chapter 7.

Markov Chains

Markov chains represent a special class of dynamic systems that evolve probabilistically. This class of models, which can be regarded in part as a special subclass of positive linear systems, has a wide variety of applications and a deep but intuitive body of theory. It is an important branch of dynamic systems.

A finite Markov chain can be visualized in terms of a marker that jumps around among a finite set of locations, or conditions. The transition from one location to another, however, is probabilistic rather than deterministic. A classic example is weather, which can be classified in terms of a finite number of conditions, and which changes daily from one condition to another. The possible positions for the process are termed "states." Since there are only a finite number of states, the structure of finite Markov chains appears at first to differ substantially from the standard dynamic system framework in which the state is defined over an n-dimensional continuum. However, the probabilistic evolution of a Markov chain implies that future states cannot be inferred from the present, except in terms of probability assessments. Thus, tomorrow's weather cannot be predicted with certainty, but probabilities can be assigned to the various possible conditions. Therefore, in general, future evolution of a Markov process is described by a vector of probabilities (for occurrence of the various states). This vector, and its evolution, is really the essential description of the Markov chain, and it is governed by a linear dynamic system in the sense of earlier chapters. The first part of this chapter develops this framework.

The vector of probabilities is a positive vector, and, accordingly, the dynamic system describing a Markov chain is a positive linear system. The results on positive linear systems, particularly the Frobenius–Perron theorem, thus imply important limiting properties for Markov chains. If the Frobenius–Perron eigenvector is strictly positive, all the states are visited infinitely often and with probabilities defined by the components of this eigenvector.

In many Markov chains, the Frobenius–Perron eigenvector is not strictly positive, and perhaps not unique. This has important implications in terms of the probabilistic context, and raises a series of important issues. To analyze such chains, it is necessary to systematically characterize the various possible chain structures. This leads to new insights in this useful class of models.

7.1 FINITE MARKOV CHAINS

A finite Markov chain is a discrete-step process that at any step can be in one of a finite number of conditions, or states. If the chain has n possible states, it is said to be an nth-order chain. At each step the chain may change from its state to another, with the particular change being determined probabilistically according to a given set of *transition probabilities*. Thus, the process moves stepwise but randomly among the finite number of states. Throughout this chapter only *stationary* Markov chains are considered, where the transition probabilities do not depend on the number of steps that have occurred.

Definition. An nth-order Markov chain process is determined by a set of n states $\{S_1, S_2, \ldots, S_n\}$ and a set of transition probabilities p_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$. The process can be in only one state at any time instant. If at time k the process is in state S_i, then at time $k+1$ it will be in state S_j with probability p_{ij}. An initial starting state is specified.

Example 1 (A Weather Model). The weather in a certain city can be characterized as being either sunny, cloudy, or rainy. If it is sunny one day, then sun or clouds are equally likely the next day. If it is cloudy, then there is a fifty percent chance the next day will be sunny, a twenty-five percent chance of continued clouds, and a twenty-five percent chance of rain. If it is raining, it will not be sunny the next day, but continued rain or clouds are equally likely.

Denoting the three types of weather by S, C, and R, respectively, this model can be represented by an array of transition probabilities:

	S	C	R
S	$\frac{1}{2}$	$\frac{1}{2}$	0
C	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
R	0	$\frac{1}{2}$	$\frac{1}{2}$

This array is read by going down the left column to the current weather condition. The corresponding row of numbers gives the probabilities associated

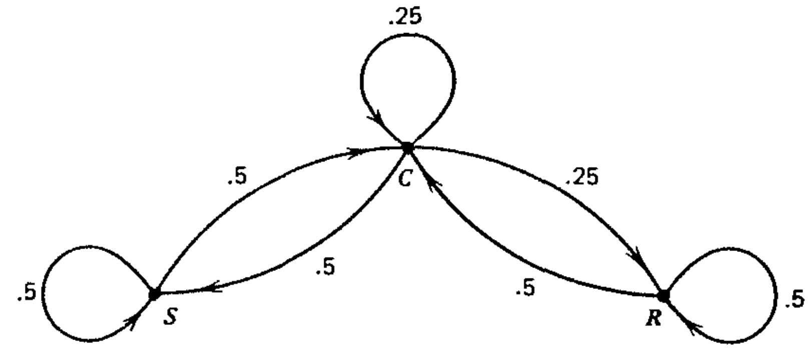

Figure 7.1. The weather chain.

with the next weather condition. The process starts with some weather condition and moves, each day, to a new condition. There is no way, however, to predict exactly which transition will occur. Only probabilistic statements, presumably based on past experience, can be made.

The weather model can be alternatively described in terms of a diagram as shown in Fig. 7.1. In general in such diagrams, nodes correspond to states, and directed paths between nodes indicate possible transitions, with the probability of a given transition labeled on the path.

Example 2 (Estes Learning Model). As a simple model of learning of an elementary task or of a small bit of information, it can be assumed that an individual is always in either of two possible states: he is in state L if he has learned the task or material, and in state N if he has not. Once the individual has learned this one thing, he will not forget it. However, if he has not yet learned it, there is a probability α, $0<\alpha<1$ that he will learn it during the next time period. This chain is illustrated in Fig. 7.2.

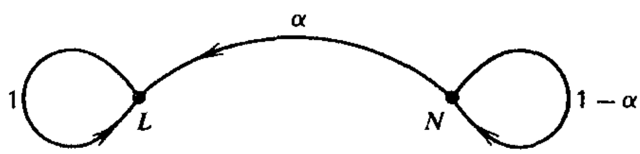

Figure 7.2. Learning model.

This idealized learning process is a two-state Markov chain having transition probabilities

	L	N
L	1	0
N	α	$1-\alpha$

Example 3 (Gambler's Ruin). The gambler's ruin problem of Chapter 2 can be regarded as a Markov chain with states corresponding to the number of coins

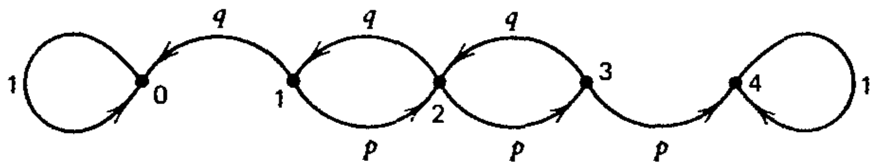

Figure 7.3. Gambler's ruin.

or chips held by player A. As a specific example, suppose both players begin with just two chips, and suppose the probability that A wins in any turn is p, while the probability that B wins is $q=1-p$. There are five possible states, corresponding as to whether player A has 0, 1, 2, 3, or 4 chips. The transition probabilities are

	0	1	2	3	4
0	1	0	0	0	0
1	q	0	p	0	0
2	0	q	0	p	0
3	0	0	q	0	p
4	0	0	0	0	1

and this structure is shown in Fig. 7.3. The process is initiated at the state corresponding to two chips.

Stochastic Matrices and Probability Vectors

The transition probabilities associated with a Markov chain are most conveniently regarded as the elements of an $n \times n$ matrix

$$\mathbf{P}=\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

It is clear that all elements of a $\mathbf{P}$ matrix associated with a Markov chain are nonnegative. Furthermore, it should be observed that the sum of elements along any row is equal to 1. This is because if the process is in state i at a given step, the probability that it goes *somewhere* during the next step must be 1. A square matrix $\mathbf{P}$ with these properties is often referred to as a *stochastic matrix.*

A vector is a *probability vector* if all its components are nonnegative and sum to 1. A fundamental relation between stochastic matrices and probability vectors is that if $\mathbf{x}^T$ is a row probability vector and $\mathbf{P}$ is a stochastic matrix, then the row vector $\mathbf{x}^T\mathbf{P}$ is also a probability vector. (The reader is asked to verify this in Problem 1.) Thus, stochastic matrices can be thought of as natural transformations in the realm of probability vectors.

The Multistep Transition Process

If an nth-order Markov chain with transition matrix $\mathbf{P}$ is initiated in state S_i, then after one step it will be in state S_j with probability p_{ij}. This fact can be characterized by stating that the probabilities of the various states after one step are the components of the row vector

$$[p_{i1} \quad p_{i2} \quad \cdots \quad p_{in}]$$

which is a probability vector. This vector is itself obtained by multiplication of $\mathbf{P}$ on the left by the special degenerate row probability vector

$$[0 \quad \cdots \quad 1 \quad \cdots \quad 0]$$

where the 1 is in the ith place.

