AE353: Notes on Solutions with Input

(to be treated as an appendix to the presentation)

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1 Solutions to state-space systems

Consider the state-space system

$$\begin{aligned} \dot{x}_m &= A_m x_m + B_m u_m \\ y_m &= C_m x_m. \end{aligned} \tag{1}$$

We are subscripting everything by "m" to make it easier for us to apply the results we will derive to other systems of this form. For example, consider the system

$$\begin{aligned} \dot{x} &= (A - BK)x + Bk_{\text{reference}} r \\ y &= Cx. \end{aligned}$$

If we define

$$x_m = x \qquad y_m = y \qquad u_m = r \qquad A_m = A - BK \qquad B_m = Bk_{\text{reference}} \qquad C_m = C$$

then we have a system that looks like (1).

1.1 Solution with zero input

Suppose $u_m(t) = 0$ for all t. We claim that the solution to (1) is

$$x_m(t) = e^{A_m t} x_m(0) \qquad\qquad y_m(t) = C_m e^{A_m t} x_m(0)$$

If our guess at $x_m(t)$ is correct, it's obvious that our guess at $y_m(t)$ is also correct. So, we'll focus on the $x_m(t)$ part. To prove our claim, we plug this solution into (1) and show that the equations are satisfied. First, the left-hand side:

$$\begin{aligned} \dot{x}_m &= \frac{d}{dt}\left(e^{A_m t} x_m(0)\right) \\ &= A_m e^{A_m t} x_m(0) \end{aligned}$$

Then, the right-hand side:

$$\begin{aligned} A_m x_m + B_m u_m &= A_m \cdot e^{A_m t} x_m(0) + B_m \cdot 0 \\ &= A_m e^{A_m t} x_m(0) \end{aligned}$$

The left- and right-hand side are the same, so the solution satisfies (1).

1.2 Solution with non-zero input

Suppose $u_m(t) \neq 0$ for some t. We claim that the solution to (1) is

$$x_m(t) = e^{A_m t} x_m(0) + \int_0^t e^{A_m(t-\tau)} B_m u_m d\tau$$
$$y_m(t) = C_m \left(e^{A_m t} x_m(0) + \int_0^t e^{A_m(t-\tau)} B_m u_m d\tau \right)$$

Again, if our guess at $x_m(t)$ is correct, then out guess at $y_m(t)$ is also correct. Again, to prove our claim, we plug this solution into (1). First, the left-hand side:

$$\begin{aligned}
\dot{x}_m &= \frac{d}{dt} \left(e^{A_m t} x_m(0) + \int_0^t e^{A_m(t-\tau)} B_m u_m d\tau \right) \\
&= \frac{d}{dt} \left(e^{A_m t} x_m(0) \right) + \frac{d}{dt} \left(\int_0^t e^{A_m(t-\tau)} B_m u_m d\tau \right) && \text{by linearity of the derivative} \\
&= A_m e^{A_m t} x_m(0) + \frac{d}{dt} \left(\int_0^t e^{A_m(t-\tau)} B_m u_m d\tau \right)
\end{aligned}$$

To differentiate the term with the integral, we take advantage of the following rule:

$$\frac{d}{dt} \left(\int_{\alpha(t)}^{\beta(t)} f(t,\tau) d\tau \right) = f(t, \beta(t)) \dot{\beta}(t) - f(t, \alpha(t)) \dot{\alpha}(t) + \int_{\alpha(t)}^{\beta(t)} \frac{d}{dt} \left(f(t,\tau) \right) d\tau$$

