AE353: Additional Notes on Linearization

(to be treated as an appendix to the presentation)

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1 Recap of Method

Follow these steps to linearize a nonlinear system and put the result in state-space form:

- Rewrite the nonlinear system as a set of first-order ordinary differential equations (nonlinear state space model) :

$$\begin{aligned} \dot{w} &= f(w, p) \\ r &= h(w, p). \end{aligned}$$

- Find an equilibrium point w_e, p_e of the nonlinear system by solving

$$f(w_e, p_e) = 0.$$

  This equation may have no solutions (in which case no equilibrium point exists for the nonlinear system) or many solutions (in which case you have to make a choice). If required, calculate $r_e = h(w_e, p_e)$.

- Define the state, input, and output as follows:

$$x = w - w_e \qquad u = p - p_e \qquad y = r - r_e$$

- Compute A, B, C, and D as follows:

$$A = \left.\frac{\partial f}{\partial w}\right|_{(w_e,p_e)} \qquad B = \left.\frac{\partial f}{\partial p}\right|_{(w_e,p_e)} \qquad C = \left.\frac{\partial h}{\partial w}\right|_{(w_e,p_e)} \qquad D = \left.\frac{\partial h}{\partial p}\right|_{(w_e,p_e)}$$

  Recall that

$$\left.\frac{\partial f}{\partial w}\right|_{(w_e,p_e)}$$

  is the Jacobian (i.e., matrix of partial derivatives) of f with respect to w (https://en.wikipedia.org/wiki/Jacobian_matrix_and_determinant), evaluated at the equilibrium point,

$$A_{ij} = \left.\frac{\partial f_i}{\partial w_j}\right|_{(w_e,p_e)}.$$

The resulting state-space model is

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

The behavior of this linear system and of the original, nonlinear system will be approximately the same as long as x and u are small (i.e., as long as w and p stay close to w_e and p_e, respectively).

2 Frequently Unasked Questions

1. Q — Why are we even linearizing the system?

   A — Almost all real world systems are inherently nonlinear. As we will see in the rest of this course, most of the theory we developed over the years is only applicable to linear systems. When we have to work with a nonlinear system, one can always ask the question "How can I relate this system to the linear class of systems that I know a lot about?". One possible answer is to approximate the nonlinear system with a linear model that, at least at and near some points, behaves the way the nonlinear system would.

2. Q — Why linearize only about equilibrium point? Why not about some arbitrary point?

   A — Recall that approximation of any function using Taylor's theorem needs an infinite number of terms; we only use the linear terms while approximating since we ultimately seek a linear model. If we linearize about some arbitrary point, the linear model obtained will not be a good approximation the original nonlinear system anymore.

3. Q — What if I have many (possibly infinitely many) equilibrium points? Which linearized model is then the correct one?

   A — All of the points that satisfy $f(w_e, p_e) = 0$ are indeed equilibrium points of the given nonlinear system. For different equilibrium points, you might end up with different linearized models and all of them are correct. As you'll see in the projects of this course, you can always exploit the existence of multiple equilibrium points while designing a controller.

4. Q — Are you saying the state variables, control inputs and outputs in the linearized model we obtain are in fact not the true states, control inputs, and outputs anymore?

   A — Correct. While deriving the approximate linear state space model of the nonlinear system, we redefined our states, inputs, and outputs by offsetting them with appropriate terms to get the linearized model form $\dot{x} = Ax + Bu$ and $y = Cx + Du$. In practice, this means that the true state of the system would be whatever you obtain by solving your linearized model plus the value of that state at equilibrium point (similarly for control inputs and outputs).

5. Q — How can we work with a linearized model that approximates nonlinear system only in the neighborhood of an equilibrium point? Do we just assume that it is always a good approximate linear model when we clearly know its not?

   A — We work with the assumption that our states and control inputs x and u are small, meaning the true states and control inputs are small perturbations from the equilibrium point of the system, where we know the linearized model is a good approximation.

6. Q — Last question, I swear! There are way too many partial derivatives to calculate and also solving the for the equilibrium point is not straight forward, right?