Suppose now that we look at the Markov process after two steps. Beginning in a given initial state S_i the process will after two steps end up at some state S_j. The overall transition from S_i to S_j is governed by two applications of the underlying transition matrix. To work out the details, let $p_{ij}^{(2)}$ be the probability that starting at state S_i the process will move to state S_j after two steps. If it were known that the process would go to state S_k after the first step, we would have

$$p_{ij}^{(2)} = p_{kj}$$

However, the probability that the state is S_k after one step is p_{ik}. Summing over all possible first steps we obtain

$$p_{ij}^{(2)} = \sum_{k=1}^{n} p_{ik}p_{kj} = [\mathbf{P}^2]_{ij}$$

This calculation shows that the probability $p_{ij}^{(2)}$ is equal to the ijth element of the matrix $\mathbf{P}^2$. Thus, the two-step transition matrix is $\mathbf{P}^2$.

In a similar way, the transition probabilities for m steps are defined by the elements of the matrix $\mathbf{P}^m$. We write, for notational convenience, $p_{ij}^{(m)}$ for the ijth component of $\mathbf{P}^m$, and recognize that it is also the probability of going from S_i to S_j in m steps.

Much of the above discussion can be expressed more directly in terms of a natural association between a Markov chain and a standard dynamic system. Let $\mathbf{x}(k)^T$ be an n-dimensional *row* vector with component x_j, $j = 1, 2, \ldots, n$ corresponding to the probability that the state at step k will be equal to S_j. If the process is initiated in state S_i, then $\mathbf{x}(0)^T$ is the unit vector with a one in the ith coordinate position. Successive probability vectors are generated by the recursion

$$\mathbf{x}(k+1)^T = \mathbf{x}(k)^T\mathbf{P} \tag{7-1}$$

We recognize (7-1) as a standard, linear time-invariant system, except that it is expressed in terms of row rather than column vectors. (By using $\mathbf{P}^T$ instead of $\mathbf{P}$ the system obviously could be expressed in terms of columns. However, it is standard convention to work with the row formulation in the context of Markov chains.)

It must be emphasized that the $\mathbf{x}(k)^T$ vector is not really the state of the Markov process. At each step the state is one of the n distinct states $S_1, S_2, \ldots, S_n$. The vector $\mathbf{x}(k)^T$ gives the probabilities that the Markov process takes on specific values. Thus, the sequence of $\mathbf{x}(k)^T$ does not record a sequence of actual states, rather it is a projection of our probabilistic knowledge.

It is possible to give somewhat more substance to the interpretation of the vector $\mathbf{x}(k)^T$ by imagining a large number N of independent copies of a given Markov chain. For example, in connection with the Estes learning model we might imagine a class of N students, each governed by identical transition probabilities. Although the various chains each are assumed to have the same set of transition probabilities, the actual transitions in one chain are not influenced by those in the others. Therefore, even if all chains are initiated in the same corresponding state, they most likely will differ at later times. Indeed, if the chains all begin at state S_i, it can be expected that after one step about Np_{i1} of them will be in state S_1, Np_{i2} in state S_2, and so forth. In other words, they will be distributed among the states roughly in proportion to the transition probabilities. In the classroom example, for instance, the $\mathbf{x}(k)^T$ vector, although not a description of the evolution of any single student, is a fairly accurate description of the whole class in terms of the percentage of students in each state as a function of k. From this viewpoint Markov chains are closely related to some of our earlier models, such as population dynamics, promotions in a hierarchy, and so forth, where groups of individuals or objects move into different categories.

Analytical Issues

When viewed in terms of its successive probability vectors, a Markov chain is a linear dynamic system with a positive system matrix. Thus, it is expected that the strong limit properties of positive systems play a central role in the theory of Markov processes. Indeed this is true, and in this case these properties describe the long-term distribution of states.

In addition to characterization of the long-term distribution, there are some important and unique analytical issues associated with the study of Markov chains. One example is the computation of the average length of time for a Markov process to reach a specified state, or one of a group of states. Another is the computation of the probability that a specified state will ultimately be reached. Such problems do not have direct analogies in standard deterministic system analysis. Nevertheless, most of the analysis for Markov chains is based on the principles developed in earlier chapters.

7.2 REGULAR MARKOV CHAINS AND LIMITING DISTRIBUTIONS

As pointed out earlier, the Frobenius-Perron theorem and the associated theory of positive systems is applicable to Markov chains. For certain types of Markov chains, these results imply the existence of a unique limiting probability vector.

To begin the application of the Frobenius–Perron theorem we first observe that the Frobenius–Perron eigenvalue, the eigenvalue of largest absolute value, is always 1.

Proposition. *Corresponding to a stochastic matrix* $\mathbf{P}$ *the value* $\lambda_0 = 1$ *is an eigenvalue. No other eigenvalue of* $\mathbf{P}$ *has absolute value greater than* 1.

Proof. This is a special case of the argument given in Sect. 6.2, since for a stochastic matrix all row sums are equal to 1. ∎

Definition. A Markov chain is said to be *regular* if $\mathbf{P}^m > \mathbf{0}$ for some positive integer m.

This straightforward definition of regularity is perhaps not quite so innocent as it might first appear. Although many Markov chains of interest do satisfy this condition, many others do not. The weather example of the previous section is regular, for although $\mathbf{P}$ itself is not strictly positive, $\mathbf{P}^2$ is. The Estes learning model is not regular since in this case $\mathbf{P}^m$ has a zero in the upper right-hand corner for each m. Similarly, the Gambler's Ruin example is not regular. In general, recalling that $\mathbf{P}^m$ is the m-step probability transition matrix, regularity means that over a sufficiently large number of steps the Markov chain must be strictly positive. There must be a positive probability associated with every transition.

The main theorem for regular chains is stated below and consists of three parts. The first part is simply a restatement of the Frobenius–Perron theorem, while the second and third parts depend on the fact that the dominant eigenvalue associated with a Markov matrix is 1.

Theorem (Basic Limit Theorem for Markov Chains). *Let* $\mathbf{P}$ *be the transition matrix of a regular Markov chain. Then:*

(a) *There is a unique probability vector* $\mathbf{p}^T > \mathbf{0}$ *such that*

$$\mathbf{p}^T\mathbf{P} = \mathbf{p}^T$$

(b) *For any initial state i (corresponding to an initial probability vector equal to the ith coordinate vector $\mathbf{e}_i^T$) the limit vector*

$$\mathbf{v}^T = \lim_{m\to\infty} \mathbf{e}_i^T \mathbf{P}^m$$

exists and is independent of i. Furthermore, $\mathbf{v}^T$ is equal to the eigenvector $\mathbf{p}^T$.

(c) $\operatorname{Lim}_{m\to\infty} \mathbf{P}^m = \bar{\mathbf{P}}$, *where $\bar{\mathbf{P}}$ is the $n \times n$ matrix, each of whose rows is equal to $\mathbf{p}^T$.*

Proof. Part (a) follows from the Frobenius–Perron theorem (Theorem 2, Sect. 6.2) and the fact that the dominant eigenvalue is $\lambda_0 = 1$. To prove part (b) we note that since $\lambda_0 = 1$ is a simple root, it follows that $\mathbf{e}_i^T\mathbf{P}^m$ must converge to a scalar multiple of $\mathbf{p}^T$. However, since each $\mathbf{e}_i^T\mathbf{P}^m$ is a probability vector, the multiple must be 1. Part (c) is really just a restatement of (b) because part (b) shows that each row of $\mathbf{P}^m$ converges to $\mathbf{p}^T$. ∎

This result has a direct probabilistic interpretation. Parts (a) and (b) together say that starting at any initial state, after a large number of steps the probability of the chain occupying state S_i is p_i, the ith component of $\mathbf{p}^T$. The long-term probabilities are independent of the initial condition.

There are two somewhat more picturesque ways of viewing this same result. One way is to imagine starting the process in some particular state, then turning away as the process moves through many steps. Then after turning back one records the current state. If this experiment is repeated a large number of times, the state S_i will be recorded a fraction p_i of the time, no matter where the process is started.

The second way to visualize the result is to imagine many copies of the Markov chain operating simultaneously. No matter how they are started, the distribution of states tends to converge to that defined by the limit probability vector.

Finally, part (c) of the theorem is essentially an alternative way of stating the same limit property. It says that the m-step transition matrix ultimately tends toward a limit $\bar{\mathbf{P}}$. This probability matrix transforms any initial probability vector into the vector $\mathbf{p}^T$.