We can apply this rule by defining:

$$\alpha(t) = 0 \qquad \beta(t) = t \qquad f(t,\tau) = e^{A_m(t-\tau)} B_m u_m$$

In particular, we find:

$$\begin{aligned}
\frac{d}{dt} \left(\int_0^t e^{A_m(t-\tau)} B_m u_m d\tau \right) &= \left(e^{A_m(t-t)} B_m u_m \right) \cdot 1 - \left(e^{A_m(t-0)} B_m u_m \right) \cdot 0 \\
&\qquad + \int_0^t A_m e^{A_m(t-\tau)} B_m u_m d\tau \\
&= e^0 B_m u_m - 0 + \int_0^t A_m e^{A_m(t-\tau)} B_m u_m d\tau \\
&= B_m u_m + \int_0^t A_m e^{A_m(t-\tau)} B_m u_m d\tau && \text{because } e^0 = I \\
&= B_m u_m + A_m \int_0^t e^{A_m(t-\tau)} B_m u_m d\tau && \text{because integration is linear} \\
&= A_m \int_0^t e^{A_m(t-\tau)} B_m u_m d\tau + B_m u_m && \text{just rearranging terms}
\end{aligned}$$

So, plugging this in, the left-hand side is:

$$\dot{x}_m = A_m e^{A_m t} x_m(0) + A_m \int_0^t e^{A_m(t-\tau)} B_m u_m d\tau + B_m u_m$$

Then, the right-hand side:

$$\begin{aligned}A_m x_m + B_m u_m &= A_m \cdot \left(e^{A_m t} x_m(0) + \int_0^t e^{A_m(t-\tau)} B_m u_m d\tau \right) + B_m u_m \\ &= A_m e^{A_m t} x_m(0) + A_m \int_0^t e^{A_m(t-\tau)} B_m u_m d\tau + B_m u_m\end{aligned}$$

The left- and right-hand side are the same, so the solution satisfies (1).

1.3 Solution with constant input

Finally, suppose $u_m(t) = v$ for all t, for some constant v. Suppose also that A_m is invertible. We claim that the solution to (1) is

$$\begin{aligned}x_m(t) &= e^{A_m t} x_m(0) + A_m^{-1} e^{A_m t} B_m v - A_m^{-1} B_m v \\ y_m(t) &= C_m \left(e^{A_m t} x_m(0) + A_m^{-1} e^{A_m t} B_m v - A_m^{-1} B_m v \right)\end{aligned}$$

Yet again, we need only focus on proving our claim about $x_m(t)$. Note that the first term is exactly as we derived in Section 1.2. So what we are trying to show is that:

$$\int_0^t e^{A_m(t-\tau)} B_m v d\tau = A_m^{-1} e^{A_m t} B_m v - A_m^{-1} B_m v$$

We take advantage of the following rule:

$$\int_\alpha^\beta \frac{d}{d\tau} \left(g(\tau) \right) d\tau = g(\beta) - g(\alpha) \tag{2}$$

In other words, to integrate

$$\int_0^t e^{A_m(t-\tau)} B_m v d\tau$$

we need only find a function $g(\tau)$ for which

$$\frac{d}{d\tau} \left(g(\tau) \right) = e^{A_m(t-\tau)} B_m v$$

This process is sometimes referred to as "antidifferentiation" — you are trying to find a function whose derivative is equal to another function. This process is something that is traditionally part of high school calculus. There is no magic way to antidifferentiate — indeed, this process is sometimes impossible. It's a mixture of rules with "guess and check." In this particular case, let's try

$$g(\tau) = -A_m^{-1} e^{A_m(t-\tau)} B_m v$$

We find

$$\begin{aligned}\frac{d}{d\tau} \left(g(\tau) \right) &= -A_m^{-1} \left(\frac{d}{d\tau} e^{A_m(t-\tau)} \right) B_m v && \text{because derivative is linear} \\ &= -A_m^{-1} \left(-A_m e^{A_m(t-\tau)} \right) B_m v \\ &= e^{A_m(t-\tau)} B_m v\end{aligned}$$

This is what we wanted, so we can apply (2):

$$\begin{aligned}\int_0^t e^{A_m(t-\tau)}B_m v d\tau &= g(t) - g(0) \\ &= \left(-A_m^{-1}e^{A_m(t-t)}B_m v\right) - \left(-A_m^{-1}e^{A_m(t-0)}B_m v\right) \\ &= A_m^{-1}e^{A_m t}B_m v - A_m^{-1}e^{0}B_m v \\ &= A_m^{-1}e^{A_m t}B_m v - A_m^{-1}B_m v\end{aligned}$$

This is what we were trying to show, so the solution does, indeed, solve (1).

