AE353: Additional Notes on Reference Tracking

(to be treated as an appendix to the presentation)

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1 Reference Tracking

Consider the state-space system

$$\dot{x} = Ax + Bu \qquad y = Cx,$$

where u and y are both scalars. We've seen that by choosing $u = -Kx$, the system is asymptotically stable if the eigenvalues of the matrix $A - BK$ have negative real parts. This means that $y \to 0$ as $t \to \infty$. Now we would like for $y \to r$ as $t \to \infty$, where r is some constant.

For asymptotic stability, we chose the input to be linear in the state. Let's also choose the input to be linear in the reference, r:

$$u = -Kx + k_{ref} r.$$

Plugging this input into our state space system gives

$$\dot{x} = (A - BK)x + Bk_{ref} r \qquad y = Cx. \tag{1}$$

We are interested in the behavior of y as $t \to \infty$. If we have chosen K so that the system is asymptotically stable, then x will approach a constant value as $t \to \infty$. We'll call this value x_{ss}, the steady-state value of x. At steady-state, $\dot{x}_{ss} = 0$. Using this in the expression (1) gives

$$0 = (A - BK)x_{ss} + Bk_{ref} r.$$

Solving this equation for x_{ss} gives

$$x_{ss} = -(A - BK)^{-1} Bk_{ref} r.$$

The steady-state output is $y_{ss} = Cx_{ss}$, or

$$y_{ss} = -C(A - BK)^{-1} Bk_{ref} r.$$

Since we want $y_{ss} = r$, we must have $-C(A - BK)^{-1} Bk_{ref} = 1$, and therefore

$$k_{ref} = \frac{1}{-C(A - BK)^{-1} B}.$$

Note that we can write k_{ref} as a fraction because $-C(A-BK)^{-1}B$ is a scalar, since we assumed y and u are scalars.

If k_{ref} is not chosen in this way, how different are y and r as $t \to \infty$? We can answer this by computing $y_{ss} - r$. Using our calculations above, we have

$$\begin{aligned} y_{ss} - r &= -C(A-BK)^{-1}Bk_{ref}r - r \\ &= (-C(A-BK)^{-1}Bk_{ref} - 1)r. \end{aligned}$$

2 Example and MATLAB Snippets

In this section, we look at an example for reference tracking. Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where given

```
>> A = [-0.1 0.4 0.2; -0.5 -0.6 -0.1; -0.6 -0.7 0.7];
>> B = [0.6; -0.2; -0.9];
>> C = [-0.6 0.2 -0.5];
>> D = [0.0];
```

Consider the input

$$u = -Kx + K_{ref}r.$$

where r is a known reference signal that is assumed to be constant. We are supposed to find the value of K and k_{ref} such that the system is asymptotically stable and $y_{ss} = r$. To find K, we start with some negative eigenvalues (which is a required condition for the closed-loop system to be asymptotically stable) and find the corresponding gains K:

```
% desired eigenvalues
p = [-1, -0.5, -0.8];


% calculate K
>> K = acker(A,B,p)

K =

    0.4270    0.8736   -2.4650
```

These gain values ensure asymptotic stability. We now calculate K_{ref} using the formula dervied above to ensure that $y_{ss} = r$:

```
% calculate K_ref
>> K_ref = 1/(-C*inv(A-B*K)*B)

K_ref =

    1.2300
```