Example 1 (The Weather Model). The weather example of Sect. 7.1 has transition matrix

$$\mathbf{P} = \begin{bmatrix} .500 & .500 & 0 \\ .500 & .250 & .250 \\ 0 & .500 & .500 \end{bmatrix}$$

This Markov chain is certainly regular since in fact $\mathbf{P}^2 > \mathbf{0}$. Indeed, computing a few powers of the transition matrix, we find

$$\mathbf{P}^2 = \begin{bmatrix} .500 & .375 & .125 \\ .375 & .438 & .187 \\ .250 & .375 & .375 \end{bmatrix}$$

$$\mathbf{P}^4 = \begin{bmatrix} .422 & .399 & .179 \\ .398 & .403 & .199 \\ .359 & .399 & .242 \end{bmatrix}$$

$$\mathbf{P}^6 = \begin{bmatrix} .405 & .401 & .194 \\ .400 & .401 & .199 \\ .390 & .400 & .210 \end{bmatrix}$$

$$\mathbf{P}^8 = \begin{bmatrix} .401 & .401 & .198 \\ .400 & .401 & .199 \\ .397 & 401 & .202 \end{bmatrix}$$

$$\mathbf{P}^{16} = \begin{bmatrix} .400 & .400 & .200 \\ .400 & .400 & .200 \\ .400 & .400 & .200 \end{bmatrix}$$

This behavior of the powers of the probability matrix is in accordance with part (c) of Theorem 1. It follows that the equilibrium probability vector is

$$\mathbf{p}^T = [.400 \quad .400 \quad .200]$$

Indeed, as an independent check it can be verified that this is a left eigenvector of the transition matrix, corresponding to the eigenvalue of 1.

The interpretation of this vector is that in the long run the weather can be expected to be sunny 40% of the days, cloudy 40% of the days, and rainy 20% of the days.

Example 2 (Simplified Monopoly). A simple game of chance and strategy for two to four players is played on the board shown in Fig. 7.4. Each player has a marker that generally moves clockwise around the board from space to space. At each player's turn he flips a coin: if the result is "heads" he moves one space, if it is "tails" he moves two spaces. A player landing on the "Go to Jail" square, goes to "Jail" where he begins at his next turn. During the game, players may acquire ownership of various squares (except "Jail" and "Go to Jail"). If a player lands on a square owned by another player, he must pay rent in the amount shown in that square to the owner. In formulating strategy for the game it is useful to know which squares are most valuable in terms of the amount of rent that they can be expected to generate. Without some analysis, the true relative values of the various squares is not apparent.

Movement of players' markers around the board can be considered to be a

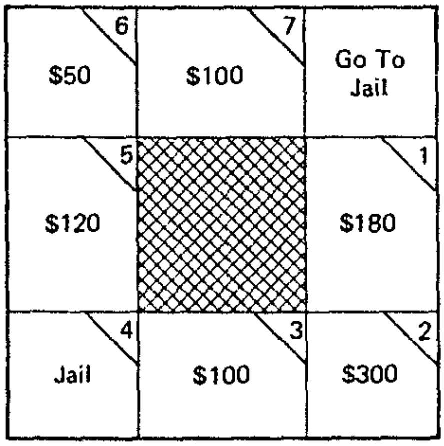

Figure 7.4. A board game.

Markov chain. The chain has seven states, corresponding to the possible landing positions on the board. The "Go to Jail" square is not counted since no piece ever really stays there, but goes instead to square number 4. The transition matrix is

$$\mathbf{P}=\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

After a bit of experimentation, it is seen that there is a finite probability of moving from any square to any square in seven steps; that is, $\mathbf{P}^7 > \mathbf{0}$. Therefore, it is quite clear that this Markov chain is regular, and there is an equilibrium probability vector that gives the long-term landing probabilities. To find the equilibrium probabilities we must solve the equation $\mathbf{p}^T\mathbf{P}=\mathbf{p}^T$ with $\sum_{i=1}^{n} p_i = 1$. Written out in detail the equations for the eigenvector* are

$$\begin{array}{lllllll@{\;}l}
 & & & & & & \tfrac{1}{2}p_7 & = p_1 \\
\tfrac{1}{2}p_1 & & & & & & & = p_2 \\
\tfrac{1}{2}p_1 & +\tfrac{1}{2}p_2 & & & & & & = p_3 \\
 & \tfrac{1}{2}p_2 & +\tfrac{1}{2}p_3 & & & +\tfrac{1}{2}p_6 & +\tfrac{1}{2}p_7 & = p_4 \\
 & & \tfrac{1}{2}p_3 & +\tfrac{1}{2}p_4 & & & & = p_5 \\
 & & & \tfrac{1}{2}p_4 & +\tfrac{1}{2}p_5 & & & = p_6 \\
 & & & & \tfrac{1}{2}p_5 & +\tfrac{1}{2}p_6 & & = p_7
\end{array}$$

* Remember that the coefficient matrix is $\mathbf{P}^T$ since $\mathbf{p}^T$

These equations could be solved successively, one after the other, if p_6 and p_7 were known. If we temporarily ignore the requirement that $\sum p_i = 1$, the value of one of the p_i's can be set arbitrarily. Let us set $p_7 = 1$. Then we find

$$\begin{aligned}
p_1 &= .50\\
p_2 &= .250\\
p_3 &= .375\\
p_4 &= .8125 + \tfrac{1}{2}p_6\\
p_5 &= .59375 + \tfrac{1}{4}p_6\\
p_6 &= .703125 + \tfrac{3}{8}p_6\\
p_7 &= 1.0
\end{aligned}$$

At this point p_6 can be found from the sixth equation and substituted everywhere else yielding

$$\begin{aligned}
p_1 &= .50\\
p_2 &= .250\\
p_3 &= .375\\
p_4 &= 1.375\\
p_5 &= .875\\
p_6 &= 1.125\\
p_7 &= 1.0
\end{aligned}$$

The actual equilibrium probability vector is obtained by dividing each of these numbers by their sum, 5.5. Thus,

$$\mathbf{p}^T = [.0909 \ \ .0455 \ \ .0682 \ \ .2500 \ \ .1591 \ \ .2045 \ \ .1818]$$

Not surprisingly we find that "Jail" is visited most frequently. Accordingly, it is clear that spaces 1–3 are visited relatively infrequently. Thus, even though these spaces have high associated rents, they are not particularly attractive to own. This is verified by the relative income rates for each square, normalized with state S_7 having an income of \$100, as shown below.

State	Rent	Relative Income	Rank
S_1	\$180.	\$ 90.00	3
S_2	\$300.	\$ 75.00	4
S_3	\$100.	\$ 37.50	6
S_4	—	—	
S_5	\$120	\$105.00	1
S_6	\$ 50.	\$ 56.25	5
S_7	\$100.	\$100.00	2

7.3 CLASSIFICATION OF STATES

Many important Markov chains have a probability transition matrix containing one or more zero entries. This is the case, for instance, in all of the examples considered so far in this chapter. If the chain is regular, the zero entries are of little fundamental consequence since there is an m such that a transition between any two states is possible in m steps. In general, however, the presence of zero entries may preclude some transitions, even over an arbitrary number of steps. A first step in the development of a general theory of Markov chains is to systematically study the structure of state interconnections.

Classes of Communicating States

We say that the state S_j is *accessible* from the state S_i if by making only transitions that have nonzero probability it is possible to begin at S_i and arrive at S_j in some finite number of steps. A state S_i is always considered to be accessible from itself.

Accessibility can be determined by taking powers of the probability transition matrix. Let $p_{ij}^{(m)}$ be the ijth element of the matrix $\mathbf{P}^m$ If $p_{ij}^{(m)}>0$, then it is possible to go from S_i to S_j in m steps, since there is a positive probability that the Markov chain would make such a transition. Thus, S_j is accessible from S_i if and only if $p_{ij}^{(m)}>0$ for some integer $m \geq 0$.

The property of accessibility is not symmetric since S_j may be accessible from S_i while S_i is not accessible from S_j. The corresponding symmetric notion is termed communication and it is this property that is used to classify states.

Definition. States S_i and S_j are said to *communicate* if each is accessible from the other.

As the following proposition shows, the concept of communicating states effectively divides the states of a Markov chain into distinct classes, each with its own identity. That is, the totality of n states is partitioned into a group of classes; each state belonging to exactly one class. Later we study the classes as units, and investigate the structure of class interconnections.

