse in this expression—like all matrix inverses—has the form

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \cdot (\text{a matrix of cofactors})\,.$$

In other words, the denominator of $T(s)$ is *exactly* the same as the characteristic polynomial

$$\det(sI - A),$$

the roots of which are the eigenvalues of A. Since the state-space system is stable if and only if the eigenvalues of A have negative real part, it is equivalent to say that this system is stable if and only if the roots of the denominator of $T(s)$ all have negative real part.

As an example, let's again consider the closed-loop transfer function from r to e, which we already showed was

$$\frac{F(s)}{1 + H(s)G(s)}.$$

Let's also suppose that we are given

$$F(s) = 2 \qquad G(s) = 5s + 2 + \frac{4}{s} \qquad H(s) = \frac{1}{s+6} \tag{5}$$

Note that the denominator of

$$\frac{F(s)}{1 + H(s)G(s)}$$

is not $1 + H(s)G(s)$, since $H(s)$ and $G(s)$ are themselves ratios of polynomials. We must plug in the expressions for $F(s)$, $G(s)$, and $H(s)$ and simplify, which is easy to do in Matlab.

```
>> syms s
>> F = 2;
>> G = 5*s+2+4/s;
>> H = 1/(s+6);
>> Tre = F/(1+H*G);
>> simplify(Tre)

ans =

(s*(s + 6))/(3*s^2 + 4*s + 2)
```

The roots of the denominator are

```
>> roots([3 4 2])

ans =

  -0.6667 + 0.4714i
  -0.6667 - 0.4714i
```

Since all of the roots have negative real part, the closed-loop transfer function from r to e is stable (and so, the closed-loop system described by this transfer function is stable).

5 Finding $F(s)$, $G(s)$, and $H(s)$ in the standard block diagram from a state-space system

Let's suppose that we're given a linear state-space model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{6}$$

and the controller and observer

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - L(C\hat{x} - y) \\ u &= -K\hat{x}. \end{aligned} \tag{7}$$

The transfer function from u to y is

$$C(sI - A)^{-1}B + D$$

This is an open-loop transfer function, since the output in the system (6) doesn't affect the input u. To find the open-loop transfer function from u to y in Figure 1, we start at u and work forwards until we reach y. In this case, the only transfer function between u and y is $H(s)$, so the open-loop transfer function from u to y is $H(s)$. We therefore conclude that

$$H(s) = C(sI - A)^{-1}B + D. \tag{8}$$

Next, we can rewrite the observer and controller equation in (7) to be

$$\begin{aligned} \dot{\hat{x}} &= (A - BK - LC)\hat{x} + Ly \\ u &= -K\hat{x}. \end{aligned} \tag{9}$$

Noting the similar structure between this system and the state-space system (6), we can say that the transfer function from y to u is

$$-K(sI - (A - BK - LC))^{-1}L.$$

Again, this is an open-loop transfer function. Now, let's compute the open-loop transfer function from y to u in Figure 1. The only transfer functions between y and u is $-G(s)$. Therefore, we have

$$G(s) = K(sI - (A - BK - LC))^{-1}L. \tag{10}$$

Since $u = -K\hat{x}$, u does not depend upon r, so the open-loop transfer function from r to u is 0. From Figure 1, the open-loop transfer function from r to u is $G(s)F(s)$, so we must have $G(s)F(s) = 0$. We already know $G(s)$, so we must have

$$F(s) = \frac{0}{G(s)} = 0.$$