AE353: Additional Notes on Diagonalization and Complex Numbers

(to be treated as an appendix to the presentation)

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1 Diagonalization

In class, we saw that being able to express the system

$$\dot{x} = Ax$$

in the form

$$\dot{x} = VDV^{-1}x$$

where D is a diagonal matrix allows us to easily compute matrix exponential required to find a solution to the system. We then saw that by choosing the columns of V to be the eigenvectors of A, and D to be a diagonal matrix with eigernvalues of A, we can sometimes diagonalize A as $A = VDV^{-1}$. We can now ask if any matrix A can be diagonalized by applying this procedure. The answer, sadly, is no. There are matrices that cannot be diagonalized. As an example, suppose

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Both eigenvalues of A are 2, and both eigenvectors of A are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so the matrix V is

$$V = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

We cannot invert the matrix V since it is singular, so we cannot compute VDV^{-1}. If a matrix is not diagonalizable, then it must have at least two repeating eigenvalues. (So, if a matrix of size $n \times n$ has n distinct eigenvalues, then you know for sure it can be diagonalized.) You can find more properties of diagonalizability at this link. If a matrix is not diagonalizable, we're not completely out of luck. It is always possible to put a matrix into a particular form called Jordan normal form (`https://en.wikipedia.org/wiki/Jordan_normal_form`). Computing the exponential of a matrix in Jordan normal form is easy, just not as easy as for a diagonal matrix.

2 Complex Numbers

You might have noticed that the entries in the matrix V can be complex. Here, we give a brief review of complex numbers. A complex number is a number of the form

$$z = a + bi,$$

where a and b are real numbers and the "imaginary unit" i is defined to be $i = \sqrt{-1}$. Another common symbol for the imaginary unit is j (MATLAB accepts either one). The number a is called the real part of z, and the number b is called the imaginary part of z. To add two complex numbers, we add the real and imaginary parts:

$$z_1 = a_1 + b_1 i \qquad z_2 = a_2 + b_2 i \qquad z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i.$$

To multiply two complex numbers, we follow the same steps as when we multiply real numbers:

$$\begin{aligned} z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i. \end{aligned}$$

The last step is true since $i^2 = \left(\sqrt{-1}\right)^2 = -1$. Finally, the complex conjugate of z is $\bar{z} = a - bi$, or in other words, it has the same real part but negative the imaginary part. If you check, you'll notice that complex eigenvalues always come in conjugate pairs.