Proposition. *The set of states of a Markov chain can be divided into communicating classes. Each state within a class communicates with every other state in the class, and with no other state.*

Proof. Let C_i be the set of states that communicate with S_i. If S_k and S_j belong to C_i, they also communicate, since paths of transition between them can be found in each direction by first passing through S_i. See Fig. 7.5a. Thus, all states in C_i communicate with each other.

Suppose that a state S_l outside of C_i communicated with a state S_j within C_i. Then a path from S_l to S_j could be extended to a path from S_l to S_i by

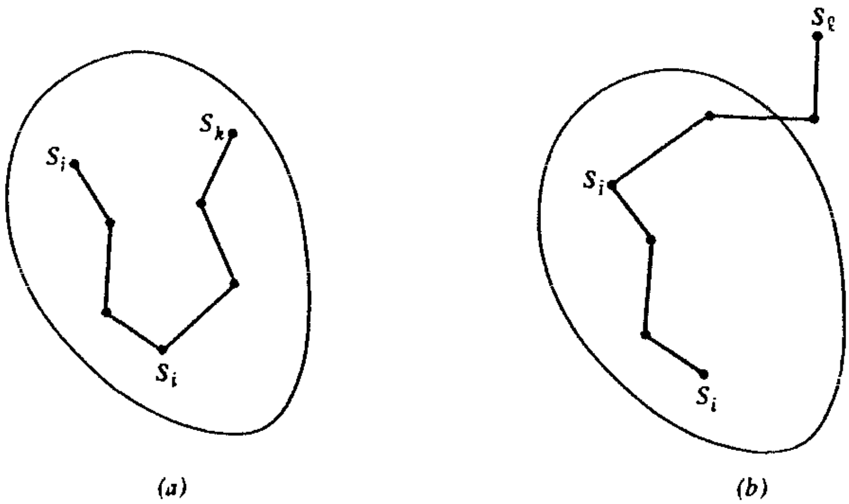

Figure 7.5. Construction for proof.

appending a path from S_j to S_i. See Fig. 7.5b. Likewise paths in the other direction could be appended. Thus, S_ℓ would communicate with S_i as well. Hence, by contradiction, no state outside C_i can communicate with any state in C_i. Therefore different communicating classes have no common states. Every state belongs to one and only one class—the class of all states that communicate with it. ∎

An important special case is when all states communicate, in which case there is only one communicating class. If a Markov chain has only one communicating class the chain is said to be *irreducible*. Otherwise, it is *reducible*.

A regular Markov chain is irreducible, since all states communicate. However, not all irreducible Markov chains are regular. An example is the chain defined by the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The chain goes from S_1 to S_2 or from S_2 to S_1 in one step. It can go from either state back to itself in two steps. However, every power of $\mathbf{P}$ contains two zero entries.

Let us apply the definitions of this section to the examples presented earlier. The weather example is irreducible, since it is possible to go from any state of weather to any other within two days. In fact, as shown earlier this chain is regular. The learning model has two states, and each is a different communicating class. Although the "*learned*" state is accessible from the "*unlearned*" state, the reverse is not true, and hence, the states do not communicate. The Gambler's Ruin chain has three communicating classes. One is the state corresponding to player A having zero chips. This state corresponds to an end of the game and no other state is accessible from it.

Similarly, a second class is the state corresponding to player A having all of the chips. Finally, the third class consists of all the other states. It is possible to go from any one of these to any other (or back) in a finite number of steps.

Closed Classes

A communicating class C is said to be *closed* if there are no possible transitions from the class C to any state outside C. In other words, no state outside C is accessible from C. Thus, once the state of a Markov chain finds its way into a closed class it can never get out. Of course, the reverse is not necessarily true. It may be possible to move into the closed class from outside.

A simple example of a closed class is provided by an irreducible Markov chain. An irreducible chain has a single communicating class consisting of all states, and it is clearly closed.

Closed classes are sometimes referred to as *absorbing classes* since they tend to ultimately absorb the process. In particular, if a closed class consists of just a single state, that state is called an *absorbing state.*

In the Estes learning model. the state corresponding to "learned" is an absorbing state. In the Gambler's Ruin problem, the two end-point states are each absorbing states.

Transient Classes

A communicating class C is *transient* if some state outside of C is accessible from C. There is, therefore, a tendency for a Markov chain process to leave a transient class.

There may be allowable transitions *into* a transient class from another class as well as out. However, it is not possible for a closed path to exist that goes first outside the class and then returns, for this would imply that there were states outside of C that communicate with states in C. The connection structure between communicating classes must have an ordered flow, always terminating at some closed class. It follows of course that every Markov chain must have at least one closed class. A possible pattern of classes together with interconnections is shown in Fig. 7.6. In this figure the individual states within a class and their individual connections are not shown; the class connections illustrated are, of course, between specific states within the class. All classes in the figure are transient, except the bottom two, which are closed.

By definition, a transient class must have at least one path leading from one of its member states to some state outside the class. Thus, if the process ever reaches the state that is connected to an outside state, there is a positive probability that the process will move out of that class at the next step. Furthermore, no matter where the process begins in the transient class, there is a positive probability of reaching that exit state within a finite number of steps.

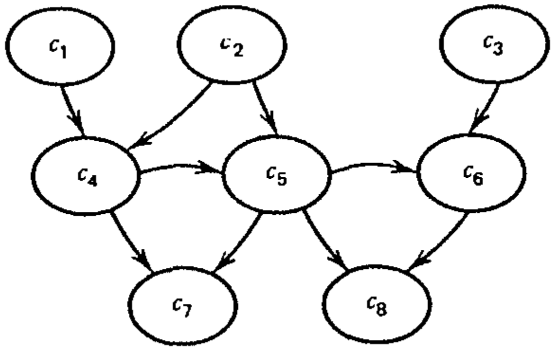

Figure 7.6. A collection of classes.

All together, over an infinite sequence of steps it seems that there is a good chance of leaving the transient class. In fact, as shown below the probability of eventually leaving is 1. (The reader can skip over the proof of this theorem without much loss of continuity.) In view of this result all states within transient classes are themselves referred to as *transient states.* The process leaves them, in favor of closed classes.

Theorem 1. *The state of a finite Markov chain is certain (with probability equal to one) to eventually enter some closed communicating class.*

Proof. From each state S_i in a transient class it is possible to reach a closed class in a finite number of steps. (See Problem 7.) Let m_i be the minimum number of steps required, and let p_i be the probability that starting at state S_i the chain will *not* reach a closed class in m_i steps. We have $p_i < 1$. Now let m be the maximum of all the m_i's, and let p be the maximum of all the p_i's. Then, starting at any state, the probability of not reaching a closed class within m steps is less than or equal to p. Likewise, the probability of not reaching a closed class within km steps, where k is a positive integer, is less than or equal to p^k. Since $p<1$, p^k goes to zero as k goes to infinity. Correspondingly, the probability of reaching a closed class within km steps is at least $(1-p^k)$, which goes to 1. ▮

Relation to Matrix Structure

The classification of states as presented in this section leads to new insight in terms of the structure of the transition probability matrix $\mathbf{P}$ and the Frobenius–Perron eigenvectors. As an example, suppose that the state S_i is an absorbing state; once this state is reached, the process never leaves it. It follows immediately that the corresponding unit vector $\mathbf{e}_i^T$ (with all components zero, except the ith, which is 1) is an eigenvector. That is, $\mathbf{e}_i^T\mathbf{P}=\mathbf{e}_i^T$. It represents a (degenerate) equilibrium distribution. If there are other absorbing states, there are, correspondingly, other eigenvectors. More generally, the equilibrium eigenvectors of $\mathbf{P}$ are associated with the closed communicating classes of $\mathbf{P}$.

These relations can be expressed in terms of a *canonical form* for Markov chains. We order the states with all those associated with closed classes first, followed by those associated with transient classes. If the states are ordered this way, the transition matrix can be written in the partitioned form

$$\mathbf{P}=\begin{bmatrix}\mathbf{P}_1 & \mathbf{0}\\ \mathbf{R} & \mathbf{Q}\end{bmatrix} \tag{7-2}$$

Assuming there are r states in closed classes and $n-r$ in transient classes, the matrix $\mathbf{P}_1$ is an $r\times r$ stochastic matrix representing the transition probabilities within the closed classes; $\mathbf{Q}$ is an $(n-r)\times(n-r)$ *substochastic* matrix (at least one row sum is less than 1) representing the transition probabilities among the transients states, and $\mathbf{R}$ is an $(n-r)\times r$ matrix representing the transition probabilities from transient states to states within a closed class.

