AE353: Additional Notes on LQR

(to be treated as an appendix to the presentation)

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1 LQR

1.1 Statement of the problem

Here is the *linear quadratic regulator (LQR)* problem:

$$\begin{aligned} \underset{u_{[t_0,t_f]}}{\text{minimize}} \quad & \int_{t_0}^{t_f} \left(x(t)^T Q x(t) + u(t)^T R u(t) \right) dt + x(t_f)^T F x(t_f) \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \end{aligned}$$

It is an optimal control problem—if you define

$$f(x,u) = Ax + Bu \qquad g(x,u) = x^T Q x + u^T R u \qquad h(x) = x^T F x$$

then you see that this problem has the same form as defined in previous notes. It is called "linear" because the dynamics are those of a linear (state space) system. It is called "quadratic" because the cost is quadratic (i.e., polynomial of order at most two) in x and u. It is called a "regulator" because the result of solving it is to keep x close to zero (i.e., to keep errors small).

The matrices Q, R, and F are parameters that can be used to trade off error (non-zero states) with effort (non-zero inputs). These matrices have to be symmetric ($Q = Q^T$, etc.), have to be the right size, and also have to satisfy the following conditions in order for the LQR problem to have a solution:

$$Q \geq 0 \qquad R > 0 \qquad F \geq 0.$$

What this notation means is that Q and F are *positive semidefinite* and that R is *positive definite* (https://en.wikipedia.org/wiki/Positive-definite_matrix). We will ignore these terms for now, noting only that this is similar to requiring (for example) that $r > 0$ in order for the function ru^2 to have a minimum.

1.2 Solution to the finite-horizon LQR problem

Companion notes on HJB equation tells us to solve the LQR problem in two steps. First, we find a function $v(t,x)$ that satisfies the HJB equation. Here is that equation, with the functions f, g, and h filled in from Section 1.1:

$$0 = \frac{\partial v(t,x)}{\partial t} + \underset{u}{\text{minimum}} \left\{ \frac{\partial v(t,x)}{\partial x}(Ax + Bu) + x^T Q x + u^T R u \right\}, \qquad v(t_1,x) = x^T F x.$$

What function v might solve this equation? Look at the boundary condition. At time t_f,

$$v(t_f, x) = x^T F x.$$

This function has the form

$$v(t, x) = x^T P(t) x$$

for some symmetric matrix $P(t)$ that satisfies $P(t_f) = F$. So let's "guess" that this form is the solution we are looking for, and see if it satisfies the HJB equation. Before proceeding, we need to compute the partial derivatives of v:

$$\frac{\partial v}{\partial t} = x^T \dot{P} x \qquad\qquad \frac{\partial v}{\partial x} = 2x^T P$$

This is matrix calculus (e.g., see https://en.wikipedia.org/wiki/Matrix_calculus or Chapter A.4.1 of http://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf). The result on the left should surprise no one. The result on the right is the matrix equivalent of $\partial(px^2)/\partial x = 2px$ (you could check that this result is correct by considering an example). Plug these partial derivatives into HJB and we have

$$\begin{aligned} 0 &= x^T \dot{P} x + \underset{u}{\text{minimum}} \left\{ 2x^T P(Ax + Bu) + x^T Q x + u^T R u \right\} \\ &= x^T \dot{P} x + \underset{u}{\text{minimum}} \left\{ x^T(2PA + \; Q)x + 2x^T PBu + u^T R u \right\} \end{aligned} \tag{1}$$

To evaluate the minimum, we apply the first-derivative test (more matrix calculus!):

$$\begin{aligned} 0 &= \frac{\partial}{\partial u} \left(x^T(2PA + Q)x + 2x^T PBu + u^T R u \right) \\ &= 2x^T PB + 2u^T R \\ &= 2\left(B^T P x + R u \right)^T. \end{aligned}$$

This equation is easily solved:

$$u = -R^{-1} B^T P x. \tag{2}$$

Plugging this back into (1), we have

$$\begin{aligned} 0 &= x^T \dot{P} x + \underset{u}{\text{minimum}} \left\{ x^T(2PA + Q)x + 2x^T PBu + u^T R u \right\} \\ &= x^T \dot{P} x + \left(x^T(2PA + Q)x + 2x^T PB(-R^{-1}B^T P x) + (-R^{-1}B^T P x)^T R(-R^{-1}B^T P x) \right) \\ &= x^T \dot{P} x + \left(x^T(2PA + Q)x - 2x^T PBR^{-1}B^T P x + x^T PBR^{-1}B^T P x \right) \\ &= x^T \dot{P} x + x^T(2PA + Q)x - x^T PBR^{-1}B^T P x \\ &= x^T \dot{P} x + x^T(PA + A^T P + Q)x - x^T PBR^{-1}B^T P x \end{aligned}$$

$\ldots$ because $x^T(N + N^T)x = 2x^T N x$ for any N and $x \ldots$

$$= x^T \left(\dot{P} + PA + A^T P + Q - PBR^{-1}B^T P \right) x.$$

In order for this equation to be true for any x, it must be the case that

$$\dot{P} = PBR^{-1}B^T P - PA - A^T P - Q$$

In summary, we have found that

$$v(t, x) = x^T P x$$

solves the HJB equation, where P is found by integrating the matrix differential equation

$$\dot{P} = PBR^{-1}B^T P - PA - A^T P - Q$$

backward in time, starting from

$$P(t_f) = F.$$

Now that we know v, we can find u. Wait, we already did that! The minimizer in the HJB equation is (2). This choice of input has the form

$$u = -Kx$$

for

$$K = R^{-1}B^T P.$$

1.3 Solution to the infinite-horizon LQR problem

The infinite-horizon case arises when $t_f = \infty$ and $F = 0$. As discussed in the lecture 13, it is possible to obtain a state feedback controller of form $u = -Kx$ for the infinite-horizon version of the LQR problem where the K values can be found directly using the MATLAB `LQR` which takes A, B, Q and R as inputs.