Optimal Controller

Problem

$$\begin{aligned} \underset{u_{[t_0,t_1]}}{\text{minimize}} \quad & x(t_1)^T M_c x(t_1) + \int_{t_0}^{t_1} \left(x(t)^T Q_c x(t) + u(t)^T R_c u(t) \right) dt \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu(t) \\ & x(t_0) = x_0 \end{aligned}$$

Solution

- Solve backward in time on the interval $[t_0, t_1]$:

$$\begin{aligned} \dot{P}(t) &= P(t)BR_c^{-1}B^T P(t) - P(t)A - A^T P(t) - Q_c \\ P(t_1) &= M_c \end{aligned}$$

- Apply the control input:

$$u(t) = -K(t)x(t)$$

  where

$$K(t) = R_c^{-1}B^T P(t)$$

Solution as $t_1 \to \infty$ ("infinite-horizon" or "steady-state")

- Solve:

$$0 = PBR_c^{-1}B^T P - PA - A^T P - Q_c$$

- Apply the control input:

$$u(t) = -Kx(t)$$

  where

$$K = R_c^{-1}B^T P$$

How to find the "infinite-horizon" or "steady-state" solution in MATLAB

```
K = lqr(A,B,Qc,Rc)
```

Optimal Observer

Problem

$$\begin{aligned} \underset{x(t_1), d_{[t_0,t_1]}, n_{[t_0,t_1]}}{\text{minimize}} \quad & n(t_1)^T M_o n(t_1) + \int_{t_0}^{t_1} \left(n(t)^T Q_o n(t) + d(t)^T R_o d(t) \right) dt \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu(t) + d(t) \\ & y(t) = Cx(t) + n(t) \end{aligned}$$

Solution

- Solve forward in time on the interval $[t_0, t_1]$:

$$\begin{aligned} \dot{P}(t) &= -P(t)C^T Q_o C P(t) + AP(t) + P(t)A^T + R_o^{-1} \\ P(t_0) &= M_o \end{aligned}$$

- Compute the state estimate:

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(t) - L(t) \left(C\widehat{x}(t) - y(t) \right)$$

  where

$$L(t) = P(t) C^T Q_o$$

Solution as $t_0 \to -\infty$ ("infinite-horizon" or "steady-state")

- Solve:

$$0 = -PC^T Q_o C P + AP + PA^T + R_o^{-1}$$

- Compute the state estimate:

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(t) - L \left(C\widehat{x}(t) - y(t) \right)$$

  where

$$L = PC^T Q_o$$

How to find the "infinite-horizon" or "steady-state" solution in MATLAB

```
L = lqr(A',C',inv(Ro),inv(Qo))'
```