The (left) eigenvectors corresponding to the eigenvalue of 1 must have the form $\mathbf{p}^T=[\mathbf{p}_1^T, \mathbf{0}]$, where p_1^T is r-dimensional, representing the fact that only states in closed classes can occur with positive probability in equilibrium. (See Problem 8.) The closed classes act like separate Markov chains and have equilibrium distributions. Transient classes cannot sustain an equilibrium.

7.4 TRANSIENT STATE ANALYSIS

Many Markov chains of practical interest have transient classes and are initiated at a transient state. The Gambler's Ruin problem and the Estes learning model are two examples, which we have already discussed. When considering such chains, it is natural to raise questions related to the process of movement within the transient class before eventual absorption by a closed class. Examples of such questions are: the average length of time that the chain stays within a transient class, the average number of visits to various states, and the relative likelihood of eventually entering various closed classes.

The analysis of transient states is based on the canonical form described at the end of Sect. 7.3. We assume that the states are ordered with closed classes first, followed by transient states. The resulting canonical form is

$$\mathbf{P}=\begin{bmatrix}\mathbf{P}_1 & \mathbf{0}\\ \mathbf{R} & \mathbf{Q}\end{bmatrix} \tag{7-3}$$

We assume, as before, that there are r states in closed classes and $n-r$ transient states.

The substochastic matrix $\mathbf{Q}$ completely determines the behavior of the Markov chain within the transient classes. Thus, it is to be expected that analysis of questions concerning transient behavior is expressed in terms of $\mathbf{Q}$. Actually, a central role in transient analysis is played by the matrix $\mathbf{M}=[\mathbf{I}-\mathbf{Q}]^{-1}$—this is called the *fundamental matrix* of the Markov chain when expressed in the canonical form (7-3). As demonstrated below, it is easily established that the indicated inverse exists, so that $\mathbf{M}$ is well defined.

Proposition. *The matrix* $\mathbf{M}=[\mathbf{I}-\mathbf{Q}]^{-1}$ *exists and is positive.*

Proof. It follows from Theorem 1, Sect. 7.3, that $\mathbf{Q}^m \to \mathbf{0}$ as $m \to \infty$ since elements of $\mathbf{Q}^m$ are the m-step transition probabilities within the transient classes. Thus, the dominant eigenvalue of the nonnegative matrix $\mathbf{Q}$ is less than 1. The statement of the proposition is then a special case of Theorem 2, Sect. 6.3. ∎

The elements of the fundamental matrix have a direct interpretation in terms of the average number of visits to various transient states. Suppose that the Markov chain is initiated at the transient state S_i. Let S_j be another (or the same, if $i=j$) transient state. The probability that the process moves from S_i to S_j in one step is q_{ij}. Likewise, for any k the probability of a transition from S_i to S_j in exactly k steps is $q_{ij}^{(k)}$, the ijth element of the matrix $\mathbf{Q}^k$. If we include the zero-step transition probability $q_{ij}^{(0)}$, which is the ijth element of $\mathbf{Q}^0=\mathbf{I}$, then the sum of all these transition probabilities is

$$q_{ij}^{(0)}+q_{ij}^{(1)}+q_{ij}^{(2)}+\cdots+q_{ij}^{(k)}+\cdots$$

This sum is the average number of times that starting in state S_i the process reaches state S_j before it leaves the transient states and enters a closed class. This summation can be expressed as the ijth element of the matrix sum

$$\mathbf{I}+\mathbf{Q}+\mathbf{Q}^2+\cdots+\mathbf{Q}^k+\cdots$$

However, this in turn is equal to the fundamental matrix through the identity

$$\mathbf{M}=[\mathbf{I}-\mathbf{Q}]^{-1}=\mathbf{I}+\mathbf{Q}+\mathbf{Q}^2+\cdots+\mathbf{Q}^k+\cdots$$

(See the Lemma on Series Expansion of Inverse, Sect. 6.3.) Therefore we may state the following theorem.

Theorem 1. *The element* m_{ij} *of the fundamental matrix* $\mathbf{M}$ *of a Markov chain with transient states is equal to the mean number of times the process is in transient state* S_j *if it is initiated in transient state* S_i.

Next we observe that if we sum the terms across a row of the fundamental matrix $\mathbf{M}$, we obtain the mean number of visits to all transient states for a given starting state. This figure is the mean number of steps before being absorbed by a closed class. Formally, we conclude:

Theorem 2. *Let* $\mathbf{1}$ *denote a column vector with each component equal to* 1. *In a Markov chain with transient states, the ith component of the vector* $\mathbf{M1}$ *is equal to the mean number of steps before entering a closed class when the process is initiated in transient state* S_i.

Finally, if a chain is initiated in a transient state, it will (with probability one) eventually reach some state within a closed class. There may, however, be several possible closed class entry points. We therefore turn to the question of computing the probability that, starting at a given transient state, the chain first enters a closed class through a particular state. In the special case where the closed classes each consist of a single absorbing state, this computation gives the probabilities of terminating at the various absorbing states.

Theorem 3. *Let b_{ij} be the probability that if a Markov chain is started in transient state S_i, it will first enter a closed class by visiting state S_j. Let* $\mathbf{B}$ *be the $(n-r)\times r$ matrix with entries b_{ij}. Then*

$$\mathbf{B}=\mathbf{MR}$$

Proof. Let S_i be in a transient class and let S_j be in a closed class. The probability b_{ij} can be expressed as the probability of going from S_i to S_j directly in one step plus the probability of going again to a transient state and then ultimately to S_j. Thus,

$$b_{ij}=p_{ij}+\sum_k p_{ik}b_{kj}$$

where the summation over k is carried out over all transient states. In matrix form we have $\mathbf{B}=\mathbf{R}+\mathbf{QB}$ and hence, $\mathbf{B}=[\mathbf{I}-\mathbf{Q}]^{-1}\mathbf{R}=\mathbf{MR}$. ∎

Example 1 (Learning Model). The simple Estes learning model is described by a Markov chain with transition matrix

$$\mathbf{P}=\begin{bmatrix}1 & 0\\ \alpha & 1-\alpha\end{bmatrix}$$

This matrix is already in canonical form with S_1, the "learned" state, being an absorbing state, and S_2, the "unlearned" state, being a transient state.

The $\mathbf{Q}$ matrix in this case consists of the single element $1-\alpha$. Accordingly, the fundamental matrix $\mathbf{M}$ is the single number $[1-(1-\alpha)]^{-1}=1/\alpha$.

It follows from Theorem 1 that $1/\alpha$ is the mean number of steps, starting from the unlearned state, before entering the learned state. This can vary from 1, if $\alpha=1$, to infinity, if $\alpha=0$. Theorem 2 is identical with Theorem 1 in this example, since $\mathbf{Q}$ is one-dimensional.

Theorem 3, in general gives the probabilities of entering closed classes through various states. Since in this example the closed class consists of just a single (absorbing) state, the probability of absorption by that state should be 1. Indeed the formula of the theorem specifies this probability as $\alpha(1/\alpha)=1$.

Example 2 (A Production Line). A certain manufacturing process consists of three manufacturing stages and a completion stage. At the end of each

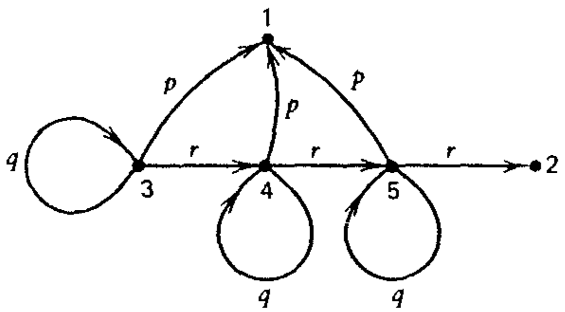

Figure 7.7. The production line chain.

manufacturing stage each item is inspected. At each inspection there is a probability p that the item will be scrapped, q that it will be sent back to that stage for reworking, and r that it will be passed on to the next stage. It is of importance to determine the probability that an item, once started, is eventually completed rather than scrapped, and to determine the number of items that must be processed through each stage. (See Fig. 7.7.)

The process can be considered to have five states:

(1) Item scrapped.
(2) Item completed.
(3) Item in first manufacturing stage.
(4) Item in second manufacturing stage.
(5) Item in third manufacturing stage.