1.4 Summary

The solution to

$$\begin{aligned}\dot{x}_m &= A_m x_m + B_m u_m \\ y_m &= C_m x_m\end{aligned}$$

is one of three things:

- when $u_m(t) = 0$ for all t

$$\begin{aligned}x_m(t) &= e^{A_m t}x_m(0) \\ y_m(t) &= C_m e^{A_m t}x_m(0)\end{aligned}$$

- when $u_m(t) \neq 0$ for some t

$$\begin{aligned}x_m(t) &= e^{A_m t}x_m(0) + \int_0^t e^{A_m(t-\tau)}B_m u_m d\tau \\ y_m(t) &= C_m\left(e^{A_m t}x_m(0) + \int_0^t e^{A_m(t-\tau)}B_m u_m d\tau\right)\end{aligned}$$

- when $u_m(t) = v$ for all t, where v is a constant, and when A_m is invertible

$$\begin{aligned}x_m(t) &= e^{A_m t}x_m(0) + A_m^{-1}e^{A_m t}B_m v - A_m^{-1}B_m v \\ y_m(t) &= C_m\left(e^{A_m t}x_m(0) + A_m^{-1}e^{A_m t}B_m v - A_m^{-1}B_m v\right)\end{aligned}$$

One quick point about the last result is that A_m will, indeed, be invertible in almost all cases we care about this semester. The reason is that A_m will often describe a closed-loop system that we have designed to be stable. If the system is stable, then every eigenvalue of A_m has negative real part. As a consequence, no eigenvalue of A_m is identically zero. It is a fact that the determinant of a square matrix is equal to the product of all the eigenvalues of that matrix—if no eigenvalue is identically zero, then the determinant is non-zero. But if A_m has non-zero determinant, then it is invertible. Bottom line, stable system means invertible A_m.

2 MATLAB Snippets

In this section, we find solutions to a given state-space system for different input conditions using MATLAB. Note that while we provide different approaches for different input conditions, all of them can be numerically solved using the numerical integration method provided in subsection 2.2. Consider the system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

with the input

$$u = -Kx + K_{ref}r.$$

where given

```
>> A = [-0.06 1.05; 0.03 0.02];
>> B = [0.01; 0.98];
>> C = [1.05 -0.05];
>> D = [0.00];
>> K = [95.17 14.70];
```

2.1 Solution with zero input

Consider the case where $r = 0$ and initial conditions $x(0) = [0.5, 0.5]^T$. We know the expression for exact solution with zero input and use that to calculate the output y at different time instances using MATLAB.

```
% initial state
x_init = [0.5; 0.5];

% exact solution for zero input
syms t
x_soln = matlabFunction(expm((A-B*K)*t)*x_init);

% calculate x and y at particular time instances
time = 0:0.001:10;
x_values = feval(x_soln, time);
y_values = C*x_values;

% plot output
plot(time, y_values);
```

2.2 Solution with non-zero constant input

Now consider a reference $r = 1$ with the initial conditions $x(0) = [0, 0]^T$. Then, the component $K_{ref}r$ in the control input is non-zero. We know that the exact solution for an general non-zero input case has an integral and we calculate it numerically using MATLAB. Note that you need to have 'odedynref.m' file provided in 'Lecture9Files.zip' in your working directory for this code to work.

```
% initial state
x_init = [0; 0];

% calculate Kref
Kref=-1/(C*inv(A-B*K)*B);

% solve for x at different time instances in the duration [0,5]
[t,x]=ode45(@(t,x)(odedynref(A,B,K,1,Kref,x,t)),[0 5],x_init);

% calculate and plot the output y
plot(t,C*x');
```