AE353: Additional Notes on Observer Design

(to be treated as an appendix to the presentation)

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1 Observers

Consider the linear state-space system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{1}$$

When designing state feedback controllers of the form $u = -Kx$ for the system (1), we have thus far assumed that we know the state x at the current time t. However, this is often not the case. In most situations, we only know the output y, and not the full state x. The output y can be thought of as a set of measurements that we have access to. These measurements, however, may not be sufficient to describe the full state of the system. If we don't know x, how can we implement state feedback control?

Our approach will be to estimate the state x by using the output y. Our estimate of the state will be denoted by $\hat{x}$. Note that $\hat{x}$ is a variable that we have introduced. The system (1) doesn't care what $\hat{x}$ is. We are introducing $\hat{x}$ to help us choose what control input to apply. Using our estimate $\hat{x}$, we will apply the input $u = -K\hat{x}$.

We still need to decide how $\hat{x}$ will estimate the state x. One approach would be to let $\hat{x}$ satisfy the same differential equation as x, and we would therefore have

$$\dot{\hat{x}} = A\hat{x} + Bu. \tag{2}$$

However, if $\hat{x}(t_0)$ is not close to $x(t_0)$ at the initial time t_0, then we shouldn't expect $\hat{x}$ to give us a good estimate of x at a later time.

Let's think back to our discussions on controller design for the system (1). Without input, the system (1) is $\dot{x} = Ax$. We wanted the state of this system to approach 0, so we added a term that is proportional to the error, i.e., we added $-BK(x - 0) = -BKx$, and the system became $\dot{x} = Ax - BKx = (A - BK)x$. We can apply the same logic to the observer. Let's add a term to (2) that is proportional to the error $C\hat{x} - y$, which gives

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y) \tag{3}$$

for some matrix L.

It's not immediately obvious that we should expect $\hat{x}$ to approach x, since we are only penalizing $C\hat{x} - y$. However, let's look at what happens to $\hat{x} - x$ by taking its derivative.

$$\frac{d}{dt}(\hat{x} - x) = \dot{\hat{x}} - \dot{x} = (A\hat{x} + Bu - L(C\hat{x} - y)) - (Ax + Bu)$$

After canceling terms, we find that

$$\frac{d}{dt}(\hat{x} - x) = (A - LC)(\hat{x} - x) \tag{4}$$

Note that this differential equation has the same structure as many equations we've already dealt with this semester, namely $\dot{x} = (A - BK)x$. We therefore know that $\hat{x} - x$ approaches 0 if and only if the eigenvalues of $A - LC$ have negative real parts. We've shown that by only penalizing $C\hat{x} - y$, we can make the difference between the estimate $\hat{x}$ and the state x approach 0 if we correctly choose the matrix L.

2 Observer Design

We can choose L using the same tools we've used earlier this semester to choose K. Let's choose L using the Matlab function `acker`. Recall that to place the eigenvalues of $A - BK$ at locations designated by the row vector p, we write

```
K = acker(A,B,p);
```

However, note that $A - BK$ is different than $A - LC$, specifically because the matrices that we must choose, K and L, appear in different orders in each of the expressions. This problem goes away after we recall the following fact from linear algebra: A matrix M and its transpose M^T have the same eigenvalues. Therefore, rather than placing the eigenvalues of the matrix $A - LC$, we can equivalently place the eigenvalues of $(A - LC)^T = A^T - C^T L^T$. The matrix $A^T - C^T L^T$ has the same structure as the matrix $A - BK$. We can place the eigenvalues of $A^T - C^T L^T$ by using the following Matlab command:

```
L = acker(A',C',p)';
```

Note that `acker(A’,C’,p)` gives the matrix L', and we therefore must take the transpose of `acker(A’,C’,p)` to find L.

3 Principle of Separation

We can now ask how our choice of L affects the state x, and similarly, how does our choice of K affect the estimate $\hat{x}$. In other words, can we choose L and K independently and still place the eigenvalues of the state-space system (1) and the observer (4) at desired locations? The answer turns out to be yes. To see that this is true, let's plug the controller $u = -K\hat{x}$ into the state-space system (1). We can write this system along with the observer (4) together as

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} - \dot{x} \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} - x \end{bmatrix} \tag{5}$$

The matrix in (5) is an upper triangular block matrix. A fact from linear algebra is that the eigenvalues of the matrix in (5) are the eigenvalues of $A - BK$ and $A - LC$. Therefore, we can place the eigenvalues of $A - BK$ and $A - LC$ independently. This fact is known as the principle of separation.