The corresponding transition matrix is

$$\mathbf{P}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ p & 0 & q & r & 0 \\ p & 0 & 0 & q & r \\ p & r & 0 & 0 & q \end{bmatrix}$$

The first two states are absorbing states and the other three are transient states The transition matrix is in canonical form and the fundamental matrix is

$$\mathbf{M}=\begin{bmatrix} 1-q & -r & 0 \\ 0 & 1-q & -r \\ 0 & 0 & 1-q \end{bmatrix}^{-1}$$

It is easy to verify that

$$\mathbf{M}=\frac{1}{(1-q)^3}\begin{bmatrix} (1-q)^2 & r(1-q) & r^2 \\ 0 & (1-q)^2 & r(1-q) \\ 0 & 0 & (1-q)^2 \end{bmatrix}$$

The elements of the first row of this matrix are equal to the average number of times each item is passed through the first, second, and third manufacturing stages, respectively.

The probability of entering the two absorbing states is given by the elements of $\mathbf{B}=\mathbf{MR}$. In this case

$$\mathbf{B}=\mathbf{MR}=\frac{1}{(1-q)^3}\begin{bmatrix}(1-q)^2 & r(1-q) & r^2\\ 0 & (1-q)^2 & r(1-q)\\ 0 & 0 & (1-q)^2\end{bmatrix}\begin{bmatrix}p & 0\\ p & 0\\ p & r\end{bmatrix}$$

For example, the probability that an item is eventually completed rather than scrapped, starting from the first stage, is the first element of the second column of $\mathbf{B}$. That is, it is equal to $r^3/(1-q)^3$.

Example 3 (Gambler's Ruin). When there are a total of n coins or chips between the two players in a Gambler's Ruin game, it can be considered to be a Markov process with $n+1$ states $S_0, S_1, S_2, \ldots, S_n$, where S_i corresponds to player A having i coins. The states S_0 and S_n are each absorbing, while all others form one transient class. The kinds of questions one asks when studying this chain are those typical of transient state analysis—for example, the probability of player A winning, or the average duration of the game.

The transition matrix can be put in the canonical form

$$\mathbf{P}=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & 0 & 0 & & & 0\\ q & 0 & 0 & p & 0 & & & \cdot\\ 0 & 0 & q & 0 & p & & & \cdot\\ \cdot & 0 & & q & 0 & p & & \cdot\\ \vdots & \vdots & & & & \ddots & & \\ & & & & & & \ddots & \\ 0 & 0 & & & & & & p\\ 0 & p & \cdots & & & 0 & q & 0\end{bmatrix}$$

In this representation the states are ordered $S_0, S_n, S_1, S_2, \ldots, S_{n-1}$. The matrix $\mathbf{Q}$ is the $(n-1)\times(n-1)$ matrix

$$\mathbf{Q}=\begin{bmatrix}0 & p & 0 & \cdots & 0\\ q & 0 & p & & \\ & q & 0 & \ddots & \\ & & \ddots & \ddots & p\\ & & & q & 0\end{bmatrix}$$

The fundamental matrix is $\mathbf{M}=[\mathbf{I}-\mathbf{Q}]^{-1}$, but an explicit formula for this inverse is somewhat difficult to find. It is, however, not really necessary to have an explicit representation for many purposes. Indeed, for this example, just as for many other highly structured problems, it is possible to convert the general expressions of Theorems 2 and 3 into alternative and much simpler dynamic problems.

Let us, for example, compute the probability of player A winning, starting from various states S_k. According to Theorem 3 the vector $\mathbf{x}=\mathbf{M}\mathbf{r}_2$, where $\mathbf{r}_2^T=[0\ 0\ 0\ 0\ \cdots\ p]$, has components equal to these probabilities. The vector $\mathbf{x}$ satisfies the equation

$$[\mathbf{I}-\mathbf{Q}]\mathbf{x}=\mathbf{r}_2$$

where $\mathbf{x}=(x_1, x_2, \ldots, x_{n-1})$. Written out in greater detail, this vector equation is

$$x_1-px_2=0$$

$$x_k-qx_{k-1}-px_{k+1}=0, \qquad 2\le k\le n-2$$

$$x_{n-1}-qx_{n-2}=p$$

Defining the additional variables $x_0=0$, $x_n=1$, the first and last equations can be expanded to have the same form as the second. In this way the above system can be expressed as the single difference equation

$$x_k-qx_{k-1}-px_{k+1}=0, \qquad k=1,2,\ldots,n-1$$

This is the difference equation identical with that used to solve this problem in Chapter 2; and it can be solved as shown there.

In a similar fashion the average length of the game, starting from various states, can be found by application of Theorem 2. These lengths are the components of the vector $\mathbf{y}=\mathbf{M}\mathbf{1}$. Equivalently, the vector $\mathbf{y}$ can be found as the solution to the equation $[\mathbf{I}-\mathbf{Q}]\mathbf{y}=\mathbf{1}$. Again with $\mathbf{y}=(y_1, y_2, \ldots, y_{n-1})$ and defining $y_0=0$, $y_n=0$, the vector equation for $\mathbf{y}$ can be written as the difference equation

$$y_k-qy_{k-1}-py_{k+1}=1, \qquad k=1,2,\ldots,n-1$$

This equation can be solved by the techniques of Chapter 2. The characteristic equation is

$$\lambda-q-p\lambda^2=0$$

which has roots $\lambda=1$, $\lambda=q/p$. Assuming $p\ne q\ne\frac{1}{2}$, the general solution to the difference equation has the form

$$y_k=A+Bk+C(q/p)^k$$

The constants A and C are arbitrary, since they correspond to the general solution of the homogeneous equation. The constant B is found by temporarily setting $A=0$, $C=0$ and substituting $y_k=Bk$ into the difference equation, obtaining

$$Bk-qB(k-1)-pB(k+1)=1$$
$$-pB+qB=1$$
$$B=\frac{1}{q-p}$$

The actual values of A and C are found by setting $y_0=0$ and $y_n=0$ yielding

$$0=A+C$$
$$0=A+\frac{n}{q-p}+C\left(\frac{q}{p}\right)^n$$

Solving for A and C and substituting into the general form leads to the final result

$$y_k=\frac{n}{(q-p)[1-(q/p)^n]}\left[-1+\left(\frac{q}{p}\right)^k\right]+\frac{k}{q-p}$$

*7.5 INFINITE MARKOV CHAINS

In some applications it is natural to formulate Markov chain models having a (countably) infinite number of states $S_1, S_2, \ldots, S_k, \ldots$. An infinite Markov chain often has greater symmetry, and leads to simpler formulas than a corresponding finite chain obtained by imposing an artificial termination condition. This simplicity of structure justifies the extension of concepts to infinite Markov chains.

Example (Infinite Random Walk with Reflecting Barrier). An object moves on a horizontal line in discrete unit steps. Its possible locations are given by the nonnegative integers $0, 1, 2, \ldots$. If the object is at position $i>0$, there is a probability p that the next transition will be to position $i+1$, a probability q that it will be to $i-1$, and a probability r that it will remain at i. If it is at position 0, it will move to position 1 with probability p and remain at 0 with probability $1-p$. The transition matrix for this infinite chain is

$$\mathbf{P}=\begin{bmatrix} 1-p & p & 0 & 0 & \cdots \\ q & r & p & 0 & \cdots \\ 0 & q & r & p & \cdots \\ 0 & 0 & q & r & \cdots \\ \vdots & & & & \end{bmatrix}$$

Although the technicalities associated with infinite Markov chains are somewhat more elaborate than for finite chains, much of the essence of the finite theory is extendible to the infinite chain case. Of particular importance is that there is an extended version of the basic limit theorem. This extension is presented in this section, without proof.

The concepts of accessibility, communicating classes, and irreducibility carry over directly to infinite chains. The definitions given earlier apply without change in the infinite case. For example, S_j is accessible from S_i if there is a path (of finite length) from S_i to S_j. In addition to these definitions, it is useful to introduce the concept of an aperiodic Markov chain. To illustrate this concept, consider again the two-dimensional chain with transition matrix

$$\mathbf{P}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

This chain is irreducible since each of the two states is accessible from the other. However, in this chain a transition from either state back to the same state always requires an even number of steps. A somewhat more complex example is represented by the matrix

$$\mathbf{P}=\begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ 1 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}$$

Starting at S_1 on Step 0, S_1 will be visited on all even numbered steps, while either S_2 or S_3 are visited on odd number steps. In general, if some state in a finite or infinite Markov chain has the property that repeated visits to that state are always separated by a number of steps equal to a multiple of some integer greater than 1, that state is said to be *periodic*. A Markov chain is *aperiodic* if it has no periodic states.