   A — You don't have to solve all the partial derivatives by hand. You can use MATLAB; sample code is shown the code snippets section.

3 Examples

Example 1

Consider the nonlinear system with dynamics

$$\dot{\mu} + 3\mu\nu = \tau$$
$$\dot{\nu} - 5\mu\nu = 0$$

and measurement

$$\theta = \cos(\mu).$$

We know that the system has two states, μ and η, and the highest order derivative for each of the two states in the given system of equations is one. Further, we have a single output θ. We now linearize this nonlinear system using the method described in Section 1:

- Rewrite as a set of first-order ODEs: since we only have first order derivatives in the given nonlinear system, this step is straightforward and gives us the nonlinear statespace model (the nonlinear terms with $\mu\nu$ still exist).

$$\begin{bmatrix} \dot{\mu} \\ \dot{\nu} \end{bmatrix} = \begin{bmatrix} -3\mu\nu + \tau \\ 5\mu\nu \end{bmatrix}$$
$$\begin{bmatrix} \theta \end{bmatrix} = \begin{bmatrix} \cos(\mu) \end{bmatrix}$$

- Find an equilibrium point by solving

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3\mu_e\nu_e + \tau_e \\ 5\mu_e\nu_e \end{bmatrix}.$$

In this case, there are many solutions. Suppose we pick this one:

$$\mu_e = 0 \qquad \nu_e = 5 \qquad \tau_e = 0.$$

- Define the state, input, and output:

$$x = \begin{bmatrix} \mu - \mu_e \\ \nu - \nu_e \end{bmatrix} \qquad u = \begin{bmatrix} \tau - \tau_e \end{bmatrix} \qquad y = \begin{bmatrix} \theta - \cos(\mu_e) \end{bmatrix}$$

- Compute A, B, C, and D:

$$A = \begin{bmatrix} -3\nu & -3\mu \\ 5\nu & 5\mu \end{bmatrix}\Bigg|_{\left(\begin{bmatrix} \mu_e \\ \nu_e \end{bmatrix}, \tau_e\right)} = \begin{bmatrix} -15 & 0 \\ 25 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\Bigg|_{\left(\begin{bmatrix} \mu_e \\ \nu_e \end{bmatrix}, \tau_e\right)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -\sin(\mu) & 0 \end{bmatrix}\Big|_{\left(\begin{bmatrix} \mu_e \\ \nu_e \end{bmatrix}, \tau_e\right)} = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}\Big|_{\left(\begin{bmatrix} \mu_e \\ \nu_e \end{bmatrix}, \tau_e\right)} = \begin{bmatrix} 0 \end{bmatrix}$$

The resulting state-space model is, as always,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

What if we picked a different equilibrium point, say $\mu_e = 7, \nu_e = 0, \tau_e = 0$? As mentioned earlier, following the steps with this equilibrium point would still yield a linear model, in this case different from the one above, which approximates the nonlinear system in the vicinity of the new equilibrium point.

4 Code Snippets

We solve the above example problem of linearizing given system of nonlinear equations using MATLAB.

```
%% define variables and system of equations

% define symbolic variables
syms mu nu tau theta

% write down given system of equations and output expression
mudot = -3*mu*nu + tau;
nudot = 5*mu*nu;
theta = cos(mu);

%% selected equilibrium point
mu_e = 0; nu_e = 5; tau_e = 0;

%% generate linear statespace form
```

```
% calculate jacobian at equilibirium point
A_sym = jacobian([mudot; nudot], [mu, nu]); %calculate jacobian
A = subs(A_sym, {mu, nu, tau}, {mu_e, nu_e, tau_e}); % substitute ...
    equilibrium point

% similarly generate B, C, D matrices
B = subs(jacobian([mudot; nudot], tau), {mu, nu, tau}, {mu_e, nu_e, ...
    tau_e});
C = subs(jacobian(cos(mu), [mu, nu]), {mu, nu, tau}, {mu_e, nu_e, tau_e});
D = subs(jacobian(cos(mu), tau), {mu, nu, tau}, {mu_e, nu_e, tau_e});
```