The first important result for infinite chains is an extension of part (b) of the Basic Limit Theorem of Sect. 7.2.

Theorem. *For an irreducible, aperiodic Markov chain the limits*

$$v_i=\lim_{m\to\infty} p_{ii}^{(m)}$$

exist and do not depend on the initial state i.

This theorem tells us that, just as in the finite case, the process settles down with each state having a limiting probability. It is quite possible, however, in the infinite case, that the limits might all be zero.

As an example, let us refer to the infinite random walk described above. Provided that $p>0$, $q>0$, it is clear that every state communicates with every other state since there is a path of nonzero transition probabilities from any state to any other. Thus, the chain is irreducible. The chain is aperiodic since one can return to a state by resting at 0 indefinitely. Assuming $p>0$, $q>0$, the conclusion of Theorem 1 must hold. However, if $p>q$, there is a strong tendency for the moving object to drift toward the right, to higher integer points. The chance that the object returns to a specific point, say point 0, infinitely often is likely to be quite small. Indeed in this case, the limits v_j are all zero. the process drifts continually to the right so that each state has the character of a transient state.

Definition. An irreducible aperiodic Markov chain is said to be *positive recurrent* if

(a) $v_j = \lim_{m\to\infty} p_{ij}^{(m)} > 0$ for all j, and
(*b*) $\sum_j v_j = 1$.

According to this definition, a chain is positive recurrent if the limit probabilities form a legitimate infinite-dimensional probability vector. The next theorem establishes the relation between these limit probabilities and the existence of an infinite-dimensional eigenvector of the transition matrix.

Theorem. *Given an irreducible aperiodic Markov chain.*

(a) *It is positive recurrent if and only if there is a unique probability distribution* $\mathbf{p} = (p_1, p_2, \ldots)$ *(satisfying $p_i > 0$ for all i, $\sum_i p_i = 1$), which is a solution to*

$$p_j = \sum_i p_{ij} p_i$$

In this case,

$$p_j = v_j = \lim_{m\to\infty} p_{ij}^{(m)}$$

for all j.

(b) *If the chain is not positive recurrent, then*

$$v_j = \lim_{m\to\infty} p_{ij}^{(m)} = 0$$

for all j.

Example (continued). In the random walk, suppose $p>0$, $r>0$, and $q>0$. Let us attempt to find the v_i's by seeking an eigenvector of the transition matrix.

The v_i's should satisfy the equations

$$pv_0 - qv_1 = 0$$
$$(1-r)v_j - pv_{j-1} - qv_{j+1} = 0, \qquad j = 1, 2, 3, \ldots$$

The characteristic equation of the difference equation is $(1-r)\lambda - p - q\lambda^2 = 0$, which has roots $\lambda = 1$, p/q. If $p < q$, a solution is

$$v_j = (1 - p/q)(p/q)^j$$

and this satisfies $\sum_{i=1}^{\infty} v_j = 1$. Therefore, for $p < q$ the solution is positive recurrent. If $p > q$, no solution can be found, and hence, as suspected (because of the drift toward infinity) the chain is not positive recurrent.

7.6 PROBLEMS

1. Let $\mathbf{P}$ be an $n \times n$ stochastic matrix and let $\mathbf{1}$ denote the n-dimensional column vector whose components are all 1. Show that $\mathbf{1}$ is a right eigenvector of $\mathbf{P}$ corresponding to an eigenvalue of 1. Conclude that if $\mathbf{y}^T = \mathbf{x}^T\boldsymbol{P}$, then $\mathbf{y}^T\mathbf{1} = \mathbf{x}^T\mathbf{1}$.

2. *Social Mobility.* The elements of the matrix below represents the probability that the son of a father in class i will be in class j. Find the equilibrium distribution of class sizes.

Upper class	.5	.4	.1
Middle class	.1	.7	.2
Lower class	.05	.55	.4

3. *Ehrenfest Diffusion Model.* Consider a container consisting of two compartments A and B separated by a membrane. There is a total of n molecules in the container. Individual molecules occasionally pass through the membrane from one compartment to the other. If at any time there are j molecules in compartment A, and $n-j$ in compartment B, then there is a probability of j/n that the next molecule to cross the membrane will be from A to B, and a probability of $(n-j)/n$ that the next crossing is in the opposite direction.
   (a) Set up a Markov chain model for this process. Is it regular?
   (b) Show that there is an equilibrium probability distribution such that the probability p_j that j molecules are in compartment A is

$$p_j = \frac{1}{2^n}\binom{n}{j}$$

4. *Languages.* The symbols of a language can be considered to be generated by a Markov process. As a simple example consider a language consisting of the symbols A, B, and S (space). The space divides the symbol sequence into words. In this language two B's or two S's never occur together. Three A's never occur together. A word never starts with AB or ends with BA. Subject to these restrictions, at any point, the next symbol is equally likely to any of the allowable possibilities.

Formulate a Markov chain for this language. What are the equilibrium symbol probabilities? (*Hint*: Let some states represent pairs of symbols.)

5. *Entropy*. The *entropy* of a probability vector $\mathbf{p}^T = (p_1, p_2, \ldots, p_n)$ is $H(\mathbf{p}) = -\sum_{i=1}^{n} p_i \log_2 p_i$. Entropy is a measure of the uncertainty associated with a selection of n objects, when the selection is made according to the given probabilities.

   (a) For a fixed n show that $H(p)$ is maximized by $\mathbf{p}^T = [1/n, 1/n, \ldots, 1/n]$.
   (b) For a regular finite Markov chain with transition matrix $\mathbf{P}$, the entropy is

$$H(\mathbf{P}) = \sum_{i=1}^{n} p_i H_i$$

   where p_i is the equilibrium probability of state S_i and where H_i is the entropy of the ith row of $\mathbf{P}$. Thus, H is a weighted average of the entropy associated with the choice of the next state of the chain. Find the entropy of the weather model in Sect. 7.2.

*6. *Equivalence Classes.* Consider a set X and a relation R that holds among certain pairs of elements of X. One writes xRy if x and y satisfy the relation. The relation R is said to be an *equivalence relation* if

$$\begin{aligned} &xRx \quad \text{for all } x \text{ in } X \\ &xRy \quad \text{implies} \quad yRx \\ &xRy \quad \text{and} \quad yRz \quad \text{implies} \quad xRz \end{aligned}$$

   (a) Let $[x]$ denote the collection of all elements y satisfying xRy, where R is an equivalence relation. This set is called the equivalence class of x. Show that X consists of a disjoint collection of equivalence classes.
   (b) Let X be the set of states in a Markov chain and let R be the relation of communication. Show that R is an equivalence relation.

7. Show that from any state in a Markov chain it is possible to reach a closed class within a finite number of transitions having positive probability.

8. Suppose the probability transition matrix of a finite Markov chain is in the canonical form (7-2). Show that any left eigenvector corresponding to an eigenvalue of magnitude of 1 must be of the form $[\mathbf{p}_1^T, \mathbf{0}]$, where $\mathbf{p}_1^T$ is r dimensional.

*9. An $n \times n$ matrix $\mathbf{P}$ is a *permutation matrix* if for all vectors $\mathbf{x}$ the components of the vector $\mathbf{P}\mathbf{x}$ are simply a reordering of the components of $\mathbf{x}$. Show that all elements of a permutation matrix $\mathbf{P}$ are either zero or one, and both $\mathbf{P}$ and $\mathbf{P}^T$ are stochastic matrices.

*10. *Theory of Positive Matrices.* An $n \times n$ matrix $\mathbf{A}$ is said to be *reducible* if there is a nonempty proper subset J of $\{1, 2, \ldots, n\}$ such that

$$a_{ij} = 0 \quad \text{for} \quad i \notin J, j \in J$$

(a) Show that $\mathbf{A}$ is reducible if and only if there is a permutation matrix $\mathbf{T}$ (see Problem 9) such that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{0} & \mathbf{A}_{22}\end{bmatrix}$$

where $\mathbf{A}_{11}$ is square.

(b) Let $\mathbf{P}$ be a stochastic matrix associated with a finite Markov chain. Show that the chain has a single communicating class if and only if $\mathbf{P}$ is irreducible.

(c) Let $\mathbf{A}$ be a nonnegative irreducible matrix, with Frobenius–Perron eigenvalue and eigenvector λ_0, $\mathbf{x}_0$, respectively. Show that $\lambda_0>0$, $\mathbf{x}_0>\mathbf{0}$, and that λ_0 is a simple root of the characteristic polynomial of $\mathbf{A}$.

11. *Periodic Positive Matrices.* Let $\mathbf{A}$ be an irreducible positive $n\times n$ matrix (see Problem 10). Then it can be shown that there is a permutation matrix $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T}=\begin{bmatrix}\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{G}_r\\ \mathbf{G}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{G}_2 & \ddots & & \\ & & & & \\ & & & \mathbf{G}_{r-1} & \mathbf{0}\end{bmatrix}$$

where the zero matrices on the main diagonal are square. Let $\lambda_0>0$ be the Frobenius–Perron eigenvalue of $\mathbf{A}$. Show that $\mathbf{A}$ has r eigenvalues of magnitude λ_0. [*Hint:* Let ω be an rth root of unity (that is, $\omega^r=1$). Show that $\lambda=\omega\lambda_0$ is an eigenvalue.]

12. *Finite Random Walk.* An object moves on a horizontal line in discrete steps. At each step it is equally likely to move one unit to right or one unit to the left. The line is a total of five units long, and there are absorbing barriers at either end. Set up the Markov chain corresponding to this random walk process. Characterize each state as transient or absorbing. Calculate the canonical form and find the fundamental matrix.

13. *First Passage Time.* Suppose $\mathbf{P}$ is the probability transition matrix of a regular Markov chain. Given an initial state $S_i\neq S_1$, show how by modifying $\mathbf{P}$ the average number of steps to reach S_1 can be computed. For the weather model, given that today is rainy, what is the expected number of days until it is sunny?

14. *Markov Chains with Reward.* You might consider your automobile and its random failures to be a Markov chain. It makes monthly transitions between the states "running well" and "not running well." When it is not running well it must be taken to a garage to be repaired, at a cost of \$50. It is possible to improve the likelihood that the automobile will continue to run well by having monthly service at a cost of \$10. Depending on your policy your automobile transitions will be governed

by either of two Markov chains, defined by

$$\mathbf{P}_1 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{7}{8} & \frac{1}{8} \end{bmatrix}, \qquad \mathbf{P}_2 = \begin{bmatrix} \frac{7}{8} & \frac{1}{8} \\ \frac{7}{8} & \frac{1}{8} \end{bmatrix}$$

where S_1 = running well, S_2 = not running well, and where $\mathbf{P}_1$ corresponds to no monthly service, and $\mathbf{P}_2$ corresponds to having monthly service.

(a) For each of the two policies, what is the equilibrium distribution of states?
(b) In equilibrium, what is the average monthly cost of each of the two policies?

15. *Simplified Blackjack.* A game between a "dealer" and a "customer" is played with a (very large) mixed deck of cards consisting of equal numbers of ones, twos, and threes. Two cards are initially dealt to each player. After looking at his cards the customer can elect to take additional cards one at a time until he signals that he will "stay." If the sum of the values of his cards exceeds six, he loses. Otherwise the dealer takes additional cards one at a time until his sum is five or more. If his sum exceeds six, the customer wins. Otherwise, the player with the highest sum (under seven) wins. Equal values under seven result in a draw.

    (a) Set up a Markov chain for the process of taking cards until a value of five or more is obtained. Identify the absorbing states. Find the probabilities of entering the various absorbing states for each initial sum. Find the probability of entering various absorbing states.
    (b) If the customer follows the strategy of taking cards until his value is five or more, at what rate will he lose?
    *(c) If the second card dealt to the dealer is face up, the customer can base his strategy on the value of that card. For each of the three possible cards showing, at what sum should the customer stay? What are the odds in this case?

16. *Periodic States.* Show that if one state in a given communicating class is periodic, then all states in that class are periodic.

17. For the following special cases of the infinite random walk, determine if (i) it is aperiodic, (ii) irreducible, and (iii) there is a solution to the eigenvector problem.

    (a) $r=0, p>q>0$.
    (b) $r>0, p=q>0$.

18. Suppose a (nonfair) coin is flipped successively with the probability of heads or tails on any trial being p and $1-p$, respectively. Define an infinite Markov chain where state S_j corresponds to a landing of the coin that represents a run of exactly j heads on the most recent flips. Show that this Markov chain is aperiodic and irreducible. Is it positive recurrent?

19. *Dynamics of Poverty and Wealth.* Income distribution is commonly represented by the distribution function $D(y)$, which measures the number of individuals with incomes exceeding y. It has been observed that with surprising regularity, in various

countries over long histories, the upper end of the distribution closely approximates the form

$$D(y) = Cy^{-\alpha}$$

for some parameters C and α. This is referred to as *Pareto's law.*

One theoretical explanation for the regularity encompassed by Pareto's law is based on a Markov chain model of individual income developed by Champernowne. In this model the income scale is divided into an infinity of income ranges. The income ranges are taken to be of uniform proportionate length; that is, they might be \$50 to \$100, \$100 to \$200, \$200 to \$400, and so on. At any one time a given individual's income falls within one of these ranges. At the next period (and periods might correspond to years), his or her income makes a transition, either upward or downward, according to a set of probabilities that are characteristic of the particular economy in which the person lives.

It is assumed that no income moves up by more than one or down by more than n income ranges in a period, where $n \geq 1$ is a fixed integer. Specifically, it is assumed that there are $n+2$ positive numbers, $p_{-n}, p_{-n+1}, \ldots, p_0, p_1$ such that

$$1 = \sum_{u=-n}^{1} p_u$$

Then, the transition probabilities for the process are defined as

$$p_{ij} = p_{j-i} \quad \text{for} \quad -n \leq j-i \leq 1, j \geq 1$$

$$p_{i0} = 1 - \sum_{r=0}^{i} p_{1-r} \quad \text{for} \quad 0 \leq i \leq n$$

$$p_{ij} = 0 \text{ otherwise}$$

The pattern of transition probabilities is illustrated in Fig. 7.8 for $n = 2$. (This model can be viewed as an extension of the infinite random walk.)

Finally, it is assumed that the average number of income ranges moved in one step is negative when at the upper levels. That is,

$$-np_{-n} + (-n+1)p_{-n+1} + \cdots + (-1)p_{-1} + 0 \cdot p_0 + p_1 < 0$$

(a) Verify that this Markov chain is irreducible and aperiodic.

(b) Let

$$F(\lambda) = -\lambda + \sum_{u=-n}^{1} p_u \lambda^{1-u}$$

and show that $F(1) = 0$, $F'(1) > 0$, and $F(0) = p_1 > 0$. Conclude that there is a root λ to the equation $F(\lambda) = 0$ in the range $0 < \lambda < 1$.

(c) Find an equilibrium probability vector.

(d) Show that the equilibrium income distribution satisfies Pareto's law.

$i \backslash j$	0	1	2	3	4	5	6	
0	$1-p_1$	p_1	0	0	0	0	0	$\cdots$
1	$1-p_0-p_1$	p_0	p_1	0	0	0	0	$\cdots$
2	$1-p_{-1}-p_0-p_1$	p_{-1}	p_0	p_1	0	0	0	$\cdots$
3	0	p_{-2}	p_{-1}	p_0	p_1	0	0	$\cdots$
4	0	0	p_{-2}	p_{-1}	p_0	p_1	0	$\cdots$
5	0	0	0	p_{-2}	p_{-1}	p_0	p_1	$\cdots$
6	0	0	0	0	p_{-2}	p_{-1}	p_0	$\cdots$
7								
.								

Figure 7.8. Champernowne model

NOTES AND REFERENCES

Sections 7.1–7.4. For general introductions to finite Markov chains and additional applications see Bhat [B8], Howard [H7], Kemeny and Snell [K11], or Clarke and Disney [C4], which is close to our level and order of presentation and considers some of the same examples. Bartholomew [B2] has several examples from the social sciences. The learning model is based on Estes [E1]. The landing probabilities for real game of *Monopoly* are available in tabulated form; see, for example, Walker and Lehman [W1].

Section 7.5. For more advanced treatments of Markov chains, including infinite chains, see Karlin [K8] and Feller [F1].

Section 7.6. Entropy of a Markov process (Problem 5) forms the basis of information theory as developed by Shannon; see Shannon and Weaver [S4]. For further discussion of the theory of positive matrices as presented in Problems 10 and 11, see Karlin [K7], Nikaido [N1], or Gantmacher [G3]. A full theory of Markov processes with reward (Problem 14) has been developed by Howard [H6]. The optimal strategy for real blackjack was found (by computer simulation) by Thorpe [T2]. The model of Problem 19 together with various modifications is due to Champernowne [C2].